EXERCISE

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Abstract. A write up of the exercise for Eric's class.

1. Hi

This is my note. This is KKK's note. Guess what this is.

2. Problem

Suppose you have a particle moving according to the following SDE:

$$dX_t = \sqrt{\epsilon} dW_t,$$

with W_t Brownian motion. The process starts at $x \in (0,1)$. We are interested in

$$p^\epsilon = \Pr(|X_1| > 1) =: \Pr(A).$$

Note that

$$p^\epsilon = 2 \int_1^\infty e^{-\frac{1}{2\epsilon}(y-x)^2} \frac{dy}{\sqrt{2\pi\epsilon}} \tag{1}$$

3. Naiive estimator

The naive estimator will require an exponential number of samples since $X_1 \sim \mathcal{N}(x, \epsilon)$ and A is a rare event.

Date: April 30, 2017.
Me!!!

4. Second Estimator

We can use the instanton

$$\phi(t) = (1-t)x + t$$

and generate the biased process

$$dY_t = \dot{\phi}(t)dt + \sqrt{\epsilon}dW_t$$

which also starts at x. The process is

$$Y_t = (1-t)x + t + \sqrt{\epsilon}W_t$$

and

$$Y_1 = 1 + \sqrt{\epsilon}W_1.$$

4.1. **Girsanov.**

Girsanov's theorem (Oksendal, Theorem 8.6.6) states (in 1D, without details) the following. Let Y_t an Ito process of the form

$$dY_t = \beta(t,\omega)dt + \theta(t,\omega)dB_t$$

and

$$\theta(t,\omega)u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega).$$

Take

$$M_t = \exp\left(-\int_0^t u(s,\omega)dB_s - \frac{1}{2}\int_0^t u^2(s,\omega)ds \right)$$

and $d\mathbb{Q}(\omega) = M_T d\mathbb{P}(\omega)$. Then

$$\hat{B}_t := \int_0^t u(s,\omega)ds + B_t$$

is BM wrt $\mathbb{Q}$ and

$$dY_t = \alpha(t,\omega)dt + \theta(t,\omega)d\hat{B}_t.$$

4.2. **In our case.**

In our case $dX_t = \sqrt{\epsilon}dW_t$, so $\beta \equiv 0$ and $\theta \equiv \sqrt{\epsilon}$. We want to finish with a new process with $\alpha = \dot{\phi}$ so this can only mean $u = -\frac{\dot{\phi}(t)}{\sqrt{\epsilon}}$. Thus

$$\begin{aligned} M_t &= \exp\left(-\int_0^t -\frac{\dot{\phi}(t)}{\sqrt{\epsilon}}dW_s - \frac{1}{2}\int_0^t \frac{\dot{\phi}^2(t)}{\epsilon}ds \right) \\ &= \exp\left(\frac{1}{\sqrt{\epsilon}}\int_0^t \dot{\phi}(t)dW_s - \frac{1}{2\epsilon}\int_0^t \dot{\phi}^2(t)ds \right). \end{aligned}$$

We want to calculate the following:

$$p^\epsilon = \mathbb{P}(A) = \int_A d\mathbb{P}(\omega) = \int_A M_{T=1}^{-1} d\mathbb{Q}(\omega) = \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A M_{T=1}^{-1}].$$

Under $\mathbb{Q}$, we know that $dX_t = \dot{\phi}(t) + \sqrt{\epsilon} d\hat{W}_t$, so we can just call it Y_t and drop the hat on W (as we did above). If we do that, then $dY_t = \dot{\phi}(t) + \sqrt{\epsilon} dW_t$ and:

$$
\begin{aligned}
p^\epsilon &= \mathbb{E}\left[\mathbf{1}_{|Y_t|>1} \exp\left(-\frac{1}{\sqrt{\epsilon}}\int_0^1 \dot{\phi}(s)dW_s + \frac{1}{2\epsilon}\int_0^1 \dot{\phi}^2(s)ds\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{|Y_t|>1} \exp\left(-\frac{1}{\sqrt{\epsilon}}\int_0^1 (1-x)dW_s + \frac{1}{2\epsilon}\int_0^1 (1-x)^2 ds\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{|Y_t|>1} \exp\left(-\frac{1-x}{\sqrt{\epsilon}}W_1 + \frac{(1-x)^2}{2\epsilon}\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_{|Y_t|>1} \exp\left(\frac{x-1}{\sqrt{\epsilon}}W_1 + \frac{(1-x)^2}{2\epsilon}\right)\right]
\end{aligned} \tag{2}
$$

4.3. **First Moment.** Let $A = \{Y : |Y| \geq 1\}$. Noting that $Y_1 = 1 + \sqrt{\epsilon}W_1$ we have that:

(3)

$$\mathbb{E}\left[\mathbf{1}_{|Y_t|>1} \exp\left(\frac{x-1}{\sqrt{\epsilon}}W_1 + \frac{(1-x)^2}{2\epsilon}\right)\right] = \int_{\mathbb{R}} \mathbf{1}_A(1+\sqrt{\epsilon}z) \exp^{-(\frac{(1-x)z}{\sqrt{\epsilon}} + \frac{(1-x)^2}{2\epsilon} + \frac{z^2}{2})} \frac{dz}{\sqrt{2\pi}}$$

(4)

Letting $z = \frac{y-1}{\sqrt{\epsilon}} \implies dz = \frac{dy}{\sqrt{\epsilon}} \implies$

$$
\begin{aligned}
p^\epsilon &= \int_{\mathbb{R}} \mathbf{1}_A(y) \exp^{-(\frac{(1-x)(1-y)}{\epsilon} + \frac{(1-x)^2}{2\epsilon} + \frac{(y-1)^2}{2\epsilon})} \frac{dz}{\sqrt{2\pi\epsilon}} \\
&= \int_{\mathbb{R}} \mathbf{1}_A(y) \exp^{-\frac{(y-x)^2}{2\epsilon}} \frac{dz}{\sqrt{2\pi\epsilon}} \\
&= 2\int_1^\infty \mathbf{1}_A(y) \exp^{-\frac{(y-x)^2}{2\epsilon}} \frac{dz}{\sqrt{2\pi\epsilon}} \\
&= \Pr(|X_1| > 1)
\end{aligned} \tag{5}
$$

Thus $\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A(Y_1)M_{T=1}^{-1}]$ is an unbiased estimator.

4.4. **Second Moment.** Now we compute the second moment for $\mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A(Y_1)M_{T=1}^{-1}]$.

$$
\begin{aligned}
(p^\epsilon)^2 &= \mathbb{E}_{\mathbb{Q}}[\mathbf{1}_A(Y_1)M_{T=1}^{-1}] \\
&= \int_{\mathbb{R}} \mathbf{1}_A(1+\sqrt{\epsilon}z) \exp^{-\frac{2(1-x)z}{\sqrt{\epsilon}} - \frac{(1-x)^2}{\epsilon} - \frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}
\end{aligned} \tag{6}
$$

Letting $y = \sqrt{\epsilon} z \implies dz = \frac{dy}{\sqrt{\epsilon}}$

$$
\begin{aligned}
(p^\epsilon)^2 &= \int_{\mathbb{R}} \mathbf{1}_A(1+y) \exp^{-\frac{1}{2}(\frac{4(1-x)y}{\epsilon} + \frac{2(1-x)^2}{\epsilon} + \frac{y^2}{\epsilon})} \frac{dz}{\sqrt{2\pi\epsilon}} \\
&= \int_{\mathbb{R}} \mathbf{1}_A(y) \exp^{-\frac{(y-x)^2}{2\epsilon}} \frac{dz}{\sqrt{2\pi\epsilon}} \\
&= \int_{\mathbb{R}} \mathbf{1}_A(y) \exp^{\frac{-(y+2(1-x))^2}{2\epsilon} + \frac{(1-x)^2}{\epsilon}} \frac{dy}{\sqrt{2\pi\epsilon}} \\
&= (\int_{-\infty}^{-2} + \int_0^\infty) \exp^{\frac{-(y+2(1-x))^2}{2\epsilon} + \frac{(1-x)^2}{\epsilon}} \frac{dy}{\sqrt{2\pi\epsilon}}
\end{aligned} \tag{7}
$$

Where the second to last equality comes from the fact that $y^2 + 4(1-x)y + 2(1-x)^2 = (y + 2(1-x))^2 - 2(1-x)^2$

Letting $z = y + 2(1-x) \implies$

$$
\begin{aligned}
(p^\epsilon)^2 &= (\int_{-\infty}^{-2x} + \int_{2-2x}^\infty) \exp^{\frac{-z^2}{2\epsilon} + \frac{(1-x)^2}{\epsilon}} \frac{dz}{\sqrt{2\pi\epsilon}} \\
&= (\int_{2x}^\infty + \int_{2-2x}^\infty) \exp^{\frac{-z^2}{2\epsilon} + \frac{(1-x)^2}{\epsilon}} \frac{dz}{\sqrt{2\pi\epsilon}}
\end{aligned} \tag{8}
$$

Need to add plot

Looking at the plot of $z = 2x$ and $z = 2 - 2x$, we can see that for small enough ϵ, the second moment is dominated by different values depending on x. There are two cases to consider:

For $1 > x > \frac{1}{2}$, $(p^\epsilon)^2(x) \asymp \exp^{\frac{(1-x)^2}{\epsilon} - \frac{(2-2x)^2}{2\epsilon}} = \exp^{-\frac{(x-1)^2}{\epsilon}} \implies \lim_{\epsilon \to 0} (p^\epsilon)^2 = 0$

For $0 < x < \frac{1}{2}$, $(p^\epsilon)^2(x) \asymp \exp^{\frac{(1-x)^2}{\epsilon} - \frac{(2x)^2}{2\epsilon}} = \exp \frac{1-2x-x^2}{\epsilon}$. For $x \in (0, \sqrt{2}-1]$, $1 - 2x - x^2 > 0 \implies \lim_{\epsilon \to 0} (p^\epsilon)^2(x) = \infty$

Thus, the relative error of the estimator, which is Rel. error $= \frac{(p^\epsilon)^2}{N \int_{\mathbb{R}} \mathbf{1}_A(Y) M_T^{-1}(Y) dY} \to \infty$ as $\epsilon \to 0$. This is not generally a good method to approximate $P(A)$ in practice.

5. Better Estimator

Consider the process:

$$dZ_t = (1 - Z_t)dt + \sqrt{\epsilon} dW_t$$

Anyone want to start this part up? i.e., use Girsanov to write out M_T for this process?