Stat Bayes:

EXO1:

$$\pi(\cdot/x) = \frac{f(x,\theta)}{m_\pi(x)} = \frac{\pi(x/\theta)\cdot f(x)}{m_\pi(x)}$$

$$\propto \cancel{\frac{1}{\sqrt{2\pi}}}$$

$$\propto \frac{1}{\sqrt{2\pi\sigma_0^2}} \exp\left(\frac{-1}{2\sigma_0^2}(\mu-\mu_0)^2\right) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$$

$$\propto \exp\left(\frac{-1}{2\sigma_0^2}(\mu-\mu_0)^2 - \frac{1}{2\sigma^2}(x-\mu)^2\right)$$

$$\propto \exp\left(-\frac{\eta}{2}(x-\mu)^2 - \frac{\eta_0}{2}(\mu-\mu_0)^2\right)$$

$$\propto \exp\left(-\frac{\eta}{2}(x^2-2x\mu+\mu^2) - \frac{\eta_0}{2}(\mu^2-2\mu\mu_0+\mu_0^2)\right)$$

$$\propto \exp\left(-\frac{\eta x^2}{2} + x\mu\eta - \frac{\eta\mu^2}{2} - \frac{\eta_0\mu^2}{2} + \eta_0\mu\mu_0 - \frac{\eta_0\mu_0^2}{2}\right)$$

$$\propto \exp\left(-\frac{(\eta+\eta_0)}{2}\mu^2 + (x\eta+\eta_0\mu_0)\mu - \frac{\eta x^2}{2} - \frac{\eta_0\mu_0^2}{2}\right)$$

$$\propto \exp\left(-\frac{(\eta+\eta_0)}{2}\mu^2 + (x\eta+\eta_0\mu_0)\mu\right)$$

$$\propto \exp\left(-\frac{(\eta+\eta_0)}{2}\right)\left[\mu^2 - \frac{2(x\eta+\eta_0\mu_0)}{\eta+\eta_0}\mu\right]$$

$$\pi(\cdot/x) \sim N\left(\frac{\eta x+\eta_0\mu_0}{\eta+\eta_0}, \frac{1}{\eta+\eta_0}\right)$$

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2) $E(\pi(\cdot/x)) = \dfrac{nx + n_0\mu_0}{n + n_0}$ $E(\pi(\theta)) = \mu_0$

$\& \ Var(\pi(\theta/x)) = \dfrac{1}{n+n_0} < \dfrac{1}{n_0} = Var(\pi(\hat{\theta}))$

3) Fonction de perte quadratique

$\delta^{\pi} = E(\pi(\theta/X)) = \dfrac{nx + n_0\mu_0}{n + n_0}$

4) 5)

a) $\hat{\theta}_{MAP} = \text{Argmax}\, f(\theta/X_1 \_ X_n) = E(\hat{\theta}/X) = \dfrac{n n\bar{X} + n_0\mu_0}{n n + n_0}$

$\lim_{n \to +\infty} \hat{\theta}_{MAP} = \lim_{+\infty} \bar{X}_n = E(X) = \mu.$

Ex 2

1) $f(x, p) = C_n^x p^x (1-p)^{n-x} \times \dfrac{1}{\beta(\frac{1}{2}, \frac{1}{2})} p^{-\frac{1}{2}} (1-p)^{-\frac{1}{2}}$

$= \dfrac{C_n^x}{\beta(\frac{1}{2}, \frac{1}{2})} p^{x-\frac{1}{2}} (1-p)^{n-x-\frac{1}{2}}$

2) $m_{\pi}(x) = \int_0^1 f(x, t)\, dt$

$= \dfrac{C_n^x}{\beta(\frac{1}{2}, \frac{1}{2})} \underbrace{\int_0^1 t^{x-\frac{1}{2}} (1-t)^{n-x-\frac{1}{2}}\, dt}_{\beta(x+\frac{1}{2},\, n-x+\frac{1}{2})}$

$= \dfrac{C_n^x}{\beta(\frac{1}{2}, \frac{1}{2})} \beta(x+\frac{1}{2}, n-x+\frac{1}{2}).$

3) $\pi(\cdot/x) = \dfrac{f(p,x)}{m_\pi(x)} = \dfrac{p^{\overbrace{x+\frac{1}{2}-1}^{a-1}}(1-p)^{\overbrace{n-x-\frac{1}{2}+1-1}^{b-1}}}{\beta(\underbrace{x+\frac{1}{2}}_{a}, \underbrace{n-x+\frac{1}{2}}_{b})}$

$= \beta\left(x+\frac{1}{2}, n-x+\frac{1}{2}\right)$

$\delta^{\pi} = \dfrac{a}{a+b} = \dfrac{x+\frac{1}{2}}{x+\frac{1}{2}+n-x+\frac{1}{2}} = \dfrac{x+\frac{1}{2}}{n+1}$

EXO 4

1) $\mathcal{L}(x/\theta=\theta) = \cancel{P(x/\theta)}\ f(x_1 \ldots x_n/\theta)$

$= \prod f(x_i/\theta)$

$= \prod_{i=1}^{n}\left[\dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{1}{2\sigma^2}(x-\theta)^2\right)\right]$

$= \dfrac{1}{\sqrt{2\pi\sigma^2}^{\,n}} \exp -\dfrac{1}{2\sigma^2}\left[\sum (x_i-\theta)^2\right].$

$f(x_1 \ldots x_n, \theta) = \left(\dfrac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(\dfrac{-1}{2\sigma^2}\sum(x_i-\theta)^2\right)\exp\left(\dfrac{-1}{2b^2}(\theta-a)^2\right) \times \left(\dfrac{1}{2\pi b^2}\right)^{1/2}$

EXO 6

$\mathcal{L}(\theta, \delta) = |\theta - \delta|$

$$3)\ \pi(\cdot/x) = \frac{f(p,x)}{m_\pi(x)} = \frac{p^{\overbrace{x+\frac{1}{2}-1}^{a-1}}(1-p)^{\overbrace{n-x-\frac{1}{2}+1-1}^{b-1}}}{\beta(\underbrace{x+\frac{1}{2}}_{a}, \underbrace{n-x+\frac{1}{2}}_{b})}$$

$$= \beta\left(x+\frac{1}{2}, n-x+\frac{1}{2}\right)$$

$$\delta^\pi = \frac{a}{a+b} = \frac{x+\frac{1}{2}}{x+\frac{1}{2}+n-x+\frac{1}{2}} = \frac{x+\frac{1}{2}}{n+1}$$

EXO 4

$$1)\ L(x/\theta=\theta) = \cancel{f(x/\theta)}\ f(x_1 \ldots x_n/\theta)$$

$$= \prod f(x_i/\theta)$$

$$= \prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\theta)^2\right)\right]$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}^{\,n}} \exp -\frac{1}{2\sigma^2}\left[\sum (x_i-\theta)^2\right].$$

$$f(x_1 \ldots x_n, \theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left(\frac{-1}{2\sigma^2}\sum(x_i-\theta)^2\right)\exp\left(\frac{-1}{2b^2}(\theta-a)^2\right) \times \left(\frac{1}{2\pi b^2}\right)^{1/2}$$

EXO 6

$$L(\theta, \delta) = |\theta - \delta|$$

$$r(\theta,\delta/x) = \int_{\Theta=\mathbb{R}} L(\theta,\delta)\,.\,\pi(\theta/x)\,d\theta$$

$$= \int_{\mathbb{R}} |\theta-\delta|\,\pi(\theta/x)\,d\theta$$

$$= \int_{-\infty}^{\delta} (\delta-\theta)\,\pi(\theta/x)\,d\theta + \int_{\delta}^{+\infty} (\theta-\delta)\,\pi(\theta/x)\,d\theta$$

$$= \delta\left(\int_{-\infty}^{\delta} \pi(\theta/x) - \int_{\delta}^{+\infty} \pi(\theta/x)\,d\theta\right) - \int_{-\infty}^{\delta} \theta\,\pi(\theta/x)\,d\theta$$

$$+ \int_{\delta}^{+\infty} \theta\,\pi(\theta/x)\,d\theta.$$

$$\frac{\partial r(\theta,\delta/x)}{\partial\delta} = \int_{-\infty}^{\delta} \pi(\theta/x) - \int_{\delta}^{+\infty} \pi(\theta/x) + \delta\left[\pi(\delta/x) + \pi(\delta/x)\right]$$

$$- \delta\,\pi(\delta/x) - \delta\,\pi(\delta/x)$$

$$= \int_{-\infty}^{\delta} \pi(\theta/x) - \int_{\delta}^{+\infty} \pi(\theta/x)\,d\theta - \int_{-\infty}^{\delta} \pi(\theta/x)$$

$$+ \int_{-\infty}^{\delta} \pi(\theta/x)$$

$$= 2\int_{-\infty}^{\delta} \pi(\theta/x) - \int_{\mathbb{R}} \underbrace{\pi(\theta/x)}_{=1}$$

$$\frac{\partial r}{\partial\delta} = 0 \Rightarrow \int_{-\infty}^{\delta} \pi(\theta/x) = \frac{1}{2}$$

δ est la mediane de la Loi à posteriori $\pi(\theta/x)$

$$r(\theta,\delta/x) = \int_{\Theta=\mathbb{R}} L(\theta,\delta)\cdot\pi(\theta/x)\,d\theta.$$

$$= \int_{\mathbb{R}} |\theta-\delta|\,\pi(\theta/x)\,d\theta$$

$$= \int_{-\infty}^{\delta} (\delta-\theta)\,\pi(\theta/x)\,d\theta + \int_{\delta}^{+\infty} (\theta-\delta)\,\pi(\theta/x)\,d\theta$$

$$= \delta\left(\int_{-\infty}^{\delta}\pi(\theta/x) - \int_{\delta}^{+\infty}\pi(\theta/x)\,d\theta\right) - \int_{-\infty}^{\delta}\theta\,\pi(\theta/x)\,d\theta$$

$$+ \int_{\delta}^{+\infty}\theta\,\pi(\theta/x)\,d\theta.$$

$$\frac{\partial r(\theta,\delta/x)}{\partial\delta} = \int_{-\infty}^{\delta}\pi(\theta/x) - \int_{\delta}^{+\infty}\pi(\theta/x) + \delta\left[\pi(\delta/x) + \pi(\delta/x)\right]$$

$$- \delta\,\pi(\delta/x) - \delta\,\pi(\delta/x)$$

$$= \int_{-\infty}^{\delta}\pi(\theta/x) - \int_{\delta}^{+\infty}\pi(\theta/x)\,d\theta - \int_{-\infty}^{\delta}\pi(\theta/x)$$

$$+ \int_{-\infty}^{\delta}\pi(\theta/x)$$

$$= 2\int_{-\infty}^{\delta}\pi(\theta/x) - \int_{\mathbb{R}}\underbrace{\pi(\theta/x)}_{=1}$$

$$\frac{\partial r}{\partial\delta} = 0 \Rightarrow \int_{-\infty}^{\delta}\pi(\theta/x) = \frac{1}{2}$$

δ est la mediane de la Loi à posteriori $\mathcal{L}(\hat{\theta}/x)$

Loi de Jeffreys : unidim. $\pi(\theta) \propto \sqrt{I_n(\theta)} \propto \sqrt{I(\theta)}$

multi $\pi(\theta) \sim \sqrt{\det(I_n(\theta))} \propto \sqrt{\det I(\theta)}$

Ex 1.

1) $I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \log f(x/\theta)\right]$

$f(x/\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\theta)^2\right)$

$\log f(x/\theta) \propto -\frac{1}{2\sigma^2}(x-\theta)^2$

$\frac{\partial}{\partial\theta} \log(f(x/\theta)) \propto -\frac{1}{2\sigma^2}(2\theta-2x)$

$\frac{\partial^2}{\partial\theta^2} \log(f(x/\theta)) \propto -\frac{1}{\sigma^2} \Rightarrow I(\theta) = \frac{1}{\sigma^2}$

Loi de Jeffreys $\pi(\theta) \propto \sqrt{\frac{1}{\sigma^2}} = \frac{1}{\sigma}$

2) $\pi(\mu/x_1 \ldots x_n) \propto f(x/\theta) \cdot \pi(\theta)$

$\propto \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-1}{2\sigma^2}(x-\theta)^2\right) \cdot \frac{1}{\sigma}$

$\propto \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^n \exp\left(\frac{-1}{2\sigma^2}\sum(x_i-\theta)^2\right)$

$\propto \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left(\frac{-n}{2\sigma^2}(\bar{x}_n-\theta)^2\right)$

$\propto \exp\left(\frac{-n}{2\sigma^2}(\bar{x}_n-\theta)^2\right)$

$\pi(\theta/x_1 \ldots x_n) \sim N\left(\bar{x}_n, \frac{\sigma^2}{n}\right)$

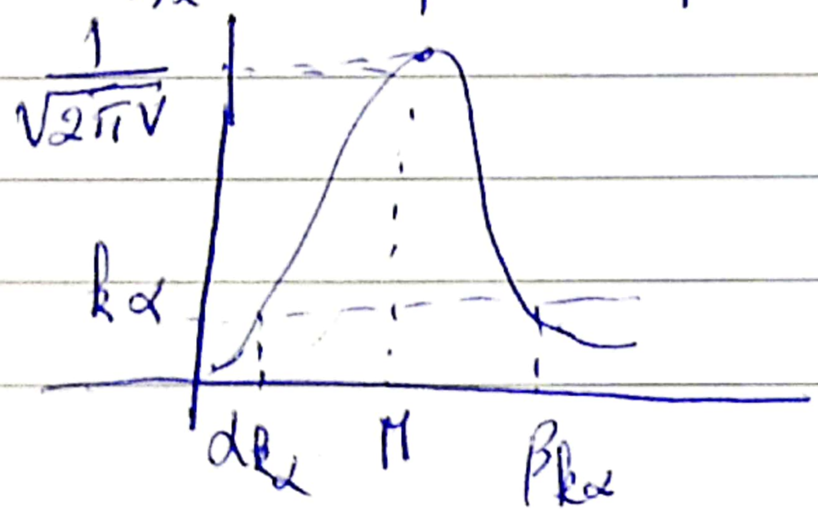
Exo. Région Crédible :

$$\mathcal{L}(\theta/x_1 \ldots x_n) \sim \mathcal{N}\left(\frac{1}{\frac{n}{\sigma^2}+\frac{1}{b^2}}\left(\frac{1}{\sigma^2}\sum x_k + \frac{a}{b^2}\right) ; \frac{1}{\frac{n}{\sigma^2}+\frac{1}{b^2}}\right)$$

Mq $\mathcal{L}(\tilde{\theta}/x_1 \ldots x_n) \sim \mathcal{N}(M, V)$; $M = \frac{n\tau \bar{x}_n + \tau_0 \mu_0}{n\tau + \tau_0}$

$\tau = \frac{1}{\sigma^2}$; $\tau_0 = \frac{1}{b^2}$ $\qquad V = \frac{1}{n\tau + \tau_0}$;

$C_{\theta/x}(k) = \{\theta \in \Theta \text{ tq } \pi(\theta/x) \geq k_\alpha\}$.



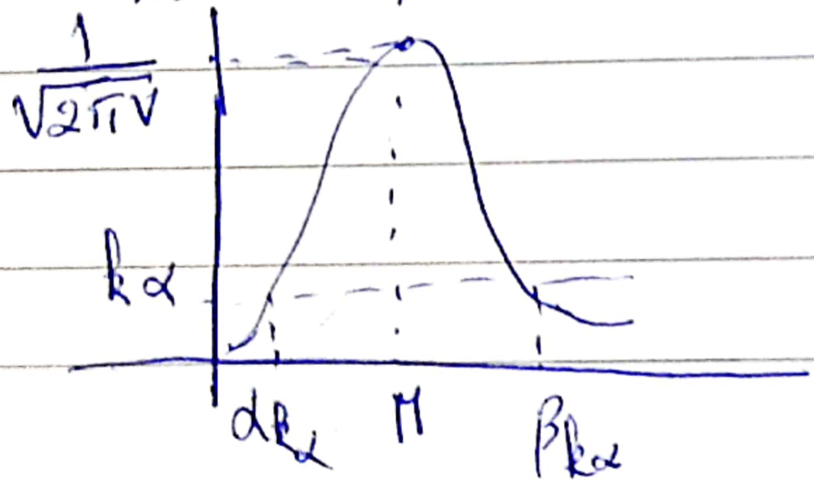
Exo. Région Crédible :

$$\mathcal{L}(\hat{\theta} / x_1 \ldots x_n) \sim \mathcal{N}\left(\frac{1}{\frac{n}{\sigma^2} + \frac{1}{b^2}} \left(\frac{1}{\sigma^2} \sum x_k + \frac{a}{b^2} \right) ; \frac{1}{\frac{n}{\sigma^2} + \frac{1}{b^2}} \right)$$

Mq $\mathcal{L}(\tilde{\theta} / x_1 \ldots x_n) \sim \mathcal{N}(M, V)$; $M = \dfrac{n\tau \bar{x}_n + \tau_0 \mu_0}{n\tau + \tau_0}$

$\tau = \dfrac{1}{\sigma^2}$; $\tau_0 = \dfrac{1}{b^2}$ $\qquad V = \dfrac{1}{n\tau + \tau_0}$;

$C_{\theta/x}(k) = \{ \theta \in \Theta \text{ tq } \pi(\theta/x) \geq k_\alpha \}$.



$$\int_{\alpha_{k\alpha}}^{\beta_{k\alpha}} \pi(\theta / x_1 \_\_ x_n)\, d\theta = 1-\alpha$$

$$F_{N(M,V)}(\alpha_{k\alpha}) = \frac{\alpha}{2}$$

$$P(u < \alpha_{k\alpha}) = \frac{\alpha}{2}$$

$$P\left(\frac{u-M}{\sqrt{V}} < \frac{\alpha_k - M}{\sqrt{V}}\right) = \frac{\alpha}{2} \rightarrow \frac{\alpha_k - M}{\sqrt{V}} = q^{N(0,1)}_{\frac{\alpha}{2}} \Rightarrow \frac{M-\alpha_k}{\sqrt{V}} = q^{N(0,1)}_{1-\frac{\alpha}{2}}$$

$$\alpha_{k\alpha} = M - \sqrt{V} q_{1-\frac{\alpha}{2}}$$

$$F(\beta_{k\alpha}) = 1-\frac{\alpha}{2} \rightarrow P\left(\frac{u-M}{\sqrt{V}} < \frac{\beta_k - M}{\sqrt{V}}\right) = 1-\frac{\alpha}{2}.$$

$$\Rightarrow \beta_k = M + \sqrt{V}\, q^{N(0,1)}_{1-\frac{\alpha}{2}}$$

$[\alpha_{k\alpha}, \beta_{k\alpha}]$ Région HPD.

Ex HPD p44

$$\pi(\mu / X_1 \_\_ X_n) \sim N\left(\bar{X}_n, \frac{\sigma^2}{n}\right)$$

$$HPD = \left[\bar{X}_n - \sqrt{\frac{\sigma^2}{n}}\, q^{1-\frac{\alpha}{2}} ; \bar{X}_n + \sqrt{\frac{\sigma^2}{n}}\, q^{1-\frac{\alpha}{2}}\right.$$