A Geometry of Non-Homogenous and Non-Isotropic (Deformed) Cellular Spaces

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Abstract

Euclidian geometry is the geometry of homogenous and isotropic continuous spaces. Riemannian geometry is the geometry of n dimensional homogenous and continuous but non-isotropic curved manifolds in more than n dimensional hyper-spaces. The new Barak geometry is the geometry of n dimensional Cellular (Lattice) spaces that are non-homogenous and non-isotropic (**deformed)**. This kind of space, according to General Relativity, is isomorphic to the real space (explanation will follow). In this geometry we use the terms and concepts of the current **Differential Geometry**.

Our contribution to Mathematics and Physics is our attribution of curvature to points in a deformed cellular space - a Lattice. All the rest, in our theory, are merely details.

Note that curvature is not an intrinsic attribute of a point but that of its surrounding topology.

Note also, that this curvature is the sole parameter of **space density** (to be defined in the introduction).

Keywords: Euclidian geometry; Riemannian geometry; Differential geometry; Barak geometry.

1. Introduction

Space Density and Curvature

We define **Space Density** ρ for an **elastic space lattice,** at a point in space, as follows:

Space Density ρ **at a point within a singular space cell** is the inverse of

the volume of this particular cell.

Space Density ρ **at a point on the boundary between cells** is the inverse of the average

volume of the cells that share this boundary.

For physicists, that consider the linear dimension of a space cell to be the **Planck Length**

$1.6 \cdot 10^{-33}$ cm, **Space Density,** is approximately the number of space cells per unit volume.

The density of an un-deformed (uniform) space is denoted ρ_0.

If space is uniform, all of its elementary cells are of the same size, and Euclidian geometry is valid. On the cellular space (lattice).

An **observer** that resides in a specific zone of space and necessarily is subjected to the deformations in this zone is an **Internal Observer**.

When the density is not uniform, i.e., cells are of different sizes, an **Internal observer** discovers that Euclidian geometry **is not** valid. An observer that resides far away from the above specific zone of space is considered an **External Observer**.

Measurements we take, being both an **Internal Observer** and an **External Observer,** are with

standard tools of length and time like the centimeter and the second.

The reality is granular. Even space is made of cells. Hence continuous equations and parameters are only approximations, appropriate for phenomena with scales that interest us and are much larger than the granularity scale.

The parameter value at a point, in these cases, is therefore the average value in the close vicinity of the point. This is especially relevant to the term curvature that is not an intrinsic parameter of a point but a parameter that relates to the topology of its surroundings.

When an internal observer measures circles, their circumferences and radii, they find an **Excess Radius** δr that differs from zero. This happens since the observer and their yardstick are both deformed exactly as the space in which they are immersed.

Note that the term Curvature and Radius of Curvature are used for both manifolds and deformed spaces.

In the new geometry **Gaussian Curvature K at a point P**, equation, (21) in Section 8, is a function of the space density ρ at the point and its gradient $\nabla\rho$.

$$K = \frac{4\pi}{45}\left(\frac{\nabla\rho}{\rho}\right)^2$$

The curvature equation represents an absolute value, whereas **the sign of K** is

determined by the gradient in space density.

Note that in different directions the gradient can have different values. In this case curvature is not a scalar but a tensor.

Sections 2, 3 are introductory sections for those who are **not** familiar with differential geometry.

Note that we know nothing about the structure or subtense of space. This, however, is not relevant to our discussions on geometry or physics.

Note also, that we symbolize a space cell by a circle in 2D and a sphere in 3D.

General Relativity and Curvature

Einstein was led to his theory of General Relativity – GR – by the need to account for the bending of light rays. This meant for him that Euclidian geometry **is not** valid. Hence, Einstein was compelled to use Riemannian geometry, the mathematics of n-dimensional bent manifolds in hyper-spaces with more dimensions than n, as his friend Marcel Grosman suggested. In Riemannian geometry, our three-dimensional space, is an elastic three-dimensional manifold that can be curved within a four-dimensional hyperspace (this additional dimension has nothing to do with the dimension of Time). Here we are getting lost, our imagination just can't handle it. But the mathematical formalism is valid and working - we can make calculations. In the case of a **cellular space,** the kind of space physicists have adopted (see appendix A**)** all the cells in a Riemannian manifold are of **equal size.** If, however, the manifold is bent how can a cell size be retained?

This argument and others led Einstein [1] and Feynman [2] to consider the possibility that our

3D space is elastic, and its deformation yields the required bending of light. We have adopted [3] this idea and present it here in this paper. Steane in his recent book “Relativity Made Relatively Easy” [4] elaborates and clarifies these points. Rindler [5] uses elastic spaces to enable visualization of bent manifolds, whereas Callahan [6] declares: “…**in physics we associate curvature with stretching rather than bending.**”

The first paper published on the new geometry [0] appeared in 2017

2. The New Geometry

In a deformed space a yardstick does not retain its length but is contracted or dilated (stretched) like its local space.

2.1 Positive Symmetric Curvature at a Point P in a 2D Deformed Space

Consider Fig.1, in which the circles represent space cells, or that a circle's diameter represents the length of a yardstick. Here, the yardstick at position P is the smallest and from P the yardstick increases in size symmetrically. This situation is analogous to a metallic plate, where the temperature increases from the center of the surface outwards, and thus the density decreases, i.e., the cell size increases. The 2D, inside observer, in Figure 4, finds that the ratio of the circumference C of the circle to the radius r, as measured by the intrinsic yardstick, is:

$C/r < 2\pi$.

The **Excess Radius** δr, in this case, is the same as for a curved manifold around a point P:

$$\delta r \equiv r_{measured} - c_{measured}/2\pi. \tag{1}$$

For a positive curved 2D space, as Figure (3) shows, $\delta r = r_{meas} - r_{cal} > 0$.

The 2D observer, therefore, concludes that there are two possibilities: they live in a two-dimensional space with a variable density, or on a curved two-dimensional surface "manifold" bent in a three-dimensional hyper-space.

The 2D observer cannot imagine a three-dimensional space, but may be able to accept the necessary abstraction. In any case, they are not able to decide between the two possibilities just by examining their **locality**.

$r_{measured} = 8$

$c_{measured} = 36$

$c/r = 36/8 = 4.5 < 2\pi$

$r_{measured} > r_{calculated}$

The Excess Radius

$\delta r = r_{meas} - r_{cal} > 0$

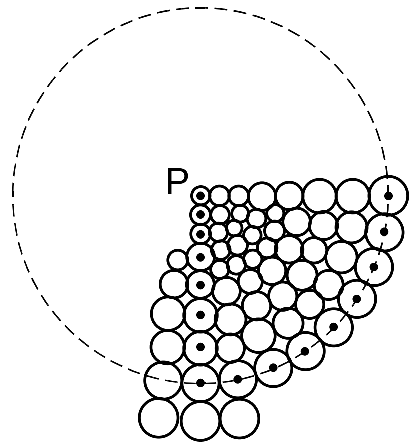

Fig. 1 Positive Curvature

2.2 The Gaussian Curvature

Gaussian curvature is defined as:

$$K = \lim_{r \to 0} \frac{6\delta r}{r^3} \qquad (2)$$

$$K = 1/R_c^2 \qquad (3)$$

R_c is defined as the “radius of curvature” at the point P. Note, however, that the term “radius”, in here, does not stand for the radius as measured by an internal or external observer.

According to Steane (2013), for the case of symmetry, (6) for small r gives:

$$K \approx 6\delta r/r^3 \qquad (4)$$

2.3 Asymmetric Positive Curvature at a Point P

For no radial symmetry, K is taken as the geometric mean (average) of the largest and smallest curvatures, k_1, k_2; as if we have two surfaces with their corresponding excess radii δr_1 and δr_2. A more accurate K should be taken based on the space density around P.

3. Negative Curvature in a 2D Deformed Space

3.1 Symmetric Curvature at a Point P

Fig. 2 shows a cell at point P, which is the largest, and from P outwards the cells decrease in size. In this case, $\frac{C}{r} > 2\pi$ and $\delta r = r_{meas} - r_{cal} < 0$. In this case, the curvature is negative

$K < 0$.

3.2 Asymmetric Curvature at a Point P

For no radial symmetry, K is taken, with a minus sign, as the geometric mean (average) of the excess radii δr_1 and δr_2. An accurate K should be taken based on the space density around P.

$r_{meas} = 8$

$C_{meas} = 64$

$\frac{C}{r} = 10.2 > 2\pi$

$r_{meas} < r_{cal}$

The Excess Radius

$\delta r = r_{meas} - r_{cal} < 0$

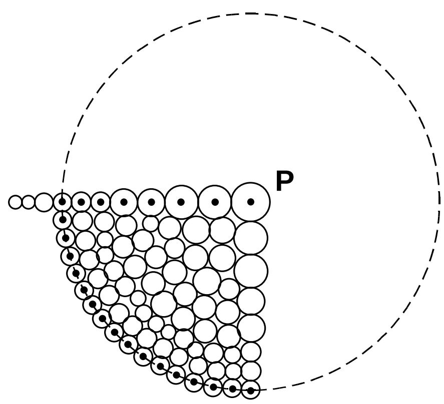

Fig. 2 Negative Curvature

4. Combined Curvature and Its Geometric Mean Radius

A point P is located in a zone of space, with an approximately symmetric radius of curvature R_L at P. By the introduction of a mass M at P, space is curved an additional curvature due to the presence of M. Let this symmetric curvature have a radius R_S. We can take the overall combined curvature at P as:

$$K = 1/(R_L R_S) \tag{5}$$

The Geometric Mean Radius of the Combined Curvature is thus:

$$R_c = \sqrt{(R_L R_S)} \tag{6}$$

This understanding is related to the long-standing issue of Dark Matter. In GR central acceleration is related to the curvature of space (Barak 2016). A star at point P curves space locally around it symmetrically. If the star is located at the skirt of a galaxy, it is also exposed to the curvature of space around the galaxy, due to the non- homogeneous expansion of space around it, (Barak 2017). This exposure contributes an additional general central acceleration, wrongly interpreted as due to the presence of additional matter – Dark Matter. The geometric mean of the compound local and general accelerations, which is the Milgrom phenomenological equation (Barak 2017), is related to the Geometric Mean Radius of the Compound Curvature.

5. A Saddle

At point P in the center, see Fig. 3, in $1/4\ \pi$ direction, the outward dilation of space means that the curvature is positive, with a radius of curvature R+. In the $3/4\ \pi$ direction the outward contraction of space means that the curvature is negative, with a radius of curvature R-.

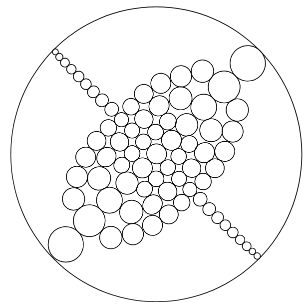

Fig.3. Saddle-like Elastic 2-D Space

Thus, we can define the overall curvature K as:

$$K = 1/(R_+ R_-) \tag{7}$$

If $R_+ - R_- > 0$ then approximately $K > 0$, and if $R_+ - R_- < 0$ then $K < 0$.

Note that an accurate definition is related to space density in all the area.

6. Curvature at a Point P in a 3D Deformed Space

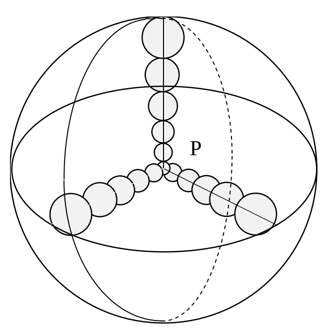

Fig.4 Three-dimensional Positive Curvature

Fig. 4 shows a small cell located at point P, and cells of increasing size radiating from P with spherical symmetry. If we measure the circumference of a great circle whose center is P, in any direction, a measurement of the Excess Radius yields:

$\delta r = r_{meas} - r_{cal} > 0$

Measurements of circumferences and radii can be taken around any point inside the sphere. But for any point, except point P in Fig. 4, there is no symmetry.

For the case of no symmetry, we must determine the degree of deformation of circles around point P, in three orthogonal planes through P. For each of the orthogonal planes we must determine the largest and smallest curvatures, k_1, k_2, called the principal curvatures. Thus, to specify a deformed three-dimensional space around a point P we need 3 x 2 numbers. The average of these six principal curvatures is the average Gaussian curvature of the deformed space at the locality of P.

According to the above, a complete definition of curvature in close proximity to a point in three-dimensional space requires six “curvature numbers”. These represent three pairs of curvature numbers for each of the three intersecting planes perpendicular to each other. These curvature numbers are components of a symmetric tensor of 2nd rank called the contracted Riemannian tensor of curvature, or the Ricci tensor.

Geodesics

Fig.5 shows the shortest distance between two points, A and B, in a deformed two-dimensional space. This figure shows that the shortest distance between points A and B for an internal observer, with their changing yardstick, is the solid line path that passes through the centers of seven cells, and not the “straight” dashed line through A and B, that passes through nine cells.

The shortest distance between points A and B, solid line, is the **geodesic line.** There is no difficulty in imagining this in three dimensions.

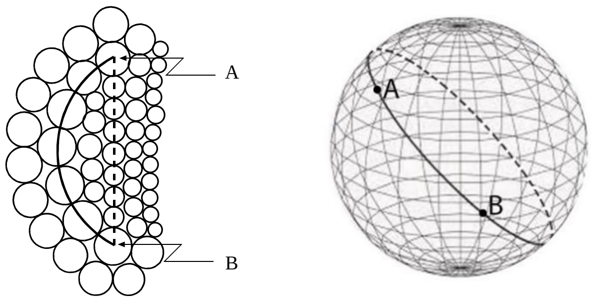

Fig. 5 A Geodesic on a Spherical Surface (Right) and on a Deformed 2D Space (Left)

On the Difference Between the Riemannian and our Geometry

On a curved manifold, several points close to each other can have the same curvature.

On the surface of a ball, for example, all the points on it have the same curvature. This characteristic is known as **global curvature.**

In a deformed space, if there is contraction or dilation around a point, then around adjacent points the deformation is different and so is the curvature of these points. This is known as a **local curvature.**

7. Curvature in Deformed Spaces as a Function of Space Density

7.1 Morgan (1998): On Riemannian Geometry [7]

"Remark. An intrinsic definition of ***the scalar curvature R at a point p*** *in an m-dimensional surface S could be based on the formula for the volume of a ball of intrinsic radius r about p:*

$$\text{volume} = \alpha_m r^m - \alpha_m \frac{R}{3\cdot(m+2)} r^{m+2} + \ldots, \tag{8}$$

where α_m is the volume of a unit ball in R^m. When m = 2…… The analogous formula for spheres played a role in R. Schoen's solution of the Yamabe problem of finding a conformal deformation of a given Riemannian metric to one of constant scalar curvature (see Schoen [Sch, Lemma 2])." This Lemma is presented by Bray and Minicozzi (2018).

From the above equation (8), for any dimension, we can obtain the approximated scalar curvature of two-dimensional space (9) and that of three-dimensional space (10):

Note that we notate the scalar curvature R, at a point P in the quotation (8), by the letter K. Note also that the term, volume, in (8) relates to a n-dimensional volume.

For a 2D space:

$$K = \frac{12}{\pi}\left\{\frac{\pi r^2 - S}{r^4}\right\} \tag{9}$$

And for a 3D space:

$$K = \frac{45}{4\pi}\left\{\frac{\frac{4\pi}{3}r^3 - V}{r^5}\right\} \qquad \textbf{(10)}$$

7.2 Space Density and the Volume Change

The relative change in space density is: $\dfrac{\delta\rho}{\rho}$

and the relative volume change is: $\dfrac{\delta V}{V}$ but: $\rho \propto \dfrac{1}{V}$

and therefore: $\dfrac{\delta\rho}{\rho} = \dfrac{\delta\frac{1}{V}}{\frac{1}{V}} = \dfrac{-\frac{\delta V}{V^2}}{\frac{1}{V}} = -\dfrac{\delta V}{V}$ and therefore:

$$\frac{\nabla\rho}{\rho} = -\frac{\nabla V}{V} \qquad (11)$$

In the case of spherical symmetry: $\dfrac{\nabla\rho}{\rho} = -\dfrac{\frac{\partial V}{\partial r}}{V}\dfrac{\mathbf{r}}{r}$

7.3. The Scalar Curvature K and the Space Density ρ(r)

This section integrates the results of the previous sections to reach the goal of relating Curvature K to Space Density $\rho(r)$.

The scalar Riemannian curvature in a three-dimensional space (17) is:

$$K = \frac{\frac{4\pi}{3} r^3 - \left(\text{Volume of Intrinsic Radius}\right)}{5r^5}$$

For the spherical symmetric case:

$$K = \frac{\frac{4\pi}{3} r^3 - \frac{4\pi}{3}(r-u)^3}{5r^5} \cong \frac{4\pi}{5}\frac{u}{r^3}$$ and for the simple case u = cr:

$$K = \frac{\frac{4\pi}{3} r^3 - \frac{4\pi}{3}(1-c)^3 r^3}{5r^5} \cong \frac{\frac{4\pi}{3} 3c^2 r^3}{5r^5} = \frac{4\pi}{5}\frac{c^2}{r^2} \tag{12}$$

On the other hand:

$$\frac{\rho_0 - \rho}{\rho} = \frac{V_0 - V}{V} = \frac{(r+u)^3 - r^3}{(r+u)^3} \cong 3\frac{u}{r}$$ Therefore:

$$\frac{\nabla\rho}{\rho} = 3\frac{\nabla u}{r}$$ For the spherical symmetric case: $$\frac{\frac{\partial\rho}{\partial r}}{\rho} = 3\frac{\frac{\partial u}{\partial r}}{r}$$ and if:

u = cr: $$\frac{\frac{\partial\rho}{\partial r}}{\rho} = \frac{3c}{r}$$ then:

$$\left(\frac{\nabla\rho}{\rho}\right)^2 = 9\frac{c^2}{r^2} \tag{13}$$

Comparing (19) to (20) gives for 3D:

$$K = \frac{4\pi}{45}\left(\frac{\nabla\rho}{\rho}\right)^2 \tag{14}$$

This is the scalar curvature, K, as the sole function of space density ρ(r). **the sign** of K is determined by the gradient in space density.

A complete definition of curvature in a three-dimensional space requires six "curvature numbers". These numbers represent three pairs of curvature numbers for each of the three intersecting planes perpendicular to each other. These curvature numbers are components of a symmetric tensor of 2nd rank called the contracted Riemannian tensor of curvature, or the Ricci tensor.

A more detailed discussion of this geometry appears in [9].

8. Summary

Our geometry enables us to visualize the various solutions of the General Relativity field equation. This is made possible since, instead of the Riemannian geometry of bent manifolds in higher dimensions, we relate to deformed spaces using the same mathematical terminology and formalism.

Tangibility inspires imagination. "Imagination is more important than knowledge" A. Einstein.

Acknowledgments

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References

[0] S. Barak: On Bent Manifolds and Deformed Spaces hal-01498435 (2017) https://hal.archives-ouvertes.fr/hal-01498435

[1] A. Einstein: Ideas and Opinions, Wings Books NY, "On the Method…" (1933)

[2] R. Feynman: Six Not So Easy Pieces, Addison Wesley (1963)

[3] S. Barak: The Geometrodynamics of Space hal-01435685v2 (2017)

[4] A.M. Steane: Relativity made relatively easy. Oxford (2013)

[5] W. Rindler: Relativity Oxford (2004)

[6] J. J. Callahan: The Geometry of Spacetime, Springer (1999)

[7] F. Morgan: Riemannian geometry, A.K. Peters (1998)

[8] Ta-Pei Cheng: Relativity and Cosmology (2005) Chapter 11.3.

[9] S. Barak: The New Geometry of Non-Homogenous and Non-Isotropic Cellular Spaces, ResearchGate

Appendix A Parallel Transport of Vectors on a Curved Manifold and in a Deformed Space

Parallel Transport on a Manifold

Fig. A1 shows the **parallel transport** of a vector on the surface of a sphere, where parallel transport means that both the tip and the back of the vector are equally displaced along the geodesic. We now transport a vector from O to P, and back through P'. The vector turns through an angle $\delta\vartheta$ so that $\mathbf{v}' = \mathbf{v} + \delta\mathbf{v}$. The curvature, K, of the surface is defined in terms of $\delta\mathbf{v}$ as the vector, **v**, is moved around an infinitesimal closed path with an infinitesimal area, σ.

$$\delta v = K v \sigma \tag{A1}$$

It can be shown that for a spherical surface with radius R, $K = 1/R^2$. Note that when a vector, **v**, is parallel transported along a geodesic, the angle subtended by the vector and the geodesic (i.e., the tangent of the geodesic) is unchanged.

. Parallel Transport in an Elastic Space

Fig. A2 shows a deformed two-dimensional space or a two-dimensional cross-section of a higher dimension elastic space. The transport from O to P directly, or via P', yields an angle $\delta\vartheta$, and thus:

$$\delta v = K v \sigma \tag{A2}$$

(A2) is the same as (A1).

The next Section relates to the vectorial nature of these equations.

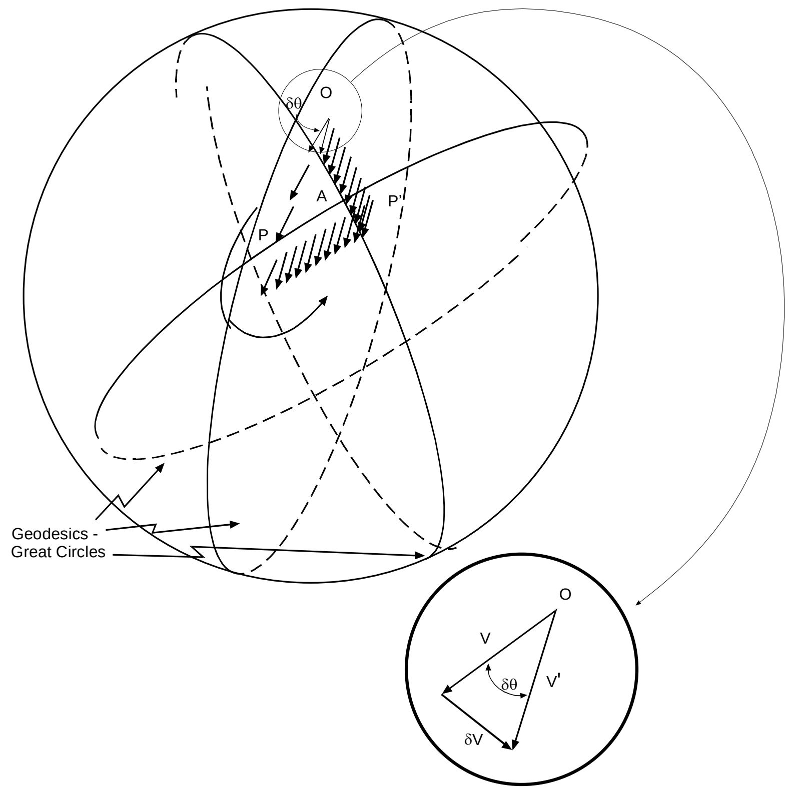

Fig. A1 Parallel Transport

The Riemannian Tensor of Curvature

This section is based on [8] Relativity and Cosmology, Ta-Pei Cheng (2005) Chapter 11.3.

We represent the vector, **v**, and the resulting vector, $\mathbf{v}'$, after parallel transportation, by **A** and **B** respectively. We also generalize the discussion to an n-dimensional space.

The area spanned by the two vectors **A** and **B** can be designated by a vector product:

$$\boldsymbol{\sigma} = \mathbf{A} \times \mathbf{B}$$

It can also be expressed by the anti-symmetric Levi-Civita symbol in the index notation:

$\sigma_k = \in_{ijk} \mathbf{A}^i \mathbf{B}^j$ or by a two-index object σ^{ij}:

$\sigma^{ij} \equiv \in^{ijk} \sigma_k = \in^{ijk} \in_{mnk} A^m B^n = \frac{1}{2}\left(A^i B^j - A^j B^i\right)$ where we have used the identity:

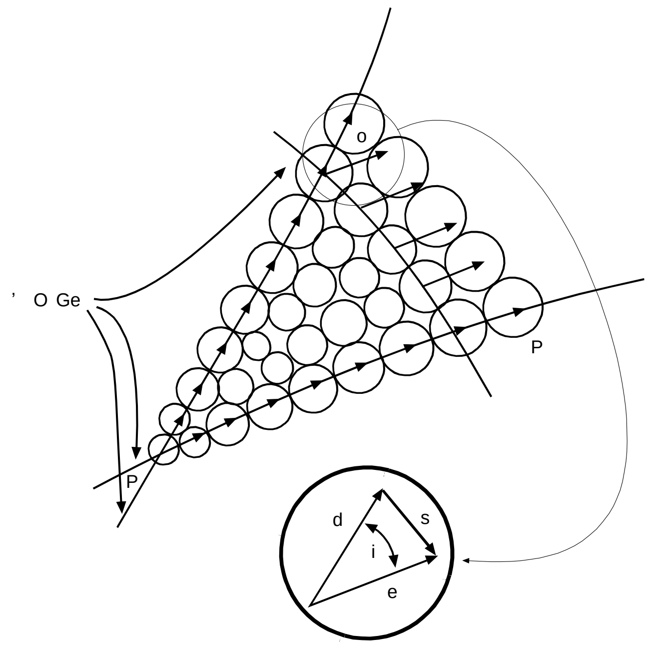

Fig. A2 Parallel Transport

$\in^{ijk} \in_{nmk} = \frac{1}{2}\left(\delta^i_m \delta^j_n - \delta^i_n \delta^j_m\right)$ For an area in an n-dimensional space, we can represent the area $\sigma^{\mu\nu}$ with $\mu = 1,2......n$ $\sigma^{\lambda\rho} = \frac{1}{2}\left(A^\lambda B^\rho - B^\lambda A^\rho\right)$ Hence:

$$dV^\mu = R^\mu_{\nu\lambda\rho} V^\lambda \sigma^{\lambda\rho}$$

dV^μ is proportional to the vector V^ν itself and to the area $\sigma^{\lambda\rho}$ of the closed path.

$R^\mu_{\nu\lambda\rho}$, the coefficient of proportionality is defined as the **Riemannian Curvature Tensor** of the above n-dimensional space.

A detailed calculation of the parallel transport of a vector around an infinitesimal parallelogram leads to the expression:

$$R^{\mu}_{\lambda\alpha\beta} = \partial_{\alpha}\Gamma^{\mu}_{\lambda\beta} - \partial_{\beta}\Gamma^{\mu}_{\lambda\alpha} + \Gamma^{\mu}_{\nu\alpha}\Gamma^{\nu}_{\lambda\beta} - \Gamma^{\mu}_{\nu\beta}\Gamma^{\nu}_{\lambda\alpha}$$

The **Christofell symbol**, Γ, being first derivative, the Riemannian curvature, $R = d\Gamma + \Gamma\Gamma$, is then a non-linear, second derivative [$\partial^2 g + (\partial g)^2$] of the metric tensor.