Note on Sheaf Theory

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1. Introduction

The theory of sheaves has come to play a central role in the theories of several complex variables and holomorphic differential geometry. The theory is also essential to real analytic geometry. The theory of sheaves provides a framework for solving "local to global" problems of the sort that are normally solved using partitions of unity in the smooth case. Their profound importance extends throughout modern mathematics, particularly in algebraic geometry, where they serve as an indispensable framework. In this note, we develop the essential theory of sheaves from first principles. The theory of sheaves emerged in the 1940s through the groundbreaking work of Jean Leray while he was a prisoner of war. Initially conceived to study the topology of function spaces, sheaf theory was subsequently refined and generalized by Henri Cartan and Jean-Pierre Serre in the early 1950s. A decisive transformation occurred when Alexandre Grothendieck revolutionized algebraic geometry by introducing schemes and developing sheaf cohomology in the late 1950s and early 1960s, leading to the modern framework we present here. We begin by introducing the twin concepts of presheaves and sheaves on topological spaces, carefully examining their definitions and distinguishing characteristics. This naturally leads us to study morphisms between presheaves - the structure-preserving maps that relate these objects. We explore their key properties, including injectivity and surjectivity, and develop the crucial notion of exact sequences, which provides a powerful tool for analyzing relationships between sheaves. A central focus will be the direct image and inverse image functors - sophisticated operations that enable us to transport sheaves between different topological spaces. These functors are particularly vital in scheme theory, where they facilitate the transfer of local information between geometric objects. The note culminates in an exploration of bundle gluing, a technique that allows us to construct global objects from compatible local data - a theme that epitomizes the sheaf-theoretic approach.

2. Presheaves

Presheaves form the foundational structure in sheaf theory. Although a categorical approach would provide a more elegant presentation, we adopt a concrete perspective to emphasize the geometric intuition. We begin by introducing presheaves in their most basic form, then systematically develop their various types and properties.

Notation. *Let X be a topological space. We denote by $\mathcal{T}_X$ the category having for objects the open subsets of X and for morphisms identity maps and inclusions.*
$\mathcal{C}$ will denote a category, which can be the category of sets (also denoted by $\mathcal{S}et$), that of groups (also denoted by $\mathcal{G}p$), that of R-modules (also denoted by $\mathcal{R}ing$), that of R-modules (also denoted by R-$\mathcal{A}lg$), for some ring R.

Definition 2.1. *Let X be a topological space. A presheaf $\mathcal{F}$ on X consists of the following deta :*

i) For every open subset U of X, a set $\mathcal{F}(U)$.

ii) Whenever $U \subseteq V$ are two open subsets of X, a map

$$res_{V,U} : \mathcal{F}(V) \longrightarrow \mathcal{F}(U)$$

called the restriction map, which satisfies the following conditions :

a) $res_{U,U} = id_{\mathcal{F}(U)}$.

b) Having three open subsets $U \subseteq V \subseteq W$ of X, then $res_{V,U} \circ res_{W,V} = res_{W,U}$

Remarks 2.1. *1) We will mostly write $s_{|U}$ for s when $s \in \mathcal{F}(U)$. The elements of $\mathcal{F}(U)$ are usually called sections of (the presheaf $\mathcal{F}$) over U.*

2) By considering $\mathcal{F}(U)$ as object in some category $\mathcal{C}$ and assuming that $res_{V,U}$ is a morphism between the objects $\mathcal{F}(V)$ and $\mathcal{F}(U)$, we may define more generally a presheaf $\mathcal{F}$ on X into $\mathcal{C}$.

3) Note that we can state definition 2.1 in another way: Let X be a topological space. A presheaf $\mathcal{F}$ on X (into a category $\mathcal{C}$) is a contravariant functor from $\mathcal{T}_X$ into $\mathcal{C}$.

$$\begin{array}{cccc} \mathcal{F}: & \mathcal{T}_X & \longrightarrow & \mathcal{C} \\ & U & \longmapsto & \mathcal{F}(U) \end{array}$$

Examples 2.1. *1) For a topological space, a presheaf $\mathcal{C}_X$ of $\mathbb{R}$-algebras on X is defined by assigning to every open $U \subseteq X$ the set of continuous functions $U \longrightarrow \mathbb{R}$.*

2) *Let X be a variety, we previously considered the presheaf of k-algebras $\mathcal{O}_X$. For any open $U \subseteq X$, $\mathcal{O}_X(U)$ is the k-algebra of regular functions. If X be an affine variety we have $\mathcal{O}_X(U) = k[U]$.*

3) *Let X be a topological space, the formula :*

$$U \longmapsto \begin{cases} \mathbb{Z} & \text{if} \qquad U = X \\ \{0\} & \text{otherwise} \end{cases}$$

defines a presheaf of abelian groups on X.

Although it is possible to define a presheaf of a topological space X into an arbitrary category $\mathcal{C}$, we will be interested in what follows only in cases where the objects of $\mathcal{C}$ are sets (that could have an additional structure) and the morphisms $res_{V,U}$ are maps (which are morphisms for the extra structure on $\mathcal{F}(V)$ and $\mathcal{F}(U)$.

Definition 2.2. *Let $\mathcal{F}$ be a presheaf on X, a subpresheaf $\mathcal{G}$ (of $\mathcal{F}$) is a presheaf on X such that $\mathcal{G}(U) \subseteq \mathcal{F}(U)$ for every open $U \subseteq X$, and such that the restriction maps of $\mathcal{G}$ are induced by those of $\mathcal{F}$.*

Example 2.1. *If U is an open subset of X, every presheaf $\mathcal{F}$ on X induces, in an obvious way, a presheaf $\mathcal{F}_U$ on U by setting $\mathcal{F}_{|U}(V) = \mathcal{F}(V)$ for every open subset V of U. This is the restriction of $\mathcal{F}$ to U.*

2.1 Morphisms of presheaves

Definition 2.3. *Let $\mathcal{F}$ and $\mathcal{G}$ be two presheaves on X. A morphism of presheaves ψ from $\mathcal{F}$ to $\mathcal{G}$ consists of the datum, for all open U of X, of a morphism $\psi(U)$ from $\mathcal{F}(U)$ to $\mathcal{G}(U)$, so that the diagram*

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\psi(V)"] \arrow[d, "res_{V,U}"'] & \mathcal{G}(V) \arrow[d, "res_{V,U}"] \\ \mathcal{F}(U) \arrow[r, "\psi(U)"'] & \mathcal{G}(U) \end{tikzcd}$$

commutes for any pair (U, V) of open subsets of X such that $U \subseteq V$.

Remarks 2.2. 1) *The commutativity of the diagram is written : $\psi(V)(s)_{|U} = \psi(U)(S_{|U})$, where $s \in \mathcal{F}(V)$.*

2) *Morphisms of presheaves can be composed. So that presheaves on the topological space X form a category, that we will denote by $\mathcal{P}reSh_X$.*

3) *A morphism $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ between two presheaves $\mathcal{F}$ and $\mathcal{G}$ is an isomorphism if it has a two-sided inverse i.e, a morphism $\phi : \mathcal{G} \longrightarrow \mathcal{F}$ such that $\psi \circ \phi = id_{\mathcal{G}}$ and $\phi \circ \psi = id_{\mathcal{F}}$.*

Definition 2.4. *Assume $\mathcal{C}$ has direct limits. The stalk of a presheaf $\mathcal{F}$ at a point $x \in X$ is*

$$\mathcal{F}_x := \varinjlim_{x \in U} \mathcal{F}(U)$$

The direct limit is taken over open neighborhoods of x, and restriction maps between them. Given a section $s \in \mathcal{F}(U)$, and a point $x \in U$, we let $s_x \in \mathcal{F}_x$ denote the image of s under the natural morphism

$$\begin{array}{ccc} \mathcal{F}(U) & \longrightarrow & \mathcal{F}_x \\ s & \longmapsto & s_x \end{array}$$

An element of the stalk is called a germ.
More generally, if $Y \subseteq X$ is a closed and irreducible subset. Then, we set

$$\mathcal{F}_Y := \varinjlim_{U \cap Y \neq \emptyset} \mathcal{F}(U)$$

Notation. *Let X be a topological space and $x \in X$, we denote by $\mathcal{V}$ the set of open neighborhoods of x, which is filtering for the opposite order to inclusion i.e, for all $U, V \in \mathcal{V}$ we have*

$$U \leq V \Longleftrightarrow V \subseteq U.$$

Remark 2.1. *We can identify $\mathcal{F}_x$ as the quotient of the set of pairs (U, s), where $U \in \mathcal{V}$ and where s is a section of $\mathcal{F}$ on U, by the relation of equivalence defined as follows :*

$$(U, s) \sim (V, t) \text{ if and only if there exists an open neighborhood } W \text{ of } x \text{ in } U \cap V \\ \text{such that } s_{|W} = t_{|W}.$$

Moreover, we can see $\mathcal{F}_x$ as the set of sections of $\mathcal{F}$ defined in the neighborhood of x. Two sections belonging to $\mathcal{F}_x$ being considered as equal if they coincide in some neighborhood of x.

Example 2.2. *Let $\mathcal{F}(U) = \Big\{$ Continuous functions $U \longrightarrow \mathbb{R}\Big\}$. Then $\mathcal{F}_x$ the set of germs of continuous functions at x.*

Proposition 2.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves, then ψ induces for every point $x \in X$ a morphism $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ between the stalks, where ψ_x is defined by $\psi_x(s_x) = \big(\psi(U)(s)\big)_x$ for any open subset U of X, $s \in \mathcal{F}(U)$, and $x \in U$.*

Proof. *If $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ are such that $s_x = t_x$, then there exists an open neighborhood W of x such that $s_{|W} = t_{|W}$. So $\psi(U)(s)_{|W} = \psi(W)(s_{|W})$ and $\psi(V)(t)_{|W} = \psi(V)(t_{|W})$. Hence $\big(\psi(U)(s)\big)_x = (\psi(V)(t))_x$.*

Note that if $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ and $\phi : \mathcal{G} \longrightarrow \mathcal{Z}$ are two morphisms of sheaves we have $(\psi \circ \phi)_x = \psi_x \circ \phi_x$ and $(id_{\mathcal{F}})_x = id_{\mathcal{F}_x}$. Moreover, $\psi \longrightarrow \psi_x$ define a functor from the category of sheaves over X to the category $\mathcal{C}$.

Definition 2.5. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves*

i) We say that ψ is injective if for any open subset U of X, $\psi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ is injective.

ii) We say that ψ is surjective if for all $x \in X$, $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is surjective.

3. Sheaves

Definition 3.1. *We say that a presheaf $\mathcal{F}$ is a sheaf if we have the following properties:*

i) (Uniqueness) Let U be an open subset of X, $s \in \mathcal{F}(U)$, $\{U_i\}_{i \in I}$ a covering of U by open subsets U_i. If $s_{|U_i} = 0$ for every $i \in I$, then $s = 0$.

ii) (Gluing axiom) If $U = \bigcup_{i \in I} U_i$, and if $s_i \in \mathcal{F}(U_i)$ is a collection of sections matching on the overlaps; that is, they satisfy

$$s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$$

for all $i, j \in I$, then there exists a section $s \in \mathcal{F}(U)$ so that $s_{|U_i} = s_i$, for all $i \in I$

Remarks 3.1. *1) When $\mathcal{F}$ is a presheaf of groups or of an algebraic structure that is in particular a group, we can replace i) by: for all $s, t \in \mathcal{F}(U)$ such that for $i \in I$, $s_{|U_i} = t_{|U_i}$ then $s = t$.*

2) The section s in ii) is unique by condition i).

Examples 3.1. *1) Let X be a topological space, $U \longmapsto \mathcal{C}^0(U, \mathbb{R})$ is a sheaf of $\mathbb{R}$-algebras over X.*

2) In example 2.1, if moreover, $\mathcal{F}$ is a sheaf then $\mathcal{F}_{|U}$ is still a sheaf.

3.1 Morphisms of sheaves

Definition 3.2. *A morphism of sheaves is just a morphism of the underlying presheaves.*

Remarks 3.2. *1) The sheaves of X form a full subcategory $\mathcal{S}h_X$ of category of the presheaves on X.*

2) The notions injective, surjective and isomorphism for sheaves are defined in the same way as for presheaves.

Lemma 3.1. *Let X be a topological space and let U be an open subset of X.*

1) Let $\mathcal{F}$ be a sheaf on X and let $s, t \in \mathcal{F}(U)$ be two sections such that $s_x = t_x$ for every $x \in U$. Then $s = t$.

2) Let $\mathcal{F}, \mathcal{G}$ be presheaves on X and let $\psi, \phi : \mathcal{F} \longrightarrow \mathcal{G}$ be morphisms of presheaves on X such that $\mathcal{F}_x = \mathcal{G}_x$ for every $x \in X$. If $\mathcal{G}$ is a sheaf, then $\mathcal{F} = \mathcal{G}$.

Proof. *1) Let $x \in U$, since $s_x = t_x$, there exists an open subset W_x of U containing x such that $s_{|W_x} = t_{|W_x}$. Since $(W_x)_x$ is an open covering of U, according to condition i) in definition 3.1, it comes that $s = t$.*

2) Let W be an open subset of X and let $s \in \mathcal{F}(W)$. We need to prove that s has the same image under the maps $\psi(W)$ and $\phi(W)$, let $t = \psi(U)(s)$ and $l = \phi(U)(s)$. For all $x \in W$, we have $t_x = \psi_x(s_x) = \phi_x(s_x) = l_x$. Since $\mathcal{G}$ is a sheaf, so by 1) we get that $t = l$.

Proposition 3.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then ψ is injective if and only if $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is injective for every $x \in X$.*

Proof. *Suppose ψ is injective. Let $x \in X$ and $s_x \in \mathcal{F}_x$ such that $\psi_x(s_x) = 0$, where $s \in \mathcal{F}(U)$ and U is an open neighborhood of x, so $\big(\psi(U)(s)\big)_x = 0$. Then, there exists an open neighborhood W of x such that $\psi(U)(s)_{|W} = 0$ or that $\psi(W)(s_{|W}) = 0$. From the injectivity of ψ it comes that $s_{|W}$, thus $s_x = 0$. Conversely, suppose that for all $x \in X$, ψ_x is injective, we fix an open subset V of X and $s \in \mathcal{F}(V)$ such that $\psi(V)(s) = 0$, locally we have, for all $x \in V$, $\psi_x(s_x) = \big(\psi(U))(s)\big)_x = 0$, it comes from local injectivity, that for all $x \in V$, $s_x = 0$. Hence $s = 0$.*

Remark 3.1. *Proposition 3.1 gives a local characterization of the injectivity.*

Theorem 3.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. The following assertions are equivalent :*

1) ψ is an isomorphism.

2) For every $x \in X$, $\psi_x : \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is an isomorphism.

3) ψ is both injective and surjective.

Proof. 1) $\Rightarrow$ 2) *Let ϕ be the inverse morphism of ψ. Plainly, for every $x \in X$, we have $\phi_x \circ \psi_x = id_{\mathcal{F}_x}$ and $\psi_x \circ \phi_x = id_{\mathcal{G}_x}$. So ψ_x is an isomorphism.*
2) $\Rightarrow$ 3) *Immediate, according to proposition 3.1 and definition 2.5, 2)*
3) $\Rightarrow$ 1) *We will construct the inverse ϕ of ψ. Let W be an open subset of X and $t \in$*

$\mathcal{G}(W)$, *for every* $x \in W$, *there exists* U_x *an open neighborhood of* x *and* $s^x \in \mathcal{F}(U_x)$ *such that* $t_x = \psi_x(s^x_x) = \big(\psi(U_x)(s^x)\big)_x$. *Hence there exists* $V_x \subseteq U_x \cap W$ *neighborhood of* x *such that* $t_{|V_x} = \big(\psi(V_x)(s^x_{|V_x})\big)_{|V_x}$. *If* $y \in W$, *then* $\psi(V_x \cap V_y)(s^x_{|V_x \cap V_y}) = \psi(V_x \cap V_y)(s^y_{|V_x \cap V_y})$, *so* $s^x_{|V_x \cap V_y} = s^y_{|V_x \cap V_y}$, *as the family* $(V_x)_{x \in U}$ *forms a covering of* U, *then* $(s^x)_x$ *rises to a section* s *of* $\mathcal{F}$ *on* U, *and we have* $\psi(U)(s) = t$, *the uniqueness of* s *follows from the injectivity of* ψ. *We set* $\phi(U)(t) = s$, *then* ϕ *is the inverse of* ψ.

4. Sheafification

In this section, we answer the following question : How to build a sheaf from a presheaves?

Definition 4.1. *Let* $\mathcal{F}$ *be a presheaf on a topological space* X. *We call associated sheaf with* $\mathcal{F}$ *any sheaf* $\mathcal{F}^\dagger$ *equipped with a morphism of presheaves* $\beta : \mathcal{F} \longrightarrow \mathcal{F}^\dagger$ *satisfying the following universal property:*
For any morphism of presheaves $\psi : \mathcal{F} \longrightarrow \mathcal{G}$, *where* $\mathcal{G}$ *is a sheaf, there exists a unique morphism of sheaves* $\overline{\psi} : \mathcal{F}^\dagger \longrightarrow \mathcal{G}$ *such that the following diagram is commutative :*

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\psi"] \arrow[d, "\beta"'] & \mathcal{G} \\ \mathcal{F}^\dagger \arrow[ur, "\overline{\psi}"'] & \end{tikzcd}$$

Remark 4.1. *The uniqueness of* $\mathcal{F}^\dagger$ *when it exists is an immediate consequence of the universal property.*

Proposition 4.1. *Let* $\mathcal{F}$ *be a presheaf on a topological space* X. *Then the sheaf* $\mathcal{F}^\dagger$ *associated with* $\mathcal{F}$ *exists and is a unique up to isomorphism. Moreover, using the above notation, all* $x \in X$, *the induced morphism* $\beta : \mathcal{F}_x \longrightarrow \mathcal{F}^\dagger_x$ *is an isomorphism.*

Proof. *Let* $\mathcal{F}$ *be a presheaf on* X. *Consider* $Z := \coprod_{x \in X} \mathcal{F}_x$ *(disjoint union) and consider the map* $\pi : Z \longrightarrow X$ *defined by : for all* s_x, $\pi(s_x) = x$. *For any open* V *of* X *and* $s \in \mathcal{F}(V)$, *let* π_s *be the map* $\pi_s : V \longrightarrow X$ *defined by* $\pi_s(x) = s_x$. *Note that* $\pi\big(\pi_s(x)\big) = x$ *i.e* $\pi \circ \pi_s = id_U$ *(*π_s *is a section and* π *is a retraction). We now endow* Z *with the topology which makes all maps* $\pi_s : V \longrightarrow Z$, V *open subset of* X *and* $s \in \mathcal{F}(V)$, *continuous.*
For any open subset V *of* X, *we define* $\mathcal{F}^\dagger(V) := \{g : V \longrightarrow Z / g \text{ continuous and } \pi \circ g = id_V\}$ *it is the set of sections of* Z *on* V.

- *For every* $W \subseteq V$, *the restriction* $\mathcal{F}^\dagger(V) \longrightarrow \mathcal{F}^\dagger(W)$ *is the usual restriction, i.e* $g \longrightarrow g_{|W}$. *In particular* $\mathcal{F}^\dagger$ *is a presheaf.*

* *Condition i) in definition 3.1 is immediate.*

* *If $(W_j)_j$ is a covering of V and $g_j \in \mathcal{F}^\dagger(W_j)$ are such that for all i, j, $g_{i|W_i \cap W_j} = g_{j|W_i \cap W_j}$, then as the g_j are continuous, and coincide on the intersections, there exists $g : V \longrightarrow X$ which is continuous such that for all j, $g_{|W_j} = g_j$. Moreover g is a section in fact : for all $x \in V$, there is some j such that $x \in W_j$, $\pi \circ g(x) = \pi(g(x)) = \pi(g_j(x)) = x$.*
  $\mathcal{F}^\dagger$is a sheaf.

* *Definition of $\beta : \mathcal{F} \longrightarrow \mathcal{F}^\dagger$: For any open subset V of X and $s \in \mathcal{F}(V)$, we define $\beta(V)(S) := \pi_s \in \mathcal{F}^\dagger(V)$.*

* *Compatibility with restrictions : let $W \subseteq V$ two open subsets of X, $s \in \mathcal{F}(V)$ and $x \in W$, we have $\beta(V)(s)_{|W}(x) = \pi_s(x) = s_x = (s_{|W})(x) = \pi_{s|W}(x)$. So $\beta(V)(s)_{|W} = \beta(W)(s_{|W})$.*

* *Let $\mathcal{G}$ be a sheaf, and $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves. We cut a section g of $\mathcal{F}^\dagger(V)$ into small sections (sections of $\mathcal{F}$) on a covering W_j of V, then by sending them to the $\mathcal{G}(W_j)$, then we stick back into $\mathcal{G}$. Sections of $\mathcal{F}^\dagger$ are obtained by gluing sections of $\mathcal{F}$, so $\mathcal{F}_x = \mathcal{F}^\dagger_x$.*

Remark 4.2. *If $\mathcal{F}$ is a sheaf, it follows from the universal property that $\mathcal{F} \simeq \mathcal{F}^\dagger$.*

Example 4.1. *Let A be a group (or a ring, an algebra,...), then*

$$U \longmapsto \begin{cases} A & \text{if} \qquad U \neq \emptyset \\ \{0\} & \text{otherwise} \end{cases}$$

is a presheaf and the associated sheaf is called the constant sheaf associated to A. We denoted by $\underline{A}$. For any $x \in X$, we have $\underline{A}_x = A$.

5. Subsheaves and Quotient sheaves

Throughout, we fix a category of objects that have an algebraic structure which are in particular groups, say e.g., $\mathcal{C} = \mathcal{G}p$ or R-$\mathcal{M}od$.

5.1 Subsheaves

Definition 5.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X, we say that $\mathcal{F}$ is a subsheaf of $\mathcal{G}$, if for any open subset U of X, $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and such that we have compatibility with the restrictions induced from $\mathcal{F}$ and $\mathcal{G}$, i.e., For every open subsets $U \subseteq V$ of*

X, the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{F}(V) \arrow[r, hook] \arrow[d, "res_{V,U}"'] & \mathcal{G}(V) \arrow[d, "res_{V,U}"] \\
\mathcal{F}(U) \arrow[r, hook] & \mathcal{G}(U)
\end{tikzcd}$$

Remark 5.1. *$\mathcal{F}$ is a subsheaf of $\mathcal{G}$ if, the canonical injection $\imath : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of sheaves.*

Definition 5.2. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ a morphism of presheaves on X. We define the presheaf $ker(\psi)$ by the formula :*

$$U \longrightarrow ker(\psi(U))$$

for any open subset U of X. $ker(\psi)$ is said to be the kernel of ψ, it's a subpresheaf of $\mathcal{F}$. and ψ is injective if and only if its kernel is the trivial presheaf.

Using the notation of Definition 5.2, one can easily see that ψ is injective if and only if its kernel is the trivial presheaf.

Lemma 5.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then the presheaf $ker(\psi)$ is a sheaf.*

Proof. *Let U be an open of X , $(U_j)_j$ be a covering of U and $s_j \in ker(\psi(U_j))$ such that for i, j, $s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$. Since $s_j \in \mathcal{F}(U_j)$, then $(s_j)_j$ rises to a section s of $\mathcal{F}$ over U, but for every $x \in U$, there exists i such that $x \in U_j$, and we have $\big(\psi(U)\big)(s)_x = \big(\psi(U_j)\big)(s_j)_x = 0$. So $\psi(U)(s) = 0$. Hence $s \in ker(\psi(U))$. On the other hand, if $s \in ker(\psi(U))$ such that for every j, $s_{|U_j} = 0$, then $s = 0$ (because $s \in \mathcal{F}(U)$ and $\mathcal{F}$ is a sheaf).*

Definition 5.3. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of presheaves on X. We define the $im(\psi)$ presheaf by the formula :*

$$U \longmapsto im\big(\psi(U)\big)$$

for any open set U of X. One can easily see that $im(\psi)$ is indeed a subpresheaf of $\mathcal{G}$. We say that $im(\psi)$ is the image presheaf of ψ.

Remark 5.2. *Note that the presheaf $im(\psi)$ is not in general a sheaf. In the same way we define the presheaf $U \longmapsto coker - pr(im(\psi))$ which too is not in general a sheaf. This justifies the following definition.*

Definition 5.4. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaf. The sheaf associated with the image presheaf $im - pr(\psi)$ called the image sheaf of ψ is denoted $im(\psi)$. In the same way we define the cokernel sheaf and that we denote by $coker(\psi)$*
Note that in general $\big(im(\psi)\big)(U) \neq im\big(\psi(U)\big)$. The first term is section of the sheaf $im(\psi)$ on the open set U, while the second is the image of the morphism $\psi(U)$. More precisely, we have :

Theorem 5.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then, the following assertions hold :*

1) *For any open subset U of X, and $s \in \mathcal{G}(U)$. $s \in (im(\psi)(U))$ if and only if there exists an open covering (U_j) of U and $t_j \in \mathcal{F}(U_j)$ such that, for any j, $s_{|U_j} = \psi(U_j)(t_j)$.*

2) *ψ is surjective if and only if, for any open subset U of X and $s \in \mathcal{G}(U)$, there exists an open covering $(U_j)_j$ of U and $t_j \in \mathcal{F}(U_j)$ such that, for any j, $s_{|U_j} = \psi(U_j)(t_j)$.*

3) *ψ is surjective if and only if $\mathcal{G} = im(\psi)$.*

Proof. *1) $im(\psi)$ is a the sheaf associated with presheaf $U \longmapsto im\big(\psi(U)\big)$, hence the result.*

2) *If ψ is surjective, let U an open subset of X and $s \in \mathcal{G}(U)$, for all $x \in U$, by theorem 3.1, the map ψ_x is surjective. So there exists $t_x \in \mathcal{F}_x$ such that $\psi_x(t_x) = s_x$. Therefore, there there exists an open neighborhood $U_x \subseteq U$, and $t^x \in U_x$ such that $s_{|U_x} = \psi(U_x)(t^x)$. The covering $(U_x)_{x \in U}$ answers the question. Conversely, let $x \in X$ and $s \in \mathcal{G}(U)$. Let $(U_j)_j$ be covering of U and $t_j \in \mathcal{F}(U_j)$ such that $s_{|U_j} = \psi(U_j)(t_j)$ for all j. Since $\mathcal{F}$ is a sheaf then there is $t \in \mathcal{F}(U)$ such that $t_{|U_j} = t_j$ for all j. In particular, for every j such that $x \in U_j$, $s_x = (s_{|U_j})_x = \big(\psi(U_j)(t_j)\big)_x = \psi_x(t_x)$. Hence ψ is surjective.*

3) *Immediate from 1) and 2).*

5.2 Quotients sheaves

Assume that $\mathcal{F}$ is a subsheaf of the sheaf $\mathcal{G}$. Then we can define a presheaf whose sections over U are the quotient $\mathcal{G}(U)/\mathcal{F}(U)$. The restriction maps of $\mathcal{F}$ and $\mathcal{G}$ are compatible the inclusions $\mathcal{F}(U) \subseteq \mathcal{G}(U)$ and hence pass to the quotient $\mathcal{G}(U)/\mathcal{F}(U)$. This presheaf, i.e., $U \longmapsto \mathcal{G}(U)/\mathcal{F}(U)$, is called quotient presheaf of $\mathcal{G}$ by $\mathcal{F}$.

Definition 5.5. *The quotient sheaf $\mathcal{G}/\mathcal{F}$ is the sheafification of the quotient presheaf of $\mathcal{G}$ by $\mathcal{F}$.*

Proposition 5.1. *Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}$, $x \in X$. Then $(\mathcal{G}/\mathcal{F})_x = \mathcal{G}_x/\mathcal{F}_x$.*

Proof. *$\mathcal{G}/\mathcal{F}$ is the sheaf associated with the presheaf $U \longmapsto \mathcal{G}(U)/\mathcal{F}(U)$ whose stalks at x is clearly isomorphic to $\mathcal{G}_x/\mathcal{F}_x$.*

6. Continuous maps and sheaves

So far, we have only talked about sheaves defined on a single topological space. We are going to study in this paragraph some transformations of sheaves via continuous mappings between topological spaces. Let $f : X \longrightarrow Y$ be a continuous map of topological spaces. We will define the pushforward and pullback functors for presheaves and sheaves.

6.1 Pushforward

Definition 6.1. *Let $f : Y \longrightarrow X$ be a continuous map between topological spaces. Let $\mathcal{F}$ be a presheaf on X. We define the pushforward of $\mathcal{F}$ by the formula :*

$$f_*\mathcal{F}(V) = \mathcal{F}\big(f^{-1}(V)\big)$$

for any open $V \subseteq Y$.
Given opens $W \subseteq V$ of Y open the restriction map is given by the commutativity of the diagram

$$\begin{tikzcd} f_*\mathcal{F}(V) \arrow[r, equal] \arrow[d] & \mathcal{F}\big(f^{-1}(V)\big) \arrow[d, "res_{f^{-1}(V),f^{-1}(W)}"] \\ f_*\mathcal{F}(W) \arrow[r, equal] & \mathcal{F}\big(f^{-1}(W)\big) \end{tikzcd}$$

It is clear that this defines a presheaf on Y.

Remark 6.1. *The construction is clearly functorial in the presheaf $\mathcal{F}$ and hence we obtain a functor*

$$\begin{array}{cccc} f_* : & \mathcal{P}reSh_X & \longrightarrow & \mathcal{P}reSh_Y \\ & \mathcal{F} & \longmapsto & f_*\mathcal{F} \end{array}$$

Proposition 6.1. *Let $f : X \longrightarrow Y$ be a continuous map and $\mathcal{F}$ be a sheaf on X. Then $f_*\mathcal{F}$ is a sheaf on Y.*

Proof. *This immediately follows from the fact that if $(W_j)_j$ is an open covering of some open subset W of Y then, $\bigcup_j f^{-1}(W_j)$ is an open covering of the open $f^{-1}(W)$. Consequently, we obtain a functor*

$$f_* : \mathcal{S}h_X \longrightarrow \mathcal{S}h_Y$$

This is compatible with composition in the following strong sense :

Lemma 6.1. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps of topological spaces. Then, the functors $(g \circ f)_*$ and $g_* \circ f_*$ are equal.*

Proof. *Immediate.*

6.2 Pullback

We saw in example 2.1 that if $\mathcal{F}$ is a sheaf on X, then for any open subset U of X $\mathcal{F}_{|U}$ is a sheaf on U. Now if we take an arbitrary subset Z of X. the restriction of $\mathcal{F}$ on Z is not necessarily a sheaf because an open set W of Z is not necessarily an open set of X. Next definition gives the meaning of $\mathcal{F}_{|Z}$, when Z is a closed subset of X. This will be generalized in Definition 6.3 to give the meaning of the pullback presheaf defined by a continuous map. For this purpose, note that if $f : X \longrightarrow Y$ is a continuous map between topological spaces and V is an open of Y, then the family $(U)_{f(U)\subseteq V}$ consisting of all open subsets U of X satisfying $f(U) \subseteq V$, is an inductive system for the inverse of the inclusion relation.

Definition 6.2. *If $\imath : Z \longrightarrow X$ is the inclusion of a closed subset Z of X, and V is an open subset of Z. We define the restriction $\mathcal{F}_{|Z}$ as the sheafification of the following presheaf*

$$V \longmapsto \varinjlim_{V\subseteq U} \mathcal{F}(U)$$

Definition 6.3. *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces and $\mathcal{G}$ be a presheaf on Y. We define the pullback presheaf of $\mathcal{G}$ by the formula :*

$$f_p\mathcal{G}(U) = \varinjlim_{f(U)\subseteq V} \mathcal{G}(V).$$

Remark 6.2. *In the language of categories. The pullback presheaf $f_p\mathcal{G}$ of $\mathcal{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words, $f_p\mathcal{G}$ will be a presheaf on X such that*

$$Mor_{\mathcal{P}reSh_X}(f_p\mathcal{G}, \mathcal{F}) = Mor_{\mathcal{P}reSh_Y}(G, f_*\mathcal{F})$$

Proposition 6.2. *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces, x be a point of X and $\mathcal{G}$ be a presheaf on $\mathcal{Y}$. Then, up to an isomorphism, we have $(f_p\mathcal{G})_x = \mathcal{G}_{f(x)}$.*

Proof.

$$\begin{aligned}
(f_p\mathcal{G})_x &= \varinjlim_{x\in U} f_p\mathcal{G}(U) \\
&= \varinjlim_{x\in U} \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) \\
&= \varinjlim_{f(x)\in V} \mathcal{G}(V) \\
&= \mathcal{G}_{f(x)}
\end{aligned}$$

Definition 6.4. *Let $f : X \longrightarrow Y$ be a continuous map between topological spaces and $\mathcal{G}$ be a sheaf on Y. The pullback sheaf $f^{-1}\mathcal{G}$ is defined by the formula :*

$$f^{-1}\mathcal{G} = (f_p\mathcal{G})^\dagger$$

$f^{-1}\mathcal{G}$ is also called the inverse image along the map f.

Remark 6.3. *f^{-1} defines a functor :*

$$\begin{array}{rccc} f^{-1}: & \mathcal{S}h_Y & \longrightarrow & \mathcal{S}h_X \\ & \mathcal{G} & \longmapsto & f^{-1}\mathcal{G} \end{array}$$

The pullback f^{-1} is a left adjoint of pushforward on sheaves.

$$Mor_{\mathcal{S}h_X}(f^{-1}\mathcal{G}, \mathcal{F}) = Mor_{\mathcal{S}h_Y}(\mathcal{G}, f_*\mathcal{F}).$$

Example 6.1. *Let $\mathcal{F}$ be a sheaf on X and $x \in X$. Let $\imath : \{x\} \longrightarrow X$ be the inclusion map, then $\imath^{-1}\mathcal{F} = \mathcal{F}_x$*

Lemma 6.2. *Let $f : X \to Y$ be a continuous map between topological spaces, $x \in X$ and $\mathcal{G}$ be a sheaf on Y, then the stalks $(f^{-1}\mathcal{G})_x$ and $\mathcal{G}_{f(x)}$ are equals.*

Proof. *This a combination of proposition 4.1 and proposition 6.2.*

Lemma 6.3. *Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly, $(g \circ f)_p = f_p \circ g_p$, for presheaves.*

Proof. *This follows from the fact that adjoint functors are unique up to unique isomorphism, and Lemma 6.1.*

7. Exact sequences of sheaves

In this section, we will define what is an exact sequence of sheaves, and we will study some of their properties. For this we will restrict our study to the case of sheaves of groups.

Definition 7.1. *A sequence of presheaves with presheaves morphisms*

$$\cdots \longrightarrow \mathcal{F}^{j-1} \xrightarrow{\psi^{j-1}} \mathcal{F}^j \xrightarrow{\psi^j} \mathcal{F}^{j+1} \xrightarrow{\psi^{j+1}} \cdots$$

is said to be exact if for all i, $Im(\psi^{j-1}) = ker(\psi^j)$. In particular the following exact sequence is call a short exact sequence when it is exact :

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

Remark 7.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then by,*

i) ψ is injective if and only if

$$0 \longrightarrow \mathcal{F} \xrightarrow{\psi} \mathcal{G}$$

is an exact sequence.

ii) ψ is surjective if and only if

$$\mathcal{F} \xrightarrow{\psi} \mathcal{G} \longrightarrow 0$$

is an exact sequence.

Example 7.1. *Let $X = \mathbb{C}$, and $\mathcal{O}_X$ the sheaf of holomorphic functions and consider the map $d : \mathcal{O}_X \longrightarrow \mathcal{O}_X$, sending $f(z)$ to $f^{'}(z)$. There is an exact sequence*

$$0 \longrightarrow \mathbb{C}_X \longrightarrow \mathcal{O}_X \xrightarrow{d} \mathcal{O}_X \longrightarrow 0$$

Indeed,

* *A function whose derivative vanishes identically is locally constant, so $ker(d)$ is the constant sheaf $\mathbb{C}_X$.*

* *In small open disks any holomorphic function is a derivative.*

Lemma 7.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves on X. Then for any $x \in X$, we have $(ker\psi)_x = ker(\psi_x)$ and $(im\psi)_x = im(\psi_x)$.*

Proof. *Let $s_x \in \big(ker(\psi)\big)_x$, and let U an open neighborhood of x such that $s \in \big(ker(\psi)\big)(U) = ker\big(\psi(U)\big)$, so $\psi(U)(s) = 0$, hence $\psi_x(s_x) = \big(\psi(U)(s)\big)_x = 0$, so $s_x \in ker(\psi_x)$. Conversely, if $\psi_x(s_x) = 0$, then $\big(\psi(U)(s)\big)_x = 0$ (U is an open neighborhood of x and $s \in \mathcal{F}(U)$), then there exists an open neighborhood $W \subseteq U$ of x such that $\psi(U)(s)_{|W} = 0$, it comes while $\psi(W)(s_{|W}) = 0$ and therefore $s_{|W} \in ker\big(\psi(W)\big)$ whence $s_x = (s_{|V})_x \in \big(ker\big(\psi\big)\big)_x$. One can proceed similarly for the image.*

Theorem 7.1. *A sequence of sheaves with sheaves morphisms*

$$\cdots \longrightarrow \mathcal{F}^{j-1} \xrightarrow{\psi^{j-1}} \mathcal{F}^{j} \xrightarrow{\psi^{j}} \mathcal{F}^{j+1} \xrightarrow{\psi^{j+1}} \cdots$$

is an exact sequence if and only if for any $x \in X$

$$\cdots \longrightarrow \mathcal{F}_x^{j-1} \xrightarrow{\psi_x^{j-1}} \mathcal{F}_x^{j} \xrightarrow{\psi_x^{j}} \mathcal{F}_x^{j+1} \xrightarrow{\psi_x^{j+1}} \cdots$$

is an exact sequence.

Proof.

$$\cdots \longrightarrow \mathcal{F}^{j-1} \xrightarrow{\psi^{j-1}} \mathcal{F}^{j} \xrightarrow{\psi^{j}} \mathcal{F}^{j+1} \xrightarrow{\psi^{j+1}} \cdots$$

is exact sequence if and only if, for any j, $im(\psi^{j-1}) = ker(\psi^{j})$ if and only if, for any $x \in X$ and for any j, $im(\psi_x^{j-1}) = ker(\psi_x^{j})$ if and only if,

$$\cdots \longrightarrow \mathcal{F}_x^{j-1} \xrightarrow{\psi_x^{j-1}} \mathcal{F}_x^{j} \xrightarrow{\psi_x^{j}} \mathcal{F}_x^{j+1} \xrightarrow{\psi_x^{j+1}} \cdots$$

is exact sequence.

Proposition 7.1. *Let $\mathcal{F}$ be a subsheaf of $\mathcal{G}$ on X. Then*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{F} \longrightarrow 0$$

is exact sequence.

Proof. *By proposition 5.1, for any $x \in X$,*

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{G}_x/\mathcal{F}_x = (\mathcal{G}/\mathcal{F})_x \longrightarrow 0$$

is exact sequence. Hence the result.

Remark 7.2. *If*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is an exact sequence over X, then $\mathcal{F}$ identified with a sub-sheaf of $\mathcal{G}$ and $\mathcal{G}/\mathcal{F} \simeq \mathcal{H}$.

Corollary 7.1. *Let $\psi : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of sheaves. Then*

1) *$im(\psi) \simeq \mathcal{F}/ker(\psi)$.*

2) *$coker(\psi) \simeq \mathcal{G}/im(\psi)$.*

Proof. 1) *It is easy to check that for all $x \in X$, we have*

$$0 \longrightarrow \big(ker(\psi)\big)_x \longrightarrow \mathcal{F}_x \longrightarrow im(\psi)_x \longrightarrow 0$$

It follows by theorem 7.1, that

$$0 \longrightarrow ker(\psi) \longrightarrow \mathcal{F} \longrightarrow im(\psi) \longrightarrow 0$$

is an exact sequence. Also by remark 7.2 we have $im(\psi) \simeq \mathcal{F}/ker(\psi)$

2) *Similar to 1).*

8. Glueing sheaves

In this section, we fix a topological space X, and we consider an open covering $\{U_i\}_{i \in I}$ of X with a sheaf $\mathcal{F}_i$ on each subset U_i. Our goal is to "glue" the $\mathcal{F}_i$ together, that is we search for a global sheaf $\mathcal{F}$ such that $\mathcal{F}_{|U_i} = \mathcal{F}_i$ for all $i \in I$.

Notation. *i) For $i, j \in I$, we denote by U_{ij} the intersection $U_i \cap U_j$.*

ii) For $i, j, k \in I$, we denote by U_{ijk} the intersection $U_i \cap U_j \cap U_k$.

Definition 8.1. *A Gluing Datum consists of a family of sheaves $\mathcal{F}_i$ over U_i and a family of morphisms $\delta_{ij} : \mathcal{F}_{i|U_{ij}} \longrightarrow \mathcal{F}_{j|U_{ij}}$ such that*

i) $\delta_{ii} = id_{\mathcal{F}_i}$.

ii) $\delta_{ji} = \delta_{ij}^{-1}$.

iii) $\delta_{ik} = \delta_{jk} \circ \delta_{ij}$ *on* U_{ijk}.

A morphism of gluing datum $(\mathcal{F}_i, \delta_{ij}) \longrightarrow (\mathcal{G}_i, \eta_{ij})$ is a family of morphism of sheaves $\psi_i : \mathcal{F}_i \longrightarrow \mathcal{G}_i$ such that the following diagram

$$\begin{tikzcd}
\mathcal{F}_i \arrow[r, "\psi_i"] \arrow[d, "\delta_{ij}"'] & \mathcal{G}_i \arrow[d, "\delta_{ij}"] \\
\mathcal{F}_j \arrow[r, "\psi_i"] & \mathcal{G}_j
\end{tikzcd}$$

is commutative.

Theorem 8.1. *(Gluing sheaves) There exists a sheaf $\mathcal{F}$ on X, unique up to isomorphism such that there are isomorphisms $\theta_i : \mathcal{F}_{U_i} \longrightarrow \mathcal{F}_i$ such that there are satisfying*

$$\theta_j = \delta_{ij} \circ \theta_i.$$

Proof. *Let W be an open subset of X. We write $W_i = U_i \cap W$, and $W_{ij} = U_{ij} \cap W$. We are going to define the sections of $\mathcal{F}$ over W by gluing sections of the $\mathcal{F}_i'$s over W_i's along the W_{ij}'s using the isomorphisms δ_{ij}. We define*

$$\mathcal{F}(W) := \left\{(s_i)_{i\in I} | \delta_{ji}(s_{i|W_{ij}}) = \delta_{j|W_{ij}}(s_{j|W_{ij}})\right\} \subseteq \prod_{i\in I} \mathcal{F}_i(W_i). \tag{8.1}$$

The δ_{ij}'s are morphisms of sheaves and therefore are compatible with all restrictions maps (see definition 2.3). So if $V \subseteq W$ is another open subset we have

$$\delta_{ij}(s_{i|W_{ij}}) = s_{j|W_{ij}}.$$

Because of this, the defining condition (8.1) is compatible with componentwise restrictions, and they can therefore be used as the restriction maps in $\mathcal{F}$. So We have defined a presheaf on X.

* *The first step: is to establish the isomorphisms $\theta_i : \mathcal{F}_{|U_i} \longrightarrow \mathcal{F}_i$. To avoid getting confused by the names of the indices, we shall work with a fixed index $j \in I$. Suppose $W \subseteq U_j$ is an open set. We have $W = W_j$, and projecting from the product $\prod_{i\in I} \mathcal{F}_i(W_i)$ onto the component*

$$\mathcal{F}_j(W) = \mathcal{F}_j(W_j)$$

gives us a map $\theta : \mathcal{F}_{j|W_j} \longrightarrow \mathcal{F}_j$. Moreover, $\theta\big((s_i)_{i\in I}\big) = s_j$. The situation is summarized in the following commutative diagram

$$\begin{tikzcd} \mathcal{F}(W) \arrow[r, hook] \arrow[dr, "\theta_j"'] & \prod_{i\in I}\mathcal{F}_i(W_i) \arrow[d, "\pi_j"] \\ & \mathcal{F}_j(W) \end{tikzcd}$$

Now, we want to show that θ_j's give the desired isomorphisms. We note that on the restrictions $W_{jj'}$, the requirement in the proposition, that

$$\theta_{j'} = \eta_{j'j} \circ \theta_j$$

is fulfilled. This follows directly from the (8.1) that

$$s_{j|W_{jj'}} = \delta_{jj'}(s_{j'|W_{jj'}}).$$

- θ_j *is surjective: Let α a section of $\mathcal{F}_j(W)$ over some $W \subseteq U_j$, and pose $s = (\delta_{ij}(\alpha_{|W_{ij}})_{i\in I}$. Then s satisfies (8.1) and is an element $\mathcal{F}(W)$. Indeed, by definition 8.1 iii) we obtain*

  $$\delta_{ki}(\delta_{ij}\big(\alpha_{|W_{kij}}\big)\big) = \delta_{kj}(\alpha_{|W_{kij}}).$$

  for each $i, k \in I$, and that is just the condition (8.1). As $\delta_{jj}(\alpha_{|W_{jj}}) = \alpha$ by the first gluing request, the element s projects to the section α of $\mathcal{F}_j$.

- θ_j *is injective : Since $s_j = 0$ if follows that $s_{i|W_{ij}} = \delta_{ij}(s_j) = 0$ for each $i \in I$. Now $\mathcal{F}_j$ is a sheaf, and the $\{V_{ij}\}_{i\in I}$ constitute an open covering of W_j, so we may conclude that $s = 0$ by definition 3.1 i).*

- *The final step: To show that $\mathcal{F}$ is a sheaf. Let $\{W_j\}_{j\in I}$ be an open covering of $W \subseteq U$, and $s_j \in \mathcal{F}(W_j)$ is a bunch of sections matching on the intersections $W_{jj'}$. Since $\mathcal{F}_{|U_i\cap W}$ is a sheaf patch together to give sections s_i in $\mathcal{F}_{U_i\cap W)}$ matching on the overlaps $U_{ij}\cap W$. This last condition means that $\delta ij(s_i) = s_j$. By definition $(s_i)i \in I$, then is a section in $\mathcal{F}(W)$ restricting to s_i. Hence the result.*
  The Gluing axiom (see definition 3.1) is easier : Let $s = (s_i)_{i\in I}$ in $\mathcal{F}(W)$, and a covering $\mathcal{L} = \{V_j\}_{j\in J}$ of W such that $s_{|V_j} = 0$ for all $j \in J$, then also $s_{|V_j\cap W_i} = 0$, and since $\{V_j \cap W_i\}_{j\in J}$ forms a covering of W_i, we must have $s_{|W_i} = 0$ as well, since $\mathcal{F}_{W_i} = \mathcal{F}_i$ is a sheaf. But from the (8.1) we thus see that $s = 0$.

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