MKDE DOA estimation with unknown phase and gain and known noise covariance

Given the following model of observations from $\mathbb{C}^p$:

$$\boxed{\mathbf{y} \triangleq s\mathbf{a}(\varphi) + \mathbf{w}}$$

Where s and $\mathbf{w}$ are statistically independent, $\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is an unknown random azimuth, $\mathbf{a}(\varphi) \triangleq \left[1, e^{-i\pi\sin(\varphi)}, \ldots, e^{-i\pi(p-1)\sin(\varphi)}\right]^T$ is a steering verctor, $\mathbf{w} \sim CN(\mathbf{0}, \mathbf{\Sigma}_{\mathrm{w}})$ with known covariance $\mathbf{\Sigma}_{\mathrm{w}}$ and $s = \rho e^{j\vartheta} s$ where $\rho > 0$ and $\vartheta \in [0, \pi]$ are unknown random gain and phase

$$s \sim \begin{cases} 1 \text{ ,w.p. } \frac{1}{2} \\ -1 \text{ ,w.p. } \frac{1}{2} \end{cases}$$

Define:

$$\mathbf{x} \triangleq \left[\mathbf{y}_R^T, \mathbf{y}_I^T\right]^T$$
$$\mathbf{y}_R \triangleq \mathrm{Re}\{\mathbf{y}\},\ \mathbf{x}_I \triangleq \mathrm{Im}\{\mathbf{y}\}$$
$$\mathbf{b}(\varphi) \triangleq \left[\mathbf{a}_R^T(\varphi), \mathbf{a}_I^T(\varphi)\right]^T$$
$$\mathbf{a}_R(\varphi) \triangleq \mathrm{Re}\{\mathbf{a}(\varphi)\} = \left[1, \cos(\pi\sin(\varphi)), \ldots, \cos(\pi(p-1)\sin(\varphi))\right]^T$$
$$\mathbf{a}_I(\varphi) \triangleq \mathrm{Im}\{\mathbf{a}(\varphi)\} = -\left[0, \sin(\pi\sin(\varphi)), \ldots, \sin(\pi(p-1)\sin(\varphi))\right]^T$$
$$\mathbf{v} \triangleq \left[\mathbf{w}_R^T, \mathbf{w}_I^T\right]^T$$
$$\mathbf{w}_R \triangleq \mathrm{Re}\{\mathbf{w}\},\ \mathbf{w}_I \triangleq \mathrm{Im}\{\mathbf{w}\}$$

and therefore we formulate the model:

$$\boxed{\mathbf{x} \triangleq \rho u \mathbf{G}(\vartheta)\mathbf{b}(\varphi) + \mathbf{v}}$$

Where now the dimension is $\mathbb{R}^{2p}$ and $\mathbf{w} \sim N\left(\mathbf{0}, \frac{1}{2}\tilde{\mathbf{\Sigma}}_{\mathrm{w}}\right)$ where $\tilde{\mathbf{\Sigma}}_{\mathrm{w}} \triangleq \begin{bmatrix} \mathrm{Re}\{\mathbf{\Sigma}_{\mathrm{w}}\} & -\mathrm{Im}\{\mathbf{\Sigma}_{\mathrm{w}}\} \\ \mathrm{Im}\{\mathbf{\Sigma}_{\mathrm{w}}\} & \mathrm{Re}\{\mathbf{\Sigma}_{\mathrm{w}}\} \end{bmatrix}$ and $\mathbf{G}(\vartheta) \triangleq \begin{bmatrix} \cos(\vartheta) & -\sin(\vartheta) \\ \sin(\vartheta) & \cos(\vartheta) \end{bmatrix}$.

Under this model, the p.d.f is given by:

$$
\boxed{\begin{aligned}
&f(\mathbf{x}\,|\,\boldsymbol{\theta}) \triangleq \frac{1}{2}\phi\left(\mathbf{x};\rho\mathbf{G}(\vartheta)\mathbf{b}(\varphi),\frac{1}{2}\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}\right)+\frac{1}{2}\phi\left(\mathbf{x};-\rho\mathbf{G}(\vartheta)\mathbf{b}(\varphi),\frac{1}{2}\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}\right)\\
&\mathbf{G}(\vartheta) \triangleq \begin{bmatrix}\cos(\vartheta) & -\sin(\vartheta)\\ \sin(\vartheta) & \cos(\vartheta)\end{bmatrix},\ \tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} \triangleq \begin{bmatrix}\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}\} & -\mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}\}\\ \mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}\} & \mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}\}\end{bmatrix}
\end{aligned}}
$$

where $\boldsymbol{\theta} \triangleq [\varphi,\rho,\vartheta]^T$, $\mathbf{x}\in\mathbb{R}^{2p}$, $\phi(\mathbf{r};\boldsymbol{\mu},\boldsymbol{\Sigma}) \triangleq \frac{1}{(2\pi)^{\frac{p}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(\mathbf{r}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{r}-\boldsymbol{\mu})\right)$ is a Gaussian p.d.f . In Bayesian estimation, $\boldsymbol{\theta}$ is considered as a random parameter with prior density function $\pi(\boldsymbol{\theta})$.

We estimate the parameter $\boldsymbol{\theta}$ by the The K-posterior-mean-estimator (KPME):

$$
\boxed{\tilde{\boldsymbol{\theta}}_h \triangleq \int_{\boldsymbol{\Theta}} \boldsymbol{\theta}\pi_K\left(\boldsymbol{\theta}\,|\,\mathbf{x}_1,\ldots,\mathbf{x}_N\right)d\boldsymbol{\theta}}
$$

where:

$$
\boxed{\pi_k\left(\boldsymbol{\theta}\,|\,\mathbf{x}_1,\ldots,\mathbf{x}_N;h\right) \triangleq \frac{e^{NJ_h(\boldsymbol{\theta})}\pi(\boldsymbol{\theta})}{\int_{\boldsymbol{\Theta}} e^{NJ_h(\boldsymbol{\theta})}\pi(\boldsymbol{\theta})d\boldsymbol{\theta}}}
$$

is the posterior density function and

$$
\boxed{J_h(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} w(\mathbf{x}_n;h)\log f(\mathbf{x}_n\,|\,\boldsymbol{\theta}) - \log\int_{\mathbb{R}^p}\hat{g}(\mathbf{r};h)f(\mathbf{r}\,|\,\boldsymbol{\theta})d\lambda(\mathbf{r})}
$$

$$
\boxed{\begin{aligned}
&w(\mathbf{r};h) = \frac{\tilde{g}(\mathbf{r};h)}{\sum_{n=1}^{N}\tilde{g}(\mathbf{x}_n;h)}\\
&\tilde{g}(\mathbf{r};h) \triangleq \hat{g}(\mathbf{r};h) - \frac{1}{N}K_h(\mathbf{0})\\
&\hat{g}(\mathbf{r};h) \triangleq \frac{1}{N}\sum_{n=1}^{N}K_h(\mathbf{r}-\mathbf{x}_n)
\end{aligned}}
$$

and $K_h(\mathbf{x})$ is the kernel function. Note that, for a Gaussian kernel function $K_h(\mathbf{x}) \triangleq \phi(\mathbf{x};\mathbf{0},h^2\mathbf{I})$:

$$\text{(2)} \log \int_{\mathbb{R}^p} \hat{g}(\mathbf{r};h) f(\mathbf{r}\,|\,\boldsymbol{\theta}) d\lambda(\mathbf{r}) =$$
$$\log \frac{1}{2N} \sum_{n=1}^{N} \left(\phi\left(\mathbf{x}_n; \rho \mathbf{G}(\vartheta) \mathbf{b}(\varphi), \frac{1}{2}\left(\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} + \tau^2 \mathbf{I}\right) \right) + \phi\left(\mathbf{x}_n; -\rho \mathbf{G}(\vartheta) \mathbf{b}(\varphi), \frac{1}{2}\left(\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} + \tau^2 \mathbf{I}\right) \right) \right) =$$
$$\log \frac{1}{N} \sum_{n=1}^{N} \tilde{f}_h(\mathbf{x}_n \,|\, \boldsymbol{\theta})$$
$$\tilde{f}_h(\mathbf{x}_n \,|\, \boldsymbol{\theta}) \triangleq \frac{1}{2} \phi\left(\mathbf{x}_n; \rho \mathbf{G}(\vartheta) \mathbf{b}(\varphi), \frac{1}{2}\left(\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} + \tau^2 \mathbf{I}\right) \right) + \frac{1}{2} \phi\left(\mathbf{x}_n; -\rho \mathbf{G}(\vartheta) \mathbf{b}(\varphi), \frac{1}{2}\left(\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} + \tau^2 \mathbf{I}\right) \right)$$
$$\tau \triangleq \sqrt{2}h$$

Therefore, we can write the objective:

$$J_h(\boldsymbol{\theta}) = \sum_{n=1}^{N} w(\mathbf{x}_n; h) \log f(\mathbf{x}_n \,|\, \boldsymbol{\theta}) - \hat{u}(\boldsymbol{\theta}, h)$$
$$\hat{u}(\boldsymbol{\theta}, h) \triangleq \log \sum_{n=1}^{N} \tilde{f}_h(\mathbf{x}_n; \boldsymbol{\theta})$$

Real-complex translation of the nominal p.d.f:

$$f(\mathbf{x}\,|\,\boldsymbol{\theta}) \triangleq \frac{1}{2} \phi\left(\mathbf{x}; \rho \mathbf{G}(\vartheta) \mathbf{b}(\varphi), \frac{1}{2} \tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} \right) + \frac{1}{2} \phi\left(\mathbf{x}; -\rho \mathbf{G}(\vartheta) \mathbf{b}(\varphi), \frac{1}{2} \tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} \right) =$$
$$\frac{1}{2} \phi_c\left(\mathbf{y}; \rho e^{j\vartheta} \mathbf{a}(\varphi), \boldsymbol{\Sigma}_{\mathbf{w}} \right) + \frac{1}{2} \phi_c\left(\mathbf{y}; -\rho e^{j\vartheta} \mathbf{a}(\varphi), \boldsymbol{\Sigma}_{\mathbf{w}} \right) = f(\mathbf{y}\,|\,\boldsymbol{\theta})$$

$$\tilde{f}_h(\mathbf{x}; \boldsymbol{\theta}) \triangleq \frac{1}{2} \phi\left(\mathbf{x}; \rho \mathbf{G}(\vartheta) \mathbf{b}(\varphi), \frac{1}{2}\left(\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} + \tau^2 \mathbf{I}\right) \right) + \frac{1}{2} \phi\left(\mathbf{x}_n; -\rho \mathbf{G}(\vartheta) \mathbf{b}(\varphi), \frac{1}{2}\left(\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}} + \tau^2 \mathbf{I}\right) \right) =$$
$$\frac{1}{2} \phi_c\left(\mathbf{y}; \rho e^{j\vartheta} \mathbf{a}(\varphi), \bar{\boldsymbol{\Sigma}}_h \right) + \frac{1}{2} \phi_c\left(\mathbf{y}; -\rho e^{j\vartheta} \mathbf{a}(\varphi), \bar{\boldsymbol{\Sigma}}_h \right) = \tilde{f}_h(\mathbf{y}\,|\,\boldsymbol{\theta})$$
$$\bar{\boldsymbol{\Sigma}}_h \triangleq \boldsymbol{\Sigma}_{\mathbf{w}} + \tau^2 \mathbf{I}$$

Where $\phi_c(\mathbf{r}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \triangleq \frac{1}{(\pi)^p |\boldsymbol{\Sigma}|} \exp\left(-(\mathbf{r} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{r} - \boldsymbol{\mu}) \right)$ is a symmetric proper complex Gaussian p.d.f.

Therefore, we can rewrite the objective as:

$$J_h(\boldsymbol{\theta}) = \sum_{n=1}^{N} w(\mathbf{y}_n; h) \log f(\mathbf{y}_n \,|\, \boldsymbol{\theta}) - \log \sum_{n=1}^{N} \tilde{f}_h(\mathbf{y}_n \,|\, \boldsymbol{\theta})$$

Computing the empirical asymptotic MSE matrix

$$h_{opt} \triangleq \arg\min_{h \in I} \left\{ tr\left[\mathbf{W}\hat{\mathbf{R}}(\tilde{\boldsymbol{\theta}}_h, h) \right] \right\}$$

where

$$\hat{\mathbf{R}}(\boldsymbol{\theta}, h) \triangleq \frac{1}{N}\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}, h)$$

$$\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}, h) \triangleq \hat{\mathbf{C}}^{-1}(\boldsymbol{\theta}, h)\hat{\mathbf{D}}(\boldsymbol{\theta}, h)\hat{\mathbf{C}}^{-1}(\boldsymbol{\theta}, h)$$

$$\hat{\mathbf{C}}(\boldsymbol{\theta}, h) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n; h)\nabla_{\boldsymbol{\theta}}^2 \log f\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right) - \nabla_{\boldsymbol{\theta}}^2 \log \hat{u}\left(\boldsymbol{\theta}, h\right) = \nabla_{\boldsymbol{\theta}}^2 J_h\left(\boldsymbol{\theta}\right)$$

$$\hat{\mathbf{D}}(\boldsymbol{\theta}, h) \triangleq \frac{1}{N}\sum_{n=1}^{N} \hat{\mathbf{v}}\left(\mathbf{y}_n \mid \boldsymbol{\theta}, h\right)\hat{\mathbf{v}}^T\left(\mathbf{y}_n \mid \boldsymbol{\theta}, h\right)$$

$$\hat{\mathbf{v}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \hat{\psi}_G(\mathbf{r} \mid \boldsymbol{\theta})\hat{\mathbf{c}}(\mathbf{r} \mid \boldsymbol{\theta}, h) + \hat{\mathbf{d}}(\mathbf{r} \mid \boldsymbol{\theta}, h) - \hat{\mathbf{z}}(\mathbf{r} \mid \boldsymbol{\theta}, h)$$

$$\hat{\mathbf{c}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \nabla_{\boldsymbol{\theta}} \log f\left(\mathbf{r} \mid \boldsymbol{\theta}\right) - \nabla_{\boldsymbol{\theta}} \log \hat{u}(\boldsymbol{\theta}, h)$$

$$\hat{\psi}_G(\mathbf{r}; \boldsymbol{\eta}) \triangleq (N-1)w\left(\mathbf{y}_n; h\right),\ \hat{\varphi}_h(\mathbf{r}) \triangleq \frac{K_h(\mathbf{r})}{N\sum_{n=1}^{N}\tilde{g}\left(\mathbf{y}_n; h\right)} = \frac{K_h(\mathbf{r})}{\sum_{n=1}^{N}\sum_{m \neq n} K_h\left(\mathbf{x}_n - \mathbf{x}_m\right)}$$

$$\hat{\mathbf{d}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq (N-1)\sum_{n=1}^{N}\left(\hat{\varphi}_h(\mathbf{y}_n - \mathbf{r})\hat{\mathbf{c}}(\mathbf{y}_n \mid \boldsymbol{\theta}, h) - N^{-1}\hat{\varphi}_h(\mathbf{0})\hat{\mathbf{c}}(\mathbf{r} \mid \boldsymbol{\theta}, h)\right)$$

$$\hat{\mathbf{z}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \frac{v(\mathbf{r} \mid \boldsymbol{\theta}, h)}{\hat{u}(\boldsymbol{\theta}, h)}\nabla_{\boldsymbol{\theta}} \log \frac{v(\mathbf{r} \mid \boldsymbol{\theta}, h)}{\hat{u}(\boldsymbol{\theta}, h)} = \frac{\dot{v}(\mathbf{r} \mid \boldsymbol{\theta}, h) - v(\mathbf{r} \mid \boldsymbol{\theta}, h)\nabla_{\boldsymbol{\theta}} \log \hat{u}(\boldsymbol{\theta}, h)}{\hat{u}(\boldsymbol{\theta}, h)}$$

$$v(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \left(K_h * f\right)(\mathbf{r} \mid \boldsymbol{\theta}) = \tilde{f}_h(\mathbf{r} \mid \boldsymbol{\theta})$$

Note that:

$$\nabla_{\boldsymbol{\theta}} \log f(\mathbf{y}|\boldsymbol{\theta}) = \frac{\nabla_{\boldsymbol{\theta}} f(\mathbf{y}|\boldsymbol{\theta})}{f(\mathbf{y}|\boldsymbol{\theta})} = \frac{\nabla_{\boldsymbol{\theta}}\phi_1 + \nabla_{\boldsymbol{\theta}}\phi_2}{\phi_1+\phi_2} = \lambda_1(\mathbf{y}|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log\phi_1 + \lambda_2(\mathbf{y}|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log\phi_2$$

$$\nabla^2_{\boldsymbol{\theta}} \log f(\mathbf{y}|\boldsymbol{\theta}) = \frac{\nabla^2_{\boldsymbol{\theta}} f(\mathbf{y}|\boldsymbol{\theta})}{f(\mathbf{y}_n|\boldsymbol{\theta})} - \frac{\nabla_{\boldsymbol{\theta}} f(\mathbf{y}|\boldsymbol{\theta})}{f(\mathbf{y}|\boldsymbol{\theta})} \times \frac{\nabla^T_{\boldsymbol{\theta}} f(\mathbf{y}|\boldsymbol{\theta})}{f(\mathbf{y}|\boldsymbol{\theta})} = \frac{\nabla^2_{\boldsymbol{\theta}} f(\mathbf{y}|\boldsymbol{\theta})}{f(\mathbf{y}|\boldsymbol{\theta})} - \nabla_{\boldsymbol{\theta}}\log f(\mathbf{y}|\boldsymbol{\theta})\nabla^T_{\boldsymbol{\theta}}\log f(\mathbf{y}|\boldsymbol{\theta})$$

$$\nabla^2_{\boldsymbol{\theta}} f(\mathbf{y}_n|\boldsymbol{\theta}) = \nabla_{\boldsymbol{\theta}}\left(\nabla^T_{\boldsymbol{\theta}}\phi_1 + \nabla^T_{\boldsymbol{\theta}}\phi_2\right) = \nabla_{\boldsymbol{\theta}}\left(\phi_1\nabla^T_{\boldsymbol{\theta}}\log\phi_1 + \phi_2\nabla^T_{\boldsymbol{\theta}}\log\phi_2\right)$$
$$= \left(\nabla_{\boldsymbol{\theta}}\phi_1\nabla^T_{\boldsymbol{\theta}}\log\phi_1 + \nabla_{\boldsymbol{\theta}}\phi_2\nabla^T_{\boldsymbol{\theta}}\log\phi_2\right) + \left(\phi_1\nabla^2_{\boldsymbol{\theta}}\log\phi_1 + \phi_2\nabla^2_{\boldsymbol{\theta}}\log\phi_2\right)$$
$$= \left(\phi_1\nabla_{\boldsymbol{\theta}}\log\phi_1\nabla^T_{\boldsymbol{\theta}}\log\phi_1 + \phi_2\nabla_{\boldsymbol{\theta}}\log\phi_2\nabla^T_{\boldsymbol{\theta}}\log\phi_2\right) + \left(\phi_1\nabla^2_{\boldsymbol{\theta}}\log\phi_1 + \phi_2\nabla^2_{\boldsymbol{\theta}}\log\phi_2\right)$$
$$\Downarrow$$
$$\nabla^2_{\boldsymbol{\theta}}\log f(\mathbf{y}_n|\boldsymbol{\theta}) =$$
$$\left(\lambda_1(\mathbf{y}|\boldsymbol{\theta})\nabla^2_{\boldsymbol{\theta}}\log\phi_1 + \lambda_2(\mathbf{y}|\boldsymbol{\theta})\nabla^2_{\boldsymbol{\theta}}\log\phi_2\right) + \left(\lambda_1(\mathbf{y}|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log\phi_1\nabla^T_{\boldsymbol{\theta}}\log\phi_1 + \lambda_2(\mathbf{y}|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log\phi_2\nabla^T_{\boldsymbol{\theta}}\log\phi_2\right)$$
$$-\nabla_{\boldsymbol{\theta}}\log f(\mathbf{y}_n|\boldsymbol{\theta})\nabla^T_{\boldsymbol{\theta}}\log f(\mathbf{y}_n|\boldsymbol{\theta})$$

$$\nabla_{\boldsymbol{\theta}}\log\hat{u}(\boldsymbol{\theta},h) = \frac{\nabla_{\boldsymbol{\theta}}\hat{u}(\boldsymbol{\theta},h)}{\hat{u}(\boldsymbol{\theta},h)} = \frac{\sum_{n=1}^{N}\left(\nabla_{\boldsymbol{\theta}}\phi_{3,n} + \nabla_{\boldsymbol{\theta}}\phi_{4,n}\right)}{\hat{u}(\boldsymbol{\theta},h)} = \sum_{n=1}^{N}\left(\lambda_3(\mathbf{y}_n|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log\phi_{3,n} + \lambda_4(\mathbf{y}_n|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log\phi_{4,n}\right)$$

$$\nabla^2_{\boldsymbol{\theta}}\log\hat{u}(\boldsymbol{\theta},h) = \frac{\nabla^2_{\boldsymbol{\theta}}\hat{u}(\boldsymbol{\theta},h)}{\hat{u}(\boldsymbol{\theta},h)} - \nabla_{\boldsymbol{\theta}}\log\hat{u}(\boldsymbol{\theta},h)\nabla^T_{\boldsymbol{\theta}}\log\hat{u}(\boldsymbol{\theta},h)$$

$$\nabla^2_{\boldsymbol{\theta}}\hat{u}(\boldsymbol{\theta},h) = \sum_{n=1}^{N}\nabla_{\boldsymbol{\theta}}\left(\nabla^T_{\boldsymbol{\theta}}\phi_{3,n} + \nabla^T_{\boldsymbol{\theta}}\phi_{4,n}\right) = \sum_{n=1}^{N}\left(\phi_1\nabla_{\boldsymbol{\theta}}\log\phi_1\nabla^T_{\boldsymbol{\theta}}\log\phi_1 + \phi_2\nabla_{\boldsymbol{\theta}}\log\phi_2\nabla^T_{\boldsymbol{\theta}}\log\phi_2\right) +$$
$$\sum_{n=1}^{N}\left(\phi_1\nabla^2_{\boldsymbol{\theta}}\log\phi_1 + \phi_2\nabla^2_{\boldsymbol{\theta}}\log\phi_2\right)$$
$$\Downarrow$$
$$\nabla^2_{\boldsymbol{\theta}}\log\hat{u}(\boldsymbol{\theta},h) = \sum_{n=1}^{N}\left(\lambda_3(\mathbf{y}_n|\boldsymbol{\theta})\nabla^2_{\boldsymbol{\theta}}\log\phi_{3,n} + \lambda_4(\mathbf{y}_n|\boldsymbol{\theta})\nabla^2_{\boldsymbol{\theta}}\log\phi_{4,n}\right) +$$
$$\sum_{n=1}^{N}\left(\lambda_3(\mathbf{y}_n|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log\phi_{3,n}\nabla^T_{\boldsymbol{\theta}}\log\phi_{3,n} + \lambda_4(\mathbf{y}_n|\boldsymbol{\theta})\nabla_{\boldsymbol{\theta}}\log\phi_{4,n}\nabla^T_{\boldsymbol{\theta}}\log\phi_{4,n}\right) - \nabla_{\boldsymbol{\theta}}\log\hat{u}(\boldsymbol{\theta},h)\nabla^T_{\boldsymbol{\theta}}\log\hat{u}(\boldsymbol{\theta},h)$$

Hence:

$$
\begin{aligned}
&\nabla^2_{\boldsymbol{\theta}} J_h(\boldsymbol{\theta}) \triangleq \mathbf{T}_1(\boldsymbol{\theta}) + \mathbf{T}_2(\boldsymbol{\theta}) - \mathbf{T}_3(\boldsymbol{\theta}) - \mathbf{T}_4(\boldsymbol{\theta}) - \mathbf{T}_5(\boldsymbol{\theta}) + \mathbf{T}_6(\boldsymbol{\theta}) \\
&\mathbf{T}_1(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n; h)\left(\lambda_1(\mathbf{y}_n \mid \boldsymbol{\theta}) \nabla^2_{\boldsymbol{\theta}} \log \phi_1(\mathbf{y}_n \mid \boldsymbol{\theta}) + \lambda_2(\mathbf{y}_n \mid \boldsymbol{\theta}) \nabla^2_{\boldsymbol{\theta}} \log \phi_2(\mathbf{x}_n \mid \boldsymbol{\theta})\right) \\
&\mathbf{T}_2(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n; h)\left(\lambda_1(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_1(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_1^T(\mathbf{y}_n \mid \boldsymbol{\theta}) + \lambda_2(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_2(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_2^T(\mathbf{y}_n \mid \boldsymbol{\theta})\right) \\
&\mathbf{T}_3(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n; h) \mathbf{v}(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{v}^T(\mathbf{y}_n \mid \boldsymbol{\theta}) \\
&\mathbf{T}_4(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} \lambda_3(\mathbf{y}_n \mid \boldsymbol{\theta}) \nabla^2_{\boldsymbol{\theta}} \log \phi_3(\mathbf{y}_n \mid \boldsymbol{\theta}) + \lambda_4(\mathbf{y}_n \mid \boldsymbol{\theta}) \nabla^2_{\boldsymbol{\theta}} \log \phi_4(\mathbf{x}_n \mid \boldsymbol{\theta}) \\
&\mathbf{T}_5(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} \lambda_3(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_3(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_3^T(\mathbf{y}_n \mid \boldsymbol{\theta}) + \lambda_4(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_4(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_4^T(\mathbf{y}_n \mid \boldsymbol{\theta}) \\
&\mathbf{T}_6(\boldsymbol{\theta}) \triangleq \mathbf{g}_2(\boldsymbol{\theta}) \mathbf{g}_2^T(\boldsymbol{\theta})
\end{aligned}
$$

Where:

$$
\begin{aligned}
&\phi_1(\mathbf{y} \mid \boldsymbol{\theta}) \triangleq \phi_c\left(\mathbf{y}; \rho e^{j\vartheta} \mathbf{a}(\varphi), \boldsymbol{\Sigma}_{\mathbf{w}}\right),\ \phi_2(\mathbf{y} \mid \boldsymbol{\theta}) \triangleq \phi_c\left(\mathbf{y}; -\rho e^{j\vartheta} \mathbf{a}(\varphi), \boldsymbol{\Sigma}_{\mathbf{w}}\right) \\
&\phi_3(\mathbf{y} \mid \boldsymbol{\theta}) \triangleq \phi_c\left(\mathbf{y}; \rho e^{j\vartheta} \mathbf{a}(\varphi), \bar{\boldsymbol{\Sigma}}_h\right),\ \phi_4(\mathbf{y} \mid \boldsymbol{\theta}) \triangleq \phi_c\left(\mathbf{y}; -\rho e^{j\vartheta} \mathbf{a}(\varphi), \bar{\boldsymbol{\Sigma}}_h\right),\ \bar{\boldsymbol{\Sigma}}_h \triangleq \boldsymbol{\Sigma}_{\mathbf{w}} + \tau,\ \tau \triangleq \sqrt{2}h \\
&i = 1,2:\ \lambda_i(\mathbf{y} \mid \boldsymbol{\theta}) \triangleq \frac{\phi_i(\mathbf{y} \mid \boldsymbol{\theta})}{\phi_1(\mathbf{y} \mid \boldsymbol{\theta}) + \phi_2(\mathbf{y} \mid \boldsymbol{\theta})} \\
&j = 3,4:\ \lambda_j(\mathbf{y} \mid \boldsymbol{\theta}) \triangleq \frac{\phi_j(\mathbf{y} \mid \boldsymbol{\theta})}{\sum_{n=1}^{N}\left(\phi_3(\mathbf{y}_n \mid \boldsymbol{\theta}) + \phi_4(\mathbf{y}_n \mid \boldsymbol{\theta})\right)} \\
&k = 1,\ldots,4:\ \mathbf{b}_k(\mathbf{y} \mid \boldsymbol{\theta}) \triangleq \nabla_{\boldsymbol{\theta}} \log \phi_k(\mathbf{y} \mid \boldsymbol{\theta}) \\
&\mathbf{v}(\mathbf{y} \mid \boldsymbol{\theta}) \triangleq \nabla_{\boldsymbol{\theta}} \log f(\mathbf{y} \mid \boldsymbol{\theta}) = \lambda_1(\mathbf{y} \mid \boldsymbol{\theta}) \mathbf{b}_1(\mathbf{y} \mid \boldsymbol{\theta}) + \lambda_2(\mathbf{y} \mid \boldsymbol{\theta}) \mathbf{b}_2(\mathbf{y} \mid \boldsymbol{\theta}) \\
&\mathbf{g}_2(\boldsymbol{\theta}) \triangleq \nabla_{\boldsymbol{\theta}} \hat{u}(\boldsymbol{\theta}, h) \triangleq \sum_{n=1}^{N}\left(\lambda_3(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_3(\mathbf{y}_n \mid \boldsymbol{\theta}) + \lambda_4(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_4(\mathbf{y}_n \mid \boldsymbol{\theta})\right)
\end{aligned}
$$

Derivatives calculations:

$$
\begin{aligned}
&\log \phi_c\left(\mathbf{y}; \rho e^{j\vartheta} \mathbf{a}(\varphi), \boldsymbol{\Sigma}\right) = -\log|\pi\boldsymbol{\Sigma}| - \left(\mathbf{y} - \rho e^{j\vartheta}\mathbf{a}(\varphi)\right)^H \boldsymbol{\Sigma}^{-1}\left(\mathbf{y} - \rho e^{j\vartheta}\mathbf{a}(\varphi)\right) = \\
&-\log|\pi\boldsymbol{\Sigma}| - \|\mathbf{y}\|^2 + 2\rho \operatorname{Re}\left\{e^{j\vartheta}\mathbf{y}^H \boldsymbol{\Sigma}^{-1}\mathbf{a}(\varphi)\right\} - \rho^2 \mathbf{a}^H(\varphi) \boldsymbol{\Sigma}^{-1} \mathbf{a}(\varphi) = -\log|\pi\boldsymbol{\Sigma}| - \|\tilde{\mathbf{y}}\|^2 + 2\rho \operatorname{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\mathbf{t}(\varphi)\right\} - \rho^2\ \|\mathbf{t}(\varphi)\|^2 \\
&\tilde{\mathbf{y}} \triangleq \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{y},\ \mathbf{t}(\varphi) \triangleq \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{a}(\varphi)
\end{aligned}
$$

$$\frac{\partial \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \varphi} = 2\rho\,\mathrm{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\dot{\mathbf{t}}(\varphi)\right\} - \rho^2\eta(\varphi),\ \ \eta(\varphi) \triangleq \frac{\partial \left\|\mathbf{t}(\varphi)\right\|^2}{\partial \varphi}$$

$$\frac{\partial \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \rho} = 2\,\mathrm{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\mathbf{t}(\varphi)\right\} - 2\rho\left\|\mathbf{t}(\varphi)\right\|^2$$

$$\frac{\partial \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \vartheta} = 2\rho\frac{\partial}{\partial \vartheta}\left\{\mathrm{Re}\left\{e^{j\vartheta}\underbrace{\tilde{\mathbf{y}}^H\mathbf{t}(\varphi)}_{\triangleq C(\varphi)}\right\}\right\} = 2\rho\frac{\partial}{\partial \vartheta}\left\{\mathrm{Re}\left\{e^{j\vartheta}C(\varphi)\right\}\right\} =$$

$$-2\rho\,\mathrm{Im}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\mathbf{t}(\varphi)\right\}$$

$$e^{j\vartheta}C(\varphi) = \mathrm{Re}\left\{\left(\cos(\vartheta) + j\sin(\vartheta)\right)\left(C_R(\varphi) + jC_I(\varphi)\right)\right\}$$

$$= \cos(\vartheta)C_R(\varphi) - \sin(\vartheta)C_I(\varphi) + j\left(\cos(\vartheta)C_I(\varphi) + j\sin(\vartheta)C_R(\varphi)\right)$$

$$\frac{\partial}{\partial \vartheta}\left\{\mathrm{Re}\left\{e^{j\vartheta}C(\varphi)\right\}\right\} = -\sin(\vartheta)C_R(\varphi) - \cos(\vartheta)C_I(\varphi) = -\mathrm{Im}\left\{e^{j\vartheta}C(\varphi)\right\}$$

$$\frac{\partial^2 \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \varphi^2} = 2\rho\,\mathrm{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\ddot{\mathbf{t}}(\varphi)\right\} - \rho^2\psi(\varphi),\ \ \psi(\varphi) \triangleq \frac{\partial^2 \left\|\mathbf{t}(\varphi)\right\|^2}{\partial \varphi^2}$$

$$\frac{\partial^2 \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \varphi \partial \rho} = 2\,\mathrm{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\dot{\mathbf{t}}(\varphi)\right\} - 2\rho\eta(\varphi)$$

$$\frac{\partial^2 \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \varphi \partial \vartheta} = -2\rho\,\mathrm{Im}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\dot{\mathbf{t}}(\varphi)\right\}$$

$$\frac{\partial^2 \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \rho^2} = 2\,\mathrm{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\mathbf{t}(\varphi)\right\} - \rho^2\left\|\mathbf{t}(\varphi)\right\|^2$$

$$\frac{\partial^2 \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \rho \partial \vartheta} = -2\,\mathrm{Im}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\mathbf{t}(\varphi)\right\}$$

$$\frac{\partial^2 \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \vartheta^2} = -2\rho\frac{\partial}{\partial \vartheta}\left\{\mathrm{Im}\left\{e^{j\vartheta}C(\varphi)\right\}\right\} = -2\rho\,\mathrm{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\mathbf{t}(\varphi)\right\}$$

$$e^{j\vartheta}C(\varphi) = \mathrm{Re}\left\{\left(\cos(\vartheta) + j\sin(\vartheta)\right)\left(C_R(\varphi) + jC_I(\varphi)\right)\right\}$$

$$= \cos(\vartheta)C_R(\varphi) - \sin(\vartheta)C_I(\varphi) + j\left(\cos(\vartheta)C_I(\varphi) + \sin(\vartheta)C_R(\varphi)\right)$$

$$\frac{\partial}{\partial \vartheta}\left\{\mathrm{Im}\left\{e^{j\vartheta}C(\varphi)\right\}\right\} = -\sin(\vartheta)C_I(\varphi) + \cos(\vartheta)C_I(\varphi) = \mathrm{Re}\left\{e^{j\vartheta}C(\varphi)\right\}$$

$$\frac{\partial \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \varphi} = 2\rho\,\mathrm{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\dot{\mathbf{t}}(\varphi)\right\} - \rho^2\eta(\varphi)$$

$$\frac{\partial \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \rho} = 2\,\mathrm{Re}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\mathbf{t}(\varphi)\right\} - 2\rho\left\|\mathbf{t}(\varphi)\right\|^2$$

$$\frac{\partial \log \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\mathbf{\Sigma}\right)}{\partial \vartheta} = 2\rho\frac{\partial}{\partial\vartheta}\left\{\mathrm{Re}\left\{e^{j\vartheta}\underbrace{\tilde{\mathbf{y}}^H\mathbf{t}(\varphi)}_{\triangleq C(\varphi)}\right\}\right\} = 2\rho\frac{\partial}{\partial\vartheta}\left\{\mathrm{Re}\left\{e^{j\vartheta}C(\varphi)\right\}\right\} =$$

$$-2\rho\,\mathrm{Im}\left\{\tilde{\mathbf{y}}^H e^{j\vartheta}\dot{\mathbf{t}}(\varphi)\right\}$$

$$e^{j\vartheta}C(\varphi) = \mathrm{Re}\left\{\left(\cos(\vartheta)+j\sin(\vartheta)\right)\left(C_R(\varphi)+jC_I(\varphi)\right)\right\}$$

$$=\cos(\vartheta)C_R(\varphi)-\sin(\vartheta)C_I(\varphi)+j\left(\cos(\vartheta)C_I(\varphi)+j\sin(\vartheta)C_R(\varphi)\right)$$

$$\frac{\partial}{\partial\vartheta}\left\{\mathrm{Re}\left\{e^{j\vartheta}C(\varphi)\right\}\right\} = -\sin(\vartheta)C_R(\varphi)-\cos(\vartheta)C_I(\varphi) = -\,\mathrm{Im}\left\{e^{j\vartheta}C(\varphi)\right\}$$

For the calculation of $\eta(\varphi)\triangleq\frac{\partial\left\|\mathbf{t}(\varphi)\right\|^2}{\partial\varphi}$ and $\psi(\varphi)\triangleq\frac{\partial^2\left\|\mathbf{t}(\varphi)\right\|^2}{\partial\varphi^2}$ Note that:

$$(a)\ \mathbf{a}^H\mathbf{A}^{-1}\mathbf{a} = \|\mathbf{a}\|_{\mathbf{A}}^2 = \mathrm{Re}\left\{\left(\mathbf{a}_R - j\mathbf{a}_I\right)^T\left(\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}+j\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}\right)\left(\mathbf{a}_R+j\mathbf{a}_I\right)\right\} =$$

$$\mathrm{Re}\left\{\left(\mathbf{a}_R^T\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}+j\mathbf{a}_R^T\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}-j\mathbf{a}_I^T\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}+\mathbf{a}_I^T\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}\right)\left(\mathbf{a}_R+j\mathbf{a}_I\right)\right\} =$$

$$\mathbf{a}_R^T\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}\mathbf{a}_R + \mathbf{a}_I^T\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}\mathbf{a}_R - \mathbf{a}_R^T\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}\mathbf{a}_I + \mathbf{a}_I^T\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}\mathbf{a}_I$$

$$(b)\frac{1}{2}\begin{bmatrix}\mathbf{a}_R^T & \mathbf{a}_I^T\end{bmatrix}\begin{bmatrix}\frac{1}{2}\mathrm{Re}\{\mathbf{A}\} & -\frac{1}{2}\mathrm{Im}\{\mathbf{A}\}\\ \frac{1}{2}\mathrm{Im}\{\mathbf{A}\} & \frac{1}{2}\mathrm{Re}\{\mathbf{A}\}\end{bmatrix}^{-1}\begin{bmatrix}\mathbf{a}_R\\ \mathbf{a}_I\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\mathbf{a}_R^T & \mathbf{a}_I^T\end{bmatrix}\begin{bmatrix}2\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\} & -2\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}\\ 2\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\} & 2\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}\end{bmatrix}\begin{bmatrix}\mathbf{a}_R\\ \mathbf{a}_I\end{bmatrix} =$$

$$\begin{bmatrix}\mathbf{a}_R^T\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}+\mathbf{a}_R^T\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\} & -\mathbf{a}_R^T\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}+\mathbf{a}_I^T\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}\end{bmatrix}\begin{bmatrix}\mathbf{a}_R\\ \mathbf{a}_I\end{bmatrix} =$$

$$\mathbf{a}_R^T\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}\mathbf{a}_R + \mathbf{a}_I^T\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}\mathbf{a}_R - \mathbf{a}_R^T\,\mathrm{Im}\left\{\mathbf{A}^{-1}\right\}\mathbf{a}_I + \mathbf{a}_I^T\,\mathrm{Re}\left\{\mathbf{A}^{-1}\right\}\mathbf{a}_I$$

Therefore, (a)=(b) and:

$$\eta(\varphi)=\frac{\partial\left\{\mathbf{a}^H(\varphi)\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}}{\partial\varphi}=\frac{1}{2}\frac{\partial\left\{\mathbf{b}^T(\varphi)\left(\frac{1}{2}\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}\right)^{-1}\mathbf{b}(\varphi)\right\}}{\partial\varphi}\underset{\text{Eq.93 in MatrixCookBook}}{=}$$

$$\frac{1}{2}\dot{\mathbf{b}}^T(\varphi)\left(\frac{1}{2}\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}\right)^{-1}\mathbf{b}(\varphi)+\frac{1}{2}\dot{\mathbf{b}}^T(\varphi)\left(\left(\frac{1}{2}\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}\right)^{-1}\right)^T\mathbf{b}(\varphi)$$

$$\mathbf{b}(\varphi)=\left[\mathbf{a}_R^T(\varphi),\mathbf{a}_I^T(\varphi)\right]^T,\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}=\begin{bmatrix}\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}\} & -\mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}\}\\ \mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}\} & \mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}\}\end{bmatrix}$$

$$\mathbf{a}_R(\varphi)\triangleq\mathrm{Re}\{\mathbf{a}(\varphi)\}=\left[1,\cos(\pi\sin(\varphi)),...,\cos(\pi(p-1)\sin(\varphi))\right]^T$$

$$\mathbf{a}_I(\varphi)\triangleq\mathrm{Im}\{\mathbf{a}(\varphi)\}=-\left[0,\sin(\pi\sin(\varphi)),...,\sin(\pi(p-1)\sin(\varphi))\right]^T$$

$$\dot{\mathbf{b}}(\varphi)\triangleq\frac{\partial\mathbf{b}(\varphi)}{\partial\varphi}$$

Note that:

$$\left(\frac{1}{2}\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}\right)^{-1}=\begin{bmatrix}2\,\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\} & -2\,\mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\}\\ 2\,\mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\} & 2\,\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\}\end{bmatrix}$$

Hence:

$$\eta(\varphi)=\dot{\mathbf{b}}^{\mathbf{T}}(\varphi)\mathbf{B}_{\mathbf{w}}\mathbf{b}(\varphi)$$

$$\mathbf{B}_{\mathbf{w}}\triangleq\frac{1}{2}\left(\left(\frac{1}{2}\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}\right)^{-1}+\left(\left(\frac{1}{2}\tilde{\boldsymbol{\Sigma}}_{\mathbf{w}}\right)^{-1}\right)^T\right)=\frac{1}{2}\left(\begin{bmatrix}2\,\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\} & -2\,\mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\}\\ 2\,\mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\} & 2\,\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\}\end{bmatrix}+\begin{bmatrix}2\,\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\} & 2\,\mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\}\\ -2\,\mathrm{Im}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\} & 2\,\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\}\end{bmatrix}\right)=$$

$$\begin{bmatrix}2\,\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\} & \mathbf{0}\\ \mathbf{0} & 2\,\mathrm{Re}\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\}\end{bmatrix}$$

and note that:

$$(a')\ \mathrm{Re}\left\{\mathbf{a}^H\,\mathrm{Re}\{\mathbf{A}^{-1}\}\mathbf{b}\right\}=\mathrm{Re}\left\{(\mathbf{a}_R-j\mathbf{a}_I)^T\,\mathrm{Re}\{\mathbf{A}^{-1}\}(\mathbf{b}_R+j\mathbf{b}_I)\right\}=$$

$$\mathrm{Re}\left\{\left(\mathbf{a}_R^T\,\mathrm{Re}\{\mathbf{A}^{-1}\}-j\mathbf{a}_I^T\,\mathrm{Re}\{\mathbf{A}^{-1}\}\right)(\mathbf{b}_R+j\mathbf{b}_I)\right\}=\mathbf{a}_R^T\,\mathrm{Re}\{\mathbf{A}^{-1}\}\mathbf{b}_R+\mathbf{a}_I^T\,\mathrm{Re}\{\mathbf{A}^{-1}\}\mathbf{b}_I$$

$$(b)\begin{bmatrix}\mathbf{a}_R^T & \mathbf{a}_I^T\end{bmatrix}\begin{bmatrix}\mathrm{Re}\{\mathbf{A}^{-1}\} & \mathbf{0}\\ \mathbf{0} & \mathrm{Re}\{\mathbf{A}^{-1}\}\end{bmatrix}\begin{bmatrix}\mathbf{b}_R\\ \mathbf{b}_I\end{bmatrix}=\begin{bmatrix}\mathbf{a}_R^T\,\mathrm{Re}\{\mathbf{A}^{-1}\} & \mathbf{a}_I^T\,\mathrm{Re}\{\mathbf{A}^{-1}\}\end{bmatrix}\begin{bmatrix}\mathbf{b}_R\\ \mathbf{b}_I\end{bmatrix}=$$

$$\mathbf{a}_R^T\,\mathrm{Re}\{\mathbf{A}^{-1}\}\mathbf{b}_R+\mathbf{a}_I^T\,\mathrm{Re}\{\mathbf{A}^{-1}\}\mathbf{b}_I=(a')$$

To conclude:

$$\boxed{\eta(\boldsymbol{\varphi}) = 2\,\mathrm{Re}\left\{\dot{\mathbf{a}}^H(\boldsymbol{\varphi})\,\mathrm{Re}\left\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\right\}\mathbf{a}(\boldsymbol{\varphi})\right\}}$$

The second derivative:

$$\Psi(\varphi) \triangleq \frac{\partial^2 \left\|\mathbf{t}(\varphi)\right\|^2}{\partial \varphi^2} = \frac{\partial\{\eta(\varphi)\}}{\partial \varphi} = \frac{\partial\left\{\dot{\mathbf{b}}^{\mathrm{T}}(\varphi)\mathbf{B}_{\mathbf{w}}\mathbf{b}(\varphi)\right\}}{\partial \varphi} \underset{\text{Eq.(93) MatrixCookBook}}{=}$$
$$\ddot{\mathbf{b}}(\varphi)\mathbf{B}_{\mathbf{w}}\mathbf{b}(\varphi) + \dot{\mathbf{b}}^{\mathrm{T}}(\varphi)\mathbf{B}_{\mathbf{w}}\dot{\mathbf{b}}(\varphi) = 2\,\mathrm{Re}\left\{\ddot{\mathbf{a}}^H(\varphi)\,\mathrm{Re}\left\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\right\}\mathbf{a}(\varphi)\right\} + 2\,\mathrm{Re}\left\{\dot{\mathbf{a}}^H(\varphi)\,\mathrm{Re}\left\{\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\right\}\dot{\mathbf{a}}(\varphi)\right\}$$

Note that:

$$\phi_1(\mathbf{y}\,|\,\boldsymbol{\theta}) \triangleq \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\boldsymbol{\Sigma}_{\mathbf{w}}\right),\ \phi_2(\mathbf{y}\,|\,\boldsymbol{\theta}) \triangleq \phi_c\left(\mathbf{y};-\rho e^{j\vartheta}\mathbf{a}(\varphi),\boldsymbol{\Sigma}_{\mathbf{w}}\right)$$
$$\phi_3(\mathbf{y}\,|\,\boldsymbol{\theta}) \triangleq \phi_c\left(\mathbf{y};\rho e^{j\vartheta}\mathbf{a}(\varphi),\overline{\boldsymbol{\Sigma}}_h\right),\ \phi_4(\mathbf{y}\,|\,\boldsymbol{\theta}) \triangleq \phi_c\left(\mathbf{y};-\rho e^{j\vartheta}\mathbf{a}(\varphi),\overline{\boldsymbol{\Sigma}}_h\right),\ \overline{\boldsymbol{\Sigma}}_h \triangleq \boldsymbol{\Sigma}_{\mathbf{w}} + \tau,\ \tau \triangleq \sqrt{2}h$$
$$i=1,2:\ \lambda_i(\mathbf{y}\,|\,\boldsymbol{\theta}) \triangleq \frac{\phi_i(\mathbf{y}\,|\,\boldsymbol{\theta})}{\phi_1(\mathbf{y}\,|\,\boldsymbol{\theta}) + \phi_2(\mathbf{y}\,|\,\boldsymbol{\theta})}$$
$$j=3,4:\ \lambda_j(\mathbf{y}\,|\,\boldsymbol{\theta}) \triangleq \frac{\phi_j(\mathbf{y}\,|\,\boldsymbol{\theta})}{\sum_{n=1}^{N}\left(\phi_3(\mathbf{y}_n\,|\,\boldsymbol{\theta}) + \phi_4(\mathbf{y}_n\,|\,\boldsymbol{\theta})\right)}$$
$$\lambda_1(\mathbf{y}\,|\,\boldsymbol{\theta}) + \lambda_2(\mathbf{y}\,|\,\boldsymbol{\theta}) = 1$$
$$\lambda_2(\mathbf{y}\,|\,\boldsymbol{\theta}) - \lambda_1(\mathbf{y}\,|\,\boldsymbol{\theta}) = \frac{\phi_2(\mathbf{y}\,|\,\boldsymbol{\theta}) - \phi_1(\mathbf{y}\,|\,\boldsymbol{\theta})}{\phi_1(\mathbf{y}\,|\,\boldsymbol{\theta}) + \phi_2(\mathbf{y}\,|\,\boldsymbol{\theta})} = \frac{e^{2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}^H\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}} - e^{-2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}^H\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}}}{e^{-2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}^H\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}} + e^{2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}^H\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}}}$$
$$= \tanh\left(2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}^H\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}\right)$$
$$\sum_{n=1}^{N}\left(\lambda_3(\mathbf{y}_n\,|\,\boldsymbol{\theta}) + \lambda_4(\mathbf{y}_n\,|\,\boldsymbol{\theta})\right) = 1$$
$$\sum_{n=1}^{N}\left(\lambda_3(\mathbf{y}_n\,|\,\boldsymbol{\theta}) - \lambda_4(\mathbf{y}_n\,|\,\boldsymbol{\theta})\right) = \frac{\sum_{n=1}^{N} e^{\mathbf{y}_n^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{y}_n}\left(e^{2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{a}(\varphi)\right\}} - e^{-2\rho\,\mathrm{Re}\left\{2\rho e^{j\vartheta}\mathbf{y}_n^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{a}(\varphi)\right\}}\right)}{\sum_{m=1}^{N} e^{\mathbf{y}_m^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{y}_m}\left(e^{2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_m^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{a}(\varphi)\right\}} - e^{-2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_m^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{a}(\varphi)\right\}}\right)}$$
$$= \frac{\sum_{n=1}^{N} e^{\mathbf{y}_n^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{y}_n}\sinh\left(2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{a}(\varphi)\right\}\right)}{\sum_{m=1}^{N} e^{\mathbf{y}_m^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{y}_m}\cosh\left(2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_m^H\overline{\boldsymbol{\Sigma}}_h^{-1}\mathbf{a}(\varphi)\right\}\right)}$$

Therefore,

$$\mathbf{T}_1(\boldsymbol{\theta}) = \begin{bmatrix} 2\alpha\,\mathrm{Re}\left\{\mathbf{h}^H(\boldsymbol{\theta})e^{j\vartheta}\ddot{\mathbf{a}}(\varphi)\right\} - \rho^2\Psi(\varphi) & 2\,\mathrm{Re}\left\{\tilde{\mathbf{h}}^H(\boldsymbol{\theta})e^{j\vartheta}\dot{\mathbf{a}}(\varphi)\right\} - 2\rho\eta(\varphi) & -2\rho\,\mathrm{Im}\left\{\mathbf{h}^H(\boldsymbol{\theta})e^{j\vartheta}\dot{\mathbf{a}}(\varphi)\right\} \\ & -2\left\|\mathbf{t}(\varphi)\right\|^2 & -2\,\mathrm{Im}\left\{\mathbf{h}^H(\boldsymbol{\theta})e^{j\vartheta}\mathbf{a}(\varphi)\right\} \\ & & -2\rho\,\mathrm{Re}\left\{\mathbf{h}^H(\boldsymbol{\theta})e^{j\vartheta}\mathbf{a}(\varphi)\right\} \end{bmatrix}$$

$$\mathbf{h}(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n;h)\tanh\left(2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}\right)\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{y}_n$$

For $\mathbf{T}_2(\boldsymbol{\theta})$:

$$\mathbf{T}_2(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n;h)\left(\lambda_1(\mathbf{y}_n|\boldsymbol{\theta})\mathbf{b}_1(\mathbf{y}_n|\boldsymbol{\theta})\mathbf{b}_1^T(\mathbf{y}_n|\boldsymbol{\theta}) + \lambda_2(\mathbf{y}_n|\boldsymbol{\theta})\mathbf{b}_2(\mathbf{y}_n|\boldsymbol{\theta})\mathbf{b}_2^T(\mathbf{y}_n|\boldsymbol{\theta})\right)$$

$$\mathbf{b}_1(\mathbf{y}_n|\boldsymbol{\theta}) \triangleq \begin{bmatrix} 2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\dot{\mathbf{a}}(\varphi)\right\} - \rho^2\eta(\varphi) \\ 2\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\} - 2\rho\left\|\mathbf{t}(\varphi)\right\|^2 \\ -2\rho\,\mathrm{Im}\left\{\mathbf{y}_n^H e^{j\vartheta}\mathbf{a}(\varphi)\right\} \end{bmatrix},\ \mathbf{b}_2(\mathbf{y}_n|\boldsymbol{\theta}) \triangleq \begin{bmatrix} -2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\dot{\mathbf{a}}(\varphi)\right\} - \rho^2\eta(\varphi) \\ -2\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\} - 2\rho\left\|\mathbf{t}(\varphi)\right\|^2 \\ 2\rho\,\mathrm{Im}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\} \end{bmatrix}$$

For $\mathbf{T}_3(\boldsymbol{\theta})$:

$$\mathbf{T}_3(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n;h)\mathbf{v}(\mathbf{y}_n|\boldsymbol{\theta})\mathbf{v}^T(\mathbf{y}_n|\boldsymbol{\theta})$$

$$\mathbf{v}(\mathbf{y}_n|\boldsymbol{\theta}) = \begin{bmatrix} 2\rho\tanh\left(2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}\right)\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\dot{\mathbf{a}}(\varphi)\right\} - \rho^2\eta(\varphi) \\ 2\tanh\left(2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}\right)\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\} - 2\rho\left\|\mathbf{t}(\varphi)\right\|^2 \\ -2\rho\tanh\left(2\rho\,\mathrm{Re}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\}\right)\mathrm{Im}\left\{e^{j\vartheta}\mathbf{y}_n^H(\boldsymbol{\theta})\boldsymbol{\Sigma}_{\mathbf{w}}^{-1}\mathbf{a}(\varphi)\right\} \end{bmatrix}$$

For $\mathbf{T}_4(\boldsymbol{\theta})$:

$$\mathbf{T}_4(\boldsymbol{\theta}) \triangleq \begin{bmatrix} 2\rho\,\mathrm{Re}\left\{\mathbf{s}^H(\boldsymbol{\theta})e^{j\vartheta}\ddot{\mathbf{a}}(\varphi)\right\} - \rho^2\Gamma(\varphi) & 2\,\mathrm{Re}\left\{\mathbf{s}^H(\boldsymbol{\theta})e^{j\vartheta}\dot{\mathbf{a}}(\varphi)\right\} - 2\rho\zeta(\varphi) & -2\rho\,\mathrm{Im}\left\{\mathbf{s}^H(\boldsymbol{\theta})e^{j\vartheta}\dot{\mathbf{a}}(\varphi)\right\} \\ & -2\left\|\mathbf{k}(\varphi)\right\|^2 & -2\,\mathrm{Im}\left\{\mathbf{s}^H(\boldsymbol{\theta})e^{j\vartheta}\mathbf{a}(\varphi)\right\} \\ & & -2\rho\,\mathrm{Re}\left\{\mathbf{s}^H(\boldsymbol{\theta})e^{j\vartheta}\mathbf{a}(\varphi)\right\} \end{bmatrix}$$

where

$$\boxed{\begin{array}{l} \mathbf{s}(\boldsymbol{\theta}) \triangleq \dfrac{\sum_{n=1}^{N} e^{\mathbf{y}_n^H \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{y}_n} \sinh\left(2\rho \operatorname{Re}\left\{e^{j\vartheta} \mathbf{y}_n^H \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{a}(\varphi)\right\}\right) \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{y}_n}{\sum_{m=1}^{N} e^{\mathbf{y}_m^H \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{y}_m} \cosh\left(2\rho \operatorname{Re}\left\{e^{j\vartheta} \mathbf{y}_m^H \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{a}(\varphi)\right\}\right)},\ \mathbf{k}(\varphi) \triangleq \bar{\boldsymbol{\Sigma}}_h^{-\frac{1}{2}} \mathbf{a}(\varphi), \\ \zeta(\varphi) \triangleq \dfrac{\partial \left\|\mathbf{k}(\varphi)\right\|^2}{\partial \varphi},\ \Gamma(\varphi) \triangleq \dfrac{\partial^2 \left\|\mathbf{k}(\varphi)\right\|^2}{\partial \varphi^2} \end{array}}$$

For $\mathbf{T}_5(\boldsymbol{\theta})$:

$$\boxed{\begin{array}{l} \mathbf{T}_5(\boldsymbol{\theta}) \triangleq \sum_{n=1}^{N} \left(\lambda_3(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_3(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_1^T(\mathbf{y}_n \mid \boldsymbol{\theta}) + \lambda_2(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_2(\mathbf{y}_n \mid \boldsymbol{\theta}) \mathbf{b}_2^T(\mathbf{y}_n \mid \boldsymbol{\theta})\right) \\ \mathbf{b}_3(\mathbf{y}_n \mid \boldsymbol{\theta}) \triangleq \begin{bmatrix} 2\alpha \operatorname{Re}\left\{e^{j\vartheta} \mathbf{y}_n^H(\boldsymbol{\theta}) \bar{\boldsymbol{\Sigma}}_h^{-1} \dot{\mathbf{a}}(\varphi)\right\} - \rho^2 \zeta(\varphi) \\ 2 \operatorname{Re}\left\{e^{j\vartheta} \mathbf{y}_n^H(\boldsymbol{\theta}) \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{a}(\varphi)\right\} - 2\rho \left\|\mathbf{k}(\varphi)\right\|^2 \\ -2\alpha \operatorname{Im}\left\{e^{j\vartheta} \mathbf{y}_n^H(\boldsymbol{\theta}) \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{a}(\varphi)\right\} \end{bmatrix}, \\ \mathbf{b}_4(\mathbf{y}_n \mid \boldsymbol{\theta}) \triangleq \begin{bmatrix} -2\alpha \operatorname{Re}\left\{e^{j\vartheta} \mathbf{y}_n^H(\boldsymbol{\theta}) \bar{\boldsymbol{\Sigma}}_h^{-1} \dot{\mathbf{a}}(\varphi)\right\} - \rho^2 \zeta(\varphi) \\ -2 \operatorname{Re}\left\{e^{j\vartheta} \mathbf{y}_n^H(\boldsymbol{\theta}) \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{a}(\varphi)\right\} - 2\rho \left\|\mathbf{k}(\varphi)\right\|^2 \\ 2\alpha \operatorname{Im}\left\{e^{j\vartheta} \mathbf{y}_n^H(\boldsymbol{\theta}) \bar{\boldsymbol{\Sigma}}_h^{-1} \mathbf{a}(\varphi)\right\} \end{bmatrix} \end{array}}$$

For $\mathbf{T}_6(\boldsymbol{\theta})$:

$$\boxed{\begin{array}{l} \mathbf{T}_6(\boldsymbol{\theta}) \triangleq \mathbf{g}_2(\boldsymbol{\theta}) \mathbf{g}_2^T(\boldsymbol{\theta}) \\ \mathbf{g}_2(\boldsymbol{\theta}) = \begin{bmatrix} 2\rho \operatorname{Re}\left\{\mathbf{s}^H(\boldsymbol{\theta}) e^{j\vartheta} \dot{\mathbf{a}}(\varphi)\right\} - \rho^2 \zeta(\varphi) \\ 2 \operatorname{Re}\left\{\mathbf{s}^H(\boldsymbol{\theta}) e^{j\vartheta} \mathbf{a}(\varphi)\right\} - 2\rho \left\|\mathbf{k}(\varphi)\right\|^2 \\ -2\rho \operatorname{Im}\left\{\mathbf{s}^H(\boldsymbol{\theta}) e^{j\vartheta} \mathbf{a}(\varphi)\right\} \end{bmatrix} \end{array}}$$

Calculation of $\hat{\mathbf{c}}(\mathbf{r} \mid \boldsymbol{\theta}, h)$:

$$\boxed{\hat{\mathbf{c}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \nabla_{\boldsymbol{\theta}} \log f(\mathbf{r} \mid \boldsymbol{\theta}) - \nabla_{\boldsymbol{\theta}} \log u(\boldsymbol{\theta}) = \hat{\mathbf{v}}(\mathbf{y}_n \mid \boldsymbol{\theta}) - \mathbf{g}_2(\boldsymbol{\theta})}$$

Calculation of $\hat{\mathbf{z}}(\mathbf{r} \mid \boldsymbol{\theta}, h)$:

$$
\begin{aligned}
&\hat{\mathbf{z}}(\mathbf{r}\,|\,\boldsymbol{\theta},h) \triangleq \frac{v(\mathbf{r}\,|\,\boldsymbol{\theta},h)}{\hat{u}(\boldsymbol{\theta},h)} \nabla_{\boldsymbol{\theta}} \log \frac{v(\mathbf{r}\,|\,\boldsymbol{\theta},h)}{\hat{u}(\boldsymbol{\theta},h)} = \frac{\dot{v}(\mathbf{r}\,|\,\boldsymbol{\theta},h) - v(\mathbf{r}\,|\,\boldsymbol{\theta},h)\nabla_{\boldsymbol{\theta}} \log \hat{u}(\boldsymbol{\theta},h)}{\hat{u}(\boldsymbol{\theta},h)} \\
&\hat{u}(\boldsymbol{\theta},h) \triangleq \frac{1}{N}\sum_{n=1}^{N} \tilde{f}_h(\mathbf{y}_n\,|\,\boldsymbol{\theta}) = \frac{1}{2N}\sum_{n=1}^{N}\left(\phi_3(\mathbf{y}_n\,|\,\boldsymbol{\theta}) + \phi_4(\mathbf{y}_n\,|\,\boldsymbol{\theta})\right) \\
&v(\mathbf{r}\,|\,\boldsymbol{\theta},h) \triangleq \left(K_h * f\right)(\mathbf{r}\,|\,\boldsymbol{\theta}) = \tilde{f}_h(\mathbf{r}\,|\,\boldsymbol{\theta}) = \frac{\phi_3(\mathbf{r}\,|\,\boldsymbol{\theta}) + \phi_4(\mathbf{r}\,|\,\boldsymbol{\theta})}{2} \\
&\dot{v}(\mathbf{r}\,|\,\boldsymbol{\theta},h) = \frac{\phi_3(\mathbf{r}\,|\,\boldsymbol{\theta})\mathbf{b}_3(\mathbf{r}\,|\,\boldsymbol{\theta}) + \phi_4(\mathbf{r}\,|\,\boldsymbol{\theta})(\mathbf{r}\,|\,\boldsymbol{\theta})}{2} \\
&\Downarrow \\
&\hat{\mathbf{z}}(\mathbf{r}\,|\,\boldsymbol{\theta},h) = N\left(\lambda_3(\mathbf{r}\,|\,\boldsymbol{\theta})\mathbf{b}_3(\mathbf{r}\,|\,\boldsymbol{\theta}) + \lambda_3(\mathbf{r}\,|\,\boldsymbol{\theta})\mathbf{b}_3(\mathbf{r}\,|\,\boldsymbol{\theta}) - \lambda_3(\mathbf{r}\,|\,\boldsymbol{\theta})\mathbf{g}_2(\boldsymbol{\theta}) - \lambda_4(\mathbf{r}\,|\,\boldsymbol{\theta})\mathbf{g}_2(\boldsymbol{\theta})\right) \\
&N\left(\lambda_3(\mathbf{r}\,|\,\boldsymbol{\theta})\left(\mathbf{b}_3(\mathbf{r}\,|\,\boldsymbol{\theta}) - \mathbf{g}_2(\boldsymbol{\theta})\right) + \lambda_4(\mathbf{r}\,|\,\boldsymbol{\theta})\left(\mathbf{b}_4(\mathbf{r}\,|\,\boldsymbol{\theta}) - \mathbf{g}_2(\boldsymbol{\theta})\right)\right)
\end{aligned}
$$

$$h_{opt} \triangleq \arg\min_{h\in I}\left\{tr\left[\mathbf{W}\hat{\mathbf{R}}(\hat{\boldsymbol{\theta}}_h, h)\right]\right\}$$

where

$$\hat{\mathbf{R}}(\boldsymbol{\theta}, h) \triangleq \frac{1}{N}\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}, h)$$

$$\hat{\boldsymbol{\Sigma}}(\boldsymbol{\theta}, h) \triangleq \hat{\mathbf{C}}^{-1}(\boldsymbol{\theta}, h)\hat{\mathbf{D}}(\boldsymbol{\theta}, h)\hat{\mathbf{C}}^{-1}(\boldsymbol{\theta}, h)$$

$$\hat{\mathbf{C}}(\boldsymbol{\theta}, h) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n; h)\nabla^2_{\boldsymbol{\theta}} \log f\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right) - \nabla^2_{\boldsymbol{\theta}} \log \hat{u}\left(\boldsymbol{\theta}, h\right) = \nabla^2_{\boldsymbol{\theta}} J_h\left(\boldsymbol{\theta}\right)$$

$$\nabla^2_{\boldsymbol{\theta}} J_h\left(\boldsymbol{\theta}\right) \triangleq \mathbf{T}_1\left(\boldsymbol{\theta}\right) + \mathbf{T}_2\left(\boldsymbol{\theta}\right) - \mathbf{T}_3\left(\boldsymbol{\theta}\right) - \mathbf{T}_4\left(\boldsymbol{\theta}\right) - \mathbf{T}_5\left(\boldsymbol{\theta}\right) + \mathbf{T}_6\left(\boldsymbol{\theta}\right)$$

$$\mathbf{T}_1\left(\boldsymbol{\theta}\right) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n; h)\left(\lambda_1\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\nabla^2_{\boldsymbol{\theta}} \log \phi_1\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right) + \lambda_2\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\nabla^2_{\boldsymbol{\theta}} \log \phi_2\left(\mathbf{x}_n \mid \boldsymbol{\theta}\right)\right)$$

$$\mathbf{T}_2\left(\boldsymbol{\theta}\right) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n; h)\left(\lambda_1\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{b}_1\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{b}_1^T\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right) + \lambda_2\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{b}_2\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{b}_2^T\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\right)$$

$$\mathbf{T}_3\left(\boldsymbol{\theta}\right) \triangleq \sum_{n=1}^{N} w(\mathbf{y}_n; h)\mathbf{v}\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{v}^T\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)$$

$$\mathbf{T}_4\left(\boldsymbol{\theta}\right) \triangleq \sum_{n=1}^{N} \lambda_3\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\nabla^2_{\boldsymbol{\theta}} \log \phi_3\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right) + \lambda_4\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\nabla^2_{\boldsymbol{\theta}} \log \phi_4\left(\mathbf{x}_n \mid \boldsymbol{\theta}\right)$$

$$\mathbf{T}_5\left(\boldsymbol{\theta}\right) \triangleq \sum_{n=1}^{N} \lambda_3\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{b}_3\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{b}_3^T\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right) + \lambda_4\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{b}_4\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)\mathbf{b}_4^T\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right)$$

$$\mathbf{T}_6\left(\boldsymbol{\theta}\right) \triangleq \mathbf{g}_2\left(\boldsymbol{\theta}\right)\mathbf{g}_2^T\left(\boldsymbol{\theta}\right)$$

$$\hat{\mathbf{D}}(\boldsymbol{\theta}, h) \triangleq \frac{1}{N}\sum_{n=1}^{N} \hat{\mathbf{v}}\left(\mathbf{y}_n \mid \boldsymbol{\theta}, h\right)\hat{\mathbf{v}}^T\left(\mathbf{y}_n \mid \boldsymbol{\theta}, h\right)$$

$$\hat{\mathbf{v}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \hat{\psi}_G(\mathbf{r} \mid \boldsymbol{\theta})\hat{\mathbf{c}}(\mathbf{r} \mid \boldsymbol{\theta}, h) + \hat{\mathbf{d}}(\mathbf{r} \mid \boldsymbol{\theta}, h) - \hat{\mathbf{z}}(\mathbf{r} \mid \boldsymbol{\theta}, h)$$

$$\hat{\mathbf{c}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \nabla_{\boldsymbol{\theta}} \log f\left(\mathbf{r} \mid \boldsymbol{\theta}\right) - \nabla_{\boldsymbol{\theta}} \log u\left(\boldsymbol{\theta}\right) = \hat{\mathbf{v}}\left(\mathbf{y}_n \mid \boldsymbol{\theta}\right) - \mathbf{g}_2\left(\boldsymbol{\theta}\right)$$

$$\hat{\psi}_G(\mathbf{r}; \boldsymbol{\eta}) \triangleq (N-1)w\left(\mathbf{y}_n; h\right), \;\; \hat{\varphi}_h(\mathbf{r}) \triangleq \frac{K_h(\mathbf{r})}{N\sum_{n=1}^{N}\tilde{g}\left(\mathbf{y}_n; h\right)} = \frac{K_h(\mathbf{r})}{\sum_{n=1}^{N}\sum_{m\neq n} K_h\left(\mathbf{x}_n - \mathbf{x}_m\right)}$$

$$\hat{\mathbf{d}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq (N-1)\sum_{n=1}^{N}\left(\hat{\varphi}_h(\mathbf{y}_n - \mathbf{r})\hat{\mathbf{c}}(\mathbf{y}_n \mid \boldsymbol{\theta}, h) - N^{-1}\hat{\varphi}_h(\mathbf{0})\hat{\mathbf{c}}(\mathbf{r} \mid \boldsymbol{\theta}, h)\right)$$

$$\hat{\mathbf{z}}(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \frac{v(\mathbf{r} \mid \boldsymbol{\theta}, h)}{\hat{u}(\boldsymbol{\theta}, h)}\nabla_{\boldsymbol{\theta}} \log \frac{v(\mathbf{r} \mid \boldsymbol{\theta}, h)}{\hat{u}(\boldsymbol{\theta}, h)} = \frac{\dot{v}(\mathbf{r} \mid \boldsymbol{\theta}, h) - v(\mathbf{r} \mid \boldsymbol{\theta}, h)\nabla_{\boldsymbol{\theta}} \log \hat{u}(\boldsymbol{\theta}, h)}{\hat{u}(\boldsymbol{\theta}, h)}$$

$$v(\mathbf{r} \mid \boldsymbol{\theta}, h) \triangleq \left(K_h * f\right)(\mathbf{r} \mid \boldsymbol{\theta}) = \tilde{f}_h(\mathbf{r} \mid \boldsymbol{\theta})$$