Electromagnetism is Space Geometrodynamics Part 1

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Abstract

Part 1: **A Theoretical Derivation of Maxwell Equation of Electrostatics**

The fields of Quantum Field Theory QFT, one for every Elementary Particle, reside in space, follow its topology, but are alien to it. These fundamental fields, according to QFT, are all there is, and the particles themselves are merely their quantized excitations.

In QFT there are separate fields for electromagnetism and for electrons / positrons. In this paper we show that these two fields and space are the same entity. This result is achieved as follows:

We define **electric charge density**, based on **space density**. This definition alone, **without any phenomenology**, yields the Maxwell equation of Electrostatics. Together with the Lorentz Transformation it yields (a known procedure), Part 3 of this paper, the entire Maxwell Electromagnetic theory as the geometrodynamics of Space.

Keywords: Electric charge, Electromagnetism, Maxwell equations, Space, Geometrodynamics

1. Introduction

1.1 The 4 Parts of this Paper

The fields of Quantum Field Theory QFT, one for every Elementary Particle, reside in space, follow its topology but alien to it. The particles themselves are merely quantized excitations of these fields. As such they are point-like and structureless, and their masses cannot be derived and calculated. Necessarily and wrongly these masses are considered constants of nature. In QFT there are separate fields for electromagnetism and for electrons / positrons. In this paper, Part 1, we show that these two fields and space are the same entity. Thus, Electromagnetism becomes Space Geometrodynamics.

We define electric charge density, based on space density. This definition alone, without any phenomenology, yields the Maxwell equation of Electrostatics. Together with the Lorentz Transformation it yields (a known procedure), Part 3 of this paper, the entire Maxwell Electromagnetic theory as the Geometrodynamics of Space.

The extension of the General Relativity field equation to incorporate the contribution of charge to curvature appear in part **2** of this paper. Thus, the electromagnetic field is simply space itself. So far, no theory has been able to derive the attributes of electric charge, which are: bivalency, stability, quantization, equality of the absolute values of the bivalent charges, the electric field it creates and the radii of the bivalent charges. In this paper, Part 1, we resolve these issues.

In Part 4 we derive, for the first time, simple equations for the radii and masses of the electron/positron muon/anti-muon and that of quarks/anti-quarks. These equations contain only the constants G, c, $\hbar$ and α (the fine structure constant). The calculated results based on these equations comply accurately with the experimental results.

The **ground state photon (**photom as we call it) has the smallest discrete amount of electromagnetic field energy, which is 1/2hυ. The **next level of excitation,** with the energy hυ, is the **photon.** Models of the photon and photom, which are wavepackets appear elsewhere.

1.2 On Our Electromagnetism (EM)

In this paper we derive the Maxwell equation of Electrostatics. The entire theory of electromagnetism and the extension of the General Relativity field equation, to incorporate the contribution of charge to curvature, appear in Part 3 and Part 2 respectively.

Our EM utilizes the dimensions of length L (cm) and time T (sec) only. Thus, each physical quantity of our EM has a dimensionality, which is L^xT^y. Using General Relativity, we can establish quantitative equivalence between our system and the conventional system of units.

By relating the field energy density to charge density our EM becomes non-linear, which is its **specificity**. In the weak field approximation, it becomes the Maxwell theory. Note that QED is also a non-linear theory.

Issues of structure, stability and quantization of an elementary charge are discussed in Part 4. Note that our EM is applicable for both a single elementary charge and an ensemble of elementary charges. Note also that EM waves, in our theory, are space transversal vibrations like Gravitational Waves, which are also space transversal vibrations, but composed of gravitons and not photons.

1.3 Space as a Lattice

Our definition of Electric Charge Density is based on the concept of space density. Space density is related to the cellular structure of space; relating it to a continuum is not prohibitive but problematic. Attributing a cellular structure (a lattice) to space explains its Hubble expansion, its elasticity (see 1.4) and introduces a cut-off in the wavelength of the vacuum-state spectrum of vibrations. Without this cut-off, infinite energy densities arise. The need for a cut-off is addressed by Sakharov [1] and Misner et al [2]. The Bekenstein Bound [3] sets a limit to the information available about the other side of the horizon of a black hole. Smolin [7] argues that: “There is no way to reconcile this with the view that space is continuous for that implies that each finite volume can contain an infinite amount of information”. A review, relevant to our discussion, appears in a paper by Amelino-Camelia [8].

1.4 The Elastic Space

We relate to space not as a passive static arena for fields and particles but as an active elastic entity. Physicists have different, sometimes conflicting, ideas about the physical meaning of the mathematical objects in their models. The mathematical objects of General Relativity, as an example, are n-dimensional manifolds in hyper-spaces with more dimensions than n. These are not necessarily the physical objects that General Relativity accounts for and n-dimensional manifolds can be equivalent to n-dimensional elastic spaces. This equivalence allows us to use General Relativity, and also relate to our own space as an elastic 3D space. Rindler [6] uses this equivalence to enable visualization of bent manifolds, whereas Steane [7] considers this equivalence to be a real option for a presentation of reality. Callahan [8], being very clear about this equivalence, declares: “…in physics we associate curvature with stretching rather than bending”. After all, in General Relativity gravitational waves [9] are space waves and the attribution of elasticity to space is thus a must.

The deformation of space is the change in size of its cells. The terms positive deformation and negative deformation, around a point in space, are used to indicate that space cells grow or shrink, respectively, from this point outwards. Positive deformation is equivalent to positive curving and negative deformation to negative curving. This is discussed in Part 2.

1.5 On Our Electric Charge and its Field

In our model a positive elementary electric charge is a contracted zone of space, whereas a negative elementary electric charge is a dilated zone of space. Relating to space as a lattice (cellular structure), we define Space Density ρ as the number of space cells per unit volume (denoted ρ_0 for space with no deformations). Based on this we define (postulate, invent) **Electric Charge Density** as:

$q = 1/4\pi \cdot (\rho - \rho_0)/\rho$ $[q] = 1$. This charge density is positive if $\rho > \rho_0$ and negative if $\rho < \rho_0$.

Electric charge, in a given zone of space τ, is then: $Q = \int_\tau q d\tau$ $[Q] = L^3$.

Our definition of electric charge density alone yields electrostatics, without any phenomenology, and together with the Lorentz Transformation - the entire Maxwell theory. This result encourages us to further pursue our idea of the essence of electric charge and, as Part 4 shows, it yields the important results presented below.

Note that our definition of “charge density” is **axiomatic**. This approach is in the spirit of Einstein [10] that: …. **the axiomatic basis of theoretical physics cannot be extracted from experience but must be freely invented**…

We consider charge to be a deformed zone of space, and since the geometry of both deformed spaces and bent manifolds is Riemannian, we can attribute curvature to an electric charge. This new idea, of charge as curved space, enables us to use the theory of General Relativity (GR) in our derivations. These derivations, in Part 2 of this paper, yield the attributes of matter. Some of which are presented below:

The proton charge radius, r_p is:

r_p(calculated) = $0.877428549 \cdot 10^{-13}$ cm. well within the experimental error range [11].

r_p(measured) = $0.8768(69) \cdot 10^{-13}$ cm.

Based on r_p, we calculate the electron radius r_e:

$r_e = 1.409858587 \cdot 10^{-13}$ cm.

Based on this r_e we derive and calculate the mass M of the electron:

M(calculated) = $0.910366931 \cdot 10^{-27}$gr.

A deviation of 0.06% from the measured CODATA 2014 value:

M(measured) = $0.910938356(11) \cdot 10^{-27}$gr.

Based on this M and our model of quarks, we derive and calculate the masses M_d and $M_{\tilde{u}}$ of the d and $\tilde{u}$ quarks:

M_d = 4.5 MeV recent experimental value [12]: M_d=4.8 +/− 0.5 MeV

$M_{\tilde{u}}$=2.25 MeV recent experimental value [12]: $M_{\tilde{u}}$=2.3 +/− 0.8 MeV

These results speak for themselves, and justify our axiomatic approach.

1.6 Recent Papers on Electric Charge

"Nonlinear models of electric charge and magnetic moment" [13] (2015)

"The enigmatic electron" [14] (2013)

"Singularity-free model of electric charge in physical vacuum: non-zero spatial extent and mass generation" [15] (2013)

"Duality and 'particle' democracy" [16] (2016)

Although these papers relate to different aspects of our subject, none present a similar idea to ours.

2. Electric Charge

2.1 Electric Charge Density

We define the electric charge density as:

$$q = \frac{1}{4\pi}\frac{\rho - \rho_0}{\rho} \qquad [q] = 1 \tag{1}$$

The factor $1/4\pi$ is introduced for no other reason, than to ensure resemblance to the Gaussian system.

The charge density is **positive** if $\rho > \rho_0$.

The charge density is **negative** if $\rho < \rho_0$.

Necessarily, only two types of electric charge exist, positive and negative. Let n be the number of space cells in a given volume V. Since $n = \rho_0 V$ and also $n = \rho V'$ we get:

$V = n/\rho_0 \quad V' = n/\rho$ Hence:

$$(V' - V)/V = -(\rho - \rho_0)/\rho \tag{2}$$

$V' > V$ is dilation, $V' < V$ is contraction.

2.2 Electric Charge Q in a Given Volume τ

We define Electric Charge in a given zone of space τ as:

$$Q = \int_\tau q\, d\tau$$

Electric charge has the dimensions of volume $[Q] = L^3$

For clarity, in this section alone, we omit the factor $1/4\pi$ in (1). For the spherical symmetric case where $d\tau = 4\pi r^2 dr$, for a given r, the radius of the charge Q, we get the result:

$Q = \int_0^r q 4\pi r^2\, dr = 4\pi/3 \cdot r^3(1 - \rho_0/\rho)$. Thus $\rho > \rho_0$ gives $Q > 0$ whereas $\rho < \rho_0$ gives $Q < 0$.

For us, outside observers, positive charge in a given spherical zone of space with radius r, means more space cells in the zone than in an un-deformed space (contraction), whereas negative charge means less space cells in the zone than in an un-deformed space (dilation). In Part 2, using Riemannian geometry, we relate curvature to this space deformation and open the way to the application of GR in issues related to charge.

Note that the equality $|Q_+| = |Q_-|$, of the absolute values of the bivalent elementary charges means, according to the integral above, $(1 - \rho_0/\rho_+) = -(1 - \rho_0/\rho_-)$ and hence $2/\rho_0 = 1/\rho_+ + 1/\rho_-$. Note that both ρ_+ and ρ_- are, probably, functions of r and not just constants.

Fig. (1) suggests simplistic models of positive and negative charges, both as spheres of "radius" r_{0+} and r_{0-}. The contracted space in the sphere with radius r_{0+}, our Q_+, contracts space around it (its field) whereas the dilated space in the sphere with radius r_{0-}, our Q_-, dilates space around it (its field). In this model, of charge and its field, **there is no physical separation between the particle and its field** (Einstein Vision)**,** and the integral of ρ over the entire space, for two bivalent elementary charges together, is zero. Note that in the field of a positive charge space is also curved positively. Similarly, in the field of a negative charge, space is curved negatively. Hence the field equation is non-linear, as is the field equation of gravitation.

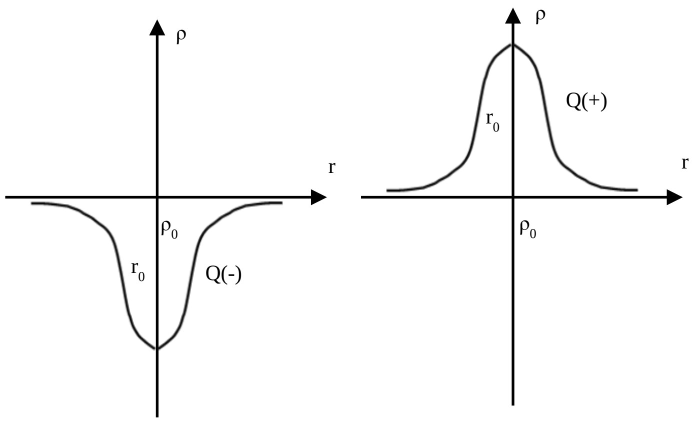

Fig. (1) A Charge and its Field

2.3 On the “border” between the charge and the Field

In our model there is no separation between the charge and its field; they are a continuous deformation of space – Einstein’s vision. This model solves also the known issue: Is the elementary charge Electrostatic Energy in the charge or in the Field?

This long-standing open issue, addressed by Feynman [17] and others, is: The energy needed to create a charge Q of radius r, by bringing in from infinity infinitesimal amounts of charge, despite repulsion, is:

$U = Q^2/2r$

The energy density of the electrostatic field $E = Q/r^2$ is:

$\epsilon = 1/4\pi\, E^2$

and the entire energy in the field is also: $U = \int_r^{\infty} \epsilon d\tau = Q^2/2r$

Thus, we ask where is the energy, in the field or in the charge?

Our model of the elementary electric charge solves this issue.

In our model there is no separation between the charge and its field; they are a continuous deformation of space. The above calculated energy, in both cases, is the same **total energy** of the charge and its field.

Thus, r can only be considered as an artificial “border” between the two. We **define** - freely invent (Einstein’s expression) - this r to be a virtual border for which on both of its sides resides the same **half** of the above U.

Hence, for r calculation we take $r = Q^2/4U$.

Note that U is the space energy of strain, be it contraction or dilation.

3. The Elastic Spatial Vector u and the Electric Field E

By relating to space as an elastic media we can use the theory of elasticity and its Elastic Displacement Vector $\mathbf{u} = \mathbf{r'} - \mathbf{r}$. In Appendix A we show that:

$$\nabla \cdot \mathbf{u} = -\frac{\rho - \rho_0}{\rho} \tag{A2}$$

Thus, according to (1):

$$\nabla \cdot \mathbf{u} = -4\pi q \tag{3}$$

By defining the Electric field vector **E** as:

$$\mathbf{E} = -H\mathbf{u} \quad H = 1 \quad [H] = T^{-2} \quad [E] = LT^{-2} \tag{4}$$

equation (3) becomes the **Maxwell** known **equation of Electrostatics**:

$$\nabla \cdot \mathbf{E} = 4\pi Hq \tag{5}$$

Note that any deformation (strain) in space is related to a stress; hence the introduction of H.

E expresses, therefore, the tension in space due to a deformation in it.

For a positively charged particle, **E** points outwards and for a negatively charged particle inwards, as it is in the Maxwell theory of electrostatics (see Fig. 2 in Appendix A).

4. Coulomb's Law

Gauss's theorem is: $\int_\tau \nabla \cdot \mathbf{u}\, d\tau = \int_\sigma \mathbf{u} \cdot \mathbf{d\sigma}$ For a spherical surface with radius r we get:

$\int_\sigma \mathbf{u} \cdot \mathbf{d\sigma} = u_r \cdot 4\pi r^2$ and since $\int_\tau \nabla \cdot \mathbf{u}\, d\tau = 4\pi H \int_\tau q d\tau = 4\pi HQ$ we get $\mathbf{u} = \frac{Q}{|\mathbf{r}|^3}\mathbf{r}$ or:

$$\mathbf{E} = \frac{HQ}{|\mathbf{r}|^3}\mathbf{r} \quad \textbf{Coulomb's Law} \tag{6}$$

5. The Electric Field E and Scalar Potential φ

Every vectorial field can be decomposed into a field that is a gradient of a **scalar potential** (the **polar** part) and a field that is a **vector potential** (the **axial** part), subject to the boundary condition $\mathbf{E} \to 0$ at infinity. Hence:

$$\mathbf{E} = -\nabla\varphi + \nabla \times \mathbf{A} \tag{7}$$

In the simple static case for the electric field:

$$\mathbf{E} = -\nabla\varphi \tag{8}$$

and, in case of spherical symmetry, in spherical coordinates:

$$E_r = -(\nabla\varphi)_r = \frac{\partial\varphi}{\partial r} \tag{9}$$

and since: $E_r = \dfrac{HQ}{r^2}$ we get:

$$\varphi = \frac{HQ}{r} \qquad [\varphi] = L^2T^{-2} \tag{10}$$

From: $\nabla \cdot \mathbf{E} = 4\pi Hq$ and $\mathbf{E} = -\nabla\varphi$ we get $\nabla \cdot \nabla\varphi = -4\pi Hq$ or:

$$\Delta\varphi = -4\pi Hq \qquad \textbf{Poisson's equation} \tag{11}$$

In the absence of charge:

$$\Delta\varphi = 0 \qquad \textbf{Laplace equation} \tag{12}$$

We can modify equation (11) to incorporate the charge density of the field, which is equivalent to the field energy density. This modification turns (11) into a non-linear equation that resembles the non-linear equation of gravitation but it is out of the scope of this paper.

6. The Electrostatic Force

From (3) $\nabla \cdot \mathbf{u} = 4\pi q$ we get $q = \frac{1}{4\pi}\nabla \cdot \mathbf{u}$, or in tensor notation $q = \frac{1}{4\pi}\frac{\partial u_j}{\partial x_j}$ (Appendix B relates u_{ij} to the metric element g_{ij}). Multiplying both sides of (3) by **u** gives:

$$q\mathbf{u} = \frac{\mathbf{u}}{4\pi}\nabla \cdot \mathbf{u} \tag{13}$$

The left-hand term multiplied by H is denoted by **f**.

$$\mathbf{f} = qH\mathbf{u} \quad \text{or} \quad f_i = qHu_i \tag{14}$$

At this stage, **f** is just a symbol. After a few steps, it is identified as the **electrostatic force density**. Substituting the tensor notation for **u**, in Equations (13) and (14) gives:

$$f_i = \frac{H}{4\pi}\frac{\partial u_j}{\partial x_j}u_i = \frac{H}{4\pi}u_i\frac{\partial u_j}{\partial x_j} = \frac{H}{4\pi}\left(\frac{\partial u_i u_j}{\partial x_j} - u_j\frac{\partial u_i}{\partial x_j}\right) \qquad \text{hence:}$$

$$f_i = \frac{H}{4\pi}\frac{\partial}{\partial x_j}\left(u_i u_j - \frac{1}{2}u^2\delta_{ij}\right) \tag{15}$$

where δ_{ij} is the Kroneker Delta defined by $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$, and $\mathbf{u}^2 = u_1^2 + u_2^2 + u_3^2$. Hence f_i may be regarded as derived from a tensor:

$P_{ij} = \frac{H}{4\pi}\left(u_i u_j - \frac{1}{2}\mathbf{u}^2\delta_{ij}\right)$. And indeed: $f_i = \frac{H}{4\pi}\frac{\partial P_{ij}}{\partial x_j}$ is identified as the force per unit volume and P_{ij} as the strain tensor.

If the x-axis is chosen parallel to a line of force at any point, then $u_y = u_z = 0$ and $u_x = u$, and:

$$P_{xx} = -P_{yy} = -P_{zz} = \frac{H}{8\pi}\mathbf{u}^2 .$$

Thus, the pressure perpendicular to the surface is equal to the energy density.

From (4), **E** = -H**u**, we get the expression for the energy density in the field:

$$\in = \frac{1}{8\pi H}\mathbf{E}^2 \qquad (16)$$

Since **f** is identified as the force per unit volume, we can return to the expression **f** = qH**u**, and recognize the **electrostatic force density**:

$\mathbf{f} = q\mathbf{E}$ [f] = LT^{-2} and the **electrostatic force**: $\mathbf{F} = Q\mathbf{E}$ [F] = L^4T^{-2}

7. Coulomb's Force Law

Equation (16) expresses the field energy density of a system of charges. Hence:

$U = \frac{1}{8\pi H}\int_\tau \mathbf{E}^2 \cdot d\tau$ where **E** is the field produced by these charges, and the integral goes over all space. Substituting E = $-\nabla\varphi$, U can be expressed as follows:

$$U = \frac{1}{8\pi H}\int_\tau \mathbf{E}^2 d\tau = \frac{1}{8\pi H}\int_\tau \nabla\cdot\varphi\mathbf{E}d\tau + \frac{1}{8\pi H}\int_\tau \varphi\nabla\cdot\mathbf{E}d\tau$$

According to Gauss's theorem, the first integral is equal to the integral of $\nabla\cdot\varphi\mathbf{E}$ over the surface bounding the volume of integration, but since the integral is taken over all space and since the field is zero at infinity, this integral vanishes. Substituting (5), $\nabla\cdot\mathbf{E} = 4\pi Hq$, in the second integral, gives the expression for the energy of a system of charges: $U = \frac{1}{2}\int_\tau q\varphi\cdot d\tau$ For a system of point charges, Q_i we can replace the integral with a sum over the charges $U = \frac{1}{2}\sum_i Q_i\varphi_i$ where φ_i is the potential of the field produced by all the charges, at the point where the charge Q_i is located. From Coulomb's law:

$\varphi_i = \sum_{i\neq j}\frac{HQ_j}{r_{ij}}$ where r_{ij} is the distance between the charges Q_i, Q_j we get:

$U = \frac{1}{2}\sum_{i\neq j}\frac{HQ_iQ_j}{r_{ij}}$ In particular, the energy of interaction of two charges is:

$$U = \frac{HQ_i\,Q_j}{r_{ij}}$$ and the force is: $$F = \frac{\partial U}{\partial r} = \frac{HQ_1\,Q_2}{r_{12}^{\;2}}$$ or:

$$\mathbf{F} = H\frac{Q_1\,Q_2}{\left|r_{12}\right|^3}\,\mathbf{r}_{12} \qquad \textbf{Coulomb's force law} \qquad (17)$$

8. Summery

Our definition of electric charge density, based on the density of the elastic space lattice, enables us to relate to an elementary charge not as point-like and not as a string, which are alien to space, but as a finite zone of contracted or dilated space. Necessarily, elementary particles are also of finite size and have a structure. This understanding enables us to do the following

Part 1: Derive Maxwell equation of electrostatics, without any phenomenology, as Space Geometrodynamics.

Part 2: Extend Einstein General Relativity Field Equation to incorporate Electromagnetism

Part 3: Derive the entire Maxwell theory, without any phenomenology, as Space Geometrodynamics.

Part 4: Derive and calculate the elementary charge/particles attributes.

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Appendix A: Contraction and Dilation of Space, and the Strain Tensor u_{ij}

The aim of this appendix is to prove that: $\nabla \cdot \mathbf{u} = -\frac{\rho - \rho_0}{\rho}$

Fig. (2) shows the position vector **r** of a spot, p, in space, with no strain. When stress is applied on space and a deformation occurs, the location of p becomes p’ with a position vector **r’**. The vector **u** is the Elastic Displacement Vector (theory of Elasticity). The origin in Fig. (2) is arbitrary and does not play any role in our discussion.

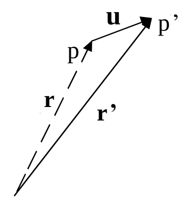

Fig. (2) The Displacement Vector u in the Elastic Space

In this Appendix, we show that the divergence of the elastic displacement vector $\nabla \cdot \mathbf{u}$ in an elastic medium equals the relative change in the volume $\frac{dV' - dV}{dV}$ of a strained medium. The following discussion is based on a derivation made by Landau and Lifshitz [18].

$\mathbf{u} = \mathbf{r}' - \mathbf{r}$ can be denoted by its components:

$u_i = x_i' - x_i$ Let dl' be the deformed distance between adjacent points, since:

$$dx_i' = dx_i + du_i \qquad dl^2 = dx_i^2 \qquad dl'^2 = dx_i'^2 = (dx_i + du_i)^2$$

by the substitution of $du_i = \left(\frac{\partial u_i}{\partial x_k}\right) dx_k$ above we get:

$$dl'^2 = dl^2 + 2\frac{\partial u_i}{\partial x_i} dx_i dx_k + \frac{\partial u_i \partial u_i}{\partial x_k \partial x_l} dx_k dx_l$$

Since the summation is taken over both suffixes i and k in the second term on the right, we get:

$$\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)dx_i dx_k$$

In the third term, we interchange the suffixes i and l. Then dl'^2 takes the final form:

$dl'^2 = dl^2 + 2u_{ik}dx_i dx_k$ where the **strain tensor** u_{ik} is defined as:

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}+\frac{\partial u_l \partial u_l}{\partial x_i \partial x_k}\right) \quad \text{(A1)}$$

If u_i and their derivatives are small, we can neglect the last term as being of the second order of smallness. Thus, for small deformations, the strain tensor is given by:

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\right)$$ We see that it is symmetrical: $u_{ik} = u_{ki}$

u_{ik}, can be diagonalized, like any symmetrical tensor, at any given point. Thus, at any given point, we can choose coordinate axes, the principal axes of the tensor, in such a way that only the diagonal components u_{11}, u_{22}, u_{33} of the 3D tensor u_{ik} are different from zero. These components, the principal values of the strain tensor, are denoted by $u^{(1)}$, $u^{(2)}$, $u^{(3)}$. We should remember that, if the tensor u_{ik} is diagonalized at a specific point in the body, it is not, in general, diagonal at any other point.

If this strain tensor is diagonalized at a given point, the element of length near it becomes:

$$\begin{aligned} dl'^2 &= \left(\delta_{ik} + 2u_{ik}\right)dx_i dx_k \\ &= \left(1-2u^{(1)}\right)dx_1{}^2 + \left(1-2u^{(2)}\right)dx_2{}^2 + \left(1-2u^{(3)}\right)dx_3{}^2 \end{aligned}$$

We see that the expression is the sum of three independent terms. This means that the strain in any volume element may be regarded as composed of independent strains in three mutually perpendicular directions, namely those of the principal axes of the strain tensor. Each of these strains is a simple dilation, or contraction, in the corresponding direction: the length dx_1 along the first principal axis becomes $dx'_1 = \sqrt{\left(1+2u^{(1)}\right)}dx_1$, and similarly for the other two axes. The quantity $\sqrt{\left(1+2u^{(1)}\right)} - 1$ is consequently equal to the relative extension $(dx'_i - dx_i)/dx_i$ along the i^{th} principal axis. The relative extension of the elements of length along the principal axes of the strain tensor, at a given point, is, to within higher-order quantities $\sqrt{\left(1+2u^{(i)}\right)} - 1 \approx u^{(i)}$, i.e., they are the principal values of the tensor u_{ik}.

Let us consider an infinitesimal volume element dV, and find its volume dV' after a deformation. To do so, we take the principal axes of the strain tensor, at the point considered, as the coordinate's axes. Then the elements of length dx_1, dx_2, dx_3 along these axes become, after the deformation, $dx'_1 = \left(1+u^{(1)}\right)dx_1$, etc. The volume dV is the product $dx_1dx_2dx_3$, while dV' is $dx'_1dx'_2dx'_3$. Thus $dV' = dV\left(1+u^{(1)}\right)\left(1+u^{(2)}\right)\times\left(1+u^{(3)}\right)$.

Neglecting higher-order terms, we therefore have $dV' = dV\left(1+u^{(1)}+u^{(2)}+u^{(3)}\right)$. The sum $u^{(1)}+u^{(2)}+u^{(3)}$ of the principal values of a tensor is well known to be invariant, and is equal to the sum of the diagonal components $u_{ii} = u_{11} + u_{22} + u_{33}$ in any coordinate system. Thus:

$dV' = dV\,(1 + u_{ii})$ or: $\frac{dV' - dV}{dV} = u_{ii}$ or: $\frac{dV' - dV}{dV} = \nabla \cdot \mathbf{u}$ and according to (2) we get:

$$\nabla \cdot \mathbf{u} = -\frac{\rho - \rho_0}{\rho} \qquad \text{(A2)}$$

Appendix B: The Small Deformation Strain Tensor and the Fundamental Metric Tensor

In the peer reviewed paper [19] the authors demonstrate that the small deformation strain tensor, see (A1), could be used as a fundamental metric tensor, instead of the usual fundamental metric tensor. We quote their **Conclusion**:

“Through the present paper, it was possible to demonstrate that **the small deformation strain tensor could be used as a fundamental metric tensor, instead of the usual fundamental metric tensor.** Also, it was possible to prove that from that tensor, not only other mathematical structures could be constructed, but also another fundamental tensor was obtained; that was to say, we had constructed two of them, $u_{\mu\upsilon}$, and $B_{\mu\upsilon\sigma\rho}$. It is through these tensors that the gap between pure geometry and physics is bridged. In particular, $u_{\mu\upsilon}$ relates the observed interval ds to the mathematical coordinate specification dx_{μ}. Also, the $u_{\mu\upsilon}$ appear as the potentials of the inertial field [6]. Therefore, it is reasonable to assume that, in the presence of a gravitational field, the $u_{\mu\upsilon}$ is again the potential which determines the accelerations of free bodies; in other words, the $u_{\mu\upsilon}$ is the potential of the gravitational field. Thus, a stage has been reached at which the results obtained can be applied to the theory of gravitation {4}. However, that task that would not be repeated here was established by *Albert Einstein*, and finally formulated by him in 1916, as probably the most beautiful of the physical theories.”

Note however that space deformation is a local feature whereas curvature can also be a global feature.