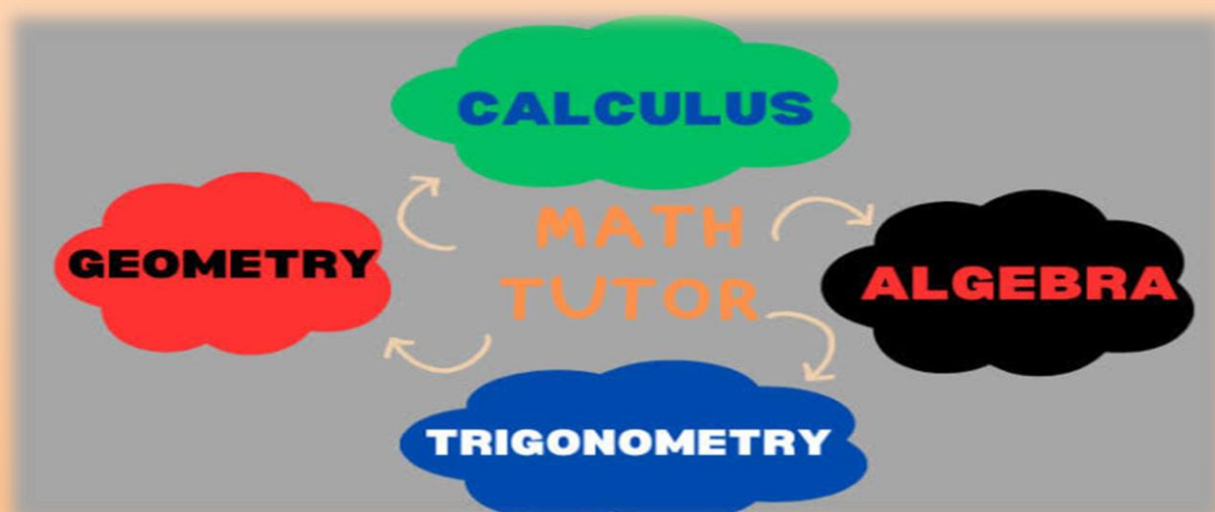
Virtual Study

Learn, Connect, Succeed

Mathematics

Grade 10



Prepared by: Virtual Study

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Relation s and Function

1.1Relations

Revision Of Pattern

A pattern is regularity in the world in human made design or Abstract Ideas. As such the elements of a pattern repeat in a predictable manner.

- ✓ Patterns are defined as regular, repeated recurring form or designs relationships, finding logic to form generalizations and make predictions.
  Example: even numbers 2, 4, 6, 8, 10- - - - - -
  Odd numbers 1, 3, 5, 7, 9, - - - - - -

Arithmetic pattern :- is also known as the algebraic pattern in an arithmetic pattern, the sequences are based on the addition or subtraction of the terms.

If two or more terms in the sequence are given, we can use addition or subtraction on to find the arithmetic pattern.

For example:- consider the pattern

2, 4, 6, 8, 10, ________ 14 ___________ Now we need to find the missing term in the pattern the difference of this pattern is 2 so that $10 + 2 = 12$ and $14+2 = 16$

12 and 16 respectively

2, 4, 6, 8, 10, 12 14, 16

Geometric Pattern

The Geometric Pattern is defined as the sequence of numbers that are based on the multiplication and division operation. similar to the arithmetic pattern , If two or more numbers in the sequence are provided we can easily find the unknown terms in the pattern using multiplication and division operation.

Example:- 2, 4, 8, __________ 32 ___________

$\frac{4}{2} = 2$ $\frac{8}{2} = 4$ so

Multiply each by 2 so that $8x2 = 16$ and $32x2 = 64$ The pattern is 2, 4, 8, 16, 32, 64

Exercise

1. determine the value of a and b in the following patterns.

65, 60, 55, 50, 45, a, 35, b.

the pattern is decreased by 5 it is Arithmetic Sequences

as – 5 = 40 and 35 – 5 = 30 a = 40 b = 30

65, 60, 55, 50, 45, 40, 35, 30.

2. Identify the pattern for the sequence 4, 8, 12, 16, 20, - - - - Arithmetic sequence the difference is 4

3. Determine the value of A and B in the following pattern.

15, 22, 29, 36, 43, A, 57, 64, 71, 78, 85, B. This sequence is Arithmetic pattern their different is 7

43 + 7 = A A = 50 85 + 7 = B, B = 92

15 , 22 , 29 , 36 , 43 , 50 , 57 , 64 , 71 , 78 , 85 , 92

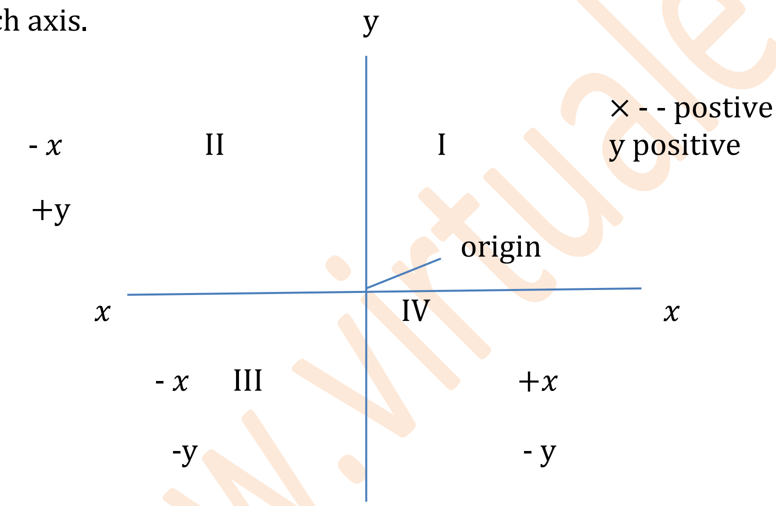
4. Find the missing value for the geometric pattern ; 96, 48, 24, ________ 6 ____________

This pattern is Geometrical sequence $\frac{96}{2} = \frac{48}{2} = \frac{24}{2} = \frac{12}{2} = \frac{6}{2} = 3$

The sequence is 96, 48, 24, 12, 6, 3

Cartesian Coordinate System in two dimensions

The Cartesian coordinate system in two dimensions (also called a rectangular coordinate system is defined by an ordered pair of perpendicular line (axes). It is single unite of length for both and used as a turning point for each axis.



Basic concept Of relations

Relation is a set of ordered pairs it is denoted by R.

Exercise

1. If R is a relation of a set ordered pair (x,y) of real numbers such that y = 3x - 2 is a relation, then list some ordered pairs belong to R. y = 3x - 2

x	0	1	2	3	4	5
y	-2	1	4	7	10	13

(0, - 2), (1, 1), (2, 4), (3, - 7), (4, 10), (5, 13)

2. Let R denote the set of ordered pairs (×,y) of real numbers , where y= x^3

a, Find the ordered pairs belog to R which have the following frist entries: 0, 1, 2 -2, 8 , $\frac{1}{5}$, 3, -3

x	-2	-3	0	1	$\frac{1}{5}$	8	
y	-8	-512	0	1	$\frac{1}{45}$	512	

(-2, - 8), (-3, -512), (0, 0), (1, 1), (2, 8), ($\frac{1}{5}$, $\frac{1}{45}$) , (8, 512)

Domain and range Of a relation

Any set of ordered pair (x,y) is called a relation in x and y

- ✓ The set of first components in the ordered pairs is called the domain of the relation.
- ✓ The set of second components in the ordered pairs is called the range of the relation.

Example

1. Determine the domain and range of the relation with ordered pairs.
   (-2, 1), (-1, 0), (0, 0), (4, 2), (3, 5)

Solution

The domain of the relation are , -2, -1, 0, 4, 3
The range of the relation are 1, 0, 2, and 5

Exercise

1. Write some ordered pairs that belongs to the following relation R, and find the domain and range of R.

a. The set of ordered pair (x,y) such that

Y = 3x , x and y are member of Integers

x	-3	-2	-1	0	1	2	3
y	-9	-6	-3	0	3	6	9

Domains of y = 3x , 0, 1, 2, 3, -1, -2, -3
range of y = 3x , 9, -6, -3, 0, 3, 6, 9

b. The set of ordered pair (× , y) such that y = -2× : × and y are members of integers.

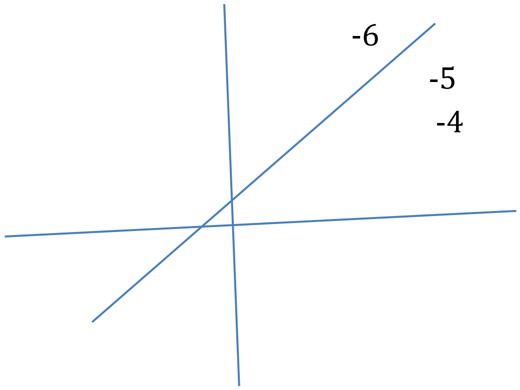
x	-3	-2	-1	1	1	2	3
y	6	4	2	-2	3	-4	-6

The domain of y = -2x is {-3, -2, -1, 0, 1, 2, 5 }
The range of y = -2x is { 6, 4, 2, 0, -2, -4 , -6, }

<u>Graphs of Relations:</u> y < ax + b or y > ax+b

Example 1

Sketch the graph of the relation R, If R be the set of ordered pair (× , y) of real numbers $x$and y such that y ≤ x+2



<u>Exercise</u>

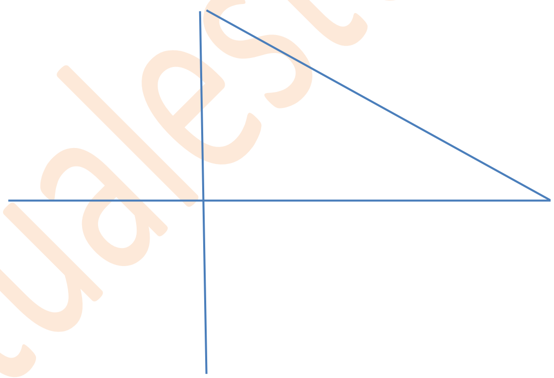
1. For each of the following relations, sketch the graph and determine the domain and the range.

a. R : (x , y): y < -x + 7.

y = - x + 7

y = 7

x = 7

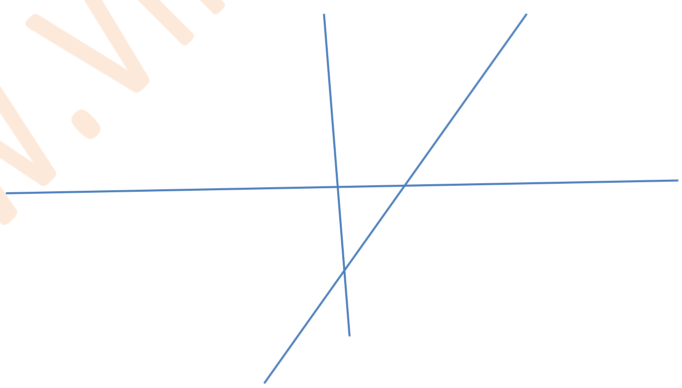


b. R : (x , y): y ≤ 3× - 4.

y = 3 × - 4

yin = -4

×in = $\frac{4}{3}$ = 1.3

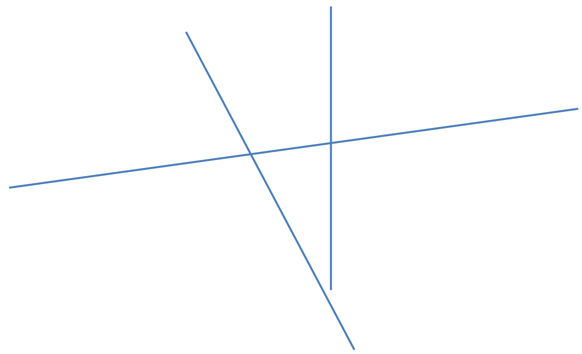


C. R : (× , y): y > 5x - 6.

y = 5 × - 6

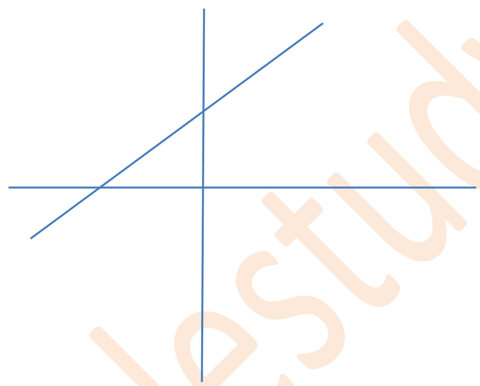
y in = - 6

x in = $\frac{6}{5}$



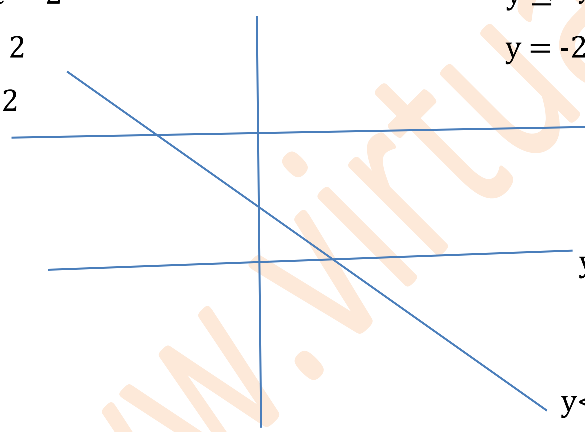
d. R is (x , y): y ≥ x + 2 and y < - x

y = x + 2	y = - x
y in = 2	y = 0
x in = -2	x = 0



e. R is (x , y): y < - −2 and y ≤ - 2

y = - x − 2	y ≤ - x
y in = - 2	y = -2
y in = -2	



y ≤ - 2

y< - x - 2

1.2 Functions

1.2.1The notation of Functions

A function f is a set of ordered pair with the property that whenever (x, y) and (x , Z) belong to F, then y = Z or it is a relation in which no two distinct ordered pairs have the same first element .

Example 1

1. R_1 = (4, 5), (6, 6), (3, 1) , (9, 7) and (5, 2) no first element is the sam it is function

2. R_2 = (5, 3), (2, 3), (3, 6) , (7, 6) , (5, 8), (5, 3) and (5, 8) belongs the same element with relation R and $3 \neq 8$ the relation R is not a function.

Exercise

1. Determine whether each of the following relation is a function or not and give reasons for those that are not function.

a. R : (6, 7) , (1, 9), (-1 , 7) , (0, 0) , (4, -4)

It is function b/c no frist element that have the sam element.

b. R : (-3, 7) , (-5, 9), (-1 , 4) , (2, 0) , (-5, -3)

not function (- 5, 9) and (-5, 3) belongs the same element on the relation.

C. The relation R is a set of an ordered pair (x : y) : $y^2 = x$ not a function b/c thire is the same element that the sam ordare paire -2,

2. Is every function a relation ? jusity your answer. Yes relation means any order pairs on a function. But not any relation is function.

Exercise

1. Determine whether each of the following relation is a function or not , and given reason for those that are not functions.

a. R is the set of ordered pair (x : y): y is the area of triangle x. Not function because ther was the sam length on a triangle.

b. The relation R is an order pair (x : y) ; y is the father of x . it is function

y is father x is Chaild

C. The relation R, with an ordered pair (: y) y is a grandmother of $x$not function b/c thir was one chiled have two grandmother.

2. Is R of ordered pair (x : y) : y = $|x|$ a function y = $|x|$ not a function b/c thire was two first order pairs.

Example: -2, = x = -

y= -2 $x = 2$

y= 2 $x = 2$

y= -3 $x = 3$

y= 3   = 2

(2, - 2) , (2, 2) , (3, - 3) , (3, 3)

Not function the same element belong this relation.

Domain and range Of a function

The domain of a function f is the set of all values of x for which f is defined and this correspond on s to all of the x- values on the graph in the xy- plane. The range of the function f is the set of all values f(x) which corresponding to the y- values on the graph in the xy-plane.

Example 1 $f(x) = 3x+4$ find the domain and the range of f

$f(x) = 3x+4$ every real number of x the domain of the function is the set of

y is pre Image of y= $f(x) = 3x+4$

Exercise

1. For each of the following functions, find the domain and range.

a. $f(\times) = x+1$ it is function b/c it has no two element have the same value of "x" range is the set of real number

b. $f(x) = \frac{x}{2} \Longrightarrow y = f(x) = \frac{y}{2}$ range the set of all real number and the domine also the set of real number

C. $f(\ ) = \frac{1}{2\,x}$ $x : \times \in \neq 0$

$y: x \in \neq 0$

d. $f(x) = 2x$-3 $x : x \in$ R and y: $y \in$ R

e. . $f(x) = x^2+1$ $x : x \in$ R and y: $y > 1$

f. $f(x) = \frac{1}{2x}$ = $x : x \in$ R $\neq 0$ and y: $y \in$ R

- ✓ If f: A ⟶ B is a function, then , for any x is included in A (the frist coordinate) the image of x under f, $f(x)$ is called the functional value of f at x

Example

$f(x) = 1- 3x$,

a. determine the domain and range of f. every real number x i the domain and range of the function f is the set domain and range of the function f is the set all of real numbers.

b. find the value of f (2) and f(- 1)

$f(x) = 1- 3x$ f(2)= 1-2x3 = 1 – 6= <u>**-5**</u>

f(-1)= 1-3x = 1+3 = <u>**4**</u>

Exercise

1. Find the domain and range of each of the following function.

a. $f(x)$ =1- x^2 domain $x : x \in$ R

Range y: $y \leq 0$

b. $f(x) = |x| + 1$ $x : x \in$ R

y: $y \geq 1$

C. $f(x) = \sqrt{2 - x}$ $x : x \in R \leq 2$ $y: y \in -R$

d. $f(x) = x^2 - 1$ $x : x \in R$

$y : y \in R \geq 0$

2. If $f(x) = 2 + \sqrt{4 - x}$, then evaluate

a. f (-5) $f(x) = 2x) = 2x(-5) + \sqrt{4 - (-5}$

f (-5) = $-10 + \sqrt{4 + 5}$ $= -10 + \sqrt{9} = -10 + 3 = -\underline{7}$

b. $f(2) = 2x2 + \sqrt{4 - 2} = \underline{\mathbf{4 + \sqrt{2}}}$

Combination of Functions

A. Sum Of functions

Let f and g be two functions with over lapping domains. Then for all x common to both domain the sum of f and g defind as follows.

$(f + g)(x) = 2x+4$ $f(x) + g(x)$

Example

$f(x) = 2x+4$ $g(x) = 3x-1$

Find $(f + g)(x)$ and the sum when $x = 2$

$f(x) + g(x) = 2x+4+3x-1$

$2x+3x+4-1$

$\underline{\mathbf{5x+3}}$

$(f+g)(2) = 2x5+3 = 10+3 = \underline{\mathbf{13}}$

B. Difference Of Functions

Let f and g be two functions with over lapping domains. Then for all x common to both domains the difference of f and g is defined as follows.

$(f - g)(x) = f(x) - f(x)$

Example

$f(x) = 2x+1$ and $g(x) = x^2+2x-1$. Find

$(f - g)(x)$ and evaluate the difference when $x = 2$

The difference of the function f and g is $(f - g)(x) = f(x) - g(x)$

$(2x+1) - (x^2+2x-1)$

$2x+1 - x^2 - 2x+1$

$- x^2 + 2x - 2x+1$

$\underline{\mathbf{-x^2 + 2}}$

$(f - g)(2) = x^2 + 2 = -2^2 + 2 = \underline{\mathbf{-2}}$

C. Produce Of Functions

Let f and g be two functions with over lapping domain. Then for all x common to both domains the product of F and g is defined as follow.

$(f . g)(x) = f(x) . g$

Example

f $(x)=x^2$ and $= x$ - 3 find (f . g) (x) and then evaluate the product of when $x = 4$

(f . g) (x) $=$ f (x) . g (x)

$= x^2\ (x - 3)$

$\underline{\underline{x^3 - 3x^2}}$

D. Quotient Of Functions

Then for all x common to both domains the quotient of f and g is defined as follows.

$\frac{f}{g}(x) = \frac{f\ x}{g\ (x)}$, g $(x) \neq 0$

Example f (x)= 4 g $(x) =$ the quotient of f and g is $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{4}{x-3}$ $x \neq 3$

f $(x)= x$ and g $(x) = \sqrt{4 - x^2}$ find $\frac{f}{g}$ and $\frac{g}{f}$

$\frac{f}{g}(x) = \frac{\sqrt{x}}{\sqrt{4 - x2}}$

$\frac{g}{f}(x) = \frac{\sqrt{4-x2}}{\sqrt{x}}$

Exercise

Given f $(x)= x$ - 1 and g $(x) = x^{2,}$

a. find (f . g) (x) and then evaluate the product when $x = 3$

f $(x)=x$ - 1 g $(x) = x^2$

f . g $(x)=x^2$ $= \underline{\underline{x^3 - x^2}}$

f . g $(x)=$ 3^3 - 3^2

27 – 9 $=$ $\underline{\underline{18}}$

b. find $\frac{f}{g}(x)$ and evaluate the product $= x = 3$

$\frac{f}{g}(x) = \frac{x-1}{x2} = \frac{x-1}{x2}$

$\frac{f}{g}(x) = \frac{3 - 1}{3^2} = \underline{\underline{\frac{2}{9}}}$

C. $\frac{f}{g}(x)$ and evaluate the Quotient $x = 3$

$\frac{f(x)}{g(x)} = \frac{x^2}{x - 1}$ $x \neq 1$

f $(3) = \frac{x^2}{x - 1} = \frac{9}{3-1} = \underline{\underline{\frac{9}{2}}}$

d. the domain of f . g, $\frac{f}{g}$ and $\frac{g}{f}$

- ✓ f . g = x^3 - x^2 domain $x : x \in R$
- ✓ $\frac{f}{g}(x) = \frac{x-1}{x^2} = x : x \in R \neq 0$
- ✓ $\frac{g}{f} = \frac{x^2}{x-1} = x : x \in R \neq 1$

Exercise

1. Given f (x)=2 - x and g (x)= -2 x+3

a. deter min f + 2 g and evaluate (f + 2 g) (2)

f (x)=2 - x 2 g = -4x+6

f + 2 g = 2 - x + (- 4 x + 6

2 - × - 4x+6 = **- 5x+8**

(f + 2g) (2) = -5 (2) + 8 = -10+8 = **- 2**

b. Deter mine 2f – g and evaluate (2f – g) (2).

2f = 4 - 2x and g (x) = -2x+3

2f – g = 4 - 2x - (- 2x+3)

4 – 2 x + 2 x-3 = **1**

C. Deter mine $\frac{3f}{2g}$ and evaluate $\frac{3f}{2g}$ (2)

3f = 3(2 - x and 2g = 2(- 2x+3) = - **4x+6**

$\frac{3f}{2g} = \frac{6-3x}{-4x+6} = \frac{3\,(2-x)}{-2\,(x} \quad \frac{6-3x}{-4\,x+6}$

$\frac{3f}{2g}$ (2) $= \frac{6-3x2}{-4\,x2)+6} = \frac{0}{-8+6} = \frac{0}{-2} = \mathbf{0}$

2. Let f (x)=3 x -3 g (x)= $\frac{2}{x-1}$ Evalute,

a. (2f . g) (2) = 6x - 6 $\left(\frac{2}{x-1}\right) = \frac{12x-12}{x-1} = \frac{12(x-1}{x-1}$

= **12**

b. $\left(\frac{f}{g} - 2f\right)(3)$

$\frac{3x-3}{\frac{2}{x-1}}$ = 3x-3 + $\frac{x-1}{2}$ = $\frac{3x^2-6x+3}{2}$

$\frac{3x^2-6x+3}{2}$ - $(6x-6)$ = $\frac{3x^2}{2}$ - 3^3 x+ $\frac{3}{2}$ - 6x+6

$\frac{3x^2}{2}$ - 9 x + $\frac{15}{2}$ = ($\frac{3x^2}{2}$ -9x+ $\frac{15}{2}$) (2)

$\frac{27}{2}$ - 27 + $\frac{15}{2}$ = $\frac{12}{2}$ = $\underline{\underline{6}}$

3. Is it always possible to deduce the domain of f + g, f – g, f . g and $\frac{f}{g}$ differ from the domain of f and g? If your answer yes, how? **No**

4. Deterimne which of the following pairs of formulas define the sam function.

a. $y = x^2$; $y = x^2$, $x \geq 0$ The sam function

b. $y = (x + 1)^2 (x-2)$; $y = x^3 - 3x - 2$

$x(+1) + 1 (\times+1) = \times^2 + 2\times+1$

$(x^2 + 2x+1) (x-2)$

$\times (x^2 + 2x+1) - 2 (x^2 + 2x+1) = x^3 - 3x - 2$

$\underline{x^3 - 3x - 2} = \underline{x^3 - 3x - 2}$ They are the same function

C. $y = \frac{1}{\times}$; $y = \frac{x+1}{x^2+x}$

$\frac{1}{x} = \frac{x+1}{x^2+x} \Rightarrow \frac{1}{x} = \frac{x+1}{x(x+1)}$

$\frac{1}{x} = \frac{x+1}{x(x+1)} \Rightarrow \frac{1}{x} = \frac{1}{x}$ They are the sam function

d. $y = 2^x$ j $y = x$

$y = 2^x$ j $y = x$ They are not the sam function

e. $y = x+1$ j $y = \frac{x^2 - 1}{x - 1}$

$x + 1 = \frac{x^2 - 1}{x - 1} \Rightarrow x + 1 = \frac{(x+1)(x-1)}{x-1}$

$\underline{x + 1} = \underline{x + 1}$ The same function

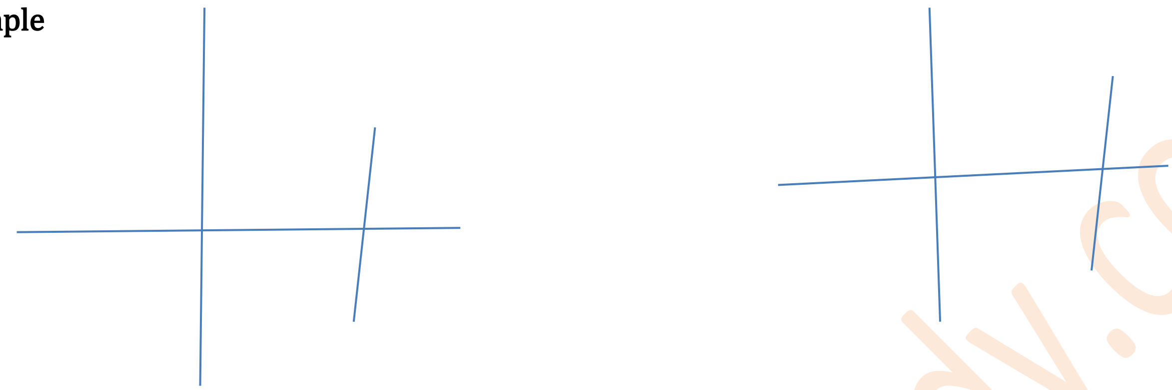
f. $y = x$ j $y = \sqrt{x}^2$

$x = \sqrt{x}^2$ $x = x$ They are the sam

<u>Vertical Line test</u>

Vertical line test is used to determine whether a graph of a curve is a function or not. If any curve cuts a vertical line at more the an one points then the carve is not a function.

Example



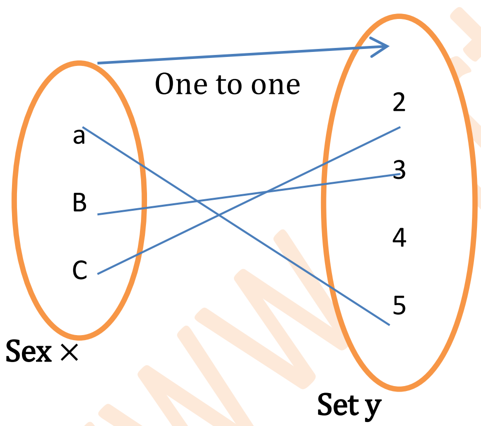
1.2.2 Types of Functions

There are different types of functions.

- ✓ One – to – one function (Injective function)
- ✓ Onto function (Subjective function)
- ✓ One – to – one correspondence (Bijective)

<u>One – to – One Function</u>

If each element in the domain of a function has a distinict Image in the Co- domain then the function is said to be one- to – one function.



A function f : A ⟶ b is called one-to-one If and only If for all x_1 , $x_2 \in$ A, f (x_1) = f (x_2)

Implies $x_1 = x_2$

Example

Let f : R ⟶ R given by f $(x)=3x+5$. Show that f is one to one

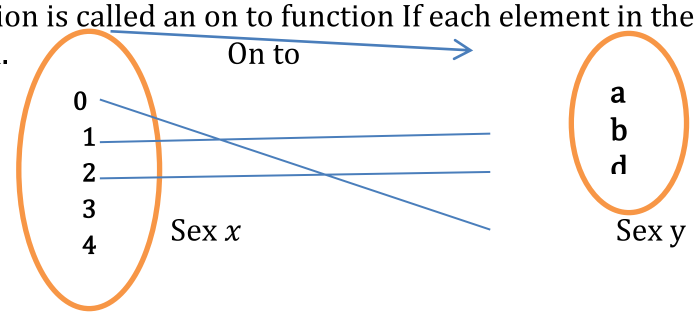
$f(x_1) = f(x_2) \Longrightarrow 3x_1 + 5 = 3x_2 + 5$

$3 = 3x_2 + 5 - 5$

$\dfrac{3x_1}{3} = \dfrac{3x_2}{3}$ $\qquad x_1 = x_2$ one to one

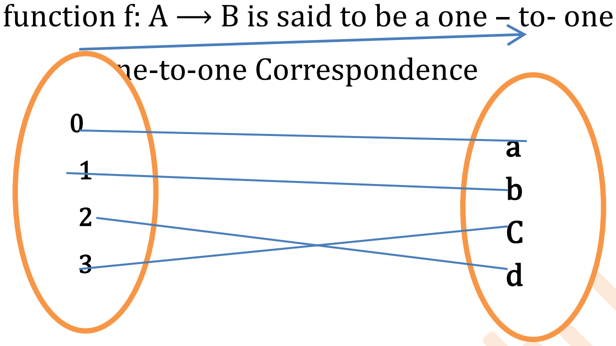
One to –Function (Surjective function)

A function is called an on to function If each element in the Co- domain has at least one pre – Image in the domain.



One to – one Correspondence

A function f: A ⟶ B is said to be a one – to- one correspondence If f is both one to one and on to.



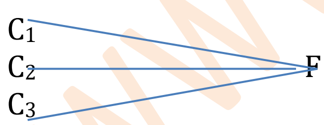
Example

f $(x)=x_2$

Exercise

1. Which of the following are one-to-one functions ?

a. A is the set of ordare pair (, y) : y is the father of x . it is not one – to one function b/c one father have mony chileds.



b. B is the set of order pair (, y) :x is a man and y is its nose . yes it is one to one function b/c no one person have two nose.

C. B is the set of order pair (, y) = |x-2|. Not one to one function This is.

$x = -1 \qquad y = 3$

$x = 5 \qquad y = 3$

2. Which one of the following is on to ?

a. f : R ⟶ R, f (x) = 2x-3

f (x)=2x-3

x	-3	-2	-1	0	1	2	3
y	-9	-7	-5	-3	-1	1	3

⟹ f (x)=2x-3 is not on to function but it is one to one function

b. g : [0, ∞) ⟶ R, g (x) = x^2

g (x) = x^2

x	0	1	2	3	4
y	0	1	4	9	16

not on to function it is one to one function

3. Evalut if the following function is an injective, Subjective, bijective function

a. f : R ⟹ f (x)=x+ 1

f (x_1) = f(x_2 ⟹ $x_1 + 1 = x_2$+1

$x_1 = x_2 + 1 - 1$

$\underline{x_1 = x_2}$ one to one function

b. f : R, f (x)=2x

x	-3	-2	-1	0	1	2	3
y	-6	-4	-2	0	2	4	6

It is one to one function

1.3 Graphs of functions

In this section you will learn how to draw graphs of functions such as- y = ax+b and y = ax^2 +b×+C. with special emphasis on liner and quadratic functions. You will study some of the Important properties of graph as in the following.

If f is a function with domain A, then the graph of f is the set of all ordered pairs

{ (x, f (x), x ∈ A }

Graphs of linear functions

If a and b are fixed real numbers, a ≠ 0, then f (x)=ax +b for every real numbers x is called a linear function. If a = 0, then f (x)= b is called a constant function. Some tines linear functions are written in the form

y = ax +b

Example – 1

f (x)= 2 x+1 is a linear function with a = 2 and b = 1 f (x)= 2 is constant function

Exercise

1. Construct tables of values of the following functions for the given domains

a. f (x)= 4 x+1 ; x = -3, -2, -1, 0, 1, 2, 3

x	-3	-2	-1	0	1	2	3
f (x)	-11	-7	-3	1	5	9	13

b. f (x)= 7- 3 x ; x = -1, 0, 1, 2, 3, 4

x	-1	0	1	2	3	4
f (x)	10	7	4	1	-2	-5

C. f (x)= $\frac{x}{4}$+1 x = -8, -4, -2, 0, 2, 4

x	-8	-4	-2	0	2	4
f (x)	-1	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2

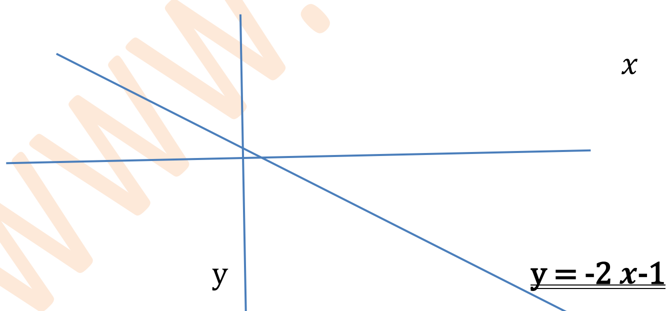
2. Draw the graph of each of the following by constructing a table of values for -4 ≤ x ≤ 4:

a. 2y +4 x +7 = 5

2y= -4x - 2

y = -2x - 1

x	-4	-3	-2	-1	0	1	2	3	4
y	7	5	3	1	-1	-3	-5	-7	-9

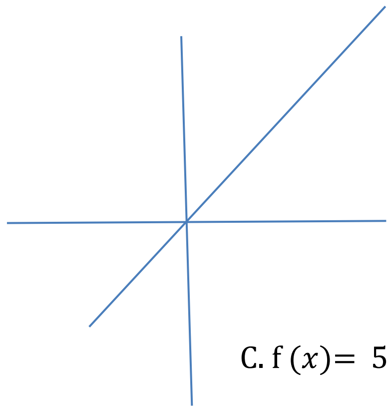


b. y – 3 x+7 = 2

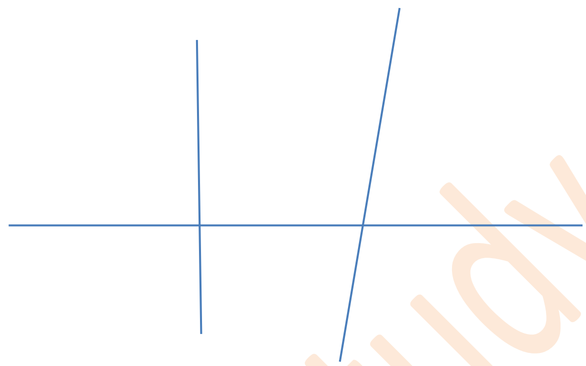
y = 3 x +2 -7 y = 3x – 5

x	-3	-2	-1	0	1	2	3
y	-14	-11	-8	-5	-2	1	4

y = 3×+7



C. f $(x)=$ 5



Properties of Linear Function

a. graphs of linear functions are straight lines

b. If a $>$ 0 then the graph of the linear function f $(x)=$ ax +b is Increasing

C. If a $<$ 0 then the graph of the linear function f $(x)=$ ax +b is decreasing

d. If a $=$ 0 then the graph of the linear function f $(x)=$ b is horizontal line.

e. If $x = 0$, then g (0) = b. This is means (0, b) lies on the graph of the function, and the graph passes through the ordered pair (0, b) This point is called the y – intercept . it is the point at which the graph intersect the y- axise

f. If g(x) = 0, then 0= ax +b Implies $x=\frac{-b}{a}$ this means $\left(\frac{-b}{a}, 0\right)$ lies on the graph of the function and the graph passes through the ordered pair $\left(\frac{-b}{a}, 0\right)$. It is a point at which the graph intersects the x - axis and it is called x- intercept.

Example - 1

If f $(x)=$ 3x -6 then finde x and y intercept

y = 3x -6 $\Longrightarrow$ y = 3 (0) -6 = -6

$x$in $= 0 = 3x$ -6 $= \frac{6}{2} = \frac{3x}{3}$ $\underline{\underline{\boldsymbol{x = 2}}}$

x intercipt $= (2, 0)$

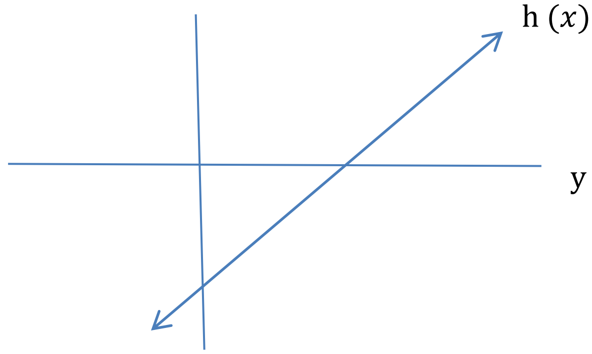
y intercept is $= (0, 6)$

Example – 2

Draw the graph of the function h (x) = $2x$-3

x in = 0 = $2x$ - 3 ⟹ $x = \frac{3}{2}$

y in = y = $2x$ - 3 ⟹ y = 2(0) -3 = -3



Exercise

1. Deter min the slope, y- intercept and x - intercept of the following linear function.
   a. $x + y = 2$ x in = 2 y in = 2
   b. $2x2$y = 3 x in = $\frac{3}{2}$ y in = $\frac{3}{2}$
   C. f (x) + 7 = 2 x ⟹ f x = 2 x-7 x in = $\frac{7}{2}$ y in = -7
   d. f (x) = -3x -5 ⟹ x in = -$\frac{5}{3}$ y in = -5
2. Given the following functions
   3x -1 g(x)= -x +2 h(x) -2x k(x) = 1
   a. Which graphs are decreasing function ?
   g(x)= -x +2 and h(x) -2x are decreasing
   b. Find the slope and x - intercept and y – Intercept of each functions

f (x) = -3x - 1 Slop is 3
x in = $\frac{1}{3}$
y in = - 1

g(x)= -x +2 Slop is - 1
x in = -2
y in = 2

h(x) -2x Slop is = 2
x in = 0
y in = 0

k(x) = 1 x in = No
y in = 0

Graphs Of quadratic Function

A function defined by f $(x) = ax^2 + bx + C$ wher a, b, C are real numbers and a $\neq 0$ is called is called quadratic function. The point a is the leading Coefficient of F.

Example

f $(x) = 3x^2 - 2x + 5$ a = 3,   b = -2   C= 5

The graph of a paqrabola. Parabolas may open upward or downward and vary in" Width " or Steepness" but they all have the same basic " U " shape. All parabla are symmetric with respect to ;one called the a xise of *Symmetry* . a parabola intersects its axis of symmetry at a point called the *Vertex* of the parabola.

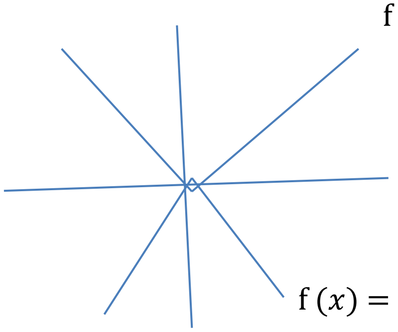
Exercise

Draw the graph of the following function by constricting tables of values x= -2 – 1,01 2)

a. f $(x) = \frac{3}{2}x^2$

x	-2	-1	0	1	2	
y	6	$\frac{3}{2}$	0	$\frac{3}{2}$	6	

f $(x) = \frac{3}{2} \times {}^2$

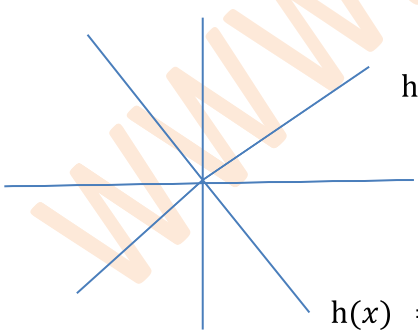


f $(x) = -\frac{3}{2}x^2$

b. h$(x) = \frac{1}{4}x^2$

h$(x) = \frac{1}{4}x^2$

x	-2	-1	0	1	2	
y	1	$\frac{1}{4}$	0	$\frac{1}{4}$	1	



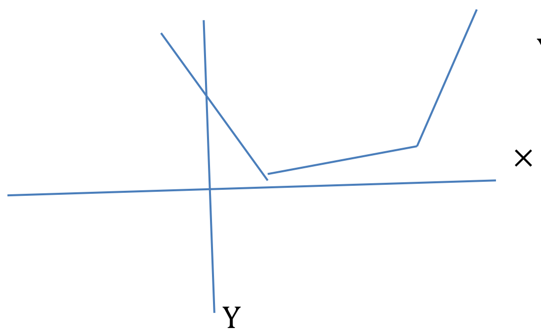
h$(x) = \frac{1}{4}x^2$

h$(x) = \frac{1}{4}x^2$

Graphs Of $y = ax^2 + C$

This type of quadratic function is similar to the basic ones of the previous function d is cussed but with a constant "C" added in the function y= a$\times^2$ i.e having the general form y $ax^2 + c$, As an example of this

y x^2+2. Graph of $y = a(\ -K)^2$ in this examples considered so far, the a xise of symmetry is the y – a xis, i.e the line $x = 0$. The next possibility is a quadratic function which has it's a xis of symmetry not on the y- a xis . A case in point to this function: $y = (x\text{-}3)^2$ has the sam shape and the same orthogonal a xise as $y = x^2$ but the axis of symmetry is the line x= shifte the graph of y= x^2 to the right by 3 units. The points x= 0 and $\times = 6$ are equidistant from 3.



From the graph of quadratic functions of the form $f(x) = ax^2$, $y = ax^2 + c$ $a \neq 0$ C. is any real number , we can summarize.

1. If $a > 0$ the graph opens up wared and

   If $a < 0$ the graph opens up down wared

2. The vertex is (0, 0) for $f(x) = ax^2$ and

   (0, C) for y= ax^2 +C

3. The domain is all real numbers

4. The vertical line that passes through the vertex is the axis of the parabola (or the axis of symmetry)

5. If $a > 0$, the range is the set of non- negative real number y for $f(x) = ax^2$ and the set of real number such that $y \geq C$ for $y = ax^2 + C$

6. if $a < 0$, the range is the set of non- positive real number y for $f(x) = ax^2$ and the set of real number such that $y \leq c$ for $y = ax^2 + C$

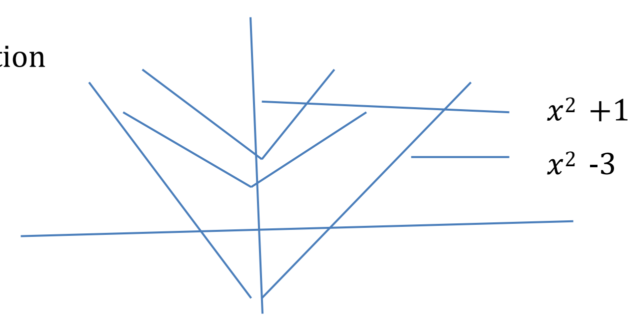
Exercise

1. Draw the graph of the following function

   a. $f(x) = x^2 + 1$

   b. $g(x) = x^2 - 3$

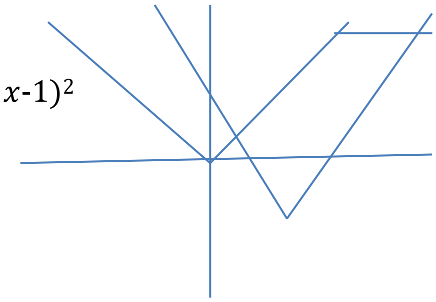
   C. $h(x) = 2x^2 + 2$



d. m $(x) = (x\text{-}1)^2$

e. K$(x) = (x\text{-}1)^2$

$(x\text{-}1)^2$



$(x\text{-}1)^2$

Graphs Of $y = a\,(x - k)^2 + m$

So fare, two separate cases have been discussed: first a standared quadratic function has its orthogonal axis shifted up or down, second a standared quadratic function has its axise of symmetry shifted left or right . the next step is to consider quadratic functions incorporate all shifts.

Example

Y = x^2 is shifted so that its axise of symmetry at $x = 3$ and its orthogonal axis is at y = 2.

a. write down the equation of the new curve.

Y = $(x\text{-}3)^2$ is symmetric about $x = 3$ and is shifted up by 2 units, so the equation is y = $(x\text{-}3)^2 + 2$

b. Find the coordinates of the point where it crosses the y – axis.

The curv crosses the y – axis when $x = 0$ put into the equstion y = $(x\text{-}3)^2 = 2$

$= (0\text{-}3)^2 + 2 = \underline{\underline{+1}}$

Exercise

1. The curve y = $-2x^2$ is shifted so that its axis of symmetry is the line x= -2 and its orthogonal axise is y= 8.
   a. write down the equation of the new curve.
   Y = $-2\,(x+2)^2 + 8$
   b. Find the coordinate of the point where this new curve cuts the × and y axes .
   y = $-2\,(x+2)^2 + 8$
   y = $-2\,(02)^2 + 8 = 0$ y = 0
2. Repat the above for each of the following

I. The curve y x^2 is shifted so that its axis of symmetry is the line x= 1 and its orthogonal axise y= -4

a. equation of new curve.

$(x\text{-}1)^2 - 4 \Longrightarrow (x\text{-}1)^2 - 4$

b. The coordinates of the points where this new curve cuts x and y

y = $(x\text{-}1)^2 - 4 \Longrightarrow y = (0\text{-}1)^2 - 4$ $\underline{\underline{-3}}$

Important Notes quadratic Function

1. The graph of f$(x) = (x+\mathrm{k})^2 + \mathrm{c}$ opens up ward.
2. The graph of f$(x) = -\,(x+\mathrm{k})^2 + \mathrm{c}$ opens down ward.
3. The vertex of the graph of f$(x) = -\,(x+\mathrm{k})^2 + \mathrm{c}$ is (- k , C) and the vertex of the graph of f$(x) = -\,(x\text{-}\mathrm{k})^2 - \mathrm{C}$

   Is (k, - C). similarly, the vertex of the graph of f$(x) = -\,(x\text{-}\mathrm{k})^2 - \mathrm{C}$ is (- k, - C) and the vertex of the graph of

$f(x) = -(x - k)^2 + C$ is (k, C)

1.4 Applications of Relation and Functions

1.4.1 Application Involving relation

Example

The data depicts the length of a woman's femur and her corresponding height. Based on these data, a forensic specialist can find a linear relationship between heights y and farmur x cm: $y = 0.96x + 24.3 \quad 40 \le x \le 55$.

Length of Farmur (Cm) x	Height (liches
45.5	65.5
48.2	68.0
41.8	62.2
46.0	66.0
50.4	70.5

From this type of relation, the height of a woman can be inferred Based on Skeletal remain

a. find the height of the woman whose femur is 46.0 cm.

$y = 0.9.6\ x + 24.3$

$y = 0.906\ (46.0) + 24.3$

$= 65.\ 976$

The woman is approximately 66.0 inches tall

b. find the height of the woman whose femurs 51.0 cm.

$y = 0.9.6\ x + 24.3$

$= 0.906\ (51.0) + 24.3$

$= 70.\ 506$

The woman is approximately 66.0 inches tall

2. If the equstion $x^2 + 18x + 81$ represents the area of the squre, what is the perimeter of the squre If $x = 10$?

Solution

$x^2 + 18x + 81$ factrite in to $(x+9)$

$(x+9)$ = Length $(x+9)$ = width

Per meter of the squre = 2 (Length + width)

$2\ (x+9) + x+9) = 2\ (+x+18 = 4x+36$

For $x = 10$ per mater = $4(10)+30= 40+36 =$ <u>**76**</u>

<u>Minimum and maximum Value of Qudratic Functions.</u>

Suppose you throw a ston upward. The stone turns down after it reaches it maximum height. Similarly, a parabola turns after it reaches a maximum or a minimum y, value

Example

1. The minimum value of a quadratic function expressed as $f(x) = (x+k)^2 + c$ is C.

   Similarly, the maximum value of $f(x) = -(x+k)^2 + c$ is C.

2. Find the maximum value of the function

   $f(x) = -x^2 + 6x - 8$, and sketch its graph.

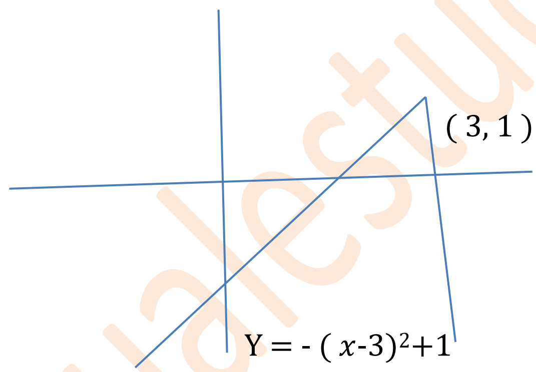
   $f(x) = -x^2 + 6x - 9 + 9 - 8$

   $= -(x^2 - 6x + 9) + 1$

   $f(x) = -(x-3)^2 + 1$

The graph of $f(x) = -(x-3)^2 + 1$ has vertex (3, 1) and hence the maximum value of f is 1

In this case , the range of the function is $\{ y : y \leq 1 \} = (-\infty, 1\}$



Exercise

1. find the vertex and the axis of symmetry of the following functions.

   a. $f(x) = -(x-4)^2 - 3$

   vertex = (4, - 3)

   Axis of symmetry $x = 4$ and shifted 3 unit to wards

   b. $f(\times) = x^2 - 5\times + 8 \quad = \left(x - \frac{5}{2}\right)^2 + 8 - \frac{25}{4}$

   $= \left(x - \frac{5}{2}\right)^2 + \frac{7}{4}$

   Vertex is $\left(\frac{5}{2}, \frac{7}{4}\right)$ and the axis of symmetry

   $x\ \frac{5}{2}$. or

2. Deter min the minimum or the maximum value of each following function and draw the graphs,

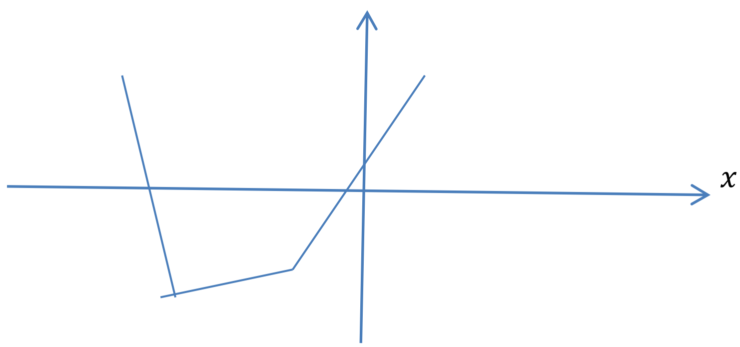
a. f(x) =$(x^2$= + 4 x + 1 = $(x + 2)^2$ -3

$(x^2$= + 4 x + 4) -4 +1 =

<u>$(x + 2)^2 - 3$</u>

Vertix is (-2 , -3) and y – intercept is 1 the function has minimum value at vertex – 2, - 3 which is 3.

The has minimum value at f $\left(-\frac{b}{2a}\right)$ = f $\left(-\frac{4}{2}\right)$ = f (-2) = - **<u>3</u>**



b. f (x)= 4 x^2+ 2 x +4

4 $(x^2 + \frac{1}{2}x)$ + 4

4 ($+\frac{1}{4})^2$ + 4 - $\frac{1}{4}$

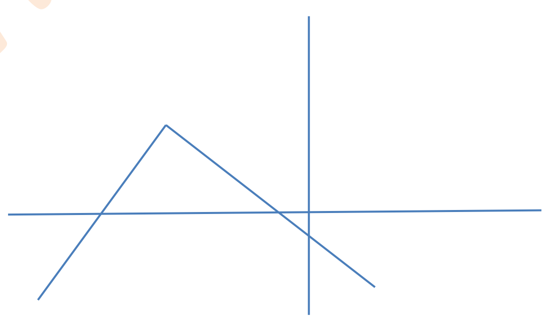
4 ($+\frac{1}{2})^2 + \frac{15}{4}$

Minimum value at vertex $\left(-\frac{1}{4}, \frac{15}{4}\right)$Which is $\frac{15}{4}$

C. f (x)= - x^2 - 4 x = - $(x + 2)^2$ + 4

- $(x^2 + 4x + 4)$ = - <u>$(x + 2)^2 + 4$</u>

The graph of the function opens down ward so the function has maximum value at vertex (-2, 4) , which is 4 .



3. A metal wire 40 cm long is cut in to two and each piece is bent to form a squre. If the sum of their areas is 58

squre cm. how long is each piece ?

Solution - 1

Let the length of one squre and the other squre be x and y respectively (y > x).

From the given condition

$$\begin{cases} 4x + 4y = 40 \\ x^2 + y^2 = 58 \end{cases}$$

From the first equation x + y = 10

Then , y = 10 – x

Substitute this to the second equation

$x^2 + (10 - x)^2 = 58$

$2x^2 - 20x + 42 = 0$

$x^2 - 10x + 21 = 0$

$(x - 3)(x - 7) = 0$

$x = 3$, $x = 7$, since y > x then $x = 3$, y = 7 ther for the length of each wire is 12 cm and 28 cm.

Solution – 2

Let the x and y be the larger and smaller lengths respectively.

$x + y = 40$ - - - 1 equation the area of each square is $\left[\frac{x}{4}\right]^2$ and $\left[\frac{y}{4}\right]^2$ respectively

$\frac{1}{16}(x^2 + y^2) = 58$ which implies $x^2 + y^2 = 928$

$x^2 + y^2 = 928$ - - - - - 2 equation consider , $(x + y)^2 + (x - y)^2 = x^2 + y^2 + 2xy + = x^2 + y^2$ -.

Ther fore, $(x + y)^2 + (x - y)^2 = 2(x^2 + y^2)$. Using the values from equations 1 and 2

$(40)^2 + (x - y)^2 = 2(928)$

$(x - y)^2 = 256 \Longrightarrow x - y = 16$ since $x > y$ the difference $x - y$ cannot negative

$x - y = 16$ - - - - - - 3

Solving simultaneous equation (1) and (3)

$x = 28$ and $y = 12$ Length of pieces are 28cm and 12cm

Important Notes

For any Quadratic equation
$f(x) = ax^2 + bx + C$ In this case
$f(x) = a\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b2-4ac}{4a}\right)$ the coordinate
of the vertex is $\left(-\frac{b}{2a}, \frac{b2-4aC}{4a}\right)$.
If $a > 0$, the range is $\{y: \geq \frac{b2-4ac}{4a}\}$
If $a < 0$ the range is $\{y: \leq \frac{b2-4ac}{4a}\}$

Review exercises

1. Let $f(x) = 2x^2 - 5x - 3$ and $(x) = -2x^2 + x + 7$

Find $F + g = f(x) + g(x) = 2x^2 - 5x - 3 - 2x^2 + x + 7$

$F + g = -\underline{4x + 4}$

ii) $F - g = 2x^2 - 5x - 3 - (2x^2 + x + 7)$

$= 2x^2 - 5x - 3 + 2x^2 - x - 7$

$F - g = \underline{4x^2 - 6x - 10}$

iii) $(2F - g)(3) = 2f = 4x^2 - 10x - 6 - (2x^2 + x + 7)$

$(2F - g) = + 2x^2 - x - 7$

$= 6x^2 - 12x - 13$

$(2F - g)(3) = 6 x 9 - 36 - 13 = \underline{\mathbf{5}}$

2. Let $f(x) = 2x^2 - 1$ and $g(x) = x - 3$, then

A evalut i f . g

$f.g = (2x^2 - 1)(x - 3) = x(2x^2 - 1) - 3(2x^2 - 1)$

$= 2x^3 - x - 6x^2 + 3$

$= 2x^3 - 6x^2 - x + 3$

ii) $f g(2) = 2(2)^3 - 6(2)^2 - 2 + 3$

$= 16 - 24 + 1$

$\underline{\underline{-7}}$

iii) $\frac{f}{g}(x) = \frac{2x^2-1}{x-3} = \frac{2x^2-1}{x-3}$

$\frac{f}{g}(x) = \frac{f}{g}(5) = \frac{2(25)-1}{5-3} = \underline{\underline{49/2}}$

3. If $f(x) = \frac{2x-3}{x-1}$ and $g(x) = \frac{x+8}{x}$ then find f . g

i) $f.g = \frac{(2x-3)}{(x-1)} \left(\frac{x+8}{x} = \frac{x(2x-3)+8(2x-3)}{x2-x}\right.$

$\frac{2x^2-3x+16x-24}{x^2-x} \qquad = \mathbf{\frac{2x^2-13x-24}{\underline{x^2-x}}}$

ii) $\frac{f}{g} = \frac{2x-3}{x-1} \div \frac{x+8}{x}$

$= \frac{2x-3}{x-1}\ x\ \frac{x}{x+8} = \frac{2x^2-3x}{x(x-1)+8(x-1)}$

$= \frac{2x^2-3x}{x^2-x+8x-8} \qquad = \frac{2x^2-3x}{x^2+7x-8}$

iii) $\frac{f}{g}(3) = \left[\frac{2x^2-3x}{x^2+7x-8}\right](3) =$

$\frac{2(3)^2-3(3)}{(3)^2+7(3)-8} = \frac{18-9}{9+12-8} = \underline{\underline{\frac{9}{22}}}$

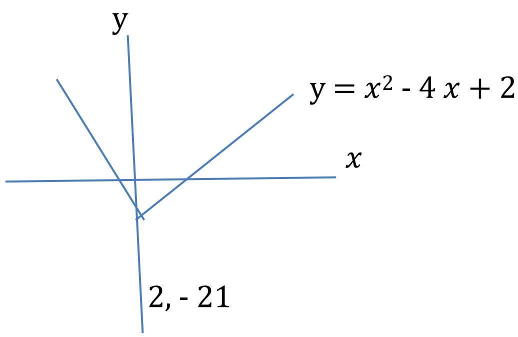
4. By using shifting rule, sketch the graph of each of the following.

a. $y = x^2 - 4x + 2$

$y = x^2 - 4x + 4 - 4 + 2$

$= (x-2)^2 - 2$

Vertex (2, - 2) and y – intercept is **2**



b. y = - x^2 - 6 x - 8

- (x^2 + 6 x+ 8) = - (x^2 + 6 x+9+9+8)

- (x^2 +6 x +9) – 9+8

- (x + 3)^2 + 1

Vertex = (- 3, 1) and y intercept is **- 8**

5. A mobile phone technician uses linear function C(t) = 2t + 10 determine the cost of repair where the time in the hours is t and C (t) is the cost Birr. How much will you pay If it takes him 3 hours to repair your mobile ?

C(t) = 2 t + 10 then

C(3) = 2 x 3 + 100 = 106 Birr is the cost you pay for the repair.

Unit Two / 2 /

Polynomial Functions

2.1 Definition Of polynomial Function

Let " n" be non – negative integer and let a_n , $a_n - 1$ - - - - a_2 , a_1 , a_0 be real numbers with $a_n \neq 0$. The function $P(x) = a_n x^n + a_n - 1x^{n-1}$ - - - - $+ a_2 x^2 + a_1 x + a_0$ is called a polynomial function in one variable x of degree n .

✳ in the above definition of polynomial function

- ✓ a_n , a_n -1 - - - - a_2 , a_1 , a_0 are called the coefficient of the polynomial function (Or simply the polynomial)

ii) The number a_n is called the leading coefficient of the polynomial and $a_n x^n$ is the leading term

iii) The number n (the exponent of the highest power of x) is the degree of the polynomial.

iv) The number $a_0$is called the constant term of the polynomial

Domain of polynomial function : The

✳ domain of a polynomial function is all real numbers

Example

1. Find the degree, leading coefficient and constant term of the following

a, $f(x)= 3x^3 - 9x^2 + 5x + 3/2$

degree, 3, leading coefficient 3, constant term $\frac{3}{2}$

b. $2(x4 + \frac{1}{2}x^2 - 2) + x + x^4 + 1$

degree = 4, leading coefficient = 2, constant term = 1

C. $- 2 + 3x^3 + \frac{2}{5}x^2 - x^4 + 4x^5 + 5x$

First ordered $4x^5 - x^4 + 3x^3 + \frac{2}{5}x^2 + 5x - 2$

Degree = 5, leading coefficient = 4, constant term = **- 2**

Exercise

1. find the degree, leading coefficient and constant term of the following polynomial

1. $f(x)= 12x^3 - 9x + 3x + 4$

degree = 3, leading coefficient = 2, constant term = 4

2. $2(x) = -3x^4 + x^2 + 3\left(2x^2 + 4x^4 + 5x^3 + \frac{2}{3}\right)$

First ordered $= -3x^4 + x^2 + 6x^2 + 12x^4 + 15x^3 + 2$

$f(x) = 9x^4 + 15x^3 + 7x^2 + 2$

degree = 4, coefficient = 9, constant = 2

3. $h(x) = 2x^3 + 6x^2 - 5 \qquad + x^3 + 2$

3 3

$= \underline{2x^3} + 2x^2 - \frac{5}{2} + x^3 + 2$

3 $\underline{2x^3} + 2x^2 - \frac{5}{2} + 2$

3

$= \underline{5x^3} + 2x^2 - \frac{1}{2}$

3

Degree = 3, leading coefficient $= \frac{5}{3}$, constant term $= \frac{-1}{2}$

Not that

- ✓ A polynomial expression is an expression of the form : $a_n x^n + a_{n-1} x^{n-1} + \text{- - -} \; a_2 x^2 + a_1 x + a_0$ where n is non-negative integer and $a_n \neq 0$ each Individual expression ak x^k make up the polynomial is called term.

2.2 Operations On Polynomial Functions

Addition Of polynomial Functions

- ✓ The sum of two polynomial functions F and g is written as f + g and is defined as :

  $(f + g)(x) = f(x) + g(x)$ for all real numbers x .
- ✓ The sum of two polynomial functions is found by adding the coefficients of like terms.

  Example – 1 find the sum of f + g

  $f(x) = 2x^4 + 5x^3 + x^2 + 9x + 4$ and $2(x) = -3x^3 + 5x^2 - 9$

  $f + g(x) = 2x^4 + 5x^3 + x^2 + 9x + 4 + (-3x^3) + 5x^2 - 5x - 9$

  $= 2x^4 + 2x^3 + 6x^2 + 4x - 5$

Not that

1. If $f(x)$ and $g(x)$ have different degrees, the degree of $f(x) + g(x)$ is the sam as the degree of $f(x)$ or the degree of $g(x)$ which ever has the highest degree.
2. If $f(x)$ and $g(x)$ have the same degree. The of the sum may be lower or equal to the common degree
3. The sum of two polynomial functions is a polynomial function.

Exercise

1. Find the sum of the polynomial function $f(x)$ and $g(x)$ where

a. f(x) = 2 x^3 - 3 x -5 and g (x) 5 x^4 - 7 x^2 +3

f(x) + g (x) = 2 x^3 - 3 x -5 + 5 x^4 - 7 x^2 +3

= 5 x^4 + 2 x^3 - 7 x^2 – 2 3 x -2

b. f(x) = - x^4 + 2 x^3 - 3 x^2 - 3 x + 2 and g (x) = 5 + 7 x - 2 x^2 - x^3 + x^4

f + g (x) = x^4 + 2 x^3 - 3 x^2 - 3 x + 2 + x^4 - x^3 - 2 x^2 + 7 x + 5

= 2 x^4 + x^3 - 5 x^2 + 4 x+7

Subtraction Of Polynomial function

The difference of two polynomial f and g is written f – g , and is defined as

(f – g) (x) = f(x) = g (x) for the all real number

Example

In each of the following find f – g

a, f(x) = - 2 x^3 + 5 x^2 + 3 x+2 and g (x) = - 2 x^3 + 4 x^2 + 8 x-7

f(x) - g (x) - 2 x^3 + 5 x^2 + 3 x+2 – (- 2 x^3 + 4 x^2 + 8 x-7)

= - 2 x^3 + 5 x^2 + 3 x+2 + 2 x^3 - 4 x^2 - 8 x-7

$\underline{\boldsymbol{x^2 - 5x + 9}}$

Exercise

a. find f(x) - g (x) where.

f(x) = 3$\sqrt{3}$ x^4 + 2 x^3 - 5 x^2 + x – 5 $\sqrt{3}$ and

g (x) $\sqrt{3}$ – 3 x - x^2 + 2 x^3 – 2 $\sqrt{3\,x4}$

f – g = $\sqrt{3}$ x^4 + 2 x^3 - 5 x^2+ x – 5 $\sqrt{3}$ – (-2 $\sqrt{3}$ x^4 + 2 x^3 - x^2 - 3 x + $\sqrt{3}$)

= $\sqrt{3}$ x^4 + 2 x^3 - 5 x^2+ x – 5 $\sqrt{3}$ + 2 $\sqrt{3}$ x^4 + 2 x^3 + x^2 + 3 x - $\sqrt{3}$

= 3 $\sqrt{3}$ x^4 + 0 - 4 x^2+ 4 x – 6 $\sqrt{3}$

= $\underline{\boldsymbol{3\sqrt{3}\,x^4 - 4x^2 + 4x - 6\sqrt{3}}}$

b. f(x) = -$\frac{1}{3}$ x^4 + 3 x^3 - 3 x^2 - $\frac{3}{5}$ x + 2 and

g (x)= $\frac{1}{5}$ x^4 - x^3 - 2 x^2 + $\frac{7}{5}$ x + 5

f – g = -$\frac{1}{3}$ x^4 + 3 x^3 - 3 x^2 - $\frac{3}{5}$ x + 2 – ($\frac{1}{5}$ x^4 - x^3 - 2 x^2 + $\frac{7}{5}$ x + 2)

= -$\frac{1}{3}$ x^4 + 3 x^3 - 3 x^2 - $\frac{3}{5}$ x + 2 –$\frac{1}{5}$ x^4 - x^3 - 2 x^2 + $\frac{7}{5}$ x + 2

= - $\boldsymbol{\frac{8}{15}x^4 + 4x^3 - x^2 - 2x}$

Multiplication Of polynomial function

The product of two polynomial functions f(x) and g(x) is written as f.g and is defined as

(f.g) (x) = f(x) . g (x) for all real numbers x .

The product of two polynomials f(x) and g(x) is found by multiplying each term of one by every term of the other as shown in the following .

Example

1. Find f . g (x) where f(x) = 2 x+3 and g(x) = 3 x^2 - 5 x+6

f . g (x) (2 x+3) (3 x^2 - 5 x+6

= 2 x (3 x^2 - 5 x+6) +3 (3 x^2 - 5 x+6)

= 6 x^3 - 10 x^2 + 12 x + 9 x^2 - 15 x + 18

= 6 x^3 - x^2 - 3 x+ 18

2. Let f(x) = x^2 - 2 x+2 and g(x) = 2 x^3 - 4 x^2 - 5 x+1

f . g (x) = (x^2 - 2 x+2) (2 x^3 - 4 x^2 - 5 x+1)

= x^2 (2 x^3 - 4 x^2 - 5 x+1) - 2 x (2 x^3 - 4 x^2 - 5 x+1) +2 (2 x^3 - 4 x^2 - 5 x+1)

= 2 x^5 - 4 x^4 - 5 x^3 + x^2 - 4 x^4 + 8 x^3 + 10 x^2 - 2 x +4 x^3 -8 x^2- 10 x+2

= **$\underline{2 x^5 - 8 x^4 + 7 x^3 + 3 x^2 - 12 x + 2}$**

Exercise

1. find the product of f(x) and g(x) using

a. f(x) = 2 x^2 - 2 x – 1 and g(x) 3 x+5

2 x^2 - 2 x – 1

$\underline{3 x+5}$

10 x^2 - 10 x – 1

6 x^3 - 6 x^2 - 3 x

$\underline{4 x^2 - 13 x - 5}$ = 6 x^3 + 4 x^2 - 13 x – 5

b. f(x) = 3 x^3 - x^2+ x – 1 and g(x) 5 x- x^2

f . g= 3 x^3 - x^2+ x – 1

$\underline{5 x - x^2}$

- 15 x^5 + 2 x^4 + 2 x^3 + 2 x^2

15 x^4 - 5 x^3 + 5 x^2 - 5 x

+ = $\underline{- 15 x^5 - 17 x^4 - 7 x^3 + 7 x^2 - 5 x}$

Important Notes

1. for any two non – zero polynomial function f and g, the degree of f . g is m + n If the degree of f is m and the degree of g is n .

2. If either f or g is the zero polynomial then f . g becomes the zero polynomial and has no degree.

3. The product of two polynomial functions is a polynomial function.

Division Of Polynomial Function

A number that takes the form $\frac{a}{b}$ where a and b are Integers and b ≠ 0 is called a rational number.

If b is positive Integer, we can divide a by b to find two other Integers q and r with 0 ≤ r < b such that $\frac{a}{b} = q + \frac{r}{b}$ here, a is called the divided, b is called divisor, q is called the quotient and r is called the remainder.

- ✓ The divison (Quotient) of two polynomial functions f and g is written as f ÷ g and is defined as (f ÷ g(x) = f(x) = f(x) ÷ g(x) for all real numbers x and g$(x) \neq 0$ (zero polynomial

Example

Find f(x) ÷ g(x) . where f$(x) = 4x^3 + 4x^3 + 4x^2 - x + 4$

G$(x) = 2x - 1$

$$
\begin{array}{r|l}
 & 2x^2 + 3x + 1 \\
\hline
2x - 1 & 4x^3 + 4x^2 - x + 4 \\
 & 4x^3 - 2x^2 \\
 & \quad 6x^2 - x + 4 \\
 & \quad - 6x^2 - 3x \\
 & \qquad 2x + 4 \\
 & \qquad 2x - 1 \quad \text{reminder } 5
\end{array}
$$

$$\frac{f(x)}{g(x)} = \frac{4x^3 + 4x^2 - x + 4}{2x - 1} = 2x^2 + 3x + 1 + \frac{5}{2x-1} \text{reminder}$$

Exercise

In each of the following find the quotient f (x) and remainder r (x) when f (x) defided by g(x)

a. f$(x) = x^3 + 4x^2 + x + 1$ and g $(x) = x + 2$.

$\frac{f}{g}(x) =$

$$
\begin{array}{r|l}
 & x^2 + 2x - 3 \\
\hline
x + 2 & x^3 + 4x^2 + x + 1 \\
 & x^3 + 2x^2 \\
 & \quad - 2x^2 + x + 1 \\
 & \quad 2x^2 + 4x \\
 & \qquad - 3x + 1 \\
 & \qquad - 3x - 6
\end{array}
$$

$$\frac{f}{g} = \frac{x^3 + 4x^2 - x + 1}{x + 2} = \overset{q}{x^2 + 2x - 3} + \overset{r}{\frac{7}{x+2}}$$

c. f$(x) = 4x^3 + 6x^2 - 8x + 5$ and g $(x) = 2x - 1$.

$\frac{f}{g} = \frac{4x^3 + 6x^2 - 8x + 5}{2x - 1}$

$$\begin{array}{r|l} & 2x^2 + 4x - 2 \\ \hline 2x - 1 & 4x^3 + 6x^2 - 8x + 5 \\ & -4x^3 - 2x^2 \\ & 8x^2 - 4x \\ & -4x + 5 \\ & -4x + 5 \\ & 3 \quad R \end{array}$$

$\frac{f}{g} = 2x^2 + 4x - 2 + \frac{+3}{2x-1}$

2.3 Theorems on Polynomials

<u>Polynomial Division The orem</u>

If f(x) and d(x) are polynomials such that d(x) $\neq$ 0 and the degree of d(x) is less than or equal to the degree of f(x) , then there exist unique polynomials f(x) and r(x) such that

F(x) = d(x) . q(x) + r (x)

Where, r (x) = 0 or the degree of r (x) is less than the degree of d (x).

If the reminder r (x) is zero , d (x) divides f (x) exactly.

Example

For each of the following find

F(x) and r (x)

F(x) = d(x) f (x) + r (x).

a. f(x) $= x^3 - 1$ d(x) = $x - 1$

$$\begin{array}{r|l} & x^2 + x + 1 \\ \hline x - 1 & x^3 - 1 \\ & x^3 - x^2 \\ & x^2 - 1 \\ & x^2 - x \\ & x - 1 \\ & x - 1 \\ & 0 \quad R \end{array}$$

f(x) = $\frac{x^3 - 1}{x - 1} = x^2 + x + 1$

reminder - Zero

Exercise

1. For each of the following pair of polynomial find the quotient f(x) and reminder r (x)

a. f(x) $=6x^2 - 2x + 3$, d(x) $= x - 1$

$\frac{f(x)}{dx} =$ $6x+4$

$x - 1 \;\big|\; 6x^2 - 2x + 3$

$4x^2 - 6x$

$4x - 4$ reminder

$+ 7$

$\frac{f(x)}{dx} = \frac{6x^2 - 2x + 3}{x - 1} = 6x+4 + \frac{7}{x-1}$

b. f(x) $= x^3 + 4x^2 + 8x + 6$, d(x) $= x^2 + 2x - 1$

$\frac{f(x)}{dx} =$ $x + 2$

$x^2 + 2x - 1 \;\big|\; x^3 + 4x^2 + 8x + 6$

$x^3 + 2x^2 - x$

$2x^2 + 9x + 6$

$- \; 2x^2 + 4x - 2$

$5x + 8$ reminder reminder

$\frac{f(x)}{g\,x} = \frac{x^3 + 4x^2 + 8x + 6}{x^2 + 2x - 1} = x + 2 + \frac{5x + 8}{x^2 + 2x - 1}$

C. f(x) $= -x^4$ d (x) $= x + 2$

$-x^3 + 2x^2 - 4x + 8$

$x + 2 \;\big|\; -x^4$ $= -x^3 + 2x^2 - 4x + 8 + \frac{16}{x+2}$

$-x^4 - 2x^3$

$-2x^3$ $-4x^2$

$+2x^3$ $-4x^2 + 8x$ $8x$

$+2x^3 + 4x^2$ $8x + \underline{16}$

Remainder Theorem

Let f(x) be a polynomial of degree greater than or equal to 1 and let C be any real number.

If f(x) is divided by the linear polynomial (x – C) then the remainder is f (C) .

Proof

F(x) is divided by x – C , the reminder is always a constant. Because we continues to divide until the degree of the reminder less than the degree of the divisor F(x) = (x – C) q(x) + k Where k is a constant. This equation hold for every real number . it hold when x = C.

F (C) = (C – C) q(x) + k

= 0 . q (C) + K

0 + k = k

Example

a. f(x) 2 x^3 + 5 x^2 +3 x+2 , d(x) = x+1

d(x) = x+1 = x – (- 1) , ther for C = - 1 and

remainder is f (C) = f (- 1) = 2

b. f(x) = x^4 +3, d(x)= x – 2

d(x) = x-2 ther for C = 2 the remainder f(2) = 19

Exercise

1. In each of the following use the remainder theorem to find the remainder when f(x) is devided by d(x)

a. f(x)= x^3 - 3 x^2+4 d(x)= x – 1

d(x)= x – 1 ther for C = 1 = f (1) = 2

b. f(x)= - 2 x^3+4 x^2 + 5 x - 2 , x+2

d(x)= x+2 C = - 2 f(- 2) =

-2 x (- 8) + 4 (4) – 10 – 2 = **20**

2. When 5 x^3 - b x^2 + 8 x – 1 is divided by x + 1 the remainder is 15. Find the value of b .

5(- 1)^3 – b – 8 – 1 = 15

- 5- 8 – 1 – b = 15 ⟹ - 14 – b = 15

- b = 15 + 14

- b = 29 ⟹ **b = - 29**

3. find the value of a and b such that when a and b such that when a x^3 - b x + 5 x – 2 is divide by x – 1 and x + 1 the remainder is 4 and 6 respectively

$\dfrac{a x^3 - b x + 5 x - 2}{x = 1}$ = r = 4

$a\,x^3 - b\,x + 5\,x - 2$ by $(x - 1)$

$d(x) = x - 1$ $C = 1$

$f(C) = a - b + 5 - 2 = 4$

$a - b + 3 = 4$

$a - b = 1$ - - - - - equal 1

$a\,x^3 - b\,x^2 + 5\,x - 2$ by $(x + 1)$

$d(x) = x + 1$ $C = -1$

$f(C) =$ $-a + b - 5 - 2 = 6$

$-a + b - 7 = 6$

$-a + b = 13$ equal 2

$-a - b = 13$

$a - b = 1$

$a = 1 + b$ from equal 1

$-(1 + b) + b = 13$

$-1 - b + b = 13$

$-1 - 2b = 13$

$\frac{-2b}{-2} = \frac{14}{-2}$ $\underline{b = -7}$

$\Longrightarrow a - (-7) = 1$

$a + 7 = 1$

$a = 1 - 7$

$\underline{a = -6}$

Factor Theorem

Remember that in the case of multiplication of polynomials, we multiply two or more polynomials to find another polynomial

For example, $(x+1)(2x - 1) = 2x^2 - x - 1$.

The polynomial $2x^2 - x - 1$ is called product or multiple and $(x+1)$ and $(2x - 1)$are called factor let $f(x)$ be a polynomial of degree greater than or equal to one and let C be any real number , then

1. If $f(C) = 0$ then $x - C$ is a factor of $f(x)$
2. If $x - C$ is factor of $f(x)$ then $f(C) = 0$

Example **Show that**

$x+2$ is a factor of $f(x) = x^2 + 5x + 6$

$C = -2$ $f(-2) = 4 - 10 + 6 = -6 + 6 = \underline{0}$

Show that $x+1$ and $(x - 2)(x + 2)$ not factor of

$F(x) = x^4 - x^3 - x^2 - x - 2$

$F(-1) = 1 + 1 - 1 + 1 - 2 = 2 - 2 = 0$ factor

$F(2) = 16 - 8 - 4 - 2 - 2 = 16 - 16 = 0$ factor

$F(-2) = 16 - (-8) - 4 + 2 - 2 = 20$ not factor

Exercise

1. in each of the following , find a number k satisfying the given condition

   a. $x - 2$ is factor of $2x^2 + kx^2 - 5x - 1$

   $2(2)^3 + 4k - 10 - 1 = 0$

   $16 + 4k - 11 = 0 \Longrightarrow 4k + 5 = 0$

   $4k = -5 \qquad k = \frac{-5}{4}$

   b. $x + 3$ is factor of $x^4 + 2kx^3 - x^2 - 5kx + 6$

   $f(-3) = (-3)^4 + 2k(-3)^3 - (-3)^2 - 15k + 6 = 0$

   $= 81 - 54k - 9 - 15k + 6 = 0$

   $= 78 - 69k = 0$

   $K = \frac{78}{69}$

2. find the value of a and b in the polynomial

   $ax^4 + x^3 - 2bx^2 - 11x + 6$ such that

   $x+1$ and $x - 2$ are its factors

   $F(x) = ax^4 + x^3 - 2bx^2 - 11x + 6$

   $d(x) = x - 2 \qquad C = 2$

   $f(C) = f(2) = 16a + 8 - 8b - 22 + 6 = 0$

   $16a - 8b - 8 = 0$ equal 1

   $d(x) = x - 1 \qquad f(1) = a + 1 - 2b - 11 + 6$

   $a - 2b + 1 + 6 - 11$

   $a - 2b - 4 = 0$ - - - - equal 2

   $16a - 8b - 8 = 0$

   $8a - b - 1 = 0$ divid ed by 8

   $a = \frac{b+1}{8}$ from equal 1

   Or **$b = 8a - 1$**

   $a - 2(8a - 1) - 4 = 0$

   $a - 16a + 1 - 4 = 0 \Longrightarrow -15a - 3 = 0$

   $-\frac{15a}{15} = \frac{3}{15} \qquad a = \frac{-1}{5}$

   $\frac{-1}{5} - 2b - 4 = 0$

   $-\frac{1-20}{5} - 2b = 0$

   $-\frac{21}{5} - 2b = 0 \Longrightarrow \frac{-2b}{2} = \frac{21}{5} \div 2$

   $b = \frac{21}{10}$

Zeros of a polynomial Function

For a polynomial function $f(x)$ and areal number C, if $f(C) = 0$ then c is a zaro of f

Not that

1. If $x - C$ is factor of f(x), then C is the zero of f(x)
2. Determine the zero of f(x) the $x - C$ is factor of f (x)
3. If C is the zero f(x), then C is the root or solution of the equation f(x) = 0

Example

1. Determine the zero of f (x) = x^4 - 13 x^2+36

F(x) = 0 $\Longrightarrow$ x^4 - 13 x^2+36 = 0

(x^2)^2 - 13 x^2 + 36 = 0 (Let y = x^2

Y^2 – 13y +36 = 0

(y – 4) (y – 9)

(x^2 – 4) (x^2 – 9) = 0

(x – 2) (x +2) (x -3) = 0

x = 2 or x= -2 or x = 3 or x = -3

Ther for x = 2, x = -2 x = 3 and x= - 3 are the zeros of f(x).

Exercise

Find the zeros of the following function

a. x^4 - 5 x^2 +4

(x^2)^2 5 x^2 +4 = 0 let y = x^2

Y^2 + 5y +4 = 0 $\Longrightarrow$ y^2 – 5 y+4 = 0

(y+4) (y -1) = 0

(x^2 +4) (x^2 – 1) = 0

x^2 = x^2 = 1

x^2 = 4 or x^2 = 1

x = 2 Or x = 1

b. f (x) = x^4 - x^2 – 2

(x^2)^2 - x^2 – 2 = 0 let y = x^2

Y^2 - y – 2 = 0

(y – 2) (y + 1) = 0

(x^2 – 2) (x^2 + 1

x^2 = 2 Or x^2 = - 1

x = $\sqrt{2}$ or x^2 = - 1 not root inset of real number

Zeros Of a polynomial function and their multiplicities

Consider f (x) = x^3 - 2 x^2 - x – 2 = (x – 1) (x+1) (x+2) and g (x) = x^3 $^+$ 4 x^2 - 3 x – 18 = (x+3)^2 (x-2)

F(x) have three distinct factors (x – 1), (x+1) and (x+2), that hase three distinct zeros – 1 , 1 and – 2

- ✓ If $(x - C)^k$ is a factor of polynomial function f(x) but $(x - C)^{k+1}$ is not, then C is said to be a zero of multiplicity K of f (x).

Example

x = - 1 is a zero of f (x) x^3 - x^2 - 5x – 3 find the multicity.

Exercise

1. for each of the following polynomial, list the zeros and state the multiplicity of each zero

a. f (x) = x^3 - 4 x^2 + 5x – 2 (hent x = 1 is one factor

$(x-1)\ (x^2 - 3x + 2)$

$(x-1)(x-1)(x-2) = ((x-1)^2(x-2)$

There for x = 1 and x = 2 are the zeros

b. k (t) = $(t + \frac{2}{3})^3 + 10$

t = $\frac{2}{3}$ and t = 0 are the zeros

C. h (t) = 2t³ + 5t² + 4t + 1=0

= $2t^3 + 5t^2 + 4t + 1 = 0$

h(t) = (t+1) (2t+1) There for

t = - 1 and t = $-\frac{1}{2}$ are the zeros with multiplicity 2 and 1 respectively.

Location Theorem

Let a and b be real numbers such that a<b.

If f is a polynomial function such that f(a) and f(b) have opposite signa, then there is at least one zero of f between the number a and b .

It is sometimes possible to estimate the zeros of a polynomial function from table ralues.

E.g:- Let f(x) = x^4 - 2 x^3 + 4 x+4 . locate the zeros of Integer $-3 \le x \le 3$

x	-3	-2	-1	0	1	2	3
F(x)	9 1	12	-1	4	3	-4	7

Since f (-2) = 12 > 0 and f (-1) -1< 0 we have see that the value of f(x) changes sign from positive to negative between – 2 and – 1 hince the location thermo thire is a zero of f(x) between x = -2 and x= -1

Rational Root test

Suppose that all the copficent of the polynomial function described by.

F (x) = $a_n x^n + a_n - 1\, x^{n-1}$ - - - - - $+ a_2 x^2 + a_1 x + a_0$ are integer with $a_n \neq 0$ and $a_0 \neq 0$.

If $\frac{p}{q}$ is a root of f (x) in lowest term , then P is a factor of a_0 and 9 is a factor of a_n . Steps to find the rational zeros of a polynomial function f (x)

1. Arrange the polynomial in descending order of the form , f (x) = $a_n x^n + a_n - 1\, x^{n-1}$ - - - - - $+ a_2 x^2 + a_1 x + a_0$

1. Write down all the factors of the constant term a_0 this are all the possible value of P.
2. Write down all the factor of the leading coefficient a_0 . These are all possible value of q.
3. Write down all the possible value of $\frac{p}{q}$. Remember that since factor can be negative , $\frac{p}{q}$ and $\frac{p}{q}$ must both be included. Simplify each value cross out any duplicates.
4. Identify those values of $\frac{p}{q}$ for which $= f\left(\frac{p}{q}\right) = 0$ These are all the ratinal root of f (x) .

Example

a. $f(x) = x^2 + x - 2$ The leading cofficentis $a_2 = 1$ and constant term $a_0 = -2$ possible value of P are factor of – 2, These are $\pm 1, \pm 2$ possible value of q are 1 This are ± 1 the possible rational zeros $\frac{p}{q}$ are $\pm 1, \pm 2$

x	-2	-1	1	2
F(x)	0	-2	0	4

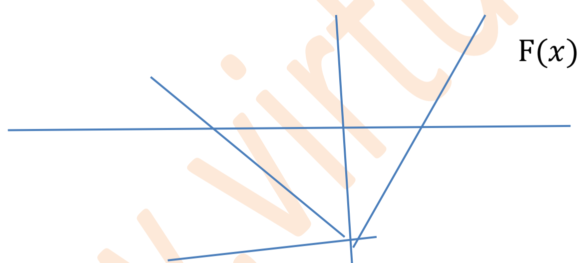
There for , the zeros of $f(x)$ = are **<u>– 2 and 1</u>**

2.4 Graph Of polynomial Function

Example

$F(x) = x^2 + 2x - 3 = y = x^2 + 2x + 3 = (x^2 + 2x + 1) - 1 - 3 = (x+1)^2 - 4$

$= -4 + (x+)^2$ since $((x+1)^2 \geq 0$ for all real numbers x and -4 is the minimum value of f. The minimum value of f is attended when $x = -1$ the point – 1, - 4 is called turing point vertex of the graph of f.

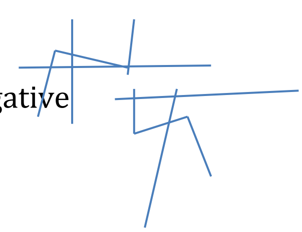


$F(x) = x^2 + 2x - 3$

Turning point (-1, 4)

<u>**For all Polynomial**</u>

1. degree of $f(x)$ is odd and its leading coefficient is positive
2. The degree of $g(x)$ is add and its leading coefficient is negative



2.5 <u>Applications</u>

Example

Let the cost required for a cell phone manufacture to manufacture of cell phones be c $(x) = 14500 - 120x + 9x^2$, $(x \geq 0)$. If the manufacture sells the phones for 600 birr each then find

a. the revenge function R (x) , total income produced by selling the phones

$R(x) = 600\,x$

b. The profit function $P(x)$

$P(x) = C(x) - R(x) = 14500 - 120\,x + 9\,x^2 - 600\,x$

$= 14500 - 720\,x + 9\,x^2$.

There fore $P(x) = 14500 - 720\,x + 9\,x^2$ (polynomial function)

C. The profit obtained when the manufacture sell x is called phone where

i) $x = 2$ ii) $x = 40$ iii) $x = 100$

i) $P(2) = 14500 - 720(2) + 9(2)^2 = 13096$ birr

ii) $P(40) = 14500 - 720(40) + 9(40)^2 = 100$ birr

iii) $P(100) + 9(100)^2 = 32500$ birr

d. The minimum profit of the manufacture and the number of phones that could be sold to get the minimum profit.

$P(x) = 14500 - 720\,x + 9\,x^2 = 9\,x^2 = 9\,x^2 - 720\,x + 14500$

$= 9(x^2 - 80\,x) + 14500$

$= 9(x^2 - 80\,x + 1600) - 14400 + 14500$

$= 9(x - 40)^2 + 100$

$P(x) = 100 + 9(x - 40)^2$

Since $9(x - 40)^2 \geq 0$

$P(x) = 100 + 9(x - 40)^2 \geq 100$

There for the minimum profit is 100 birr and this minimum profit is obtunded when the manufacturer sell $x = 40$ cell phones.

Unit Three / 3 /

Exponential and logarithmic Functions

3.1 Exponential

For any natural number n and areal number a the symbol a^n , read as " the n^{th} power of a " or " a raised to n " , is defined as follows. $a^n = \underbrace{a \, x \, a \, x \text{ - - - - - } x a}_{n - \text{factors}}$

in a^n a is call the base and n is called exponent

Example evalut

a. 2^3 = B. – 2^3 C. C – (- 2)^3

a = 2 x 2 x 2 = 8

b. = - 2 x – 2 x – 2 = - 8

C. = (- 2)^3 = - 2 x – 2 x – 2 (= -(- 8) = 8

Laws Of Exponents

If a is any real number and n is a positive integer then a^n means a x a x a x - - - x a . The laws for the behaviors of exponents follow naturally from this meaning a^n . If a is a real number and ma and n are natural number.

1. $a^m \, x \, a^n = a^{m+n}$

2. $\frac{a^m}{a^n} = a^{m-n}$ E.g $\frac{3^6}{3^3} = 3^{6-3} = 3^3 = 27$

3. $(a^m)^n = a^{m \times n}$ $(3^2)^3 = 3^6$

4. $(a\,b)^n = a^n b^n$ - E.g $(2 \, x \, 3)^3 = 2^3 \, x \, 3^3$

5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ E.g $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4}$

evaluate

i) $a^3 \, x \, a^2 = a^{3+2} = a^5$

ii) $\frac{a^5}{a^2} = a^{5-2} = a^3$

Zero and Negative Exponent

$\frac{a^n}{a^n} = a^{n-n} = a^0 \Longrightarrow 1$

If a ≠ then we defined a^0 to be equal to 1 we do not attempt to give any meaning to the expression 0^0 , it remains undefined.

Zero and Negative Exponent

If n is a positive integer and a ≠ 0 then

1. $a^0 = 1$ and 0^0 undefined
2. $a^{-n} = \frac{1}{a^n}$

Example

2. $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
3. $\left(\frac{2}{3}\right)^{-3} = \frac{1}{(2/3)^3} = \left(\frac{3}{2}\right)^3 - \frac{27}{8}$

The laws for integer exponent

For real numbers a and b and integers m and n , the following laws of exponents hold true.

1. $a^m \; x \; a^n = a^{m+n}$ - laws of multiplication
2. $\frac{a^m}{a^n} = a^{m-n}$ = laws of division
3. $(a^m)^n = a^{m\,n}$ laws of power of power
4. $(a\, x b)^n = a^n \; x \; b^n$ laws of power product
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ laws of a power of quotient

Example - Simplify each of the following

A. $A^{-2} \; x \; a^5 = a^{-2+5} = a^3$
b. $(3a)^4 \; x \; (3a)^{-2} = (3a)^{-2+4} = (3a)^2 = 9a^2$
C. $x^2 \; x \; x^{-3} \; x \; x^4 = x^{2-3+4} = x^3$

Exercise

1. Simplify the exponential expression using laws of exponents.

a. $2^t \; x \; 2^{3t} \; x \; 2^{2t} = 2^{t+3t+2t} = \underline{\underline{2^{6t}}}$

b. $\frac{2^5}{2^7} = 2^{5-7} = 2^{-2} = \frac{1}{4}$

C. $(4y)^2 \; x \; (8y) = 4^2 \; y^2 \; x \; \frac{1}{8^2 \; y^2} = \frac{1}{4}$

d. $\dfrac{(a^2)^{-3} x (a^3)^4}{a^{10}} = \dfrac{(a^{3+2})^{4-3}}{a^{10}} = \dfrac{a^5}{a^{10}} = a^{5-10} = a^{-5} = \dfrac{1}{a^5}$

e. $\left[\dfrac{(m^{-5}\ n^2)}{n^{-2}\ m^6}\right]^{-2} = \left[\dfrac{m^{-5}\ n^2}{m^6\ n^{-2}}\right]^{-2} = \left[m^{-5-6}\ n^{2+2}\right]^{-2}$

$\left[m^{-11}\ n^4\right]^{-2} = 3^{22}\ n^{-8} = \dfrac{m^{22}}{n^8}$

f. $(a^y)^{-1} = a^{-y} = \dfrac{1}{a^y}$

g. $(3^2)^{2n} = \underline{\underline{3^{4n}}}$

The Rational Exponent

In the general at sidelined to be the positive square root of a which can also be written as $\sqrt{a}$. so $a^{1/2} = \sqrt{a}$

- ✓ If a is positive , then $a^{1/n}$ is defined a to as to be positive number whose n^{th} power is equal to a. this number is called the n^{th} root of a and sometimes written as $\sqrt[n]{a}$.
- ✓ If n is even and a is negative , $a^{1/n}$ cannot be defined because rasing any number to an even power result in a positive number.
- ✓ If n is odd and a is negative, $a^{1/n}$ can be defined . it is negative number whose n^{th} power is equal to a.

Example

a. $\sqrt[4]{16} = 2$ C. $\sqrt[3]{-8} = -2$

b. $\sqrt[3]{8} = 2$ d. $\sqrt[4]{-16}$ = not real number

Exercise

1. express in the form $a^{1/n}$ and evalut the following

a. $\sqrt[4]{81} = 3$ b. $\sqrt[5]{32} = 2$ C. $\sqrt[3]{-27} = -3$

d. $-\sqrt[3]{-1000} = 10$ e. $\sqrt[4]{-1000}$ = not real root

Rational Exponents $a^{m/n}$

If $a^{1/n}$ is areal number , then $a^{m/n} = (a^{1/n})^m$ (that is the n^{th} root of a raised to the m^{th} power.

We can also defined negative rational exponents $a^{m/n} = \dfrac{1}{a^{m/n}}$ $(a \neq 0)$

The laws of exponent discussed earlier for integral exponents hold true for rational exponents.

Example

A. $4^{1/3}\ x\ 16^{1/3} = 2^{2/3}\ x\ 2^{4/3} = 2^{2/3+4/3} = 2^{6/3} = \underline{\underline{2^2}} = \underline{\underline{4}}$

B. $\dfrac{3^{1/2}}{27^{3/2}} = \dfrac{3^{1/2}}{(3^3)^{3/2}} = \dfrac{3^{1/2}}{3^{9/2}} = 3^{1/2-9/2} = 3^{-8/2}$

$3^{-8/2} = 3^{-4} = \frac{1}{3^4} = \frac{1}{3^4} = \frac{1}{\underline{81}}$

The n^{th} root

If $a^{1/n}$ is a real number and m an integral then. $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$ or $\frac{m}{a^n} = (a^{1/n})^m = (\sqrt[n]{a})^m$

Example

a. $\sqrt[3]{4} = (4)^{\frac{1}{3}} = 2^{2/3}$

b. $\sqrt[5]{27} = (27)^{\frac{1}{5}} = (3^3)^{\frac{1}{5}} = 3^{\frac{3}{5}}$

Exercise

Express in the form of $a^{m/n}$ with a being prim number

a. $\sqrt[3]{81} = 8^{1/3} = 3^{4\ 1/3} = 3^{4/3}$

b. $^{-}\sqrt[4]{32} = (32)^{\frac{1}{4}} = (2^5)^{\frac{1}{4}} = 2^{5/4}$

C. $\sqrt[3]{25} = \sqrt[3]{\frac{25}{5}} = \left(\frac{25}{5}\right)^{\frac{1}{3}} = 5^{1/3}$

d. $\frac{\sqrt[5]{40}}{\sqrt[5]{5}} = \sqrt[5]{\frac{40}{5}} = \sqrt[5]{8} = (8)^{\frac{1}{5}} = (2^3)^{\frac{1}{5}} = \underline{\underline{\mathbf{2^{3/5}}}}$

Irrational Exponents

expression $3\sqrt{2}$, $2\sqrt[2]{3}$, 5^{II} are powers with Irrational exponents.

Using calculator the value $\sqrt{2} = 1, 4, 42, 5$ Therefore, $3\sqrt{2} = 3^{1,4,42,5----}$

$3^1 = 3$

$3^{1.4} = 3^{14/10} = 4.65553672$

Simplify each of the following

a. $3\sqrt{2}\ x\ 3\sqrt{2} = 3\sqrt{2} + \sqrt{2} = 3\sqrt[2]{2} = 9\sqrt{2}$

b. $(4\sqrt{2})^3 = (4^3)\sqrt{2} = 64\sqrt{2}$

C. $2\sqrt{3}\ x\ 2\sqrt{3} = 2\sqrt{3} + \sqrt{3} = 2\sqrt[2]{3} = \mathbf{\underline{4\sqrt{3}}}$

d. $\frac{(5\sqrt{5})^2\ x\ 5 - \sqrt{12}\ x\ 25\sqrt{3}}{5\sqrt{27}} = \frac{(5\sqrt[2]{5})^2\ x\ 5\ ^{-2}\sqrt{3}\ x\ 5\sqrt[2]{3}}{5\sqrt{27}} = \frac{5\sqrt[2]{5}x\ 5^0}{5\sqrt{27}}$

$\underline{\mathbf{5\sqrt{27}}}$

$\underline{\underline{\mathbf{5\sqrt{27}}}}$

Log arithms

$Log_3{}^9 = 2$ because $3^2 = 9$

$Log_3, \left(\frac{1}{9}\right) = -2$ because $3^{-2} = \frac{1}{9}$

$log_S{}^1 = 0$ because $S^0 = 1$

For $a > 0$, $a \neq 1$, and $C > 0$ $Log_a{}^C = b$ If and only if $a^b = C$

3.2 Properties Of Log arithms

The following properties flow directly from the definition of the logartithm with base $a > 0$ and $a \neq 1$

Properties Of Log arithms

1. $\text{Log}_a^1 = 0$ because $a^0 = 1$
2. $\text{Log}_a^a = 1$ because $a^1 = 1$
3. $\log_{Sa}^{aP} = p$ and $a \log^{aP} = p$ = invers property
4. If $\log_a^M = \log_a^N$ then $M = N$ - - - - one to one property.

   Example

Using properties of logarithms for the following questions.

a. Simplfy $\log_2^{2P} = \underline{2}$

Find the values of the following logarithms.

a. $\text{Log}_{12}^{144} = \text{Log}_{12}\, 12^2 = 2\,\text{Log}_{12}^{12} = \underline{2}$

b. $\text{Log}_5\, 25^P = \text{Log}_S\, S^{2P} = 2\,\text{Log}S^S = \underline{2P}$

C. Find P such that $\log 6^6 = P = \log_6^6 = P \Longrightarrow \underline{P = 1}$

<u>Low Of Log arithms</u>

We now establish law of logarithms, the laws are represented by theorems and we prove the theorms based on the corresponding laws of exponents.

<u>Log arithms Of products</u>

For any positive numbers M , N and $a > 0$ and $a \neq 1$ $\log_a^{MN} = \log_a^m + \log_a^N$

(The logarithm of a product is the sum of the logarithms of the factors .)

This property of logarithms corresponding to the product law for exponent

$a^m\ a^N = a^{M+N}$

Example

$\text{Log}_2^{(4 \times 8)}$ as a sum of logarthims

$\text{Log}_2^{(4 \times 8)} = \log_2^4 + \log_2^8$

$= \log_2 2^2 = \log_2 2^2 = 2 \log 2^2 + 3 \log_2 2^2$

$2 + 3 = \underline{5}$

$\text{Log}_3^5 + \log_3^8$

$\text{Log}_3^5 + \log_3^8 = \log_3^{(5 \times 8)} = \log_3^{40}$

<u>logarithms Of Powers</u>

For any positive number M, any real number r, and $a > 0$ and $a \neq 1$

$\text{Log}_a(m)^r = r \log_a^m$

(The logarthim of a power of x is the exporrem times the logarithms of x .)

This property of logarithms corresponds to the power law for exponents

$(a^m)^r = a^{mr}$

a. $\log_2\sqrt{8} = \log_2 8^{1/2} = \frac{1}{2}\log 2^3 = \frac{1}{2}(3\log_2 2^2) = \frac{3}{2}$
2

b. $\log_3 81 = \log_3 4 = 4\log_3 3 = 4x1 = 4$
3

C. $\log_4 8 + \log_4 2 = \log_4 18\ x2 = \log_4 16 = \log_4 2 = \underline{\underline{2}}$
4

Exercise

1. Use low of logarithm to find the law of

a. $\log_3\sqrt{3} = \log_3(3)\frac{1}{2} = \frac{1}{2}\log_3 3 = \frac{1}{2}$

b. $\log_2\frac{1}{4} = \log_2{}^{-2} = -2\log_2 2 = -2$
2

C. $\log\frac{1}{3}(\frac{1}{8}) = \log\frac{1}{3}\left(\frac{1}{3}\right)^4 = 4\log\frac{1}{3} = \underline{\underline{4}}$

d. $\log 8^3{}_2 + \log 8^2 = \log 8^{(32x2)} = \log 8^{64} = \log_8 2$
8

$2\log 8^8 = \underline{\underline{2}}$

f. $\log_2 6 + \log_2\frac{1}{2} = \log_2\left(\frac{6}{12}\right) = \log_2\frac{1}{2} = \log_2{}^{-1} = \underline{\underline{-1}}$
2

g. $\log_3 10 + \log_3\left(\frac{6}{5}\right) + \log_3\frac{9}{4} =$

$\log_3(10\ x\frac{6}{5}) + \log_3\frac{9}{4} = \log_3 12 + \log_3\left(\frac{9}{4}\right)$

$= \log_3\left(12x\frac{9}{4}\right) = \log_3 27 = \log_3 3 = 3\log_3 3 = \underline{\underline{1\ x\ 3 = 3}}$
3

h. $\frac{1}{2}\log_4 8 + \log_4\sqrt{2} = \frac{1}{2}\log_4 8 + \log_4 2\frac{1}{2}$

$= \frac{1}{2}\log_4 8 + \frac{1}{2}\log_4 2 = \frac{1}{2}\left(\frac{1}{2}\right)\log_4{}^{8x2} = \frac{1}{4}\log_4 16 =$

$\frac{1}{4}\log_4 2 = \frac{2}{4}\log_4 4 = \frac{1}{2}x1 = \frac{1}{2}$
4

logarithms Of quotients

for any positive numbers M , N and a > 0 and a ≠ 1

$\log_a\frac{M}{N} = \log_a m - \log_a N$

Example

evaluate

a. $\log_3 54 - \log_3 2 = \log_3\frac{54}{2} = \log_3 27 = \log_3 3$
3

$= \underline{\underline{3\ x\ 1 = 3}}$

b. $\log_{10}\sqrt{2000} - \log_{10}\sqrt{20} = \log_{10}2000\frac{1}{2} - \log_{10}20\frac{1}{2}$

$= \frac{1}{2}\log_1{}^{2000}$ - $\log_{10}{}^{20} = \frac{1}{4}$ ($\log_{10}{}^{2}\,\frac{2000}{20}$

$\frac{1}{4}\ \log_{10}{}^{2}\ = \frac{2}{4}\ \log\ _{10}{}^{10} = \frac{2}{4}\ x\ 1 = \mathbf{\frac{1}{2}}$

10

Exercise

1. use laws of logarithms to find the value of

a. $\log_5{}^{50} - \log_5{}^{2} = \log_5\left(\frac{50}{2}\right) = \log_5{}^{2} = 2\log_5{}^{5} = \underline{\mathbf{2}}$

2

b. $\log_5{}^{2} + \log_5{}^{50} - \log_5{}^{4} = \log_5{}^{100} - \log_5{}^{4} =$

$= \log_5\left(\frac{100}{4}\right) = \log_5{}^{25} = \log_5{}^{2} = 2\ \log_5{}^{5} = 2$

2

C. $\log_6{}^{9} - \log_6{}^{15} + \log_6{}^{10} = \log_6\frac{9}{5} + \log\ _6{}^{10}$

$\text{Log}_6\left(\frac{3}{5}x10\right) = \log_6\frac{6}{5}\ = \log_6{}^{6} = 1\ x1\ = \underline{\mathbf{1}}$

d. $\log_{10}{}^{24} - 2\log_{10}{}^{6} + \log_{10}{}^{15}$

$2\ \log_{10}\left(\frac{24}{6}\right) + \log_{10}{}^{15} = 2\ \log_{10}{}^{4} + \log_{10}{}^{15} =$

$2\ \log_{10}{}^{(4x15)} = \underline{\mathbf{2\log_{10}{}^{60}}}$

Chang Of base

For any positive real number M, a > 0 b > 0 a ≠ 1 and b ≠ 1,

$\log_a{}^{M}\ = \frac{\log_b{}^{M}}{\log_b{}^{a}}$

let $p = \log_a{}^{M}$. Then

$a^p\ =\ a\ \log_a{}^{M}\ = M$

$\log_a{}^{P}\ = \log_a{}^{M}$ - - - taking logarithm to the base b of both sides

b

$P\ \log_b{}^{a}\ = \log_b{}^{M}$ - - - Using power law

$P = \frac{\log_b{}^{M}}{\log_b{}^{a}}$

Therefore , $P\ \log\ _a{}^{M}\ = \frac{\log_b{}^{M}}{\log b^{a}}$

Example

a. $\log_{\sqrt{2}} 4 = \log_{\sqrt{2}} 4\ = \frac{\log_2{}^{4}}{\text{Log}_2\sqrt{2}}\ = \frac{\log_2{}^{2}}{\log_2 2^{\frac{1}{2}}}$ 2

$= \frac{2}{\frac{1}{2}}\ \log_2{}^{2}\ = \frac{2}{\frac{1}{2}}\ = 2x\,\frac{2}{1}\ = \underline{\mathbf{4}}\ \ \text{Log}_2{}^{2}$

b. $\log 0.1^{100} = \frac{\log_{10}^{100}}{\log_{10}^{0.1}} = \frac{\log_{10}^{10^{2}}}{\log_{10}^{10^{-1}}}$ b/c $100 = 10^2$ and $0.1 = \frac{1}{10} = 10^{-1}$

$= \frac{2\log_{10}^{10}}{-1\ \log_{10}^{10}} = \underline{\underline{-2}}$

Exercise

Use the law $\log_a^x = \frac{\log_b^x}{\log_b^a}$ to find the value of the following expressions.

a. $\log_{\sqrt{3}}^{9} = \frac{\log_3^{3^2}}{\log_3^{3^{\frac{1}{3}}}} = \frac{2}{\frac{1}{3}} \frac{\log_3^3}{\log_3^3} = 2\ x\ \frac{3}{1} = \underline{\underline{6}}$

b. $\log_{\sqrt{2}}^{128} = \frac{\log_2^{2^8}}{\log_2^{2^{\frac{1}{2}}}} = \frac{8\log_2^2}{\frac{1}{2}\log_2^2} = 8x\frac{2}{1} = \underline{\underline{16}}$

C. $\log_{\left(\frac{1}{3}\right)}^{243} = \frac{\log_5^3}{\mathrm{Log}_3^{-1}} = \frac{5\log_3^3}{-1\ ;og_3^{3-1}} = \underline{\underline{-5}}$

e. $\log_4^{\left(\frac{1}{2}\right)} = \frac{\log_2^{2-1}}{\log_2^2} = \frac{-1\log_2^2}{2\log_2^2} = \underline{\underline{-\frac{1}{2}}}$

Remember that

1. $\log_a^{MN} \neq (\log_a^M)(\log_a^N)$ - - - - The logarithm of a product is not the product of logarithm
2. $\log_a^{(M+N)} \neq \log_a^M + \log_a^N$ - - - The logarithm of a sum is not the sum of the logarithm
3. $\log_a\left(\frac{M}{N}\right) \neq \frac{\log_a^M}{\log_a^N}$ - - - - The logarithm of a quotient is not the quotient of the logarithm
4. $(\log_a^M)^r \neq r\log_a^M$ - - - The power of a logarithm is not the exponent times the logarithm

Logarithm to base 10 (Common logarithm)

- ✓ The logarithm to the base 10 is called common logarithm or decadic logarithm and written as $\log_{10}^M$
- ✓ Common logarithm is usually written without indicating its base. **for example** $\log_{10}^M$ is simply denoted by log M.

Example

- ✓ Find the value of each of the common logarithms.

a. $\log_{10}\sqrt[3]{10} = \log_{10}^{10^{\frac{1}{3}}} = \frac{1}{3}\log_{10}^{10} = \frac{1}{3}\ x\ 1 = \frac{1}{3}$

b. $\log^{200} - \log 2 = \log\left(\frac{200}{2}\right) = \log_{10}^{10} = 2\log_{10}^{10} = \underline{\underline{2}}$

Exercise

Find the values of the following common logarithm

a. $\log \sqrt[3]{0.1} = \log(0.1)^{\frac{1}{3}} = (\log_{10}10^{-1})\frac{1}{3} = \frac{1}{3} \quad \log_{10}10^{-1}) \frac{-1}{3}$

b. $\log_{10}10\sqrt{10} = \log_{10}10 + \log_{10}10^{\frac{1}{2}} = 1+\frac{1}{2} = \frac{3}{2}$

C. $\log \left(\frac{10^m}{10^n}\right) = \log_{10}{}^m - \log_{10}{}^n$

Suppose P can be written as $P = m x 10^C$, $1 \leq m < 10$ then the logarithm of P can be read from the common logarithm table (a table that contains the common logarithm value of a number m such that $1 \leq m < 10$)

$\text{Log } P = \log (m\, x 10^C) = \log m + \log 10^C = \log m + C$

That is, $\log P = \log m + C$

The common logarithm of ,m, $\log_m$ is called the mantissa (fractional part) of the common logarithm of P and C is called the characteristic of the logarithm.

Anti Logarithms

Find the antilog of the following numbers.

In 3.0913 the character tic is 3. There for after finding the antilog of 0.0913 we multiply it by 10^3 .

to find antilog (0.0913).

1. Separate the number 0.0913 in to 0.09, 1 and 3
2. From the antilogarithm part read the number at the intersection of row 0.09 and column 1, this gives 1.233'
3. from the mean difference part read the number at the intersection of row 0.09 and column 3 this gives 1 we write this as 0.001.
4. Add the values obtained in step 2 and 3 to get $1.233 + 0.001 = 1.234$
5. $1.234\, x\, 10^3 = 1234.00$. Therefore the atilog (3.0913) = 1234.00

3.3 The Exponential functions and their graphs

Exponential functions

The exponential function f with base a is denoted by $f(x)= a^x$ where $a > 0$, $a\neq1$ and x is any real number.

Example

$f(x)= 3^x$, Evalute the following

a. $f(2) = 3^2 = 9$

b. $f(0) = 3^0 = 1$

C. $f(-1) = 3^{-1} = \frac{1}{3}$

1. in the definition of exponential function , $a\neq1$ because If $a = 1$, $f(x)= 1^x = 1$ is constant function
2. The exponential function $f(x)= a^x$, where $a>0$, $a\neq1$ is different from all the functions that you have studied in the previous chapters because the variable x is an exponent
3. A distinct characteristic of an exponential function $f(x)= a^x$ is showing a rapid increase as x increases for $a>1$ and showing a rapid decreases as x increases for $a<1$

4. many real – life phenomena with patterns of rapid growth (or decline) can be modeled by exponential functions.

Exercise

1. Given $f(x) = \left(\frac{1}{4}\right)^x$ find the values of

   a. $f(2) = \left(\frac{1}{4}\right)^2 = \frac{1}{16}$

   b. $f(-2) = \left(\frac{1}{4}\right)^{-2} = 16$

   C. $f(\frac{1}{2}) = \left(\frac{1}{4}\right)^{½} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

   d. $f\left(\frac{-1}{2}\right) = \left(\frac{1}{4}\right)^{\frac{-1}{2}} = 4^{½} = 2$

2. Write each of the following functions in the form $f(x) = 2^{KX}$ or $f(x) = 3^{KX}$ for a suitable constant K.

   a. $f(x) = 8^X = 2^{3X}$

   b. $f(x) = \sqrt[3]{2}^X = 2^{\frac{1}{3}X}$

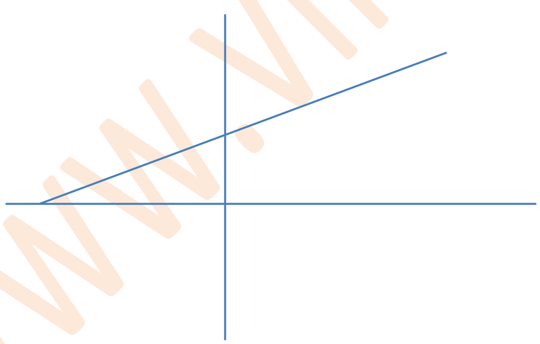
   C. $f(x) = \left(\frac{1}{27}\right)^{\frac{x}{3}} = (3^{-3})^{\frac{x}{3}} = 3^{-X}$

Graph Of exponential functions

Draw the graph of the exponential function $f(x) = 2^X$

x - 3	-2	-1	0	1	2	3
$F(x) = 2^X$, $\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

- ✓ The function $f(x) = 2^X$ is positive for all value of x .
- ✓ As x increase, the value of the function gets larger and larger
- ✓ As x decrease, the value of the function gets smaller and smaller approaching Zero.

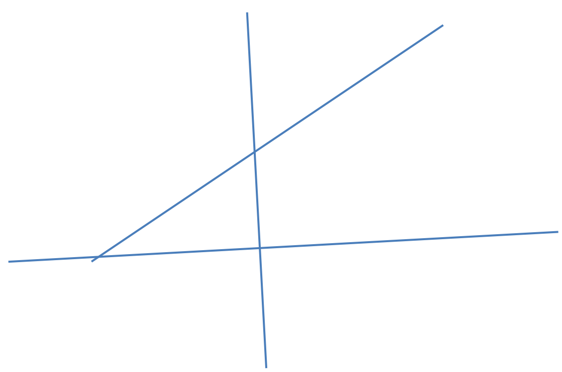


Exercise

For the function $f(x) = 3^x$

$f(-3) = \frac{1}{27}$, $f(-2) = \frac{1}{9}$, $f(-1) = \frac{1}{3}$, $f(0)= 1$ $f(1) = 3$ $f(2) = 9$ $f(3) = 27$

x	-3	-2	-1	0	1	2	3
$F(x) = 3^x$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27



Characteristics of Graph of $f(x) = a^x$, a>1

a. Domain : R = The set of all real numbers

b. Range : R^+ = The set of all positive real number

C. y – intercept, The point (0, 1) .

d. has no x – intercept.

e. it is increasing on R = (- ∞ , ∞)

f. The graph goes up ward without bound as x gets closer to the negative

g. The graph gets closer to the negative x – axis when x is negative and axise larger

h. Horizontal a asymptote : The x – axis (the line y = 0) is a horizontal a asymptote

Characteristics of Graph of $f(x) = a^x$, 0< a < 1

a. Domain : R = The set of all real numbers.

b. Range R^+ = The set of all positive real number

C. y – intercept : The point (0 , 1).

d. has no x – intercept

e. it is decreasing function. The value of f decreases whenever the value of x increases

f. The graph goes upward without bound when x is negative and axis is larger

g. The graph gets closer to the positive x – axis when x – gets larger and positive

h. Horizontal asymptote, the x- axis (The line y = 0) is horizontal asymptote

The Natural exponential function

- ✓ Any positive number can be used as the base for an exponential function, but for the bases the number denoted by the letter e and 10 are used more frequently. The number e is the most important base and convenient for certain applications.
- ✓ The number "e" is defined as the value that $\left(1+\frac{1}{n}\right)^n$ for increasingly large values of n. it appears that $e \approx 2.71828$
- ✓ The natural exponential function is the exponential function $f(x) = e^X$ with base.

The logarithmic function and Their graphs

- ✓ every exponential function $f(x) = a^X$ with a>0 and a≠1 is a one-to-one function and hence it has an inverse function. The inverse function f^{-1} is called the logarithmic function with base a and is denoted by $g(x) = \log a^X$ where $g = f^{-1}$. This leads us to the following definition of the logarithmic function.
  Let a>0 and a≠1, The logarithmic function with base a denonected by $y = \log a^X$ is defined by
  $Y = \log a^X$ If and only If $x = a^Y$.

Example

a. $f(x) =, x = 4$
   $f(4) = \log 2^4 = \log 2^2 = 2 \log 2^2 = \underline{2}$

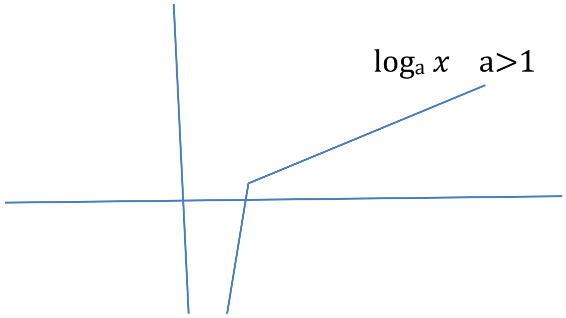
b. $f(x) = \log {}_4{}^X, x = 1$ b/c $2^2 = 4$
   $f(1) = \log 4^1 = 0$ b/ C $4^0 = 2$

Graph Of logarithmic function

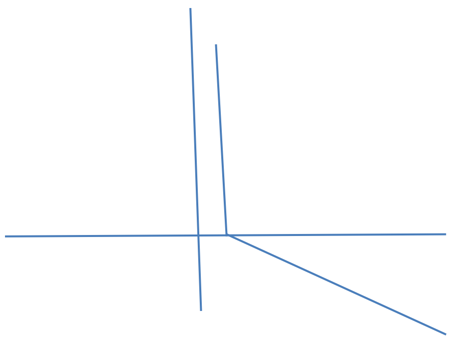
- ✓ If a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A ,
- ✓ The exponential function $f(x) = a^X$ with a>0 and a≠1 has domain R and Range (0, ∞), we see that Inverse function $g(x) = \log a^X$ has domain (0, ∞)

Basic Characteristics Of the graph of $f(x) = \log a^X$, a>1

1. Domain : R^+= The set of all positive real numbers.
2. Range: R= The set of all real numbers
3. x – intercept: (1, 0)
4. it has no y – intercept. It does not intersect the y – axise
5. it has increasing function. The value of f increases whenever the value of x increases
6. The graph goes upward as x gets larger and positive.
7. The graph gets closer to the negative y – axise whene x – gets closer to 0 from the right



Basic characteristics Of the graph of f(x) = $\log_a x$, 0<a<1.

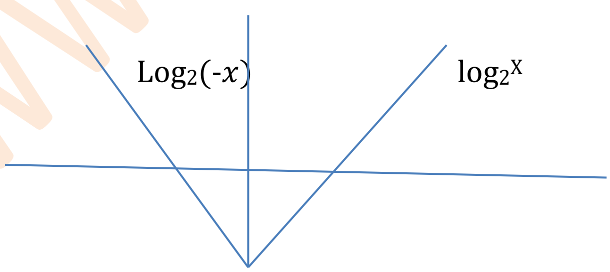


1. The Domain R^+ = the set of all positive real number
2. Range : R= The set of all real numbers.
3. x – intercept : (1, 0)
4. It has no y – intercept. It does not intersect the y – axis
5. It is decreasing function, the value of decrease when never the value of x increases.
6. The graph goes down ward as x gets larger and positive.
7. The graph gets closer to the positive y – axis when x – gets closer to 0 from the right

✶ **Not that**

For the logarithmic function f(x) = log a^X

- ✓ The graph of – f(x) is the reflection of the graph of f(x) along the x – axise
- ✓ The graph of f($-x$) is the reflection of the graph of f(x) along the y – axis.



Natural Logarithms

The logarithm of a number to the base e is called Natural logarithm and it is written as

$\text{Log}_e^X = \text{M}x$

Example

Find the value of each of the following Natural logarithms

a. $\text{M } 1 = \log_e^1 = 0$
b. $\text{M e} = \log_e^1 = 1$
c. $\text{M e}^3 = 3\log e^e = 3$
d. $\text{M } \sqrt[5]{e} = \log e^e = \frac{1}{5}$
e. $\text{M}(\frac{1}{5}) = -3 \log_e^e = \underline{\underline{-3}}$

Solving Logarithmic equation

- ✓ Logarithmic equation is an equation that involve the logarithm of an expression containing a variable. For instance, $\log_2^X+3 = 4$ is logarithmic equation.
- ✓ For any positive real number x , y, a > 0 and a ≠ 1
  $\text{Log}_a^X = \log_a^y$ If and only If $x = y$

We use the following procedures to solve logarithm equation

1. State the Universe
2. Collect the logarithmic term on one side of the equation
3. Write the equation in exponential form
4. Solve for the variable

   Example

   $\text{Log}_2(x+3) = 4$

   If $x+3 > 0$ the universe is $x > -3$

   a. $\text{Log}_2(x+3) = 4$

   $x+3 = 2^4$

   $x+3 = 16$

$x = 13 > -3$

$x = 13$

b. $\text{Log}_2(3x\text{-}1) = 5$

$3x - 1 = 2^5$

$3x - 1 = 3^2$

$3x = 33 \Rightarrow \underline{\underline{x = 11}}$

The Relation Between Exponential and Logarithmic Function With The Same Base

1. The domain of y = a^x is the set of all real numbers, that is the range of y = loglog$_a$ x and 2.
2. The range of y = a^x is the set of all positive real numbers that is the domain of y = $\log_a x$.

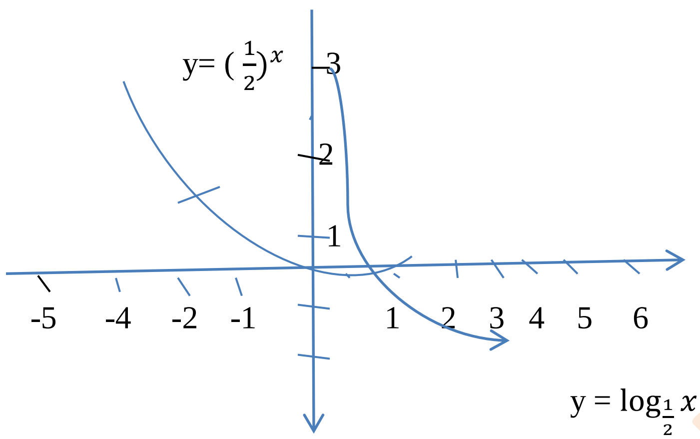
   a. Domain of y = a^x= range of y = $\log_a x$

   b. Range of y = a^x=domain of y = $\log_a x$.
3. The x-axis is the horizontal asymptote of the graph of y = a^x the y- axis is a vertical asymptote of the graph of y = x$\log_a x$
4. The point (0,1) is the y- intercept of the graph of y = a^x the point (1,0) is the x-intercept of the graph of y= $\log_a x$

   Example: - draw y = $(\frac{1}{2})^x$ and the logarithm y =$\log_{\frac{1}{2}} x$

x	-3	-2	-1	0	1	2	3
f(x) = $(\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
f(x) =$\log_{\frac{1}{2}} x$	3	2	1	0	-1	-2	-3



3.3 Applications

3.3.1Compound Interest

- Compound interest is calculated by the formula A(t) = p(1 + $\frac{r}{n}$)^{nth}

  Where A(t) = amount after t-year

  P = principal

  r= interest rate per year

  n = number of times interest is compound per year and

  t = number of year

Example:- a total of birr 20,000 is investigated at an interest rate of 7 % per year find the amounts in the account after 5 year if the interest is compound

a. Annually
b. Semi-annually
c. Quarterly
d. Monthly
e. Daily

Solution

a). p = 100 r = 7 % = 0.07 n= 1 and t = 5

$$A(t) = p\left(1+\frac{r}{n}\right)^{nt}$$

$$= 20{,}000\left(1+\frac{0.07}{1}\right)^{1(5)}$$

$$= 20{,}000(1.07)^5 = 28051.03$$

b). for semi-annually compounding n = 2 hence after 5 year at 7% rate the amount in the

$A(s) = 20{,}000\left(1+\frac{0.07}{2}\right)^{2(5)} = 20{,}000(1.035)^{10}$

= 28211.98//

c). for quarterly compounding , n= 4

$A(s) = 20{,}000\left(1+\frac{0,07}{4}\right)^{4(5)} = 20{,}000(1.0175)^{20}$

= 28295.56//

d). for monthly n= 12

$A(s) = 20{,}000\left(1+\frac{0.07}{12}\right)^{12(5)} = 20{,}000(1.00583)^{60}$

= 28352.51//

Population growth

The exponential function

$P(t) = p0e^{kt}$, k > is mathematical model of many kinds of population growth.

In this function, p0 is the population at initial time t0 ,p(t) is the population after time t and k- is called the exponential growth rate.

Example :- a culture contain 10,000 bacteria initially after an hour , the bacteria count is 25,000

a. Find the doubling period

   $P(t) = p0e^{kt}$, $p(1) = 25,000 = 10,000e^{k}$

   $e^{k} = \frac{25,000}{10,000} = \frac{5}{2}$

   $\ln e^{k} = kn(\frac{5}{2}) = \ln 2.5$

   $k= \ln 2.5 \sim 0.9163$

   $p(t) = 10,000e^{0.9163}$

b. Find the number of bacteria after 5 hours .

   $P(t) = p0e^{kt}$

   $20,000 =10,000e^{0.9163t}$

   $2= e^{0.9163t}$

   $0.9163t = \ln^{2}$

   $t=0.7565$//

c. For daily compounding n= 365

   $A(5)= 20,000(1+ \frac{0.07}{365})^{365(5)} = 20,000(1.00019)^{1825}$

   $20,000(1.00019)^{1825} = 28380.40$//

- The interest paid increases as the number of compounding period n increases continuously compound interest is calculated by the formula

  $A(t) = p\, e^{rt}$

  Where A(t) = amount after t-years

  P = principal , r= interest rate per year and t= number of year

  Example:- 1. If birr 40,000 is invested in an account for which interest is compounded continuously , find the amount of the investment at the end of 10 years for the following interest.

  a. $6\ \% = A(t0) = 40,000\ e^{(0.06)10}$

     $= \underline{\underline{\mathbf{40,000\ e^{0.6}}}}$

  b. $7.5\ \% = A(10) = 40,000\ e^{(0.075)10}$

     $= \underline{\underline{\mathbf{40,000\ e^{0.75}}}}$

The PH Scale

PH= - $\log\lfloor H^+ \rfloor$

Example: - 1. The hydrogen ion concentration of a sample of each substance is given. Calculate the PH of the substance and determine whether it is acidic or basic

2. lemon juice :- $\lfloor H^+ \rfloor = 5 \times 10^{-3}$M

PH = - $\log\lfloor H^+ \rfloor$

= - $\log\lfloor 5 \times 10^{-9} m \rfloor$)

= - log5 + log 10^{-9}

= - (0.6990 -9)

= 8.301 – it is basic

Review exercise

1.Find the values of the given logarithm

a. $\log_3 3 = 1$

b. $\log_3 3^4 = 4 \log_3 3 = 4$

c. $\log_{\frac{1}{27}} 3 = \log_{3^{-3}} 3 = -3$

d. $\log_5 15 = \log_{5^{-1}} 5 = -1$

2. let f(x) = $\log_4 x$. then find f(4) , f(1) f($\frac{1}{4}$) =

f(x) = $\log_4 4 = 1$

f(1) = $\log_4 1 = 0$

f($\frac{1}{4}$) = $\log_4 \frac{1}{4}$ = -1

3. Find the appropriate logarithmic or exponential form of the equation

Logarithm form	exponential form
$\log_7 7 = 1$	$7^1 = 7$
$\log_8 64 = 2$	$8^2 = 64$

$\log_8 4 = \frac{2}{3}$ $8^{2/3} = 4$

$\log_8 512 = 3$ $8^3 = 512$

$\log_8(\frac{1}{8}) = -1$ $8^{-1} = \frac{1}{8}$

$\log_8(\frac{1}{64}) = -2$ $8^{-2} = \frac{1}{64}$

4. Express the logarithmic statement in to exponential statement
   a. $\log_5 125 = 3$ $5^3 = 125$
   b. $\log_5 1 = 0$ $5^0 = 1$
   c. $\log_8 2 = \frac{1}{3}$ $8^{1/3} = 2$
   d. $\log_2 \frac{1}{8} = -3$ $2^{1/3} = \frac{1}{8}$
5. Express the exponential statements in to logarithmic statement .
   a. $3^3 = 27$ $\log_3 27 = 3$
   b. $8^{-1} = \frac{1}{8}$ $\log_8 \frac{1}{8} = -1$
   c. $10^{-3} = 0.001$ $\log_{10} 0.001 = -3$
6. Using the definition of algorithmic function to find x

a. $\log_{\sqrt{2}} x = 6$ x= 8

b. $\log_2 x$ = x x= 5

c. $\log_5 x$= 4 x= 625

d. $\log_{\frac{1}{2}} 2$ = x x = -1

7. Given $\log_{10} 2 = 0.3010$ and $\log_{10} 2$ and $\log_{10} 3 = 0.4771$. then find the following algorithms.

$a.\log_2 \sqrt{3} = \frac{1}{2}\log_2 3$ = x

$$\frac{1}{2} \times \frac{\log_{10} 3}{\log_{10} 2} = \frac{1}{2} \times \frac{0.3010}{0.4771} = 0.7925$$

$b.$ $\log_2 0.3 = \frac{\log_{10} 3 \times 10^{-1}}{\log_{10} 2} = \frac{\log_{10} 3 + (-1)}{\log_{10} 2}$

$$= \frac{0.4771+(1)}{0.3010} = -1.7372$$

c. $\log_2 108 = \frac{\log_{10}(2^2 \times 3^3)}{\log_{10} 2} = \frac{2\log_{10} 2 + 3\log_{10} 3}{\log_{10} 2}$

$= \frac{2(0.4771)+3(0.3010)}{0.4771} = 6.7551//$

d. $\log_3 5 = \frac{\log_{10}(\frac{10}{2})}{\log_{10} 3} = \frac{\log_{10} 10 - \log_{10} 2}{\log_{10} 3}$

= 1.4651//

e. $\log_4 75 = \frac{\log_{10}(3\times5^2)}{\log_{10} 2^2} = \frac{\log_{10} 3 + \log_{10}\frac{10}{2}}{2\ \log_{10} 2}$

$= \frac{\log_{10} 3 + 2(1 - \log_{10} 2)}{2\ \log_{10} 2} = 3.1148//$

8. Solve each of the following equations .

a. $(\frac{1}{4})^{x-1} = 4^{2-3x} = 4^{(x-1)-1} = 4^{2-3x}$

-(x-1) = 2-3x

-x+1 =2-3x =-x+3x =2-1

2x=1 X=$\frac{1}{2}$//

b. $9^{x+1} + 3^{x+2} - 18=0$

$= 9^{x+1} + 3^{x+2}-18 = 0$

$= (3^2)^{x+1} + 3^{x+2} - 18 =0$

$= 3^2.(3^x)^2 + 3^2.3^x - 18 =0$

$= 3^{x(2)} + 3^2 - 2 = 0$

= X at 3^x = x(x>0)

Then x^2 +-2 =0(x+2) (x-1) = 0

X= -2, x= 1

Since x >0 x =1

Hence , 3^x =1 , x= 0

c. $\frac{81^{5-2x} \times 243^{x-2}}{9^{5x-1}} = \frac{1}{3}$

$= \frac{3^{4(5-2x)} \times 3^{5(x-2)}}{3^{2(5x-1)}} = 3^{-1}$

= 20-8x +5x -10 -10x-2=-1

= -3x-10x+8 = -1

$= \frac{-13x}{-13} = \frac{-9}{-13}$

x = $\frac{\mathbf{9}}{\mathbf{13}}$//

9. State the universe and solve each of the following equations

a. $\log_2(x+2)$. + $\log_2(x-1)$ = 2

x+2 >0 and x-1 >0 x >-2 and x> 1 so the universe is { x:x >1} the solution is x =2.

b. $\log_3(x^2 - 8x) = 2$ the universe is $(-\infty, 0) \cup (8, \infty)$ and the solution is x1 = -1 x2 = 9

c. $\frac{2+\log x}{3-\log x} = 5$

10. Solve each of the following equations.

a. $(\frac{1}{4})^{x-1} = 4^{2-2x}$

$= 4^{-1(x-1)} = 4^{2-2x}$

$= -x+1 = 2-2x$

$-x+2x = 2-1$

$\underline{X = 1}$

b. $9^{x+1} + 3^{x+2} - 18 = 0$

$= \quad 3^{2(x+1)} + 3^{x+2} - 9x^2 = 0$

$= 2x+2 + x+2 = 18$

$= \quad 3x+4 = 18$

$= 3x+4 = 18$

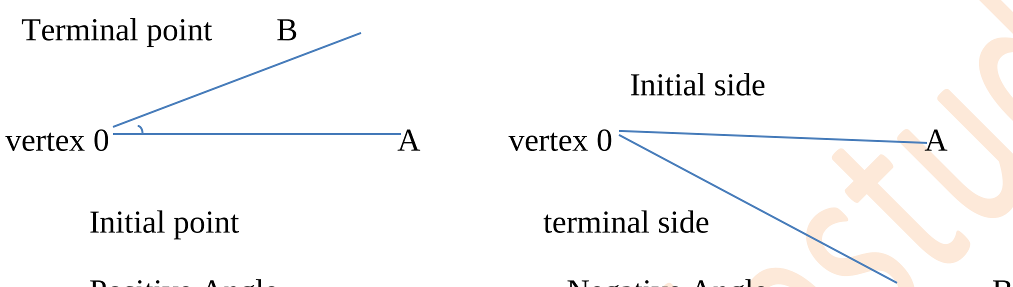
$= \frac{3x}{3} = \frac{14}{3}$, $x = \frac{14}{3}$//

Chapter 4

Trigon Metric Functions

4.1 Radian Measure of Angle Conversion between Radian and Degree Measures.

Angle:- is a measure of rotation of a given ray about its initial point . the original ray is called the initial side and the final position of the ray after rotation is called the terminal side of the angle the point of rotation is called the vertex.



Positive Angle Negative Angle

Angle in Standard Position

Angle in the coordinate system is side to be in standard position if

- ✓ It vertex is at the origin
- ✓ The initial side lies on the positive x- axis

Example :- The following are measures of different angles. Put the angles in standard position.

a. 200^0 b. $1125\ ^0$ c. -900^0

Solution

a. $200^0 = 180^0 + 20^0$
b. $1125^0 = 3 \times 360^0 + 45^0$
c. $-900^0 = (\ 2\times 360^0) + (-180^0)$

Degree measured and radian measure

1. Degree measure.

If a rotation from the initial side to terminal side is $(\frac{1}{360})^{th}$ of a revolution , the angle is side to have a measure of one degree written as 1^{0}. a degree is divided into $60'$ Minutes, and 2 minute is divided into $60'$ second one sixteen of a degree is called a minute .

1^0	$60'$
$1'$	$60''$

Exercise

1. The following are measures of different angles put the angles in standard in standard position.
   a. $765^0 = 2 \times 360^0 + 45^0$
   b. $245^0 = \theta \times 360^0 + 245^0$
   c. $-740^0 = 2 \times (-360^0) + (-20)$
2. Radian Measure :- There is another unit for measurement of an angle called the radian measure an angle at the center a circle with radius r which is subtended by an arc of length r unite in a circle is said to have a measure of 1 radian .

Radiation between degree and radian measure

Since a circle subtended at the central angle whose radian measure is 2π and its degree measure is 360^0, it follow that 2π radian = 360^0 or $\pi r = 180^0$. The relation enable us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure

$\theta = \frac{1}{r}$ l= rθ

when an angle is expressed in radians the word "radian' is frequently omitted. Thus , π $180^{\,0}$ and $\frac{\pi}{4}$= 45^0

$radian$ measure = $\frac{180}{\pi} \times radian\ measure$

Exercise

1. Convert each of the following degree into radians.
   a. $270^0 = 270 \times \frac{\pi}{180} = \frac{27\pi}{18} = \frac{3\pi}{2}$
   b. $-330^0 = -330^0 \times \frac{\pi}{180} = \frac{-33\pi}{18} = \frac{-11\pi}{6}$
   c. $220^0 = 220 \times \frac{\pi}{180} = \frac{11\pi}{9}$
2. Find the degree measure of angles which have the following radian measure.

a. $\frac{5\pi}{4} = \frac{5\pi}{4} \times \frac{180}{\pi} = 45 \times 5 = \underline{\underline{\mathbf{225^0}}}$

b. $\frac{-3\pi}{12} = \frac{-3\pi}{12} \times \frac{180}{\pi} =$ $45 \times (-3) = \underline{\underline{\mathbf{-135^0}}}$

$c. \frac{-\pi}{12} = \frac{-\pi}{12} \times \frac{180}{\pi} =$ $\underline{\underline{\mathbf{-15}}}$

3. Convert each of the following radian into degrees (use $\pi = \frac{22}{7}$)

a. $\frac{110}{7}$ radian = $\frac{110}{7} \times \frac{180}{\pi}$

$5 \times \frac{22}{7} \times \frac{180}{\pi} = \underline{\underline{\mathbf{900}}}$

$find$ the radius of the circle in which a central angle of 30^0 intercept an arc of length 11cm($\pi = \frac{22}{7}$)

Solution

An arc length L = 11 cm and $\theta = 30^0 = \frac{30\pi}{180} = \frac{\pi}{6}$ since $r = \frac{l}{\theta} =$ we have radius of the circle.

$r = \frac{11}{\pi} \times 6 = \frac{11}{22} \times 6 \times 7 = 2\text{cm}$

Hence the required distance travelled l is calculated as fallows $l = r\theta = 1.5 \times \frac{4\pi}{3} = \underline{\underline{\mathbf{2\pi\ unite}}}$

Exercise

1. Find the radius of the circle in which a central angle of 60^0 intercept an arc of length 37.4 cm(use $\pi = \frac{22}{7}$)

   Solution :- $\theta = 60^0$ l = l 37.4

   $r = \frac{l}{\theta} = \frac{37.4}{60^0} = \frac{37.4 \times 21}{22} = \underline{\underline{\mathbf{35.7}}}$

   r = 35.7 cm

2. The minute hand of a watch is 1.5 long . how long does its tip move in 15 minute.

Solution :-

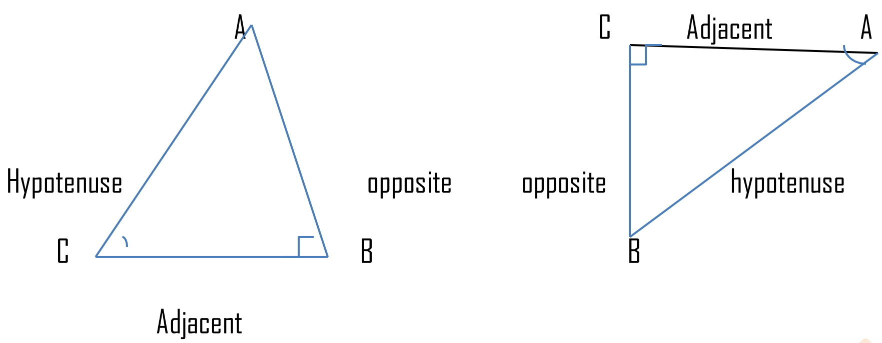
l = 1.5 $\qquad$ 1 $revolution$ = 60 min

t = 15 minute $\qquad$ $x = 15\ min$

$\frac{1}{4}$ revolution $\qquad$ x = $\frac{1}{4}$ revolution

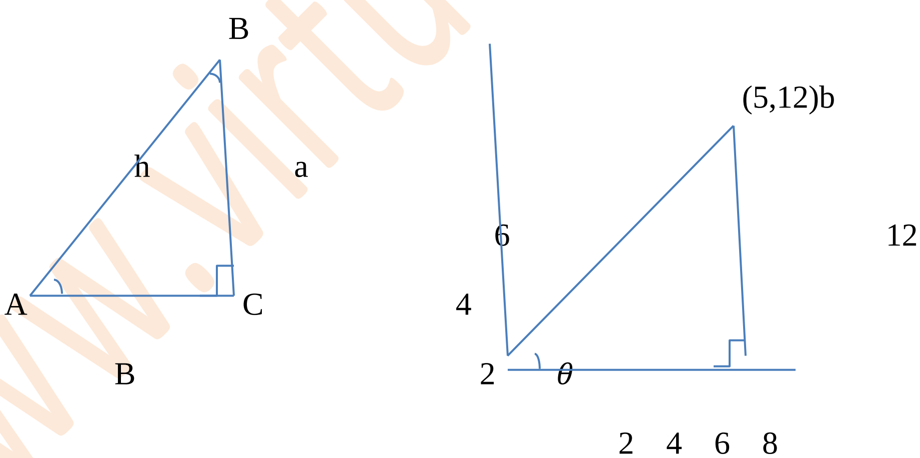
$\frac{1}{4} \times 360^0 = 90^0 \times \frac{\pi}{180} = \frac{\pi}{2}$ // $l = r\theta = 1.5 \times \frac{\pi}{2} = \underline{\mathbf{0.75\pi}}$

4.2 Basic Trigonometric Function



<u>The sine , cosine and tangent function</u>

- Trigonometric functions are originally used to relate the angles of a triangular yto the length of the side of triangle .
- Sin a = $\frac{opposite\ side\ to\ a}{hypotenuse\ side\ to\ a} = \frac{BC}{AB} = \frac{a}{h}$
- Cos a = $\frac{adjecent\ side\ to\ a}{hypotnuse\ side\ to\ a} = \frac{AC}{AB} = \frac{b}{h}$
- Tan a = $\frac{opposite\ side\ to\ a}{adjecent\ side\ to\ a} = \frac{BC}{AC} = \frac{a}{b}$



r = $\sqrt{x^2 + y^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \underline{\underline{\mathbf{13}}}$ now we can find the value of the three trigonometric ratio. Hence

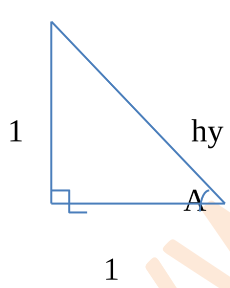
$\sin\theta = \frac{opposite\ side\ to\ \theta}{hypotnuse\ to\ \theta} = \frac{12}{13}$,

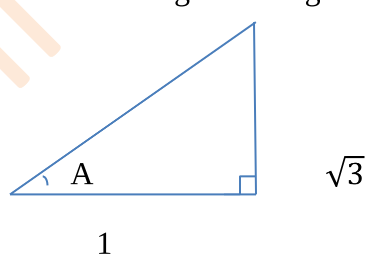
$\cos\theta = \frac{adjecent\ side\ to\ \theta}{hypotnuse} = \frac{5}{13}$

$\tan\theta = \frac{opp}{adj} = \frac{12}{5}$

Exercise

1. Evaluate the sin, cosine and tangent of angle θ is in standard position and its terminal side contain the given point(x, y)
   a. (4,3) x= 4 y= 3
      r = $\sqrt{4^2+3^2}$, r = $\sqrt{12+9} = \sqrt{21}$
      $\sin\theta = \frac{y}{r} = \frac{3}{\sqrt{21}}$
      $\cos\theta = \frac{x}{r} = \frac{4}{\sqrt{21}}$
      $\tan\theta = \frac{x}{y} = \frac{4}{3}$
   b. $(1,\sqrt{2})$
      $r^2 = x^2 + y^2$
      $r = \sqrt{1+2} = \sqrt{3}$//
      $\sin\theta = \frac{0pp}{hyp} = \frac{y}{x} = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{6}}{3}$//
      $\cos\theta = \frac{adj}{hyp} = \frac{x}{hyp} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$//
      $\tan\theta = \frac{opp}{hyp} = \frac{y}{x} = \frac{\sqrt{2}}{1} = \sqrt{2}$//
2. Find the values of the trigonometric ratio of the angle A in figure 4.9



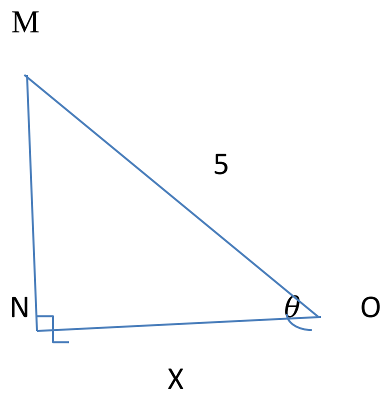


$1^2 + 1^2 = hyp^2 = r^2$ $r^2 = 1^2 + \sqrt{3}^2$

$r^2 = 2$ $r = \sqrt{2}$ $r^2 = 1+3$

$r^2 = 4$ **r = 2**

3. Almaz want to find the value of x in the given $\triangle$ MNO, where $\cos\theta = \frac{1}{2}$ how you help Almaz?



Sin $\theta = \frac{opp}{hyp} = \frac{1}{2} = \frac{y}{5} = 2y = 5$

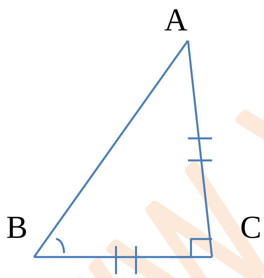
y = $\frac{5}{2}$ = 2.5

cos $\theta = \frac{adj}{hyp} = \frac{x}{5} = \frac{1}{2} = \frac{x}{5}$ x = $\frac{5}{2}$

x = 2.5

Exercise

1. Calculate all angles and sides if the hypotheses of an isosceles right angled triangle ABC are equal to 4.

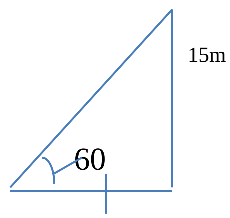


$AB^2 = BC^2 + AC^2$

$H^2 = 4^2 + 4^2$ hyp = $\sqrt{16+16}$

HYP = $\sqrt{32}$ = $\sqrt[4]{2}$

2. Ali wants to find the exact Length of the shadow cast of a15m lamppost when the angle of elevation of the sun is 60°. What is the length of the shadow cast?

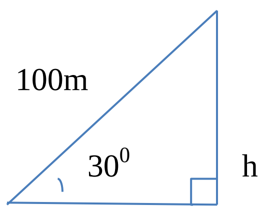


cos60 =tan60 = $\frac{opp}{x}$

tan60 = $\frac{15}{x} = \sqrt{3} = \frac{x\sqrt{3}}{\sqrt{3}} = \frac{15}{\sqrt{3}}$

x = $\frac{15}{\sqrt{3}} = \frac{15}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = \mathbf{5\sqrt{3}}$//

3. A kite in the air has a string tied to the ground as shown in figure 4.14 . if the length of the string is 100m, find the height of the kite above the ground when the string is taut and its inclination is 30^0 to the horizontal.



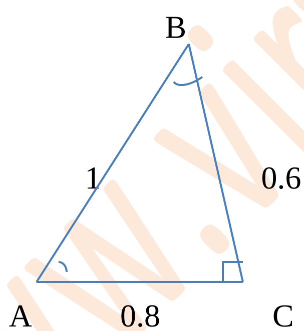
sin $30^0 = \frac{h}{100}$

h = 100 sin 30^0 = 100× $\frac{1}{2}$ = **$\underline{\underline{50m}}$**

Sin $30^0 = \frac{opp}{hyp} = \frac{h}{100}$

4.3 The Unit Circles

- The unit circle is in the x y- plan it is circle with a radius of 1 and centered at the origin.
- Now let us draw a right –angled triangle with the same acute angles a hypotenuse of unites long. We find that side opposite to angle A is $\frac{3}{5}$ = 0.6 long and the side adjacent to angle A is $\frac{4}{5}$ = 0.8 long.

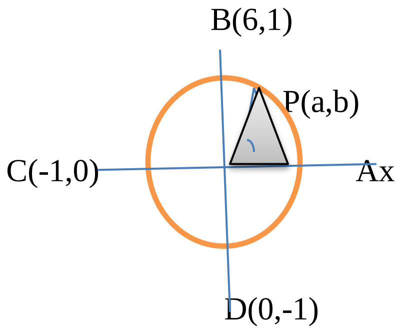


Cos A= $\frac{0.8}{1}$ = 0.8

Sin A= $\frac{0.6}{1}$ = 0.6 , thus the length of the side adjacent is numerical equal to the cosine of the angle. Whose center is the origin and whose radius is 1 unit long to help us visualize the values of the cosine and sine of the central angle.

✓ Plot the points (1,0) , (0,1) ,(-1,0) and (0,-1) on the x y coordinate system.

- ✓ Let p(a, b) be any point on the circle with angle A op = x radian i.e length of arc AP=x as show in the figure.
- ✓ We define cosx = $\frac{a}{1}$ = a and sin x = $\frac{b}{1}$ = b



So, the point (a,b) = cosx,sin x

Since, Δ OMP is a right –angled triangle we have $(OM)^2 + (Mp)^2 = (op)^2$

$a^2 + b^2 = 1$, Thus for any point on the unit circle we have $a^2 + b^2 = 1$ or $\cos^2 x + \sin^2 x = 1$

Since one complete revolution subtend an angle of 2π radian at the center of the circle m($< AoB$) = $\frac{\pi}{2}$, m($< Aoc$) =π, and m(<AOD) = $\frac{3\pi}{2}$,

All Angles which are integral multiplies of $\frac{\pi}{2}$ are called Qudrantal angle.

The coordinates of the point a, b, c and D are respectively (1,0) ,(0,1) ,(-1,0) and (0,-1). Therefore qudarantine angle we have cos 0 = 1 Sin 0 = 0

Cos $\frac{3\pi}{2}$ = 0 Sin $\frac{3\pi}{2}$ = 1

Cos $\frac{3\pi}{2}$ = 0 Sin $\frac{3\pi}{2}$ = -1

Cos 2π = 1 $\text{Sin}^2\,\pi = 0$ and the n Sin($2n\pi$ + x) = Sin x $n \in z$. $\text{Cos}\,(2n\pi + x) = \cos x \;\; n \in z$.

Degree	0^0	30^0	45^0	60^0	90^0	180^0	270^0	360^0
Radian	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin x	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
Cos x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1

Tan x	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined	0	not defined	0

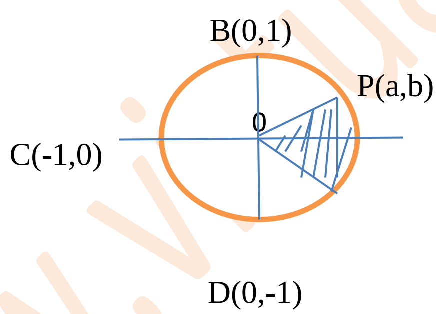
<u>Exercise</u>

1. Find the value of the sin, cosine and tangent function of the following quadrant angles.
   a. 450^0
   Sin $(1 \times 360^0 + 90^0)$ = Sin 90^0 = 1
   Cos $(1 \times 360^0 + 90^0)$ = Cos 90^0 = 0
   Tan $(1 \times 360 \times 90)$ = Cos 90^0 = undefined
   b. $540^{\,0}$ Sin ($1\times 360^0 + 180^0$) = Sin 180^0 = 0
   Cos $(1 \times 360^0\ \ 180^0)$ = tan 180^0 = 0

4.4 Sign of trigonometric function

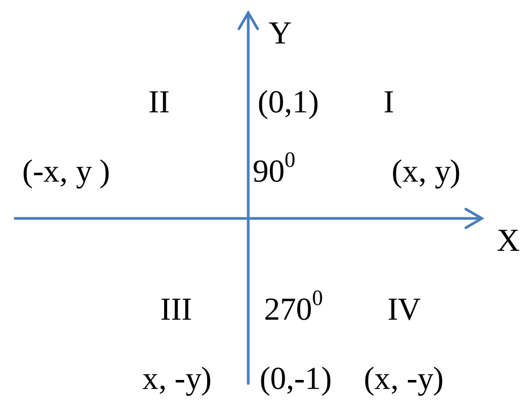
Let p(a,b) be a point on the unite circle with center at the origin such that m(<AOP) = x. if M(<AOQ) = - x the coordinate of the point Q(a,-b) therefore , cos (-x) = cos x and Sin(-x) = Sin x

Since for every point p(a,b) on the unit circle where $-1 \le a \le 1$ and $-1 \le b \le 1$. we have $-1 \le \cos x \le 1$ and $-1 \le \sin x \le 1$for all x.



Table

y =	Quadratic I	II	III	IV
Sin x	+	+	-	-
Cos x	+	-	-	+
Tan x	+	-	+	-



All Students take chemistry

I	II	III	IV
+	+	+	+
	Sin	tan	cos

Exercise

1. Find the quadrant where angle x is located for the following condition
   a. Sin x < 0 and cos x > 0
      Quadrant four (IV)
   b. Sin x > 0 and tan x < 0
      Quadrant two(II)
   c. Cos x > 0 and tan x < 0
      Quadrant four (IV)

Reciprocal Trigonometric Function

NOTE :- We can define other trigonometric functions in terms of sine and cosine it is convenient to have a name for the reciprocal of the sine, cosine and tangent of a given angle θ . we call these reciprocal functions the secant (sec) cosecant (csc) and cotangent (cot) and define them as follow.

$\mathrm{Csc}\theta = \frac{1}{\sin\theta}$

Sec $\theta = \frac{1}{cos\theta}$

Cot $\theta = \frac{1}{\tan\theta}$

Example: - evaluate sec 30^0

Solution: - sec 30^0 is the reciprocal of Cos 30^0 therefore , we have

Cos $30^0 = \frac{\sqrt{3}}{2}$ so sec $30^0 = \frac{1}{\cos 30^0} = \frac{2}{\sqrt{3}}$

Exercise

1. Evaluate the following
   a. Sec $45^0 = \sqrt{2}$
   b. Sec $-\frac{\pi}{6} = \frac{2}{\sqrt{3}}$
   c. Csc $\frac{3}{4}\pi = \sqrt{2}$
   d. Cot $\frac{5\pi}{6}$ = - $\sqrt{3}$
   e. Cot $(\frac{-5\pi}{4})$ = -1
   f. Csc $(-300^0) = \frac{2}{\sqrt{3}}$
2. Evaluate the following trigonometric expression.
   a. Sec $\frac{10\pi}{3}$ + csc $(\frac{-7\pi}{2})$

      Sec $\frac{10\pi}{3}$ = - sec $\frac{\pi}{3}$ = - $\frac{1}{cos\frac{\pi}{3}}$ = -2 and csc $(\frac{-7\pi}{2})$ = csc $(\frac{-7\pi}{2})$ = csc $\frac{\pi}{2} = \frac{1}{\sin\frac{\pi}{2}} = 1$

Therefore , sec $\frac{10\pi}{3}$ + csc $(\frac{-7\pi}{2})$ = -2 + 1 = -1

   b. Sec 330^0 + cot 480^0

      Sec 330^0 = sec $30^0 = \frac{1}{\cos 30^0} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

Cot 480^0 = - cot 60^0 = - $\frac{1}{\tan 60^0}$ = - $\frac{1}{\sqrt{3}}$ therefore sec 330^0 + cot 480^0 = sec 30^0 + (- cot 60^0)

$= \frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$//

Trigonometric values of Angles.

- ✓ Trigonometric angles are the angle given by the ratios of the trigonometric function.
- ✓ Trigonometric deals with the study of the relationship between angles and sides of a triangle .
- ✓ The angle value ranges from 0^0 to 360^0 , then important angles in trigonometry are 0^0 , 30^0 45^0 , 60^0 ,90^0,180^0 ,270^0 and 360^0 are important six trigonometric ratios of functions are sine, cosine, tangent, cosecant, secant, cotangent.

Complementary Angle

- ✓ If two angles are complementary they are added up to 90^0.

Example :- 60 + 30 = 90 find the complementary angle of 57^0

$90- 57 = 33^0$

Note :- The following on a cute angle a and its complementary angle ($\frac{\pi}{2}$, a).

- $\sin(\frac{\pi}{2} - a) = \cos a$
- $\cos(\frac{\pi}{2} - a) = \sin a$
- $\tan(\frac{\pi}{2} - a) = \cot a$
- $\cot(\frac{\pi}{2} - a) = \tan a$

Example: - If sin 3A= Cos $(A-26^0)$ Where 3A is an acute angle, find the value of A.

Solution :-Given that , sin 3A= $\cos(A-26^0)$-------1

Since 3A= cos $90^0 – 3A)$,we can write 1 as

$\cos(90-3A) = \cos(A-26^0)$

$90^0 – 3A = A- 26^0$

$90^0 + 26 = 3A+A$

$\frac{116}{4} = \frac{4A}{4}$ $\underline{\underline{\mathbf{A= 29^0}}}$ Therefore the value of A is $\mathbf{29^0}$

Exercise

1. find the numerical value of
   a. sin 30^0 and cos 60^0

   sin 30^0 = $\frac{1}{2}$ and cos 60 =$\frac{1}{2}$

   b. sin 45^0 and cos 45^0

   sin 45^0 = $\frac{\sqrt{2}}{2}$
2. if sin 31^0 = 0.515, then what is cos 59^0

   cos 59^0 = 0.515 because complementary angles are cogent.
3. If sin θ. = $\frac{3}{5}$, then what is cos(90^0 – θ)

Sin $= \frac{3}{5}$, then cos(90 –θ) = $\frac{3}{5}$

4. if cos a= $\frac{4}{5}$, then what is sin$(\frac{\pi}{2} - a) = \frac{4}{5}$

Reference angle (θR)

- The reference angle of any angle always lies between 0^0 and 90^0.
- It is the angle between the terminal side of the angle and the x- axis . the reference angle depends on the quadrant terminal side.

The steps to find the reference angle of an angle depend on the quadrant of the terminal sides,

- ✓ We first determine its conterminal angle which lies between 0 and 360^0
- ✓ We then see the quadrant of the co terminal angle
- ✓ If the terminal sides is in the first quadrant (0^0 to 90^0), then the reference angle is the same as our given angle.

  If the terminal side is in the second quadrant (90^0 to 180^0) then the reference angle (θR)is 180^0 minus the given.

  Example :- 100

  180 -100 =80 $(\theta R) = 80$

  If the terminal side is in the third (180 to 270^0) then the reference angle $(\theta R) is$ the given angle minus 180^0.

  Example: - 215 215- 180^0 = 35^0. if the terminal side is in the fourth quadrant (270^0 to 360^0) , then the reference angle (θR) is 360 ^{0} minus the given angle.

Example:- 330

$360^0 - 330^0 = 30^0$

Note that :-

- The value of the trigonometric function of a given angle θ and the values of the corresponding trigonometric functions of the reference angle θR, are the same in absolute value

  Exercise

  1. Find the reference angle θR for the angles.
     a. $\theta = 109^0$

        $\theta R = 180 - \theta = \theta R = 180 - 109 = 71^0$

     b. $\theta = 345^0$

        $\theta R = 360^0 - 345^0 = \underline{\underline{15^0}}$

     c. $\theta = \frac{5\pi}{3}$ Fourth quadrant

        $\theta R = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$//

     d. $\theta = \frac{7\pi}{4}$ $\theta R = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$ //

     e. $\theta = \frac{4\pi}{3}$ $\theta R = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$ //

Supplementary Angles

- The supplementary angles are angles that exist in pairs summing up to 180^0 so supplementary angle of an angle x is 180^0 minus x.

  Example: - 1. Determine the following pairs of angles are supplementary or not

  a. 50^0 and 130^0 50 + 130 = 180 they are supplement

  b. 70^0 and 100^0

     $70^0 + 100^0 = 170^0$ not supplement

  Conterminal angle

- Co-terminal angle angles are that have the same initial side and share the terminal sides .

- The formula to find the co-terminal angles of an angle θ depends upon whether it is in terms of degrees radians.
  1. Degrees : $\theta \pm 360n$ where n is an integer.
  2. Radian : $\theta \pm 2n\pi$,where n is an integer. So 45^0, -315^0 , 405^0 , -675^0 , 765^0 are all conterminal angle.

Example :- find two co-terminal angle of 30^0 .

$\theta=30^0$

$30 + 360(n) = 30 + 360(1) =$ **$\underline{\underline{390^0}}$**

To find the second co-terminal angle when n= -2(clockwise)

$\theta + 360(n)$

$30 + (-2) = 30 + (-720) = \mathbf{-690}$

From the above explanation we can find the co-terminal angles of any angle either by adding or subtracting multiple of 360(or 2π) from the given angle. So we actually do not need to use the conterminal angles formula to find the co-terminal angle. Find a co-terminal angle of $\frac{\pi}{4}$

$\theta = \frac{\pi}{4}$ $\frac{\pi}{4}$ $-2\pi = \frac{-7\pi}{4}$// thus one of the co-terminal angles of $\frac{\pi}{4}$ is $\frac{-7\pi}{4}$ co-terminal angles can be positive or negative . in one of the above examples , we found that 390^0 and – 690 are co- terminal angles of $30^{0.}$

Note that:- $\theta \pm 360n$, where n takes a positive value when the rotation is anticlockwise and takes a negative value when the rotation is clockwise so we can decide that add or subtract multiples of 360^0 (or 2π) to get positive or negative co-terminal angles, respectively .

<u>Exercise</u>

1. Determine whether the following pair of angles are supplementary or not.
   a. 90^0 and 100^0

      90 +100 = 190 not supplementary
   b. $\frac{\pi}{6}$ and $\frac{2\pi}{3}$

      $\frac{\pi}{6} + \frac{2\pi}{3} = \frac{3\pi+12\pi}{18} = \frac{15\pi}{18} = \frac{5\pi}{6}$// it is supplementary

2. Find a positive and a negative angle which are co-terminal with angle 55^0.

   $55^0 + 360^0 = 415^0$ is positive co terminal angle with 55^0

   $55^0 + 360^0 = 415^0$ is positive co-terminal angle with 55^0.

   $55^0 – 360^0 = -305^0$ is negative co-terminal with 55^0.

3. Find a positive and a negative angle which are co-terminal with angle $\frac{\pi}{3}$

   $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$ is positive conterminal angle with $\frac{\pi}{3} - 2\pi = \frac{-5\pi}{3}$ is negative co-terminal angle with $\frac{\pi}{3}$.

4. evaluate the following expression

   a. $\sin 360^0 = \sin(360 + 30) = \sin 30$ $\sin 360^0 = \sin 30 = \frac{1}{2}$

   b. $\cos \frac{10\pi}{3} = -605\frac{\pi}{3} = -\frac{1}{2}$//

   c. $\tan(-420^0) = -\tan 60^0 = -\sqrt{3}$

   d. $\sin(-660^0) = \sin(-300^0) = \sin 60^0 = \frac{\sqrt{3}}{2}$

   e. $\cos(\frac{41\pi}{4}) = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

   f. $-2\cos(-150^0) = \theta = 150 = 360 – 150$

   $= 210^0$ and $\theta R = 210^0 – 180^0 = 30^0$

   $-2\cos(-150^0) = -2(\cos 210^0) = -2(-\cos 30^0) = -2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$//

4.5Graphs of The Sine, Cosine and Tangent Functions

4.5.1 The graph of the sine function

Draw the graph of $y = \sin\theta$. To determine the graph of $y = \sin\theta$ we construct a table of values for $y = \sin\theta$ where $-360^0 \leq \theta \leq 360^0$.

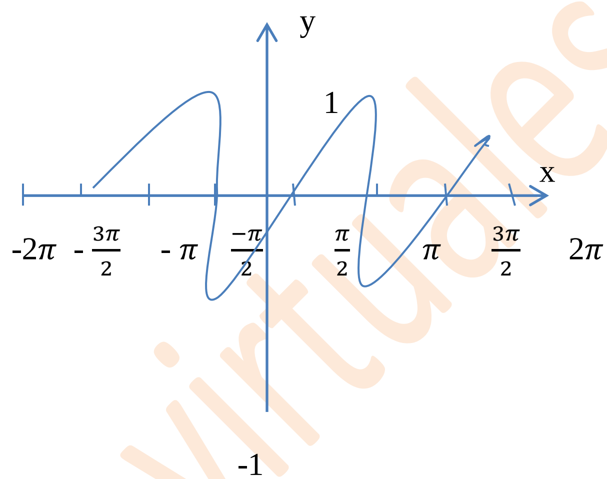
θ *in degree*	-360	-330	-270	-240	-210	-180	-150	-120	-90	-60	-45	-30	0
θ *in radian*	-2π	$-\frac{11\pi}{6}$	$-\frac{3\pi}{2}$	$-\frac{4\pi}{3}$	$-\frac{7\pi}{6}$	$-\pi$	$-\frac{5\pi}{6}$	$\frac{-2\pi}{3}$	-	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0
y = sin θ	0	0.5	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.71	-0.5	0

θ **in degree**	30^0	45^0	60^0	90^0	120^0	150^0	180^0	210^0	240^0	270^0	300^0	330^0	360^0
θ **in radian**	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
y = sin θ	0.5	0.71	0.87	1	0.87	0.5	0	-0.5	-0.87	-1	-0.87	-0.5	0

After a complete revolution (every 360^0 or 2π). The values of the sine function repeat themselves. Thus means

- ✓ Sin $0^0 = \sin(0^0 \pm 360^0) = \sin(0^0 \pm 2 \times 360^0) = \sin(0^0 \pm 3 \times 360^0)$, etc..
- ✓ Sin $90^0 = \sin(90^0 \pm 360^0) = \sin(90^0 \pm 2 \times 360^0) = \sin(90^0 \pm 3 \times 360^0)$, etc..
- ✓ Sin $180^0 = \sin(180^0 \pm 360^0) = \sin(\theta \pm 2 \times 360^0) = \sin(\theta^0 \pm 3 \times 360^0)$,etc..

A function that repeats its value at regular intervals is called a periodic function. The sine function repeats after every 360^0 (or 2π). Therefore, 360^0 (or 2π) is called the periodic of the sine function.



The graph of y= sin θ for $-2\pi \le \theta \le 2\pi$

Exercise

1. Draw graph of the following function.
   y = sin x for $-2\pi < x \le 2\pi$

θ	-360	-270	-90	-30	0^0	30^0	90^0	270^0	360^0
y = sin θ	0	1	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	-1	0

Domain and range

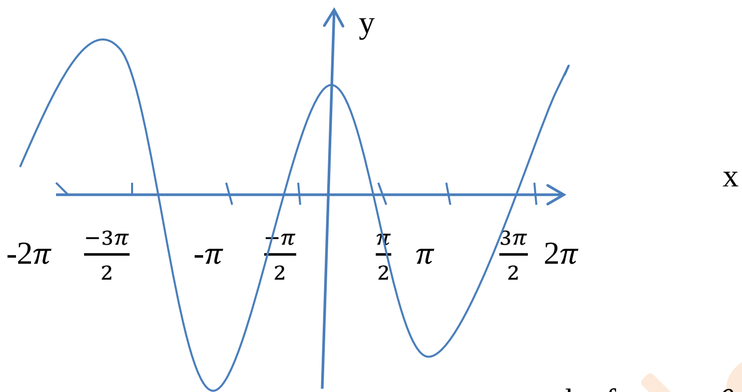
For any angle θ taken on the unite circle, there is some point p(x, y) on its terminal side. Since y = sinθ , the function y = sin θ is defined for every angle θ taken on the unite circle.

Therefore , the domain of the sine function is the set of all real numbers.

Note :- the domain of the sine function is $\{\ \theta : \theta \in R\}$. the range of the sine function is $\{\ y(\theta) : -1\ \leq y \leq 1, \theta \in R\}$.

<u>The graphs of cosine function</u>

Just like the sine function, the cosine function is product at every 360^0(or 2π) radians . Therefore 360^0(or 2π) is called the period of the cosine function.

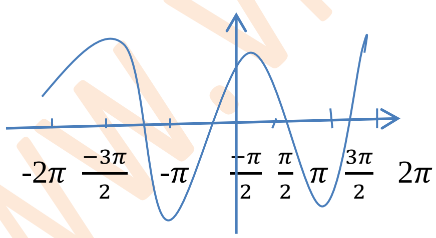


graph of y =cos θ

<u>Exercise</u>

1. Draw graph of the following function.
   y = cos x, for $-2\pi \leq x \leq 2\pi$

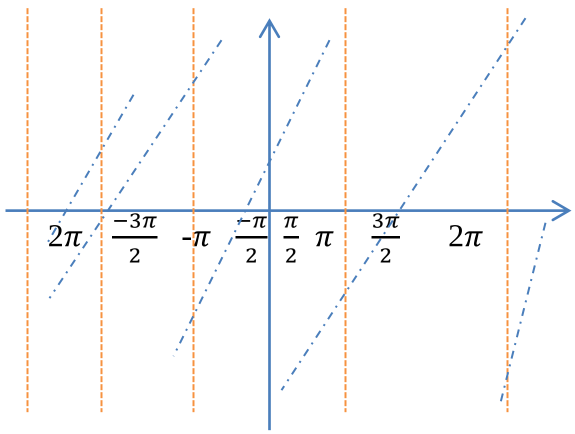
θ	360^0	-270^0	-90^0	-30^0	0^0	30^0	90^0	270^0	360^0
y = cos θ	1	0	0	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	0	0	1



<u>The graph of then tangent function</u>

θ	-300	-315	-270	-225	-180	-135	-90	-45	0
θ in radian	-2π	$\frac{7\pi}{4}$	$\frac{-3\pi}{2}$	$\frac{-5\pi}{4}$	$-\pi$	$\frac{-\pi}{2}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0
y = tan θ	0	1	Undefined	-1	0	1	Undefined	1	0

θ	30	45	9	135	180	225	270	315	360
θ in radian	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
y = tan θ	0.56	1	Undefined	-1	0	1	Undefined	-1	0



The graph of y=tanθ for - $2\pi \leq \theta < 2\pi$

Trigonometric identities and equation

There are basically three trigonometric identities, which we learn in this topic they are:-

1. $\text{Cos}^2\,\theta + \sin^2\theta = 1$
2. $1 + \tan^2\theta = \sec^2\theta$
3. $1 + \cot^2\theta = \csc^2\theta$

   Proof of trigonometric identities

   (Perpendicular sin)^2 + (base)^2 = (hypotenuse)^2

   $x^2 + y^2 = (\text{hyp})^2$

   $\frac{AB^2}{(AC)^2} + \frac{BC^2}{AC^2} = 1$

$\sin^2\theta + \text{Cos}^2\,\theta = 1$ (since $\sin\theta = \frac{AB}{AC}$ and $\cos\theta = \frac{BC}{AC}$

$\sin^2\theta + \text{Cos}^2\,\theta = 1$

Example:- x is the angle of the third quadrant , when sin x = $\frac{-3}{5}$, find the value of cos x.

$\sin^2 x + \cos^2 x = 1$ and $\cos^2 x = 1 - \sin^2 x$

$= 1-(\frac{-3}{5})^2 = 1- \frac{9}{25} = \frac{16}{25}$ but x is the angle of the third quadrant , so cos x $<$ 0 thus cos x $= \frac{\sqrt{16}}{25} = \frac{-4}{5}$//

Exercise

a. If x is the angle in the second quadrant and cos x = $-\frac{\sqrt{5}}{3}$, find the value of sin x.

$\sin^2 x = 1 - \cos^2 x = 1-(-\frac{\sqrt{5}}{3})^2 = 1- \frac{5}{9} = \frac{4}{5}$// x is the angle of the second quadrant so sin x $>$ 0 thus sin x $= \sqrt{\frac{4}{9}} = \frac{2}{3}$//

b. If x is the angle in the fourth quadrant and sin x = $-\frac{1}{3}$ find the value of cos x.

$\sin^2\theta + \cos^2\theta = 1$

$\cos^2 x = 1 - \sin^2 x = 1-(-\frac{1}{3})^2 = 1- \frac{1}{9} = \frac{8}{9}$// x is the angle of the third quadrant so cos x < 0

thus cos x $= \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$//

c. x is the angle of the third quadrant . if tan x =3 , find the values of sec x and cos x.

$\sec^2 x = 1 + \tan^2 x$, so $\sec^2 x = 1 + 3^2 = 10$ and since x is the third quadrant

sec x = $-\sqrt{10}$ and cos x = $-\frac{1}{\sqrt{10}}$//

d. x is the angle of the second quadrant . if tan x = 3 find the values of sec x and cos x

$\sec^2 x = 1 + \tan^2 x$. so $\sec^2 x = 1 + 3^2 = 10$ and x is second quadrant

sec x = $-\sqrt{10}$ and cos x = $\frac{-1}{\sqrt{10}}$//

Addition and Subtraction of Identities

The formulas for the addition and subtraction theorems of sine and cosine are expressed as follow

- ✓ sin (a + B) = sin a Cos B + Cos a sin B
- ✓ sin (a – B) = sin a Cos B – Cos a Sin B
- ✓ Cos (a + B) = Cos a Cos B – sin a Sin B

✓ Cos (a – B) = Cos a Cos B + Sin a Sin B

Exercise

1. Find the values of the following trigonometric expression.

a. Sin 105^0

Sin 105^0 = Sin (60^0 + 45^0)

Sin (60^0 + 45^0) = Sin 60^0Cos 45^0 + Cos 60^0 Sin 45^0

Sin 60 Cos 45 + Cos 60 Sin 45 = $\frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + \frac{1}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{b}+\sqrt{2}}{4}$//

b. Cos 105^0

Cos 105^0 = Cos (45^0 + 60^0)

Cos (60^0 + 45^0) = Cos 60 Cos 45 – Sin 60 Sin 45

$= \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-\sqrt{6}}{4}$//

c. Sin 15

Sin 15 = Sin (45-30)

Sin (45- 30) = Sin 45 Cos 30 – Sin 30 Cos 45

$= \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$//

Double Angle Identities

Formulas expressing trigonometric functions of an angle 2θ in terms of an angle θ:

Sin $2\theta = 2\, Sin\theta\, Cos\theta$

Cos 2θ = Cos $^2\theta - Sin^2\theta = 1 - 2Sin^2\theta = 2Cos^2\theta - 1$

Tan 2θ = $\frac{2tan\theta}{1- tan^2\theta}$

Example :- if x is an angle of the second quadrant and Sin x = $\frac{3}{5,}$ find the following value

a. Cos x

$Sin^2x + Cos^2x = 1$

$\text{Cos}^2x = 1 - \text{Sin}^2 x = 1-(\frac{3}{5})^2 = 1 - \frac{9}{25} = \frac{16}{25}$ x is second quadrant thus $\text{Cos } x = \frac{\sqrt{16}}{25} = \frac{-4}{5}$//

b. $\text{Sin } 2x = 2\text{Sin } x \text{ Cos } x = 2 \times \frac{3}{5}(-\frac{4}{5}) = \frac{24}{25}$//

Half Angle identities

The half – angle identities for the sine and cosine are derived from two of the cosine identities described earlier.

$\text{Cos } 2\theta = 2 \text{ Cos}^2 \theta - 1$

Let $\theta = \frac{a}{2}$, then $\text{Cos}(2 \times \frac{a}{2}) = 2 \text{ Cos}^2 \frac{a}{2} - 1$

$\text{Cos a} = 2\text{Cos}^2 \frac{a}{2} - 1$

$2 \text{ Cos}^2 \frac{a}{2} = \text{Cos a} + 1$

$\text{Cos}^2 \frac{a}{2} = 1 + \frac{Cosa}{2}$, so $\text{Cos} \frac{a}{2} = \pm \frac{\sqrt{1+Cosa}}{2}$

Similarly $\text{Sin}^2 \frac{a}{2} = \frac{1-Cosa}{2}$ and $\text{Sin} \frac{a}{2} = \pm \frac{\sqrt{1-Cos\, a}}{2}$//

Example: - find the exact value for Cos 15^0 using the half-angle identity

$\text{Cos } 15^0 = \text{Cos} \frac{30^0}{2} = \pm \frac{\sqrt{1+Cos\, 30}}{2}$

$= \pm \frac{\sqrt{1+\frac{\sqrt{3}}{2}}}{2} = \pm \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \pm \sqrt{\frac{2+\sqrt{3}}{2}}$ first quadrant

Exercise

1. Assume x is the angle in the first quadrant when $\text{Cos } x = \frac{2}{3}$, find the following value.

   a. Sin x

   $\text{Sin}^2x + \text{Cos}^2 x = 1$

   $\text{Sin}^2x = 1 - \text{Cos}^2x$

   $\text{Sin}^2x = 1 - (\frac{2}{3})^2$

$\text{Sin}^2 x = 1 - \frac{4}{9} = \frac{5}{9} \quad x^2 = \frac{5}{9} \quad x = \sqrt{\frac{5}{9}}$

$x = \frac{\sqrt{5}}{3}$//

b. Sin 2x

$\text{Sin } 2x = 2 \text{ Sin } x \text{ Cos } x = 2.\frac{\sqrt{5}}{3}.\frac{2}{3} = \frac{4\sqrt{5}}{9}$//

2. Find the exact values using the half-angle identity

a. $\text{Sin}\frac{\pi}{8} = \text{Sin}\frac{\frac{\pi}{4}}{2} = \pm\sqrt{\frac{1-Cos\frac{\pi}{4}}{2}}$

$= \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}$ but 15^0 is first quadrant

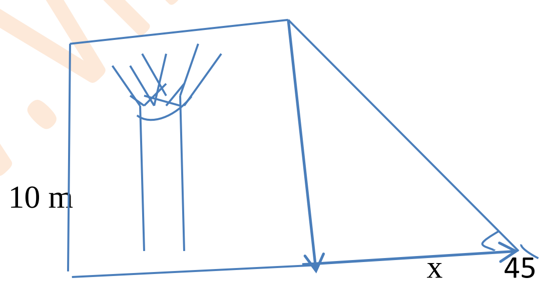
$= \frac{\sqrt{2-\sqrt{3}}}{2}$//

b. $\text{Sin}\frac{\pi}{8} = \text{Sin}\frac{\frac{\pi}{4}}{2} = \pm\sqrt{\frac{1-Cos\frac{\pi}{4}}{2}}$

$= \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}$ $\frac{\pi}{8}$ is first quadrant $= \sqrt{\frac{2-\sqrt{2}}{2}}$//

4.6 Application of trigonometric function

Example: - 1. from the top of a vertical tree 10 m height the angle depression of an object that is on the ground is 45^0 as shown in figure below . How far is the object from the base of the tree?



Solution

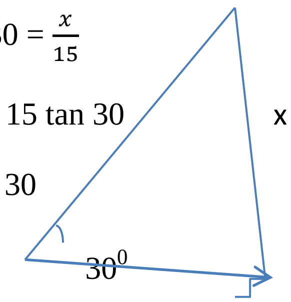
$\text{Tan } 45^0 = \frac{10}{x}$ which implies $x = \frac{10}{\tan 45}$

$x = \frac{10}{\tan 45} = 10\text{m}$//

1. Dana is standing on the ground and looking at the top of the tower with an angle of elevation of 30^0 . if he is standing 15m away from the foot of the tower can you determine the height of the tower?

$\tan 30 = \frac{x}{15}$

x = 15 tan 30



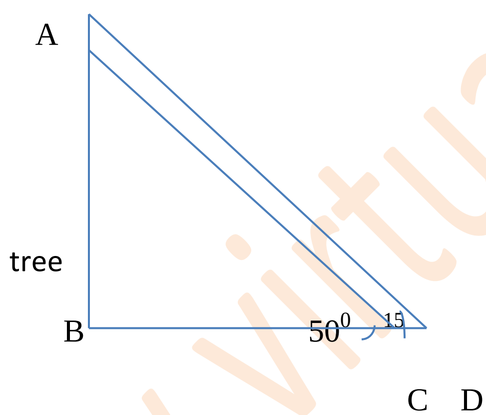
x= 15 tan 30

$x = \frac{15\sqrt{3}}{3}$

$= 5\sqrt{3}$ m//

Review exercise

1. If A is an obtuse angle and Sin A = $\frac{4}{5}$ then evaluate
   a. Cos A A = 30^0
2. Find the height of the tree if the angle of the elevation of its top changes from 25^0 to 50^0 and the observer advance 15 meter towards its base.



Tan $50^0 = \frac{h}{x}$

$x = \frac{h}{\tan 50}$ tan 25 = $\frac{h}{x+15}$ tan 25 = $\frac{h}{\frac{h}{\tan 50+15}}$

$0.4663 = \frac{h}{\frac{h}{1.1917+15}}$

$0.4663 \times (\frac{h}{1.1917+15}) = h$

0.4663 h + 8.3353 = 1.1917 h

$\frac{0.4663h}{0.4663} = \frac{8.3353}{0.4663}$

<u>h = 11.49m</u>

3. A man observed a pole of height 60ft. according to his measurement , the pole cast a 20ft, long shadow. Find the angle of elevation of the shadow using trigonometry.

   <u>Solution</u> :-

   Let x be the angle of elevation of the sign , then tan x = $\frac{60}{20}$ = 3// x = tan $^{-1}$(3) or x = 71.56 degree. therefore the angle of elevation of the sum is **<u>71.56</u>**

UNIT 5

CIRCLE

5.1 Symmetrical Properties of Circles

A circle is the locus of points (set of points) in a plane each of which is equidistant from a fixed point in the plane. The fixed point is called the center of the circle and the constant distance is called its radius. Thus, the circle is defined by its center O and radius r.

A circle is also defined by two of its properties such as area and perimeter. Recall that area of a circle, $A = \pi r2$ and perimeter of the circle, $P = 2\pi r$.

Observe that in a symmetrical figure the length of any line segment or the size of any angle in one half of the figure is equal to the length of the corresponding line segment or the size of the corresponding angle in the other half of the figure.

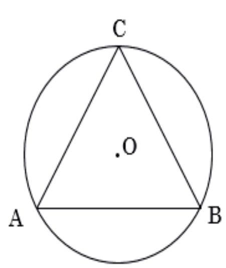
If we fold a circle over any of its diameters, then the parts of the circle on each side of the diameter will match up and the parts of the circle on each side of the diameter must have the same area. Thus, any diameter of a circle can be considered as a line of symmetry for the circle.

The line segment joining the center of a circle to the midpoint of a chord is perpendicular to the chord.

Exercise

1. ΔABC is an equilateral triangle and circle O is its circumcircle. How many lines

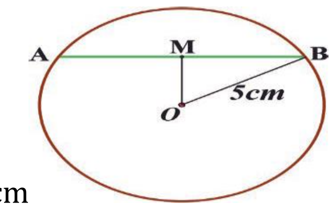
of symmetry does



Characteristics of Chord (1)

- ✓ The line segment drawn from the center of a circle perpendicular to a chord bisects the chord.
- ✓ Equal chords of a circle are equidistant from the center of the circle.

Example a chord of a circle of radius 5 cm is 8 cm long. Find the distance of the chord from the center.



Solution:Given: $AB = 8$cm and $OB = 5$cm

$OM^2 + MB^2 = OB^2$

$OM = \sqrt{OB^2 - MB^2}$

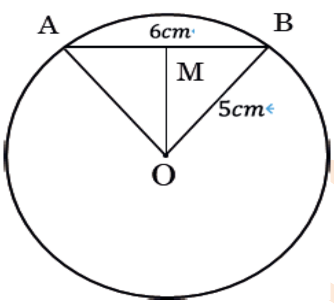
$= \sqrt{52 - 42}$

$= \sqrt{25 - 16}$

$= \sqrt{9}$ cm $= 3$cm.

Exercise

In figure , a chord of a circle of radius 5 cm is 6 cm long. Find the distance of the chord from the center.



Given $\overline{AB} = 6$cm and $r = 5$cm

$OM^2 + MB^2 = OB^2$

$OM = \sqrt{OB^2 - MB^2}$

$= \sqrt{52 - 32}$

$= \sqrt{25 - 9}$

$= \sqrt{16}$ cm $= 4$cm

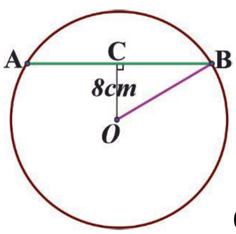
Characteristics of Chord (2)

✓ If the angles subtended by the chords of a circle are equal in measure, then the length of the chords are equal.

✓ Chords which are equal in length subtend equal angles at the center of the circle.

Exercise

1. A chord of length 20cm is at a distance of 8cm from the center of the circle. Find the radius of the circle.



Given: $AB = 20$cm and $OC = 8$cm. From theorem, $\overline{OC}$ is the perpendicular bisector of $\overline{AB}$ at C. So, ΔBCO is a right-angled triangle. By Pythagoras Theorem we observe,

$OB = r = \sqrt{OC^2 + CB^2} = \sqrt{8^2 + 10^2}$

$= \sqrt{164} = 2\sqrt{41}$cm

2. A chord of a circle of radius 8cm is 10cm long. Find the distance of the chord from the center of the circle.

Given: $\overline{PR} = 10$cm and radius$= \overline{OR} = 8$cm. Let the mid-point of $\overline{PR}$ be M.

$OM^2 + MR^2 = OR^2$

$OM = \sqrt{OR^2 - MR^2}$

$= \sqrt{8^2 - 5^2}$

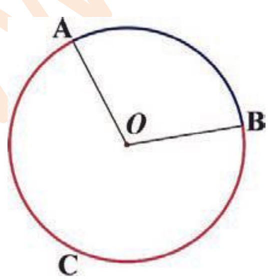
$= \sqrt{64 - 25}$

$= \sqrt{39}$cm.

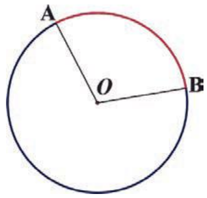
5.2 Angle Properties of Circles

Central Angles and Inscribed Angles (1)

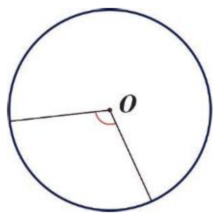
❖ A major arc is an arc connecting two endpoints on a circle and its measure is greater than180°or π.



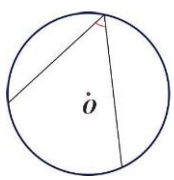
- Minor arc is an arc connecting two endpoints on a circle and its measure is less than 180° or π.



- A major arc is usually referred to with three letters and a minor arc is usually referred to with only two letters.
- A central angle is an angle formed by two radii with vertex at the center of the circle.



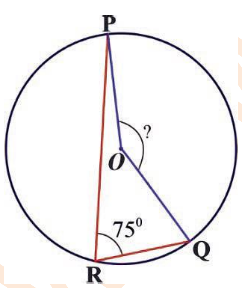
- An inscribed angle is an angle with vertex on the circle formed by two intersecting chords.



If an inscribed and a central angle intercept the same arc, then the measure of an inscribed angle is half of the measure of a central angle.

Example 1

if O is the center of a circle with $m(\angle PRQ)$ = 75° , what is the size of $\angle POQ$?

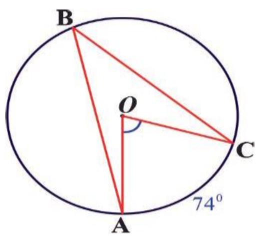


Solution:

$m(\angle POQ) = 2 \times m(\angle PRQ) = 2 \times 75° = 150°$.

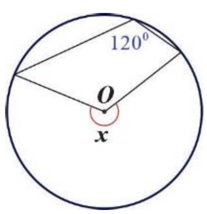
Exercise

1. In the figure 5.24, O is the center of a circle. Find the measure of $\angle ABC$



<ABC=1/2<AOC=1/2(74)=37

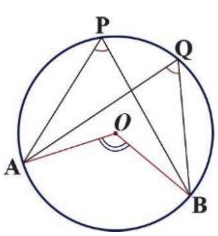
2. In figure 5.25, O is the center of a circle. Find the measure of $\angle x$.



<X=240,<O=360-240=120

Central Angles and Inscribed Angles (2)

Inscribed angles subtended by the same arc have the same measure.



m($\angle A$PB) = 1/2 m($\angle A$OB) (By theorem 5.6)

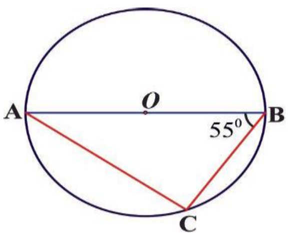
m($\angle A$QB) = 1/2 m($\angle A$OB) (By theorem 5.6)

Therefore, m($\angle A$PB) = m($\angle A$QB).

Angle in a semicircle (Thales' Theorem) An angle inscribed in a semicircle is a right angle.

Example

If AB is the diameter of the circle with center O as shown in the figure 5.29, then find the measure of $\angle BAC$.



Solution:

By Thales' Theorem m($\angle ACB$) = 90°.

By angle sum theorem we have,

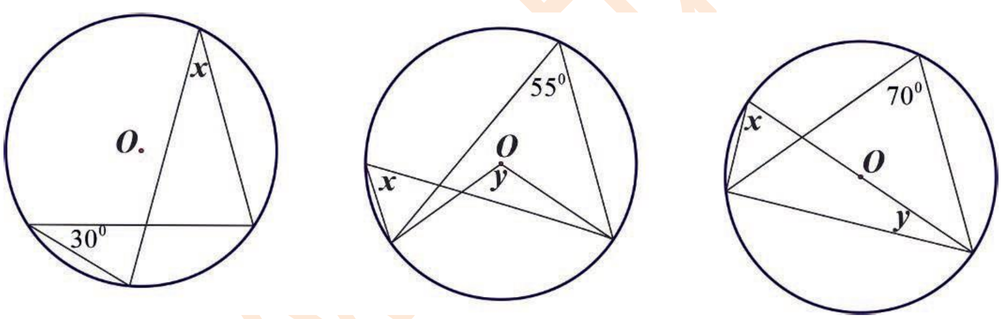
m($\angle BAC$) + m($\angle ACB$) + m($\angle ABC$) = 180°,

m($\angle BAC$) + 90° + 55° = 180°.

Hence m($\angle BAC$) = 35°.

Exercise

In figure below, O is the center of a circle. Find the measure of $\angle x$ and $\angle y$.



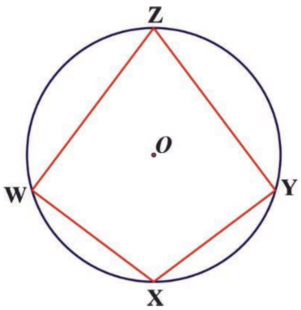
A b c

a. x = 30° b. x = 55° and y = 110° c. x = 70° and y = 20°

Cyclic Quadrilateral

A quadrilateral is said to be a cyclic quadrilateral if there is a circle passing through all its four vertices.

In a cyclic quadrilateral WXYZ in figure below, the sum of either pair of opposite angles is 180°.



Example

If the measures of all four angles of a cyclic quadrilateral are given as $(4y + 2)$, $(y + 20)$, $(5y - 2)$, and $7y$ respectively, find the value of y.

Solution:

The sum of all four angles of a cyclic quadrilateral is 360°. So, to find the value of y, we need to equate the sum of the given four angles to 360°.

$(4y + 2) + (y + 20) + (5y - 2) + 7y = 360°$

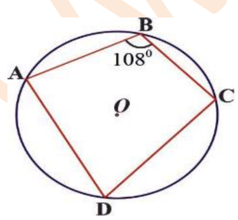
$17y + 20 = 360°$

$17y = 340°$.

Therefore, $y = 20$.

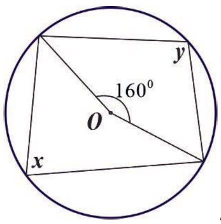
Exercise

1. In figure below, $ABCD$ *is* a cyclic quadrilateral drawn inside a circle with center O and m(∠ABC) = 108°. What is the measure of m(∠ADC)?



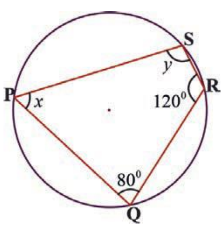
solution $m(\angle ADC) = 72°$

2. In the figure 5.33, O is the center of a circle. Find the measure of $\angle x$ and $\angle y$.



$x = 80°$ and $y = 100°$

3. Find the value of angle x and angle y as shown in figure



PQRS is inscribed in a circle, so opposite angles are supplementary by the inscribed Quadrilateral Theorem.

m(Rˆ) + x = 180°

1200 + x = 180°

x = 180° – 120° = 60°

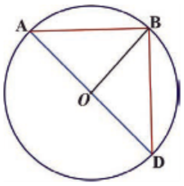
m(Ŝ) + y = 180°

180° + y = 180°

y = 180° – 80° = 100°

Therefore, the value of angle x is 60° and the value of angle y is 100

4. Given circle with center O, as shown in the figure ,



a. Name one minor arc in the circle.

b. Name one major arc in the circle.

c. Name the angle subtended by arc BD.

d. Name the inscribed angle subtended by arc BD.

e. Name an angle in a semicircle.

f. Two angles subtended by chord AB.

solution

a. Minor arc AB b. arc ADB c. $\angle BOD$ d. $\angle BAD$ e. $\angle ABD$ f. $\angle AOB$ and $\angle ADB$

5.3 Arc Lengths, Perimeters and Areas of Segments and Sectors

Length of Arc and Chord

Arc length is an important aspect to understand portions of curved lengths. As you will learn in this lesson, combining our knowledge of circumference and central angle measures, we will find arc length.

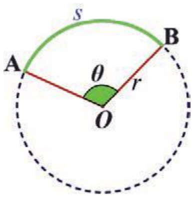
Arc length is the length of an arc which is a portion of the circumference of a circle.

For a circle of radius r subtended by an angle θ, the length s of the corresponding arc is:

$s = 2\pi r \times \theta$

3600 or $\pi d \times \theta$ 3600

where d is the diameter of a circle (see figure).



Find the arc length that a central angle of 150° subtends in a circle of radius 6 cm as shown in figure .

Solution:

The length s of the corresponding arc is

$s = 2\pi r \times \theta$

3600

Given: $\theta = 150°$, $r = 6$cm.

arc length= $s = 2\pi r \times \theta$

$3600 = 2\pi \times 6 \times 150°$

$360° = 5\pi$ cm

Length of a chord

Let the midpoint of AB be M, and the length of the chord AB and AM be l and m respectively. Using trigonometric ratios on the right-angled triangle OAM,

$m = r\sin\ (\theta/2)$

Then, $l = 2m = 2r\sin\ (\theta/2)$

In general, the length of a chord is defined as $l = 2r\sin\ (\theta/2)$, where r is the radius of the circle and θ is the angle subtended at the center by the chord.

1. Find the chord length AB if $\theta = 120°$ when a central angle subtended in a circle of radius 8cm .

Solution:

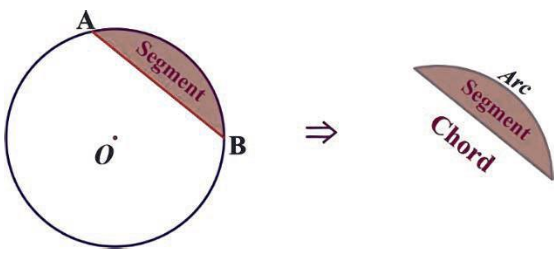
Chord length = $2r \sin(\theta/2) = 2 \times 8\text{cm} \times \sin(120°/2)$

$= 16\text{cm} \times \sin(60°)$

$= 16\text{cm} \times \sqrt{3}/2 = 8\sqrt{3}\text{cm}$

Perimeter of a segment

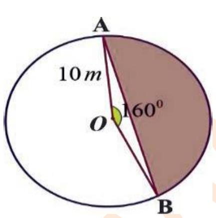
A segment is part of a circle bounded in between a chord and an arc of a circle.



• When a segment in a circle is bounded by a chord and an arc, the perimeter of a segment is given by: Perimeter of segment = Length of chord + Length of arc

Example

Find the perimeter of the segment as shown in figure below. (Use π.)



Solution:

Perimeter of segment = $2\pi \times 10 \times 160°/360° + 2 \times 10 \times \sin(160°/2)$

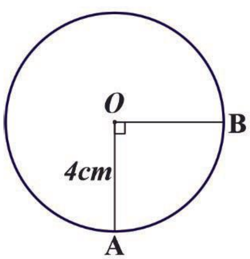
$= 20\pi \times 4/9 + 20 \times \sin(80°)$

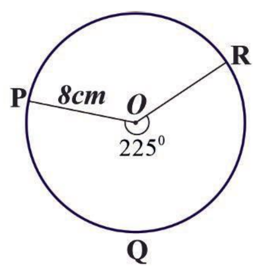
$= 80/9\pi + 20 \times 0.985$

$= (80/9\pi + 19.7)$ m

Exercise

1. Find the length of arc AB and arc PQR. (Use π.)

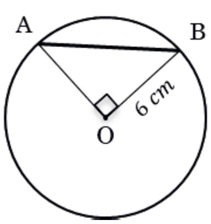




a. arc $AB = \pi r\theta/180° = \pi(4cm)90°/180° = 2\pi$cm

b. arc $PQR = \pi(8cm)225°/180° = 10\pi$cm

2. Find the length of chords AB and CD.



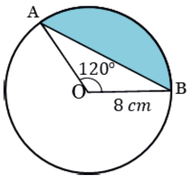
Chord length AB = 2r sin (θ/2) = 2 × 6cm × sin (90°/2)

= 12cm × sin(45°) = 12cm × √2/2 = 6√2cm.

Chord length CD = 2r sin (θ/2) = 2 × 10cm × sin (60°/2)

= 20cm × sin(30°) = 20cm × 1/2 = 10cm

3. Find the perimeter of the shaded segment with radius 8cm. (Use π.)



Perimeter of shaded segment = $2\pi \times 8 \times 120°/360° + 2 \times 8 \times \sin(120°/2)$

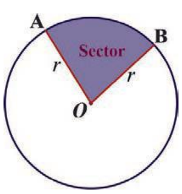
$= 16\pi \times 1/3 + 16 \times \sin(60°)$

$= 16/3\pi + 16 \times \sqrt{3}/2$

$= (16/3\pi + 8\sqrt{3})\ cm^2$.

Area and Perimeter of Sector

The sector is basically a portion of a circle enclosed by two radii and an arc. It divides the circle into **two** regions, namely major and minor Sector



Area of a Sector

Consider a circle of radius r, centre O, and m$(\angle POQ) = x = \theta$ (in degrees) as shown in figure . The area of a sector is given by:

$Asector = \pi r^2\ (\theta/360°)$

Perimeter of a Sector

Perimeter of sector = 2 radius + arc length.

Therefore, the perimeter of sector $= 2r + 2\pi r \times \theta/360°$

Example 1

If the angle of the sector with radius 6 cm is 210°, then find the area and perimeter of the sector. (Use π.)

Solution:

Given: $r = 6$ cm, $\theta = 210°$

$Asector = \pi r^2\ (\theta/360°)$

$= \pi \times (6)^2 \times (210°/360°)$

$= \pi \times 36 \times 7/12$

$= 21\pi$ cm^2

$Psec = 2r + 2\pi r \times \theta/360°$

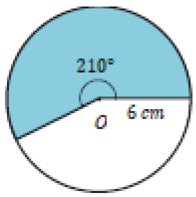
$= 2 \cdot 6 + 2\pi \cdot 6 \times 210°/360°$

$= 12 + 7\pi$

Area of a Segment

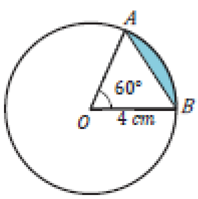
An arc and two radii of a circle form a sector. These two radii and the chord of the segment together form a triangle. Thus, the area of a segment of a circle is obtained by subtracting the area of the triangle from the area of the sector. i.e., Area of a segment of circle = area of the sector – area of the triangle.

$A = \pi r^2 \times \theta/360° - 1/2r^2\sin\theta$



Example

Find the area of the shaded region as shown in the figure below .



Solution:

Area of segment: $A = \pi r^2 \times \theta/360° - 1/2r^2\sin\theta$

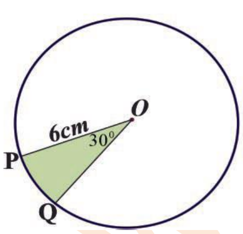
$= \pi \cdot 4^2 \times 60°/360° - 1/2 \times 4^2 \times \sin60°$

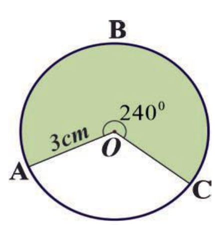
$= 16 \cdot 1/6\pi - 1/2 \cdot 16 \cdot \sqrt{3}/2$

$= 8/3\pi - 4\sqrt{3}$

Exercise

1. Find the area and perimeter of the shaded sectors (use π).





a. $Asector = \pi r^2 \;(\theta/360°)$

$= \pi \times (6\text{cm})^2 \times (30°/360°)$

$= \pi \times 36cm^2 \times 1/12$

$= 3\pi\ cm^2$.

$Psector = 2r + 2\pi r \times \theta/360°$

$= 2 \times 6cm + 2 \times \pi \times 6cm \times (30°/360°)$

$= 12cm + 2 \times \pi \times 6cm \times 1/12$

$= (12 + \pi)$ cm.

b. $Asector = \pi r^2\ (\theta/360°)$

$= \pi \times (3cm)^2 \times (240°/360°)$

$= \pi \times 9cm^2 \times 2/3 = 6\pi cm^2$.

$Psector = 2r + 2\pi r \times \theta/360°$

$= 2 \times 3cm + 2 \times \pi \times 3cm \times (240°/360°)$

$= 12cm + 2 \times \pi \times 6cm \times 2/3 = (12 + 8\pi)$ cm.

5.4 Theorems on Angles and Arcs Determined by Lines Intersecting inside, on and outside a Circle.

Angles Formed by Chords

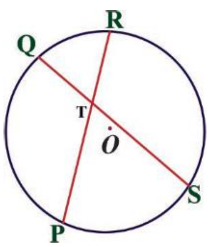
If two chords intersect inside a circle, then the measure of an angle formed between the chords is one half the sum of the measure of the arcs intercepted by the angle and its vertical angle.

In the circle, as shown in figure below, the two chords PR and QS intersect inside the

circle at point T.

m(∠PTQ) = 1/2 [m(arcPQ) + m(arcRS)]

and m(∠QTR) = 1/2 [m(arcQR) + m(arc(PS)]



Example

In the circle shown in figure below if m(arcPQ) = 68° and m(arcRS) = 128°, then find measure of ∠RTS, where $PR$and QS intersects at T.

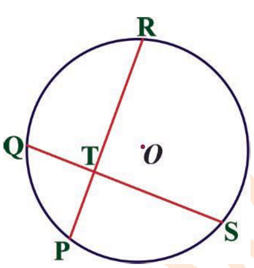
Solution:

m(∠RTS) = 1/2 [m(arcPQ) + m(arcRS)]

= 1/2 (68° + 128°)

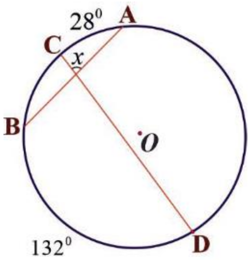
= 1/2 × 196° = 98°

Therefore, m(∠RTS) = 98°



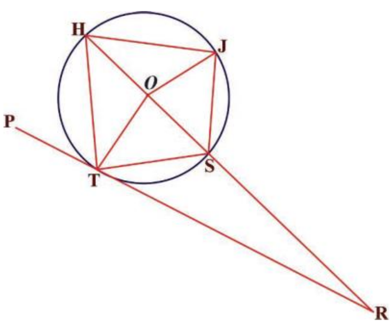
Exercise

1. In figure below, if m(arcAC) = 28° and m($arcBD$) = 132°, find x.



<x=1/2(132+28)=80

2. Given a circle with center O, as shown in figure below. If m(arc HT) = m(arc JH) and m(∠PTH) = 58^0, calculate the measure of the remaining angles ∠PRH, ∠THS, ∠TSH, ∠HTS and minor arc HT, minor arc TS.



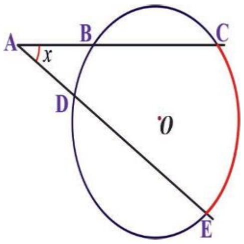
Arc TH=116

ArcTJ=360-232=28

Angles Formed by Secants and Tangents

The measure of the angle formed by two secants, two tangents, or a secant and a tangent that intersect at a point outside a circle is equal to one-half the positive difference of the measures of the intercepted arcs.

In figure below, we illustrate this result for the angle formed by the intersection of two secants, $AC$and AE.



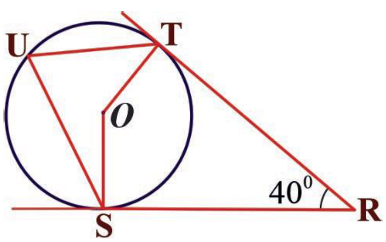
The minor arc intercepted by the two secants is BD and an arc CE. Hence, by the

theorem of angles between intersecting secants,

x = 1/2 [m(arcCE) – m(arcBD)]

Exercise

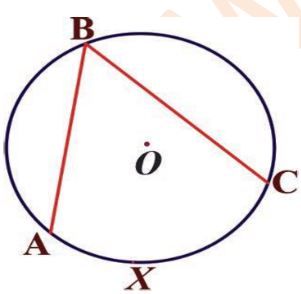
1. In figure below, $R\ S$ and RT are tangent lines to the circle with center O. If ∠SRT) = 40°, what is the m(∠TUS)?



m(∠SRT) + m(∠RTO) + m(∠OSR) + m(∠TOS) = 360°

40° + 90° + 90°+ m(TÔ S) = 360°. So, m(∠TOS) = 140° = m(arcST),

2. Theorem: The measure of an angle inscribed in a circle is half of the measure of the arc subtending. Prove it (see figure below).m(∠TUS) =1/2m(arcST) = 70°.



Proof: Given: circle O with B ˆ an inscribed angle intercepting arc AC.

To prove: m(∠ABC) = 1/2

$m(arcAXC) = 1/2m(\angle AOC)$, where x is a point as shown in the figure.

We consider three cases.

Case:1. Suppose that side of AB ˆ C is a diameter of the circle with centre O.

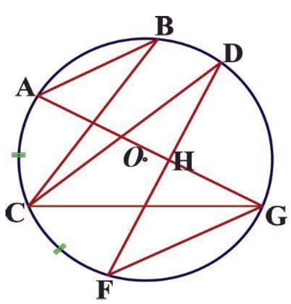
Review Exercise

1. In the figure below, if m (arc AC) = m (arc CF), m (arc AB) = 46°, m($\angle AGF$) = 58°, m($\angle BCD$) = 10° and m ($\angle GCF$) = 35°.

Calculate: a. m($\angle ABC$)

b. m($arcDG$)

c. m($\angle DHG$)



a. m($\angle ABC$) = 29° ,

b. Given: m(arcAB) = 46°, m(arcACF) = 2 × 58° = 116° , m(arcFG) = 2 × 35° = 70°

and m(arcBD) = 2 ×10° = 20°.

Hence, m(arcDG) = 360° – (46° + 116° + 70° + 20°)

= 360° – 252°

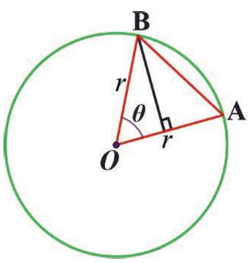
m(arcDG) = 108°

c. m(∠DHG) = 1/2

[m(arcACF) + m(arcDG)] = 1/2

(116° + 108°) = 112°

2. Find the area of the segment shown in figure if the central angle is 0.6 rad and the radius is 10 m.



First, we find the area of the sector.

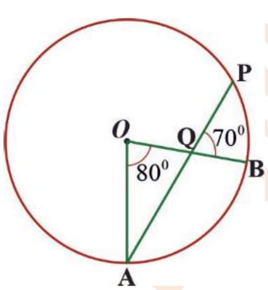
$Asec = (\pi r^2)\ (\theta/2\pi) = 100 \times 0.3 = 30\text{m}^2$,

$\sin(0.6) = h/r = h/10$. So, $h = 10\sin\ (0.6) = 5.65$

and $a(\Delta AOB) = hr/2 = 10 \times 5.65/2 = 28.25\text{m}^2$

therefore, the area of the segment is $30\text{m}^2 - 28.25\text{m}^2 = 1.75\text{m}^2$

3.In figure below, O is the Centre of a circle, m(∠ AOB) = 80° and m(∠ PQB) = 70°. Find m(∠ PBO)



Given: Central angle AOB whose measure is 80° and m(∠POB) = 70°.

Now, consider ΔPQB; m(∠P) + m(∠Q) + m(∠B) = 180° (Sum of interior angles of a triangle is 180°).

40° +70° +m(∠B) = 180° ($\angle AO$B and $\angle AP$B are intercepted by the same arc).

m(∠B) = 180° – 110° = 70°.

Therefore, m(∠PBO) = 70°.

UNIT 6

SOLID FIGURES

6.1 Revision of Prisms and Cylinders

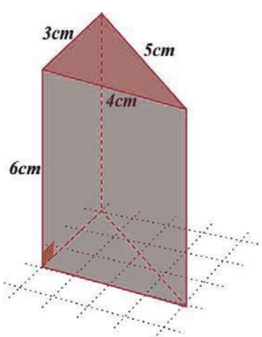
If LSA denotes the lateral surface area, BA denotes the base area and TSA denotes the total surface area of a right prism, then LSA = ph, where p represents the perimeter of the base region and h represents the altitude of the prism, TSA = LSA + 2BA.

The volume $V$of any prism equals the product of its base area BA and altitude h.

That is, V = BA·h

Example

Find the total surface area and the volume of the right triangular prism shown in figure below



LSA = ph = (4 + 3 + 5) × 6 = 72 cm$^{2.}$

Since the triangle has sides of lengths 3 cm, 4 cm and 5 cm it is a right-angled triangle with legs 3 cm and 4cm and hypotenuse 5 cm.

Observe also that $3^2 + 4^2 = 5^2$.

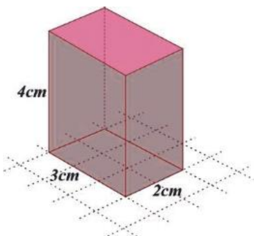
Therefore,

BA = $1/2ab = 1/2 \times 3 \times 4 = 6$ cm^2,

TSA = LSA + 2BA = 84 cm^2. Volume V = BA $\cdot h = 6 \times 6 = 36$ cm^3.

Example

Find the total surface area and volume of the rectangular prism shown in figure below.



Solution:

LSA = $ph = (2 \times 2 + 2 \times 3) \times 4 = 40$ cm^2,

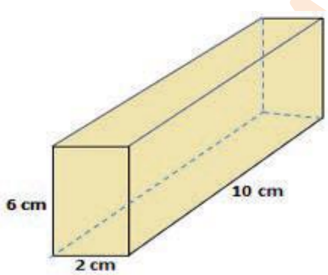
BA = $lw = 3 \times 2 = 6$ cm^2,

TSA = LSA + 2BA = 52 cm^2.

Volume V = BA $\cdot h = 6 \times 4 = 24$ cm^3.

Exercise

1. Find the total surface area and volume of the following solid figures



a. BA = $2 \times 6 = 12$ cm^2, LSA = $(2 + 6 + 2 + 6)10 = 160$ cm^2, TSA $= LSA + 2BA = 184$ cm^2

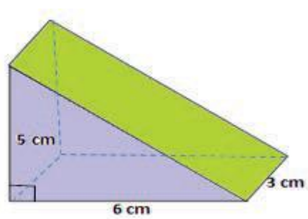
There is also another alternative,

BA = 2 × 10 = 20 cm^2, LSA = (2 + 10 + 2 + 10)6 = 144 cm^2,

TSA $= LSA + 2BA = 184$ cm^2

Volume $= BA \times h = 20 \times 6 = 120$ cm^3

B



BA = 1/2 × 5 × 6 = 15 cm^2, LSA = (5 + 6 + √61) × 3 = 3(11 + √61) cm^2

TSA $= LSA + 2BA = 3(11 + \sqrt{61}) + 30 = 3(21 + \sqrt{61})$ cm^2

Volume $= BA \times h = 15 \times 3 = 45$ cm^3

2. The base of a right prism is an equilateral triangle with a side of 4 cm and its

BA $= 1/2absin\theta = 1$

$2 \times 4 \times 4sin60° = 4\sqrt{3}$ cm^2

LSA $= ph = (4 + 4 + 4) \times 10 = 120$ cm^2

TSA = LSA + 2BA = 120 + 8√3 = 8(15 + √3) cm^2

Volume $= BA \times h = 4\sqrt{3} \times 10 = 40\sqrt{3}$ cm^{3}height is 10 cm. Find its total surface area and volume.

3. Find the perimeter of the base of a right prism for which the area of the lateral

Surface is 120 cm^2 and for which the altitude is 5 cm. LSA $= ph$ $120 = p \times 5$ implies $p = 24$ cm

The lateral surface area of a right circular cylinder is the product of the circumference of the base and altitude of the cylinder.

That is, LSA $= 2\pi rh$ where r is the radius of the base and h is the altitude of the cylinder.

The total surface area is the sum of the areas of the bases and the lateral surface area. That is,

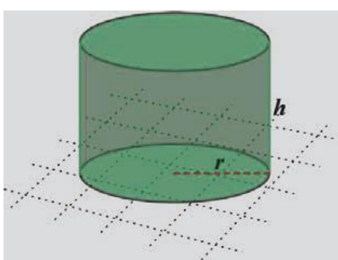
TSA = LSA + 2BA

TSA $= 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$

The volume V of a circular cylinder is the product of its base area BA and altitude h. That is,

V = BA · h

V $= \pi r^2 h$, where r is the radius of the base.



Example 4

Find the total surface area and volume of a right circular cylinder whose radius is 3cm and whose altitude is 5 cm.

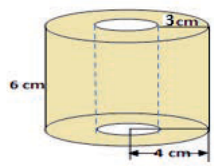
Solution:

TSA = LSA + 2BA $= 2\pi r(h + r) = (2\pi \times 3)(5 + 3) = 48\pi$ cm^2.

V $= \pi r^2 h = 45\pi$ cm^3.

Example 5

Find the total surface area and volume of the solid figure in figure below.



Solution:

The bases are annulus regions with inner radius 1cm and outer radius 4cm. Let the inner radius be r and the outer radius be R. Then, BA = Annulus area = $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$

$= \pi(4^2 - 1^2) = 15\pi$ cm^2.

Outer LSA= $2\pi Rh = 2\pi \times 4 \times 6 = 48\pi$ cm^2.

Inner LSA = $2\pi rh = 2\pi \times 1 \times 6 = 12\pi$ cm^2.

TSA = LSA + 2BA = $60\pi + 30\pi = 90\pi$ cm^2.

Volume = BA $\cdot h = 15\pi \times 6 = 90\pi$ cm^3.

Exercise

1. The radius of the base of a right circular cylinder is 4 cm and its altitude is 10 cm.

Find the lateral surface area, the total surface area and the volume of the cylinder.

LSA = 80π TSA = 112π cm^2 and V = 160π cm^3.

2. The diameter of the base of a right circular cylinder is 6 cm and its altitude is 8 cm. Find the lateral diameter = 6 cm implies $r = 3$ cm, LSA = $2\pi rh = 48\pi$,

TSA = LSA + 2BA = $2\pi r(h + r)$ = 66 cm^2 and V = $\pi r^2 h = 72\pi$ cm^3.surface area, the total surface area and the volume of the cylinder.

3. A circular hole of radius 5 cm is drilled through the center of a right circular cylinder whose base has radius 7 cm and whose altitude is 8 cm. Find the total surface area and volume of the resulting solid.

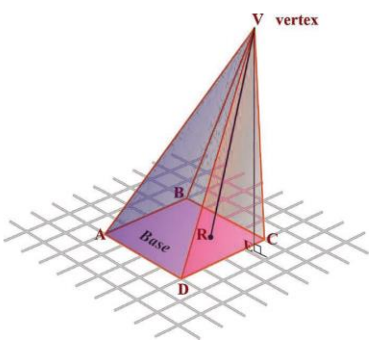
BA = $\pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(7^2 - 5^2) = 24\pi$ cm^2.

LSA = $2\pi h(r + R) = 192\pi$ cm^2

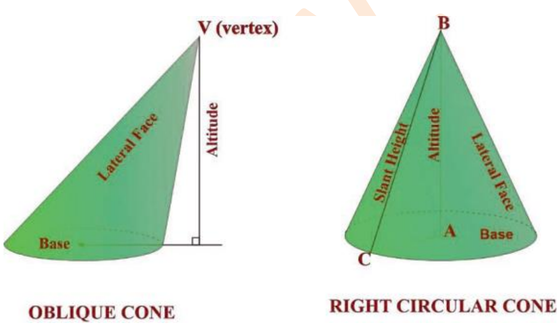
TSA = LSA + 2BA = 240π cm^2, Volume = BA. $h = 192\pi$ cm^3

6.2 Pyramids, Cones and Spheres

A pyramid is a solid figure defined by a polygonal base and a point called an apex (vertex) not on the base. It is formed when each point of the polygonal base is joined with the vertex.



A pyramid whose base is a circular region is called a circular cone



A right circular cone is a cone with a foot of its altitude at the center of the basecircle.

The line segment from the vertex to a point on the boundary of the base circle is called the slant height of the cone

Exercise

1. Determine whether each of the following statements is true or false.

a. All lateral edges of a pyramid are equal in length.

b. All lateral edges of a regular pyramid are equal in length.

c. The length of the slant height of a right circular cone is greater than the length of its altitude.

d. We can take any face of a triangular pyramid as its base.

e. All faces of a regular pyramid are congruent.

f. All lateral faces of a regular pyramid are congruent.

a. False

b. True

c. True

d. True

e. False

f. True

Surface Area of Pyramids and Cones

If LSA denotes lateral surface area, TSA denotes total surface area and BA denotes base area then

LSA = sum of areas of the lateral faces

TSA = LSA + BA

A geometry net is a two-dimensional shape that can be folded to form a three dimensional shape or a solid. When the surface of a three-dimensional figure is laid out flat showing each face of the figure, the pattern obtained is called the net of the figure.

To find the formula for the lateral surface area of a regular pyramid in terms of its base perimeter and its slant height, consider the regular pentagonal pyramid and its net

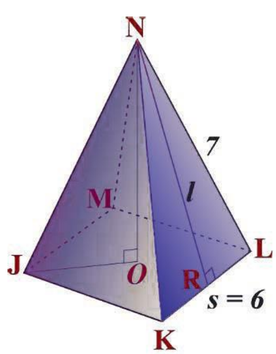
The base is a square. Therefore BA = $s^2 = 4^2 = 16$ cm^2.

TSA = LSA + BA = 40 + 16 = 56 cm^2

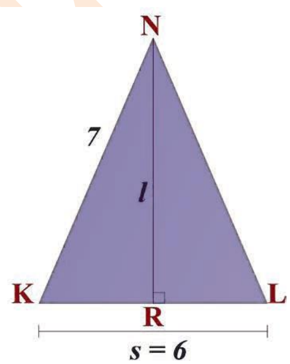
Example 2

A regular square pyramid has a base edge 6 cm and lateral edge 7 cm. Find its lateral

and total surface areas.



(a) (b



)

Figure

ΔNKL in figure 6.18b is one lateral face of the pyramid in Figure 6.18a.

The lateral edgeNK = 7 cm, base edge s = KL = 6 cm and KR = 3 cm. The slant height l = NO of the pyramid is the altitude of ∆NKL.

Since ∆NRK is a right-angled triangle,

$l^2 + (KR)^2 = (NK)^2$

$l^2 + 3^2 = 7^2$

$l^2 = 7^2 - 3^2 = 40$

$l = 2\sqrt{10}$

LSA = 1/2

pl = 1/2 × 24 × 2√10 = 24√10 cm2

BA = $s^2 = 6^2 = 36$ cm^2

TSA = LSA + BA = 24√10 + 36 = 12(2√10 + 3) cm^2

Exercise

1. Find the lateral and the total surface area of the regular hexagonal pyramid of 6cm height and a length of 4cm on one side of the base.

The base has 4 sides therefore, $n = 4$ and base perimeter

$p = ns = 4 \times 6 = 24$ cm.

LSA = $1/2pl$ = 1/2 × 24 × 8 = 96 cm^2.

The base is a square. Therefore BA = $s^2 = 6^2 = 36$ cm^2.

TSA = LSA + BA = 96 cm^2 + 36 cm^2 = 132 cm^2.

2. A right pyramid of 3m height has a square base whose diagonal is 6m. Find its lateral and total

surface area.

$h = 3$ m, square base with diagonal, $d = 6$ implies

The radius of the square $= r = d/2 = 3$.

$s^2 = r^2 + r^2 = 2r^2 = 18$ implies, length of side of the square $= s = 3\sqrt{2}$,

BA $= 1/2nr^2 \sin(3600/n) = 18$ m^2 or BA $= s^2 = (3\sqrt{2})^2 = 18$ m^2.

$r^2 = a^2 + (s/2)^2$, $9 = a2 + 9/2$ implies, Apothem $= a = 3\sqrt{2}/2$.

$h^2 + a^2 = l^2$, slant height $= l = 3\sqrt{6}/2$.

LSA $= 1/2pl = 1/2 \times 12\sqrt{2} \times 3\sqrt{6}/2 = 18\sqrt{3}$ m^2.

The lateral surface area of a right circular cone is:

LSA $= \pi rl$ or LSA $=1/2pl$, where $p = 2\pi r$

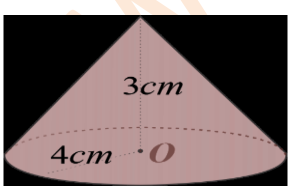
$l = \sqrt{h^2 + r^2}$, where l is the slant height, h is the height (altitude), r is the base radius and p is perimeter or circumference.

The total surface area is the sum of the area of the base and the lateral surface area. That is,

TSA $=$ LSA $+$ BA $= \pi rl + \pi r^2 = \pi r(l + r)$.

Example

A right circular cone has a base diameter 8cm and height 3cm, see Figure below. Find its LSA and TSA.



Solution:

$l = \sqrt{h^2 + r^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5cm,$

$LSA = \pi rl = \pi(4)(5) = 20\pi\ cm^2,$

$BA = \pi r^2 = 16\pi cm^2,$

$TSA = LSA + BA = 36\pi\ cm^2.$

Exercise

1. Find the total surface area of a cone of radius 6cm and height of 12cm.

$r = 6$ cm, $h = 12$ cm

$l = \sqrt{h^2 + r^2} = \sqrt{144 + 36} = \sqrt{180} = 6\sqrt{5}$ cm

LSA $= \pi rl = \pi \times 6 \times 6\sqrt{5} = 36\sqrt{5}\pi$ cm^2, $BA = \pi r^2 = 36\pi$ cm^2 and

TSA $= 36\sqrt{5}\pi + 36\pi = 36\pi(\sqrt{5} + 1)$ cm^2.

2. The area of the total surface of a cone is $64m2$ and its slant height is 5 times the radius of the base. Find the radius of the base.

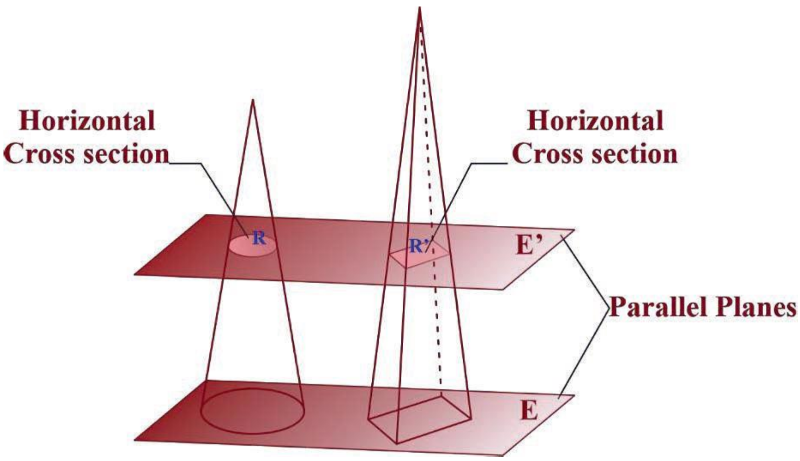
3. A conical tent is 6m high and the radius of its base is 8m. Find

a. slant height of the tent.

b. cost required to make the tent, if the cost of 1 m2 canvas is 250 birr. Use $\pi = 3.14$.

Horizontal cross-section of pyramids and cones

If a pyramid or a cone is cut by a plane parallel to the plane containing the base, the intersection of the plane and the pyramid (or the cone) is called a horizontal cross-section of the pyramid (or the cone).



Every horizontal cross-section of a triangular pyramid is a triangular region similar to the base.

Let h be the altitude of a triangular pyramid and let k be the distance from the vertex to a horizontal cross-section. Then the ratio of the area of the cross section to the area of the base is k^2/h^2.

Example 1

The area of the base of a triangular pyramid is 270 cm2. The altitude of the pyramid is 6 cm. Find the area of the horizontal cross-section of the pyramid 4 cm from the vertex.

Solution:

h = 6 cm, k = 4 cm and base area of the pyramid = 270 cm^2 area of cross section

base area=k^2/h^2

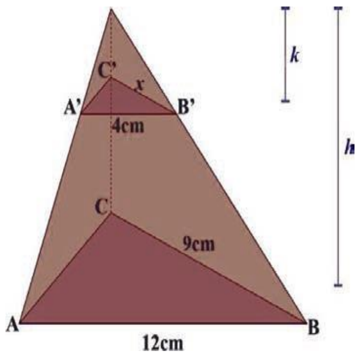
area of cross section 270=42/62 = $(2/3)^2$ =4/9

Area of the cross section = 270 × 4/9 = 120 cm^2.

Exercise

1. In the triangular pyramid shown in Figure below, ΔA'B'C' is a horizontal crosssection.

Find the values of x and k/h.



Since ΔABC and ΔA'B'C' are similar,

$A'B'/AB = B'C'/BC$

That is, $4/12 = x/9$

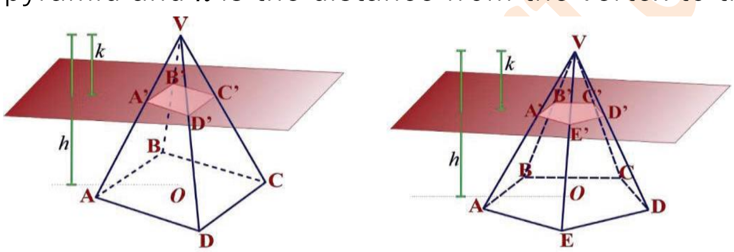
Solving for x we have $x = 3$.

$A'B'/AB = k/h$ implies $k/h = 1/3$

2. The area of the base of a triangular pyramid is 100 cm². The altitude of the pyramid is 5 cm. Find the area of the horizontal cross-section of the pyramid 2 cm from the vertex

h = 5 cm, k = 2 cm and base area of the pyramid = 100 cm^2 area of cross section/base are $= k^2/h^2$
area of cross section/100 = $2^2/5^2 = 4/25$
Area of the cross section = 100 × 4/25 = 16 cm^2

In any pyramid the ratio of a cross-section to the area of the base is k^2/h^2 where h is the altitude of the pyramid and k is the distance from the vertex to the plane of the cross section.



$$\frac{Area(A'B'C'D')}{Area(ABCD)} = \frac{k^2}{h^2} \qquad \frac{Area(A'B'C'D'E')}{Area(ABCDE)} = \frac{k^2}{h^2}$$

a *b*

Example

The altitude of a square pyramid is 10 cm and a side of the base is 5 cm long. Find the area of the horizontal cross section of the pyramid 3 cm from the vertex.

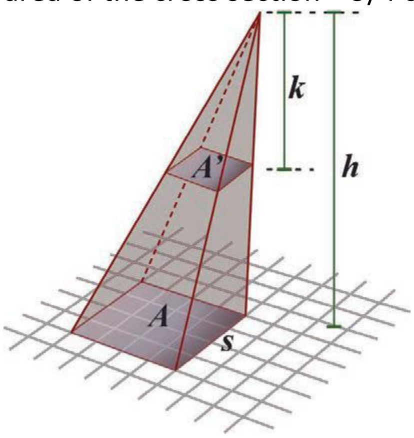
Solution:

The base is a square of side $s = 5\ cm$, its area is $A = s^2 = 25\ \text{cm}^2$,

$\frac{Area\ of\ cross\ section}{base\ area} = \frac{\text{k2}}{\text{h2}}$

area of cross section $/25 = 3^2/10^2$

area of the cross section = $9/4\ \text{cm}^2$



If two pyramids have the same base area and the same altitude then cross sections equidistant from the vertices have the same area.

Exercise

1. The base of a pyramid is a rectangle with sides 6 cm and 4 cm. If the altitude of the pyramid is 12 cm, find the area of the horizontal cross-section of the pyramid 4 cm from the vertex.

The base is a rectangle of sides 4cm and 6cm, its area is $A = 4 \times 6 = 24\ \text{cm}^2$,

area of cross section/base area $= k^2/h^2$

area of cross section/24 = $4^2/12^2$

area of the cross section = $8/3\ \text{cm}^2$

2. The area of the horizontal cross-section of a pyramid at a distance 6 cm from the base is 90 cm^2. If the area of the base of the pyramid is 160 cm^2, find its altitude.

BA = A = 160 cm^2, Cross-section area = $A' = 90\ \text{cm}^2$,

Distance from the base to the cross-section = 6 cm.

Let the altitude of the pyramid be h,

The distance from the vertex to the cross-section, $k = h - 6$ and

$A'/A = k^2/h^2$ implies $90/160 = (h-6)^2$

h^2 implies $3/4 = h-6/h$,

$3 \times h = 4 \times (h - 6)$

$3h = 4h - 24$ Hence, $h = 24$ cm.

3. The altitude of a regular hexagonal pyramid is 9 cm and the side of the base is 3 cm. What is the area of a horizontal cross-section at a distance of 5 cm from the base?

For a regular hexagon length of side = s = r = radius of the regular hexagon.

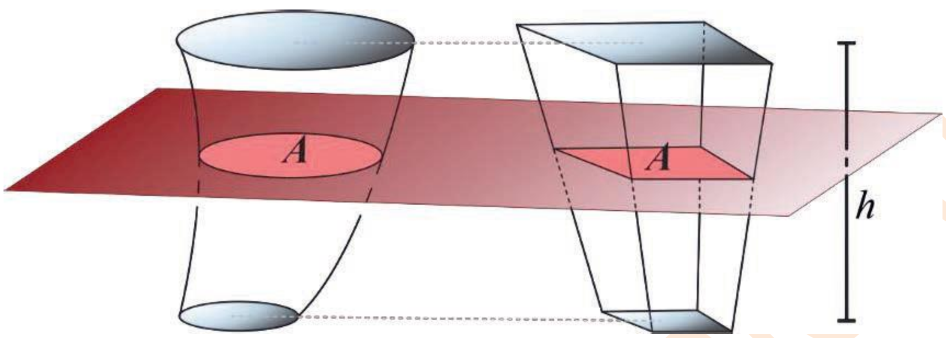
Therefore, r = 3 cm.

Base area A = $1/2nr^2$ sin (3600/60) = 27√3/2 cm2.

$A'/A = k^2/h^2$ implies $\frac{A'}{27\sqrt{3}/2}$ = 16/81 implies A' = 8√3/3 cm^2

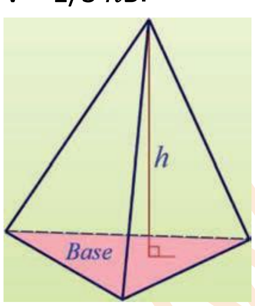
6.3 Volume of Pyramids and Cones

CAVALIERI'S PRINCIPLE: If two solids of equal height have equal cross-sectional areas at every level parallel to the respective bases, then the two solids have equal volume.

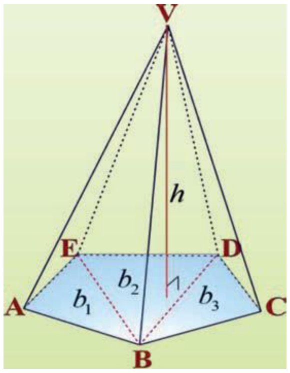


If two pyramids have equal altitudes and equal base areas, then their volumes are equal.

The volume of a triangular pyramid is one- third of the product of the height and the base area, that is, V = 1/3 hB.



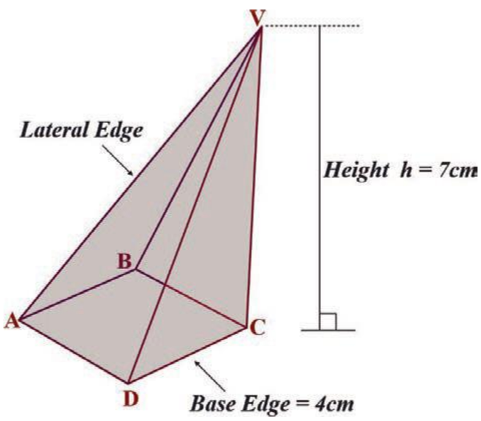
The volume of a pyramid is one third of the product of its altitude and its base area.



The volume V of a circular cone with altitude h and base radius r is,

V = $1/3\pi r^2 h$.

A regular square pyramid has a base edge of length 4 cm and altitude 7 cm. Find its volume.



Solution:

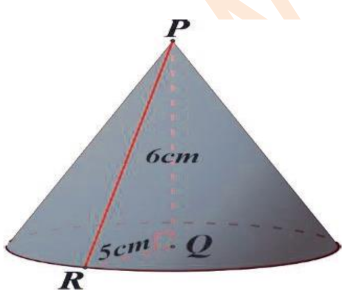
The base of the pyramid is a square with base edge of length 4 cm.

BA = 4 × 4 = 16 cm^2

V = 1/3(BA)h = 1/3 × 16 × 7 = 112/3 cm^3

Example

A circular cone has a base radius 5 cm and altitude 6 cm. What is its volume?



Solution:

V =1/3 $\pi r 2h$ = 1/3π(5^2)(6)

= 50π cm^3

Exercise

1. The altitude of a regular square pyramid is 6 cm. If one edge of the base has length 4 cm then find its volume.

The base is a square of side = s = 4 cm implies $BA = s^2$ = 16 cm^2

h = 6 cm. volume, V = 1/3$BA \times h$ = 1/3 × 16 × 6 = 32 cm^3.

2. A circular cone has an altitude 12 cm and a base radius 10 cm. What is its volume?

V = 1/3$\pi r^2 h$ = 1/3π(10^2)(12) = 400π cm^3

3. A right circular cone has height 10 cm and circumference of the base is 12π cm.

Find its volume. Circumferance = $c = 2\pi r = 12\pi$ implies r = 6 cm,

Volume, V = 1/3$\pi r^2 h$ = 1/3π × 36 × 10 = 120π cm^3.

4. The lateral edge of a regular tetrahedron is 6 cm. Find its total surface area and its volume.

All the four faces of a regular tetrahedron are equilateral triangles.

Area of one face = 1/2$s^2 sin600$ = 9√3.

TSA = 4(Area of one face) = 4 × 9√3 = 36√3 cm^2.

The altitude and the median of an equilateral triangle are the same.

The center of an equilateral triangle is the point where its altitudes intersect.

$\overline{AO}$ =2/3$\overline{AN}$ or $\overline{ON}$ =1/3$\overline{AN}$

$(AN)^2 = (AC)^2 - (CN)^2$ = 36 − 9 = 27

implies, height of ΔABC = $\overline{AN}$ = 3√3 cm.

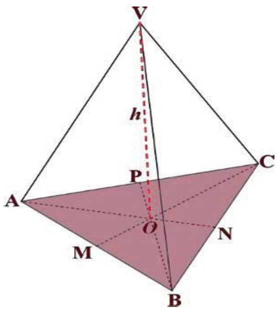
$\overline{AO}$ = 2/3 $\overline{AN}$ = 2/3 × 3√3 = 2√3 cm.

$(VO)^2 + (AO)^2 = (VA)^2$,

$h^2 + (2\sqrt{3})^2 = 6^2$,

Altitude of the tetrahedron = h = 2√6 cm.

Volume = 1/3 BA × h = 1/3 × 9√3 × 2√6 = 18√2 cm^3.



5. A right circular conical vessel of altitude 20 cm and base radius 10 cm is kept with its vertex downwards. If one liter of water is poured into it, how high above the vertex will the level of the water be? Use π = 3.14

1 liter = 1000 cm^3

$A = \pi r^2 = 100\pi$ cm^2

vlome of water = 1000 = $1/3\pi r'^2 k$, where r' is the radius of the water level and k is the height of the water.

$r'^2 = 3000\ \pi k$.

Area of the water level = $A' = \pi r'2 = 3000k$.

$A'/A = k2/h2$ implies

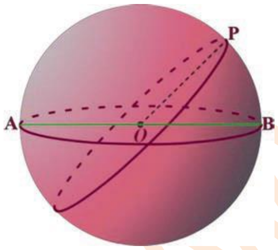
$300\ 0/k/100\pi = k2/400$ implies $3000/100\pi k = k2/400$ implies

$k3 = 12000/\pi$,

Height of the water = k = √12000/π= 10 √12/π3 cm = 15.63 cm taking π = 3.14.

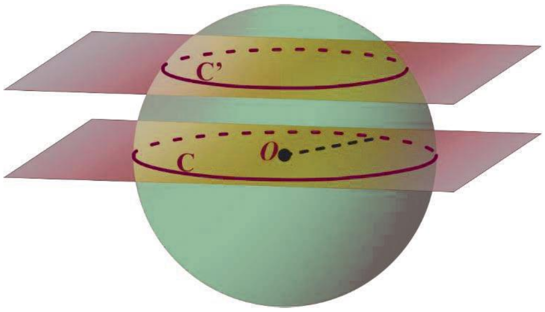
Surface Area and Volume of Sphere

A sphere is a solid bounded by a closed surface every point of which is equidistant from a fixed point called the center.



Radius of a sphere is a line segment connecting its center with any point on the sphere.

Diameter of the sphere is a line segment from the surface of the sphere passing through the center and ending at the surface.



Great and Small Circles

Every cross-section made by a plane passed through a sphere is a circle.

If the plane passes through the center of a sphere, the cross-section formed is a **great circle**; otherwise, the cross-section is a **small circle**. Clearly any plane through the center of the sphere contains a diameter. Hence all great circles of a sphere are equal and have for their common center, the center of the sphere and have for their radius, the radius of the sphere. In Figure above, C′ is a small circle and C is a great circle of the sphere.

Hemisphere

A great circle bisects the surface of a sphere. One of the two equal parts into which the sphere is divided by a great circle is called a **hemisphere**. Figure below is a hemisphere

If r is the radius, SA surface area and V volume of a sphere, then

SA = $4\pi r2$, V = $4/3\pi r3$.

Exercise

1. The radius of a sphere is 10 cm. Find its surface area and volume.

Surface area = $4\pi r^2 = 400\pi$ cm^2 , Volume = $4/3\pi r^3 = 4000/3\pi$ cm^3

2. The diameter of an iron ball is 6 cm. Find its surface area and volume

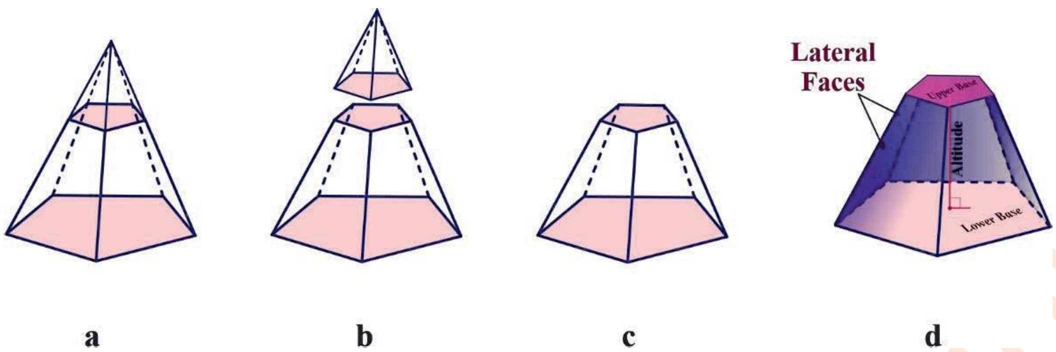
($r = d/2 = 3$ cm, A = $4\pi r^2 = 113.04$ cm^2, V = $4/3\pi r^3 = 113.04$ cm^3.use π = 3.14).

3. Find the formula for the surface area and volume of a sphere in terms of its diameter d.

A = πd^2, V = $\pi/6d^3$.

Frustum of Pyramids and Cones

A frustum of a pyramid is part of the pyramid between the base and the vertex formed when the original pyramid is cut off by a plane parallel to the plane of the base. That is, the frustum of a pyramid is part of the pyramid between the base and a cross-section of the pyramid.



The base of the pyramid and the cross section are called the bases of the frustum. The other faces are called lateral faces.

The lateral surface of the frustum is the sum of the lateral faces. The total surface is the sum of the lateral surface and the bases.

The altitude of a frustum of a pyramid is the perpendicular distance between the bases.

1. The lateral faces of a frustum of a pyramid are trapezium.
2. The lateral faces of a frustum of a regular pyramid are congruent isosceles trapeziums.
3. The slant height of a frustum of a regular pyramid is the altitude of any one of the lateral faces.
4. The lateral surface area of a frustum of a pyramid is the sum of the areas of the lateral faces

A frustum of a cone is a part of the cone between the base and the vertex formed when the original cone is cut off by a plane parallel to the plane of the base.

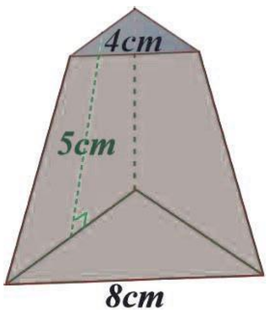
The lateral surface area (LSA) of a frustum of a regular pyramid is equal to half the product of the slant height l and the sum of the perimeter p of the lower base and perimeter p' of the upper base. That is, LSA = $1/2l(p + p')$.

Exercise

1. The lower base of the frustum of a regular pyramid is an equilateral triangle of side of length 8 cm and the upper base has a side of length 4 cm. If the slant height is 5 cm, then find

a. the lateral surface area of the frustum

b. the total surface area of the frustum



a. LSA = $1/2l(p + p') \times 3 = 1/2 \times 5(8 + 4) \times 3 = 90$ cm^2.

b. Area of an equilateral triangle of side $s = 1/2s2sin\theta$.

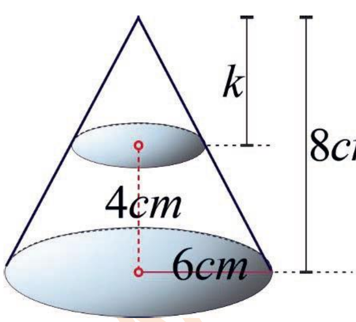
Area of the upper base = $1/2 \times 4^2 sin600 = 4\sqrt{3}$ cm^2.

Area of the upper base = $1/2 \times 8^2 sin600 = 16\sqrt{3}$ cm^2.

TA = BA + LSA = $90 + 4\sqrt{3} + 16\sqrt{3} = 10(9 + 2\sqrt{3})$ cm^2.

Example

From a right circular cone of altitude 8 cm and base radius 6 cm a frustum of height 4 cm is Formed. What is the lateral surface area of the frustum?



Solution:

In Figure above , let CA denote the cross-section area, k and h are altitudes of the smaller cone and the bigger cone, respectively. Assume also that l' and l are slant heights of the smaller and the bigger cones, respectively. Then, area of th e cross –sectio n /bas e area = CA/BA = $(k/h)^2$, $k = h - 4 = 4$ cm

$CA/36\pi = 1/4$, since BA $= \pi r^2 = 36\pi$

CA $= 36\pi/4 = 9\pi$ cm^2 . .. (∗)

CA $= \pi(r')^2$, where r' is the radius of the cross section

$9\pi = \pi(r')^2$, using CA $= 9\pi$ from (*)

$r' = 3$ cm

$l = \sqrt{r2 + h^2} = \sqrt{6^2 + 8^2} = 10$ cm and

$l' = \sqrt{r'^2 + k^2} = \sqrt{3^2 + 4^2} = 5$ cm

LSA of the smaller cone $= \pi r' l' = 3 \times 5 \times \pi = 15\pi$ cm^2.

LSA of the bigger cone $= \pi rl = 6 \times 10 \times \pi = 60\pi$ cm^2.

Hence, lateral surface area of the frustum

= (LSA of the bigger cone) - (LSA the smaller cone)

$= 60\pi$ cm^2 $- 15\pi$ cm^2 $= 45\pi$ cm^2.

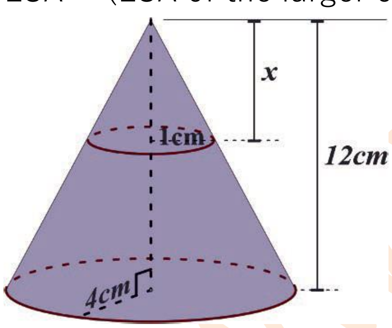
Exercise

For the right circular cone in Figure abelow,

a. Find the height of the smaller cone.

b. Find the Lateral Surface Area (LSA) of the frustum using

LSA = (LSA of the larger cone)– (LSA of the smaller cone).



a. $h'/\text{h} = r'/r$.

$x/12 = 1/4$. This implies $x = 3$. The height of the smaller cone is 3 cm.

b. Slant height of the bigger cone $= l = \sqrt{r^2 + h^2}$

$= \sqrt{4^2 + 12^2} = \sqrt{16 + 144} = 4\sqrt{10}$ cm.

LSA of the bigger cone $= \pi rl = \pi \times 4 \times 4\sqrt{10} = 16\sqrt{10}\pi$ cm^2.

Slant height of the smaller cone $= l = \sqrt{r^2 + h^2} = \sqrt{1^2 + 3^2} = \sqrt{10}$ cm.

LSA of the smaller cone = $\pi rl = \pi \times 1 \times \sqrt{10} = \sqrt{10}\pi$ cm^2.

LSA of the frustum = LSA of the bigger cone – LSA of the smaller cone.

= $(16\sqrt{10} - \sqrt{10})\pi$ cm^2 = $15\sqrt{10}\pi$ cm^2.

At this point, the formula for the lateral surface area of a frustum of a right circular cone is derived as

LSA = $l\pi(r + r')$,

where r and r' are base radii and l is the altitude of the frustum The volume of a frustum of a right circular cone is V = $1/3\pi h(r^2 + r'^2 + rr')$, where r is the radius of the bigger circle, r' is the radius of the smaller circle and h is the altitude of the frustum.

Example

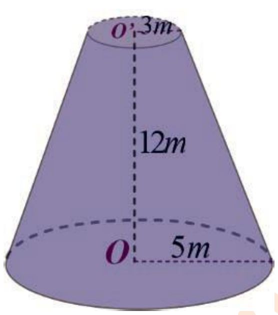
Find the volume of the frustum of a cone whose top and bottom diameters are 6 m and 10 m and the height is 12 m. (Use π=3.14)

Solution:

Since the diameter of the upper base is 6 m the radius of the upper base becomes 3 m and since the diameter of the lower base is 10 m, its radius becomes 5 m. The height is 12 m, so

Volume = $1/3\pi h(r^2 + r'^2 + rr') = 1/3\pi(12)(3^2 + 5^2 + (3)(5))$

= $196\pi = 196 \times 3.14 = 615.44$ m^3.



Exercise

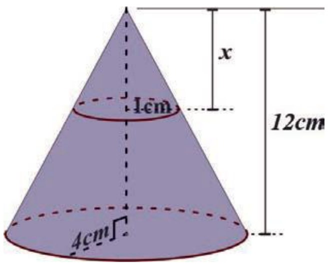
1. Given the same right circular cone as in Figure 6.46 above,

a. Calculate the volume of the larger cone.

b. Calculate the volume of the smaller cone.

c. Find the volume of the frustum by V = (volume of the larger cone) – (volume of the smaller cone).

d. Find the volume by using the formula, V = $1/3\pi h(r^2 + r'^2 + rr')$ and compare the results



$r'/4 = x/12$,

$1/4 = x/12$,

$x = 3$ cm.

a. The volume of the larger cone = 1/3
$\pi r^2 h$ = 1/3
$\pi \times 16 \times 12 = 64\pi$ cm^3.

b. The volume of the smaller cone = $1/3\pi r'^2 x$ = 1/3
$\pi \times 1 \times 3 = \pi$ cm^3.

c. Volume of the frustum V = (volume of the larger cone) – (volume of the smaller cone)
= 63π cm^3.

The volume of a frustum of a right circular cone is
V = $1/3\ h(A + A' + \sqrt{AA'})$,
Where A and A′ are base areas and h is the altitude of the cone.
The volume of a frustum of a pyramid is also, V = $1/3\ h(A + A' + \sqrt{AA'})$, where A and A′ are base areas and h is the altitude of the frustum.

Exercise

1. A right circular cone of altitude 9 cm is cut by a plane parallel to the base 7 cm from the vertex. If the area of the base of the frustum formed are 81 cm^2 and 49 cm^2 then find the volume of the frustum
Height of the frustum, h = 9 – 7 = 2 cm.

V = $1/3\ h(A1 + A2 + \sqrt{A1A2}) = 1/3 \times 2(49 + 81 + \sqrt{49 \times 81}) = 386/3$ cm^3.

2. The lower base of a frustum of a regular pyramid is a square of side of length 6 cm, and the upper base has side length 4 cm. If the slant height is 8 cm, find
a. its lateral surface area

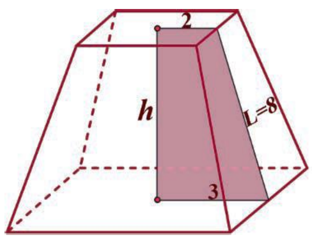
b. its total surface area

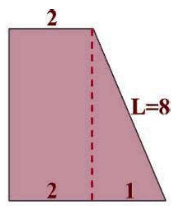
c. its volume

a. LSA = $1/2l(P + P')$, where P and P' are perimeters of the bases.

$= 1/2 \times 8 \times (24 + 16) = 160\ cm^2$.

b. BA $= 6^2 + 4^2 = 5^2\ cm^2$ and TSA = BA + LSA = $212\ cm^2$.

c. 



$h = \sqrt{82 - 12} = \sqrt{63}$ cm.

$V = 1/3h(A1 + A2 + \sqrt{A1A2}) = 1/3 \times \sqrt{63}(36 + 16 + \sqrt{36 \times 16})$

$= 76\sqrt{63}/3 = 76\sqrt{7}\ cm^3$

3. What is the lateral area of a regular pyramid whose base is a square 12 cm on a side and whose slant height is 10 cm? If a plane is passed paralle to the base and 4 cm from the vertex, what is the lateral surface area and volume of the frustum?

OC = l = 10 cm,

LSA of the pyramid = 1/2

$pl = 1/2 \times 48 \times 10 = 240\ cm^2$.

$(OB)^2 + (BC)^2 = (OC)^2$,

implies $(OB)^2 = 100 - 36 = 64$.

Height of the pyramid $= h = OB = 8$ cm.

cross – section area $Base\ area = k^2/h^2 = 16/64$

cross – section area $/144 = 16/64$

cross – section area = 36, since, cross – section area = $s^2 = 36$,

Side of the cross-section has length 6cm.

$(OA)^2 + (AD)^2 = (OD)^2$, implies $4^2 + (3)^2 = (OD)^2$,

implies OD = slant height of the smaller pyramid = 5 cm.

Or, $\Delta OAD \approx \Delta OBC$ implies OA/OB = OD/OC implies 4/8 = OD/10 implies OD = 5 cm.

Slant height of the frustum = 10 – 5 = 5 cm.

LSA of the frustum = 1/2

$l(p + p')$ = 180 cm^2.

Volume of the frustum = 1/3 $h(A + A' + \sqrt{AA'})$ = 336 cm^3.

4. A frustum of a regular square pyramid has a height of 2 cm. The lateral faces of the pyramid are equilateral triangles of side $3\sqrt{2}$ cm. Find the volume of the frustum

Frustum height = h = 2 cm.

Diagonal of the base square = $d2 = (3\sqrt{2})2 + (3\sqrt{2})2 = 36$,

d = 6 cm. BC = 3 cm,

$(OB)^2 + (BC)^2 = (OC)^2$ implies OB = 3 cm.

implies OA = 1 cm.

BA = A = $3\sqrt{2} \times 3\sqrt{2}$ = 18 cm^2.

$A'/A = (OA)^2/(OB)^2$ implies $A'/18 = 1/9$,

Area of the cross-section = A' = 2 cm^2.

Volume of the frustum = 1/3 $h(A + A' + \sqrt{AA'})$ = 52/3 cm^3

5. A cone 12 cm high is cut 8 cm from the vertex to form a frustum with a volume of 156 cm^3. Find the radius of the bases of the cone.

$A'/A = \pi r1^2/\pi r2^2 = r1^2/r2^2 = k^2/h^2 = 8^2/12^2$, where $r1$ is the radius of the upper base and $r2$ is the radius of the lower base.

$r1/r2 = 2/3$ implies $r1 = 2/3r^2$.

The volume of the frustum = V = 156 cm^3.

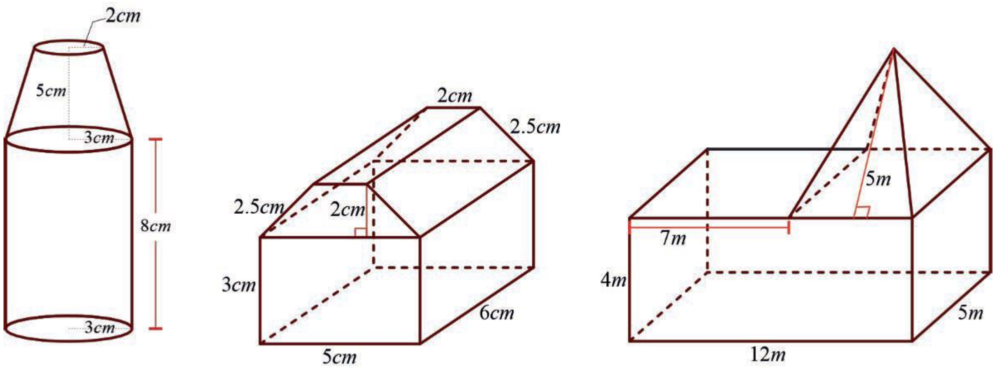
Frustum volume = $V = 1/3\pi h(r1^{\,2} + r2^2 + r1r2)$,= $1/3 \times 4\pi(4/9r2^2 + r2^2 + 2/3r2r2)$,

Surface Area and Volume of Composed Solids

A composed solid is a solid that is made up of two or more solids. In order to find the volume and surface area of a composed solid, one needs to identify the different parts it is made of. This decomposition allows working out the volume and surface area of each part independently. The volume of the composed solid is simply the sum of the volumes of its parts.

Exercise

1. Find the total surface area and volume of the following.



a **b** **c**

Solution

a. The solid is composed of a cylinder and a frustum of a cone

LSA of cylinder $= 2\pi rh = 2\pi \times 3 \times 8 = 48\pi$ cm^2

BA of the cylinder $= \pi r^2 = \pi \times 3^2 = 9\pi$ cm^2

TSA of the cylinder $= 57\pi$ cm^2

$l = \sqrt{5^2 + 1^2} = \sqrt{26}$ cm^2

LSA of the frustum of the cone $= l\pi(r + r') = \sqrt{26}\pi(3 + 2) = 5\sqrt{26}\pi$ cm^2

BA of the frustum of the cone $= \pi r^2 = \pi \times 2^2 = 4\pi$ cm^2

TSA of the frustum of the cone $= (4 + 5\sqrt{26})\pi$ cm^2

TSA of the solid = TSA of the cylinder + TSA of the frustum of the cone

$=(61 + 5\sqrt{26})\pi$ cm^2

Volume of the cylinder $= \pi r^2 h = 72\pi$ cm^3

Volume of the frustum of the cone $= 1/3\pi h(r^2 + r'^2 + rr')$

$=1/3\ \pi(5)(3^2 + 2^2 + (3)(2))$

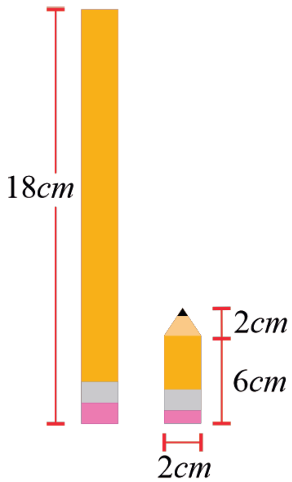
$=95/3\ \pi$ cm^3

Total volume of the solid = Volume of the cylinder + Volume of the frustum of the cone

$= 72\pi + 95/3\pi$

$= 311/3\pi$ cm^3

2. Hawi bought a new pencil like the one shown in Figure below on the right. She used the pencil every day in her mathematics class for a week, and now her pencil looks like the one shown on the right. How much of the pencil, in terms of volume did she use?



Volume of the original pencil = $\pi r^2 h$ = 18π cm^3

Volume of the used pencil is composed of cylinder and cone.

Volume cylinder = $\pi r^2 h$ = 6π cm^3

Volume cone = $1/3\pi r^2 h$ = $2/3\pi$ cm^3

Volume of the used pencil = $20/3\pi$ cm^3

The ratio of the remaining pencil = Volume of the used pencil/Volume of the original pencil

= $20/3\ \pi/18\pi$

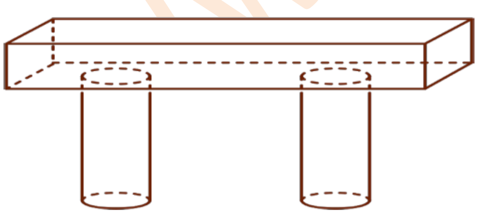
= 10/27

She used 1 – 10/27 = 17/27 = of the original pencil

6.4 Applications

Exercise

1. A concrete beam is to rest on two concrete pillars. The beam is a cuboid with sides of length 0.6 m, 4 m and 0.5 m. The pillars have diameter 0.5 m and height 2.5 m. Calculate the total volume of concrete needed to make the beam and the pillars. Use π = 3.14 and put your answer by rounding it to two decimal places



Volume of the cuboid = lwh

$= 0.6 \times 4 \times 0.5 = 1.2$ m^3

Total volume of concrete needed = $0.98 + 1.2 = 2.18$ m^3

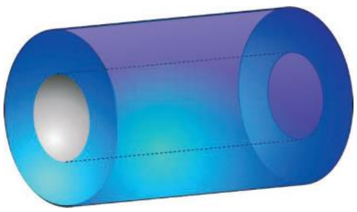
2. The diagram shows the cross-section of a pipe of length 50 m. The inner diameter of the pipe is 20 cm and the outer diameter is 28 cm.

a. Calculate the volume of metal needed to make the pipe. Use π = 3.14

b. Calculate the total surface area of the pipe, including the inside surface. Use π = 3.14

inner diameter = 20 cm = 0.2 m and ouer diameter = 28 cm = 0.28 m

inner radius = r′ = 0.1 m and ouer radius = r = 0.14 m



a. Volume of metal needed to make the pipe = $\pi(r^2 - r'^2)h$

$= \pi(0.14^2 - 0.1^2) \times 50 = 1.5072$ m^3

b. LSA = $2\pi h(r + r')$

BA = $2\pi(r^2 - r'^2)$

TSA = LSA + BA = $2\pi h(r + r') + 2\pi(r^2 - r'^2)$

$= 2\pi h(r + r') + 2\pi(r - r')(r + r')$

$= 2\pi(r + r')(h + (r - r')) = 75.42$ m^2

3. A lead bar of length 15 cm, width 8 cm and thickness 5 cm is melted down and made in five equal spherical ornaments. Find the radius of each ornament. (Use π = 3.14)

Volume of the lead bar = 5(Volume of one spherical ornament)

Volume of the lead bar = $lwh = 15 \times 8 \times 5 = 600$ cm3

5(Volume of one spherical ornament) = 600 cm3

$5\,(4/3\pi r^3) = 600$ implies $r = \sqrt[3]{90/\pi} = \sqrt[3]{90/3.14} = 3.06$ cm

Review Exercise

1. A lateral edge of a right prism is 6 cm and the perimeter of its base is 36 cm. Find the area of its lateral surface.

LSA $= hp = 6 \times 36 = 216$ cm^2

2. The height of a circular cylinder is equal to the radius of its base. Find its total surface area and its volume. Give your answer in terms of its radius.
TSA $= 4\pi r^2$ and Volume $= \pi r^3$

3. Find the total surface area of a regular hexagonal pyramid, given that an edge of the base is 8 cm and the altitude is 12 cm.
Apothem $= a = 4\sqrt{3}$ cm, slant height $= l = \sqrt{h^2 + a^2} = 8\sqrt{3}$ cm

LSA $= 1/2lp = 1$

$2 \times 8\sqrt{3} \times 48 = 192\sqrt{3}$ cm^2

BA $= 1/2nr^2 \sin(3600/n) = 96\sqrt{3}$ cm^2

TSA $= 192\sqrt{3} + 96\sqrt{3} = 288\sqrt{3}$ cm^2

4. When a lamp of stone is submerged in a rectangular water tank whose base is 20 cm by 50 cm, the water level rises by 1cm. What is the volume of the stone?

Volume of the stone = 1000 cm3

5. The altitude and base radius of a right circular cone are 5 cm and 8 cm respectively. Find the total surface area and volume of the cone.
TSA $= LSA + BA = \pi r\sqrt{r^2} + h2 + \pi r^2 = 8\pi(\sqrt{89} + 8)$ cm^2

V $= 1/3\pi r^2 h = 320/3\pi$ cm^3

6. The radii of the internal and external surfaces of a hollow spherical shell are 3 m and 5 m respectively. If the same amount of material were formed into a cube what would be the length of the edge of the cube?

$V = 4/3\pi r^3 - 4/3\pi r'\,3 = 4/3\pi(r^3 - r'^3) = 4/3$

$\pi(5^3 - 3^3) = 4 \times 98/3\pi$

$s^3 = 4 \times 98/3\pi$

$s = 2 \times \sqrt[3]{49/3\pi}$m

UNIT 7

COORDINATE GEOMETRY

7.1 Distance between Two Points

Example

1. Find the distance between the following pairs of points.

a. $A(1, 2)$ and $B(4, 2)$ **b.** $P(1, -2)$ and $Q(1, 3)$

Solution:

a. Since AB is horizontal line, distance d = à$x2 - x1$à = à4 − 1à = 3.

b. Since PQ is vertical line, Distance d = à$y2 - y1$à= à3 − (−2)à= 5.

The distance PQ can be defined as follows.

Note that ΔPQR is a right-angled triangle and by the Pythagoras Theorem, we have

$(PQ)^2 = (PR)^2 + (RQ)^2$. So, PQ = √$(PR)^2 + (RQ)^2$.

Since distance of PR = à$x2 - x1$àand distance of RQ = à$y2 - y1$à

If $P(x1, y1)$ and $Q(x2, y2)$ are endpoints of a line segment PQ, then the distance of pQ, denoted by d, is given by:

d = $\sqrt{(x2 - x1)^2 + (y2 - y1)^2}$.

Exercise

In each of the following, find the distance between the two given points.

a. $A(2, 5)$ and $B(4, 7)$

b. $P(-3, 5)$ and $Q(4, 10)$

c. $R(9, 5)$ and $S(6, -3)$

d. $M(-4, -3)$ and $N(5, 7)$

e. $T(-5, -2)$ and $S(0, -14)$

f. The origin and a point $P(\sqrt{2}, \sqrt{2})$.

Solutions

a. $d = \sqrt{(4-2)^2 + (7-5)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$
b. $d = \sqrt{(-3-4)^2 + (5-10)^2} = \sqrt{49+25} = \sqrt{74}$
c. $d = \sqrt{(9-6)^2 + (5+3)^2} = \sqrt{9+64} = \sqrt{73}$
d. $d = \sqrt{(-4-5)^2 + (-3-7)^2} = \sqrt{81+100} = \sqrt{181}$

7.2 Division of a Line Segment

The Section Formula

The point $R(x0, y0)$ dividing the line segment PQ internally in the ratio $p: q$ is given by:

$R(x0, y0) = (px2+qx1/p+ q, py2+qy1/p+ q)$, where $P(x1, y1)$ and $Q(x2, y2)$ are the end-points .This is called the section formula.

Exercise

1. Find the point dividing the line segment AB internally in the given ratio.

a. $A(1, 2)$, $B(4, 5)$, 1:2 .

b. $A(2, -3)$, $B(-1, 5)$, 3:1.

a. Given: Line segment AB with $A(1, 2)$ and $B(4, 5)$. A point divides the line segment in the ratio 1: 2 = $p1$:$p2$, and hence, $(x0, y0) = (p1x2 + p2x1/p1 + p2 ,p1y2 + p2y1/p1 + p2) = (1 \times 4 + 2 \times 1/3,1 \times 5 + 2 \times 2/3)= (2, 3)$. Therefore, the point is (2, 3).

b. Given: line segment AB with $A(2, -3)$ and $B(-1, 5)$. The first point divides the line segment in the ratio 3: 1 = $p1$: $p2$, and hence

$(x0, y0) = (p1x2 +p2x1/p1+p2 , p1y2+p2y1/p1+p2) = (3\times(-1)+1\times2/3+1 , 3\times5+1\times(-3)/3+1) = (- 1/4 , 3)$.
Therefore, the point is (− 1/4 , 3).

2. A line segment has end points $P(-1, 5)$ and Q (5, 2). Find the coordinates of the points that trisect the segment.

Solution

Given: line segment PQ with $P(-1, 5)$ and $Q(5, 2)$. A point divides the line segment in the ratio $1: 2 = p1:p2$, and hence $(x0, y0) = (p1x2 + p2x1/p1 + p2, p1y2 + p2y1/p1 + p2) = (1 \times 5 + 2 \times (-1)/3, 1 \times 2 + 2 \times 5/3) = (1, 4)$

Therefore, the point is (1,4).

The second point divides the line segment in the ratio $2: 1 = p1: p2$, and hence $(x0, y0) = (p1x2 + p2x1/p1 + p2 , p1y2 + p2y1/ p1 + p2) = (2 \times 5 + 1 \times (-1)/3, 2 \times 2 + 1 \times 5/3) = (3,3)$

Therefore, the second point is (3,3)

The Midpoint Formula

Suppose that $P(x1, y1)$ and $Q(x2, y2)$ are two distinct points on the xy-plane and

point $R(x0, y0)$ is midpoint of PQ. Then,

$R(x0, y0) = (\frac{x1+x2}{2}, \frac{y1+y2}{2})$

Example

Find the midpoint of the line segment joining the points $A(-1, -2)$ and $B(3,2)$.

Solution:

Midpoint of AB

$(x0 , y0) = (\frac{-1+3}{2}, \frac{-2+2}{2}) = (1, 0)$.

Exercise

1. Find the coordinate of the midpoint of the line segments joining the points:

a. $P(1, -3)$ and Q (4, 5) b. $P(-9, -3)$ and Q (18, 2)

A $(\frac{1+4}{2}, \frac{-3+5}{2}) = (\frac{5}{2}, 1)$

B$(\frac{-9+18}{2}, \frac{-3+2}{2}) =$

2. Find the midpoint of the sides of the triangle with vertices $A(-1, 3)$, $B(4, 6)$ and $C(3, -1)$.

Midpoint of $AB = (\frac{-1+4}{2}, \frac{3+6}{2}) = (\frac{3}{2}, \frac{9}{2})$

Midpoint of $AC = (-1+3(\frac{-1+3}{2}, (\frac{3-1}{2}) = (1,1)$,

Midpoint of $BC = (\frac{3+4}{2}, \frac{-1+6}{2}) = (\frac{7}{2}, \frac{5}{2})$

3. If $M(4, 6)$ is the midpoint of the line segment AB, point A has the coordinates

(−3, −2). Find the coordinates of point B.

Given: midpoint of AB = $M(4,6)$ and A = (−3, −2). Let the coordinate of B be (a, b), then $M(4,6) = (\frac{-3+a}{2}, \frac{-2+b}{2})$.

$4 = \frac{-3+a}{2}$=8=-3+a,a=8+3=11;a=11

and $6 = \frac{-2+b}{2}$; 12=-2+b,b=14

. So, a = 11 and b = 14

Therefore, the coordinate of B = (11,14)

4. Find the coordinates of point $C\ (x, y)$ where it divides the line segment joining (4, − 1) and (4, 3) in the ratio 3: 1 internally.

Given coordinates are A (4, −1) and B (4, 3). Let $C(x0, y0)$ be a point which divides the line segment in the ratio of 3: 1, i.e., $p: q = 3: 1$.

Now using the formula $C\ (x, y) = (\frac{px2 + qx1}{p+q}, \frac{py2+qy1}{p+q})$ as C is dividing internally.

$$C\ (x, y) = (\frac{3\times4+1\times4}{3+1}, \frac{3\times3+1\times(-1)}{3+1}) = (4,2)$$

Hence, the coordinates of $C\ (x, y)$ is (4, 2).

7.3 Equation of a Line

Gradient (slope) of a line

If we denote the gradient of a line by the letter m, then

$m = \frac{\text{change in y coordinate}}{\text{change in x coordinate}} = \frac{y2-y1}{x2-x1}$, $x1 \neq x2$.

If $P(x1, y1)$ and $Q(x2, y2)$ are points on a line with $x1 \neq x2$, then the gradient of the line, denoted by m, is given by:

$M = \frac{y2-y1}{x2-x1}$, $x1 \neq x2$.

Exercise

1. Find the gradient of the line passing through the following points.

a. $P(4, 3)$ and $Q(6, 7)$ b. $P(4, -3)$ and $Q(7, -4)$

c. $P(0, 3)$ and $Q(0, -7)$ d. $P(-6, -3)$ and $Q(-2, 5)$

solutions

a) m=$\frac{y2-y1}{x2-x1}$, $\frac{7-3}{6-4}$, 2

b) $m = \frac{-4-(-3)}{7-4}$, $\frac{-1}{3}$

2. If $A(-4, 6)$, $B(-1, 12)$ and $C(-7, 0)$ are points, then show that they are collinear.

Gradient of AB = $\frac{y2-y1}{x2-x1} = \frac{12-6}{-1-(-4)} = 2$,

Gradient of AC = $\frac{y2-y1}{x2-x1}$ $\frac{6-0}{-4-(-7)}=2$

and gradient of $BC = \frac{y2-y1}{x2-x1} = \frac{12-0}{-1-(-7)} = 2$. Therefore they

3. If $A(x1, y1)$ and $B(x2, y2)$ are distinct points on a line with $x1 = x2$, then what can be said about the gradient of the line? Is the line vertical or horizontal?

Solution

Since $x2 = x1$, then the line is vertical and it has no slope.

4. Consider the line with equation $y = x + 4$. Take three distinct points A, B and C on the line $y = x + 4$.

a. Find the gradient using A and B.

b. Find the gradient using A and C.

c. What do you observe from a and b?

If $A = (0,4)$, $B = (1,5)$ and $c = (2,6)$, then

a. gradient of $AB = \frac{y2-y1}{x2-x1} = \frac{5-4}{1-0} = 1$

b. gradient of $AC = \frac{y2-y1}{x2-x1} = \frac{6-4}{2-0} = 1$

c. they have the same gradient.

Therefore, the three points A, B, and C have the same gradient that means they lie on a straight line. Hence, they are collinear.

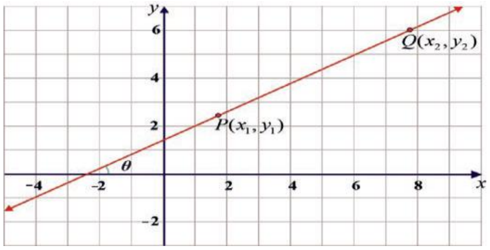
Define an angle of inclination θ of a line l to students. The Angle of inclination θ is the angle measured from the positive x-axis to the line l in counter clockwise direction.

Support students to do activity 7.5 in groups and arrive at the formula of slope of a line l in terms of its angle of inclination θ as: $slope\ m = tan\ \theta$

If l is a vertical line = 90° , $tan\ \theta$ is not defined and a vertical line has no slope. For a horizontal line = 0 , $tan\ \theta = 0$ and the slope of a horizontal line is zero.

Slope of a line in terms of angle of inclination

The angle measured from the positive x-axis to a line, in the anticlockwise direction is called the inclination of the line or the angle of inclination of the line. This angle is always less than 180°.



Find the slope of a line if its angle of inclination is:

a. 45° **b.** 120°

Solution:

a. Slope, m = tanθ = tan45° = 1

b. Since θ = 120°, the supplemental angle α = 60°.

Thus, m = −tanα.

= −tan60°

= −√3.

Exercise

1. Find the slope of a line if its angle of inclination is:

a. 60° b. 150°

solution

a) slope ,m=tan θ=tan60=√3

b) θ=150 the supplemental angle θ=30

m = tanθ = tan150°

= tan(180° – 150°)

= tan(−30°) = −tan30° = – 1/√3.

Different forms of equation of a line

The equation for the slope m of a line passing through the point $P(x1, y1)$ is called point slope form of equation of a straight line and is given by: $\boldsymbol{y - y1 = m(x - x1)}$

The slope-intercept form of equation of the line is given by: $\boldsymbol{y = mx + b}$, where b is the y-intercept.

Observe the properties of the line in relation to the slope:

1) If $m \in R$ where R is the set of real numbers, then

- $m > 0$, then the line rises from left to right
- $m < 0$, then the line goes downward from left to right
- $m = 0$, then the line is horizontal

2) A vertical line has no slope.

Two points form of equation of a line:

$\frac{y-y1}{x-x1} = \frac{y2-y1}{x2-x1}$

So, $y - y1 = \frac{y2-y1}{x2-x1}(x - x1)$.

Thus, equation of the line passing through the points $(x1, y1)$ and $(x2, y2)$ is given by: $y - y1 = \frac{y2-y1}{x2-x1}(x - x1)$.

This is called two-point form of equation of a line.

Remark: The general form of the equation of a line is given as $Ax + By + C = 0$, where A, B, and C are real numbers

Exercise

1. Find the equation of the line with slope m and y-intercept b.

a. $m = -6$; $b = 5/3$

b. $m = 0$; $b = -2$.

c. $m = -3/4$; $b = 1/6$

solution

a. $18x + 3y - 5 = 0$ or $y = -6x + 5/3$

b. $y = -2$

c. $9x + 12y - 2 = 0$ or $y = -3/4x + 1/6$

2. Find the equation of the line with slope m and passing through the given point P.

a. $m = 3$; $P(2, 4)$ b. $m = -2$; $P(-3, -1)$

c. $m = 4/5$; $P(-5, 0)$ d. $m = 0$; $P(7, -4)$

Solution

a. $y = 3x - 2$

b. $2x + y + 7 = 0$ or $y = -2x - 7$

c. $4x - 5y + 20 = 0$ or $y = 4/5x + 4$.

d. $y = -4$

3. Find the equation of the line passing through the given points.

a. $P(1, 3)$ and $Q(3, 7)$ b. $A(-1, 2)$ and $B(2, -3)$

c. $R(4, 3)$ and $S(5, -4)$ d. $P(1, 8)$ and $Q(7, -2)$

e. $C(6, 3)$ and $D(5, -5)$ f. $M(-9, 4)$ and $N(-7, -3)$

a. $2x - y + 1 = 0$ or $y = 2x + 1$

b. $5x + 3y - 1 = 0$ or $y = -5/3x + 1/3$

c. $7x + y - 31 = 0$ or $y = -7x + 31$

d. $5x + 3y - 29 = 0$ or $y = -5/3x + 29/3$

e. $8x - y - 45 = 0$ or $y = 8x - 45$

f. $7x + 2y + 55 = 0$ or $y = -7/2x - 55$

4. Suppose a line has x-intercept p and y-intercept q , for $p, q \neq 0$; Show that the equation of the line is $\frac{x}{q} + \frac{y}{p} = 1$.

Given: x-intercept= $(p, 0)$ and y-intercept= $(0, q)$. The two-point form of equation of a line

$y - y1 = \frac{0-q}{p-0}(x - x1)$

$y - 0 = -q/p(x - p)$

$py = -qx + pq$

$py + qx = pq$

$x/p + y/q = 1$

$\frac{x}{q} + \frac{y}{p} = 1$

5. For each of the following equations, find the slope and y-intercept:

a. $5x + 2y + 10 = 0$ b. $5/4x - y = 0$

c. $7x - 4y - 56 = 0$ d. $1/3x + 5/12y - 1/4 = 0$

e. $y - 5 = 0$

solution

a. m = – 5/2, y-intercept = (0, −5)

b. m = 5/4 , y-intercept = (0, 0)

c. m = 7/4, y-intercept = (0, −14).

d. m = – 4/5 , y-intercept = (0, 3/5)

e. m = 0, y-intercept = (0, 5)

6. If a line passing through the points $P(2, 5)$ and $Q(-4, 7)$, then find

a. the point-slope form of the equation of the line;

b. the slope-intercept form of the equation of the line;

c. the two-point form of equation of the line. What is its general form?

a. $y - y1 = m(x - x1)$, $y - 5 = -1/3(x - 2)$. So, $x + 3y - 17 = 0$

b. $y = mx + b$, $5 = (-1/3) \times 2 + b/5 + 2/3 = b/b = 17/3$

Therefore, $y = -1/3x + 17/3$

c. $\frac{y-5}{x-2} = \frac{7-5}{-4-2}$, and the general equation is $x + 3y - 17 = 0$

7.4 Parallel and Perpendicular Lines

Slopes of parallel and perpendicular lines

If two non - vertical lines $l1$ and $l2$ having slope $m1$ and $m2$ respectively are parallel to each other, then they have the same slope ($m1 = m2$).

Example

Find the equation of the line which is parallel to the line $y = -2x + 6$ and passing through the point $P(1,10)$.

Solution:

The slope of the line $y = -2x + 6$ is $m = -2$. Therefore, the line through the point $P(1,10)$ parallel to $y = -2x + 6$ has equation $y - y1 = m(x - x1)$

$y - 10 = -2(x - 1)$

$y = -2x + 12$

Exercise

1. Show that the line passing through the points $A(7, 5)$ and $B(6, 11)$ is parallel to the line passing through $P(5, 1)$ and $Q(3, 13)$.

Gradient of $AB = \frac{11-5}{6-7} = -6$ and gradient of $PQ = \frac{13-1}{3-5} = -6$. The two lines have the same slope and so are parallel.

2. Find the equation of a line parallel to :

a. $y = 3x - 4$ and passing through point $(2, 8)$

b. $3x + 4y = 5$ and passing through points$(1, 1)$

a. slope is 3. The equation of the line passing through the point (2, 8) is: $3 = \frac{y-8}{x-2}$, which implies $y - 8 = 3x - 6$ and $y = 3x + 2$.

b. slope is$- 3/4$. The equation of the line passing through the point (1, 1) is: $\frac{y-1}{x-1} = - 3/4$,

$y - 1 = - 3/ 4 (x - 1)$. i.e. $3x + 4y - 7 = 0$

Two non-vertical lines having slopes $m1$ and $m2$ are perpendicular if and only if $m1. m2 = -1$.

Exercise

1. Find the slope of a line that is:

a. parallel to the line : $y = -7x + 5$

b. perpendicular to the line: $y = -7x + 5$

a. slope of $L1$ parallel to $y = -7x + 5$ is $- 7$.

b. slope of $L1$ perpendicular to $y = -7x + 5$ is $1/7$.

2. Find the equation of the line that is perpendicular to

a. $y = -3x + 5$ and passes through the point $(7, 2)$.

b. $y = 4x + 5$ and passes through the point $(-7, 2)$.

b. $y - 2 = - ¼ (x - (-7))$, $y = - 1/4x - 7/4 + 2 = - 1/4x + 1/4$

7.5 Equation of a Circle

A circle is the set of all points on a plane with a fixed distance from a fixed point.

This fixed point is called the center of the circle and the fixed distance is the radius r of the circle

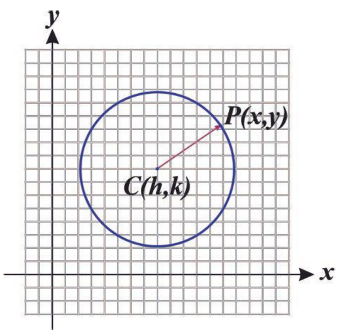
Different Forms of Equation of Circle

An equation of circle represents the position of a circle on a Cartesian plane. A circle can be drawn on a piece of paper given its center and the length of its radius. Using the equation of circle, once we find the coordinates of the center of the circle and its radius, we will be able to draw the circle on the Cartesian plane. There are different forms to represent the equation of a circle,

- ✓ **Standard form**
- ✓ **General form**

Standard Equation of a Circle

A circle is a closed curve that is drawn from the fixed point called the center, in which all the points on the curve are having the same distance from the center point of the center. The equation of a circle with (h, k) center and r radius is given by $\boldsymbol{(x - h)^2 + (y - k)^2 = r^2}$



Example

Find the equation of the circle whose center and radius are following:

a. $C(3, 4)$, $r = 2$ **b.** $C(4, -2)$, $r = 5$

Solution:

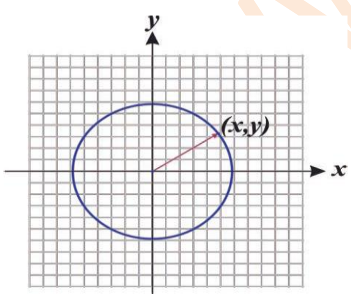
a. Given the center $(h, k) = (3, 4)$ and $r = 2$, and $(x - h)^2 + (y - k)^2 = r^2$.

$(x - 3)^2 + (y - 4)^2 = 2^2$

b. $(x - 4)^2 + (y - (-2))^2 = 5^2$

$(x - 4)^2 + (y + 2)^2 = 5^2$

Equation of a Circle When the Centre is Origin



Consider an arbitrary point $P(x, y)$ on the circle. Let 'a' be the radius of the circle which is equal to OP.

We know that the distance between the point (x, y) and origin (0,0) can be found using the distance formula which is equal to: $\sqrt{x^2 + y^2} = a$. Now, squaring on both sides to obtain, $x^2 + y^2 = a^2$.
This is the equation of the circle with the center as the origin.

Exercise

1. Find the equation of the circle in standard form for the following circles.
a. $C(2, -3)$, $r = 3$.
b. $C(-7, 4)$, $r = 4$
c. $C(0, 0)$, $r = \sqrt{10}$

a. Equation for a circle in standard form is written as:
$(x - h)^2 + (y - k)^2 = r^2$. Here,
(h, k) = (2, −3) is the center of the circle and radius $r = 3$.
$(x - 2)^2 + (y + 3)^2 = 3^2$ is the required standard form of the equation of the given circle.
b. (h, k) = (−7, 4) is the center of the circle and radius $r = 4$.
$(x + 7)^2 + (y - 3)^2 = 4^2$ is the required standard form of the equation of the given circle.
c. (h, k) = (0, 0) is the center of the circle and radius $r = \sqrt{10}$.
$(x - 0)^2 + (y - 0)^2 = \sqrt{10}^2$ or $x^2 + y^2 = 10$ is the required standard form
of the equation of the given circle.

2. Find the center and radius of the following circles.
a. $(x - 2)^2 + (y - 5)^2 = 7^2$.
b. $x^2 + y^2 = 100$.
c. $(x + 1)^2 + y^2 = 3^2$.

a. (2, 5) is the center of the circle and radius $r = 7$.
b. (0, 0) is the center of the circle and radius $r = \sqrt{100} = 10$.
c. (−1, 0) is the center of the circle and radius $r = 3$.

3. Show the point (-12, 5) is on the circle with equation $x^2 + y^2 = 169$. Given: $x = -12$ and $y = 5$. Substitute them into the equation: $x2 + y2 = 169$ $(LHS) = (-12)^2 + 5^2 = 144 + 25 = 169$.

So, $(LHS) = (RHS)$. Therefore, the point (−12, 5) is on the circle. $(-12)^2 + 5^2 = 169 \Rightarrow 144 + 25 = 169$. So, point (−12, 5) is on the circle.

General Form of Equation of a Circle

The general form of the equation of a circle is expressed as:

$x^2 + y^2 + lx + my + n = 0$

If $l^2 + m^2 - 4n = 0$, then the radius of the circle is zero which tells us that the circle is a point that coincides with the center. Such a type of circle is called a point circle.

If $l^2 + m^2 - 4n < 0$, then the radius of the circle become negative and not real (imaginary).

General form of equation of a circle is expressed as:

$\boldsymbol{x^2 + y^2 + lx + my + n = 0}$**,** where $l^2 + m^2 - 4n > 0$.

We need to make sure that the coefficients of x^2 and y^2 are 1 before applying the formula.

Example

Equation of a circle is $x2 + y2 - 12x - 16y + 19 = 0$. Find the center and radius of the circle.

Solution:

By using completing square method,

$x^2 - 12x + y^2 - 16y + 19 = 0$

$(x - 6)^2 + (y - 8)^2 - 36 - 64 + 19 = 0$

$(x - 6)^2 + (y - 8)^2 = 81$

$(x - 6)^2 + (y - 8)^2 = 9^2$

Therefore, center of the circle is $(6, 8)$ and the radius of the circle is 9 units.

Exercise

1. Find the center and radius of the circle.

a. $x^2 + y^2 - 10x + 14y + 38 = 0$

b. $x^2 + y^2 + 6x + 8y + 9 = 0$

a. $x^2 + y^2 - 10x + 14y + 38 = 0$

By using completing square method,

$(x - 5)^2 + (y + 7)^2 - 25 - 49 + 38 = 0$

Then, $(x - 5)^2 + (y + 7)^2 = 6^2$

Hence, the center is $(5, -7)$, radius is 6.

b. $x^2 + y^2 + 6x + 8y + 9 = 0$

By using completing square method, $(x + 3)^2 + (y + 4)^2 - 9 - 16 + 9 = 0$

Then, $(x + 3)^2 + (y + 4)^2 = 4^2$

Hence, the center is $(-3, -4)$, radius is 4.

2. Write the equation of the circle described below:

a. It has center at $C(3, 1)$ and pass through the point $P(7, -3)$

b. It passes through the origin and has center at $P(-3, 2)$

c. The end point of its diameter are $A(3, 4)$ and $B(-1, 2)$

a. Taking the center at $C(3, 1)$, the equation will be $(x - 3)^2 + (y - 1)^2 = r^2$

Substitute the point $P(7, -3)$ into the equation, $(7 - 3)^2 + (-3 - 1)^2 = r^2$

Then, $r = \sqrt{(7 - 3)^2 + (-3 - 1)^2} = \sqrt{3^2} = 4\sqrt{2}$

$(x - 3)^2 + (y - 1)^2 = (\sqrt{32})^2$. i.e. , $(x - 3)^2 + (y - 1)^2 = 32$.

b. Taking the center at $C(0, 0)$, the equation will be $x^2 + y^2 = r^2$

Substitute the point $(-3, 2)$ into the equation, $r = \sqrt{(-3)2 + 22} = \sqrt{13}$

Therefore, $x^2 + y^2 = 13$

c. $A(3, 4)$ and B $(-1, 2)$, $d = \sqrt{(-1 - 3)^2 + (2 - 4)^2} = \sqrt{20} = 2\sqrt{5}$.

So, $r = \sqrt{5}$.

The center is the midpoint of AB. Thus, $(3-1: \frac{3-1}{2}, : \frac{4+2}{2}) = (1, 3)$

$(x - 1)^2 + (y - 3)^2 = (\sqrt{5})^2$. i.e. , $(x - 1)^2 + (y - 3)^2 = 5$

3. Write the general form of the circle equation with center$(-1, 6)$ and radius 3 unit.

Here, $(x - h)^2 + (y - k)^2 = r^2$ is the standard form of equation of a circle.

Using the center $(-1, 6)$ and $r = 3$, $(x + 1)^2 + (y - 6)^2 = 3^2$ implies

$x^2 + y^2 + 2x - 12y + 28 = 0$ which is the general form of equation of a circle

4. Write the equation of the circle in standard form $x^2 + y^2 + 8x - 2y - 8 = 0$

$x^2 + y^2 + 8x - 2y - 8 = 0$. By using completing square method,

$x^2 + 8x + y^2 - 2y - 8 = 0$

$(x + 4)^2 - 16 + (y - 1)^2 - 1 - 8 = 0$

Hence, the standard form is $(x + 4)2 + (y - 1)^2 = 5^2$.

7.6 Applications

Find the area of the triangle having vertices at A, B, and C which are at points $(-3, 4)$, $(0, 1)$ and $(-3, -2)$ respectively. Also, mention the type of triangle.

Solution:

$A = 1/2$à$x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)$à

$= 1/2\ |(-3)(1 - (-2)) + 0(-2 - 4) + (-3)(3 - 0)|$

= 1/2à(−9) + 0 + (−9)à= 9 Sq. units

$d(AB) = \sqrt{(-3-0)^2 + (4-1)^2}$

$= \sqrt{18} = 3\sqrt{2}$

$d(BC) = \sqrt{(-3-0)^2 + (-2-1)^2} = 3\sqrt{2}$

So, $AB = BC$

And slope of $AB = -1$ and slope of $BC = 1$

Slope of $AB \cdot$ slope of $BC = -1 \times 1 = -1$

Therefore, it is an isosceles right-angled triangle.

Exercise

1. Find the area of the triangle having vertices at A, B, and C which are at points (3, 3), (−1, 0) and (3, – 5), respectively.

AC= $\sqrt{(3-3)^2 + (3+5)^2} = 8$ and

height= 4

$Area = 1/2 \times AC \times height$

$= 1/2 \times 8 \times 4 = 16$ sq.units.

2. Show that the plane figure with vertices:

a. $A(5, -1)$, $B(2, 3)$ and $C(1, 1)$ is a right-angled triangle.

b. $A(2, 3)$, $B(6, 8)$ and $C(7, -1)$ is an isosceles triangle.

c. $A(-4, 3)$, $B(4, -3)$ and $C(3\sqrt{3}, 4\sqrt{3})$ is an equilateral triangle.

a. Solution1:

Using the figure, find the slope of AC and BC.

$mAC = 1-(-1) \quad \frac{1-(-1)}{1-5} = 2/-4 = -1/2$

$mBC = \frac{1-3}{1-2} = -2/-1 = 2$

Since $mAC \cdot mBC = (-1/2) \cdot 2 = -1$,

the line AC and BC is perpendicular.

Therefore, the triangle ABC is a right-angled triangle.

Solution2:

In the figure below,

$AB = \sqrt{(2-5)^2 + (3+1)^2} = \sqrt{9+16} = \sqrt{25} = 5$

$AC = \sqrt{(1+1)^2 + (1-5)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$BC = \sqrt{(3-1)^2 + (2-1)^2} = \sqrt{4+1} = \sqrt{5}$

Thus, the longest side is AB. By using the conversion of Pythagoras theorem,

$AC^2 + BC^2 = (2\sqrt{5})^2 + (\sqrt{5})^2 = 25$
$AB^2 = 25$
Hence, $AC^2 + BC^2 = AB^2$

Therefore, ΔABC is a right-angled triangle.

b. In the figure below,

$AB = \sqrt{(6-2)^2 + (8-3)^2} = \sqrt{16+25} = \sqrt{41}$
$AC = \sqrt{(7-2)^2 + (3+1)^2} = \sqrt{25+16} = \sqrt{41}$

Slope of $AB = 5/4$ and $AC = -4/5$ and their slope product is -1, and $AB = AC$. Therefore, it is an isosceles right-triangle

c. In the figure below,

$AB = \sqrt{(4+4)^2 + (-3-3)^2} = \sqrt{100} = 10$
$AC = \sqrt{(-4-3\sqrt{3})^2 + (3-4\sqrt{3})^2} = \sqrt{100} = 10$
$BC = \sqrt{(3\sqrt{3}-4)^2 + (4\sqrt{3}+3)^2} = \sqrt{100} = 10$

Therefore, triangle ABC is an equilateral triangle

3. Find the equation of the line containing side of the triangle whose vertices are $A(3, 4)$, $B(2, 0)$ and $C(-1, 6)$.

4. Find the coordinates of a point on the x-axis, which is at a distance of 5 units from the point $(6, -3)$.

Let the coordinates of the point on the x-axis $(x, 0)$.

$25 = (x-6)^2 + (0+3)^2$, $25 = x^2 - 12x + 36 + 9$

$0 = x^2 - 12x + 20$. So, $x = 2$ or $x = 10$

Therefore, the coordinates of the points on the x-axis are $(2, 0)$ and $(10, 0)$.

5. If end points of the diameter of a circle are $(-5, 2)$ and $(3, -2)$, then find the center and equation of the circle.

$C(a, b) = (\frac{-5+1}{2}, \frac{2+(-2)}{2}) = (-1, 0)$.
$r = \sqrt{\{(-1-(-5)\}^2 + (0-2)^2} = \sqrt{4^2 + 2^2} = \sqrt{20}$
Hence, $(x+1)^2 + y^2 = 20$.

The distance d between points $P(x1, y1)$ and $Q(x2, y2)$ is given by the formula
$d = \sqrt{(x2-x1)^2 + (y2-y1)^2}$

Review Exercise

1. If the gradient of a line is -3 and the y-intercept is -7, then find the equation of the line.

$y = mx + b$, where m is the slope and b is the y-intercept. Given: $m = -3$ and y-intercept is -7. So, $y =$

$-3x - 7$

2. Find the slope of the line passing through the points $P(5, -3)$ and $Q(7, -4)$.

The slope of line $\frac{y2-y1}{x2-x1}$ $PQ = -1/2$

3. Find the equation of a line passing through $(-2, 3)$ and having a slope of -1.

   $\frac{y-3}{x+2} = -1$, $y = -(x + 2) + 3$ implies $y = -x + 1$.

4. Find the equation of the line which passes through the point $(-2, 5)$ and is perpendicular to the line whose equation is $2x - y + 5 = 0$.

   $y = -1/2(x + 2) + 5 = -1/2x + 4$

5. Determine the equation of the line that passes through the points $A(-3, 2)$ and $B(5, 4)$.

$\frac{y-4}{x-5} = \frac{4-2}{5+3}$, $y = 1/4(x - 5) + 4$

6. Find the area of the triangle having vertices at A, B, and C with coordinates $(2, 3)$, $(-1,0)$ and $(2, -3)$, respectively. What type of triangle is it?

$A = 1/2|x1(y2 - y3) + x2(y3 - y1) + x3(y1 - y2)|$
$= 1/2|2(0 + 3) - 1(-3 - 3) + 2(3 - 0)|$
$= 1/2|6 + 6 + 6| = 18/2 = 9$ sq.units.
$d(AB) = \sqrt{(-3 - 0)^2 + (4 - 1)^2}$
$= \sqrt{18} = 3\sqrt{2}$
$d(BC) = \sqrt{(-3 - 0)^2 + (-2 - 1)^2} = 3\sqrt{2}$
So, $\overline{AB} = \overline{BC}$ and further more slope of $\overline{AB} = -1$, slope of $\overline{BC} = 1$,
Slope of $\overline{AB}$ × slope of $\overline{BC} = -1 \times 1 = -1$.

Therefore, ΔABC is an isosceles right-angled triangle