Série n°2 : Estimation

Exo 1 :

1) $X_1, X_2, \ldots, X_n$ indépendantes , $X_i \rightsquigarrow N(\alpha_i+\nu, \sigma^2)$

$\theta = (\alpha_1, \ldots, \alpha_n, \nu, \sigma^2)$, $\theta' = (\alpha_1', \ldots, \alpha_n', \nu', \sigma'^2)$

$$P_\theta = P_{\theta'} \iff \begin{cases} \alpha_1+\nu = \alpha_1'+\nu \\ \vdots \\ \alpha_n+\nu = \alpha_n'+\nu \\ \sigma_2 = \sigma_2' \end{cases} \iff \begin{cases} \sum_{i=1}^n \alpha_i + n\nu = \sum_{i=1}^n \alpha_i' + n\nu' \\ \sigma_2 = \sigma_2' \end{cases}$$

D'où $\begin{cases} \nu = \nu' \\ \sigma_2 = \sigma_2' \end{cases}$. Ainsi le modèle est identifiable

2) $\left.\begin{array}{l} X \rightsquigarrow N(\mu_1, \sigma^2) \\ Y \rightsquigarrow N(\mu_2, \sigma^2) \\ X \text{ et } Y \text{ sont indépendantes} \end{array}\right\}$ $Y - X \rightsquigarrow N(\mu_2 - \mu_1, 2\sigma^2)$

$\theta(\mu_1, \mu_2)$

$\mu_2 = 1$, $\mu_1 = 0 \Rightarrow P_\theta = N(1, 2\sigma^2)$

$\mu_2 = 2$, $\mu_1 = 1 \Rightarrow P_{\theta'} = N(1, 2\sigma^2)$

D'où le modèle est non identifiable.

3) $X = X_{i,j}$, $X_{i,j} \rightsquigarrow N(\mu_{i,j}, \sigma^2)$

$i \in [\![1, n]\!]$ $X_{i,j}$ indépendantes

$j \in [\![1, p]\!]$

On a $\mu_{i,j} = \nu + \lambda_j + \alpha_i$

$$\theta = (\alpha_1, \dots, \alpha_n, \lambda_1, \dots, \lambda_p, \nu, \sigma^2)$$

$$\theta_1 = (0, \dots, 0, 1, \dots, 1, \nu, \sigma^2)$$

$$\theta_2 = (1, \dots, 1, 0, \dots, 0, \nu, \sigma^2)$$

$$\mu^1_{i,j} = \nu + 1 \quad , \quad \mu^2_{i,j} = \nu + 1$$

Ainsi le modèle est non identifiable.

Exo 2 :

1) $T_1(\underline{x}) = T_1(\underline{y}) \Leftrightarrow \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i$

$$\Leftrightarrow \ln\left(\prod_{i=1}^{n} x_i\right) = \ln\left(\prod_{i=1}^{n} y_i\right)$$

$$\Leftrightarrow \sum_{i=1}^{n} \ln x_i = \sum_{i=1}^{n} \ln y_i$$

$$\Leftrightarrow T_2(\underline{x}) = T_2(\underline{y})$$

2) Prenons $\underline{x} = (1, \dots, 1)$, $\underline{y} = (\frac{1}{2}, \frac{1}{2}, 2, 1, \dots, 1)$

$$\sum_{i=1}^{n} x_i = n$$

$$\sum_{i=1}^{n} y_i = n$$

mais $\sum_{i=1}^{n} \ln(x_i) = 0$ et $\sum_{i=1}^{n} \ln(y_i) = -\ln 2$.

Ainsi T_1 et T_2 ne sont pas équivalentes.

$$T_1(\underline{x}) = \left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n} x_i^2\right)$$

$$T_2(\underline{x}) = \left(\sum_{i=1}^{n} x_i, \sum_{i=1}^{n}(x_i - \bar{x})^2\right)$$

$$T_2(\underline{x}) = T_2(\underline{y}) \Leftrightarrow \begin{cases} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \\ \sum_{i=1}^{n}(x_i - \bar{x})^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \bar{x} = \bar{y} \\ \sum_{i=1}^{n} x_i^2 - n\bar{x}^2 = \sum_{i=1}^{n} y_i^2 - n\bar{y}^2 \end{cases}$$

$$\Leftrightarrow \begin{cases} \bar{x} = \bar{y} \\ \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 \end{cases}$$

Ainsi $T_1(\underline{x}) = T_1(\underline{y})$.

Exo3:

$X_1, \ldots, X_n \sim \mathcal{P}(\theta), \theta > 0$ iid

1) $T(\underline{X}) = \sum_{i=1}^{n} X_i$ exhaustive?

1er Cas : Calcul direct :

$\mathbb{P}[\underline{X} = \underline{x} / T(\underline{X}) = t]$ independante de θ?

$\mathbb{P}[X = x] = e^{-\theta} \dfrac{\theta^x}{x!} \mathbb{1}_{\mathbb{N}}(x)$ ⚠ une moyenne de Poisson n'est pas une poisson

Somme d'une poisson est une poisson

$$\sum_{i=1}^{n} X_i \longrightarrow \mathcal{P}(n\theta)$$

$$\mathbb{P}[\underline{X} = \underline{x} / T(\underline{X}) = t] = \frac{\mathbb{P}[\underline{X} = \underline{x}, T(\underline{X}) = t]}{T(\underline{X}) = t} \quad (*)$$

$$\mathbb{P}[T(\underline{X})=t] = e^{-n\theta}\frac{(n\theta)^t}{t!}\mathbb{1}_{\mathbb{N}}(t)$$

$$(*) = \begin{cases} \dfrac{\mathbb{P}[\underline{X}=\underline{x}]}{\mathbb{P}[T(\underline{X})=t]} & \text{si } \sum_{i=1}^{n} x_i = t \\ 0 & \text{sinon} \end{cases}$$

$$\mathbb{P}[\underline{X}=\underline{x}] = \prod_{i=1}^{n} e^{-\theta}\frac{\theta^{i}}{x_i!}\mathbb{1}_{\mathbb{N}}(x_i) = \frac{e^{-n\theta}\,\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i}\mathbb{1}_{\mathbb{N}^n}(x_i)$$

D'où

$$(*) = \begin{cases} \dfrac{\cancel{e^{-n\theta}}\,\cancel{\theta^{\sum_{i=1}^{n} x_i}}\quad t!}{\prod_{i=1}^{n}(x_i)!\;\cancel{e^{-n\theta}}\;(n\cancel{\theta})^t} & \text{si } \sum_{i=1}^{n} x_i = t \\ 0 & \text{sinon} \end{cases}$$

$$= \begin{cases} \dfrac{t!\quad \mathbb{1}_{\mathbb{N}^n}(\underline{x})}{\prod_{i=1}^{n}(x_i)!\; n^t\; \mathbb{1}_{\mathbb{N}}(t)} & \text{si } \sum_{i=1}^{n} x_i = t \\ 0 & \text{sinon} \end{cases}$$

Ainsi T est exhaustive.

2ème Cas : Théorème de factorisation :

$$\mathcal{L}(\underline{x},\theta) = \frac{e^{-n\theta}\,\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n}(x_i)!}\mathbb{1}_{\mathbb{N}^n}(\underline{x})$$

$$g(T(\underline{x}),\theta) = e^{-n\theta}\,\theta^{\sum_{i=1}^{n} x_i}$$

$$h(\underline{x}) = \frac{1}{\prod_{i=1}^{n}(x_i)!}\mathbb{1}_{\mathbb{N}^n}(\underline{x}).$$

$\dots$ mesurable tq $\forall \theta > 0,\ E_\theta(\phi(T(\underline{X}))) = 0$

Montrons que $\phi \equiv 0$.

$$E_\theta(\phi(T(\underline{X}))) = \sum_{k=0}^{\infty} \phi(k)\, e^{-n\theta} \frac{(n\theta)^k}{k!} 1\!\!1_{\mathbb{N}}(k) = 0$$

$$\Leftrightarrow \quad \sum_{k=0}^{\infty} \phi(k) \frac{(n\theta)^k}{k!} = 0 \quad \forall \phi > 0$$

Il s'agit d'une série entière ayant pour terme général nul. Ainsi $\phi \equiv 0$.

D'où T est complète.

⚠ Variante de 1) " 1er Cas"

On reprend de $\mathbb{P}[\underline{X} = \underline{x}] = \dfrac{e^{-n\theta}\, \theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n}(x_i)!}$

$$\frac{\mathbb{P}[\underline{X} = \underline{x}, T(\underline{X}) = t]}{\mathbb{P}[T(\underline{X}) = t]} = \frac{\mathbb{P}[X_1 = x_1, \dots, X_n = x_n, \sum_{i=1}^{n} X_i = t]}{\mathbb{P}[\sum_{i=1}^{n} x_i = t]}$$

$$= \frac{\mathbb{P}[X_1 = x_1, \dots, X_n = t - \sum_{i=1}^{n-1} x_i]}{\mathbb{P}[\sum X_i = t]}$$

$$= \frac{\cancel{e^{-n\theta}}\ \theta^{\cancel{\sum_{i=1}^{n-1} x_i} + t - \cancel{\sum_{i=1}^{n-1} x_i}}}{\prod_{i=1}^{n-1}(x_i)!\ (t - \sum_{i=1}^{n-1} x_i)} \cdot \frac{t!}{\cancel{e^{-n\theta}}\ \theta^t}$$

Exo 8

b/ $\mathbb{P}[X = x] = \dfrac{\theta^x\, e^{-\theta}}{x!}$

$$\mathcal{L}(\underline{x}, \theta) = \frac{\theta^{\sum_{i=1}^{n} x_i}\, e^{-n\theta}}{\prod_{i=1}^{n} x_i} 1\!\!1_{\mathbb{N}^n}(\underline{x})$$

$$= \exp\Big(\underbrace{\log \theta}_{c(\theta)} \cdot \underbrace{\sum_{i=1}^{n} x_i}_{T(\underline{x})} - \overbrace{n\theta}^{d(\theta)} - \underbrace{\log\big(\prod_{i=1}^{n}(x_i)!\big)}_{s(\underline{x})}\Big) 1\!\!1_{\mathbb{N}^n}(\underline{x})$$

$\Rightarrow T$ est exhaustive

$\Rightarrow T$ est complète

Exo4 :

1) $f(x,\theta) = \theta x^{\theta-1}$, $0 < x < 1$ et $\theta > 0$

$$\mathcal{L}(\underline{x},\theta) = \theta^n \prod_{i=1}^{n} x_i^{\theta-1} \mathbb{1}_{[0,1]^n}(x)$$

$$= \exp(n\log\theta + (\theta-1)\log(\prod_{i=1}^{n} x_i)) \mathbb{1}_{[0,1]^n}(x)$$

$$= \exp(n\log\theta + \theta\log\prod_{i=1}^{n} x_i - \log\prod_{i=1}^{n} x_i) \mathbb{1}_{[0,1]^n}(x)$$

$$= \exp(\underbrace{n\log\theta}_{d(\theta)} + \underbrace{\theta}_{c(\theta)} \underbrace{\sum_{i=1}^{n}\log x_i}_{T(\underline{x})} - \underbrace{\sum_{i=1}^{n}\log x_i}_{S(\underline{x})}) \mathbb{1}_{[0,1]}(x)$$

$T(\underline{x}) = \sum_{i=1}^{n}\log x_i$ est une statistique exhaustive.

2) $f(x,\theta) = \theta a x^{a-1} \exp(-\theta x^a)$, $0 < x$, $\theta > 0$ et $a > 0$.

$$\mathcal{L}(\underline{x},\theta) = (\theta a)^n \prod_{i=1}^{n} x_i^{a-1} \exp(-\theta\sum_{i=1}^{n} x_i^a) \mathbb{1}_{\{x>0\}^n}$$

$$= \exp(\underbrace{n\log\theta}_{d(\theta)} + \underbrace{n\ln a + a-1\sum_{i=1}^{n}\ln(x_i)}_{S(\underline{x})} \underbrace{-\theta}_{c(\theta)} \underbrace{\sum_{i=1}^{n} x_i^a}_{T(\underline{x})}) \mathbb{1}_{\{x>0\}^n}$$

$T(\underline{x}) = \sum_{i=1}^{n} x_i^a$ est une statistique descriptive.

3) $f(x,\theta) = \theta a^\theta / x^{\theta+1}$, $a < x$, $\theta > 0$ et $a > 0$

$$\mathcal{L}(\underline{x},\theta) = (\theta^n a^{n\theta} / \prod_{i=1}^{n} x_i^{\theta+1}) \mathbb{1}_{(]a,+\infty[^n)}(\underline{x})$$

$$= \frac{\theta^n a^{n\theta}}{(\prod_{i=1}^{n} x_i)^\theta \prod_{i=1}^{n} x_i} \mathbb{1}(]a,+\infty[^n) \Rightarrow g(T(\underline{x}),\theta) = \frac{\theta^n a^{n\theta}}{(\prod_{i=1}^{n} x_i)^n}$$

$$\mathcal{L}(\underline{x},\theta) = \exp\left(n\log\theta + n\theta\log a - \theta\sum_{i=1}^{n}\ln x_i - \sum_{i=1}^{n}\ln x_i\right) 1\!\!1_{]a,+\infty[^n}(\underline{x})$$

$$T(\underline{x}) = \sum_{i=1}^{n}\ln x_i$$

D'après 2) 1 : On a $\sum_{i=1}^{n}\ln x_i = \prod_{i=1}^{n} x_i$

Exo5 :

$P(V_k) = \theta_k$

$\theta = (\theta_1, \dots, \theta_{k+1})$, $\theta_i \geq 0 \quad \forall\ 1 \leq i \leq k+1$

et $\sum_{i=1}^{k+1}\theta_i = 1$.

$\boxed{N_j = \sum_i 1\!\!1_{\{x_i = v_j\}}}$: le nbre de x_i qui prennent la valeur v_j

1)
$$\begin{cases} P(X=v_1) = \theta_1 & \Rightarrow P(X=x) = \theta_1 1\!\!1_{(x=v_1)} \text{ ou bien } P(X=v_1) = \theta_1^{1\!\!1\{x=v_1\}} \\ \vdots & \\ P(X=v_{k+1}) = \theta_{k+1} & \Rightarrow P(X=v_{k+1}) = \theta_{k+1} 1\!\!1_{(x=v_{k+1})} \text{ ou bien } P(X=v_{k+1}) = \theta_{k+1}^{1\!\!1\{x=v_{k+1}\}} \end{cases}$$

$$P(X=x) = \sum_{j=1}^{k+1}\theta_j 1\!\!1_{\{x=v_j\}}$$

$$\mathcal{L}(\underline{x},\theta) = \prod_{i=1}^{n}\left(\sum_{j=1}^{k+1}\theta_j 1\!\!1_{\{x=v_j\}}\right) \text{ !! (impasse)}$$

Donc on essaie d'écrire $P(X=x)$ sous une autre forme

$$\boxed{P[X=x] = \theta_1^{1\!\!1\{x=v_1\}} \times \dots\dots \times \theta_{k+1}^{1\!\!1\{x=v_{k+1}\}} 1\!\!1_{\{v_1,\dots,v_{k+1}\}}} \text{ !!}$$

⚠ Dans le cas discret, on essaie tjrs de mettre la loi sur cette forme (Produit)

$$\mathcal{L}(\underline{x},\theta) = \prod_{i=1}^{n}\prod_{j=1}^{k+1}\theta_j^{1\!\!1\{x_i=v_j\}} 1\!\!1_{\{v_1,\dots,v_{k+1}\}^n}(\underline{x})$$

$$= \prod_{j=1}^{k+1}\prod_{i=1}^{n}\theta_j^{1\!\!1\{x_i=v_j\}} 1\!\!1_{\{v_1,\dots,v_{k+1}\}^n}(\underline{x})$$

$$= \prod_{j=1} \theta^{\sum_{i=1} \mathbb{1}\{x_i = v_j\}} \; \mathbb{1}_{\{v_1, \dots, v_{k+1}\}}(\underline{x})$$

$$= \underbrace{\prod_{j=1}^{k+1} \theta^{n_j}}_{g(T(\underline{x}),\theta)} \; \underbrace{\mathbb{1}_{\{v_1, \dots, v_{k+1}\}}(\underline{x})}_{h(\underline{x})}$$

$T(\underline{x}) = (n_1, \dots, n_{k+1})$, $k+1$- uplet.

$$((a_1, \dots, a_{k+1}), (b_1, \dots, b_{k+1})) \longmapsto \prod_{i=1}^{k+1} b_i^{a_i}$$

C.C.L : $T(\underline{X}) = N = (N_1, \dots, N_{k+1})$ est une statistique exhaustive

2) La loi de $N = (N_1, \dots, N_{k+1}) \rightsquigarrow \mathcal{M}(n ; \theta_1, \dots, \theta_{k+1})$ n- échantillon

$$\mathbb{P}\begin{pmatrix} N_1 = n_1 \\ \vdots \\ N_{k+1} = n_{k+1} \end{pmatrix} = \frac{n!}{n_1! \dots n_{k+1}!} \theta_1^{n_1} \times \dots \times \theta_{k+1}^{n_{k+1}}$$

$P(\underline{X} = \underline{x}) = \mathcal{L}(\underline{x}, \theta)$

$P(X = x) \neq P(\underline{X} = \underline{x})$

$$\mathbb{P}[\underline{X} = \underline{x} / T(\underline{X}) = t] = \frac{\mathbb{P}[\underline{X} = \underline{x}, T(\underline{X}) = t]}{\mathbb{P}[T(\underline{X}) = t]}$$

$$= \begin{cases} \dfrac{\mathbb{P}[\underline{X} = \underline{x}]}{\mathbb{P}[T(\underline{X}) = t]} & \text{si } T(\underline{x}) = t \\ 0 & \text{sinon} \end{cases}$$

$$= \begin{cases} \prod_{j=1}^{k+1} \theta_j^{n_j} \mathbb{1}_{\{v_1, \dots, v_{k+1}\}}(\underline{x}) \dfrac{n_1! \dots n_{k+1}!}{n! \, \theta_1^{n_1} \dots \theta_{k+1}^{n_{k+1}}} \mathbb{1}_{\{v_1, \dots, v_{k+1}\}}(\underline{x}) \\ 0 \quad \text{sinon} \end{cases}$$

$$= \begin{cases} \dfrac{n_1! \times \cdots \times n_{k+1}!}{n!} \mathbb{1}_{\{v_1, \ldots, v_{k+1}\}} & \text{indépendant de } \theta \\ 0 & \text{sinon} \end{cases}$$

$\Rightarrow$ N est une statistique exhaustive.