Série I : Sondage aléatoire simple

Exercice 1 :

1/ $E(Y) = \frac{1}{N} \cdot \sum_{i=1}^{4} y_i = \frac{11+8+10+11}{4} = 10$

$\sigma^2 = Var(Y) = \frac{1}{N} \sum_{i=1}^{4} (y_i - \bar{Y})^2 = \frac{1}{4} \cdot (1^2 + (-2)^2 + 1^2) = \frac{6}{4} = 1,5$

2_a/ nbr d'échantillons possible est $C_4^2 = 6$

b/ $\hat{\bar{Y}}_{pesr} = \frac{1}{m} \cdot \sum_{i=1}^{m} y_i = \bar{y}$ $\quad s_c^2 = \frac{1}{m-1} \sum_{i=1}^{m} (y_i - \bar{y})^2$

	$\{1,2\}$ (11, 10)	$\{1,3\}$ (11, 8)	$\{1,4\}$ (11, 11)	$\{2,3\}$ (10, 8)	$\{2,4\}$ (10, 11)	$\{3,4\}$ (8, 11)
$\hat{\bar{Y}}_{pesr}$	10,5	9,5	11	9	10,5	9,5
S_c^2	0,5	4,5	0	2	0,5	4,5
$P[\hat{\bar{Y}} = \theta]$	1/3	1/3	1/6	1/6		
$P[S_c^2 = P]$	1/3	1/3	1/6	1/6	1/3	

$(0,5)^2 + (0,5)^2 = 0,5$

c/ $\longrightarrow P(E=e) = 1/6$

$E(\hat{\bar{Y}}_{psr}) = \sum_{i=1}^{m} \theta_i \, P(\hat{\bar{Y}}_{psr} = \theta_i) = 10,5 \times 1/3 + 9,5 \times 1/3 + 11 \times 1/6 + 9 \times 1/6$
$= 10$

$E(\hat{\bar{Y}}_{pesr}^2) = \sum_{i=1}^{m} \theta_i^2 \, P(\hat{\bar{Y}}_{pesr} = \theta_i) = (10,5)^2 \times 1/3 + (9,5)^2 \times 1/3 + 11^2 \times 1/6$
$+ 9^2 \times 1/6 = 36,75 + 30,08 + 20,17 + 13,5 = 100,5$

$V(\hat{\bar{Y}}_{pesr}) = E(\hat{\bar{Y}}_{pesr}^2) - (E(\hat{\bar{Y}}_{pesr}))^2 = 100,5 - 100 = 0,5$

$E(S_c^2) = \sum_{i=1}^{m} P_i \, P(S_c^2 = P_i) = 0,5 \times 1/3 + 4,5 \times 1/3 + 2 \times 1/6$
$= 1/6 + 1,5 + 2/6 = 2$

$E((S_c^2)^2) = \sum_{i=1}^{m} P_i^2 \, P(S_c^2 = P_i) = (0,5)^2 \times 1/3 + (4,5)^2 \times 1/3 + 4 \times 1/6$
$= \frac{0,25}{3} + \frac{20,25}{3} + \frac{2}{3} = 7,5$

$Var(S_c^2) = 7,5 - 2^2 = 3,5$

d/ On a $\sigma_c^2 = \frac{1{,}5 \times 4}{3} = 2$ $\quad \sigma^2 = 1{,}5$

On a $E(S_c^2) = \sigma_c^2$ (prévisible car c'est un estimateur sans biais)

$E(\hat{\bar{Y}}_{pesr}) = \bar{Y}$ estimateur sans biais

$Var(\hat{\bar{Y}}_{pesr}) = 0{,}5 \quad \left((1-f)\frac{\sigma_c^2}{m} = (1-\frac{1}{2})\cdot\frac{2}{2} = 0{,}5\right)$

$Var(S_c^2) = 3{,}5$

Exercice 2 :

1/ $\pi_1 = \frac{1}{2} \quad \pi_2 = \frac{1}{2}$

$\pi_3 = \pi_4 = \pi_5 = \frac{2}{6} = \frac{1}{3}$

$\pi_{12} = \frac{1}{2}$ Les autres sont $= 0$

$\pi_{34} = \pi_{35} = \pi_{45} = \frac{1}{6}$

2/ $\hat{T}_\pi = \sum_{i \in E} \frac{y_i}{\pi_i}$

si $e = \{1,2\} \Rightarrow \hat{T}_\pi = \frac{1}{\pi_1} + \frac{1}{\pi_2} = 4$

$e = \{3,4\} \Rightarrow \hat{T}_\pi = \frac{8/3}{\pi_3} + \frac{8/3}{\pi_4} = 16$

$e = \{3,5\} \Rightarrow \hat{T}_\pi = 16$

$e = \{4,5\} \Rightarrow \hat{T}_\pi = 16$

	$\{1,2\}$	$\{3,4\}$	$\{3,5\}$	$\{4,5\}$
$\hat{T}_\pi$	4	16	16	16
$P(\hat{T}_\pi = t)$	$\frac{1}{2}$	$\frac{1}{2}$	/	/

$E(\hat{T}_\pi) = \frac{1}{2} \times 4 + \frac{1}{2} \times 16$

$= 10$

On a $T = 2 \times 1 + 3 \times \frac{8}{3} = 10$

d'où $E(\hat{T}_\pi) = T$ alors $\hat{T}_\pi$ est

le π-estimateur est sans biais

3/ $Var(\hat{T}_\pi) = \frac{-1}{2} \sum_{i \in P} \sum_{\substack{j \in P \\ i \neq j}} \left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2 \times (\pi_{ij} - \pi_i \pi_j) > 0$

4/ $\hat{Var}(\hat{T}_\pi) = \frac{-1}{2} \sum_{i \in E} \sum_{\substack{j \in E \\ j \neq i}} \left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2 \frac{\pi_{ij} - \pi_i \pi_j}{\pi_{ij}}$

$e = \{1,2\}$ alors $\hat{Var}(\hat{T}_\pi) = 0$

si $e = \{3,4\}$ ou $\{3,5\}$ ou $\{4,5\}$ alors

$\hat{Var}(\hat{T}_\pi) = 0$

$\Rightarrow$ dans tous les cas $\hat{Var}(\hat{T}_\pi) = 0$

$\Rightarrow E(\hat{Var}(\hat{T}_\pi)) = 0 \neq Var(\hat{T}_\pi)$

$\Rightarrow$ cet estimateur est biaisé et ce biais était prévisible car il existe des proba d'inclusion d'ordre 2 qui sont nulles

5/ $e = \{1,2\} \Rightarrow \sqrt{\hat{T}_\pi} = 2 \quad P(\sqrt{\hat{T}_\pi} = 2) = \frac{1}{2}$

$e = \{3,4\}$, $e = \{3,5\}$, $e = \{4,5\}$ alors

$\sqrt{\hat{T}_\pi} = 4$ et $P(\sqrt{\hat{T}_\pi} = 4) = \frac{1}{2}$

6/ $E(\sqrt{\hat{T}_\pi}) = 4 \times \frac{1}{2} + 2 \times \frac{1}{2} = 3 < \sqrt{T} = \sqrt{10}$

cet estimateur est donc biaisé

7/ $Var(\sqrt{\hat{T}_\pi}) = E(\hat{T}_\pi) - \left(E(\sqrt{\hat{T}_\pi})\right)^2$

$= 10 - 9 = 1$

alors que $Var(\hat{T}_\pi) = 1$