Mathematics

Grade 12

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Unit one

Sequences & series

Definition: A sequence is a function whose domain is the collection of all integer greater than or equal to a given integer m (usually 0 or 1)

- It is denoted by an or $\{an\}_{=0\ or\ 1}^{\infty}$
- An is called the general term a_1, a_2,---a_n are called the term of the sequence.

- We have two types of sequences
  - Finite
  - Infinite

Finite sequence: A sequence that has a last term

- Its domain is 1,2,3---,n

Infinite sequence: - A sequence that does not have a last term

- Its domain is the set of natural numbers (N)

Examples-1: List the first five terms of each of the following sequences, where ∩∈ ℕ

a) $a_n = \left(-\frac{1}{3}\right)^{n-1}$

$a_1 = \left(-\frac{1}{3}\right)^{1-1} = 1;$ $a_3 = \left(-\frac{1}{3}\right)^{3-1} = \frac{1}{9}$

$a_2 = \left(-\frac{1}{3}\right)^{2-1} = -\frac{1}{3}$ $a_4 = \left(-\frac{1}{3}\right)^{4-1} = -\frac{1}{27}$ $a_5 = \left(-\frac{1}{3}\right)^{5-1} = \frac{1}{81}$

$\therefore$ The first five terms of the sequence $a_n=\left(-\frac{1}{3}\right)^{n-1}$ $are\,1,-\frac{1}{3},\frac{1}{9},\frac{1}{27},\frac{1}{81}$

b) $a_n=\frac{n}{n+1}$; (Ex. 1.1) $a_1=\frac{1}{1+1}=\frac{1}{2}$; $a2=\frac{2}{3}$; $a_3=\frac{3}{4}$; $a4=\frac{4}{5}$; $a_5={}^5/_6$

$\therefore$ The first five terms of the sequence $a_n=\frac{n}{n+1}$ are $\frac{1}{2},\frac{2}{3},\frac{3}{4},\frac{4}{5},{}^5/_6$.

c) $a_n=n^3$(Ex. 1.1)

$a_1=1^3=1$; $a_2=2^3=8\Rightarrow a_2=8$; $a_3=3^3=27\Rightarrow a_3=27$; $a_4=4^3=64\Rightarrow a_4=64$&

$a_5=5^3=125\Rightarrow a_5=125$

$\therefore$ The first five terms of the sequence $a_n=n^3$ are 1,8,27,64,125

Q3(ex.11.)

Benut's uncle gave 130 Ethiopian birr to her in January, in the next month she shaves & has 210 Ethiopian birr & in the third month she has 290 Ethiopian birr. How much money will she have in the fourth, fifth, sixth and seventh month respectively.

Solution

a_1=130(in January)

a_2=210(in February)

a_3= 290 (in March)

$\Rightarrow$. Her saving increase by 80 birr for each consecutive month.

$\Rightarrow$. a_4=290+80=370 (in the fourth month)

$\Rightarrow$. a_5=370+80=450 (in the fifth month)

$\Rightarrow$. a_6=450+80=530 (in the sixth month)

$\Rightarrow$. $a_7=530+80=610$ (in the seventh month)

Fibonacci and Mulatu sequence.

- A sequence that relates to the general term an of a sequence where one or more of the terms that comes before it is said to be defined recursively. The domain of the recursive sequence can be the set of whole numbers.

  Eg. Fibonacci & Mulatu sequences are examples of recursion sequence.

  Eg. 1. Find the 12th term of Fibonacci's sequence {1,1,2,3,5,8,…}

  $F_n=F_{n-1}+F_{n-2}$, $n\geq 2$

  $F_1=1$ $\quad F_3=2 \Rightarrow F_{12}=F_{12}+F_{12-2}$

  $F_2=1$ $\quad F_{11}+F_{10}$

  $F_4=3$; $F_5=5$;$F_6=8$; $F_9=F_8+F_7=21+13$

  $F_7=F_6+F_5=8+5=13$

  $F_8=F_7+F_6=13+8=21$

  $F_9=F_8+F_7=21+13=34$

  $F_{10}=F_9+F_8=34+21=55$

  $F_{11}=F_{10}+F_9=55+34=89$

  $F_{12}=F_{11}+F_{10}=89+55=144$

1.2. Arithmetic and Geometric sequences.

Arithmetic sequences.

Definition:- Is a sequence in which each term except the first is obtained by adding a fixed number (positive or negative) to the preceding term. The fixed number is called the common difference of the sequence.

Examples-1

1. Given $a_1=10$ (the first term of an arithmetic sequence) and the common difference is -4, find the terms from 2nd to 5th term.

Solution

$a_1=10$ $\quad$ $a_2=a1+d=10+-4=10-4=6$

$d=-4$ $\quad$ $a_3=a_2+d=6-4=2$

$a_4=a_3+d=2-4=-2$

$a_5=a_4+d=-2-4=\underline{\underline{-6}}$

Theorem 1.1: If A_n is an arithmetic sequence with the first term A_1 and a common difference d, then the n^{th} term of the sequence is given by $A_n=A_1+(n-1)d$.

Examples 2. Find the general terms of the sequence An, when:

a. $A_1=2,d=3$

b. What is the 10^{th} term of the sequence,10,6,2,-2…?

Solution

a. $An=A1+(n-1)d$

$A_n=2+(_n-1)3$

$A_n=2+3_n-3$

$\underline{\underline{A_n=3_n-1}}$

b. 10,6,2,-2..

$d=6-10=2-6=-2-2=-4$

$An = A_1+(n-1)d$

$= 10+(n-1)-4$

$= 10-4n+4$

$= \underline{\underline{14-4n}}$

Examples-3

1. Determine whether or not the sequences with the following general terms are arithmetic.

a. $a_n=3_{n^{-2}}$ $\quad$ b. $a_n=3_{n^2-2}$

Solution

a. $a_n+1=3(n+1)-2$ * To be an arithmetic sequence the deference between

= 3n+3-2 $a_n+1=a_n$=constant.

= 3n+1

⇒. $A_{n+1}- a_n$

$=3_{n+1}- (3_{n-2})$

$= 3_{n+1}-3_{n+2}$

= $\underline{\underline{3}}$

∴ $a_n=3_{n-2}$ is a sequence.

b. an=3n2-2

an+1=3(n+1)²-2

= 3(n2+2n+1)-2

= 3n2+6n+3-2

= 3n2+6n+1

an+1-an

3n2+6n+1-3n2+2

= $\underline{\underline{6n+3}}$

⇒. An=3n2-2is not an arithmetic sequence

Arithmetic mean between two numbers

- The terms of arithmetic sequence that lie between two given terms are called the arithmetic mean

Examples

a. Given that 1,x,8, is an A.S find X.

Solution: x-1=8-x

$2x=9 \Rightarrow x= {}^{9}/_{2}$

b. Given that the sequence $\frac{1}{2},\frac{1}{x},\frac{1}{6}$ *is* an A.P (arithmetic progression fined x)

Solution

$\frac{1}{x}-\frac{1}{12}=\frac{1}{6}-\frac{1}{x}$

$\frac{1}{x}+\frac{1}{x}=\frac{1}{6}+\frac{1}{12}$

$\frac{2}{x} = \frac{3}{12}$

24=3x

X=8

c. Find the arithmetic mean of 4&14.

Solution

The A mean between 4 & 14 is $\frac{4+14}{2} = \frac{18}{2}$ =9

d. Insert four arithmetic mean between 4 & 14 to create an arithmetic sequence.

Solution

If we insert n-arithmetic mean between a and b then, the common difference d is given by:

d=$\frac{b-a}{n+1}$

d=$\frac{14-4}{4+1} = \frac{10}{5}$ =2

4,4+2,6+2,10+2,14

∴ The require numbers are 4,6,8,10,12,14

4 arithmetic between 4 & 14 are 6,8,10,12

Geometric sequences

Definition:- A geometric sequence or geometric progression is one in which the ration between consecutive terms is a non-zero constant. This constant is called the common ratio.

- $\{G_n\}$ is geometric sequence if $\gamma = \frac{G_{nH}}{G_n}$, $\gamma \in l\beta \& \gamma \neq 0$ where γ is the common ratio

Examples

1. If $G_1 - \frac{1}{2} \& \gamma = \frac{1}{2}$, then find

   G_2, G_3, G_4 & G_5

Solution

$G_1=\frac{1}{2}; \gamma=\frac{1}{2}$

$G_2=G_1\,\gamma=-\frac{1}{2}\times\frac{1}{2}=-\frac{1}{4}$

$G_3=G_2\,\gamma=-\frac{1}{4}\times\frac{1}{2}=-\frac{1}{8}$

$G_4=G3\,\gamma=-\frac{1}{8}\times\frac{1}{2}=-\frac{1}{16}$

$G_5=G4\,\gamma=-\frac{1}{16}\times\frac{1}{2}=-\frac{1}{32}$

2. For the geometric sequences 1,2,4,8,16 find the common ratio & the 6th term (ex.1.7)

   Solution

   $G_1=1; G_4=8$

   $G_2=2; G_5=16$

   $G_3=4;$

   $\gamma=\frac{G_{nH}}{G_n}=\frac{G_2}{G_1}=\frac{63}{62}$

   $=\frac{2}{1}=\underline{\underline{2}}$

   & $G_6=G_5\,\gamma$

   $=16x2=32$

<u>Determining the nth term of geometric sequence</u>

<u>Theorem 1</u>

If $\{G_n\}$ is a geometric sequence with the first term G1 & common ratio γ, then the nth terms the sequence is given by $G_n=G_1\ \gamma^{n-1}$.

Examples: 1.

For each of the following, find the nth term of the geometric sequence.

a. $G_1=2, \gamma=5$(Ex.1.8)

   $G_n=G_1\ \gamma^{n-1}$

$G_n=2(5)^{n-1}$

$\underline{G_n=2.5^{n-1}}$

b. $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}$,---.

Solution:- $\gamma=\frac{3}{4} \div \frac{3}{2}, = \frac{1}{2}$

$G_n= G_1\gamma^{n-1}$

$= \left(\frac{3}{2}\right) * \left(\frac{1}{2}\right)^{n-1}$

$= \frac{3}{2} * 2^{-n+1}$

$= 3*2^{-n}*2^{-n+1}$

$Gn= 3*2^{-n}$

Geometric mean between two numbers

When a, m & b are terms in a geometric sequence, then m is called the geometric mean between a and b ($a\neq 0, b \neq 0, m \neq 0$).

$\Rightarrow r=\frac{m}{a} = \frac{b}{m} \Leftrightarrow m^2=ab$

$m=\pm\sqrt{ab}$

Examples (Ex.1.9)

1. Find geometric mean between 2&8 let the geometric mean be m ($m\neq 0$)

$\gamma = \frac{m}{2} = \frac{8}{m}$

$\Leftrightarrow m^2=16$

$m=\pm 4$

$\therefore$, the geometric mean between 2 & 7 is -4 or 4.

2. If x, 4x+3, 7x+6 are consecutive terms of a geometric sequence, find the value (s) of x, x≠0).

Solution:-

$\frac{4x+3}{x} = \frac{7x+6}{4x+3} \Leftrightarrow (4x+3)^2=x(7x+6)$

$16x^2+24x+9=71x^2+6$

$\Rightarrow 16x^2-7x^2+24x-6x+9=0$

$\Rightarrow$ 9x2+18x+9=0

$x^2+2x+1=0$

$(x+1)^2=0$

$\underline{\underline{x=-1}}$

3. Find three consecutive terms of a geometric sequence, such that their sum is 35 & their product is 1000. (Let the terms be $\frac{a}{\gamma}$, a & aγ a≠0, γ ≠0)

$\frac{a}{\gamma} + a + a\gamma = 35$

$\frac{a}{\gamma} a * a\gamma = 1000$

$a^2=1000$

$a=\sqrt[3]{1000} \Rightarrow \gamma = 10$

$\frac{a}{10}$+a+10a=35

a+10a+100a=350

111a=350

a=$\frac{350}{111}$

$\therefore$Three consecutive terms of a given geometric saunas are: $\frac{350}{111}, \frac{350}{111}, \frac{3500}{111}$

1.3 The sigma Notation & partial sums.

Definition: Let $\{a_n\}_{n-1}^{\infty}$ be a sequence. The sum of the first n terms of the sequence, denoted by

S_n is called the partial sum of these sequences. Such summation is denoted as follows.

$S_n=\sum_{k=1}^{n} a_k=a_1+a_2+\text{---}+a_n$

Where K is the index of the summation 1 is the lower limit of summation, n is the upper limit of summation & ∈ is the sigma notation or the summation notation.

Examples:

1. Let an=3_{n+1}, find S_6

   $S_6=a_1+a_2+a_3+a_4+a_5+a_6$.

   $a_1=4$; $a_2=7$; $a_3=10$; $a_4=13$; $a_5=16$; $a_6=19$

   $\Rightarrow S6 = 4 + 7 + 10 + 13 + 16 + 19 = \underline{\underline{69}}$

2. Let an$\frac{1}{n(n+1)}$, find the sum of the first

   a. 99 terms   b. n-terms

   Solution

an=$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}$ (using parties traction decompositions)

$\Rightarrow$ A(n+1)+B(n)=1

An+A+Bn=1

A+B=0&A=1⇒B=-1

$$\Rightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

a. $S_{99}=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+---+\left(\frac{1}{99}-\frac{1}{100}\right)=1-\frac{1}{100}=0.99$

b. $Sn=1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\frac{1}{3}---+\frac{1}{n-1}-\frac{1}{n}+\frac{1}{n}-\frac{1}{n+1}=1-\frac{1}{n+1}=\frac{n}{n+1}$

- Such a sequence is said to be telescoping sequence.

3. Find the sum of:

a. The first five odd natural numbers.

Solution

1,3,5,7,9

$S_5=a_1+a_2+a_3+a_4+a_5$

$=1+3+5+7+9$

$=\underline{\underline{25}}$

b. $a_n=n^2+1$, find S_6

Solution

$Sn=\sum_{n=1}^{n} n^2+1$

$\Rightarrow S_6=2+5+10+17+26+37=\underline{\underline{97}}$

Sigma notation

- Sigma notation is a method used to write out a long sum in a concise way.

Examples:- 1. Express the following sigma notation in the form of the sum:

a. $\sum_{k=1}^{8} 3k$ b. $\sum_{k=6}^{6} k^2$

Solution

a. $\sum_{k=1}^{8} 3k=3+6+9+12+15+18+21+24=\underline{\underline{108}}$

b. $\sum_{k=6}^{6} k^2=4+9+16+25+36=90$

c. $\sum_{k=3}^{5} k^3=3^3+4^3+5^3$

d. $\sum_{k=1}^{n} 3^k=3^1+3^2+3^3+3^4+3^5---+3^n$

Properties of sigma notation

$\sum_{k=1}^{n} a_n=c+c+c+---+c=nc$

a. $\sum_{k=1}^{n} c_{ak}$=c$\sum_{k=1}^{n} ak$, c is constant.
b. $\sum_{k=1}^{n}(ak-bk)=\sum_{k=1}^{n} ak-\sum_{k=1}^{n} bk$
c. $\sum_{k=1}^{n}(ak+bk)=\sum_{k=1}^{n} ak+\sum_{k=1}^{n} bk$
d. $\sum_{k=1}^{n} ak=\sum_{k=1}^{m} ak+\sum_{k=m+1}^{n} bk$,1 ≤m≤n.

Examples

1. Evaluate each of the following
   a. $\sum_{k=1}^{3} 4k$=4+8+12=24
   b. $\sum_{k=1}^{5}(3k-2)$=$\sum_{k=1}^{5} 3k$-$\sum_{k=1}^{5} 2$
      =3+6+9+12+15-5x2
      =45-10
      =35
   c. $\sum_{k=1}^{6} 2^{k-1}$=$2^{1-1}+2^{2-1}+2^{3-1}+2^{4-1}+2^{5-1}+2^{6-1}$
      = 1+2+4+8+16+32=63

1.3.1 Sum of arithmetic sequences

Theorem 3

The sum S_n of the first n-terms of arithmetic sequence with first term A1 & common differenced is

$S_n=\sum_{k=1}^{n} Ak=\frac{n}{2}(2A,+(n-1)d)=n(\left(\frac{A_1+A_n}{2}\right)$

☯ The sum of first n consecutive natural number is $S_n=\frac{n}{2}(n+1)$.

Examples

1. Find the sum of the first 35 terms of the sequence whose general term is A_n-S_n.
   Solution
   A_n-A_n⇒A1=5, A_2=10,A_3=15 & A_{35}=175

$S_n=\frac{n}{2}(A_1+A_n)$

$S_{35}=\frac{35}{2}(5+175)=35(90)=\underline{\underline{3,150}}$

2. Find the partial sum of the following arithmetic sequences.
   a. A1=2, & last term A10=$\underline{\underline{21}}$

      Solution

      $S_{10}=\frac{10}{2}(A_1+A_{10})=5(2+21)=23x5=\underline{\underline{115}}$

3. For a given arithmetic sequence the sum S10=165 and $A_1=3$, find A_{10}

   Solution

   $S_n=\frac{n}{2}(A_1+A_n) \Longrightarrow S_{10}=10\left(\frac{A_1+A_{10}}{2}\right)$

   $165=\frac{10}{2}(A_1+A_{10})$

   $165=5(A_1+A_{10})$

   $\frac{165}{5}=3+A_{10}$

   $\Rightarrow 33=3+A_{10}$

   $\Rightarrow A_{10}=\underline{\underline{30}}$

Sum of geometric sequence

Theorem: Let $\{G_n\}_{n=1}^{\infty}$be a geometric sequence with common ratio r. then the sum of the first n-terms S_n is given by, $S_n=\begin{cases} nG_1, if\ r=1 \\ G_1\ \frac{(1-r^n)}{1-r} \end{cases} = G_1\ \frac{(r^n-1)}{r-1}$, if $r\neq 1$

Examples

1. The sum of first three terms of a geometric sequence is 7, and the sum from 4[th] to 6[th] terms is 56. Find the first term and the common ratio.

   Solution

   Let the first terms be G_1 and common ratio r, then

   $G_1+G_2+G_3=7$

   $\Longrightarrow G_1+G_1r+G_1r^2=7$

   $\& G_4+G_5+G_6=56$

$\Longrightarrow G_1r^3+G_1r^4+G_1r^4$=56

Form 2 $r^3(G_1+G_1r+G_1r^2)$=56

But $G_1+G_1r+G_1r^2$=7

$\Longrightarrow r^3$(7)=56

$7r^3$=56

$\Longrightarrow r^3$=8 $\Longrightarrow r$ =2

Then $G_1+G_1r+G_1r^2=7$

$G_1+2G_1+4G_1=7$

$\Longrightarrow$ 7G_1=7 $\Longrightarrow$ G_1=1

$\therefore$ The common ratio is 2 and the 1st term is $\underline{\underline{1}}$

2. How many terms of the sequence:

$3,\frac{3}{2},\frac{3}{4},\frac{3}{8},---$ are needed to give the sum $\frac{3069}{512}$?

Solution

$r=\frac{3}{2}\div 3=\frac{3}{4}\div\frac{3}{2}$

$=\frac{1}{2}$

$S_n=\frac{G_1(r^n-1)}{r-1}; r\neq 1$

$\frac{3069}{512}=\frac{G_1(r^n-1)}{r-1}$

$\frac{3069}{512}=3\frac{\left(\left(\frac{1}{2}\right)^n-1\right)}{-\frac{1}{2}}$

$\frac{3069}{512}=-6\left(\left(\frac{1}{2}\right)^n-1\right)$

$\frac{3069}{3072}=1-\left(\frac{1}{2}\right)^n$

$\frac{3069}{3072}-1=-\left(\frac{1}{2}\right)^n$

$\frac{-3}{3072}$=-$\left(\frac{1}{2}\right)^n$

$\frac{1}{1024}$=-2^{-n}

2^{-10}=2^{-n}

$\Leftrightarrow \underline{\underline{n=10}}$

3. Find the sum to indicate number of terms in each of the geometric sequence in question a to d:
   a. 0.15,0.015,0.0015,---n terms

      Solution: $r = \frac{G_n+2}{G_n} = \frac{0.015}{0.15}$ =$\underline{\underline{0.1}}$

      $S_n = G_1 \left(\frac{1-\ r^n}{1-r}\right)$

      $\Longrightarrow\ S_n = 0.15\frac{(1-(0.1)^n}{1-0.1}$

      $= \frac{0.15}{0.9}(1-(0.1)^n)$

      $= \frac{15}{90}(1-(0.1)^n)$

      $\Longrightarrow S_n=\frac{1}{6}(1-(0.1)^n)$

   d. $S_n=G_1\left(\frac{1-r^n}{1-r}\right) = \frac{x^3\left(1-(x^2)^n\right)}{1-x^2}$ $=x^3\left(\frac{1-{}^x2n}{1-x^2}\right)$

1.4 Infinite series

- Suppose we have the sequence: a_1,a_2,a_3, ---, an, ---
- An infinite sum of the form a_1,a_2,+a_3+---+an+--- is called an infinite series & using summation, we can write as: a_1,a_2,a_3,+--- an+---=$\sum_{1-1}^{\infty} an$

Divergence and conversance of infinite sequence

- For an infinite sequence, there are cases of convergence and divergence, When n-approaches to infinity, as follows:

Convergence:

a. Sequence an converges to α: n→ $\infty, an \to \alpha$.

Divergence

b. Sequence and diverges to ∞:n→ $\infty, an \to -\infty$.
c. Sequence and diverges to $-\infty$:n→ $\infty, an \to -\infty$.
d. Sequence and diverges: an has no limit i.e. the value oscillator vibrate back and forth between numbers.

<u>Divergence / convergence of in finite geometric sequence</u>

Consider and infinite geometric sequence

$G_n=r^n$, where $G_1=r$, and the common ratio is r,

We have the following cases:

1. $r>1$, when n→ ∞,then $G_n\to\infty$ (diverge)
2. $r=1$, when n→ ∞,then $G_n\to 1$ (converge)
3. $|r|<1$, when n→ ∞,then $G_n\to 0$ (converge)
4. $r\le -1$, when n→ ∞,then G_n vibrates (no limit diverge)

<u>Examples</u>

1. Find whether the given geometric sequences, diverge, converge or vibrate as n→ ∞

   a. $G_n = (\sqrt{2})^n$

Solution

a. $G_n = (\sqrt{2})^n=(\sqrt{2})^n$

   As $\sqrt{2}$ >1, when n→ ∞, then $G_n \to \infty$

   $\Rightarrow$ G_n diverges

b. $G_n=\left(\frac{2}{3}\right)^n$

As $\left(\frac{2}{3}\right)$<1, when n→ ∞, then $G_n \to \infty$, it diverges.

c. G_n=$(-3)^n$

As 3≤-1, when n→ ∞, then G_n vibrate.

d. $G_n = (\sqrt{3})^n$, $G_1 = \sqrt{3}$ (Ex. 1.18)

$G_2 = (\sqrt{3})^2 \Rightarrow \sqrt{3}1$

$\Rightarrow G_n = (\sqrt{3})^2$ a divergent sequence.

c/G_n=$\left(-\frac{1}{2}\right)^n$, since -$\frac{1}{2}$<1

$\Rightarrow G_n$ is a convergent sequence

Infinite series

Definition:- Let $\{an\}_{n=1}^{\infty}$ be a sequence and S_n be the n^{th} partial sum such that, as n gets larger and larger, S_n tends to S, where S is a real number, than we say the infinite series $\sum_{n=1}^{\infty} an$

Converges and written as $\sum_{n=1}^{\infty} an$=S

However, if suh an S does not exist or infinite, we say the infinite series

$\sum_{n=1}^{\infty} an$ diverges.

Examples:

1. Consider the geometric sequence 1, $^2/_3$,$\frac{4}{9}$,--- where G_n=1& $^2/_3$. We have partial sum.

Sn= $\frac{G_1(1-r^n)}{1-r}$=$\frac{1\left(1-\left(\frac{2}{3}\right)^n\right)}{1-{^2/_3}}$ = $3\left(1-\left(\frac{2}{3}\right)^n\right)$

$\left(\frac{2}{3}\right)^n$ →0, as $n \to \infty$

$\Longrightarrow$ As $n \rightarrow \infty$, $\left(\frac{2}{3}\right)^n \rightarrow 0 \Longrightarrow$S=3

Thus, for an infinite geometric sequence $G_1, G_1r,\ G_1r^2$---, if numerical value of the common ratio r is between -181, then.

S$_n$=$\frac{G_1(1-r^n)}{1-r}$=$\frac{G}{1-r} - \frac{G_1r^n}{1-r}$

As $n \rightarrow \infty$,$r^n \rightarrow 0$, since $|r| < 1$& $then\ \frac{G_1r^n}{1-r}$=0

$\therefore$ S$_n \rightarrow \frac{G_1}{1-r}$, as $n \rightarrow \infty$,

Symbolically, the sum to infinity of an infinite geometric series is denoted by S_∞, Thus we have.

$$\boxed{S_\infty = \frac{G_1}{1-r}}$$

NB: Recall that if $\sum_{n-1}^{\infty} G_1r^{n-1} = G_1 + G_1r + G_1r^2$--- is a geometric series with first term G_1& common ratio r, then S$_n$=$\frac{G_1(1-r^n)}{1-r} = \frac{G_1}{1-r} - \frac{G_1r^n}{1-r}$.

i. If $|r| < 1$, as $n \rightarrow \infty$,$r^2 \rightarrow 0$. So that S$_n$=$\frac{G_1}{1-r} - \frac{G_1r^n}{1-r} \Longrightarrow\ S_\infty = \frac{G_1}{1-r}$

ii. If $|r| \geq 1$,then the series diverges or vibrate as follows

a. If r=1, then $\sum_{n-1}^{\infty} G_k = nG_1$

When $n \rightarrow \infty$, $\sum_{k=1}^{\infty} Gk = \infty(G, > 0)$

$$\sum_{k=1}^{\infty} Gk = -\infty(G, < 0)$$

b) If $r \neq 1, then\ \ \sum_{k=1}^{\infty} Gk = \frac{G_1(1-r^n)}{1-r}$

Then, if $r \leq -1, when\ n \rightarrow \infty, then \sum_{k=1}^{\infty} Gk$, vibrates,

$if\ r > 1, when\ n \to \infty, then \sum_{k=1}^{\infty} Gk = \infty(G, > 0)$

& $\sum_{k=1}^{\infty} Gk = -\infty(G, < 0)$

Examples

1) Determine whether the series $\sum_{n=1}^{\infty} 3^n$ converges or diverges.

Solution

The series $\sum_{n=1}^{\infty} 3^n = 3 + 9 + 27 + \cdots + \cdots$ is a geometric series with $G_1 = 3 \& r = 3$

$$S_n = \frac{G_1(1 - r^n)}{1 - r} = \frac{3(1 - r^n)}{1 - 3} = \frac{-3}{2}(1 - 3^n) = \frac{-3}{2} + \frac{3}{2}3^n$$

As $n \to \infty, s_n \to \infty, \Rightarrow$ the series diverse.

2) Find the sum for each of the following, if it exists, assuming the patterns continue as in the first few terms.

Ex.1.19)

d) $\frac{1}{5} + \frac{-1}{10} + \frac{1}{20} + \frac{-1}{40} + \ldots.$

Solution

we can rewrite

$\frac{1}{5} + \frac{-1}{10} + \frac{1}{20} \pm \frac{1}{40} + \ldots\ldots$

$$r = \frac{-1}{20} \div \frac{1}{5} = \frac{-1}{2}, G = \frac{1}{5}$$

It is a geometric series

$$S_\infty = \frac{1}{5} - \frac{1}{10} + \frac{1}{20} - \frac{1}{40} = \sum_{i=1}^{\infty} \left(\frac{1}{5}\right)^{n-1} = \frac{G_1}{1 - r} = \frac{\frac{1}{5}}{1 + \frac{1}{2}} = \frac{2}{15}$$

e) $7 + \frac{7}{10} + {}^{7}/_{100} + \frac{7}{1000} + \cdots$

It is a geometric series whose first term $G^1 = 7$ & $common\ ratio\ r = \frac{1}{10}$

$$\therefore S_\infty = 7 + \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + \cdots = \sum_{i=1}^{\infty} (7)\left(\frac{1}{10}\right)^{n-1} = \frac{70}{9}$$

Exercise

1) Find the sum for each of these geometric series.

d) $\sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{k+3}\left(\frac{2}{3}\right)^{k-1}$

Solution

- It is a geometric series whose first term $G_1 = \left(\frac{3}{4}\right)^4$ & common ratio $r = \frac{1}{2}$

$$S_\infty = \sum_{k=1}^{\infty}\left(\frac{3}{4}\right)^{k+3}\left(\frac{2}{3}\right)^{k-1}$$

$$= \left(\frac{3}{4}\right)^4\left[1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\cdots\right]$$

$$= \frac{G_1}{1-r} = \frac{\left(\frac{3}{4}\right)^4}{1-\frac{1}{2}} = 243$$

$$= 512$$

c) $\sum_{k=1}^{\infty} 5^{3-k}$ It is a geometric series with $G_1 = 25$ & $r = \frac{1}{5}$

$$S_\infty = \frac{G_1}{1-r} = \frac{25}{1-\frac{1}{5}} = \frac{125}{4}$$

1.4.1. Recurring Decimals

- recurring or repeating decimals are rational numbers (fractions) whose representations as a decimal contain a pattern of digit that repeats indefinitely after decimal point

Converting purely recurring decimals to fraction

Examples

1) Convert $0.\overline{3}$ to a fraction $0.\overline{3} = 0.333333\cdots$

$$= 0.3\ to.03 + 0.003 + \cdots$$

$$= \frac{3}{10} + \frac{3}{100} + \frac{3}{100} + \cdots$$

This is the in finite geometric series, where $G_1 = \frac{3}{10}$;

$$r = \frac{3}{100} \div \frac{3}{10} = \frac{1}{10}$$

Since $|r| < 1 \Rightarrow S_\infty = \frac{G_1}{1-r}$

$$= \frac{3}{10} \div 1 - \frac{1}{10}$$

$$= \frac{3}{10} \div \frac{9}{10} = \frac{1}{3}$$

2) Ex.

c) $0.5'6'$

$= 0.565656$

$= 0.56 + 0.0056\ to\ .000056$

$$= \frac{56}{100} + \frac{56}{10000} + \frac{56}{1000000}$$

$r = \frac{Gn}{Gn-1}$ since, $|r| < 1$

$$= \frac{5.6}{10000} \div \frac{56}{100} = \frac{1}{100}$$

Then $s_n = \frac{G_1}{1-r} = \frac{56}{100} \div 1 - \frac{1}{100} = \frac{56}{100} \div \frac{99}{100}$ $= \frac{56}{100} . \frac{100}{99} = \frac{56}{99}$

1.5. Applications of sequence & series in Daily life

Examples

Q4) Sofia deposits birr 3,500 in a bank account paying an annual interest rate of 6%. Find the amount she has at the end of the;

a) First year b) second year c) third year

b) Four year e) n^{th} year

f) Do the amounts she has at the end of each year form a geometric sequence? Explain.

Solution: Let $G_1 = 3{,}500\ birr$. Then $G_2 = G_1 + \frac{6}{100}\ G, = G_1(1\ to\ 06)$

$$\Rightarrow G_n = G_1\ (1.06)n - 1$$

a) At the end of the first year, she has: $G_1 = 3{,}500(1.06)^{1-1} = 3.500$

b) At the end of the second year ….. $G_2 = G_1(1.06)^{2-1} = 3{,}500 \times 1.06 = 3710$

c) At the end of the 3rd year ……..$G_3 = G_1(1.06)^2 = 3932.60$

d) At the end of the 4th year : $G_4 = G_1(1.06)^{n-1}$

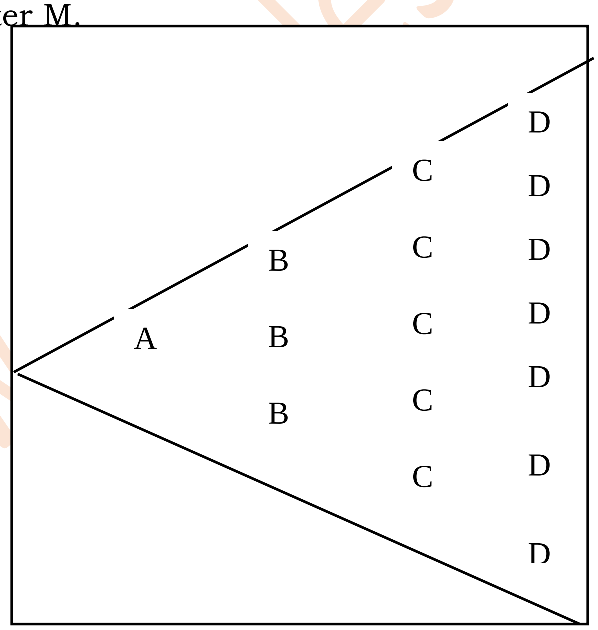
e) At the end of the nth year $G_n = G_1(1.06)^{n-1}$

f) Since the ratio any two consecutive terms is a constant, which is 1.06, this sequence is a geometric sequence.

g) **Example**

2) In the figure

a) If the pattern continues, find the number of letters. In the column containing the letter M.



b) If the total no of letters in the pattern is 361, which letter will be in the last column?

Solution

a) If we observe the structure of the letters, it is of the form:

A, 3,5,7,9, - - - -

This is an arithmetic sequence with $A_1=1$ & d=2

$\therefore A_n = A_1 + (n-1)d$

For Letter M: n=13

$Al_3 = A_1 + (n-1)d = 1 + (13-1) \times 2 = 1. +12 \times 2 = 25$

b) $s_n = \frac{n}{2}\,(2A_1 + (n-1)d)$

$\Rightarrow 361 = \frac{n}{2}\,(2(1) + (n-1)(2))$

$\Rightarrow 361 = \frac{n}{2}\,(2 + 2n - 2)$

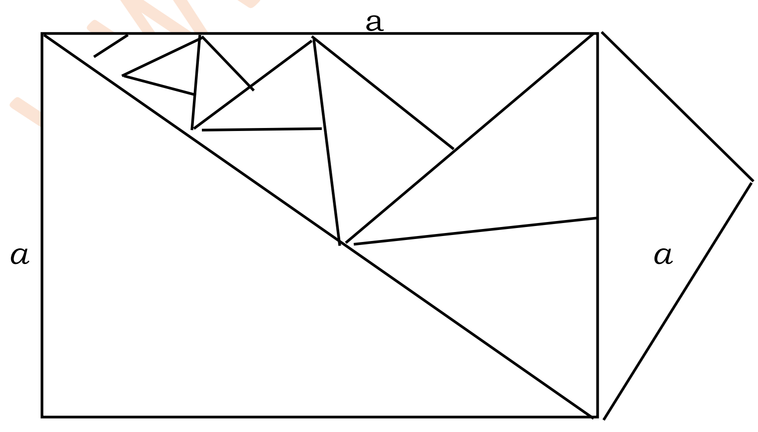
$361 = \frac{n}{2}(2n)$

$\Rightarrow 361 = n^2 \Rightarrow n = \pm 19(but\ n > 0$

$\boldsymbol{n = 19}$

So, the letter "S" will be in the last column.

3) Given a square with side length "a". The side of the second square is halt of its diagonal. The side of the 3rd square is half of the diagonal of the second square & 500 n. Find the sum of the areas of all the squares.



Let the side of square be a_n, diagonal d_n,

Then $a_0 = a$

$$a_1 = \frac{1}{2}\, do = \frac{1}{2}\sqrt{2}\, a = \frac{\sqrt{2}}{2} a$$

$$a_2 = \frac{1}{2} d_1 = \frac{1}{2}\sqrt{2}\, a_1 = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2}\, a$$

$$a_3 = \frac{1}{2} d2 = \frac{1}{2}\sqrt{2}\, a2 = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} a$$

$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$

$$s_n = ao^2 + a1^2 + a3^2 + \cdots$$

$$= a^2 + \frac{1}{2}a^2 + \frac{1}{4}a^2 + \frac{1}{8}a^2 + \cdots$$

Since, $r = \frac{G_n}{G_n - 1} = \frac{1}{2}, |r| < 1, then\ S_\infty = \frac{G_1}{1-r}$

$$= \frac{a^2}{1 - \frac{1}{2}} = 2a^2$$

Review exercise on unit -1

Q1&2

1. List the five terms of each of the sequences whose general terms are given below where n is a positive integer.

   a. $a_n = \frac{n^n}{n!}$

   $a_1 = \frac{1^1}{1!} = \frac{1}{1} = 1$

   $a_2 = \frac{2^2}{2!} = \frac{4}{2} = 2$

   $a_3 = \frac{3^3}{3!} = \frac{27}{3x2x1} = \frac{27}{6} = \frac{9}{2}$

   $a_4 = \frac{4^4}{4!} = \frac{4x4x4x4}{4x3x2x1} = \frac{256}{6x4} = \frac{256}{8x3}$

$a_5 = \frac{5^5}{5!} = \frac{5x5x5x5x5}{5x4x3x2x1} = \frac{625}{24} =$

b. $a_n = (-1)^n + (-1)^n \sin(n\pi)$

$a_1 = (-1)^1 + (-1)^1 \sin(\pi)$

$= -1 + (-1)(10)$

$= \underline{\underline{-1}}$

$a_2 = (-1)^2 + (-1)^2 \sin(2\pi)$

$= 1+1.0=\underline{\underline{1}}$

$a_3 = (-1)^3 + (-1)^3 \sin(3\pi)$

$= -1+(-1)(0)$

$=\underline{\underline{-1}}$

$a_4 = (-1)^4 + (-1)^4 \sin(4\pi)$

$= 1+(-1)(0)$

$=\underline{\underline{1}}$

$a_5 = (-1)^5 + (-1)^5 \sin(5\pi)$

$= -1+(-1)(0)$

$= \underline{\underline{-1}}$

c. $a_n = \text{sgn}(3\text{-}n)$

$a_1 = \text{sgn}(3-1)=\text{sgn}(1)=1$

$a_2 = \text{sgn}(3-2)=\text{sgn}(1)=1$

$a_3 = \text{sgn}(3-3)=\text{sgn}(0)=0$

$a_4 = \text{sgn}(3-4)=\text{sgn}(-1)=-1$

$a_5 = \text{sgn}(3-5)=\text{sgn}(-2)=-1$

d. $a_n = 3^{1/n}(-1)^n$

$a_1 = 3^{\frac{1}{1}}(-1)^1=\underline{\underline{-3}}$

$a_2 = 3^{\frac{1}{2}}(-1)^2=3^{\frac{1}{2}}=\sqrt{2}$

$a_3 = 3^{\frac{1}{3}}(-1)^3=-\sqrt[3]{3}$

$a_4 = 3^{\frac{1}{4}}(-1)^4=\sqrt[4]{3}$

$a_5 = 3^{\frac{1}{5}}(-1)^5=-\sqrt[5]{3}$

e. $a_n = ne^{-2n}$

$a_1 = 1.e^{-2} = e^{-2} = \frac{1}{e^2}$

$a_2 = 2.e^{-4} = \frac{2}{e^4}$

$a_3 = 3e^{-6} = \frac{3}{e^6}$

$a_4 = 4e^{-8} = \frac{4}{e^8}$

$a_5 = 5e^{-10} = \frac{5}{e^{10}}$

f. $a_n = \frac{(-2)^n+6}{(n-1)!}$

$a_1 = \frac{(-2)^1+6}{(1-1)1} = \frac{-2+6}{0!} = \frac{4}{1} = \underline{\underline{4}}$

$a_2 = \frac{(-2)^2+6}{(2-1)!} = \frac{4+6}{1!} = 10$

$a_3 = \frac{(-2)^3+6}{(3-1)!} = \frac{-8+6}{2!} = \frac{-2}{2} = -1$

$a_4 = \frac{(-2)^4+6}{(4-1)!} = \frac{16+6}{3!} = \frac{22}{3x2x1} = \frac{11}{3}$

$a_5 = \frac{(-2)^5+6}{(5-1)!} = \frac{-32+6}{4!} = \frac{-26}{4x3x2x1} = \frac{-26}{4x3x2} = \frac{13}{12}$

g. $a_n = (-1)^n - \frac{1}{n^2}$

$a_1 = (-1)^1 - \frac{1}{1^2} = -1-1 = -2$

$a_2 = (-1)^2 - \frac{1}{2^2} = 1 - \frac{1}{4} = \frac{3}{4}$

$a_3 = (-1)^3 - \frac{1}{3^2} = -1 - \frac{1}{9} = -\frac{10}{9}$

$a_4 = (-1)^4 - \frac{1}{4^2} = 1 - \frac{1}{16} = -\frac{15}{16}$

$a_5 = (-1)^5 - \frac{1}{5^2} = -1 - \frac{1}{25} = -\frac{26}{25}$

h. $a_n = cos\left(\frac{n\pi}{2}\right)$

$a_1 = cos\left(\frac{\pi}{2}\right) = 0$

$a_2 = cos\left(\frac{2\pi}{2}\right) = -1$

$a_3 = cos\left(\frac{3}{2}\pi\right) = 0$

$a_4 = cos\left(\frac{4}{2}\pi\right)= cos(2\pi) = 1$

$a_5 = cos\left(\frac{5}{2}\pi\right)= 0$

2. Determine whether the sequences with the following general terms are arithmetic.

   a. $a_n = \frac{n^2+5n+6}{n+2}$, To show whether the given sequence is arithmetic or not show that $a_n+1\text{-}a_n$=constant.

   $a_n = \frac{n^2+5n+6}{n+2}$, $a_n+1=\frac{(n+1)^2+5(n+1)+6}{(n+1)+2} = \frac{n^2+2n+1+5n+5+6}{n+3} = \frac{n^2+7n+12}{n+3}$

   $and\ \ a_{n+1} + a_n = \frac{n^2+7n+12}{n+3} - \frac{n^2+5n+6}{n+2}$

   $= \frac{(n^2+7n+12)(n+2)-(n^2+5n+6)(n+3}{(n+3)(n+2)}$

   $= \frac{(n^3+2n^2+7n^2+14n+12n+24)-(n^3+3n^2+5n^2+15n+6n+18}{(n+3)(n+2)}$

   $= \frac{n^3+9n^2+26n+24-n^3+8n^2-21n-18}{(n+3)(n+2)}$

   $= \frac{n^2+5n+6}{n^2+5n+6} = 1 \rightarrow$ Which is constant

   $\Longrightarrow a_n=\frac{n^2+5n+6}{n+2}$ is an arithmetic sequence.

   b. $a_n\sqrt{4a+1}$, $a_{n+1}=\sqrt{4(n+1)+1} = \sqrt{4n+4+1} = \sqrt{4n+5}$

   $\Longrightarrow a_{n+1} - a_n=\sqrt{4n+5}$ - $\sqrt{4n+1}$ which is not constant

   $\Longrightarrow a_n=\sqrt{4n+1}$ is not arithmetic sequence

$\therefore$ The three numbers are: $\frac{-50}{369}, \frac{-4}{3}, \frac{-328}{25}$

Q7 Find the sum of:-

a. $\sum_{k=1}^{8}(k^3 + 2k^2 - 3k + 5) = \sum_{k=1}^{8} k^3 + 2\sum_{k=1}^{8} k^3 - 3\sum_{k=1}^{8} k + \sum_{k=1}^{8} 5$

$= 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3 + 2(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2)$ -3(1+2+3+4+5+6+7+8)+8x5

= (1+8+27+64+125+343+512)+2(1+4+9+16+25+36+49+64)-3(36)+40

= 980+2(204)-108+40

= 980+408-108+40

= 980+340=1320

b. $\sum_{k=2}^{5} 34x3 = 12$

c. $\sum_{k=1}^{5}\left(\frac{1}{k^2+5k+6}\right)=\sum_{k=1}^{5}\left(\frac{1}{(k+2)(k+3)}\right)$

Using partial fraction decomposition, we have $\frac{A}{k+2}+\frac{B}{k+3}=\frac{1}{(k+2)(k+3)}$

=A(k+3)+B(k+2)=1

Ak+3A+Bk+2B=1

$-2\begin{cases} A+B=0 \\ 3A+2B=1 \end{cases} \Longrightarrow \begin{matrix} -2A+2B=0 \\ 3A+2B=1 \end{matrix}$ <u>A=1 & B=-1</u>

$\therefore \sum_{k=1}^{5}\left(\frac{1}{(k+2)(k+3)}\right) = \sum_{k=1}^{5}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)= \sum_{k=1}^{5}\frac{1}{k+2}$ - $\sum_{k=1}^{5}\frac{1}{k+3}$

$= \left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)$ - $\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$

$= \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$ - $\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$

$= \frac{1}{3}+\frac{1}{8} \qquad \frac{8-3}{24}=\frac{5}{24}$

$\therefore \sum_{k=1}^{5}\left(\frac{1}{k^2+5k+6}\right)=\frac{5}{24}$

d. $\sum_{m=1}^{10} ln\left(\frac{m}{m+1}\right)=\sum_{m=1}^{10}(\ln((m)-\ln(m+1))$

$= \sum_{m=1}^{10} ln(m)$- $\sum_{m=1}^{10} ln(m+1)$

$= -ln6 = ln\left(\frac{1}{6}\right)$

<u>Q8</u>. Let $\sum_{i=1}^{5} xi = 37, \sum_{i=1}^{5} yi = 127$

$\sum_{i=1}^{5} xi^2 = 303;\ \ \sum_{i=1}^{5} yi^2 = 50$ & $\sum_{i=1}^{5} xiyi = 105$

Then a valuate the following:-

a) $\sum_{i=1}^{5}(2xi-3yi) = \sum_{i=1}^{5} 2xi - \sum_{i=1}^{5} 3yi = 2\sum_{i=1}^{5} xi - 3\sum_{i=1}^{5} yi$

$= 2x37 - 3x127$

$= 74 - 381$

$\underline{\boldsymbol{= -307}}$

b) $\sum_{i=1}^{5}(2xi + 3yi)$

$$= \sum_{i=1}^{5} 2xi + \sum_{i=1}^{5} 3yi$$

$$= 2\sum_{i=1}^{5} xi + 3\sum_{i=1}^{5} yi$$

$= 2 \times 37 + 3x127$

$= 74 + 381$

$= 455$

c) $\sum_{i=1}^{5}(2xi - 3yi)^2$

$$= \sum_{i=1}^{5}(4xi^2 - 12xiyi + 9yi^2)$$

$$\sum_{i=1}^{5} 4x^2i - \sum_{i=1}^{5} 12xiyi + \sum_{i=1}^{5} 9yi$$

$$= 4\sum_{i=1}^{5} xi^2 - 12\sum_{i=1}^{5} xiyi + 9\sum_{i=1}^{i=1} yi^2$$

$= 4 \times 303 - 12 \times 105 + 9 \times 50$

$= 1212 - 1260 + 450$

$= 1662 - 1260$

$= 402$

d) $\left(\sum_{i=1}^{5} xi\right)^2 = 37^2 = 1369$

e) $\sum_{i=1}^{5}(2xi - 5yi + 3)$

$$= 2\sum_{i=1}^{5} xi - 5\sum_{i=1}^{5} yi + \sum_{i=1}^{5} 3$$

$= 2x37 - 5 \times 127 + 3 \times 5$

$= 74 - 63 + 15 = -546$

Q9. Find the sum of all the natural numbers between 100 and 1000 which are multiples of 5.

Solution:-

Natural numbers lying between 100 & 1000, which are a multiple of 5, are, 105,110,115, 120,125,---, 995, with a1=105,d=5, An=995

Bet An=$a_1 + (n-1)d$

$\Rightarrow$ 995=105+(n-1)x5 $\therefore$ $S_n = \frac{n}{2}\ (2a_1 + (n-1)d)$

$\Rightarrow$ 995=105+sn-5 $\Rightarrow$ S $=\frac{179}{2}\ (2x105 + (179-1)x5)$

$\Rightarrow$ 995=105-5+5n $\Rightarrow$ S =179 $(105 + 89x5)$

$\Rightarrow$ 995.100=5n $\Rightarrow$ S =179 $(105 + 445)$

$\Rightarrow$ 895=5n $\Rightarrow$ S =179 (550)

$\Rightarrow$ n=179 $\Rightarrow$ S = 98450

$\therefore$ The sum of all natural numbers lying between 100 & 1000 , which are multiple of 5, is 98450.

Q10. IN an arithmetic progression the first term is 7 & the sum of the 1st five terms is one quarter of the next five terms. Show that the 20th term is -112.

Solution

$a_1 = 2$

The arithmetic progressions can be expressed as:
2,2+d,2+2d,2+3d,2+4d,....

- Sum of next five terms = 2++2d+2+2d+2+3d+2+4d=10+10d
- Sum of next five terms = 2+5d+2+6d+2+7d+2+8d+2+9d=10+35d

  But the sum of the first five terms= ¼ (the sum of the next five terms)

  $\Rightarrow$ 10+10d= ¼ (10+35d)

  $\Rightarrow 10 + 10d = \frac{10}{4} + \frac{35}{4}d$

$$\Longrightarrow 10 - \frac{10}{4} = \frac{35}{4}d - 10d \Longrightarrow \ d = -8$$

$a_{20} = a_1$+(n-1)d

= 2+(20-1)x-6

= 2+19x-6

$\Longrightarrow$ a_{20} =2-114

$\Longrightarrow$ a_{20} =-112 which is the required number.

Q11. How many terms of the arithmetic progression -6, - $\frac{11}{2}$, -5, … are needed to give the sum-25.

Solution

$S_n = \frac{n}{2}\,(2a_1 + (n-1)d)$

$-25 = \frac{n}{2}\,(2x - 6 + (n-1)\frac{1}{2})$

$-50 = n\,\left(-12 + \frac{n}{2} - \frac{1}{2}\right)$

$-50 = n\,\left(-\,\frac{25}{2} - \frac{n}{2}\right)$

$-50 = -\,\frac{25}{2}n - \frac{n^2}{2}$

$-100 = -\,25n - n^2$

n^2-25n+10=0

n^2-20n-5n+100=0

n(n-20)-5(n-20)=0

(n-5)(n-20)=0

<u>n=5 or n-20</u>

Q12. The sum of a certain number of terms of the arithmetic progression 25,22,19,… is 116. Find the last term

Solution

S_n $=\frac{n}{2}$(2a,+(n-1)d) $\Longrightarrow$ 116= $\frac{n}{2}$(50+(n-1)-3)

$\Longrightarrow$ 232= n(50+3-3n)

$\Longrightarrow$ 232= n(53-3n)

$\Longrightarrow$ 232= 53n-3n^2

$\Longrightarrow$ -$3n^2 + 53n - 232 = 0$

$\Longrightarrow$ $3n^2 - 53n - 232 = 0$

$\Longrightarrow$ $3n^2 - 53n + 232 = 0$

$\Longrightarrow$ $3n(n-8) - 20(n-8) = 0$

$\Longrightarrow$ $(3n-20)(n-8) = 0$

$\Longrightarrow$ $n = 8$

The last term means A_8=a_1+(n-1)d

=25+(8-1)(-3)

=25+(7)(-3)

=25-21=4

Q13. If the sum of n- terms of an arithmetic progression is $(P_n + q_{n^2})$, where P and q are constants find the common difference.

Solution

S_n=$\frac{n}{2}(2a_1 + (n-1)d)$ =$p_n + q_{n^2}$

$\Longrightarrow \frac{n}{2}(2a_1 + nd - d)$ $=p_n + q_{n^2}$

$\Longrightarrow \frac{2na_1}{2} + \frac{n^2d}{2} - \frac{nd}{2}$ $=p_n + q_{n^2}$

$\Longrightarrow na_1 + \frac{n^2d}{2} - \frac{nd}{2}$ $=p_n + q_{n^2}$

$by\ comparing\ the\ coefficients\ we\ have$

$\frac{d}{2} = q \Longrightarrow d = 2q$

∴ The required common difference d is <u>2q.</u>

Q14. If the partial sum of an arithmetic sequence $\{A_n\}$ is $4n^2$, find $A_n\ and\ A_{10}$.

Solution

S_n=$\frac{n}{2}(2a_1 + (n-1)d)$ $=4_{n^2}$

$\Longrightarrow$ but $S_n - S_{n-1} = A_n \Longrightarrow 4_{n^2} - (4(n-1)^2) = A_n$

$= 4_{n^2} - 4(n^2 - 2n + 1) = A_n$

$= 4_{n^2} - 4n^2 + 8n - 4 = A_n$

$\Longrightarrow A_n = 8n - 4$ & A_{10}=8x10-4=<u>78</u>

Q15&16. Convert this mixed recovering decimal $0.3\overline{17}$ to traction.

a. $0.3\overline{17}$
b. $0.3\overline{7}$
c. $3.23\overline{54}$

Solution

d. $\frac{317-3}{990} = \frac{314}{990}$ b. $0.3\overline{7}$=$\frac{37-3}{90} = \frac{34}{90}$ c. $3.23\overline{54}$=$\frac{32354-23}{9900} = \frac{32331}{9900}$

Q17. The first three terms of a convergent geometric series are x+1,x-1, 2x-5.

a. Find the values of x($x \neq 1, or -1$)
b. Find sum to infinity of the series.

Solution

First find the common ratio r.

$\Longrightarrow r = \frac{2x-5}{x-1} = \frac{x-1}{x+1} = \frac{G_n+1}{G_n}$

$\Longrightarrow (2x-5)(x+1) = (x-1)(x-1)$

$\Longrightarrow 2x^2 + 2x - 5x - 5 = x^2 - 2x + 1$

$2x^2 + 3x - 5 = x^2 - 2x + 1$

$x^2 - x - 6 = 0$

$x^2 - 3x + 2x - 6 = 0$

$x(x-3) + 2(x-3) = 0$

$(x+2)(x-3) = 0$

$\Leftrightarrow$ x+2=0 or x-3=0

$\underline{x = -2 \ or \ x = 3}$

The value of x which satisfy the given condition is <u>x=3</u>.

b. $S_\infty = \frac{G_1}{1-r} = \frac{4}{1-\frac{1}{2}} = 8$

Q18. Find P: If $\sum_{k=1}^{\infty} 27p^k = \sum_{x=1}^{12}(2x-3x)$

$\sum_{x=1}^{12}(2x-3x)$=21+18+15…

This is an arithmetic sequence with $A_1 = 21 \& d = -3$.

$$S_n = \frac{n}{2}(2a_1 + (n-1)d) \Rightarrow S_{12} = \frac{12}{2}(2x21 + (12-1)(-3))$$

$$= 6(42 + (11)(-3))$$

$$= 6(42 - 33)$$

$$= 6(9)$$

$$= \underline{54}$$

And $\sum_{k=1}^{\infty} 27p^k = 27p + 27p^2 + 27p^3 + \cdots$

Is a geometric series with G1=27p, r=p(-1<p<1)

$$\Longrightarrow S_\infty = \frac{G_1}{1-r} = \frac{27p}{1-p} \Rightarrow \frac{27p}{1-p} = 54 \Rightarrow 54 - 54p = 27p$$

$$\Rightarrow -54p - 27p = -54$$

$$\Rightarrow -81p = -54$$

$$\Rightarrow p = \frac{54}{81} = {}^{2}/_{3}$$

Q19. Find the product $4 \times 4^{\frac{1}{2}} \times 4^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times \ldots 4^{\frac{1}{2n}} \ldots$

Solution

$$4^1 \times 4^{\frac{1}{2}} \times 4^{\frac{1}{4}} \times 4^{\frac{1}{8}} \times \ldots 4^{\frac{1}{2n}} \ldots \times 4^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\frac{1}{2^n}}$$

the exponents $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \frac{1}{2^n}$ is a geometric convergent services with

$$G_1 = 1, r = \frac{1}{2} \Rightarrow 4^{\frac{G_1}{1-r}} = 4^{\frac{1}{1-\frac{1}{2}}} = 4^{1 \div \frac{1}{2}} = 4^2 = 16$$

Q20. If $\sum_{k=1}^{\infty} S^{kt}=\frac{1}{24}$, find the values of t.

$\sum_{k=1}^{\infty} S^{kt}= 5^t + 5^{2t} + 5^{3t} + \cdots . + \cdots = \frac{1}{24}$ is

A geometric convergent series with

$G_1 = 5^{t,} r = 5^t$

$\Longrightarrow S_\infty=\frac{G_1}{1-r} \Rightarrow \frac{5^t}{1-5^t} = \frac{1}{24}$

$\Rightarrow 5^t.24 = 1 - 5^t$

$\Rightarrow 5^t.24 + 5^t = 1$

$\Rightarrow 5^t(24 + 1) = 1$

$\Rightarrow 5^t = \frac{1}{25}$

$\Rightarrow 5^t = 5^{-2} \Leftrightarrow t = -2$

Q21. If the product $5^k.5^{k^2}.5^{k^3} \ldots = 5, find\ k$

Solution

$5^k.5^{k^2}.5^{k^3} \ldots = 5 \Rightarrow 5^{k+k^2+k^3+\cdots+\cdots} = 5^1$

The exponents $k + k^2+k^3 + \cdots +..$ is a geometric series with $G_1 = k, r = k$

$\Longrightarrow S_\infty=\frac{G_1}{1-r} \Rightarrow 5^{\frac{G_1}{1-r}} =5$

$\Rightarrow 5^{\frac{k}{1-k}} =5^{-1}$

$\Leftrightarrow \frac{k}{1-k} = 1$ $\Rightarrow$k=1-k

$\Rightarrow$2k=1

k= $\underline{½}$

Q22. Evaluate $\sum_{k=1}^{11}(2+3^k)$

Solution

$\sum_{k=1}^{11} 2 + \sum_{k=1}^{11} 3^k$

= $2x11+3^1+3^2+3^3+\cdots+3^{11}$

= $22+G_1\frac{(1-r^n)}{1-r}$, Since $3+3^2+3^3+\cdots+3^{11}$ is a geometric series with r>1,

$\Longrightarrow S_n = G_1\left(\frac{1-r^n}{1-r}\right)$

$\Longrightarrow 22+3\left(\frac{1-r^{11}}{1-r}\right)$

$\Longrightarrow 22+3\left(\frac{1-3^{11}}{1-3}\right) = 22+3\frac{(1-3^{11})}{1-3}$

$= 22-\frac{3}{2}(1-3^{11})$

$= 22+\frac{3}{2}(3^{11}-1)$

$\Longrightarrow \sum_{k=1}^{11}(2+3^k) = 22+\frac{3}{2}(3^{11}-1)$

$= 22+\frac{3}{2}(177145)$

= 22+ 265717.5

= $\underline{265739.5}$

Q23. The sum of the first three terms of a geometric progression is $\frac{39}{10}$ & their product is 1. Find the common ratio and the three terms.

Solution

Let $\frac{a}{r}, a$be the first three terms of the geometric series.

$\frac{a}{r} + a + ar = \frac{39}{10} - 0$ and $\frac{a}{r} * a * ar = 1$

$\Rightarrow \frac{a^3 r}{r} = 1 \Rightarrow a^3 = 1 \quad \Rightarrow$ a=$\underline{1}$

$\Rightarrow \frac{1}{r} + 1 + r = \frac{39}{10}$

$\Rightarrow \frac{1+r+r^2}{r} = \frac{39}{10} \Rightarrow 10+10r+10r^2= 39r$

$\Rightarrow 10r^2+10r-39r+10= 0$

$\Rightarrow 10r^2-29r+10= 0$

$\Rightarrow 10r^2-25r-4r+10= 0$

$\Rightarrow 10r^2-25r-4r+10= 0$

$\Rightarrow 5r(2r-5)-2(2r-5)=0$

$\Rightarrow (2r-5)\ (5r-2)=0$

$\Leftrightarrow 2r-5=0$ or $5r-2=0$

$\Rightarrow r={}^5/_2$ or $r = {}^2/_5$

$\therefore$ The common ratio is ${}^5/_2$ or ${}^2/_5$ and the three terms are : $\frac{5}{2}, 1\frac{2}{5}$

Q24. How many terms of the geometric progression $3, 3^2, 3^3, \ldots$ are needed to give the sum 120?

Solution

$S_n = G_1\frac{(r^n-1)}{r-1} \Longrightarrow 120 = 3\frac{(3^n-1)}{3-1} \Longrightarrow 120 = 3\frac{(3^n-1}{2}$

$with\ G_1 = 3$ $\Longrightarrow \frac{240}{3} = 3^{n-1}$

$\Longrightarrow 80 = 3^{n-1}$

$\Longrightarrow 80 + 1 = 3^n$

$\Longrightarrow 81 = 3^n$

$\Longrightarrow 3^4=3^n \Leftrightarrow n=4$

Q25. The sum of the first three terms of a geometric progression is 16 and the sum of next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the geometric progression.

Solution

Let a, ar, a$r^2, \ldots ar^{n-1}$ be the given geometric progressions

a, ar, ar^2=16 and $ar^3 + ar^4 + ar^5 = 128$

$\Longrightarrow$ a(1+r+r^2)=16 & ar^3(1+r+r^2)=128

$\Longrightarrow \frac{ar^3(1+r+r^2)}{a(1+r+r^2} = \frac{128}{16} \Longrightarrow r^3 = 8 \Longrightarrow \boldsymbol{r = 2}$

And $a(1 + r + r^2)$ =16 a S_n=a$\frac{(r^n-1)}{r-1}$

$\Longrightarrow a\ (1 + 2 + 2^2)$ =16 =$\frac{16}{7}\frac{((2)^n-1)}{2-1}$

$\Longrightarrow a(7)$ =16 $\Longrightarrow S_n$=$\frac{16}{7}(2^n - 1)$

$\Longrightarrow$ 7a =16

$a = \frac{16}{7}$

Q26. Given a geometric progression with a= 779& 7th terms 64, determine S_7

Solution

G_1 = 779, $G_1 r^6$ =G_7=64

$\Longrightarrow$ 729r^6=64r^6=$\frac{64}{729} \Longrightarrow r = \sqrt[6]{\frac{64}{729}}$

$\Longrightarrow r = \sqrt[6]{\frac{2^6}{36^6}} \Longrightarrow r$=$\frac{2}{3}$

S_n =$G_1 \frac{(r^n-1)}{r-1} \Longrightarrow S_7$=729 $\left(\frac{\left(\frac{2}{3}\right)^7-1}{\frac{2}{3}-1}\right)$

=729x3 $\left(\frac{3^7-1}{3^7}\right)$=$(3)^7-(2)^7$ = $\underline{\underline{2059}}$

Q27. If the 4th, 10th & 16th terms of a geometric progress n are x, y & z, respectively. Prove that x, y t are in geometric progression.

Solution

Let G_1 be the first term & r be the common ratio.

$G_4 = G_1 r^3 = x \ldots 1$

$G_{10} = G_1 r^9 = y \ldots 2$

$G_{16} = G_1 r^{15} = z \ldots 3$

$\frac{y}{x} = \frac{G_1 r^9}{G_1 r^3} \Longrightarrow \frac{y}{x} = r^6$ & $\frac{z}{y} = \frac{G_1 r^{15}}{G_1 r^9} \Longrightarrow \frac{z}{y} = r^6$

$\therefore \frac{y}{x} = \frac{z}{y}$ Which in geometric progression j.e. x,y,z are in geometric progression.

Q28. Find the sum to n terms of the sequence 8,88,888,8888,...

Solution

The given sequence 8,88,888,8888, ... are not a geometric series; but, it can be changed to geometric series by writing the terms as:

S_n=8+88+888+8888+...+n terms

= $^8/_9$ (9+99+999+9999+...+ to n terms)

= $^8/_9$ $\left((10-1)+(10^2-1(10^3-)+(10^4-1)+\cdots+n\ terms\right)$

= $\frac{8}{9}$ $[(10+10^2+\cdots+n\ \ terms)-(1+1+1+\cdots+n\ terms]$

= $^8/_9\left[10\frac{(10^n-1)}{10-1}-n\right]=\frac{8}{9}\left[10\frac{(10^n-1)}{9}-n\right]=\frac{80}{81}\left[(10^n-1)-\frac{8}{9}n\right]$

Q29. Show that the products of the corresponding terms of the sequences a, ar, ar^2,ar^{n-1} $and\ A, AR, AR^2, \ldots\ AR^{n-1}$ form a geometric progression, and find the common ratio.

Solution

It has to be proved that the sequence $aA, arAR, ar^2AR^2, \ldots\ ar^{n-1}AR^{n-1}$ forms a geometric series.

$\Rightarrow \frac{Second\ term}{first\ term}=\frac{arAR}{ar}$=rR...1 and

$\Rightarrow \frac{thirs\ \ term}{second\ term}=\frac{ar^2AR^2}{arAR}$=rR...2

Since the common ratio in both equation is the same⟹ the given sequence forms a geometric series with common ratio is rR.

Q30. If the p^{th} , q^{th} & r^{th} terms of a geometric progression are a, b, &c respectively prove that $a^{q-r}b^{r-p}c^{p-q}=1$

Solution

Let A be the first term & R be the common ratio of the geometric series.

$\Longrightarrow AR^{p-1} = a$

$AR^{q-1} = b$

$AR^{r-1} = c$

$\Longrightarrow a^{q-r}b^{r-p}c^{p-q} = A^{q-r}R^{(p-1)(q-r)}xA^{r-p}R^{(q-1)(r-p)}A^{p-q}R^{(r-1)(p-q)}$

$= Aq^{-r+r-p+p-q}xR^{(pr-pr-q+r)+(rq-r+p-p-pq)+(pr-p-qr+q)}$

$= A^0R^0=\underline{\underline{1}}$ the given result is proved.

Q31. If the first and the n^{th} terms of a geometric progression a & b, respectively, & if p is the product of n terms, prove that $p^2 = (ab)^n$

Solution

The first terms of the geometric series is a and the last term is b

$\therefore$ The geometric series is: a, ar, ar^2, ar^3, … ar^{n-1} where r is the common ratio.

b=ar^{n-1}

& product of n terms.

(a)(ar)(ar^2)… (ar^{n-1})

= (axax…xa)(rxrx…xr^{n-1})

$= a^n . r^{1+2+\cdots n-1}$

Here 1,2,…(n-1) is an A.P

$\therefore$ 1+2+…+(n-1)=$\frac{n-1}{2}$(2a+(n-1-1)d)

$= \frac{n-1}{2}\,[2 + n - 2] = \frac{n(n-1)}{2}$

P=$a^n r^{\frac{n(n-1)}{2}}$

$\therefore\ P^2 = \left(a^n r^{\frac{n(n-1)}{2}}\right)^2$

$\Longrightarrow P^2 = a^{2n}\, r^{n(n-1)}$

$\Longrightarrow P^2 = \left[a^2 r^{(n-1)}\right]^n$

$\Longrightarrow P^2 = (ab)^n$, the given results proved

Q32. If a, b, c, & d are in geometric progression, show that $(a^1b^2c^3)(b^2c^2d^2) = (ab + bc + cd)^2$

Solution

A,b,c,d are in G.P

$\Longrightarrow$ bc=ad

$b^2 = ac$

$c^2 = bd$

$\Longrightarrow (a^2 + b^2 + c^2)(b^2 + c^2 + d^3) = (ab + bc + cd)^2$

RHS

$(ab + bc + cd)^2$ =$(ad + ab + cd)^3$

$= [ab + d(a + c]^2$

= $a^2b^2 + abd(a + c) + d^2(a + c)^2$

= a^2b^2 + $2a^2bd + 2acbd + 2acbs + d^2(a^2 + 2ac + c^2)$

= a^2b^2 +$2a^2c^2 + 2ab^2c^2 + d^2a^2 + 2d^2b^2 + d^2c^2$

= a^2b^2+$a^2c^2 + a^2c^2 + b^2c^2 + b^2c^2 + a^2a^2 + d^2b^2 + d^2b^2 + d^2c^2$

= a^2b^2+$a^2c^2 + a^2d^2 + b^2b^2 + b^2c^2 + b^2d^2 + c^2b^2 + c^2xc^2 + c^2d^2$

= $a^2(b^2 + c^2 + d^2) + b^2(b^2 + c^2 + d^2) + c^2(b^2 + c^2 + d^2)$

= $a^2(b^2 + c^2 + d^2)(b^2 + c^2 + d^2)$

L.H.S

$\Longrightarrow$ L.H.S=R.H.S

$\therefore$ $(b^2 + c^2 + d^2)(b^2 + c^2 + d^2)$=$(ab + bc + cd)^2$

Q33. Insert two numbers between 3& 81 so that the resulting sequence is geometric progression.

Solution

Let G_1&G_2 be two numbers between 3&81 such that the series, forms a geometric series.

Let a be the first term and r be the common ratio.

81=(3)$(r)^2$

$\Longrightarrow r^3 = 27 \Longrightarrow r = 3$

$G_1 = ar = 3x3 = 9$

$G_2 = ar^2 3x3^2 = 27$

$\therefore$ The required numbers are 9 & 27

Q34. The sum of two no s is 6 times their geometric mean, show that nos are in the ratio $(3+2\sqrt{2}):(3-2\sqrt{2})$

Solution

Let the two no s be a and b

$G.M = \sqrt{ab}$

$\Rightarrow a + b = 6\sqrt{ab} \ldots \ldots .1)$

Squaring $(a+b)^2 = 36ab$

Also $(a-b)^2 = (a+b)^2 - 4ab$

$\Rightarrow (a-b)^2 = 36ab - 4ab$

$= 32ab$

$\Rightarrow a - b = \sqrt{32ab} = \sqrt{32}.\sqrt{ab}$

$\Rightarrow a - b = 4\sqrt{2}.\sqrt{ab} \ldots \ldots .2)$

Adding & 2), we have

$2a = 6\sqrt{ab} + 4\sqrt{2}.\sqrt{ab}$

$\Rightarrow 2a = (6 + 4\sqrt{2})(ab)$

$a = (3 + 2\sqrt{2})(\sqrt{ab})$

But $a + b = 6\sqrt{ab}$

$b = 6\sqrt{ab} - a$

$= 6\sqrt{ab} - (3 + 2\sqrt{2})(\sqrt{ab})$

$\Rightarrow b = (3 - 2\sqrt{2})(\sqrt{ab})$

$$\frac{a}{b} = \frac{(3+2\sqrt{2})(\sqrt{ab})}{(3-2\sqrt{2})(\sqrt{ab})} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

$\therefore$ ***The required ratio is*** $= \mathbf{3 + 2\sqrt{2} : 3 - 2\sqrt{2}}$

Q35. If A and G are an arithmetic mean & geometric mean, respectively between two positive numbers prove that numbers A$\pm\sqrt{(A+G)(A-G)}$

Solution

Let A is an arithmetic mean between a and b.

G is a geometric mean between a and b.

$\Rightarrow$ A=$\frac{a+b}{2}$.....1 and G= $\sqrt{ab}$ 2

From 1& 2

2A=a+b....1 and ab=G^2.....2

But $(a-b)^2$= $(a+b)^2$-4ab

$\Rightarrow (a-b)^2 = 4A^2 - 4G^2 = 4(A^2 - G^2)$

$\Rightarrow (a-b)^2 = 4(A+G)(A-G)$

$\Rightarrow$ (a-b)= $\pm2\sqrt{(A+G)(A-G)}$

But a+b 2A & a-b=$\pm2\sqrt{(A+G)(A-G)}$

$\Rightarrow$ a+b=2A

a-b=$\pm2\sqrt{(A-G)(A+G)}$

2a=2A$\pm\sqrt{(A-G)(A+G)}$

$\Rightarrow$ a=A$\pm\sqrt{(A+G)(A-G)}$

$\Rightarrow$ b=2A-A$\pm\sqrt{(A+G)(A-G)}$

b=A$\pm\sqrt{(A+G)(A-G)}$

Q36. 150 workers were hired to finish a job in a certain number of days. 4 workers dropped out on second day, four more workers dropped out on third day and 50 on. It took light mare days to finish the work. Find the number of days in which the work was completed:

Solution

Let 150 workers were engaged to finish a jab in a certain number of days say k days.

Number of workers who would have worked for k days.

=(150+150+150+…+k terms)

=150k

Bet workers present on 1st day are 150 workers present on second day 150-4=146. Workers present on third day= 146-4-142.

- Number of days taken to finish the work k+8-n

150+146+142+….. to n

Terms (days)=150k

150+146+142+… to n is an arithmetic series with $a_1 = 150$

& d=146-150=-4

$\frac{n}{2}(2a_1 + (n-1)d)$=150k, but k+8=n $\Rightarrow$ k=n-8

$\Rightarrow$ $\frac{n}{2}$ (300+(n-1)(-4))=150(n-8)

$\Rightarrow$ $\frac{n}{2}$ (300-4n+4)=150(n-8)

$\Rightarrow$ $\frac{n}{2}$ (304-4n)=150(n-8)

$\Rightarrow$ n (304-4n)=300(n-8)

$\Rightarrow$ 304n-4n^2=300n-2400

$\Rightarrow$ - $4n^2 + 304n - 300n + 2400 = 0$

$\Longrightarrow$ - $4n^2 + 4n + 2400 = 0$

$\Longrightarrow\ n^2 - n - 600 = 0$

$\Longrightarrow\ n^2 25n + 24n - 600 = 0$

$\Longrightarrow\ n(n - 25) + 24(n - 25) = 0$

$\Longrightarrow\ (n + 25)(n + 25) = 0$

$\Leftrightarrow$ m=-24& n=25

But n>0

$\Longrightarrow$n=25

Unit 2

Introduction to calculus

Introduction

In our daily life, we come a cross. Things that change according to some well recognizable rules.

- Calculus is one of the components of mathematics that is concerned with change & motion; it deals with quantities that approach other quantities.
- The derivative is the exact rate at which one quantity changes with respect to another.

2.1Introduction to Derivatives

Rate of change

Definition 2.1: A rate change is a rate that decribes how one quantity changes in relation to another quantity. The unit of a rate of change are output units per input units

a) **Appositive rate of change**
When the value of the two quantities (say x and y) increases at the same time & the graph has a positive slope.

b) **Negative rate of change:** when the value of one quantity increases & the value of the other quantity decrease & the graph has anegative slope.

c) **Zero rate of change:** when the value one quantity increase & the value of the other quantity remains constant & the slope of the graph is 0.

- A function is increasing where its of change is positive & decreasing where its rate of change is negative.

  Examples 1) calculate the rate of change of the following pairs of numbers:

a) (4, 6) & (5, 9)
b) (4,7) & (5, 6)
c) (0, 9) & (9, 9)
d) (8, 0) & (8, 8)

Solution

1a) Rate of change $\frac{Change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-6}{5-4} = \frac{3}{1} = 3$

The rate of change is positive

b) Rate of change$=\frac{\Delta y}{\Delta x} = \frac{6-7}{5-4} = \frac{-1}{1} = 1$

The rate of change is Negative.

c) Rate of change $= \frac{\Delta y}{\Delta x} = \frac{9-9}{9-0} = \frac{0}{9} = 0$

The rate of change is 0.

d) Rate of change $= \frac{\Delta y}{\Delta x} = \frac{8-0}{8-8} = \frac{8}{0} = undefined\ the\ line\ is\ vertical.$

- We have two types of rate of change
  1- Average rate of change
  2- Instantaneous rate of change

1) Average Rate of change (ARC)

- Is a measure of how much the function changed per unit, on average, over the interval.
- Geometrically, the average rate of change of a function $y = f(x)$ on the interval $[a, b]$is the slope of the scant Line through $(a, f(a))$ & $(b, f(b))$

- If $y = f(x)$ is a function, the ARC of y with respect to x on the interval $[a, b]$ is given by:

$$ARC = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

Example

i) Compute the average rate of change of

a) $f(x) = x^3 - 3 \times ove\square\ the\ interval\ 1 \leq x \leq 6$.

Solution

$$ARC = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(6) - f(1)}{6 - 1} = \frac{6^3 - 3(6) - (1^3 - 3(1))}{6 - 1}$$

$$= \frac{216 - 18 - (-2)}{5}$$

$$= \frac{198 - (-2)}{5}$$

$$= \frac{198 + 2}{5} = \frac{200}{5}$$

$\underline{= \mathbf{40}}$

2) If the average rate of $f(x) = kx - 9$ over the interval $[9, 12]$ is 4, then what is the value of k?

Solution

$$ARC = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a} = \frac{f(12) - f(g)}{12 - 9}$$

$$4 = \frac{(12k - 9) - (9k - 9)}{3} \Rightarrow \frac{12k - 9 - 9k + 9}{3} = 4$$

$3k = 12 \Rightarrow \underline{\boldsymbol{k = 4}}$

2) Instantaneous Rate of change

Definition 2.3: The instantaneous rate of change is the change at that particular moment.

Instantaneous rate of change $= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$, where

Δx closes & closer to zero.

NB: Average rate of change is the rate of change over the entire period.

- The instantaneous rate of change is the change at a certain period of time.

Instantons

Rate of change of a function $f\ at\ x = x_o\ is\ \frac{\Delta y}{\Delta x}$

$= \frac{f(x_o+h)-f(x_o)}{h}$, when h gets closer & closer to zero

$$i.e\ IRC = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

Examples

1) Evaluate the instantaneous rate of change $f(x) = 2x^2 + 9, at\ x = -3$

Solution

IRC of a function $f\ at\ x = \ x_0\ is$:

$\frac{\Delta y}{\Delta x} = \frac{f(x_0+h)-f(x_0)}{h}$ as h gets closer & closer to 0.

$$= IRC = \frac{f(-3th) - f(-3)}{h}$$

$$= \frac{2(-3th)^2 + 9 - (2(-3)^2 + 9)}{h} = \frac{2(h^2 - 6 + 9) + 9 - (18 + 9}{h}$$

$$= \frac{2h^2 - 12h + 18 + 9 - (18 + 9)}{h} = \frac{2h^2 - 12h}{h} = 2h - 12$$

as $h \to ,2h - 13 \to -12$

Thus, the instantaneous rate of change is **-12**

- The instantons rate of change at $x = x_0$ of a function is the derivate of a function evaluated at x=x0

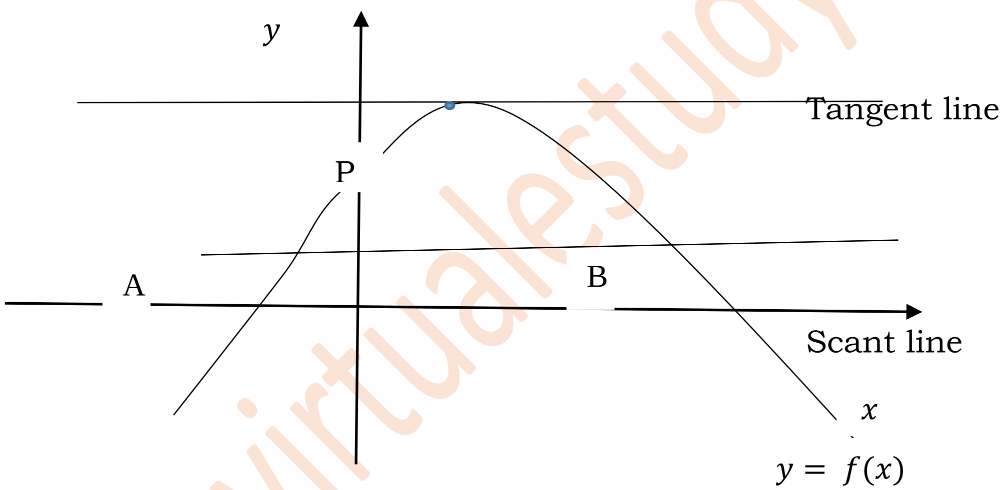
Gradient of curves & Rate of changes

Definition:- a) A tangent line to a curve or a graph of a function y= f(x) is a line that touches the curve exactly at one point but does not cross the curve.

b) Gradient (slope) is a number that describes steepness & direction of a line. If slope is **+ve** the line is ↑

If slope is **–ve**, the line is ↓

If slope is 0, the line is horizontal



- $Slope = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$, $A(x_0, x_0)$, $B(x_1, y_1$
- The slope of a scant line =average rate of change

Examples

1) Find the slope of a line passing through the two given points:

Solution

$Slope = \frac{\Delta y}{\Delta x} = \frac{11-(-5)}{3-(-3)} = \frac{16}{6} = {}^{8}/_{3}$

2) Find the slope of a scant line to the graph of $f(x) = x^3 - 4x + 6$ at [1,2]

Solution

$$slope = \frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(1)}{2-1} = \frac{(2^3-4(2)+6)-(1^2-4(1)+6)}{1}$$

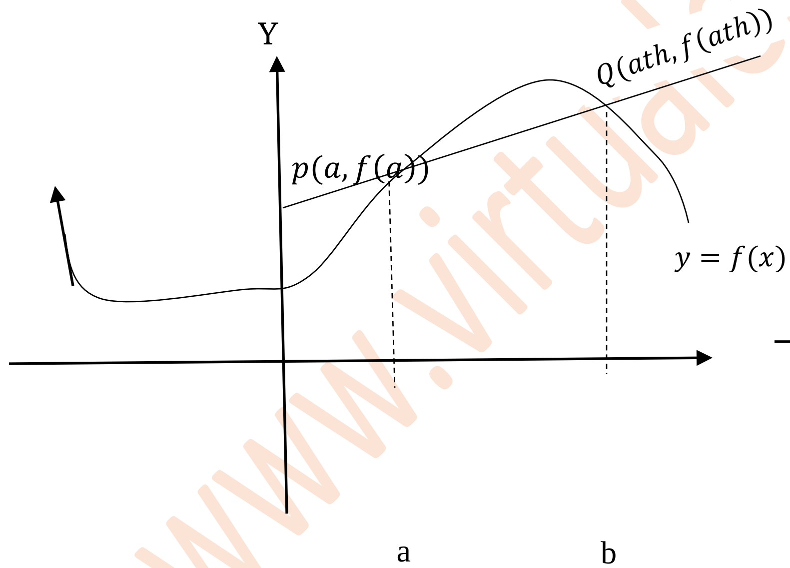
$$= (8-8+6)-(1-4+6)$$

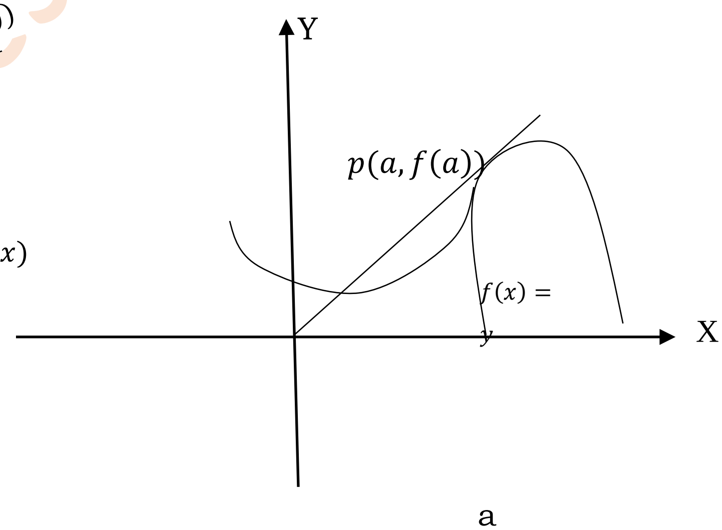
$$= 6-(3) = 3$$

Gradient at appoint on a curve

- If f is a function defined on an open interval containing a then as h gets closer & closer to zero the slope 'm' of the tangent line to the graph of $y = f(x)$at the point $p(a), f(a) is\ given\ by : m = \frac{f(a\ th)-f(a)}{h}$
- The slope of a tangent line represents the instantaneous rate of change of the function at that and point.

  i.e. The slope of a tangent line = instantaneous rate of change.





Example

1) Find the gradient (slope) of the functions $f(x) = x^2 at\ x = 2,$

Solution

Slope $= \frac{f(x)-f(2)}{x-2} = \frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$

Slope $(at\ x = 2) = {}^{x+2}/_{x=2} = 2 + 2 = 4$

2.2 Derivative

Derivative of function at a point

Definition: The derivative of a function f(x) at a number "a" in the domain of f, denoted by $f'(a)$, is the gradient of the tangent line to the graph of $y = f(x) at\ (a, f(a))$

i.e, as h gets closer & closer to zero from both directions, $\frac{f(ath)-f(a)}{h}$ becomes closer to $f'(a)$.

- $f'(a)$ is the slope of the line tangent to the graph of f at the point (a, f(a)).
- $if\ y = f'(a)$ is defined 0, then we say that f has a derivative at a or f is differentiable at a.

Example

1) Find the derivatives of the following function at the given value of a.
   a) $f(a) = 5x - 3; a = -2$ b) $f(x) = \sqrt{3x-6}\,; a = 3$

   Solution

   1a) $f'(-2) = \frac{f(-2th)-f(-2)}{h} = \frac{(5(-2th)-3)-(-2x5-3)}{h}$

$$= \frac{(-10+5h-3)-(-13)}{h} = \frac{13+5h+13}{h} = \frac{5h}{h} = 5$$

$= \boldsymbol{f'(-2) = 5}$

b) $f'(3) = \frac{f(ath)-f(a)}{h} = \frac{f(3th)-f(3)}{h} = \frac{\sqrt{3(3th)-6}-\sqrt{3x3-6}}{h}$

$$= \frac{\sqrt{9+3h-6}-\sqrt{9-6}}{h} = \frac{\sqrt{3h+3}-\sqrt{3}}{h}$$

Rationalization the numerator

$$= \frac{\sqrt{3h+3}-\sqrt{3}}{h} \times \frac{\sqrt{3h+3}+\sqrt{3}}{\sqrt{3h+3}+\sqrt{3}} = \frac{3h+3-3}{h\left(\sqrt{3h+3}+\sqrt{3}\right)} = \frac{3h}{h\left(\sqrt{3h+3}+\sqrt{3}\right)}$$

$$= f'(3) = \frac{3}{\sqrt{3h+3}+\sqrt{3}}$$

As $h \to 0, f'(3) = \frac{3}{\sqrt{3}+\sqrt{3}} = \frac{3}{2\sqrt{3}}$

$$f'(3) = \frac{3}{2\sqrt{3}}$$

The derivative as a function.

Definition:- The function $f(x)$ whose domain consists of those values of x at which f is differentiable & whose value at any such number as h gets closer & closer to zero is given by $f'(x) = \frac{f(xth)-f(x)}{h}$

- The derivative is the exact rate at which one quantity changes with respect to another.
- The can represent the derivative of a function by $f'(x), \frac{dy}{dx}, \frac{d}{dx} f(x). D\, f(x), Dx\, f, if\ f(x) = y$

Derivatives of various functions

a) Differentiation of power functions

Theorem 2.1: Let $f(x) = x^n$, where n is positive integer

Then $f'(x) = nx^{n-1}$.

- Corollary 2.1. a) If $f(x) = x^{-n}$, where n is appositive integer, then

  $f'(x) = -nx^{-n-1}$

  b) If $f(x) = kx^n, then\ f'(x) = knx^{n-1}$ where n is any non – zero integer & k is constant.

Examples: 1) Find the derivatives each of the following functions.

a) $f(x) = x^8$ b) $f(x) = x^{-27}$ c) $f(x) = 10x^{15}$ d) $f(x) = -10x^{-2/5}$

Solution

1a) $f'(x) = 8x^7$ b) $f(x) = -27x^{-28}$ c) $f'(x) = 150x^{14}$

d) $f'(x) = -10\dfrac{-2}{5}x^{-2/5} - 1 = 4x^{-7/5}$

ii) **Derivatives of combination of functions**

Theorem 2.2: If f &g are differentiable functions at

a) Then $f \pm g, fg \,\& \frac{f}{g},\;\; g(a) \neq 0$ are also differe $\left(\frac{1}{g}, \frac{1}{f}\right.$ are differentiable at a$\left.\right)$ $n+i$ able at a and $(f \pm g)'(a) = f'(a) \pm g'(a)$

b) If $f_1, f^2, f_3, \cdots f_n$are differentiable at a no, a, then $(f_1 \pm f_2 \pm f_3 \pm \cdots + f_n)'(a) = f_1^1(a) \pm f_2^1(a) \pm \cdots \pm f_n^1(a)$

c) $(kf)'(a) = k(f'(a))$

d) $(fg)'(a) = f'(a)g(a) + f(a).g'(a)$

e) $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a).g'(a)}{(g(a))^2}$

Examples

1) Find the derivative of each of the following functions.

a) $f(x) = 5x^4 \pm \sqrt{x}$

b) $f(x) = (2x+1)(x^2 + x - 5)$

c) $\frac{3x^2+3x}{x^2} = f(x)$

d) $f(x) = \frac{1}{x^2+1}$

Solution

1a) $f'(x) = \left(5x^4 \pm \sqrt{x}\right)^1 = 20x^3 \pm \frac{1}{2\sqrt{x}}$

b) $f'(x) = \left((2x+1)(x^2+x-5)\right)^1 = (2x+1)^1(x^2+x-5) + (x^2+x-5)'(2x+1)$

$$= 2(x^2+x-5) + (2x+1)(2x+1$$

$$= 2x^2 + 2x - 10 + 4x^2 + 4x + 1$$

$$\underline{= 6x^2 + 6x - 9}$$

c) $f'(x) = \left(\frac{3x^2+3x}{x^2}\right)^1 = \frac{(3x^2+3x)^1 x^2 - (x^2)^1(3x^2+3x)}{(x^2)^2}$

$$= \frac{(6x+3)x^2 - 2x(3x^2+3x)}{x^4}$$

$$= \frac{6x^3+3x^2-6x^3-6x^2}{x^4}$$

$$\frac{-3x^2}{x^4} = \frac{-3}{x^2}$$

d) $f'(x) = \left(\frac{1}{x^2+1}\right)^1 = \frac{(1)^1(x^2+1)-(x^2+1)'}{(x^2+1)^2}$

$$= \frac{-2x}{(x^2+1)^2}$$

- Derivative of polynomial functions
- If $f(x) = a_n x^n \tan -1\, x^{n-1} + a_1 \times + a_0$,then

  $f'(x) = nanx^{n-1} + (n-1)an - 1x^{n-2} + \cdots + a_1$

Examples

1) If $f(x) = x^3 + \frac{1}{2}x^2 - \frac{2}{\sqrt{x}}$, $find\ f'(x) or\ \frac{dy}{dx}; if\ f(x) = y$

Solution: $\frac{dy}{dx} = \frac{d}{dx}\left(x^3 + \frac{1}{2}x^2 - \frac{2}{\sqrt{x}}\right)$

$$= 3x^2 + \frac{2}{2}x \frac{-2}{\frac{1}{2\sqrt{x}}} = 3x^2 + x + \frac{1}{x\sqrt{x}}$$

The Chain Rule

Theorem: Let of be differentiable at a, and f be differentiable at g(a). Then fog is differentiable at a, and $(fog)'(a) = f'(g(a)).g'(a)$.

Examples

1) For each of the following functions find a formula for $f'(x)$.

   a) $f(x) = \sqrt{x^4 + 3x^2 - 5}; g(x)\sqrt{x}; \&h(x) = x^4 + 3x^2 - 5$

$$f(x) = g(h(x)) \Rightarrow f'(x) = (goh)'(x) = g'(h(x)).h'(x)$$

$$= \frac{1}{2\sqrt{x^4 + 3x^2 - 5}}$$

$$= \frac{4x^3 + 6x}{2\sqrt{x^4 + 3x^2 - 5}} = \frac{2x^3 + 3x}{\sqrt{x^4 + 3x^2 - 5}}$$

b) $f(x) = \left(\frac{3x-5}{2-7x}\right)^6 = 6\left(\frac{3x-5}{2-7x}\right)^5\left(\frac{3x-5}{2-7x}\right)^1$

$$= 6\left(\frac{3x-5}{2-7x}\right)^5 \frac{3(2-7x) - (2-7x)'3x - 5}{(2-7x)^2}$$

$$= 6\left(\frac{3x-5}{2-7x}\right)^5 \frac{6 - 21x + 7(3x - 5}{(2-7x)^2}$$

$$= 6\left(\frac{3x-5}{2-7x}\right)^5 \frac{(6 - 21x + 21x - 35)}{(2-7x)^2}$$

$$= \left(\left(\frac{3x-5}{2-7x}\right)^5 \frac{(-29)}{(2-7x)^2}\right.$$

$$= -174\left(\frac{(3x-5)^5}{(2-7x)^7}\right)$$

Maximum and minimum points

1) Increasing and Decreasing functions

Definition: Let f be a function on an interval I. If X1 & X2 are in I,

i) For $x_1 < x_2$ *if* $f(x_1) \leq f(x_2)$, *then f is said to be increasing on I*,

ii) For $x_1 < x_2$ *if* $(x_2) \geq f(x_2)$, then f is said to be decreasing on I.

iii) For $x_1 < x_2$ if $f(x_1) < f(x_2)$, then f is strictly increasing on I

iv) For $x_1 < x_2$ *if* $f(x_1) > f(x_2)$, *then f is strictly decreasing on I*.

Increasing & Decreasing Test

Suppose that $f(x_1) \geq f(x_2)$, f is differentiable in the interior of an interval I.

a) If $f'(x) \geq 0, for\ all\ x\ in\ the\ interior\ of\ I, then\ f\ is\ \uparrow on\ I.$

b) $If\ f'(x) \leq 0, for\ all\ x\ in\ the\ interior\ of\ I, then\ f\ is\ \downarrow on\ I.$

c) $If\ f'(x) > 0, \&\ f'\ (x) = 0\ on\ I, then\ f\ is\ strictly\ \uparrow on\ I.$

d) $If\ f'(x) < 0\ \&\ f'(x) = 0\ on\ I, then\ f\ is\ strictly\ \downarrow on\ I.$

- $a\ function\ that\ is\ either\ \uparrow or\ \downarrow on\ is\ known\ as\ amonotonic\ function.$
- A number 'c' in the domain of a function f is said to be a critical number of f iff either $f'(c) = 0$ or f has no derivative at c.

Example

1) Find the critical numbers of the given functions

a) $f(x) = 4x^3 - 5x^2 - 8x + 20$

b) $f(x) = 2\sqrt{x}(6 - x)$

Solution

1a) $f'(x) = 12x^2 - 10x - 8$

$f'(x) = 0 \Rightarrow 12x^2 - 10x - 8 = 0$

$6x^2 - 5x - 4 = 0$

$= 6x^2 - 8x + 3x - 4 = 0$

$2x(3x - 4) + 1(3x - 4) = 0$

$(2x + 1)(3x - 4) - 0$

$2x + 1 = 0\ or\ 3x - 4 = 0$

$x = {}^{-1}/_2\ \square r\ x = {}^{4}/_3$

∴ The critical nos of f are $x = \frac{-1}{2}\ \&\ x = {}^{4}/_3$

b) $f'(x) = \frac{2}{2\sqrt{x}}(6 - x) - 2\sqrt{x}$

$= \frac{6 - x}{\sqrt{x}} - \frac{2x}{\sqrt{x}} = \frac{6 - 3x}{\sqrt{x}}$

$$= \frac{3(2-x)}{\sqrt{x}} \Rightarrow f'(x) = 0$$

$= x = 2$ *and*

$f'(x)$ does not exist when x=0 Hence, the critical numbers of f are x=0& x=2.

2) Find the interval on which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing & decreasing.

   Solution

   $f'(x) = 12x^3 - 12x^2 - 24x$

   $\Rightarrow 12x(x^2 - x - 2)$

   $f'(x) = 0$

   $\Rightarrow 12x = 0 \; or \; x^2 - 2x + x - 2 = 0$

   $\lambda = 0 \; or \; x = -1, x = -1, x = 2$

Hence, the critical no off are x= -1, 0, 2.

12x	$X < 1$ -	$-1X < 0$ -	$0 < x < 2$	$x > 2$
X+1	- - - - - - -	+	+ +	+ +
x-2	- - - - - - - - -	- - - - - - - - -	- - - - - -	+ +
$f'(x) = 12x(x-2)(x+1$	- - - - - - - - -	+ + +	- - - - - -	+

- From the above sign chart $f'(x) \geq 0 on \; [-1, 0] \cup [2, \infty)$

  $\Rightarrow f(x) increasing \; on \; [-1, 0] \cup [2, \infty) \& \; f'(x) \leq 0 \; on \; (\infty, -1] \cup [0,2]$

  $\Rightarrow f(x) is \; decreasing \; on \; (-\infty, -1] \cup [0, 2]$

Minimum and maximum values of a function

Definition:- If there exists an open interval (a, b) containing C such that $f(x) < f(c) for$ all x in (a, b) other than C in the interval, then $f(c)$ is a relative (local) maximum value of f & appoint $(c, f(c))$ is called relative maximum point.

If f(x)> f© for all x in (a, b) other than c, the f© is a relative (local) minimum value of & the point (c, + ©) is known as relative (local) minimum point.

- Functions may have any number relative extrema
- $If\ C \in D$ Domain of f
- $If\ f(x) \leq f(c), then\ (c, f(c))$ is an absolute maximum point of f.
- $If\ f(x) \geq f(c), then\ \ (c, f(c))$ is an absolute minimum point of the function f.
- The absolute maximum and minimum values of f and called absolute extreme values.
- Absolute extrema are not necessarily unique
- At either local maximum or local minimum point the gradient is zero.

Examples

1. Find the absolute maximum & minimum values of the ff function an a given interval.

a) $f(x) = 3x^4 - 26x^3 + \ 60x^2 - 11\ on\ [1,5]$

b) $f(x) = \frac{x^2-2x+4,}{x-2}[-3, 1]$

Solution: - The question must be corrected as $f(x) = 3x^4 - 26x^3 + 60x^2 - 11$

1a) $f'(x) = 12x^3 - 78x^2 + 120x \qquad f'(x) = 0\ \ 12x^3 - 78x^2 + 120x = 0$

$12x^3 - 78x^2 + 120x = 0$

$6x(2x^2 - 13x + 20) = 0 \qquad 6x(2x^2 - 8x - 5x + 20) = 0$

$6x(2x(x-4) - 5(x-4)$

$x = 0, {}^5/_2, 4$

On the given interval the critical numbers are $x = 4\ \&\ x = {}^5/_2$

$= f(1) = 26, f\left({}^5/_2\right) = \dfrac{1199}{16} = 74.9, f(4) = 53\ \&\ f(5) = 114$

∴ The global maximum value of f is 114 & its minimum value is 26.

c) $f'(x) = \frac{\square(x-4)}{(x-2)^2}, f'(x) = 0, x = 0\ or\ x = 4\ \&\ f'(x)$ is undetinedat $x = 2$

∴ the only critical no of the given interval is $x = 0$

$f) - 3) = \frac{-19}{3}, f(0) = -2\ \&\ f(1) = -3$

Hence, the maximum value of f is -2 & the minimum valve is $\frac{-19}{5}$

The first derivative test

Let c. be a critical number of a function f on an open interval I can taining C.

a) If the signg f' changes from $-\underline{ve}$ to $+\underline{ve}$ atc, then f has a relative minimum $a + c$.

b) If the signq f' changes from $+\underline{ve}$ to $-\underline{ve}$ atc, then f has a relative maximum atc.

c) If the relative maximum or minimum atc.

Examples

1. Find the relative maximum & minimum values f:

a) $f(x) = 2x^3 + 3x^2 - 12x - 3$

$f'(x) = 6x^2 + 6x - 12$

$f'(x) = 0\ \ 6x^2 + 6x - 12 = 0$

$= x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

$x(x + 2) - 1(x + 2) = 0$

$(x - 1)(x + 2) = 0$

$x = 1\ or\ x = -2$

The critical No 1 of f are x= -2 & x=1

	$xc - 2^{-2}$	$-2x < 1$	$x > 1$
6(x+2)	0	+++++++	+++++++

x-1	--------	-------	+++++++
$f'(x) = 6(x+2)(x$	+ + +	- - -	+ + +

- f' changes its sign from $+\underline{ve}$ to $-\underline{v□}$ at $x = -2$

  f has a local maximum at x= -2 & f' changes its sign from $-\underline{ve}$ to $+\underline{ve}$ at 1

  f has a local minimum at $x = 1$

  $(-2, f(-2)) = (-2, 17)$ is evaluative maximum & $(1, f(1) = (1, -10)$ is arelative minimum point of the function.

∴ 17 is a relative maximum value & -10 is a relative minimum value.

2. If $f(x) = ax^3 + bx - 5$ has arelative minimum value 7- 6 at $x = \frac{1}{2}$ find a and b.

   Solution

   Given $f\left(\frac{1}{2}\right) = \frac{1}{8}a + \frac{1}{2}b - 5 = -6 \Leftrightarrow a + 4b = -8 \ldots..(1)$

   & $f'\left(\frac{1}{2}\right) = 0, \Leftrightarrow f'(x) = 3ax^2 + b \Rightarrow \frac{3}{4}a + b = 0 \Rightarrow 3a + 4b = 0(2)$

   From (1) & (2) $\quad -2a = -8 \Rightarrow a = 4 \; and \; a + 4b = -8$

   $\begin{cases} a + 4b = -8 \\ 3a + 4b = 0 \end{cases}$

   $$4 + 4b = -8$$
   $$4b = -12$$
   $$b = -3$$

Equations of tangents & normal lines to curves

- The equation of the tangent line to the carve

  $y = f(x) at\ x = a\ is\ given\ by\ y = f'(a)(x - a) + f(a).$

- A line that is perpendicular to the tangent line at the point of tangency is known as the normal line to the curve.
- The slope of the normal line at the point of tangency

$$(a, f(a)) is\ \frac{-1}{f'(a)}\ \&\ its\ equation\ is\ given\ by\ y = -\frac{1}{f'(a)}(x - a) + f(a)$$

Examples (1) find the equation of the tangent line & the normal line to the graph of the ff fu as at the given point:

a) $f(x) = x^3 - 2x; (1, -1)$

Solution

1a) $f'(x) = 3x^2 - 2 = f'(1) = 1$

i) Equation of the tangent line to the graph of f at (1, -1) is:

$y = f'(1)(x-1) + f(1)$

$= 1(x-1) - 1 = x - 2 = y = x - 2$

ii) The equation of the normal line to the graph of (1, -1) is:

$y = -\frac{1}{f'(11)}(x-1) + f(1) = \frac{-1}{1}(x-1) + f(1)$

$= -1(x-1) - 1$

$= -x$

$\Rightarrow y = -x$ is the equation of a normal line

c) $f(x) = \begin{cases} x, if\ x > 3 \\ x^2 - 6, if\ x \leq 3 \end{cases}$

$f'(x) = \begin{cases} 1, if\ x > 3 \\ 2x, if\ x \leq 3 \end{cases} \quad \Rightarrow f'(1) = 2$

i) The equation of the tangent line is:

$y = f'(1)(x-1) + f(1)$

$= 2(x-1) - 5 =$ **2x-7**

ii) The equation of the normal line is:

$y = \frac{-1}{f'(1)}(x-1) + f(1)$

$= -\frac{1}{2}(x-1) - 5 = -\frac{1}{2}x - \frac{9}{2}$

$(\boldsymbol{Ex. 2. 17})\boldsymbol{Q3}$

3) If $f(x) = x^2 + ax + b.d\ g(x) = x^3 - C$ have the same tangent line at (1, 2), find the values of the constants a, b & c & the equation of their normal lines.

4)

2.2.1Applications of Derivations

A) Applications of Derivative in vate of change

- $Velocity = \frac{d\vec{s}}{dt}$ & $\vec{a} = \frac{dv}{dt}$

Examples: (1) The vadius of a circle is increasing at a rate 3 cm/minute. Find the rate of change of the area when:

a) $r = 8\,cm$

b) R= 12 cm

Solution

a) The area a circle is $A = \pi r^2$

& $\frac{dr}{dt} = \frac{3cm}{min}$, $to\ find\ \frac{dA}{df}, when\ r = 3cm$

$A = \pi r^2 = \frac{dA}{dt} = \pi\frac{d(r^2)}{df} = \pi.2r\frac{dr}{dt} = \pi.2r.\frac{dr}{df}$

Since $\frac{dA}{dt} = \frac{dA}{dr}\ \frac{dr}{dt}$ $= \pi.2.3\ cm\ /60\ se$

$= 48\,\pi cm^2/60\ sec$

$= \frac{8}{10}\pi cm^2/sec$

$= \frac{4}{5}\pi cm^2/sec$

b) $\frac{dA}{dt},\ when\ r = 12cm, \frac{dr}{dt} = \begin{matrix} = 3\ cm/60\ s \\ \frac{1}{20}\ cm/se \end{matrix}$

$\frac{d(\pi r^2)}{dt} = 2\pi r\ \frac{dr}{dt}$

$= 2\pi \times 12m.\frac{1}{20}\ cm^2/se$

$\underline{\frac{6}{5}\pi cm^2/\square e}$

2. The radius r of a sphere is increasing at a rate of 3cm$/min = \frac{1}{20}cm)sec.$

Evaluate the rate of change of the volume when: a) r= 2cm b) r= 3 cm

* Volume of sphere: $V = \frac{4}{3}\pi r^3$

$$\frac{dv}{df}, = \frac{dv.}{dr}\frac{dr}{df}$$

$$= \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right).\frac{dr}{df}$$

$$= \frac{4}{3}\pi.3r^2.\frac{dr}{dt}$$

$$= \frac{4}{3}\pi B(r^2).\frac{1}{20}\ cm/sec$$

$$= 4\pi.2.\frac{1}{20}\ cm/sec = 16\pi \times 3 = 48\pi m^3/m$$

$$= \frac{16}{20}\pi cm^3/se = \frac{4}{5}\pi cm^3/se$$

$$= \frac{4}{5}\pi^3/sec$$

3. Find $\frac{dy}{dx}$ & $\frac{dx}{dy}$ (Assuming that x is differentiable w.r.t y & is differentiable w. r. t. x. w.r.t y & y is differentiable w.r.t. x.

a) $x^2 + y^2 = 25,\ y^2 = 25 - x^2 = \pm\sqrt{25 - x^2}$

$$\frac{dy}{dx} = \frac{d(x^2 + y^2)}{dx} = \frac{dy}{dx} = \frac{d}{dx}(x^2 + y^2 = 25)$$

$$= 2x + 2y\frac{dy}{dx} = 0$$

$$= \frac{\square y}{dx} = \frac{-x}{y}$$

- $\frac{d}{dy}\ (x^2 + y^2 = 25) = 2y + 2x\frac{dx}{dy} = \frac{dx}{dy} = \frac{-y}{x}$

B. Applications of Derivatives in Business & Economics

* Companies, both in & out of the financial industry have begun to use derivatives as a method of speculating & generating income.

Example

1. The profit function of a company can be represented by $p(x) = x = 0.00001x^2$, where x represents units sold. Find the optimal sales volume & amount of profit to be expected at that volume.

Solution

Marginal profit: $Mp = \frac{dp}{dx} = \frac{d}{dx}\,(x - 0.00001x^2)$

$= 1 - 0.00002x$

To get maximum profit new we put marginal profit equals to zero. So,

1-0.0002x=0 $x = \frac{1}{0.00002}$

$\underline{50{,}000\ Units}$

Now check whether the value is maximum or minimum we can use the first derivative test at $x = 50{,}0N$

		50,000	
$p'(x) = 1 - 0.00002x$	+ + + +	0	- - -

From this sign chart the signt $p'(x)$ changes from positive to negative at x=50,000.

Thus, by the first derivative test, we get maximum profit when the

Company sold $x = 50{,}000\, units\ of\ production$

Now, by putting the value of x in the profit function we get maximum profit.

$p = f(50{,}000) = 50{,}000 - 0.0000(50{,}000)^{12}$

$= 50,000 - (0.00001(2500,000,000))$

$= 50,000 - 25,000 =$**25,000**

The optimum output for the company will be 50,000 units of x & the maximum profit at that volume will be **25,000 birr**

2. The demand equation for a certain product is $p = 6 - \frac{1}{2}x$ birr, where x represents the amount of product. Find the level of production which results in maximum revenue.

Solution

The revenue function is R(x)

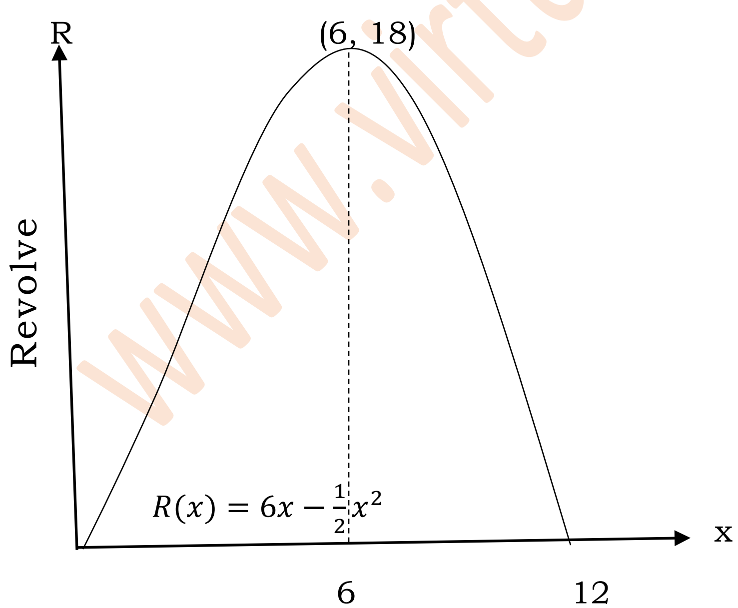
Thus, $R(x) = x\left(6 - \frac{1}{2}x\right)$ which means R(x)=px

$R(x) = 6x - \dfrac{1}{2}x^2\ Birr$

The marginal revenue is given by $R'(x) = 6 - x$

$R'(x) = 0\ 6 - x = 0$

$\underline{\boldsymbol{x = 6}}$



	x>6	x>6
$6-x$	$+\ +\ +$	$-\ -\ -\ -$
$R'(x)6-x$	$+\ +\ +$	$-\ -\ -\ -$

By the first derivative test, the rate of production resulting maximum revenue is x=6, which results in total revolve of 18 birr

3. Promoters of international fund raising can certs must walk a fine line between profit and loss, especially when determine the price to charge for admission to closed circuit TV showing in local. By keeping records, atheatre deter mines that at an admission price of 26 birr it averages 1000 people in attendance for every drop in price of 1 birr, it gains 50 customers. Each customer spends an average of 4 birr on concessions. What admission price should the threaten charge to maximize total revenue?

Solution

At x be the number of birr by which the price of 26 birr should be decreased, (If X is negative, the price is increased)

R(x)= (revenue from tickets) + revenue from concessions)

= (number of people) (Ticket price + (no of people) 4

= 1000+ 50x). (26-x)+(1000+50x):-4

$$= 26{,}000 - 1000 \times +1300 \times -50x^2 + 4000 + 200V$$

$$\underline{= \boldsymbol{-50x^2 + 500x + 30{,}000}}$$

$R'(x) = -100 \times 500$

This derivative exists for all real numbers x.

This the only acritical values are where

$R'(x) = 0 - 100x + 500 = 0$ **x=5**

This corresponds to towering the price by 5 birr

To determine whether R(x) has a maximum value are not this critical point we use the first derivative test.

	x <5	x>5
-100x+500	**+ + + 0**	**- - - -**
$R'(x)$ = 500 -100x	**+ + + 0**	**- - - -**

From the above sign chart $R'(x)$ changes from positive to negative at x=5,

By the first derivative test line order to maximize revenue, the theatre should charge 26 birr 5 birr

$= \mathbf{21\ Birr\ perticket}$

2.3. Introduction to Integration

Integral calculus

Is primarily concerned with the area below the graph of a function (specifically the area between the graph a function & the x - axis).

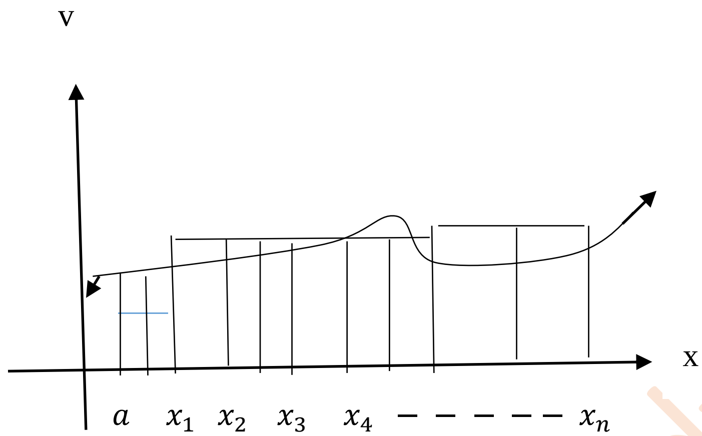
- Area is the limit of the Riemann sums.
- To define $\sum_a^b f(x)d^x$, *choose an integer* $n \geq 1$ &

Divide up $[a, b]$ in to ∩ equal parts. That is

$\left[a, a + \frac{b-q}{n}\right], \left[a + \frac{b-a}{n},\ a + 2\frac{b-a}{\cap}\right] - - - \left[a + (n-1)\left(\frac{b-a}{n}\right), b\right]$

Where $\frac{b-a}{\cap} = \Delta x$ (the width of each rectangle)

Consider the graph below. Find the area between the graph & the x – axis using Riemann sum



- The area below the curve is the sum of the areas the rectangle.
- If f is integrable an $[a,\ b]$, then

$$A = \int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(xi)\Delta x,$$

Where $\Delta x = \frac{b-a}{n}$ $xi = a + i\Delta x$ and

$\sum_{i=1}^{n} f(xi)\Delta x$ is called the Riemann sum.

<u>Steps to use Riemann sum in finding the area under a curve</u>

1) Calculate $\Delta x = \frac{b-a}{n}$
2) Find $xi = a + i\Delta x$
3) Evaluate $f(xi)$
4) Use limit laws to simplify $\sum_{i=1}^{n} f(xi)\Delta x$
5) Find $\lim_{n \to \infty} \sum_{i=1}^{n} f(xi)\Delta x$.

Examples

1. Find $\int_0^2 xdx$ using Riemann sum

Solution

i) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ ii) $xi = 0 + i\Delta x$

$= \frac{i2}{n}$

iii) $f(xi) = xi = \frac{2}{n}i$

iV) $\sum_{i=1}^{n} f(xi)\Delta x = \sum_{i=1}^{n} \left(\frac{2}{n}i\right)\left(\frac{2}{n}\right)$

$$= \frac{4}{n^2}\sum_{i=1}^{n} i = \frac{4}{n^2}\,\frac{n(n+1)}{2} = \frac{2n^2+2n}{n^2}$$

V) $\lim_{n\to\infty}\left[\frac{4}{n^2}\left(\frac{n(n+1)}{2}\right)\right] = \lim_{n\to\infty}\frac{2n+2}{n} = 2$

2. Find $\int_1^4 x^2\,dx$ using Riemann sum by dividing in to n- sub – in the values

Solution

i) $\Delta x = \frac{4-1}{n} = \frac{3}{n}$ ii) $xi = 1 + i\Delta x = 1 + \frac{3}{n}i$ iii) $f(xi) = xi^2$

$= \left(1 + \frac{3}{n}i\right)^2$

$= 1 + \frac{6}{n}i + \frac{9}{n^2}i\,2$

ii) $\lim_{n\to\infty} \sum_{i=1}^{n} f(xi)\Delta x$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\left(1 + \frac{6}{n}i + \frac{9}{n^2}i^2\right)\left(\frac{3}{n}\right)$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{3}{n} + \frac{18}{n^2}i + \frac{27}{n^3}i^2\right]$$

$$= \lim_{n\to\infty}\left[3 + \left(\frac{18}{n^2}\right)\left(\frac{n(n+1)}{2}\right) + \frac{27}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]$$

$$\lim_{n\to\infty}\left[3 + \frac{9n^2+9n}{n^2} + \frac{18n^2+27n+9}{2n^2}\right]$$

$$= 3\lim_{n\to\infty}\left[1 + \frac{3n^2+3n}{n^2} + \frac{6n^2+9n+3}{2n^2}\right] = 3(1+3+3) = 21$$

2.3.1Indefinite Integral

- The process of finding f(x) from f' is said to be anti – differentiation or integration, f(x) is said to be the anti derivative of $f'(x)$

$$\begin{tikzcd}
f(x) \arrow[r, bend left, "\text{Differentiation}"] & f'(x) \arrow[l, bend left, "\text{anti – differentiation}"]
\end{tikzcd}$$

- Integration is the reverse operation of differentiation
- The set of all anti – derivatives of a function f(x) is called the indefinite integral of f(x) denoted by $\int f(x)dx$ read as " the integral of f(x) w.r.t.x.

  Where,

  i) $\int$ *is the integral sign*

  ii) $f(x)$ is called integrand

  iii) dx denotes the variable of integration

- $If\ F'(x) = f(x), then\ \int f(x)dx = F(x) + C$, where C is called constant of integration.
- *if a function has an integral, then it is said to be integrable.*

<u>Integration of some simple functions</u>

b) $\int odx = C$, where C is a constant

c) $\int cdx = Cx + d$, where c is the given constant & d is the constant of integration

d) $\int dx = x + c,$

e) $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$

f) $\int kx^n d^x = k \int x^n\, dx = \frac{\square}{n+1} x^{n+1} + C$

g) $\int e^x dx = e^x + C$

h) $\int \frac{1}{x} dx = ln|x| + c$

i) $\int k(ax+b)^n . dx = \frac{k}{a(n+1)} (ax+b)^{n+1} + C,\ n \neq -1 \& \ a \neq 0$

j) $\int a^x dx = \frac{a^x}{lna} + c, a > 0 \ \& \ a \ \neq 1.$

Examples

1. Integrate each of the following expressions w.r.t. x.
   a. $7 = \int 7\ dx = 7x + c$
   b. $\int x^{\frac{-4}{5}} = \frac{x^{\frac{-4}{5}+1}}{\frac{-4}{5}+1} = \frac{x^{\frac{1}{5}}}{\frac{1}{5}} = 5x^{\frac{1}{5}} + C$
   c. $3x^{-4}$

      $= 3\int x^{-4}$

      $\frac{3.x^{-4+1}}{-4+1} = \frac{3}{3}x^{-3}$

      $= -x^{-3} + c$
   d. $\int 6x^2\sqrt{x} = \int x^2 . x^{\frac{1}{2}} = \int 6x^{5/2}$

      $= 6\int x^{5/2}$

      $= \frac{6.x^{5/2+1}}{\frac{5}{2}+1} = \frac{12}{7}x^{7/2} + C\int$
   e. $\int \frac{2}{x} = 2\int \frac{1}{x} = 2ln|x| + C$

2.

f(x)	x^n	$\sin x$	cos x	tan x	$cotx$	e^x	4^x	$\ln x$	lag_a^x
$f'(x)$	nx^{n-1}	$cosx$	$-sinx$	sec^2x	$-csc^2x$	e^x	$4\ {}^4x\ In$	$1/x$	$\frac{\ln x}{a\ lna}$

From the above table, evaluate each of the following integrals.

a. $\int sec^2 xdx = tanx + c$

b. $\int sin\, x\ dx = \ -cosx + c$

c) $\int \cos xdx = \sin x + c$

d. $\int csc^2\, xdx = -cotx + C$

e. $\int \frac{1}{x}$dx=$\ln|x| + C$

f. $\int 4^x dx = 4^x \ln 4$

property of the indefinite integral

1. Let k be a constant, $\int kf(x)d(x) = k\int f(x)d(x)$
2. $\int f(x) + g(x) = \int f(x) + \int g(x)$
3. $\int f(x) - g(x) = \int f(x) - \int g(x)$

Examples:- find the following indefinite integral.

a. $\int 50dx = 50x + C$

b. $\int 12x^5\, dx = 60x^4 + C$

c. $\int \frac{5}{x} dx = 5\ln|x|$

d) $\int (x^3 - x^2 + x)\, dx$

$= \int x^3\, dx - \int x^2\, dx + \int xdx$

$= \frac{x^4}{4} - \frac{x^3}{3} + \frac{x^2}{2} + C$

e) $\int \left(x^{-2} - 4x^3 + \sqrt{x}\right)dx$

$$= \int x^{-2}dx - 4\int x^3dx + \int \sqrt{x}\, dx$$

$$= \frac{x^{-2+1}}{-2+1} - \frac{-4x^4}{4} + \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C = -x^{-1} - x^4 + \frac{2x^{3/2}}{3} + C$$

$$= -\frac{1}{x} - x^4 + {}^2/_3\, x^{3/2} + C$$

2.3.2Area & Definite integrates

- We can use the anti – derivative of a function to determine the exact area under the graph of the function. This process is called integration.
- At f be a given function over the interval $\_[a, b]$ and F be any anti derivative of f. Then the definite integral of from a to b is: $\int_a^b f(x)d\,x = F(b) - f(a)$.

  This is called fundamental theorem of calculus
- The no a and b are said to be lower and upper limit of integration.

Examples a) Find the are area of a region enclosed by the graph $f(x) = x^2 + 4$ & the x –axis on $\left[-1, \frac{1}{2}\right]$.

Solution

$$\left|\int_{-1}^{\frac{1}{2}} (x^2+4)dx = \frac{x^3}{3} + 4x\right|_{-1}^{\frac{1}{2}} = \frac{\frac{1}{2}^3}{3} + 4\left(\frac{1}{2}\right) - \left(\frac{-1}{3}\right)^3 + 4(-1)$$

$$= \frac{1}{24} + 2 - \left(-\frac{1}{3} - 4\right)$$

$$= \frac{49}{24} - \left(\frac{-13}{3}\right)$$

$$= \frac{49}{24} + \frac{13}{3} = \frac{49+104}{24} = \frac{153}{24}$$

Properties of definite integrals.

If f & g are given function on $[a,b], k \in lR \,\&\, CE[a,b]$

a) $\int_a^a f(x) = 0$

b) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

c) $\int_a^b f(x)x = -\int_b^a f(x)dx$

d) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

$e)\ \int_a^b f(x) \pm g(x)dx = \int_b^a f(x)dx \pm \int_b^a g(x)dx$

2.3.3 More on Area

The area between two curves (graphs)

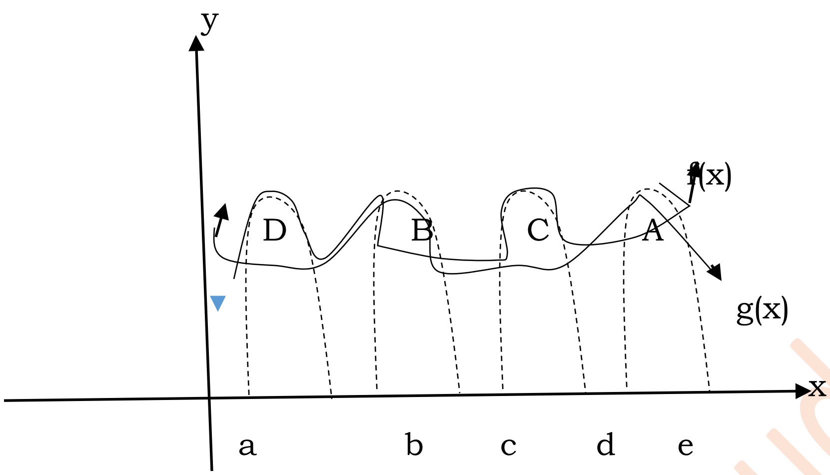
The area A bounded by two continuous curves

$y = f(x) \& y = g(x) on\ [a,b]\ with\ f(x) \geq g(x), \forall x \in [a,b]$ is given by

$A = \int_a^b (f(x) - g(x)dx,)$where f(x) is the top curve & g(x) is the bottom curve.

- The area A bounded by $x = f(y) \& x = g(y) on\ [c,d]$ with $f(y) \geq g(y), \forall y \in [c,d]$ is given by

$A = \int_c^d f(y) - g(y)dy$, where f(y)= right curve & g(y)= left curve consider the graph below:-



the area between f(x) & g(x) is: a=D+B+C+A

A. $\int_a^b (g(x) - f(x)dx$

$f\int_b^c (f(x) - g(x)dx$

$f\int_c^d (g(x) - f(\square)dx + \int_d^e (g(x) - f(x)dx$

Examples

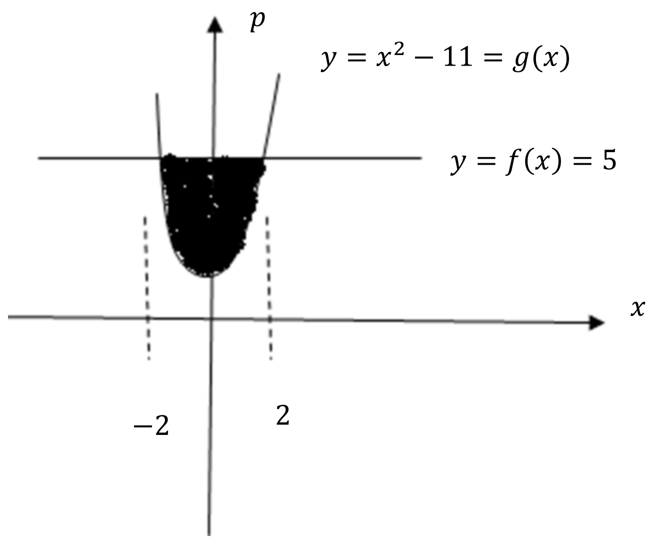
1) Find the area of a region enclosed by the graph of

a) $f(x) = 5\&g(x) = x^2 + 1$

Solution

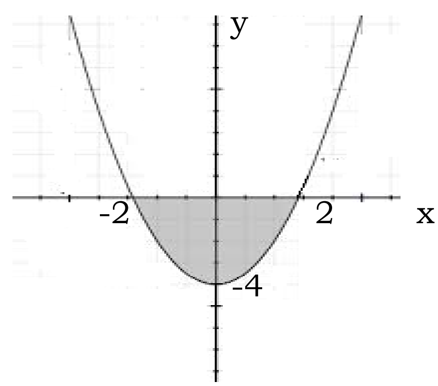
$$A = \int_{-2}^{0} (f(x) - g(x)dx\ f \int_{0}^{f} (f(x) - g(x)dx$$

$= \int_{2}^{0} (5 - (x^2 + 1)dx + \int_{0}^{2} (s - x - 1)dx = \frac{16}{3} + \frac{16}{3} = \frac{32}{3}$ square units



Exar

1. raph of f(x) and the x-axis.



   a. f(x)=x^2-4

      y=x^2-4

      A=$\int_{-2}^{2} (x^2 - 4)dx$

      A=$\int_{-2}^{2} (x^2 - 4)dx$=$\frac{32}{3}$sq units

   b. A=$\int_{-2}^{2} (4 - x^2)dx$=$\frac{32}{3}$sq units

   c. A=$\int_{0}^{2} (x^2 - 2x)dx$=$\frac{4}{3}$sq units

   d. A=$\int_{0}^{2} (x^2 - 2x)dx$=$\frac{20}{3}$sq units

2. Find the area S which is enclosed by the graph of f(x), the two straight lines and the x-axis.

   a. f(x), the two straight lines x=0 & x=4

      A=$\int_{0}^{4} x^2 dx$=$\frac{x^3}{3} \Big|_{0}^{4}$=$\frac{64}{3} - 0 = \frac{64}{3}$ sq units

   b. f(x) x^2+3, the two straight lines x=-1&x=3.

      A=$\int_{-1}^{3} (x^2 + 3)dx = \left(\frac{x^3}{3} + x\right)\Big|_{-1}^{3} = \frac{27}{3} + 9 - \left(-\frac{1}{3} - 3\right)$=$\frac{54}{3} - \left(-\frac{10}{3}\right)$=$\frac{64}{3}$ sq units

3. Find the area S which enclosed between the graph of f(x)=x(x+2)(x-1)and the x-axis.

   A= $\int_{-2}^{1}(x(x+2)(x-1))dx = \int_{-2}^{0}(x(x+2)(x-1))dx + \int_{0}^{1}(x(x+2)(x-1))dx$

   $=\int_{-2}^{0}(x^3+x^2-2x)dx + \int_{0}^{1}(x^3+x^2-2x)dx$

   $= \frac{37}{12} sq.units$

2.4Applications of integration

Integration has various applications in deferent fields of study such as business, physical science, economics etc.

Examples

c. Suppose that the velocity v(t)=4t³ and s(0)=5. Find s(t), assuming that s(t) is in meters and v(t) is in meter per second.

   Solution: s(t)=$\int v(t)dt = \int(4t^3)dt = \frac{4t^4}{4} = t^4 + c$

   $\Rightarrow$s(0)=$0^4 + c = s \Rightarrow$ c=5

   Hence, s(t)= $t^4 + 5$

Exercise

1. Pure water enterprises find that the marginal profit, in dollars, from drilling a well that is x feet deeps is given by p′(t)=$\sqrt[5]{x}$. Find the profit when a well 250 feet deeps is drilled.

   Solution:- p(t)=$\int p'(t)dx = x^{\frac{1}{5}}dx = \frac{5}{6}x^{\frac{6}{5}} + c$

   Assume that there is no any fixed charges so, the profit only depends on the drilling so at x=0 and p(0)=0

   $\Rightarrow$ p(0)=$\frac{5}{6}(0)^{6/5} + c \Rightarrow p(250) = \frac{5}{6}(250)^{6/5}$=628.56≈ 629

2. The company estimates that its sales will grow continuously at a rate given by the function s′(t)=20eᵗ, where s′(t) is the rate at which sales are increasing in dollars per day, on day t.

Solution:

a. Find the accumulate sales for the first 5 days.

b. Find the sales from the 2nd day through the 5th day

Solution:

a. s′(t)= $20e^t$, t-days

$s(5)=\int_0^5 s'(t)dt = \int_0^5 20e^t dt = 20(e^5 - 1)$ dollars

b. sales (2nd to 5th days)= $\int_2^5 20e^t dt = 20(e^5 - e^2)$ dollars

Review exercise on unit -2

1. Find the derivative of each of the following function at the given number.

a. f(x)= $\frac{x^3+3x}{x^2}$, at x=2

$$f'(t)=\left(\frac{x^3+3x}{x^2}\right)^1 = \frac{(x^3+3x)^1(x^2)-(x^2)^1(x^3+3x)}{(x^2)^2}$$

$$= \frac{(x^3+3x)(x^2)-2x(x^3+3x)}{x^4}$$

$$= \frac{3x^4+3x^2-2x^4-6x^2}{x^4}$$

$$= \frac{x^4-3x^2}{x^4}\Big|_{x=2}$$

$$\Rightarrow f'(2)= \frac{2^4-3(2)^2}{2^4} = \frac{16-12}{16} = \frac{4}{16} \text{-} \frac{1}{4}$$

b. f(x)=$\frac{5}{x^3+2}$at x=-4

$$f'(x)=\frac{(5)^1(x^3+2)-(x^3+2)^1 s}{(x^3+2)^2} = \frac{-15x2}{(x^3+2)^2}\Big|_{x=-4}$$

$$\Rightarrow f'(-4)= \frac{-15(-4)^2}{((-4)^3+2)^2} = \frac{-2402}{3844} = \frac{60}{961}$$

c. f(x)=$\frac{x^4}{x^3+2}$, at x=2

$$f'(x)=\frac{(x^4)^1(x^3+2)-(x^3+2)^1 x4}{(x^3+2)^2}$$

$$=\frac{4x^3(x^3+2)-(4(3x^2)}{(x^3+2)^2}\Big|_{x=2}$$

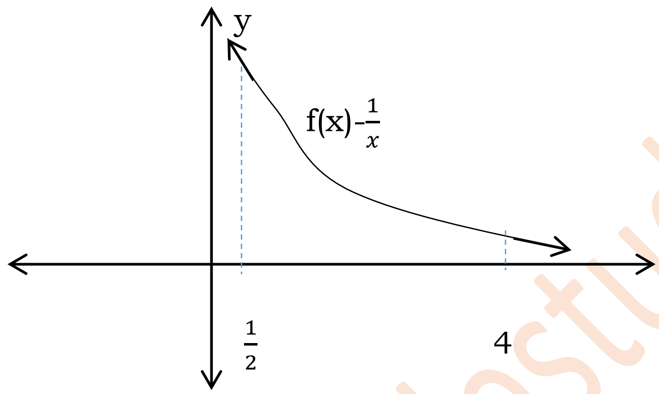
$\Longrightarrow$ f'(2)= $\frac{4(2)^3(2^3+2)-4(3(2)^3)}{(x^3+2)^2} = \frac{32}{25}$

d. f(x)=$\frac{4\sqrt{x}}{2x+1}$at x=4

f'(x)=$\frac{(4\sqrt{x})^1(2x+1)-(2x+1)^1(4\sqrt{x})}{(2x+1)^2}$

$\Longrightarrow$ f'(4)= $\frac{-4x+2}{\sqrt{x}(2x+1)^2} \Rightarrow$ f'(4)= $\frac{-4x4+2}{\sqrt{4}(2(4)+1)^2} = \frac{-7}{81}$

2. The diagram shown below is the graph of the function y=$\frac{1}{x}$ between.



3. If an object is dropped from an 80m high window its height y above the ground at the time t second is given by the formula y=f(t)=80-4.9t^2 (neglect air resistance)
   i. Find the average velocity of the falling object between
      a. t=1sec and t=1.1 sec
      b. t=1sec and t=1.01 sec
      c. t=1sec and t=1.001 sec
   ii. Find a simple formula for the average velocity of the falling object between t=1sec and t=(1+Δt)sec.
   iii. Determine what happens to this average velocity as Δt approaches 0.

Solution

Given y=f(x)=80-4.9t^2

i. a. average velocity =$\frac{F(x_2)-f(x_1)}{x_2-x_1} = \frac{f(1.1)-f(1)}{1.1-1}$

$= \frac{80-4.9(1.1)^2-(80-4.9)}{1.1-1} = \frac{-5.929+4.9}{0-1} = \underline{\underline{-10.29}}$

b. Average velocity= $\frac{f(1.01)f(1)}{1.01-1}$=$\underline{\underline{-9.849}}$

c. Average velocity= $\frac{f(1.001)f(1)}{1.01-1}$=$\underline{\underline{-9.8049}}$

ii. Average velocity= $\frac{f(1+\Delta t)-f(1)}{\Delta t}$ = $\underline{\underline{-9.8-4.9\Delta t}}$

iii. As $\Delta t \to 0$, average velocity= $\underline{-9.8m/sec}$

i. Find the slope of the secant line between:

a. x=3 and x=3.1 b. x=3 and x=3/01 c. x=3 and x=3.001

ii. Find a formula for the slope of the secant line between (3,f(3)) and (3+$\Delta x, +(3+\Delta x)$) for a function f.

iii. Determine what happens when Δx approaches 0.

Solution:

i. Slope of a secant line= average rate change

$$= \frac{\text{change in y}}{\textit{change in x}} = \frac{f(x_2)+f(x_1)}{x_2-x_1}$$

$$= \frac{\frac{1}{3.1}-\frac{1}{3}}{3.1-3} = \left(\frac{10}{31}-\frac{1}{3}\right) \div 0.1 = \left(\frac{30-31}{93}\right) \div 0.1$$

$\Longrightarrow$ Slope of a secant line= $-\frac{1}{93} \div 0.1 = 0.1 = -\frac{1}{9.3}$

b. slope of a secant line= $-\frac{\frac{1}{3.01}-\frac{1}{3}}{0.01} = -\frac{1}{9.03}$

c. slope of a secant line=$\frac{f(x_2)-f(x_1)}{x_2-x_1} = \frac{\frac{1}{3.001}-\frac{1}{3}}{3.001-3} = \frac{\frac{1000}{3001}-\frac{1}{3}}{0.001} = \frac{-1}{9.003}$

ii. slope= $\frac{f(3+\Delta x)-f(3)}{\Delta x}$

iii. slope =0 as x approaches to zero.

4. Let y=f(t)=t² where t is the time in second and y is the distance in meters that an object falls on a certain airless planet draw a graph of this function between t=0 and t=3

i. Make a table of the average speed of the falling object between.

a. T=2sec and t=sec

b. T=2sec and t=2.1sec

c. t=2sec and t=2.01sec

ii. Find a simple formula for the average speed between time t=2sec and t=(2+Δt)sec

iii. In your formula for average speed determine what happens as Δt approaches zero.

iv. Draw the straight line through the point (2,4) whose slope is the instantaneous velocity you just computed.

Solution:

i. $y=f(t)=t^2$

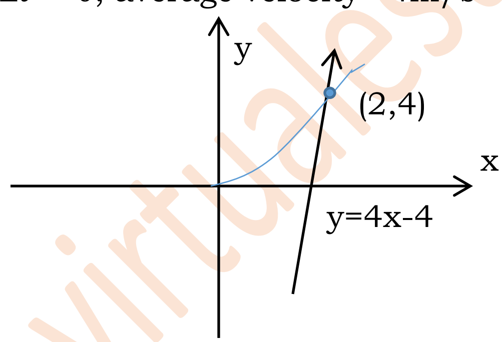
a. Average velocity= $\frac{f(3)-f(2)}{3-2} = \frac{9-4}{1} = 5m/s$

b. Average velocity = 4.1m/s

c. Average velocity = 4.01m/s

ii. Average velocity = $\frac{f(2+\Delta t)-f(2)}{\Delta t} = \frac{a+a\Delta t+\Delta t^2-4_2}{\Delta t}$= $\underline{(4+\Delta t)m/s}$

iii. As $\Delta t \to 0$, average velocity =4m/s



5. Find the equation of the tangent and normal lire's $f(x)=(1-x^3)(\sqrt{x+2})$

Solution:

a. Equation of tangent line $y-f(-1)=f'(-1)(x+1) \Rightarrow y=-2x$

b. Equation of normal line $y-f(-1)=-\frac{1}{f'(-1)}(x+1) \Rightarrow y=\frac{1}{2}x$

6. Find the derivatives of the following functions.

a. $f(x)=x^2(2x+5)=2x^3+5x^2$ b. $f(x)=\frac{x^2+x}{x^3-x+2}$

$f'(x)=6x^2+10x$ $f'(x)=\frac{x^4-2x^3-x^2+4x+2}{(x^3-x+2)^2}$

c. f(x)=$\frac{x^2}{x+1}$ ⇒ f ′(x) = $\frac{x^2+2x}{(x+1)^2}$ d. f(x)= $3x^2(1-\frac{1}{x})\left(x^2-\frac{1}{x^2}\right)$

⟹ f ′(x)= $3x^2(1-\frac{1}{x})\left(x^2-\frac{1}{x^2}\right)$+x$\left(x^2-\frac{1}{x^1}\right)+x^3\left(1-\frac{1}{x}\right)\left(2x+\frac{2}{x^3}\right)$

Solution

f ′(x)=$30x^2$-$15x^2$ = $30x^2(x^2-5)$

⟹ f ′(x)=0 ⇒ $30x^2(x^2-5)$=0

⟹ x=0 and x=$\pm\sqrt{5}$

	$x<\sqrt{5}$	$-\sqrt{5}<x<0$	$0<x<\sqrt{5}$	$x>\sqrt{5}$
$30x^2$	++	+++++	+++++	+++++
$x+\sqrt{5}$		++++	++++	+++
$x-\sqrt{5}$				+++
f′(x)	++			+++

a. f is increasing on (-∞,$-\,-\sqrt{5}$]∪[$\sqrt{5}$, ∞)

b. f has relative maximum value at x=$-\sqrt{5}$

c. f has relative minim values at x=$\sqrt{5}$

7. Consider the function f(x)= $6x^5$-$50x^3$-120
   a. Find the interval on which f is increasing.
   b. Find the interval on which f is decreasing.
   c. Find all values of x for which f have a relative maximum.
   d. Find all values of x for which f have a relative minimum.
8. A particle moves along the curve 2y=x^2+4. Find the point(s) on the curve at which the y-coordinate changes 4 times as fast as the x-coordinate.

Solution

2y=x2+4⇒ $y=\frac{1}{2}x^2+2\Rightarrow\frac{dy}{dx}=x$

Hence, $\frac{dy}{dx}\Big|_{x=4} = 4$ when x=4, y=10

Thus the point is (4,10)

9. Let f(x)=$x^2 + px + q$. Find the values of p and q such that f(1)=3 is an extreme value of f on the interval [0,2]. Is this value maximum or minimum?

Solution

f ′(1)=2x+p f has extreme value at (1,3)

⟹ f ′(1)=0 and p(1)=3 ⟹ p=-2 and 1+p+q=3 ⟹q=4

The value is minimum.

10. Find the value of a,b,c,d so that h(x)=ax^3+bx^2+cx+d will be extreme values at (1,2) and (2,3).

Solution

h′(x)=3ax2+2bx+c

h has extreme value at (1,2) and (2,3) ⇒ f ′(1)=3a+3b+c=0 and

f ′(2)=12a+4b+c=0---1

f (1)=2 ⇒a+b+c+d=2 and f (2)=3⇒8a+4b+2c+d=3---2

from 1 and 2 b=-$\frac{9}{2}a$ and 7a+3b+c=1⇒c=$\frac{13a+2}{2}$

3a+2$\left(-\frac{9}{2}a\right)$+$\frac{13a+2}{2}$=0⇒a=-2,b=9,c=-12, d=7

11. Find the absolute maximum of the minim values of the following function on the interval [0,2].

f(x)=$\begin{cases} x^3 - \frac{x}{3}; 0 \le x \le 1 \\ x^2 + x - \frac{4}{3}; 1 < x \le 2 \end{cases}$

Solution

f ′(x)=$\begin{cases} 3x^3 - \frac{1}{3}; 0 \le x \le 1 \\ 2x + 1; 1 < x \le 2 \end{cases}$ f ′(x)=0⇒x=$\pm\frac{1}{3}$ $and\ x = \frac{1}{2}$

f(0)=0, f(2)=$\frac{14}{3}$, f$\left(\frac{1}{3}\right) = -{}^{2}/_{27}$

$\therefore$ The absolute maximum value is $\frac{14}{3}$ and occur at x=2. and the absolute minimum value is $-{}^{2}/_{27}$ and occur at x=${}^{1}/_{3}$.

12. A board 5 feet long slider down a wall at the instant when the bottom end is 4 feet from the wall, the other end is moving down at a rate of 2 feet/sec. at that moment.
   a. How fast is the bottom and sliding along the ground?
   b. How fast is the area of the region between the board, the ground, and the wall changing?

   Solution

$$\frac{dy}{dt} = 2ft/sec, \frac{dx}{at} = ? \Rightarrow y^2 + x^2 = 25$$

$$\Rightarrow y = \sqrt{25 - x^2} \;\; \Rightarrow y = \sqrt{25 - 16} = 3$$

$$\Rightarrow 2y\frac{dy}{dt} + 2x\frac{dy}{dt} = 0 \;\; \Rightarrow y\frac{dy}{dt} + x\frac{dy}{dt} = 0$$

a. $x\frac{dy}{dt} = -y\frac{dy}{dt} \Rightarrow \frac{dy}{dt}\left(\frac{-y}{x}\right)\left(\frac{dy}{dt}\right)$

$\therefore \frac{dy}{dx}\Big|_{(4,3)} = \left(\frac{-3}{3}\right)$(2ft/sec)=-$\frac{3}{2}$ ft/sec

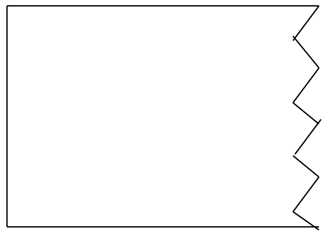
b. A=$\frac{1}{2}xy \Rightarrow \frac{dy}{dt} = \frac{1}{2}\left(y\frac{dy}{dt}\right)+\left(x\frac{dy}{dt}\right)$

$$\Rightarrow \frac{dy}{dx}\Big|_{(4,3)} = \frac{1}{2}\left(3ft + \left(\left(-\frac{3}{2}\; fect +/sec\right) + 4fact(2fect/sec)\right)\right)$$

$$= \frac{1}{2}\left(-\frac{9}{2} + 8\right) ft^2/sec = {}^{7}/_{4}\, ft^2/\text{sec}$$

13. A farmer has 240m of tensing material and wants to fence a rectangular field that borders a straight river. (No fence is needed along the river). What are the dimensions of the field that has the largest area?

Solution



y=240-2x

A(x)=x.y=x(240-2x)=240x-2x^2

A′ (x)=240-4x⇒ $A'(x) = 4(60 - x)$

A′ (x)=0⇒ $x = 60$

A′ (x) changes from positive to negative at x=0

⇒ (60)=240(60)-2(60)^2

= 7200m^2 is a maximum area.

∴ The dimensions of the rectangle that will give maximum area is x=60 and y=120.

14. The radius of the balloon is increasing at the rate of 0.5cm/sec. At what rate the surface area of the balloon is increasing when the radius is 4cm?

Solution

Given $\frac{dr}{dt} = 0.5$cm/sec, r=4cm

A= $4\pi r^2$

$\frac{dA}{dt} = 8\,\pi r\frac{dr}{dt}$

⇒ $\frac{dA}{dt}\Big|_{r=4cm}$=(8$\pi$)(4cm)(0.5cm/sec)

= 16cm^2/sec

15. A cyclist decelerates at a constant rate from 30km/hr to a standstill in 45sec.

a. How fast is the cyclist traveling after 20sec?

b. Who far has the cyclist traveler after 45sec?

Solution

a. V_0=30km/hr, V_f=0

t= 45sec=0.0125hr

$\vec{a} = \frac{\Delta v}{\Delta t} = \frac{v_t - v_0}{t} = \frac{\frac{0.30}{0.0125}km}{hr} = \frac{\frac{-30}{0.0125}km}{hr}$

$\vec{a} = \frac{dv}{dt} \Rightarrow dv = adt \quad \Rightarrow v(t) = \int adt = \int \left(\frac{-30}{0.0125}\right) dt$

$= \frac{-30}{0.0125}t + c_1$

At t=0 ⇒ v(0)=30⇒c_1=30

v(t)=-2400t+30

At t=20sec = 0.056hr

V(0.0056)=-2400x0.0056+30= 16.56km/hr

s(t)=$\int(-2400t + 30)dt$ = -1200t^2+30t+c_2

At t=0, s(t)=0 ⇒ c_1=0

b. s(t)=-1200t^2+30t

at t=45 sec=0.0125hr

s(0.0125)=0.175km=<u>187.5m</u>

16. A factor is polluting a lake in such a way that the rate of pollutants entering the lake at time t, in months, is given by

$N'(t) = 280t^{3/2}$, where

N is the total numbers of pounds of pollutants in the lake at time t.

a. how many pounds of pollutants enter the lake in 16 months?

b. An environmental board tells the factory that it must begin to clean up procedures after 50,000 pounds of pollutants has entered the lake. After what length of time will this occur?

Solution

In 16 months means time t start from 0 month to 16 month.

$\Rightarrow \int_0^{16} 280t^{3/2}dt = 280\int_0^{16} t^{3/2}$ $=280\ \frac{t^{3/2+1}}{\frac{3}{2}+1}$ $= 280t^{5/2} * \frac{2}{5} = 280\left(\emptyset t^{5/2}\right)\frac{2}{5}\Big|_{t=0}^{16}$

$280.16^{5/2} * \frac{2}{5} = 280.4^5 * \frac{2}{5} = 280(409.6)=$ <u>114688 pollutants</u>

N(t)=$280t^{5/2}\left(\frac{2}{5}\right)$ ⇒50,000=$280t^{5/2}\left({}^{2}/_{5}\right)$

b. ⇒ 125,000=$280t^{5/2}$

$\Rightarrow (446..43)^{2/5}=\left(t^{5/2}\right)\frac{2}{5}$

$\Rightarrow (446..43)^{2/5} = 11.48 \approx 12$

After 12 months the lake will have exceeded 50.000pound of pollutants.

Unit three (3)

Statistics

Introduction

- Statics as a science deals with the proper collection, organization presentation anlysis & interpretation of numerical data.
- Statics : is applicable in business, metrology, schools, economic social & political activities.

3.1Measures of Absolute Dispersions

- The degree to which numerical data tends to spread about an average.
- The scatter or variation of variables about a central value types of measures of dispersion

  There are two types.

  a) Absolute measure of dispersion

  b) Relative measure of dispersion

     a) Absolute measure of dispersion

The following are absolute measure of dispersion

- Range
- Inter – quantile range
- Mean deviation
- Standard deviation

- The absolute measure of dispersion measures the variability interims of the same units of the data.

b) **Relative measure of dispersion**

The common relative measures of dispersion are:

- Coefficient of range
- Coefficient of quartile range
- Coefficient of mean deviation
- Coefficient of standard deviation or coefficient of variation

- A relative measure of dispersion compares the variability of two or more data that are independent of the units of measurement.

Range & Inter quartile range

Range for ungrouped data: Is the difference between the larges value & smallest value in a set of data.

$i.e. R = L - S$

Where $L - Largest\ value$

$S - Smallest\ value$

Range for Grouped data

- Is the difference between the upper class boundary of the highest. Class Bu(H). and the lower class boundary of the lowest class BL(L)

$i.e.$

$$\boxed{R = Bu(H) - BL(L)}$$

Examples

1) Calculate the range of each of the following data sets:

a) 14, 17, 18, 16, 10, 18, 19, 22.

$$\left.\begin{array}{l} here, smallest\ value = 10 \\ \text{largest value} = 22 \end{array}\right\} \quad \begin{array}{l} LV - S.V \\ Range = 22 - 10 \\ = \mathbf{12} \end{array}$$

b)

x	5	7	8	9	11
f	13	10	11	12	13

here,

$S.V = 5$
$L.V = 11$ $\} R = L.V - S.V = 11 - 5 =$ **6**

x	0-20	20-40	40-60	60-80	80-100
f	3	1	2	4	1
CBS	-0.5-20.5	20.5-40.5	40.5-60.5	60.5-80.5	80.5-100.5

c)

$BU(H) = 10.5$
$BL(L) = -0.5$

$\& = BU(H) - BL(L) = 100.5 - (-0.5) = \mathbf{101}$

3.1.1Inter Quartile Range (IQR)

Definition: Inter quantile range is the difference between the upper & the lower quantiles.

i.e. $IQR = Q_3 - Q_1$

Ex. 3.2 Example Q calculate the IQR of the following data sets.

a) 25, 21,22,20,19,16.
Arranged data: 16, 19, 20, 21, 22, 25

$$Q_1 = \left(\frac{\left(\frac{kn}{4}\right) + \left(\frac{kn}{4} + 1\right)}{2}\right)^{th\ item}$$

$$\left(\frac{\left(\frac{6}{4}\right) + \left(\frac{5}{4} + 1\right)}{2}\right)^{th\ item}$$

$$\left(\frac{\left(\frac{6}{4}\right) + \left(\frac{10}{4}\right)}{2}\right)^{th\ item}$$

$$\left(\frac{1.5^{th} + 2.5^{th}}{2}\right) = \frac{4^{th}}{2} = 2^{nd}\ observation = \mathbf{19} = \boldsymbol{Q1}$$

$$Q3 = \left(\frac{\frac{3\times 6}{4} + \frac{3\times 6}{4} + 1}{2}\right)^{th} = \left(\frac{\frac{18}{4} + \frac{22}{4}}{2}\right)^{th} = 5^{th} \text{ observation } Q3 = 22$$

X	5	7	8	9	11
f	13	10	11	12	13
Cf	13	23	34	46	59

$\therefore IQR = Q3 - Q1 = 22 - 19 = \mathbf{3}$

b)

Total frequency = 59 location of Q1, $k\frac{(n+1)^{th}}{4}$

$Q1 = \left(\frac{1\times(59+1)}{4}\right)^{th} = \left(\frac{60}{4}\right)^{th}$ 15^{th} observation $\boldsymbol{Q1} = \mathbf{7}$

$$Q_3 = \left(\frac{3(59+1)}{4}\right)^{th} \; item = 3(15) = 45^{th} \; observation \; \boldsymbol{Q_3} = \mathbf{9}$$

$IQR = Q_3 - Q_1 = \mathbf{9 - 7 = 2}$

X	0-20	20-40	40-60	60-80	80-100
f	3	1	2	4	1
Cf	3	4	6	10	11

c)

$Q_1(class) = \left(\frac{kn}{4}\right)^{th} = \frac{1\times 11}{4} = (2.75)^{th}$ Which is found in the 1st class.

$$Q_1 = Bl + \left(\frac{\frac{kn}{4} - c + b}{fc}\right)xi = + \left(\frac{2.75 - 0}{3}\right) \times 20 = \mathbf{71.25}$$

3.1.2 Mean Deviation & Quartile Deviation

Definition

- Mean deviation of the data is the sum of all deviations (in absolute value) of each item from the average value divided by the total number of items.

i) Mean deviation about the mean MD ($\overline{x}$) ($for\ ungrouped\ data$

$$MD = |x_1 - \hat{x}| + |x_2 - \hat{x}| + |x_3 - \hat{x}||+\cdots+|x_n - \hat{x}| = \sum_{i=1}^{\cap} \left|xi = \overset{\cap}{x}\right|$$

Mean deviation for discrete frequency distributions

If $x_1 x_{2,} x_3, \cdots, x_n$ $are\ values\ with\ corresponding\ frequencies$ $f_1, f_2, f_3, \cdots, f_n$, then

i) **Mean deviation from the mean**

$$MD(\bar{x}) = \frac{f_1|x_1 = \bar{x}| + f_2|x_2 - \bar{x}| + \cdots + f_n|x_n - \bar{x}|}{f_1 + f_2 + \cdots + f_n}$$

$$= \frac{\sum_{i=1}^{n} fi|fi - \bar{x}|}{\sum_{i=1}^{n} fi}$$

ii) Mean deviation about the median

$$MD\left|\tilde{x}\right| = \frac{f_1\left|x_1 - \tilde{x}\right| + f_2\left|x_2 - \tilde{x}\right| + \cdots + f_n\left|x_n - \tilde{x}\right|}{f_1 + f_2 + \cdots + f_n} = \frac{\sum_{i=1}^{n} fi\ \left|x1 - \tilde{x}\right|}{\sum_{i=1}^{n} fi}$$

iii) Mean deviation about the mode

$$MD(\hat{x}) = \frac{f_1|x_1 - \hat{x}| + f_2|x_2 - \hat{x}| + \cdots + \ f_n|x_n - \hat{x}|}{f_1 + f_2 + \cdots + f_n}$$

$$= \frac{\sum_{i=1}^{n} fi\ |x_i - \hat{x}|}{\sum_{i=1}^{n} fi}$$

Mean deviation for grouped frequency distribution

i) Mean deviation about the mean

$$MD(\bar{x}) = \frac{f_1|m_1 - (\bar{x})| + f_2\left|m_2 - \tilde{x}\right| + \cdots + f_n\left|mn - \tilde{x}\right|}{f_1 + f_2 + \cdots + fn}$$

$\frac{=\sum_{i=1}^{n} fi\left|mi - \tilde{x}\right|}{\sum_{i=1}^{n} fi}$ where mi the mid point of the i^{th} class

ii) **Mean deviation about the median**

$MD(\hat{x}) = \frac{f_1|m_1-\hat{x}|+f_2|m_2-\hat{x}|+\cdots+f_n|mi-\hat{x}|}{f_1+f_2+f_3+\cdots+f_n} = \frac{\sum_{i=1}^{n} fi|mi-\hat{x}|}{\sum_{i=1}^{n} fi}$

Where mi= midpoint of the i^{th} $\hat{x}$ class

Examples

1) Calculate the mean deviation about the mean, median & mode of each of the following data sets

a) 12, 9, 15, 12, 7, 10, 12 Arranged data: 7,9,10,12,12,12,15

$$\bar{X} = \frac{7+9+10+12+12+12+15}{7} = \frac{77}{7}11$$

x	**f**	**fx**	$\mathbf{f}f\lvert x-\bar{x}\rvert$
7	1	7	4
9	1	9	2
10	1	10	1
12	3	36	3
15	1	15	4
Total	7	77	14

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} fi|xi-\bar{x}|}{n} = \frac{14}{7} = 2$$

- $MD(\hat{x}) = \frac{\sum_{i=1}^{n} fi(mi-\hat{x})}{n}$

But $\hat{x} = \left(\frac{n+1}{2}\right)^{th} = \frac{7+1}{2} = 4^{th}$ observation is the median $\tilde{x} = 12$

X	F	Fx	$f\lvert x-\tilde{x}\rvert$
7	1	7	5
9	1	9	3
10	1	10	2
12	3	36	0
15	1	15	3
Total	7	77	13

$$MD\left(\tilde{x}\right) = \sum_{i=1}^{n} fi \frac{x\left|i - \tilde{x}\right|}{n} = \frac{13}{7} = 1.86$$

- $MD(\hat{x})$, the mode for the given data is 12

X	F	Fx	F$\lvert x-\hat{x}\rvert$
7	1	7	5
9	1	9	3
10	1	10	2
12	3	36	0
15	1	15	3
Total	7	77	13

$$MD(\hat{x}) = \sum_{i=1}^{n} \frac{fi|xi - \hat{x}|}{n} = \frac{13}{7}$$
$$= 1.86$$

b)

x	12	13	14	15
f	10	12	9	11

c)

x	0-4	5-9	10-14	15-19	20-24
f	3	1	2	4	1
Cf	3	4	6	10	11

Solution

b) $\bar{x} = \frac{12\times10+13\times12+14\times9+15\times11}{42} = 13.5 \;\; MD = 13, MO(\hat{x}) = 13$

$$MD(\hat{x}) = \frac{\sum_{i=1}^{n} +i|xi - \bar{x}|}{n} = \frac{42}{42} = 1$$

$$\& \; MD(\hat{x}) = \sum_{i=1}^{n} fi\left|xi - \tilde{x}\right| = \frac{41}{42} = 0.98$$

$$MD(\hat{x}) = \frac{\sum_{i=1}^{n} fi|xi - \hat{x}|}{n} = \frac{41}{42} = 0.98$$

x	F	m	fm	$F\|m-\bar{x}\|$	$f\|m-\hat{x}\|$	$F\|m-\hat{x}\|$
0-4	3	2	6	28.5	33.75	43.5
5-9	1	7	7	4.5	6.25	9.5
10-14	2	12	24	1	2.5	9
15-19	4	17	68	22	15	2
20-24	1	22	22	10.5	8.75	5.5
Total	11		127	66.5	66.25	69.5

$$\bar{X} = \frac{127}{11} = 11.5, md(x^{\sim}) = 13, \hat{X} = \mathbf{16.6}$$

$$MD(\bar{x}) = \frac{\sum_{i=1}^{n} fi|mi-\bar{x}|}{n} = \frac{66.5}{11} = \mathbf{6.04}$$

$$MD(\tilde{x}) = \frac{\sum_{i=1}^{n} |mi-\tilde{x}|}{n} = \frac{66.5}{11} = \mathbf{6.32}$$

$$MD(\hat{x}) = \sum_{i=1}^{n} fi\,|mi-\hat{x}| = \frac{69.5}{11} = \mathbf{6.32}$$

Quartile Deviation for ungrouped data

Definition: Quartile Deviation (Q, D) is half the difference between the upper & lower Quartile i.e. $QD = \frac{Q3-Q1}{2}$

Quartile deviation of grouped data

The formula for the ith quantile is $Q_i = L + \left(\frac{\frac{in}{4}-c+b}{fc}\right)xi$

Examples: The following table gives the amount of time (in minutes) spent on the internet each evening by a group 756 students. Calculate the quartile deviation.

Time spent on internet (x)	10-12	13-15	16-18	19-21	22-24
No of sts (f)	3	12	15	24	2
Class boundary	9.5-12.5	12.5-15.5	15.5-18.5	18.5-21.5	21.5-24.5
Cf	3	15	30	54	56

$Q_1 = \left(\frac{56}{4}\right)^{th} = 14^{th}$ observation is the first quartile which is found in the second class.

$Q_1 = 12.5 + \left(\frac{14-3}{12}\right) \times 3 = \mathbf{15.25}$ minutes

$$Q_3 = \left(\frac{3 \times 56}{4}\right)^{th} = (42)^{th} \textit{which is found in the } 4^{th} \textit{ class}$$

$$Q_3 = 18.5 + \left(\frac{42-30}{24}\right) \times 3 = 20\, minutes$$

$$Q_0 = \frac{Q_3 - Q_1}{2} = \frac{20 - 15.25}{2} = 2.375\, minutes$$

3.2 Variance & Standard deviation

- **Variance for ungrouped data & discrete data**

Definition The variance of a data set is the average of the squared difference of each value from the mean of the data set.

- For row data $\delta^2 = \frac{(x_1-\bar{x})^2+(x_2-\bar{x})^2+\cdots+(x_n-\bar{x})^2}{n} = \frac{\sum_{i=1}^{n}(xi-\bar{x})^2}{n}$

  For discrete data

$$\delta^2 = \frac{f_1(f_1-\bar{x})^2 + f_2(x_2-\bar{x})^2 + \cdots + f_n(xn-\bar{x})^2}{n}$$

$$= \frac{\sum_{i=1}^{n} fi(xi-\bar{x})^2}{\sum_{i=1}^{n} fi}$$

$for\ grouped\ data$

$$\delta^2 = \frac{\sum_{i=1}^{n} fi(mi - \bar{x})^2}{\sum_{i=1}^{n} fi}$$

Standard deviation

Definition: The standard deviation of a data set is the positive square root of a variance. The standard deviation is represented by the Greek letter $\delta(sigma)$

If $x_1, x_2, \cdots, x_n$ are observed values, then the standard deviation for the ungrouped data is given by:

$\delta = \sqrt{variance} = \sqrt{\frac{\sum_{i=1}^{n}(xi - \bar{x})^2}{n}}$ for raw data

$\delta = \sqrt{\frac{\sum_{i=1}^{n} fi(xi - \bar{x})^2}{n}}$ for discrete data

$\delta = \sqrt{\frac{\sum_{i=1}^{n}(mi - \bar{x})^2}{n}}$ for grouped data

Examples

1) Find the variance & standard deviation of the following
   a) 6,7,4,11,12,13,17..
   b)

x	F	$\bar{x}$	$x - \bar{x}$	$(x - \bar{x})^2$	$f(x - \tilde{x})^2$
4	1	10.1	-9.1	82.81	82.81
6	1	10.1	-9.1	82.81	82.81
7	1	10.1	-9.1	82.81	82.81
10	1	10.1	-9.1	82.81	82.81
11	2	10.1	-8.1	65.61	131.20
12	1	10.1	-9.1	82.81	82.81
13	1	10.1	-9.1	82.81	82.81
17	1	10.1	-9.1	82.81	82.81
Total	9				

$$\delta^2 = \frac{\sum_{i=1}^{n}(xi - \bar{x})^2}{n} = \frac{124.08}{9} = 13.79$$

$$\delta = \sqrt{13.79} = \mathbf{3.71}$$

x	1	2	3	5	6	7
f	2	3	2	1	2	2

$$\bar{x} = \frac{45}{12} = 3.75, \delta^2 = \frac{\sum_{i=1}^{n} f(xi - \bar{x})^2}{\sum_{i=1}^{n} fi} = \frac{58.25}{12} = 4.85$$

$\delta = \sqrt{4.85} = \mathbf{2.20}$

C)

X	10-19	20-29	30-39	40-49
F	2	3	1	2
M	14.5	24.5	34.5	44.5
$f(m-\bar{x})^2$	378.125	42.1875	39.06	528.125

$$\bar{x} = \frac{226}{8} = \mathbf{28.25}$$

$$\delta^2 = \frac{\sum_{i=1}^{n} f(m-\bar{x})^2}{\sum_{i=1}^{n} fi} = \frac{987.5}{8} = 123.43$$

$\delta = \sqrt{123.43} = \mathbf{11.11}$

3.3 Interpretation of Relative Dispersions

3.3.1coefficient of Range

Definition coefficient of range $(CR) = \frac{L-S}{L+S}, where\ L = largest\ value\ S - Smallest\ value$

Examples Find the CR of the following data set.

a) 42,17,83,59,72,76,64,45,40,32

Solution $LV = 83; SV = 17$

$$CR = \frac{L.V - S.V}{L.V + S.V} = \frac{83-17}{83+17} = \frac{66}{100} = \mathbf{0.66}$$

b)

x	10	15	17	18	20
f	3	4	9	3	6

$L.V = 20; S.V = 10$

$$C.R = \frac{L.V - S.V}{L.V + S.V} = \frac{10}{30} = \frac{1}{3}$$

Mark	0-10	10-20	20-30	30-40	40-50	50-60
No sts	1	5	3	4	5	2

Here, upper limit of highest class L=60

Lower limit of lower class =0

The $CR = \frac{60}{60+0} = 1$

3.3.2Coefficient of Quartile Deviation (CQD)

Definition coefficient of Quartile Deviation: $CQD = \frac{Q_3-Q_1}{Q_3+Q_1}$ for raw data & discrete data For grouped data: use $Q_k = Bl = Bl + \frac{\left(\frac{kn}{4}-c+b\right)xi,}{fc}$ $k = \mathbf{1}$ &

Examples

1) Calculate the coefficient of quartile deviation

Mark	0-10	10-20	20-30	30-40	40-50	50-60
No sts	10	20	30	50	40	30
<f	10	30	60	110	150	180

$Q_1 = \left(\frac{n}{4}\right)^{th} = \left(\frac{180}{4}\right)^{th} = 45^{th}$ item which is found in the (20-30)

$$Q_1 = L + \left(\frac{\frac{n}{4} - c + b}{fc}\right)xi = 20 + \left(\frac{\frac{180}{4} - 30}{30}\right) \times 10 = \mathbf{25}$$

$Q_3 = \left(\frac{3n}{4}\right)^{th} = \left(\frac{540}{4}\right)^{th} = 135^{th}$ item which lies in 40-50

$$Q_3 = L + \left(\frac{\frac{3n}{4} - c + b}{fc}\right)xi = 40 + \left(\frac{\frac{3 \times 180}{4} - 110}{40}\right) \times 10 = \mathbf{46.25}$$

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{46.25 - 25}{46.25 + 25} = \frac{21.25}{71.25} = \mathbf{0.3}$$

Coefficient of mean Deviation (CMD)

Definition: The coefficient of mean deviation from the mean is CMD

Where MD mean deviation from the mean & $\bar{x} = mean$

- The coefficient of mean deviation from the medians $CMD = \frac{MD}{\tilde{x}}$, where MD= mean deviation from the median & $\tilde{x} = median$
- The coefficient of mean deviation from the mode is $CMD = \frac{MD}{\hat{x}}$ where MD mean deviation from the made & $\hat{x} = mode$

Coefficient of variation (CV)

The coefficient of variation is a unit less relative measure that is used to measure the degree of consistency given as a ratio of the standard deviation to the mean

$$i.e. CV = \frac{\delta}{\bar{x}} \times 100$$

Examples

1) Find the coefficient of mean deviation of the following data set a bout the mean.

x	0	1	2	3	4
f	1	9	7	3	4

Solution

X	F	$x - \bar{x}$	$f(x - \bar{x})$
0	1	2	2
1	9	1	9
2	7	0	0
3	3	1	3
4	4	2	8
Total	$Ef = 24$		$Ef\|x - \bar{x}\| = 22$

$MD = \frac{22}{24} = 0.92$

$\therefore CMD = \frac{MD}{\bar{x}} = \frac{0.92}{2} = \mathbf{0.46}$

2) Find the coefficient of quartile deviation coefficient of mean deviation about the mean, median & made & the coefficient of variation for the data set as shown below.

Ex.) Q2

Income	35-39	40-44	45-49	50-54	55-59	60-64	65-69
f	13	15	17	28	12	10	5
cf	13	28	45	73	85	95	100
$\bar{x}$	46.35	46.365	46.35	46.35	46.35	46.35	46.35
$\lvert x-\bar{x}\rvert$	9.35	4.35	0.65	5.65	10.65	15.65	20.65
Mi	37	42	47	52	57	62	67
$f\lvert mi-\tilde{x}\rvert$	121.55	65.25	11.05	158.2	127.8	156.5	103.25
$\lvert mi-\tilde{x}\rvert$	13.4	8.4	3.4	1.6	6.6	11.6	16.6

$$\sum f = 100, \quad \sum f|x-\bar{x}| = \mathbf{743.6}$$

$$CQD = \frac{Q_3 - Q_1}{Q_3 + Q_1}, but\ Q_1 = be + \left(\frac{\frac{kn}{4} - c + b}{fc}\right) xi, but\ Q_1 = \left(\frac{n}{4}\right)^{th}\ item$$

$$= \left(\frac{100}{4}\right)^{th} - 25^{th} \text{ which is in 40-44}$$

$$Q_1 = 39.5 + \left(\frac{25-13}{15}\right) \times 5 = 39.5 + \frac{12}{15} \times 5 = \mathbf{43.5}$$

$$\&\ Q_3 = be + \left(\frac{\frac{kn}{4} - c + b}{fc}\right) xi = 54.5 + \left(\frac{75-73}{12}\right) \times 5 = \mathbf{55.33}$$

$$CQD = \frac{55.33 - 43.5}{55.33 + 43.5} = \frac{11.83}{98.83} = \mathbf{0.1197}$$

$$CMD = \frac{MD}{\bar{x}}, MD = \frac{\sum f|x-\bar{x}|}{\sum f} = \frac{743.6}{100} = \mathbf{7.436}$$

$$CMD = \frac{MD}{\tilde{x}}, \tilde{x} = 50.4,$$

$$\sum f|mi - \tilde{x}| = 61.6$$

$$\sum f = 100$$

$$MD = \frac{\sum fi|mi - \tilde{x}|}{\sum fi}$$

$$MD = \frac{681}{110} = \mathbf{6.81}$$

$$CMD = \frac{MD}{\tilde{x}} = \frac{6.81}{50.4} = \mathbf{0.135}$$

$$CMD = \frac{MD}{\tilde{x}}$$

But $\hat{x} = be + \left(\frac{d_1}{d_1+d_2}\right)xi$

$$= 49.5 + \left(\frac{11}{11+16}\right) \times 5$$

$$= 49.5 + \frac{55}{27}$$

$$= 49.5 + 2.04$$

$$= 51.54$$

$$CMD = \frac{MD}{\hat{x}} = \frac{6.81}{51.54} = \mathbf{0.132}$$

$$CV = \frac{\delta}{\bar{x}} \times 100$$

But $\delta = \sqrt{\frac{(\sum_{i=0} fi(mi-\bar{x})^2}{n}}$

And

$$\delta^2 = \frac{\sum_{i=1}^{n} fi(mi - \bar{x})^2}{\sum_{i=1}^{n} fi}$$

$$\delta^2 = \frac{8263.75}{100}$$

$$\delta^2 = 82.6375$$

$$\delta = \sqrt{\delta^2}$$

$= \delta = \sqrt{82.6375}$

$\boldsymbol{\delta = 9.09}$

$C.V = \frac{\delta_{\times 100}}{\bar{x}} = \frac{9.09}{46.35} \times 100$

$= 0.1961 \times 100$

$\boldsymbol{CV = 19.61}$

3) The median and mean deviation from the median gross incomes of two companies are given below: (Ex. 3.22) Q3

Company	$\tilde{x}(md)$	$MD(md)$
A	50,000	4430
B	20,000	2200

a) Calculate the CMD of each company from the median
b) Which company has more consistent income?

Solution for company A

a) $CMD = \frac{MD(\tilde{x})}{\tilde{x}} = \frac{4430}{50,000} = \mathbf{0.089}$

$$CMD\ for\ company\ B, CMD = \frac{MD}{\tilde{x}} = \frac{2200}{20,000} = 0.1$$

b) Company a has the more consistent income because it has less CMD (less variability).

Ex.

Q3) The mean and δ of the gross incomes of two companies are given below:

Company	$\boldsymbol{\bar{x}}$	$\boldsymbol{\delta}$
A	12,000	2400
B	20,000	4,400

a) Calculate the CV of each company
b) Which company has more variable income?

Solution: for company A, $CV = \frac{\delta A}{\bar{x}A} \times 100 = \frac{2{,}400}{12{,}000} \times 100 = \mathbf{20}$

For company B, $CV = \frac{\delta B}{\bar{x}B} \times 100 = \frac{4{,}40}{20{,}000} \times 100 = 22$

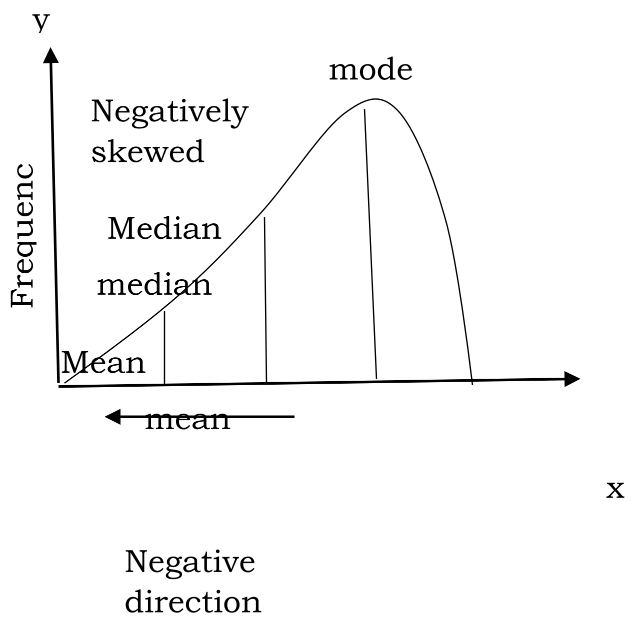
b) Company B has the more variable income

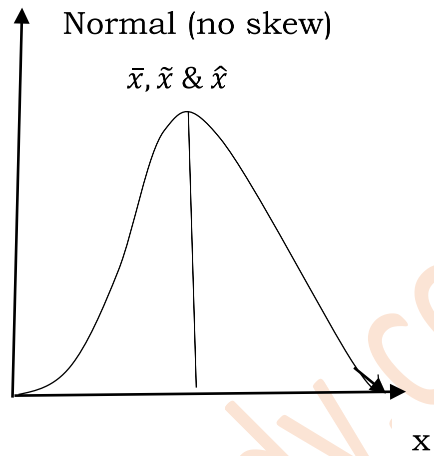
3.4 use of frequency curves

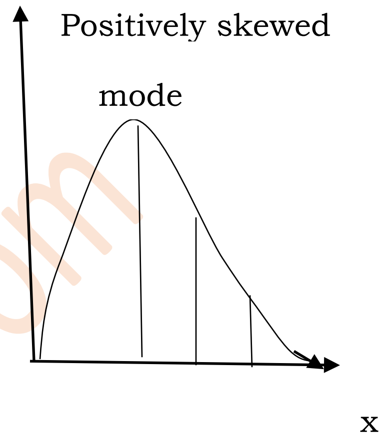
The shape of a frequency curve describes the distribution of a data set. Such a description is made possible after the frequency curve of a frequency distribution is drawn.

3.4.1 Mean, Median and mode on the frequency curve

- A measure of central tendency or a measure of dispersion alone does not tell us whether or not the distribution is symmetrical. It is the relationship between the $\bar{x}$, $\tilde{x}$ & $\hat{x}$ that tells us whether the distribution is symmetrical or skewed.
- If the mean is smaller in value, the median is larger than the mean but smaller than the mode, then the distance is negatively skewed. i.e. If $\bar{x} < \tilde{x} < \hat{x}$.(skewed to the left)
- For a unimodal distribution in which the values of mean, median and made coincide (i.e. mean=median=mode)the distribution is said to be perfectly symmetrical.
- If the mean is the larger in value, and the median is larger than the mode but smaller than the mean, then the distribution is positively skewed.
  i.e. $\bar{x} > \tilde{x} > \hat{x}. (skewed\ to\ the\ right)$.





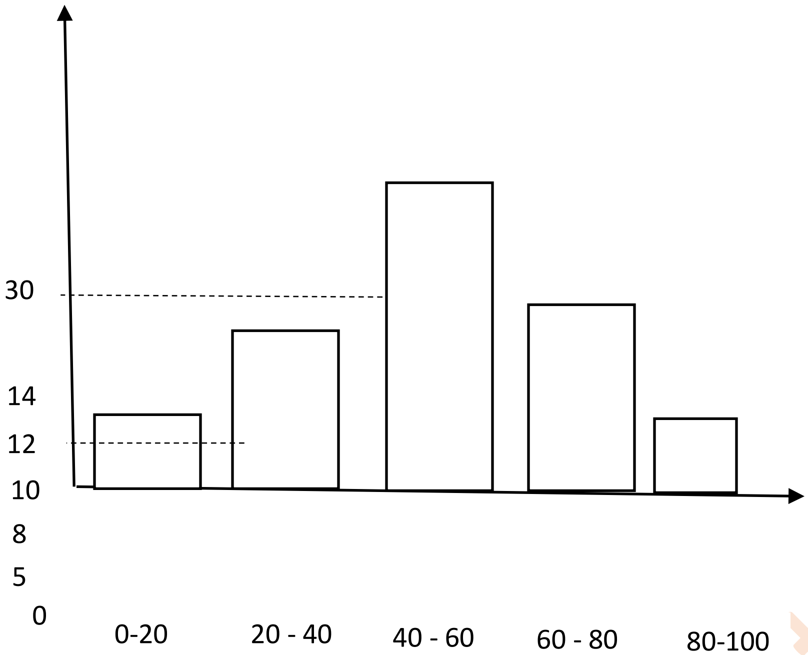


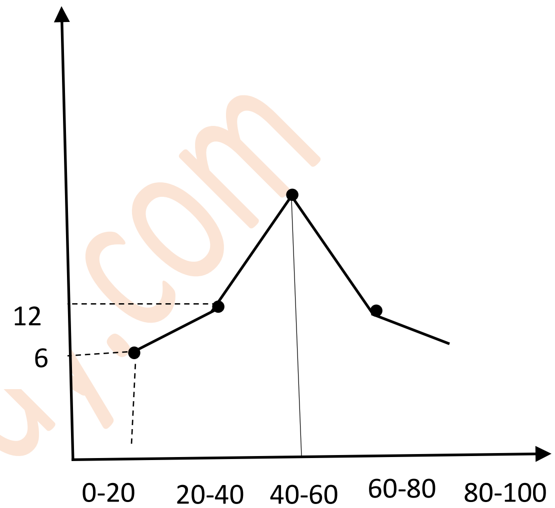
Ex.

Consider the following frequency distribution

x	0-20	20-40	40-60	60-80	80-100
f	8	12	30	14	6
cf	8	20	50	64	70

a) Draw the bar graph & frequency curve





b) Median Bet $\left(\frac{\frac{n}{2}-cfb}{fc}\right)xi\ but\ \left(\frac{n}{2}\right)^{th} = \left(\frac{70}{2}\right)^{th} = 35^{th}$ which is in 40-60

Here, $Be = L = 40, c+b = 20, fc = 30, i = 20$

$$\tilde{x} = 40 + \left(\frac{35-20}{30}\right) \times 20 = 40 + \frac{15 \times 20}{30} = \mathbf{50}$$

$$\tilde{x} = \frac{\sum_{i=i}^{n} fi\ mi}{\sum_{i=1}^{n} fi} = \frac{70 \times 8 + 30 \times 12 + 50 \times 30 + 70 \times 14 + 90 \times 6}{70}$$

$$= \frac{3460}{70} = \mathbf{49.43}$$

$$\hat{x} = Bl + \left(\frac{d_1}{d_1 + d_2}\right)xi = 40 + \frac{18}{34} \times 20 = \mathbf{50.88}$$

c) $49.43(\bar{x}) < 50(\tilde{x}) < 50.88(\hat{x})$ = median <mode.

The distribution is negatively & skewed (skewed to the left).

3.4.2 Measures of skewness

Pearson's coefficient of skewness

- With the help of central tendencies & standard deviation it is also possible to determine the skewness of a distribution

- This called a mathematical measure of skewness.
- Skewness can be measured by calculating
  i) Pearson's coefficient of skewness
  ii) Bow ley's coefficient of skewness

  i) Pearson's coefficient of skewness (α)

$$\boldsymbol{\alpha = \frac{3(mean - median)}{standard\ deviation}}$$

- If mean = median, then the distribution is symmetrical. $i.e\ \alpha = 0$
- If $\alpha > 0$, then the distribution is positively(skewed to the right).
- If $\alpha < 0$, then the distribution is negatively skewed (skewed to the left).

  ii) Bowley's coefficient of skewness (B)

$$\beta = \frac{Q_3 + Q_1 - 2(median)}{Q_3 - Q_1}$$

- $If\ \beta = 0$, then the distribution is symmetrical.
- $If\ \beta > 0$, then the distribution is skewed positively (skewed to the right)
- $if\ \beta < 0$, then the distribution is negatively skewed (skewed to the left)

Examples: Ex.

1) Calculate α & β for the data below & determine the skewness of the distribution.

X	10	20	30	40	50
F	4	6	5	3	2
Cf	4	10	15	18	20

2) Compare α & β & internet the result

Solution

1) $\bar{x} = \frac{10\times4+20\times6+30\times5+40\times3+50\times2}{4+6+5+3+2} = \frac{530}{20} = \mathbf{26.5}$

$$\tilde{x} = 25 = Q_2\left(\tilde{x}\ location\ \left(\frac{n}{2} + \left(\frac{n}{2}\right) + 1\right)\right)^{th\ item}$$

$$= \frac{10^{th} + 11^{th}}{2} = \frac{20 + 30}{2} = \mathbf{25}$$

$$Q_1 = 20, \quad \boldsymbol{Q_3} = \mathbf{35}$$

$$\delta^2 = \frac{\sum fixi}{\sum fi} = \mathbf{152.7696}$$

$$\delta = \sqrt{152.7696} = \mathbf{12.36}$$

- $\alpha = \frac{3(mean - median)}{standard\ deviation} = \frac{3(26.5-25)}{12.36} = 0.364 > 0$
- $\beta = \frac{Q_3 + Q_1 - 2(median)}{Q_3 - Q_1} = \frac{1}{3} > 0.$

2) Since, both α & β are positive, the distribution is skewed to the right (positively skewed)

Sampling Techniques

- Some examples applications of statistics are given below:
  1) Statistics in schools
  2) Statistics in business
  3) Weather fore casting
  4) Sales tracking
  5) Health insurance
  6) Traffic
  7) Investing
  8) Medical studies
  9) Manufacturing
  10) Urban planning

- Population:- is the complete set of items which are of interest in any particular situation
- It is not possible to collect information from the whole population because it is costly in terms of time, energy & resource. To overcome these problems, we take only ascertains part of the population called a sample.

- A sample is a limited number of items taken from a population which is being studied or investigated.
- A sample serves as representative of the population so that we can draw conclusion about the entire population based on the results obtained from the sample.

a) <u>Size of a sample</u>:- there is no single rule for determining the size of a sample of a given population.

- The size should be adequate in order to represent the population.
- Homogeneity or hetero genetic of the population
- If the population has a homogeneous nature a smaller size sample is sufficient.
- Availability of resources. If sufficient resources are available, it is advisable to increase the size of the sample.

b) Independence: Each item or individual in the population should have an equal chance of being selected as a member of the sample.

Ex.

Your school has 4 sections for grade 12 and the numbers of students are shown in the table below.

Section	Boys	Girls	Total
A	20	30	50
B	20	20	40
C	10	10	30
D	20	10	30
Total	**70**	**90**	**160**

Suppose you want to conduct a survey about the grade 12 student of your school. Since time is limited you have select 20 students only.

a) What is the size of the population? <u>Ans</u>: <u>160</u>

b) What is the size of the sample? Ans: 20

c) Are the following methods of selection acceptable?

If not, explain why it is not acceptable?

1) Select 20 girls ANS: Note acceptable because buys are not in the sample

2) Select 20 students from section A.

ANS: Not acceptable because section B.C & D are not represented in the sample

3) Select 10 boys & 10 girls

ANS: Not acceptable 10 boys & 10 girls are not proportional to the population.

4) Select's students each from the four sections.

ANS: Not acceptable because sections are not represented proportially in the sample.

- We have two types of sampling
  1) Random or probability sampling
  2) Non- random or non – probability sampling
     1) Select 20 girls **Ans**: Not acceptable, because boys are not in the sample.
     2) Select 20 students from section A.

        Ans Not acceptable be cause section B,C &D are not represented in the sample.
     3) Select 10 boys & 10 girls

        Ans Not acceptable 10 boys & 10 girls are proportional to the population.
     4) Select's students each from the four sections.

        Ans: Not acceptable because sections are not represented proportially in the sample.
- We have two types of sampling
  1) Random or probability sampling
  2) Non – random or non – probability sampling

1) Random or probability sampling

a) Simple random sampling: is the section of individual for which every individual has equal chance of being selected.

We have two types of simple random sampling

i) The lottery method

ii) Using a table of random numbers (Random no s table attached a the and of the text book)

Examples

1) In your class, there are 20 students whose roll no s are from 1 to 20. You want to select five students using the random numbers table attached at the end of this text book. Following the steps explained below; select five students.
2) Select one random number from the table in random way.
3) Read 20 < on secutive numbers from the table & attach them to each of the roll. No s as listed
4) Sort the random numbers (together with the roll numbers) in ascending order.
5) Take the first 5 random numbers & the corresponding roll numbers.

Roll no	Random no	Random no in order	Roll no arranged
1	49	4	9
2	56	6	7
3	31	14	6
4	28	17	14
5	72	25	19
6	14	27	13
7	06	28	4
8	39	31	3
9	04	37	12

10	78	39	8
11	48	41	17
12	37	48	11
13	27	49	1
14	17	50	20
15	68	56	2
16	71	68	15
17	41	71	16
18	80	72	5
19	25	78	10
20	50	80	18

Which roll no s did you select? 9,7,6,14 &19.

b) Systematic sampling

Systematic sampling is another random sampling technique used for selecting a sample from a population.

- If N= size of the population & n= size of the sample, then we use $k = \frac{N}{n}$ for a sampling interval.

Examples

1) In a class, there are 110 students with class list no s written from 1-110. You need to select a sample of 10 students. How can you apply the systematic sampling technique?

Solution

$$\begin{matrix} N = 110 \\ n = 10 \end{matrix}, which\ implies\ k = \frac{N}{n} = \frac{110}{10} = 11$$

- Sort the list in ascending order
- Choose one no at random from the first 11 no s
- If the selected no is 6 then the selected sample will be 6,17,28,39,50,61,72,83,94,105.

- Note that in systematic sampling, you use

$S_n = S_1 + (n-1)k$, where S_1 is the first randomly selected sample.

- Stands for n^{th} no member of a sample & k is the sampling interval.

Exercise

You want to know how many words on average are listed on one page of a dictionary. The dictionary has 1200 pages in total. You have decided to select 20 pages as a sample for your survey. Supposing you apply the systematic sampling technique, then:

a) What is the sampling interval?

b) If you chosen 13 as the first no list all the page no s to be selected as samples.

Solution

a) Sampling interval $(k) = \frac{N}{n} = \frac{1200}{20} = 60$

b) 13,73,133,193,253,313,373,433,493,553, 613,673,733,793,853,913,973,1033,1093&1153

iii) **Stratified sampling**

Is useful wherever the population under consideration has some identifiable stratum or categorical difference where, in each stratum, the data values or items are supposed to be homogenous.

Examples

In city A, there are 60 secondary schools with 250 classes in total. Suppose that all the sections have 25 boys & 25 girls each. Among the classes, 100 are social science stream & 150 are Natural science stream. You want to select 30 classes from them to conduct asurvey about the desired future occupation of the students.

a) In this case, what are the strata?

b) If you have to choose 30 classes as your sample, how many classes you should choose from social stream & Natural science stream, respectively?

Solution

a) Stream (s0cial & Natural)

b) From social science 12 sections & from Natural science 18 sections.

IV) Cluster sampling

- Cluster sampling divides the population in to sub groups but each subgroups has similar characteristics to the whole sample. Instead of sampling individuals from each sub group, you randomly select entive sub groups.

V) Multistage sampling: Is a more complex form of smaller groups are successively selected from large population to form the sample population.

Review Exercise on Unit 3

Q1) Find the range, inter- quartile range, coefficient of range & coefficient of quartile deviation of the following data set.

Class	15-20	20-25	25-30	30-35
F	8	21	15	4
Cf	8	29	44	48

Solution

- Range = L.V. S.M=35-15=20

Range=20

Q1) (position) $\left(\frac{48}{4}\right)^{th} = 12^{th}$ which in 15-20

Q1) $\left(\frac{Bl+\frac{kn}{4}-cfb}{fc}\right)xi = \text{Q1} = 20 + \left(\frac{12-8}{21}\right) \times 5$

$Q1 = 20 + \frac{20}{21}$

$\underline{\boldsymbol{Q1 = 20.95}}$

Q3 = 27.33

- IQR= Q3-Q1=27.33-20.95=6.38
- CQD=$\frac{Q3-Q1}{Q3+Q1} = \frac{27.33-20.95}{27.33+20.95} = 0.132$
- $CR = \frac{L-S}{L+S} = \frac{35-15}{35+15} = \frac{20}{40} = \mathbf{0.5}$

Q2) Find the mean deviation about the mean of mass (in kg) & the coefficient of mean deviation about the mean of 460 intants born in hospital in one year from the following table.

mass	2-2.5	2.5-3	3-3.5	3.-5-4	4-4.5	4.5-5
f	17	97	187	135	18	6
cf	17	114	301	436	454	460

Solution

xi	f	cf	$Xifi$	$\lvert xi - \bar{x}\rvert$	$\lvert xi - \bar{x}\rvert$
2.25	17	17	38.3	1.08	18.41
2.75	97	114	267	0.58	56.55
3.25	187	301	608	0.08	15.52
3.75	135	436	506	0.42	56.30
4.25	18	454	119	0.92	25.68
4.75	6	460	28.5	1.42	8.50
Total	460		1567	4.5	180.95

$$MD = \frac{\sum_{i=1}^{n} fi|xi - \bar{x}|}{\sum_{i=1}^{n} fi} = \frac{180.95}{460} = CMD = \frac{MD}{\bar{x}} \times 100$$

Q3) Find the variance & standard deviation of the data which represents the marks scored by ten students in a class test: 30, 29, 25, 33, 35, 38, 37,48,40,44.

Solution

$$\delta^2 = \frac{\sum(x-\bar{x})^2}{n} = 44.5 \;\&\; \delta = \sqrt{\delta^2} = \sqrt{44.5} = \mathbf{6.67}$$

Q4) consider the data given below: 4,8,7,10,11

a) Find the standard deviation for the data
b) Add 10 to all the values then find the standard deviation of the new values
c) Multiply all the values by 5, then find the standard deviation of the new data.
d) Compare the results you obtained in parts b & c which the results you obtained in part a & write a conclusion.

Solution

a) $\delta^2 = \frac{\sum(x-\bar{x})^2}{n} = 6, \delta = \sqrt{\mathbf{6}}$

b) $\delta^2 = \sum(x + 10 - \check{x})^x = 6 \rightarrow \delta = \sqrt{6}$

c) $\delta^2 = \frac{\sum(5x-\bar{x})^2}{n} = 150 \rightarrow \delta = \mathbf{5\sqrt{6}}$

d) By adding the same number to the all values the standard deviation is the same but multiplying value by some constant n, then the new standard deviation is n times the old standard deviation.

Q5) calculate the coefficient of variance of the amount of money (in birr) a group of children spent on food during a school trip. The amounts spent are: 10,15,5,20,25,30,35,40.

$\delta^2 = \frac{\sum(x-\bar{x})^2}{n} = 131.25, \bar{x} = 22.5, \delta 11.46$

$$CV = \frac{\delta}{\bar{x}} = \frac{11.46}{22.5} \times 100 = \mathbf{50.93}$$

Q6) 53 students where asked to write the total number of hours per week they spent to studying mathematics with this information find the mean deviation about the mode the standard deviation & the coefficient of variation.

X	4	6	8	10	12
F	8	12	15	4	5

Solution

Mean deviation about the mode= $1.74, \bar{x} = 7.13, \delta = 2.87$

$$CV = \frac{\delta}{\bar{x}} = \frac{2.87}{7.13} \times 100 = 40.25 \rightarrow coefficient\ of\ variation\ about\ the\ mean$$

$$CV = \frac{\delta}{\hat{x}} =$$

Q7) The mean & standard deviation of 15 observations are found to be 10 &5 respectively. On checking it was found that one of the observations with the value 8 was incorrect. Calculate the mean & 8 standard deviation if the value of the corrected observation was 21.

Solution

$n = 15, \bar{x} = 10, \delta = 5, Exi = n\bar{\bar{x}} = 15 \times 10 = 150$

Wrong observation value=8, correted observation value=23

Corrected total = $150 - 8 + 23 = 165$

Correct mean $\frac{165}{15} = \mathbf{11}$

$\delta = \sqrt{\frac{Ex^2}{n} - \left(\frac{Exi}{n}\right)^2}$ incorrect $\delta = 5 = \sqrt{\frac{Ex^2}{n} - (10)^2}$

$$25 = \frac{Ex^2}{n} - 100$$

$$\frac{Ex^2}{n} = 125$$

$i.e. \dfrac{Ex^2}{15} = 125 \rightarrow Ex^2 = 1875$

Corrected $Ex^2 = 1875 - 8^2 + 23^2 = \mathbf{23210}$

Corrected $\sqrt{\frac{23210}{15} - (11)^2} = \sqrt{35} \approx \mathbf{5.9}$

Q8) suppose that each day laboratory technician A completes 40 analyses with standard deviation of 5 Technician B complete 160 analyses with standard deviation 15. Which employee shows less variation?

Solution

Technician A: $CV = \frac{\delta}{\bar{x}} \times 100 = \frac{5}{40} \times 100 = \mathbf{12.5}$

Technician B: $CV = \frac{\delta}{\bar{x}} \times 100 = \frac{15}{160} \times 100 = \mathbf{9.36}$

Employee B shows less variation.

Q9) Find the range, mean, variance & δ of the first n natural no s.

Solution

Range $n-1, \bar{x} = \frac{\sum xi}{n} = \frac{1+2+3+\cdots+n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$

$$\delta^2 = \frac{\sum xi^2}{n} - \frac{(\sum xi)^2}{n} = \frac{i^2 + 2^2 + \cdots + n^2}{6n} - \left(\frac{n(n+1)}{2n}\right)^2$$

$$= \frac{2n+3n+1}{6} - \frac{n^2+2n+1}{4} = \frac{4n^2+6n+2-3n^2-6n-3}{12} = \frac{\boldsymbol{n^2-1}}{\mathbf{12}}$$

$$\therefore \ \delta = \sqrt{\frac{n^2-1}{12}}$$

Q10) A wall clock strikes the bell once at 1 O'clock, 2 at 2 O'clock & 3 at 3 O'clock & so on.

How many times will it strike in a particular day? Find variance & standard deviation of the number of strikes the bell makes a day on the hour.

Solution

The number of strikes the bell make a day

$= 2(1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12$

$= 2 \times \frac{n(n+1)}{2} = 2 \times \frac{12 \times 13}{2} = \mathbf{156}$

$\delta^2 = (6.90)^2 = \mathbf{47.61}$

Q11) 1500 visitors turned up to the city library during the first month of the academic year. They who borrowed 6750 books altogether. Find & interpret the coefficient of variation if the square of the sum of the borrowed books is 33315.

Solution

$$\delta = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum xi}{n}\right)^2} = \sqrt{\frac{33315}{1500} - \left(\frac{6750}{1500}\right)^2} = \sqrt{1.96} = \mathbf{1.4}$$

$$\delta \; \& \; \bar{x} = \frac{\sum xi}{n} = \frac{6750}{1500} = 4.5$$

$$C.V = \frac{\delta}{\bar{x}} = \mathbf{31.11\%}$$

- The higher the C.V the more variability of the data set.

Q12) The following data obtained from the police records of the accident cases in Dire Dara city police station from February 2009 –August 2013 G.C.

X(per month	Mean (x)	Standard Deviation
Death	21.20	3.83
Heavy injury	47.20	12.85
Light injury	88.40	12.70
No of accident	276.40	12.80

a) Determine the coefficient of variation of death, heavy injury, light, injury & number of accidents.

b) Based on the values obtained in part (a) give your interpretations of the data.

Solution

a) X	$\bar{x}$	δ	CV
Death	21.20	3.83	$C.V = \frac{\delta}{\bar{x}} \times 100 = \frac{3.83}{21.20} \times 100 = 18.07$
Neavy injury	47.20	12.85	$C.V = \frac{\delta}{\bar{x}} \times 100 = \frac{12.85}{47.20} \times 100 = 27.22$
Light injury	88.40	12.70	$C.V = \frac{\delta}{\bar{x}} \times 100 = \frac{12.70}{88.40} \times 100 = 14.37$
No of accident	276.40	12.80	$CV = \frac{\delta}{\bar{x}} \times 100 = \frac{12.80}{276.40} \times 100 = 4.63$

b) Number of death is more variable than no of accident.

Q13) The mean and coefficient of variation of seven observations are 8 & 50 respectively. If five of those are 12,14,10,4 & 2, then find the remaining two observations.

Solution

Let x & y be the two observations

$$\bar{x} = 8 \,\& CV = 50 \quad CV = \frac{\delta}{\bar{x}} \times 100 \rightarrow \delta = 4, \delta^2 = \mathbf{16}$$

New $\frac{\sum x}{n} = \frac{42+x+y}{7} = 8 \quad 42 + x + y = 56$

$x + y = 14 \cdots (1)$

$\sum x = 42 + x + y \;\&\; \delta = \sqrt{\left(\frac{\sum x^2}{n} - \frac{\sum xi}{n}\right)^2}$

$\delta^2 = \frac{\sum x^2}{n} - \left(\frac{\sum xi}{n}\right)^2 = \left(\frac{\sum x}{n}\right)^2 - \bar{x}^2$

$= \frac{460+x^2+Y^2}{7} - 64 = 16 \rightarrow x^2 + y^2 = 100$

But $x^2 + y^2 = (x+Y)^2 - 2xY$

$= 100 = 14^2 - 2xy \quad xy = 48 \ldots\ldots (2)$

From (1) & (2)

$x^2 - 14x + 48 = (x-8)(x-6) = 0$

$x = 8\; or\; x = 6 \;\&\; y = 6, if\; x = 8$

$y = 8\; if\; x = 8$

$\therefore The\; tow\; nos\; are\; 6 \;\&\; 8.$

Q14. For a group 100 students the mean & coefficient of variation of their marks were found to be 60 & 25, respectively. Later on it was found that the scores 45 & 70 were wrongly entered as 40 & 27. Find the corrected mean & coefficient of variation.

Solution: $\overline{\mathrm{X}} = \delta \;\& CV = 15$

$$\delta = \frac{CVx\bar{x}}{100} = \frac{25 \times 60}{100} = 15$$

- Wrong values are 40 & 27.
- Correct values are 45 & 72

- Wrong total $= \bar{x}\, xn = 60 \times 100 = \mathbf{6000}$
- Corrected total = wrong total – wrong value + correct values

  $= 6000 - 40 - 27 + 45 + 72 = \mathbf{6050}$

- Corrected mean$= \frac{correct\ total}{100} = \frac{6050}{100} = \mathbf{60.5}$
- Wrong $\delta = \sqrt{\frac{\sum x^2}{n} - \frac{(\sum x)^2}{n}} = \quad \sqrt{\frac{\sum x^2}{100} - (60)^2} = 15$

$$= \sum x^2 = 382500$$

$$correct \sum x^2 = 382\,500 - 40^2 - 27^2 + 45^2 + 72^2$$

$$= \mathbf{387380}$$

Correct $\delta = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}$

$= \sqrt{co\frac{rect\ \sum x^2}{n} - \left(\frac{correct\ \sum x}{n}\right)^2}$=$\underline{\mathbf{14.61}}$

Corrected C.V. $\frac{corrected\ \delta}{correted\ \bar{x}} = \frac{14.61}{60.5} = \mathbf{24.15}$

Q15) Solution: $\underline{\boldsymbol{f = 20}}$

Q16) $\boldsymbol{t = 20}$, $\boldsymbol{S = 4}$

Unit four

Introduction to linear programming

Introduction

- Application of the unit as a career in Mathematics
- Software engineering : A software engineer creates writes, develops & interprets different computer programs;
- Software engineers use mathematics to write a program & to solve various computing problems.

4.1. Graphical solutions of system of linear inequalities

4.1.1 Inequalities

(linear) An inequality involving a sign "<", ">" or "≤" "≥ " which are known as inequalities or an inequality of the form

$ax + b ? 0, ax + b \geq 0, ax + b < 0, ax + b < 0$ are called linear inequality.

Or $ax + by < 0, ax + by < 0, ax + by \geq 0$ are linear inequalities

Examples: $\frac{3x-5}{2} \geq \frac{x+1}{4} - 2$,$5x + 1 < 2x + 7$, $\begin{cases} 3x + 1 < 16 \\ -2x + 5 \leq 13 \end{cases}$

$\frac{2x}{3} - 3 > \frac{16x}{21} - \frac{13}{3} - \frac{2x}{15}$ etc are examples of linear inequalities.

<u>Word problem of linear inequality</u>

- To solve the word problem related & real world situations you can use the following procedures.

Step 1: Understand the problem

Step 2: Setting up the model

Step 3 solving the problem

<u>Examples</u> 1) Find all pairs of consecutive odd natural numbers both of which are larger them 20 & their sum is less than 80.

Solution let n be any natural no then in -1 & In +1 are two consecutive odd natural no s = In -1 20 & (2n-1)+(2n+1)<80

- In > 21 and 4n <80 = n< 20
- $n > \frac{21}{2}$ &

From $n > \frac{21}{2}$ & $n < 20, we\ have\ \frac{21}{2} < n < 20, n = 11,12 \ldots,19$

∴ The required possible pairs will be (21,23), (23, 25), (25, 27), (27,29), (29, 31), (31,33), (33,35), (35, 37), (37, 39)

4.1.2. Linear inequalities in two variables & their graphical solution

Definition: Half plane: The region on a side of a line in the xy – plane

Examples (Ex. 4.3)

1) Find the solution set for the following inequality graphically.

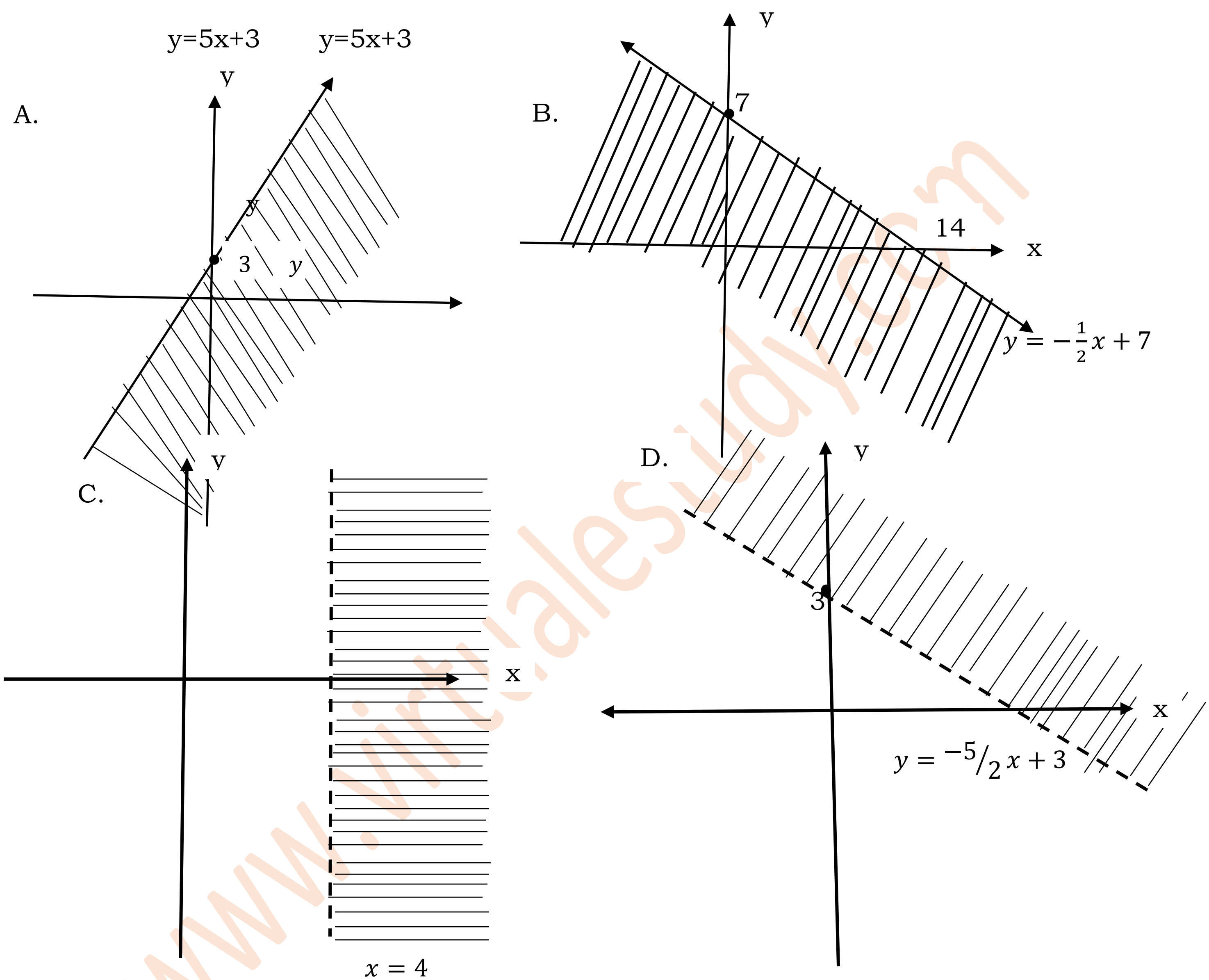
   a) $y < 5x + 3$ b) $y \leq -\frac{1}{2}x + 7$ C. $x > 4$ D. $5x + 2y > 6$

   Solution

1a) First change the inequality to equality.

Since the inequality is strict, we use a broken line & the inequality is less than type shade the region below the boundary line.

:- The solution set is all shaded region below the boundary line excluding points (ordered) pairs on the boundary line.



4.1.3. Systems of linear inequalities in two variables & Graphical solutions

Examples

1) Find a graphical solution to the system of linear inequalities

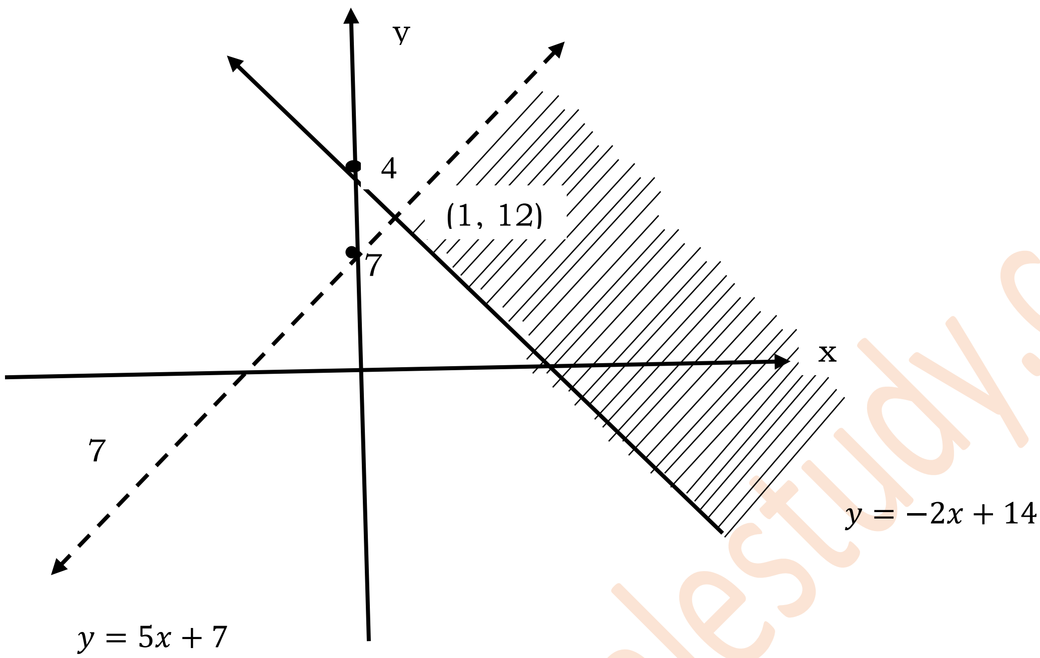
$\begin{cases} 5x - y > -7 \\ 2x + y \geq 14 \end{cases}$ $\begin{cases} y < 5x + 7 \\ y \geq -2x + 14 \end{cases}$ $\begin{cases} y = 5x + 7 \\ y = -2x + 14 \end{cases}$

$y = 5x + 7$ & $y = -2x + 14$

$y = y \Rightarrow 5x + 7 = -2x + 14$

$7x = 7$

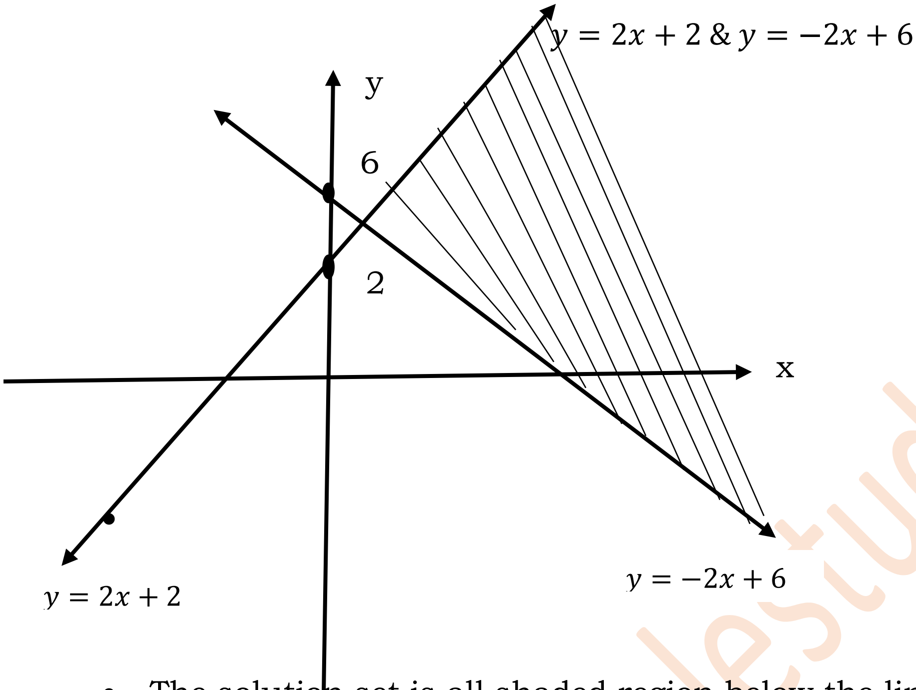
$x = 1$



The solution set is all order pairs above the boundary line $y = -2x + 14$ & on the line $y = 5x + 7$ except those points on the boundary line $y = 5x + 7$ **Ex.**

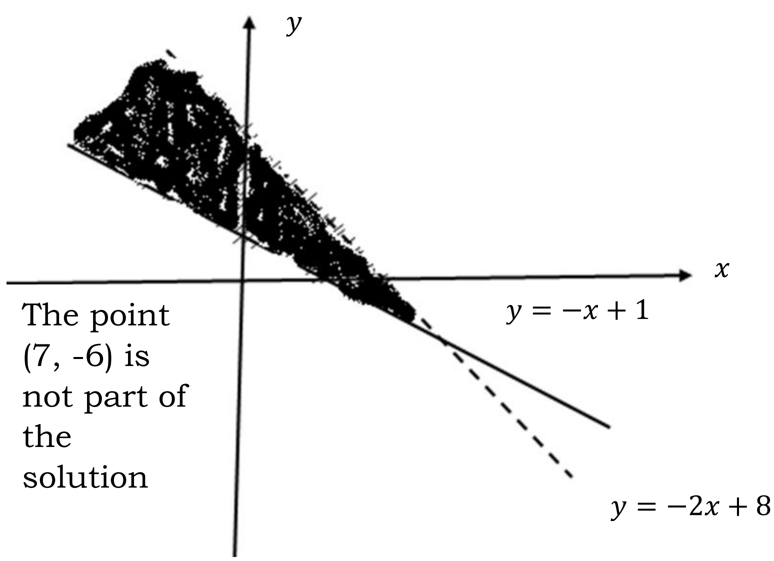
1) Solve the following systems of inequalities graphically

a) $\begin{cases} y \leq 2x+2 \\ y \geq -2x+6 \end{cases}$



- The solution set is all shaded region below the line $y = 2x + 2$ & above the line $y = -2x + 6$ & to the right of the intersection point $(1, 4)$. Including the boundaries.

b) $\begin{cases} 2x+y<8 \\ x+y>1 \end{cases} \Rightarrow \begin{cases} y<-2x+8 \\ y \geq -x+1 \end{cases}$



: - The solution set is all shaded region above the boundary line $y = -x + 1$ & on the line $y = -x + 1$, below the boundary line except those points on the line

$y = -2x + 8.$

4.1.4. Further on the systems of inequalities

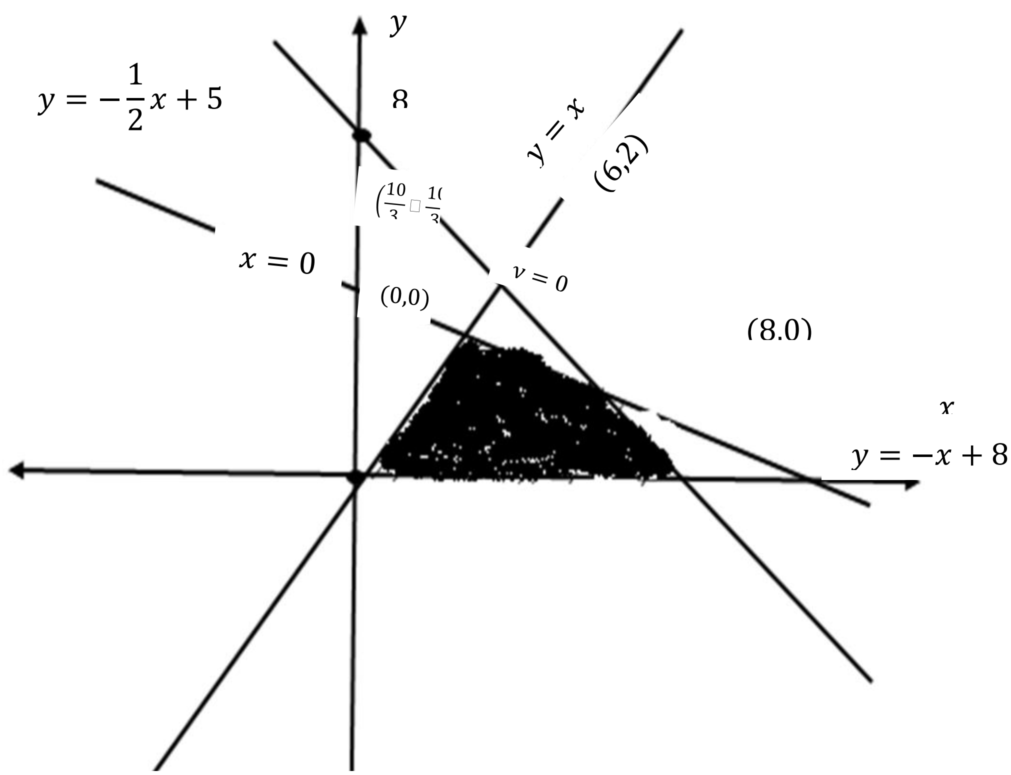
Definition: A point of intersection of two or more boundary lines of a solution region is called a vertex (or a cornered point) of the region.

Examples

2) Solve the system $\begin{cases} y \le -\frac{1}{2}x + 5 \\ y \le -x + 8 \\ y \le x \\ x \ge 0 \\ y \ge 0 \end{cases}$

To draw the graph first convert the system of inequality to a system of equations as:-

$\begin{cases} y = -\frac{1}{2}x + 5 \\ y = -x + 8 \\ y = x \\ x = 0 \\ y = 0 \end{cases}$

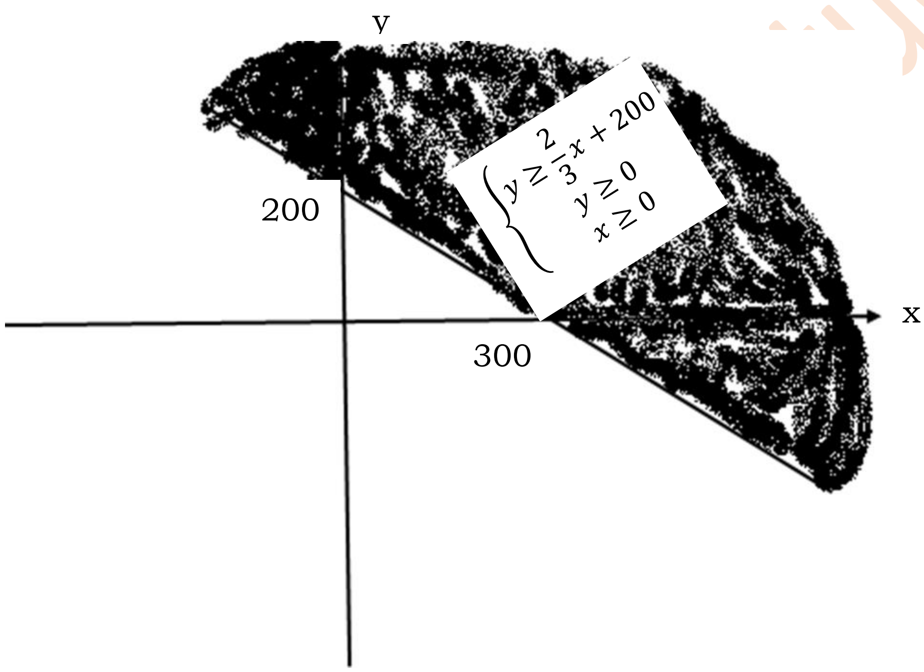


The solution set of the system is all shaded region (all ordered pairs in side) the shaded region bounded by the corner points & the boundary lines.

3) A company sells one product for birr 8 & another for birr 12. How many of each product must be sold so that revenues are at least birr 2,400? Let X represent the number of products sold at birr 8 & let y represent the no of products sold at birr 12. Write a linear inequality in terms of x & y sketch the graph of all possible solutions.

Solution

Let Revenue is R, the no of first product sold is x unit the no of second product sold is y unit. $R = 8x + 12y \geq 2400. \Rightarrow y \geq \frac{-2}{3}x + 200.$



Linear programming: a field of mathematics that deals with the problem of finding the maximum & minimum values of a given linear expression.

b) Constraints the situation in which we cannot just take any x & any y when looking for the x & they that optimize our objective function.

c) Objective function: linear function $Z = ax + by$, when a & b are constants, which has to be maximized or minimized is called a linear objective function.

d) Feasible solutions: Are points within & on the boundary of the feasible region.

e) Optimal (feasible) solution: Any point in the feasible region that gives the optimal value.

Definition 3 suppose f is a function with domain $I = \{x: a \leq x \leq b\}$

1) A number $M = f(c)$ for some c in I is called the maximum value of f on I, $M \geq f(x), for\ all\ x\ in\ I$.
2) A number $m = f(d)$for somed in I is called the minimum value of f on I, If $m \leq f(x), for\ all\ x\ in\ I$.
3) A value which is either a maximum or a minimum is called an optimum (or extremum) value of f on I.

4.2 Graphical method of solving linear programming problems

Examples

1) A furniture dealer deals in only two items tables & chairs. He has birr 50,000 to invest & has storage space at most 60 pieces. A table costs birr 2500 & a chair birr 500. Suppose that the dealer can make profit of birr 200 from the sale of one table & birr 50 from that of a chair.
   a) Find the linear objective function of the profit.
   b) What is the maximum profit can the dealer make?

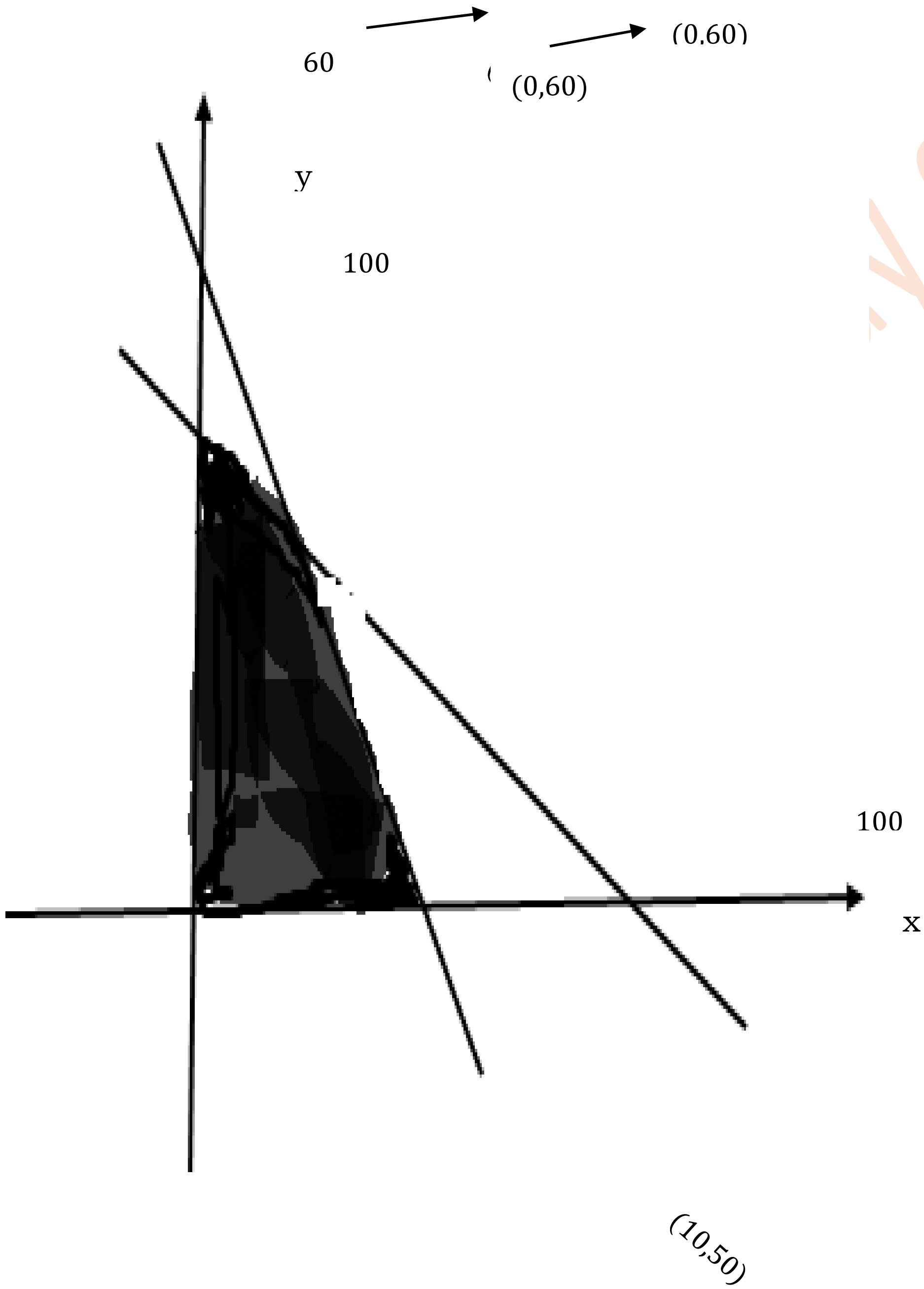
Solution

1) To solve the given problem first. We must formulate the mathematical expression as follow:
   a) Let x be the no of table produced &
      Let y be the no of chair produced & sold out.

A dealer has two limitations finance & space $x + y \leq 60$, space limitation

$2500 \times 500y \leq 50{,}000$, finance limitation. The goal of the dealer is to maximize the profit function $Z = 200x + 50y$

b) We can solve the problem graphically $y \leq -x + 60 \& \ y \leq -5x + 100$



(20,0) (20,0) x

(0,0) 20 60

$y = -5x + 100$ $y = -x + 60$

From the graph the possible optimal points are $(20, 0), (0, 60), (10, 50)$

Testing each point in the objective functions as:

- At $(2010), Z = 200x + 50y$
  $Z = 200\ (20) + 50(0)$
  $= \mathbf{400}$

$At\ (0, 60)$
$Z = 200(0) + 50 \times 60$
$= \mathbf{3000}$

- At (10, 50), Z=200(10)+ 50 x 50 = **4500**
- At $(0, 0), Z = 200(0) + 50\ (0) = \mathbf{0}$
  ∴ The dealer learns maximum profit by producing & selling 10 tables and 50 chairs.

i.e. The maximum profit is 4500, & Occurs at $x = 10$ & $y = 50$.

Steps to solve a linear programming problem

Definition

- A corner point of a feasible region is appoint in the region which is the inter section of two boundary lines.

  Definition 4.5

- A feasible region of a system of leaner inequalities is said to be bounded if it can be enclosed with in a line. Otherwise, it is called unbounded. Unbounded means that the feasible region extends indefinitely in any direction

  Theorem 4.1. Let R be the feasible region for a linear programming problem & Let $Z = ax + by$ be the objective function.

when Z has an optimal value (maximum or minimum) where the variables x and y are subject to constraints described by linear inequalities this optimal value must occur at accorder point (vertex) of the feasible region

Theorem 4.2. Let R be the feasible region for a linear programming problem, & Let $Z = ax + by$ the objective function.

If R is bounded, then the objective function Z has both a maximum & a minimum value on R & each of these occurs at a corner point (vertex) of R.

- If R is unbounded, then a maximum or a minimum value of the objective function may not exist.

Examples

1) Solve the following linear programming problems by corner point method

   a) Maximize $Z = x + y$

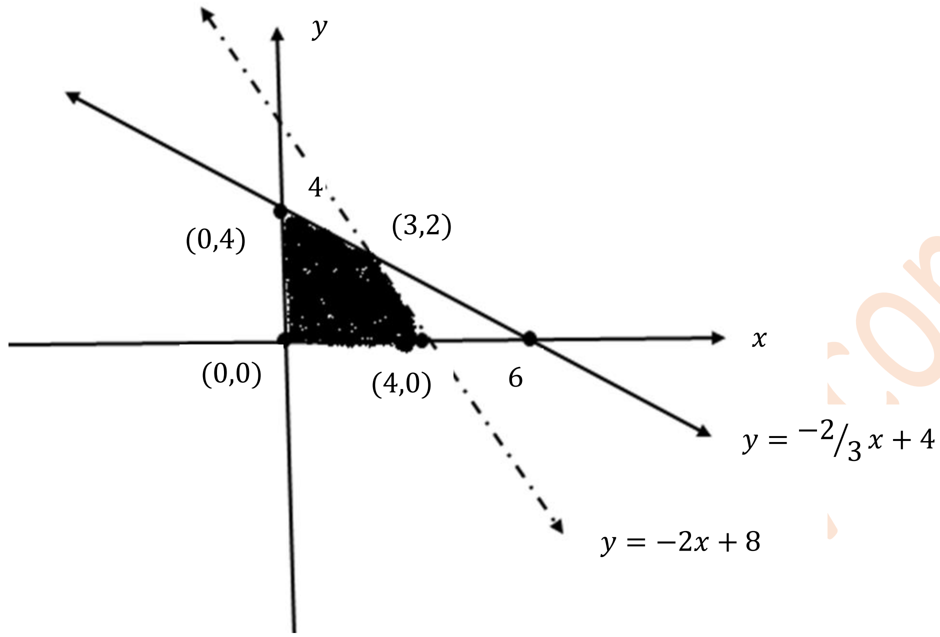
   Subject to: $\begin{cases} 2x + 3y \leq 12 \\ 2x + y < 8 \\ x \geq 0, y \geq 0 \end{cases}$

Solution

1a) $\begin{cases} y \leq \frac{-2}{3}x + 4 \\ y < -2x + 8 \\ x \geq 0, y \geq 0 \end{cases}$ $\begin{cases} y = {}^{-2}/_{3}\, x + 4 \\ y = -2 + 8 \\ y = 0, x = 0 \end{cases}$

- $y = -2x + 8, y = {}^{-2}/_{3}\, x + 4$

  $y = y \Rightarrow -2x + 8 = {}^{-2}/_{3}\, x + 4$



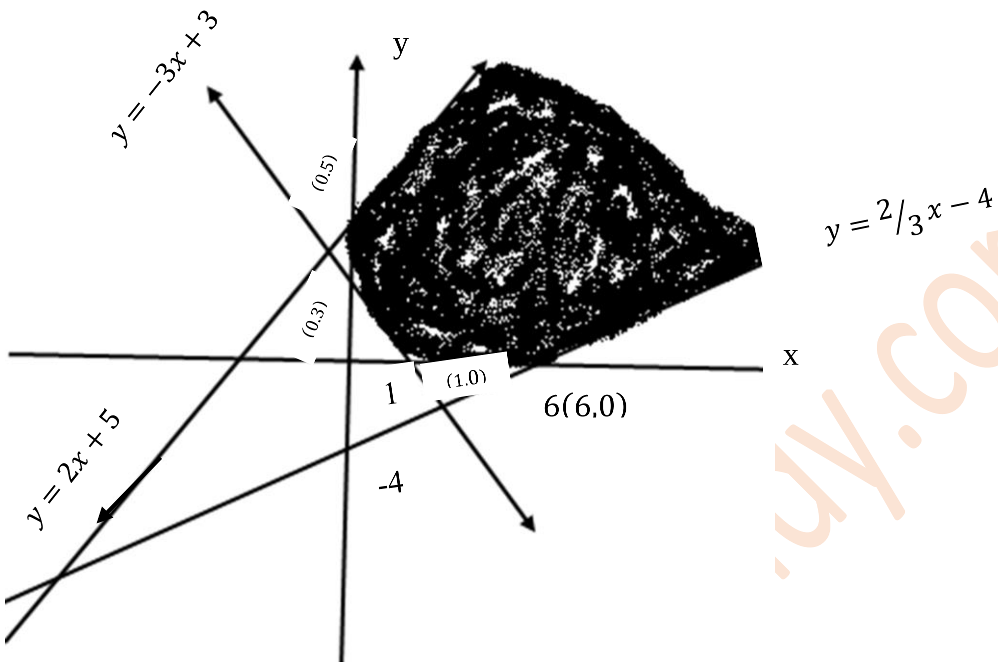
To maximize the objective function $Z = x + y$ first find the corner points

- The corner points are $(0,0), (3,2), (4,0), (0,4)$ but $(3,2)$ & $(4,0)$ are not part of the shaded region
- The only corner points are $(0,0)$ & $(0,4)$
  $Z = x + y, when\ x = 0, y = 0$
  $Z = 0\ to = 0\ \&\ Z = x + y = x + y, when\ x = 0, y = 4$
  $= 0 + 4$
  $= \mathbf{4}$
  $\therefore$ The maximum value is $\underline{\mathbf{4}}$ & occurs at $x = 0$ & $y = 4$

c) Minimize: $Z = -40x + 20y$

subject to

$$\begin{cases} 2x - y \geq -5 \\ 3x + y \geq 3 \\ 2x - 3y \leq 12 \\ x \geq 0, y \geq 0 \end{cases} \qquad \begin{cases} y \leq 2x + 5 \\ y \geq -3x + 3 \\ y \geq \frac{2}{3}x - 4 \\ x \geq 0, y \geq 0 \end{cases} \qquad \begin{cases} y = 2x + 5 \\ y = -3x + 3 \\ y = {}^{2}/_{3}x - 4 \\ x = 0, y = 0 \end{cases}$$



- The problem is maximization & bounded below. The possible solutions are the corner points (6, 0), (1, 0), (0, 3),(0,5). Testing each point in the objective.

  Functions we obtain:

- At (6, 0), Z= -40(6) +20(0)=-240
- At (1,0), Z= -40(1)+20(0)= -40
- At (0, 3), Z= -40(0)+20(3)=60
- At (0, 5), Z-40(0) + 20(5)=100

The minimum result is -240 & occurs at $(6, 0)$

i.e. $x = 6$ & $y = 0$

Further on a linear programming problem

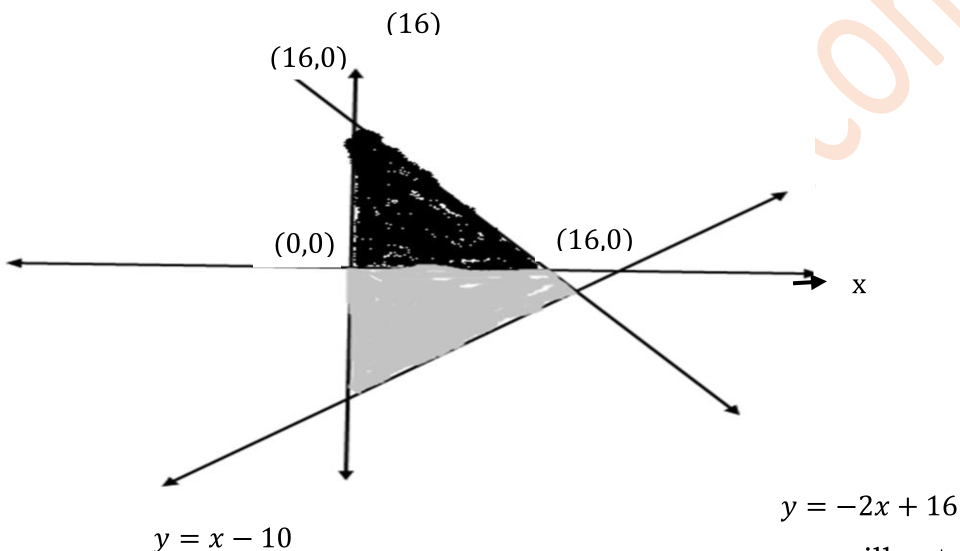
Examples

Solve the following linear. Programming problems graphically

b) Maximize or minimize Z= 5x+6y.

$$\begin{cases} 2x + y \le 16 \\ x - y \le 10 \\ x \ge 0 \\ y \ge 0 \end{cases} \qquad \begin{cases} y \le -2x + 16 \\ y \ge x - 10 \\ x \ge 0 \\ y \ge 0 \end{cases}$$

$$y = -2x + 16, y = x - 10$$
$$y = y \Rightarrow -2x + 16 = x - 10$$
$$-3x = -26$$



- Come
- Testi ... will get.
- At (0,
- At $(16,0), Z = 5(16) + 6(0) = 80$

∴ The maximum value is 96 & occurs at $x = 0$ & $y = 16$ & the minimum value is 0 & occurs at (0,)

4.3. Applications

4.3.1 Real life problems

Some of the known linear programming problems are listed below:

A) Diet problems

B) Allocation models

C) Manufacturing problems

D) Transportation problems

Examples

1) World food program (WFP) uses at least 800 kg of special feed daily in conflict areas. A special feed is a mixture of corn & soybean meal with the composition of a kilogram of corn that contains 10% protein & 2% fiber costs birr 30 per kg. A kilogram of soybean mal. Constitutes 60% protein & 6% fiber & costs birr 90 per kg. The dietary requirements for the special feed are at least 30% protein & at most 4% fiber. Which wishes the daily minimum cost feed mix.

Solution We define the two decision variables:

x- is the no of kilogram of corn in the daily mix.
y – is the no of kilogram of soybean meal in the daily mix.

	Content protein (%)	Fiber (%)	Cost (Birr/kg)	Mixture in a day (g)
corn	10	2	30	x
Soybean	60	6	90	y

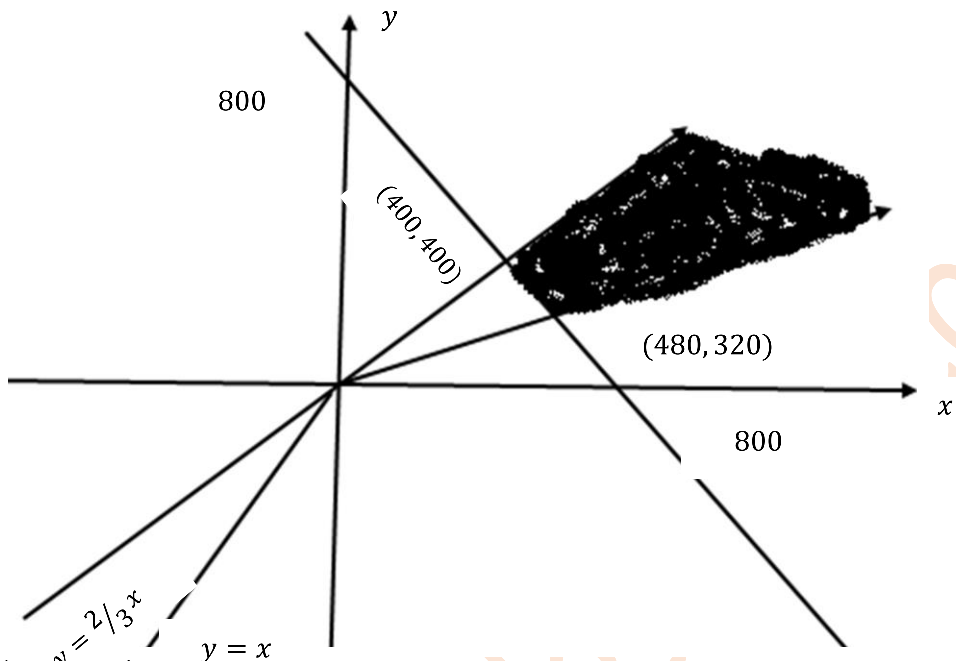
- In any linear programming problem (LPP), the decision maker wants to maximize (usually revenue or profit)or minimize (cost) which are some function of the decision variables.

The objective function to minimize the total daily cost is:

Minimize: $Z = 30x + 90y$

Subject to: $\begin{cases} x + y \geq 800 \\ 0.2x - 0.3y \leq 0 \\ 0.02x - 0.02y \geq 0 \\ x \geq 0, y \geq 0 \end{cases}$

$$\begin{cases} y \geq -x + 800 \\ y \geq {}^{2}/_{3}\, x \\ y \leq x \\ x \geq 0, y \geq 0 \end{cases}$$



at the points

- At $(480, 320), Z = 30\,x\,480 + 90\,x\,320 = \mathbf{43,200}$
- At $(400, 400), Z = 30x\,400 + 90x\,400 = \mathbf{48,000}$

$\therefore$ The minimum value is 43, 200 (the associated minimum cost of the feed mix) & occur at $x = 400$ & $y = 400$

4.3.2. Solving linear programming problems using spreadsheet

Solving < p problem in excel

The spread sheet solver approach makes solving optimization problems a tairly simple task.

Step 1: put the problem in excel put the objective function coefficients in to a row with at least 2 blank rows above it with the constraint coefficients below. label the rows down the left hand side in column. Leave one blank column after the last variable & label it sum.

Step 2: Now, label the row just above tableau variable values to manipulate. Enter 0 values above the variables.

Step 3: Now, construct Excel cell entries to addup each LP model equations. Place these in the column named sum.

Step 4: activate the solver. To do this, go to Toolsin the toolbar & click on solver.

Step 5: Define where the objective function is by defining the variable dialogue box called set Target cell as the cell number where you added the objective function up.

Step 6: Choose whether to maximize or minimize using the buttons just below the set target cell box.

Step 7: Identify the decision variables by entering the range in which they fall in the by changing cells box.

Step 8: Enter consideration of the constraint equations in to the model by clicking the Add button to the right of the subject to the constraints box.

Step 9: Review the problem & when all looks right, solve by clicking the solve button.

Step 10: Excel will solve <p problem based on the formulas you inputted solution.

Step 11: choose desired output reports. High light both the answer report & sensitivity report. Click on keep solver solution & ok then the reports will be generated.

- Main feature of spreadsheet or excel is listed below.

1) **Home:** comprises options like font size, font size, font styles font color background color, alignment, formatting options & styles, insertion & deletion of cells & editing options.
2) **Insert:** comprises options like table format & style inserting images & figures, adding graphs, charts & spark lines, header & tooter option, equation & symbols.
3) **Page layout** Themes, orientation & page set up options are available under the page layout option.
4) **Formulas:** since tables with a large amount of data can be las to your table & get quicker solutions.
5) **Data** Adding external data (from the web), filtering options & data tools are available under this category.
6) **Review:** proofreading can be done for an excel sheet (like spell check) in the review category & a reader can odd comment in this part.
7) **View:** Different views in which we want the spreadsheet to be displayed can be edited hare. Options to zoom in & out & pane arrangement are available under this category.

Review exercise on unit 4

1) Determine whether each ordered pairs is a solution to the inequality $y > x - 1$

a) (0,1) b) (-4, -1) C. (-2, 0) d) (4, 2), e) (3, 0)

Solution

1a) when x=0, y=1 = 1_____0-1 = 1>-1 which is on the graph of y>x-1

b) (-4, -1) = -1 ______-4-1 = -1>-5 which on the graph of y>x-1

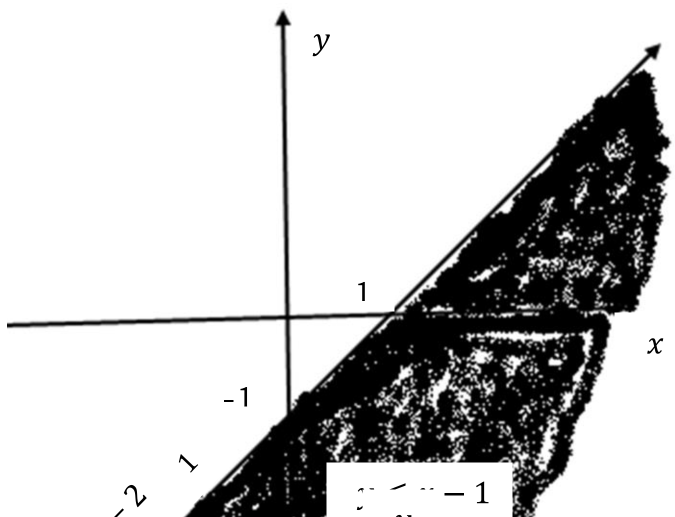
c) $(-2,0) \Rightarrow 0 - -2 - 1 \Rightarrow 0 > -3$ which is on the graph of $y > x - 1$

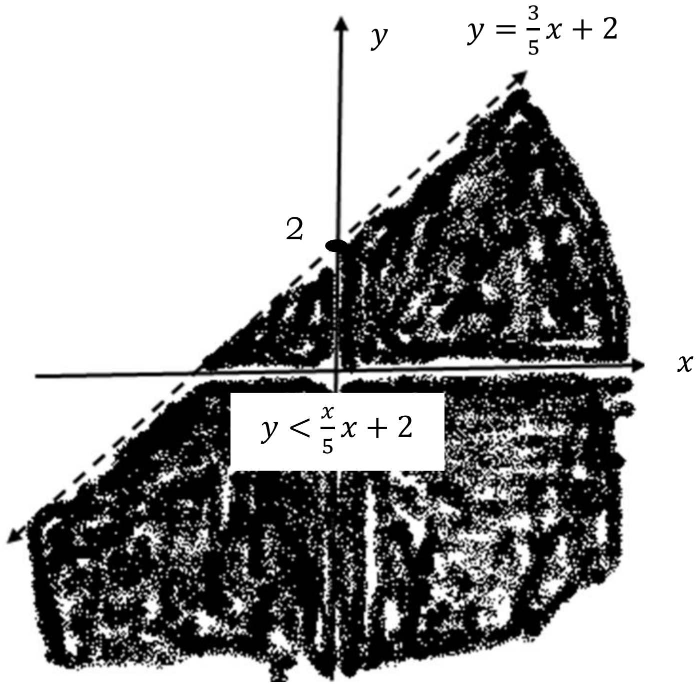
d) $(4, 2) \Rightarrow 2 - 4 - 1 \Rightarrow 2 > 3$ which is on the graph of y>x-1

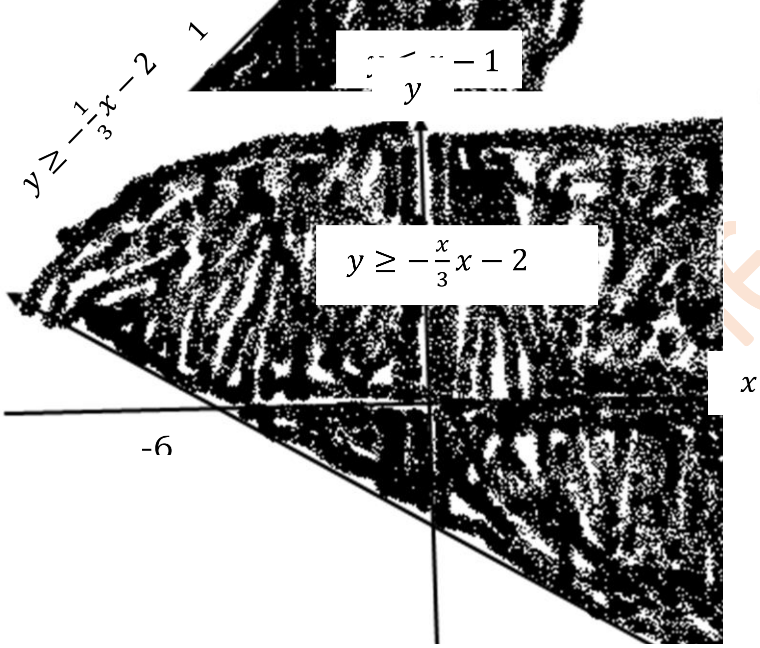
e) $(3, 0) \Rightarrow 0$_____$3 - 1 \Rightarrow 0$ >2 which is false (3,0) is not a solution to the graph of y>x-1.

2) Graph each of the following linear inequalities.

a) $y \le x - 1$ b) $y < \frac{3}{5}x + 2$

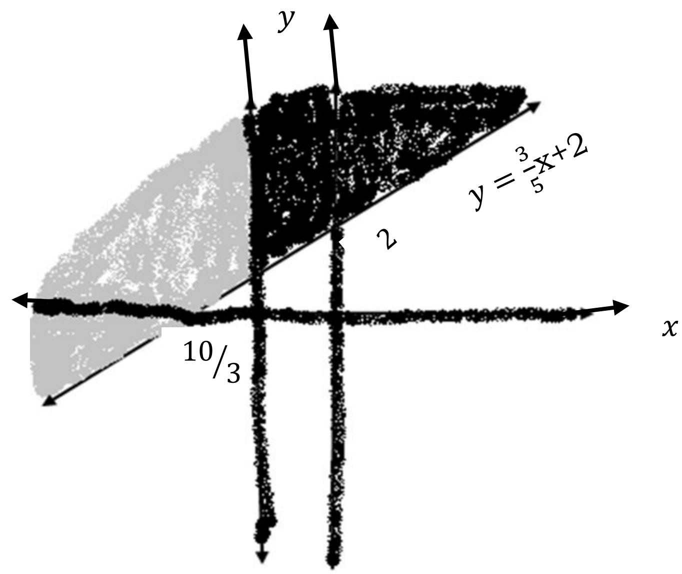






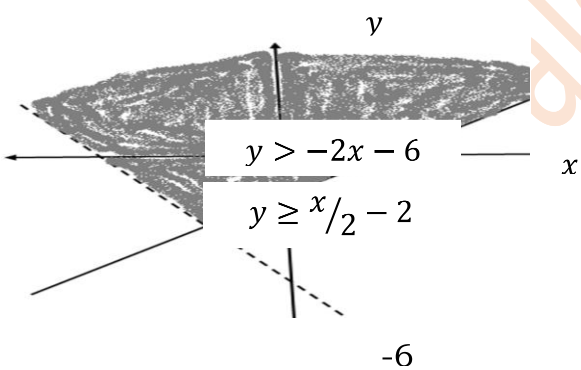
3) Solve each system by graphing.

i) $\begin{Bmatrix} -3x + 5y > 10 \\ x > 1 \end{Bmatrix}$ $\quad \left\{ y > \frac{3}{5}x + 2 \right.$



$x = -1$

iv) $\{2x + y > -6\}$
$-x + 2y \geq -4$ $\quad \begin{cases} y > -2x - 6 \\ y \geq \frac{x}{2} - 2 \end{cases}$ $\quad y = -2x$ -
$y = \frac{x}{2} - 2$



$y = \frac{x}{2} - 2$

6) Sultan works two part time jobs. One is at agas station that pays birr 11 an hour & the other is computer technician for birr 16.5 an hour. Between the two job $y = -2x - 6$:o learn at least birr 3300.00 in tour weeks. How many hours does sultan need to work each job to earn at least birr 3300. Assume sultan has 15 working hours per day & works 5 days per week. Write an inequality which models the situation. Graph the inequality & list three solutions to the inequality.

Solution

Let x be the number of hours sultan engaged in agas station job.

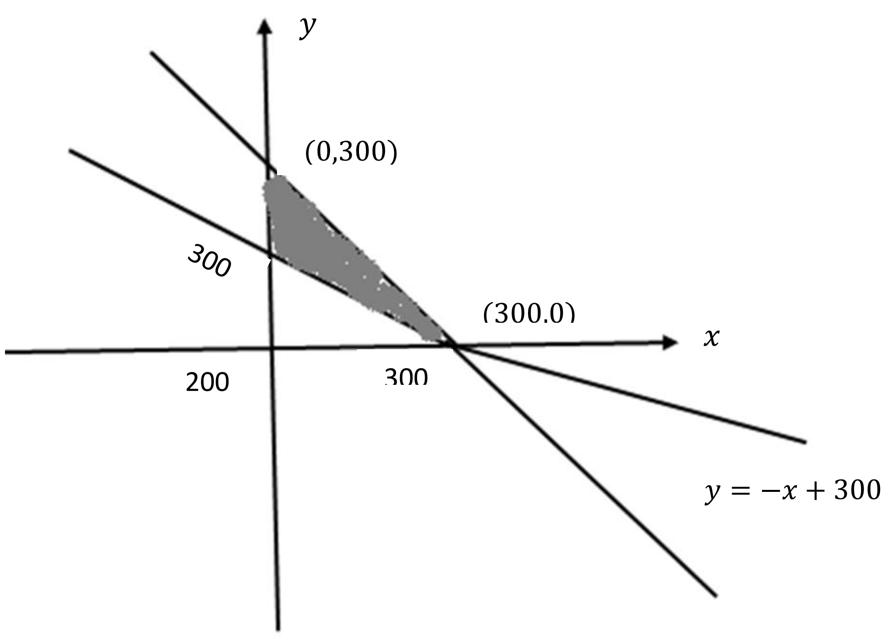
Y be the no of hours sultan works as a computer technician in four weeks.

= $11x + 16.5y \geq 3300$

$$11x + 16.5y \geq 3300$$
$$x + y \leq 300$$
$$x \geq 0, y \geq 0$$

$x + y \leq 300$

- The corner points are (300,0), (0, 200), (0, 300)
- Any points in the region meet the requirements & sultan can achieve his goal.



7) Find the maximum and minimum values of the objective functions subject to the given constraints.

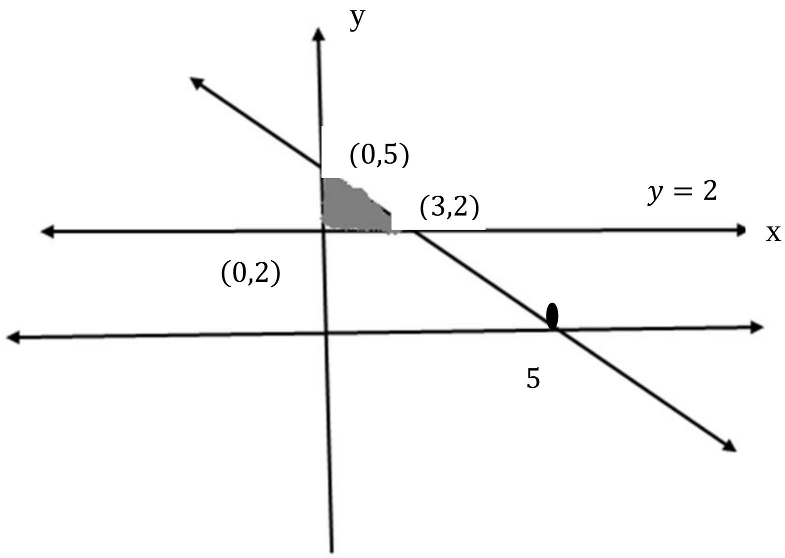
i) Objective functions $Z = 6x + 4y$

$$\text{Subject to: } \begin{cases} x + y \leq 5 \\ y \geq 2 \\ x \geq 0, y \geq 0 \end{cases}$$

- Convert the inequality to equality form

$Z = 6x + 4y$

Subject to: $\begin{cases} y \le -x + 5 \\ y \ge 2 \\ x \ge 0, y \ge 0 \end{cases} = \begin{cases} y = -x + 5 \\ y = 2 \\ x = 0, y = 0 \end{cases}$



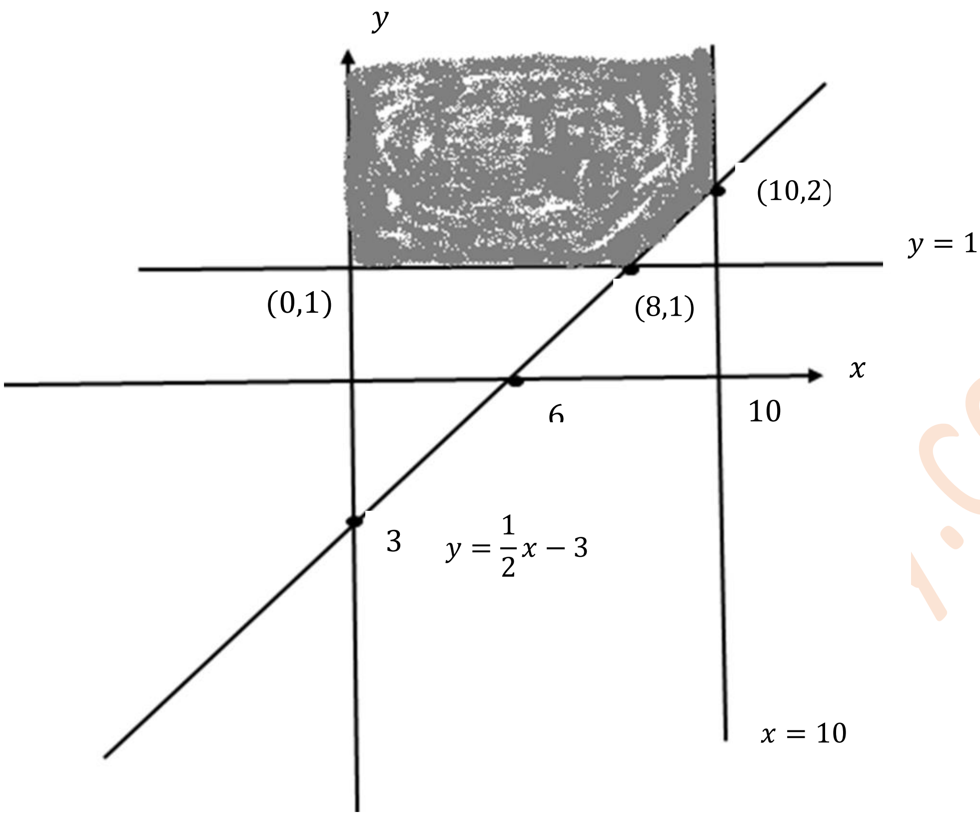
Testing each points we obtain

- $At\ (0,2), Z = 6(0) + 4(2) = 8$ $At\ (0,5), Z = 6(0) + 4(5) = 20$
- $At\ (3,2), Z = 6(3) + 4(2) = 26$

- From the result $Z = 26$ is the maximum value and $Z = 8$ is the minimum value.

ii) Objective functions $Z = 3x + 5y$ subject to $\begin{cases} x - 2y \le 6 \\ x \le 10, y \ge 1 \\ x \ge 0, y \ge 0 \end{cases}$

$$\begin{cases} y \ge \frac{1}{2}x - 3 \\ x \le 10 \\ y \ge 1 \\ x \ge 0, y \ge 0 \end{cases} \qquad \begin{cases} y = \frac{1}{2}x - 3 \\ x = 10 \\ y = 1 \\ x = 0 \\ y = 0 \end{cases}$$



- The feasible region is unbounded, so they have only minimum value. Testing the corner points (0,1), (8,1) & (10,2), weget the following result.

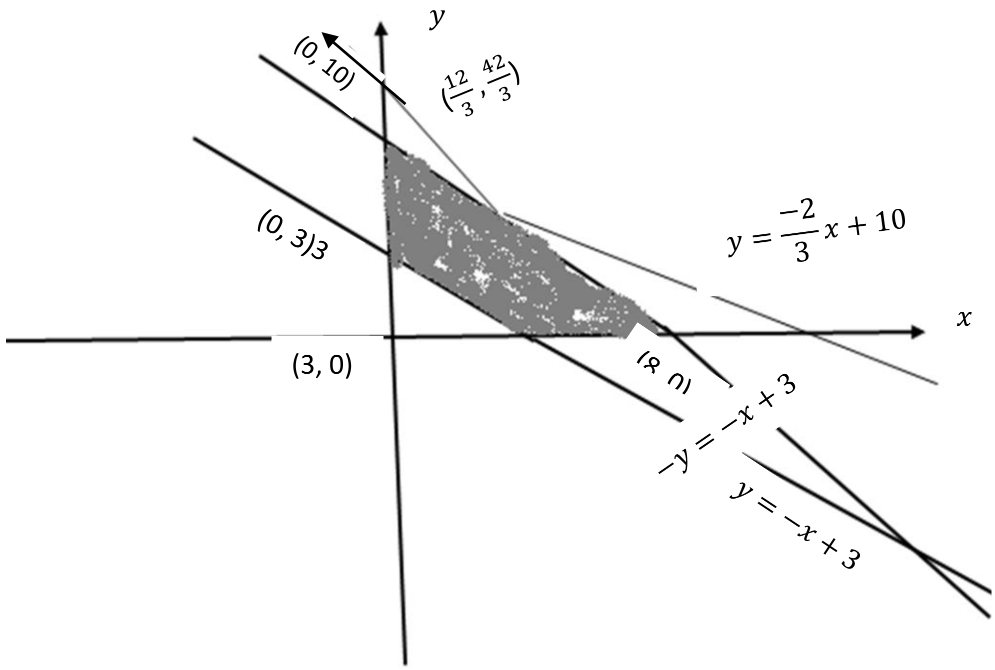
- At $(0,1), Z = 3(0) + 5(1) = 5 - At\ (10,2), Z = 3(10) + 5(2) = 40$
- At $(8,1), Z = 3(8) + 5(1) = 29$

Form the result the value of Z is has a minimum value 5 & 0ccorat (0, 1).

iii) Objective function $Z = 6x + 4y$ subject to: $\begin{cases} 2x + 3y \leq 30 \\ 3x + 2y \leq 24 \\ x + y \geq 3 \\ x \geq 0, y \geq 0 \end{cases}$

$$\begin{cases} y \leq \dfrac{-2}{3}x + 10 \\ y \leq \dfrac{-3}{2}x + 12 = \\ y \geq -x + 3 \\ x \geq 0, y \geq 0 \end{cases} \qquad \begin{cases} y = {}^{-2}/_{3}\, x + 10 \\ y = {}^{-3}/_{2}\, x + 12 \\ y = -x + 3 \\ x = 0 \\ y = 0 \end{cases}$$

The corner points are $(0, 10), (0, 3) \left(\frac{12}{3}, \frac{42}{3}\right), (8,0)$



Testing eac

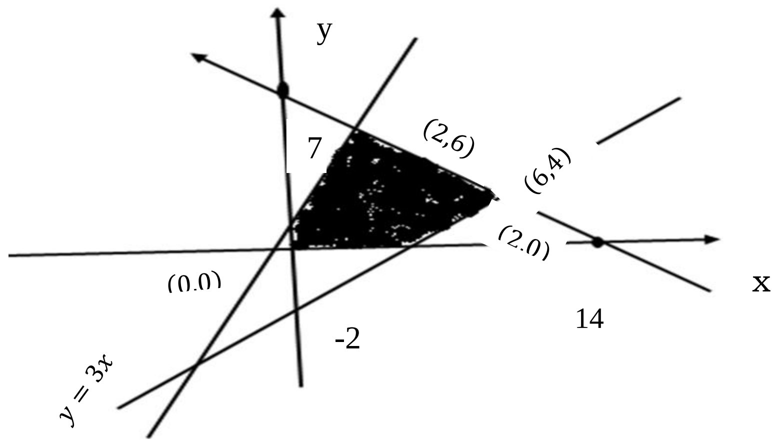
- At $(8, 0), Z = 6(8) + 4(0) = 48$
- At $\left(\frac{12}{3}, \frac{42}{3}\right), Z = 6\left(\frac{12}{3}\right) + 4\frac{(42)}{3}$

  $= 24 + \frac{168}{3} = \frac{72 + 168}{3} = \frac{240}{3} = \mathbf{80}$
- $At\ (0, 10), Z = 6(0) + 4(10) = 40$
- $At\ (0,3), Z = 6(0) + 4(3) = 12$

∴ The minimum value Z is 12 & occurs (0, 3) & the maximum value Z is 80 & occurs at $\left(\frac{12}{3}, \frac{42}{3}\right)$.

8) Maximize or Minimize, $Z = 3x + 4y$

$\begin{cases} x + 2y \leq 14 \\ 3x - y \geq 0 \\ x - y \leq 2 \\ x \geq 0, y \geq 0 \end{cases}$ $\begin{cases} y \leq \frac{-1}{2}x + 7 \\ y \leq 3x \\ y \geq x - 2 \\ x \geq 0, y \geq 0 \end{cases}$ $\begin{cases} y = \frac{-1}{2}x + 7 \\ y = 3x \\ y = x - 2 \\ x = 0, y = 0 \end{cases}$



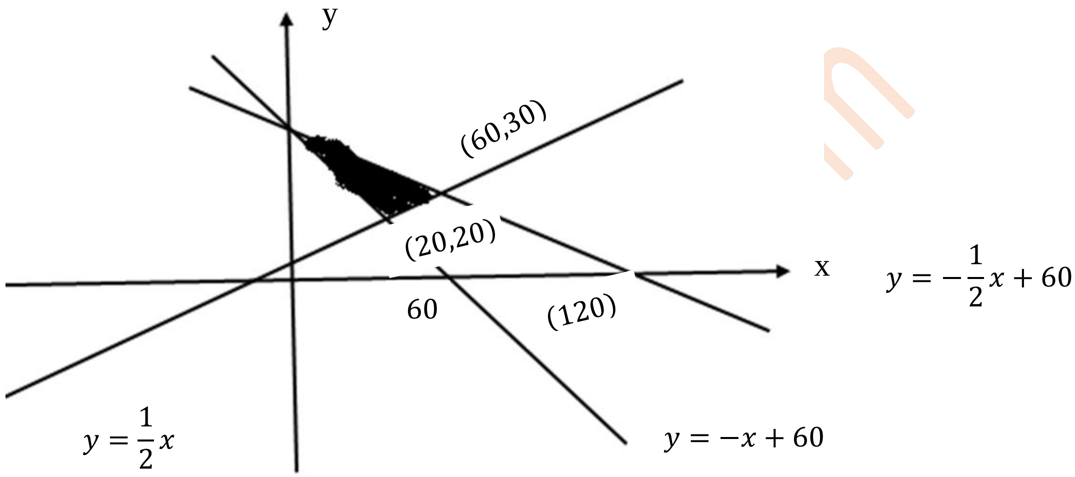
The corner points are: (2, 0), (6,4), (2, 6), (0, 0) Testing the point.

- At $(0,0), Z = 3(0) + 4(0) = 0$
- At $(2,0), Z = 3(2) + 4(0) = 6$
- At $(6,4), Z = 3(6) + 4(4) = 34$
- At $(2,6), Z = (3(2) + 4(6) = 30$

∴ The minimum value of Z is 0 & occur at (0, 0) & The maximum value of Z is 30 & occur at (2, 6)

9) Minimize & Maximize $Z = 5x + 10y$

Subject to: $\begin{cases} x + 2y \leq 120 \\ x + y \geq 60 \\ x - 2y \leq 0 \\ x \geq 0, y \geq 0 \end{cases}$ $\begin{cases} y \leq \frac{-2}{2}x + 60 \\ y \geq -x + 60 \\ y \geq \frac{1}{2}x \\ x \geq 0, y \geq 0 \end{cases}$



- The corner points are: $(20, 20), (60, 30)$ & $(0, 60)$
- Testing the corner point in $Z = 5x + 100$

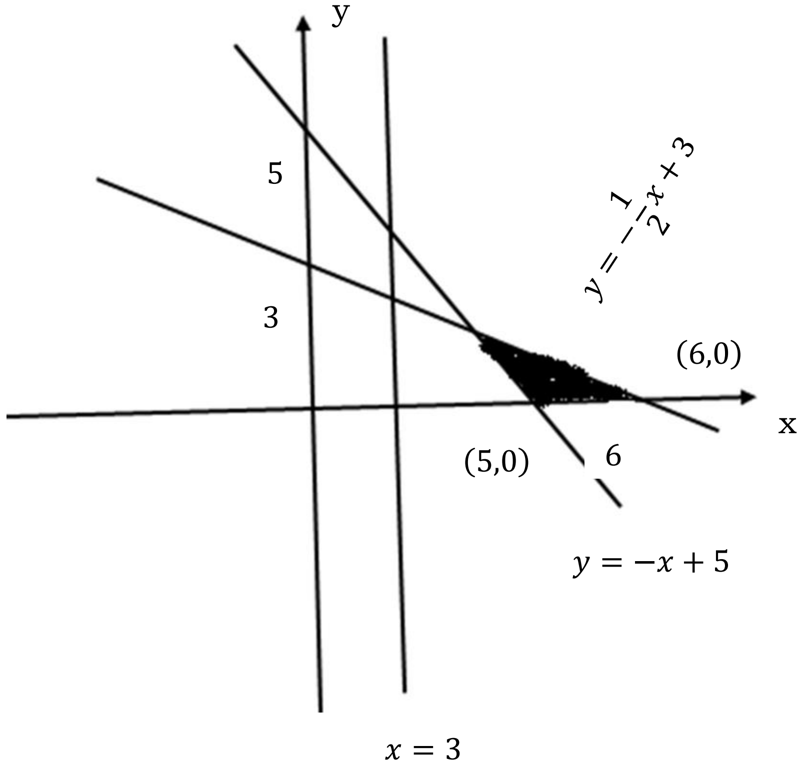
- At $(0, 60), Z = 5(0) + 10(60) = 600$
- At $(20, 20), \mathrm{Z} = 5(20) + 20(10) = 100 + 200 = 300$
- At $(60, 30), Z = 5(60) + 10(30) = 300 + 300 = 600$

$\therefore$ The minimum value of Z is 300 and occur at $(20, 20)$ and the maximum value of Z is 600 & occur at $(60, 30)$ & $(0, 60)$

11) Maximize, $Z = x + 2y$

Subject to the constraints: $\begin{cases} x \geq 3 \\ x + y \geq 5 \\ x + 2y \leq 6 \\ y \geq 0 \end{cases}$

$\begin{cases} x \geq 3 \\ y \geq -x + 5 \\ y \leq -\frac{1}{2}x + 3 \\ y \geq 0 \end{cases}$ $\begin{cases} x = 3 \\ y = -x + 5 \\ y = \frac{-1}{2}x + 3 \\ y = 0 \end{cases}$



(5, 0), (6, 0) & (4, 1)

Testing the corner point in $z = x + 2y$

- At $(5,0), Z = 5 + 2(0) = 5$
- At $(6, 0), Z = 6 + 2(0) = 6$
- At $(4,1), Z = 4 + 2(1) = 6$

$\therefore$ The maximum value of Z is 6 & occur at (6, 0) & (4, 1)

10) A health fitness center wants to produce a guide to healthy living. The center intends to produce the guide in two formats, a short video & a printed book. The center needs to decide how many of each format to produce for sale. Estimates show that no more than 10,000 copies of both items together will be sold. At least 4, 000 copies of the video.

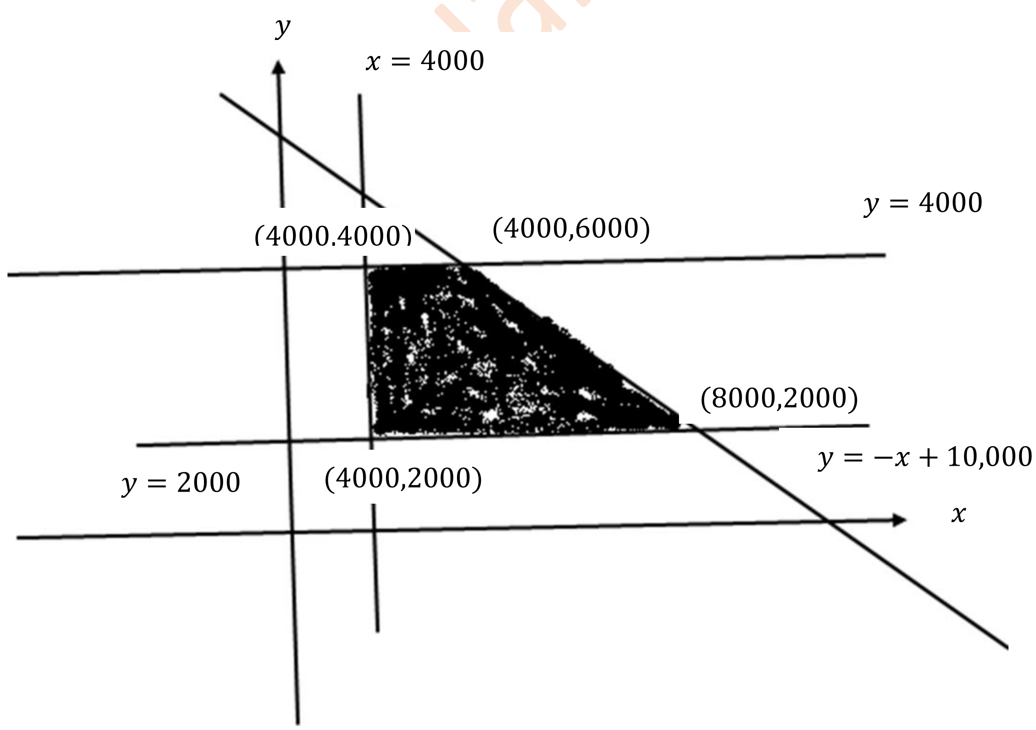
And at least 2,000 copies of the book could be sold, athough sales of the book are not expected to exceed 4000 copies.

Let X be the number of videos sold & y the number of printed books sold.

a) Write down the constraint inequalities that can be deduced from the given information.
b) Represent these inequalities graphically & indicate the feasible region clearly.
c) The health fitness center is seeking maximize the income, I, earned from the sales of the two products.
Each video will be sold for birr 50 & each book for birr 30 write down the objective function for the income.
d) What maximum income will be generated by the two guides?
e) Solve the problem by using spread sheets solver.

Solution

- Estimates show that no more than 10,000 copies of both items together will be sold:
$x + y \leq 10{,}000$
- At least 4000 copies of the video could be sold: $x \geq 4000$



Vertices	$I = 50x + 30y$	
(4000, 2000)	260,000	
(4000,4000)	320,000	
(6000, 4000)	420,000	
(8000, 2000)	460,000	

- The maximum profit is birr 460,000 & can be made if 8000 videos & 2000 books were sold.

11) To meet the requirements of a specialized dier ameal is prepared by mixing two types of ceroal, wheat & sorghum (Mashila). The mixture must contain x packets of wheat cereal & y packets of sorghum cereal. The meal requires at least 15g of protein & at least 72g of carbohydrates. Each packet of wheat cereal contains 4g of protein & 16g of carbohydrates. Each packet of sorghum cereal contains 3g of protein & 24g of carbohydrates. There are at most 5 packets of cereal available.

i) Write down the constraint inequalities
ii) Identify the feasible region.
iii) If wheat cereal costs birr 40 per packet & sorghum cereal also costs birr 40 per packet, use the graph to determine how many packets of each cereal must be used so that the total cost for the mixture is a minimum.
iv) Use the graph to determine how many packets of each cereal must be used so that the total cost for the mixture is a maximum (give all possibilities)

Solution

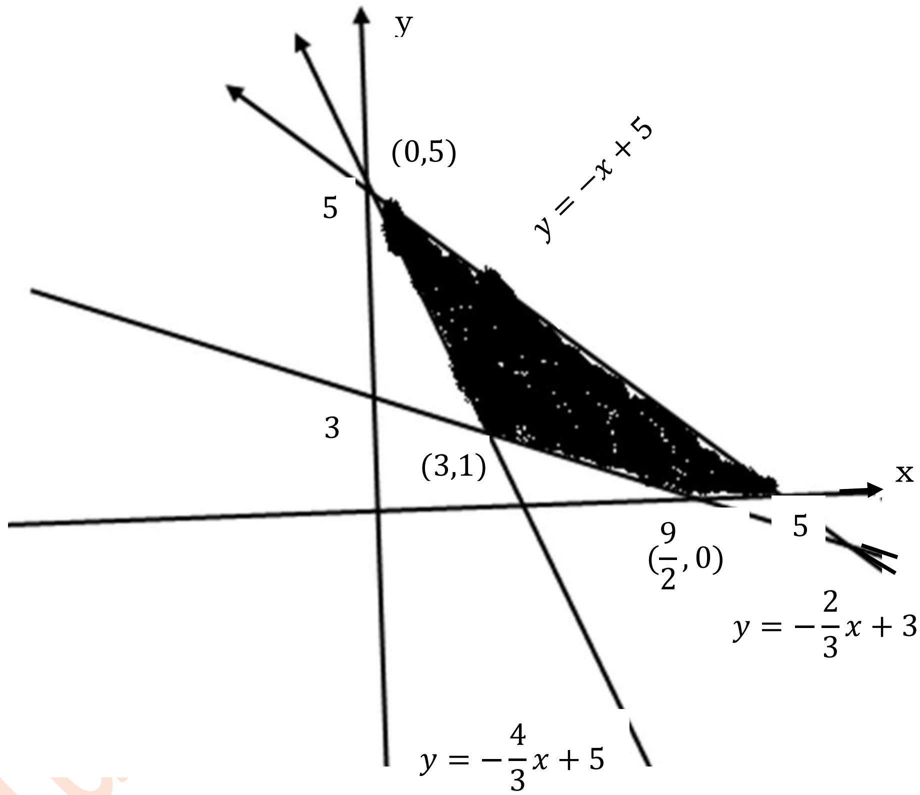
Let x be the number of packets of wheat, y be the number of packets of sorghum.

i) Adding the protein $4x + 3y \geq 15$

Carbohydrate: $16x + 24y \geq 72$

- Collectively 5 packets of food available, it is expressed as: $x + y \leq 5$

ii) The feasible region is



$$\begin{cases} 4x + 3y \geq 15 \\ 16x + 24y \geq 72 \\ x + y \leq 5 \\ x \geq, y \geq 0 \end{cases}$$

$$\begin{cases} y \geq -\frac{4}{3}x + 5 \\ y \geq -\frac{2}{3}x + 3 \\ y \leq -x + 5 \\ x \geq 0, y \geq 0 \end{cases}$$

$$\begin{cases} y = -\frac{4}{3}x + 5 \\ y = -\frac{2}{3}x + 3 \\ y = -x + 5 \\ x = 0 \\ y = 0 \end{cases}$$

iii) First you can write the objective function as:

Minimize or maximize $Z = 40x + 40y$

The vertices of the feasible region are as follows:

	Vertices	The cost at each vertex, Z=40x+40y
A	(3,1)	160
B	$\left({}^{9}/_{2},0\right)$	180
C	(5, 0)	200
D	(0,5)	200

- To minimize the objective function select point A. The minimum possible cost birr 160 can be made if 3 packets of what & 1 packets of sorghum are sold.

iv) To maximize the objective function select point C or point D. The maximum possible cost birr 200 can be made if either 5 packets of what & 0 packets sorghum are sold, or packets of wheat & 5 packets of sorghum are sold.

Other than the person responsible for paying them.

- According to the Ethiopian tax law direct taxes include all income taxes such as employment income tax, business, income tax and land use free, mining income tax and other income taxes.
- Generally direct taxes are income based taxes.

  Income tax: Income tax is a very important direct tax. It is an important and most significant source of revenue of the government.
- Taxable income: means the amount of income subject to tax after deduction of all expenses and other deductible items allowed under this proclamation 286/2002 and regulation 78/2000.

  Sources of income

  Income taxable under this proclamation shall include, but not limited to:

  - Income from employment
  - Income from business activities

- Income derived by an entertainer, musical, or sports person from his personal activities.
- Dividends distributed by a resident company e.t.c

Unit 5

Mathematical Applications in Business

5.1. Basic Mathematical concepts in Business

1) Ratio: Is a comparison of two numbers by division with the same unit.

The ratio of $a\ to\ b$ can be expressed as: $a{:}\,b\ or\ a \div b$.

- A comparison of three or more quantities in adefinite order is called continues ratio.

2) Rate: Arate like aratio is a comparison of two quantities, but the quantities may have difference units of measure.

- Arate that has adeno minatory of 1 is called aunit rate
- The rate of change of agiven quantity given by the relation:

  Rate of change = $\frac{amount\ of\ chnege}{original\ amount} = \frac{final\ amount - original\ amount}{original\ amount}$

<u>Examples Q1</u>

1) A sum of money was divided between Debora, kalid & Mesfin in the ratio of 5:3:1, respectively. Debora has received birr 3504. How much money was there to start with?

Solution:- The given part to part ratio 5:3:1 can be converted to part to whole ratio

$i.e. 5 + 3 + 1 = 9$

Debora's part to whole ratio is 5:9

Get x be the total money

$5{:}\,9 = 3504{:}\,x\ \frac{5}{9} = \frac{3504}{x} = \frac{5}{9}x = 3504$

$x = 3504\ \frac{9}{5}$

$x = 6307.20$

At the beginning was a sum of birr **6307.20**

2) **Ex. 5.3(Q_2)** A carpenter's daily production of chairs increased from 20 units to 40 units. At the same time his daily income (or revenve) increased from 1600 Birr to 2400 Birr. As are sult of this increase, what is the rate of change of income per unit?

Solution

Average rate of change of income $\frac{change\ in\ income}{change\ in\ unit\ produced}$

$$= \frac{2400-1600}{40-70} = \frac{800}{20} = 40$$

∴ The carpenter's income increase by Birr 40 for each chair produced.

3) Proportion

- A proportion is an equation that states that two ratios are equal. In the proportion, $a: b = c: d, with\ b \neq 0\ \&\ d \neq 0.$
- The four numbers are cased the terms of the proportions.
- The first term & the last terms b & C are called the extremes
- The second & the third terms b &C are called the means
- A situation in which one variable quantity depends on two or more other variable quantity is called a compound proportion
- Two quantities x and y are said to be an inverse proportion if an increase in x causes a proportional decrease in y & vice versa.
- If x and y are inverse proportion (vary inversely), $xy = k$, where k is a positive number.
- If $x\ \&\ y$ are in direct proportion (vary directly), $y = kx$, where k is a positive number.
- **i.e** $\boldsymbol{y\alpha x} \Leftrightarrow \boldsymbol{y = kx}$ & $y\alpha\frac{1}{x} \Leftrightarrow xy = k$

Examples

1) both x and y vary directly with each other. Let x is 10, & y is 15, then, write four corrects ponding pairs values of x & y.

solution: $x\alpha y \rightarrow \frac{x}{y} = k$ $is\ constant. Hene, k = \frac{10}{15} = {}^{2}/_{3}$

when $x = 12, y = \frac{3}{2}x \rightarrow y = \frac{3}{2}.12 = 18$

X	12	14	16	18
Y	18	21	24	27

4) **Percentage** A percentage is the numerator of a fraction whose denominator is 100.

Percentage = base x rate

Markup: The difference between a product selling price & its cost is called markup.

Markup= selling price – purchase price

- mark up with respect to selling price

markup percent = $\frac{Markup}{cost} \times 100\%$

discount is a reduction on the original selling price

Examples (Ex. 5.5)

1) Find the indicated percentage for each the following

$i. 2.5\% \rightarrow 400 = \frac{2.5}{100} \times 400 = 2.5 \times 4 = 10.0 = 10.$

$ii. 12.5\% \rightarrow 175 = \frac{12.5}{100} \times 175 = 0.125 \times 175 = 21.875$

$iii. 35\%\ of \rightarrow 1500 = \frac{35}{100} \times 1500 = 35 \times 15 = 525$

2) Find each number or base.

a) 8% of what number is 12?

Solution let x be the number 8% ×= 12

$$\frac{8x}{100} = 12 \rightarrow 8x = 1200$$

$$x = \frac{1200}{8} = 150$$

5.2. Time value of money

a) Simple interest

Definition Interest is what a borrower pays a lender for the temporary use of the lender's money. In other words, it is the "rent" that a borrower pays a lender to use the lender's money.

A Debtor: - Is someone who owes some one esse's money

A creditor: Is someone to whom money is owed.

Term: The amount of time for which aloan is made

- To calculate simple interest

$$\boxed{I = prt}$$

Where I- simple interest

P – the principal

R – annual interest rate

T – time of loan

- Future value of a simple in terest: A= (1+rt), where
  A- Is the future value

  p- is the principal

r- is the simple interest rate per year

t – is the time in years

<u>Compound Interest</u>

$A_n = p(1+r)^n$ where p- initial deposit r – simple interest rate per year

n – the n<u>o</u> of years or compounding period

A- Future value or ending amount

- Future value of a compound interest: $F.V = p(1 + i)^{mt}$

  Where p – principal or present value

  f.v future value

  i- r/m, r- is annual or nominal rate

  t – time in year

  m- the n<u>o</u> of conversion period per year

Examples

1) For each of the following combinations of interest rates compounding frequencies & terms, find the value of I & n that would be used in the compound interest formula.
   a) 8% quarterly compounding, 10 years
   b) 9% compounded monthly, 7 years
   c) 15%, compounded semi – annually 15 years
   d) 9% daily compounding using bankers rule, 8 years

   Solution

1. The value of I &n that would be used in the compound interest formula.

a) $r = 8\%, m = 4, so\ i = {}^{r}/_{m} = \frac{0.08}{4} = 0.02\ or\ 2\%$ & the total n<u>o</u> of compounding in ten years is $n = 10m = 10 \times 4 = 40$

b) $r = 9\%, m = 12, t = 7\ years, so, i = {}^{r}/_{m} = \frac{0.09}{12}$

$$= 0.0075$$

$$\underline{= \mathbf{0.75}\% In = txm = 7 \times 12 = 84}$$

This means the institute can compute 84 times in 0.75% interest rate.

c) $r = 15\%, m = 2, t = 15\ years, so, i = {}^{r}/_{m} = \frac{0.015}{2}$

$$= 0.0075$$

$= 0.75\%$

- $number\ of\ computation\ period, n = txm = 15x2 = 30$

d) $R = 9\%, m = 360\ \&\ t = \ 8\ years, i = {}^{r}/_{m} = \frac{0.09=0.00025}{360=0.025\%}$

$0.075\%\ daily$

The total n͟o of computation n is

$n = m \times t = 360 \times 8 = \mathbf{2880}$

2. Calculate the total amount accumulated if birr 4,500 is deposited for 7 years at 6% interest compounded monthly.

Solution

The pv (present value) is birr 4,500, but the values of i & n are not known

i .e. $i = {}^{r}/_{12} = \frac{0.06}{12} = 0.005, n = mt = 12 \times 7 = 84$

$FV = pv(1+i)^n = 4500\ (1+i)^n = 4500(1+0.005)^{84}$

$$= \mathbf{6,841.66}$$

3. For how long must you leave an initial deposit of birr 100 in a 12% savings account compounded semi – annually to see it grows to birr 1200?

Solution

$p = 100, \quad m = 2, \quad A = 1200$

$r = 12\%$

$$A = p(1+{}^{r}/_{m})^{mt} \rightarrow 1200 = 100\left(1+\frac{0.12}{2}\right)^{2t}$$

$$12 = \left(1+\frac{0.12}{2}\right)^{2t}$$

Take logarithm both sides

$log_{10}^{12} = log_{10}(1+0.06)^{2t}$

$$log_{10}^{12} = at\log(1.06)t = \frac{\log 12}{2\log(1.06)}$$

$$t = \frac{\log 12}{2\, lag\ (1.06)}$$

$$\boldsymbol{t \approx 21.32273}$$

5.2.1 Effective interest Rate

Definition The annually compounded rate which produces the same results as a given interest rated compounding is called the equivalent annual rate (EAR) or the effective interest rate. The original interest rate is called the nominal rate.

Examples Find the EAR for 9.5% compounded monthly, if 100 birr deposited.

Solution: $A = 100\left(1 + \frac{0.095}{12}\right)^{12} = 109.92$

Suppose ye is the rate compounded annually & gives a future value of birr 109.92

- $F.V = pv(1+R)^t = 109.92 = 100(1+R)^1$
  $= 109.92 = 100(1 + re$
  $= 1.0992 = 1 + re$
  $re = 1.0992 - 1 = re = 0.0992$
  $= \mathbf{9.92}\%$
- If interest is compounded m times a year, then the effective rate must satisfy the equation:
  $\boldsymbol{p\left(1+\frac{r}{m}\right)^m = p(1+re)\&\ so,}$ $\boxed{re = \left(1+\frac{r}{m}\right)^m - 1}$

  Example:

2) Hagos plans on opening a new saving account. Which of the following is offering the highest rate?

Bank	Rate	Compounding
A	2.44	Annually
B	2.44	Semi – annually
C	2.44	Monthly

Solution

All the three bank offer the same interest rate annually. For the depositor the banks with higher frequencies paid better interest.

Bank C is preferable

3) Which one of the three is offering the best CD (Certificate of Deposit) rate?

Institute	**Rate**	**Compounding**
Bank B	8.98%	Monthly
Credit & saving Associations	9.05%	Quarterly
x.y micro finance	9.1% semi- annually	

Solution:- The three institute offer difference interest rate & difference compounding frequencies.

To choose the better we must calculate the effective interest rate (re) for each.

For bank B; Let $p = 100(principal)$,

$$A = 100 \times \left(\frac{i + 0.0898}{12}\right)^{12}$$

$$= 100\left(\frac{1 + 0.0074833}{12}\right)^{12}$$

$$= 100(1.0074833)^{12}$$

$$= 100 \times 1.09358978$$

$$A \approx 109.359$$

For credit & saving Association

$$A = 100\left(1 + \frac{0.905}{4}\right)^{4}$$

$$= 100(1 + 0.022625)^{4}$$

$$100(1.022675)^{4} \approx 109.3618$$

For xy microfinance: $A = 100\left(17\frac{0.091}{2}\right)^2\ 100(1+0.0455)^2$

$$= 100 \times (1.0455)^2$$

$$\approx 109.30$$

∴ 9.05 interest rate compounded quarterly gave a good result in one year.

5.2.2. Annuity

Definition:- An annuity is any collection of equal payments made at regular time intervals.

- An ordinary annuity is an annuity whose payments are made at the end of each time period.
- An annuity due is an annuity whose payments are made at the beginning of each time period.

Annuity computation

A) The future value an ordinary annuity

- For a given interest rate, payment frequency, & number of payments, the future value annuity factor is the future value that would accumulate if each payment were Birr 1.00. we denote this factor with symbol

Solution, where n is the number of payment d is the interest rate per payment period

- Future value of annuity factor $\frac{S_n}{i} = \frac{(1+i)^n-1}{i}$

B) The future value of an Annuity Due

- With an annuity due, payments are made at the beginning each period rather than the end.
- The future value of annuities due is:
  $Fv = R \times \frac{sn}{i}(1+i)$, where R is the periodic payment.

Present values of Annuity

- Present value of ordinary annuity $PV = R \times \frac{an}{i}$

  Where $\frac{an}{i}$ present value annuity factor

  n-number of payments

  i- The interest rate per payment period.
- Present value annuity due

$$pv = R \times \frac{an}{i}(1+i)$$

- $present\ value\ annuity\ factor\ \frac{\boldsymbol{an}}{\boldsymbol{i}} = \frac{\frac{\boldsymbol{sn}}{\boldsymbol{i}}}{(\boldsymbol{1}+\boldsymbol{i})^{\boldsymbol{n}}}$

Where $\frac{sn}{i}$ future value annuity factor

n- no of periodic payment

i . interest rate of the payment period

5.2.3. Amortization
- The action or process of gradually writing off the initial cost of an asset:

5.2.4. Depreciation:
- The decline in somethings cash value is called depreciation.

- The increase in something's value is called appreciation.

* **Straight line depreciation**

- In straight line depreciation we assume that the price declines by the same birr amount (not the same percent) each year.

* Straight line $dep^n = \frac{Initial\ value - Residual\ Value}{useful\ life\ in\ years}$

5.3. Saving investing & Borrowing Money
1) **Money** Is a medium of exchange Is a unit of account Is a store of value

Investment

a) **Investment goal:** The three most common types of inv't goal are

- Retirement planning or property purchase over the very long term (15 years or mane)
- Life event such as school fees over the medium term (10-15 year)
- Rainy day or life style funds to finance goal.
- The minimum time horizon for all types investing should be at least five years.

Investment strategy

- Refers to a set principles designed to help an individual investor achieve his/her financial & investment goal

Types of Investment

A) **Stock (Equity**) Buying stock is like buying a small fraction of a company

- The stock of a corporation may be either common preferred or par & no – par

i. **Common & preferred:-** Every corporation issue common stock, the basic form of capital stock.

ii. **Preferred stock:** gives its owners certain advantages over common stock holders.

iii. **Part value:** stock may be par value stock

- Par value is an arbitrary amount assigned by a company to a share of its stock.

iv. **No par stock**:- does not have par- value but some no – par stock has a stated value, which make it similar to par value stock.

B. Bonds/Debt Instruments: Buying bonds is like having accompany takes out a load from you.

5.3.1Characteristics of Bonds

a) **Face value**- is the money amount the bond will be worth at maturity.

b) **The coupon rate:** is the rate of in terest the bond issuer will pay on the face value of the bond, expressed as apercentage.

c) **Coupon datas**: are the dates on which the bond issuer will make interest payments.

d) **The maturity data :** is the data on which the bond will mature the bond issuer will pay the bond holder the face vale of the bond

e) **The issue price:** is the price at which the bond issuer originally sells the bonds.

C) Mutual funds: pool money from several investors invest it in deference asset classes.

* **Return on investment (ROI)** is a metric that evaluates approximately how much value has been gained from aninv't relative to the cost.

* $\boldsymbol{ROI = \frac{Current\ value\ of\ inv't - cost\ inv't}{cost\ of\ inv't}}$

5.3.2. Borrowing Monoy

a) Equity Financing: Involves selling apportion of a company's equity in return for capital.

b) Debt Finaning: Involves the borrowing of money & paying it back with interest eg. Loan.

- The main sources of loan or debt financing are saving in situations like commercial banks, saving & loan associations & credit unions.

Types of loan

a) Secured loan

- The lender has the right to force the sale the asset against which the loan is secured if you fail to keep up the repayments.

b) Unsecured loan means the lender relies on your promise to pay it back.

c) **Credit union loan:** Are mutual financial organizations which are owed & run by their members for their members.

- **Money lines:** are community development finance institutions that land & invest in deprived area & un deserved markets that cannot access main stream finance.
- **Overdraft:** are like a' safety net' on your account.
- **Buying on credit:** is a form of borrowing. It can include paying for goods or services using credit cards or under some other credit agreement.

5.4. Taxation

Tax: Is a compulsory financial charge or some other type of levy imposed on a taxpayer by a government organization in order to fund gov't spending & various public expenditure.

- Government impose & collect taxes to raise revence.

Types of taxes

a) **Direct taxes** refers to the those taxes that are paid by the person who earns the income.

b) **Indirect taxes:** Is borne by some one.

Schedules of income

The proclamation 979/2016 provided for the taxation of income in accordance with four schedules.

1. Schedule 'A' income from employment
   - Every person deriving income from employment is liable to pay tax on that in come at the rate specified in schedule 'A'
   - The tax payable on income from employment shall be charged, levied and collected at the following rates.

No	Taxable monthly income	Rates of tax in percent age
1.	<Birr 600	Exemption
2.	[600-1650]	10%
3.	[1650-3200]	15%
4.	[3200-5250]	20%
5.	[5250-7,800]	25%
6.	[7,800-10,900]	30%
7.	Over 10,900	35%

Methods of employment income tax computations.

1. Progression method
   - The amount of tax if calculated for each layer of tax bracket by multiplying the given rate under schedule A for each additional income.

   a. Deduction methods

$$\text{Income tax}=\binom{taxable}{income}\binom{tax}{rate} - \text{Deduction}$$

No	Taxable monthly income (Birr)	Rates of tax in %	Deduction
1.	< 600	Exemption	0
2.	[600-1650]	10%	60
3.	[1650-3200]	15%	142.5
4.	[3200-5250]	20%	302.5
5.	[5250-7,800]	25%	565
6.	[7,800-10,900]	30%	955
7.	Over 10,900	35%	1,500

- Deduction is computed as follows:
- Deduction= upper taxable

  Income previous

  Tax bracket tax rate of given bracket cumulative threshold.

60=0.1x600

142.5=0.15x600+(15-10)%x1050.

142.5=90+52.5=<u>142.5</u>

- 302.5=20%x600+0.1x1050+0.05x1550

  $= 600x\frac{20}{100} + 1050x\frac{10}{100} + 1550x\frac{5}{100}$

  = 120+105+77.5

  = <u>302.5</u>

Examples 1

Assume Mr. Zelalem earns monthly salary of Birr 10,460.00 calculate income tax of Mr. Zelalem.

Solution

Taxable income = 10,460.00

By progression method Birr 10,460.00 divided in to six tax brackets.

No	Amount	Rate of tax in %	Taxable amount
1.	600	Exempted	0
2.	1050	10%	105
3.	1550	15%	232.5
4.	2050	20%	410
5.	2550	25%	637.5

6.	2660	30%	798
Total	10460		2183

Total income tax

1. Progression method

   2183=

   0x600+0.1x1050+0.15x1550+0.2x2050+0.25x2550+0.3x2660

2. Deduction method

   10,460.00 lay in tax bracket (7800-10900), with deduction 955.

   and rate 30%

   income = 0.3x10460=955

   tax = 3138-955= 2183

Mr. Zelalem net income after tax is birr 10,4600-2183= Birr 8277.00

Example 2

Assume Mr. Zelalem got a promotion with salary increment of Birr 2,680.00 on the previous amount of Birr 10,460.00.

a. Calculate the net income of Mr. Zelalem
b. By what percentage did the net income increase?
c. By what percentage did the income tax increase?

Solution

a. The new gross salary of 10460+2680=13140.00Br. which lay in (>10900), with tax rate 35%.

   Income tax=0.35x13140-1500=3099

Net income mr. Zelalem = 13140-3099= 10,041(Assume the only deduction is tax)

b. %↗ on net income $=\frac{10041-8277}{8277}= 0.213=$ $\underline{\underline{21.3\%}}$

c. %↗ on net income $=\frac{3099-2183}{2183}= 0.4196=$ $\underline{\underline{41.96\%}}$

2. Schedule 'B'

Tax on income from rental of building.

- Any income arising from rental of building is taxable under schedule 'B'

Rental income includes

- Income from rent of abuilding
- Income from rent of furniture and equipment of the building is fully furnished.

- The party who grants rent of the building is called the lessor.
- The one who leases the property for use is called the lessee.

Taxable income

- ✓ Gross income includes all payments, either in cash or benefited in kind, received by the lessor.
- ✓ All payments made by the lessee on the behave of the lessor.
- ✓ The value of any renovation or improvement to the land or the building is also part of taxable income.

5.4.1Deduction

Taxable income from schedule 'B' income is determined by subtracting the allowable deductions from the gross income. Allowable deductions include the following:

A. For lessors that do not maintain books of accounts.

- Taxes paid with respect to the land and buildings being leased; except income taxes.
- For taxpayers not maintaining books of account, one fifth $\left(\frac{1}{5}\right)$ of the gross income received as rent for building's furniture and equipment as an

allowance for repairs, maintenance and depreciation of such buildings, furniture and equipment.

B. For lessors that maintaining books of accounts.

- For taxpayers maintaining books of account, the expenses incurred in earning, securing, maintaining rental income, to the extent that the expenses can be proven by the taxpayer and subject to the limitations specified by the proclamation 979/2016, deductible expenses income (but are not limited to) the cost of lease (rent) of land, repairs, maintenance, and depreciation of buildings, furniture and equipment proclamation as well as interest an bank loan, insurance premiums. i.e. building 5%, computer and related as set 25%, furniture and equipment 20% and other asset 10% of depreciation base.

Tax rate

The tax payable on rental houses shall be charged, levied and collected at the following rates:

- If the lessors or owners are bodies, they pay 30% of taxable income.
- On income of persons according to the schedule b (hereunder)

Schedule 'B' tax rate and deduction

No	Taxable income from rental of building (income per year)		Tax rate in %	Dedication in Birr
	From Birr	To		
1	0	7,200	Exempted	0
2	7,201	19,800	10	720
3	19,801	38,400	15	1,710
4	38.401	63,000	20	3,630
5	63,001	93,600	25	6,630

6	93.601	130,800	30	11,460
7	Over 130,800		35	18,000

Example

1. Missa Saba has a building that is available for rent in year 2012. The following are the details of the property let out.
   - She has let out for twelve months.
   - Actual rent for a month is Birr 30,000.
   - She paid 15% of the actual rent received as land taxes and 3% as other taxes
   - She spent Birr 10,000 for maintenance of the building.

   Other information in 2012.

Type	Original cost	Additional cost	Total
Building	300,000.00		300,000
Equipment	150,000.000		150,000
Computer	10,000.00	6000.00	16,000

Compute the taxable income and tax liability

i. He does not maintain any books of accounts in this regard.
j. Assume that Mr. X has maintained books of accounts.

Solution

I. Annual rental X 12x30,000=birr 360,000 income.

Allowable deduction

- Land taxes 360,000x0.15=54,000
- Other taxes 360,000x0.03=10,800
- Maintenance $\left(\frac{1}{5}x360,00\right)$= 72,000

Total deduction=136,800

Taxable income = 360,000-136,800=223,200

Then tax liability should be calculated as: Birr 223,200 is in tax bracket over 130,800 so the rate is 35% and deduction is also Birr 18,000.

Tax liability = $\begin{pmatrix} taxable\ x\ tax \\ income\ \ \ rate \end{pmatrix}$- deduction

= 223,200x0.35-18,000=Birr 60,120

II. For the existence of a book of account depreciation schedule:

- For building 300,000x0.05=15,000.
- For equipment 15,000x0.2=3,000.
- For computer 16,000x0.25=4,000.

Annual rental Birr 360,000.00 is the income same for cases.

Allowable deduction

- Land taxes 360,000x0.15=54,000
- Other taxes 360,000x0.03=10,800
- Maintenance= 10,000

Depreciation expense. Building ….15,000 depreciation expense equipment =3,000

Depreciation expense computer and accessories. …….. 4,000

Total deduction -----96,800

Taxable income = Birr 360,000- 98,800= Birr 263,200

Birr 263,200 is in the tax bracket over 130,800 so the rate is 35% and deduction is Birr 18,000.

Tax liability= $\frac{taxable\ income}{x\ tax\ rate}$ - deduction

= 263,200x0.35-18,000= Birr 74,120

Schedule 'C' income from business

The taxable business income of a tax payer for a tax year shall be determined in accordance with the profit and loss, or income statement of the taxpayer for the year prepared.

Business income tax rates

- The rate of business income tax applicable to a body is 30%.
- The rate of business income tax applicable to an individual are the same as schedule B tax rate given above.

Business category

- The Ethiopian tax system classifies businesses in to three categories A,B,C
  1. Category A (>1,000,000 annual turnover) and face the same tax rate (30%).
  2. Category B (Their annual turnover is between 500,000 and 1,000,000)
  3. Category C (Their annual turnover is below Birr 500,000)

❖ Schedule 'D' income:
Other income

- Other taxable sources of income include
- Royalties
- Dividends
- Interest income.
- Winnings from games of chance
- Gains on the disposal inv't assets.

Income source	Applicable tax rate and tax base
Royalty	5% on the gross amount of the royalty
Dividend	10% on the gross amount of the dividend
Interest income	5% of the gross amount of interest in the case of a savings deposite an Ethiopian financial institution.
Game of chance	15%
Pains on disposal of inv't as set	15% of disposal of an immovable as set:- 30% for disposal of a share or a bond.
Casual rental income	15% on the gross amount of rental income

Examples

Workers credit and saving associations have 8,312 common shares. Each has Birr 1000.00 value of coop Bank. The bank paid Birr 310 per share in the year ended June 30,2012 E.C.

a. How much dividend is the association on titled to?

   Solution: Dividend income = 310x8,312= Birr 2,576,720

b. How much is the tax to be paid?

   Solution: tax to be paid= 2,576.720x0.1=257.672

c. How much did the association earn on the year?

   Solution: the association earn Birr 2,576,720-257,762= Birr 2,319,048

5.4.2. Indirect Taxes

- Indirect taxes are basically taxes that can be passed on to another entity or individual.
- The major types of indirect taxes in Ethiopia are
  a. Value Added Tax (VAT)
     - The standard rate VAT is 15%
  b. Turn Over Tax (TOT)
     - The standard TOT rate is 2% an goads sold locally and services rendered locally by contractors, gain mills, tractors and combine harvesters, and 10% on other services.
  c. Excise Tax
     - Excise duty rates now range from 0 to 500% (of either the ex-factory price or C/F value plus customs duties) with the tax being applied to certain demand0inelastic and luxury items, as well as to goods that are assumed to have negative externalities (of fuel, alcohol, tobacco)
  d. Customs Duty
     - Are levied on goods imported into Ethiopia.

- Standard tax rates vary between 0% and 35%

e. Stamp Duty

- The legal instrument which regulates stamp duty in Ethiopia is Stamp Duty proclamation number 110/1998 and its amendment proclamation number 612/2008.

Review exercise on unit- 5

a.

1. What is the ratio of 1.8 km to 900ms:

Solution

1km=1000m $\Rightarrow \frac{1800m}{900m} = \underline{\underline{2:1}}$

1.8km=?

2. In a family there are three daughters and as on. What is the ratio of the number of:

   i. Females to the number of people in the family

Solution:

Assume that in this family there are father and mother- Three daughters and one son.

- A total of six family
  - Females to family
    4:6= 2:3
  - Males to females
    2:4=1:2

3. Allocate a profit of Birr 21,300 of accompany among three partners in the ratio of their share of the company 1:2:3

Solution: We have : 1+2+3=6

i. The first shareholders have a share of 1/6 of the company.
ii. The second one has 1/6 of the company
iii. The third one has 3/6 of the company

- The first share holder earns a dividend of Birr 1/6x21,300= 3,550 Birr

iv. The second one earns 2/6x21,300= Birr 7,100
v. The third one earns 3/6 x 21,300= 10,650 Birr

4. A group of 15 workers can accomplish a job in 28 days. At the same rate by how many workers can the workers be accomplished in 8days less time?

Solution

To accomplished the job 15 workers used 28 days. For the job 15x28=420 days needed. Eight day less means 20 days. The number of workers needed is $\frac{420}{20}$=21. 21 workers needed to complete the job in 20 days by assuming the rate is the same.

5. What percent Birr 52 is Birr 3.12?

Solution

To know the percentage, $\frac{3.12}{52}$x100=6,50

Birr 3.12 is 6% of birr 52

6. 8.35% of what amount is 18.37?

Solution

Let x be the known number whose 8.35% is 18.37

0.0835x X=18.37

X= $\frac{18.37}{0.0835}$= $\underline{\underline{220}}$

7. A 6% tax on a pair of shoes amounts to Birr 102. What is the cost of the pair of shoes?

Solution

Let x be the price of shoes without tax. 506% tax levied on the shoe which is equal to birr 102. Thus, 0.06x=102$\Rightarrow x = \frac{102}{0.06}$= 1700

The price of the shoes is Birr 1700.00

8. If the average daily wage of a laborer increased from Birr 16.00 to Birr 21.64 in the last three years, what is the rate of ↗ ?

Solution

Rate of change= $\frac{final\ value - original\ value}{orginal\ value}$

$= \frac{21.64-16}{16} = \frac{5.64}{16} = \underline{\underline{0.3525}}$

The rate of increase in three years is 35.25%

9. A radio recorder sold for Birr 210 has a markup of 25% on the selling price. What is the cost?

Solution

Markup percent=$\frac{markup}{selling\ price}$=100%

$\Rightarrow$ 25=$\frac{markup}{210}$x100%

$\Rightarrow$ markup=210.025=52.5

$\therefore$ Cost of the radio recorder is 210-52.5=157.5.

10. Ato Alula deposited Birr 5,000 in a saving account that pays 6% interest rate per year, compounded quarterly, what is the amount of interest obtained at the end of seven years? (No deposit or withdrawal was mode in the se seven years)

Solution

Given: P=3000, r=6%, m=4xn=7x4=28

$i=\frac{r}{4}=\frac{0.06}{4}=\underline{\underline{0.015}}$

A_7 $=p(1+i)^{28}=3000\text{-}(1+0.015)^{28}$

$=3000(1.015)^{28}\approx 3000\text{x}1.517=\underline{\underline{4551.67}}$

∴ The interest earned can be

4551.67-3000=1551.67

11. If you receive 6% interest compounded monthly, about how many years will it take for a deposit at time-0 to triple?

Solution

Let p be the amount deposited at a time

0, r=6%, m=12, $i=\frac{0.06}{12}=0.005$

$3p=p(1+0.005)^n$

$\Rightarrow 3=(1.005)^n$

Take natural lagrithm both sides

$\ln 3=\ln(1.005)^n \Rightarrow \ln 3=n\ln(1.005)$

$\Rightarrow n=12xt=220.27$

$\Rightarrow t=\frac{220.27}{12}=\underline{\underline{18.35}}$

If takes about 18 years to triple the deposited amount by 6% compounded interest monthly.

12. Ato Alemu makes regular deposits of Birr 230 at the end of each month for three years. What is the future value of his deposit, if interest rate per years is 9% compounded monthly? What is the amount of interest?

Solution

This is an ordinary annuity problem. Monthly deposit R=230, n= 12x3=36.

r= 9%, $i=\frac{0.09}{12}=0.0075$

$An=R\left(\frac{(1+i)^n-n}{i}\right)$

$\Rightarrow A=230\left(\frac{(1+0.0075)^{36}-1}{0.0075}\right)=230\left(\frac{(1.0075)^{36}}{0.0075}\right)$

$= 230x41.1527 \Rightarrow A=9465.1247$

13. At the end of each month Ato Mohammed deposits 10% of his salary in a saving insertion that pays annual interest rate of 6% for one years & then 15% for the next three years. If the salary of Ato Mohammed is Birr 1800, find the future value of his deposits at the end of the 4-years.

    Solution

    Let $A_1=180\left(\frac{(1+0.005)^{12}-1}{0.005}\right)=180\left(\frac{(1.005)^{12}-1}{0.005}\right)$

    $= 180x12.3357$

    $\Rightarrow A_1=2220.40$

    For the three years A to Mohammed deposited 0.15x1800=270 monthly.

    $A_2=270\left(\frac{(1+0.005)^{36}-1}{0.005}\right)=270\left(\frac{(1.005)^{36}-1}{0.005}\right)$

    $\Rightarrow A_2$ 270x39.3361=$\underline{\underline{10620.75}}$

14. Find the future value annuity factor for a monthly annuity, assuming the term is fifteen years and the interest rate is 7.5% compounded monthly.

    Solution

    $S_{\frac{n}{i}}=\left(\frac{(1+i)^n-1}{i}\right) \Rightarrow S\left(\frac{180}{0.00675}\right) = \left(\frac{(1+0.00675)^{180}}{0.00675}\right)$

    $\Rightarrow S\left(\frac{180}{0.00675}\right) = \underline{\underline{331.11}}$

15. Find the future values of Birr 857.35 per years for 20 years at 10.5% annual interest rate.

    Solution

    $Fv=R\left(\frac{(1+i)^n-1}{i}\right)=857.35\left(\frac{(1+0.105)^{20}-1}{0.105}\right)$

    $\Rightarrow Fv=\underline{\underline{51,981.82}}$

16. Apiece of machinery costs birr 50,000 with an estimated residual value of birr 7,000 and a useful life of 8 years. It was placed in service on July 1 of the current fiscal year. Determine the accumulated depreciation and book value at the end of the following fiscal year using.
    i. The straight line method
    ii. The percentage method

Solution: initial value=50,000

Useful life= 8years, residual value=7,000

i. By straight lien method

Depreciation= $\frac{initial\ value - residual\ value}{useful\ life\ in\ years}$

= $\frac{50,000-7000}{8}$=5775

ii. Assume the machinery depreciate 10% annually. The depreciated value of the year can be: Depreciation= 0.1x50,000=<u>5000</u>

17. A share company needs Birr 10,000,000 to expand the existing company. From the following which option gives better economic advantage for the company?
    a. A share issuing 5,000 shaves with a value of Birr 1,000 and borrowing the remaining amount from commercial banks.
    b. Issuing 10,000 shares with par value Birr 1,000.
    c. Borrowing Birr 10,000 from commercial banks

    Solution

All the three cases have their own advantage and disadvantages. It depends on the performance of the company. If the company perform well and is able to pay loan without any difficult financing by loan is good. If the company will have a chance of loss selling a share is suitable**Schedules of income**

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- Every person deriving income from employment is liable to pay tax on that in come at the rate specified in schedule ‘A’
- The tax payable on income from employment shall be charged, levied and collected at the following rates.

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10.	[1650-3200]	15%
11.	[3200-5250]	20%
12.	[5250-7,800]	25%
13.	[7,800-10,900]	30%
14.	Over 10,900	35%

Methods of employment income tax computations.

2. Progression method
   - The amount of tax if calculated for each layer of tax bracket by multiplying the given rate under schedule A for each additional income.

b. Deduction methods

$$\text{Income tax}=\binom{taxable}{income}\binom{tax}{rate} - \text{Deduction}$$

No	Taxable monthly income (Birr)	Rates of tax in %	Deduction
8.	< 600	Exemption	0
9.	[600-1650]	10%	60
10.	[1650-3200]	15%	142.5
11.	[3200-5250]	20%	302.5
12.	[5250-7,800]	25%	565
13.	[7,800-10,900]	30%	955
14.	Over 10,900	35%	1,500

- Deduction is computed as follows:
- Deduction= upper taxable

  Income previous

  Tax bracket tax rate of given bracket cumulative threshold.

60=0.1x600

142.5=0.15x600+(15-10)%x1050.

142.5=90+52.5=142.5

- 302.5=20%x600+0.1x1050+0.05x1550

$$= 600\text{x}\frac{20}{100} + 1050x\frac{10}{100} + 1550x\frac{5}{100}$$

$= 120+105+77.5$

$= \underline{302.5}$

Examples 1

Assume Mr. Zelalem earns monthly salary of Birr 10,460.00 calculate income tax of Mr. Zelalem.

Solution

Taxable income = 10,460.00

By progression method Birr 10,460.00 divided in to six tax brackets.

No	Amount	Rate of tax in %	Taxable amount
7.	600	Exempted	0
8.	1050	10%	105
9.	1550	15%	232.5
10.	2050	20%	410
11.	2550	25%	637.5
12.	2660	30%	798
Total	10460		2183

Total income tax

3. Progression method

   2183=

   0x600+0.1x1050+0.15x1550+0.2x2050+0.25x2550+0.3x2660

4. Deduction method

   10,460.00 lay in tax bracket (7800-10900), with deduction 955.

   and rate 30%

   income = 0.3x10460=955

   tax = 3138-955= 2183

Mr. Zelalem net income after tax is birr 10,4600-2183= Birr 8277.00

Example 2

Assume Mr. Zelalem got a promotion with salary increment of Birr 2,680.00 on the previous amount of Birr 10,460.00.

d. Calculate the net income of Mr. Zelalem
e. By what percentage did the net income increase?
f. By what percentage did the income tax increase?

Solution

d. The new gross salary of 10460+2680=13140.00Br. which lay in (>10900), with tax rate 35%.
   Income tax=0.35x13140-1500=3099

Net income mr. Zelalem = 13140-3099= 10,041(Assume the only deduction is tax)

e. %↗ on net income $=\frac{10041-8277}{8277}$= 0.213= 21.3%

f. %↗ on net income $=\frac{3099-2183}{2183}$= 0.4196= 41.96%

3. **Schedule 'B'**

Tax on income from rental of building.

- Any income arising from rental of building is taxable under schedule 'B'

Rental income includes

- Income from rent of abuilding
- Income from rent of furniture and equipment of the building is fully furnished.

- The party who grants rent of the building is called the lessor.
- The one who leases the property for use is called the lessee.

Taxable income

- ✓ Gross income includes all payments, either in cash or benefited in kind, received by the lessor.
- ✓ All payments made by the lessee on the behave of the lessor.
- ✓ The value of any renovation or improvement to the land or the building is also part of taxable income.

Deduction

Taxable income from schedule 'B' income is determined by subtracting the allowable deductions from the gross income. Allowable deductions include the following:

C. For lessors that do not maintain books of accounts.

- Taxes paid with respect to the land and buildings being leased; except income taxes.
- For taxpayers not maintaining books of account, one fifth $\left(\frac{1}{5}\right)$ of the gross income received as rent for building's furniture and equipment as an allowance for repairs, maintenance and depreciation of such buildings, furniture and equipment.

D. For lessors that maintaining books of accounts.

- For taxpayers maintaining books of account, the expenses incurred in earning, securing, maintaining rental income, to the extent that the expenses can be proven by the taxpayer and subject to the limitations specified by the proclamation 979/2016, deductible expenses income (but are not limited to) the cost of lease (rent) of land, repairs, maintenance, and depreciation of buildings, furniture and equipment proclamation as well as interest an bank loan, insurance premiums. i.e. building 5%, computer and related as set 25%, furniture and equipment 20% and other asset 10% of depreciation base.

Tax rate

The tax payable on rental houses shall be charged, levied and collected at the following rates:

- If the lessors or owners are bodies, they pay 30% of taxable income.
- On income of persons according to the schedule b (hereunder)

Schedule 'B' tax rate and deduction

No	Taxable income from rental of building (income per year)		Tax rate in %	Dedication in Birr
	From Birr	To		
1	0	7,200	Exempted	0
2	7,201	19,800	10	720
3	19,801	38,400	15	1,710
4	38.401	63,000	20	3,630
5	63,001	93,600	25	6,630
6	93.601	130,800	30	11,460
7	Over 130,800		35	18,000

Example

2. Missa Saba has a building that is available for rent in year 2012. The following are the details of the property let out.
   - She has let out for twelve months.
   - Actual rent for a month is Birr 30,000.
   - She paid 15% of the actual rent received as land taxes and 3% as other taxes
   - She spent Birr 10,000 for maintenance of the building.
     Other information in 2012.

Type	Original cost	Additional cost	Total
Building	300,000.00		300,000
Equipment	150,000.000		150,000
Computer	10,000.00	6000.00	16,000

Compute the taxable income and tax liability

k. He does not maintain any books of accounts in this regard.
l. Assume that Mr. X has maintained books of accounts.

Solution

III. Annual rental X 12x30,000=birr 360,000 income.

Allowable deduction

- Land taxes 360,000x0.15=54,000
- Other taxes 360,000x0.03=10,800
- Maintenance $\left(\frac{1}{5}x360{,}00\right)$= 72,000

Total deduction=136,800

Taxable income = 360,000-136,800=223,200

Then tax liability should be calculated as: Birr 223,200 is in tax bracket over 130,800 so the rate is 35% and deduction is also Birr 18,000.

Tax liability = $\left(\begin{matrix} taxable & x\ tax \\ income & rate \end{matrix}\right)$- deduction

= 223,200x0.35-18,000=Birr 60,120

IV. For the existence of a book of account depreciation schedule:

- For building 300,000x0.05=15,000.
- For equipment 15,000x0.2=3,000.
- For computer 16,000x0.25=4,000.

Annual rental Birr 360,000.00 is the income same for cases.

Allowable deduction

- Land taxes 360,000x0.15=54,000

- Other taxes 360,000x0.03=10,800
- Maintenance= 10,000

Depreciation expense. Building ....15,000 depreciation expense equipment =3,000

Depreciation expense computer and accessories. 4,000

Total deduction -----96,800

Taxable income = Birr 360,000- 98,800= Birr 263,200

Birr 263,200 is in the tax bracket over 130,800 so the rate is 35% and deduction is Birr 18,000.

Tax liability= $\frac{taxable\ income}{x\ tax\ rate}$ - deduction

= 263,200x0.35-18,000= Birr 74,120

Schedule 'C' income from business

The taxable business income of a tax payer for a tax year shall be determined in accordance with the profit and loss, or income statement of the taxpayer for the year prepared.

Business income tax rates

- The rate of business income tax applicable to a body is 30%.
- The rate of business income tax applicable to an individual are the same as schedule B tax rate given above.

Business category

- The Ethiopian tax system classifies businesses in to three categories A,B,C
  4. Category A (>1,000,000 annual turnover) and face the same tax rate (30%).
  5. Category B (Their annual turnover is between 500,000 and 1,000,000)
  6. Category C (Their annual turnover is below Birr 500,000)

❖ Schedule 'D' income:
Other income

- Other taxable sources of income include
- Royalties
- Dividends
- Interest income.
- Winnings from games of chance
- Gains on the disposal inv't assets.

Income source	Applicable tax rate and tax base
Royalty	5% on the gross amount of the royalty
Dividend	10% on the gross amount of the dividend
Interest income	5% of the gross amount of interest in the case of a savings deposite an Ethiopian financial institution.
Game of chance	15%
Pains on disposal of inv't as set	15% of disposal of an immovable as set:- 30% for disposal of a share or a bond.
Casual rental income	15% on the gross amount of rental income

Examples

Workers credit and saving associations have 8,312 common shares. Each has Birr 1000.00 value of coop Bank. The bank paid Birr 310 per share in the year ended June 30,2012 E.C.

d. How much dividend is the association on titled to?

Solution: Dividend income = 310x8,312= Birr 2,576,720

e. How much is the tax to be paid?

Solution: tax to be paid= 2,576.720x0.1=257.672

f. How much did the association earn on the year?

Solution: the association earn Birr 2,576,720-257,762= Birr 2,319,048

5.4.3. Indirect Taxes

- Indirect taxes are basically taxes that can be passed on to another entity or individual.
- The major types of indirect taxes in Ethiopia are

f. Value Added Tax (VAT)

- The standard rate VAT is 15%

g. Turn Over Tax (TOT)

- The standard TOT rate is 2% an goads sold locally and services rendered locally by contractors, gain mills, tractors and combine harvesters, and 10% on other services.

h. Excise Tax

- Excise duty rates now range from 0 to 500% (of either the ex-factory price or C/F value plus customs duties) with the tax being applied to certain demand0inelastic and luxury items, as well as to goods that are assumed to have negative externalities (of fuel, alcohol, tobacco)

i. Customs Duty

- Are levied on goods imported into Ethiopia.
- Standard tax rates vary between 0% and 35%

j. Stamp Duty

- The legal instrument which regulates stamp duty in Ethiopia is Stamp Duty proclamation number 110/1998 and its amendment proclamation number 612/2008.

Review exercise on unit- 5

18. What is the ratio of 1.8 km to 900ms:

Solution

1km=1000m $\Rightarrow \frac{1800m}{900m} = \underline{\underline{2:1}}$

1.8km=?

19. In a family there are three daughters and as on. What is the ratio of the number of:

    ii. Females to the number of people in the family

Solution:

Assume that in this family there are father and mother- Three daughters and one son.

- A total of six family
  - Females to family
    4:6= 2:3
  - Males to females
    2:4=1:2

20. Allocate a profit of Birr 21,300 of accompany among three partners in the ratio of their share of the company 1:2:3

Solution: We have : 1+2+3=6

vi. The first shareholders have a share of 1/6 of the company.
vii. The second one has 1/6 of the company
viii. The third one has 3/6 of the company

- The first share holder earns a dividend of Birr 1/6x21,300= 3,550 Birr

ix. The second one earns 2/6x21,300= Birr 7,100
x. The third one earns 3/6 x 21,300= 10,650 Birr

21. A group of 15 workers can accomplish a job in 28 days. At the same rate by how many workers can the workers be accomplished in 8days less time?

Solution

To accomplished the job 15 workers used 28 days. For the job 15x28=420 days needed. Eight day less means 20 days. The number of workers needed is $\frac{420}{20}$=21. 21 workers needed to complete the job in 20 days by assuming the rate is the same.

22. What percent Birr 52 is Birr 3.12?

Solution

To know the percentage, $\frac{3.12}{52}$x100=6,50

Birr 3.12 is 6% of birr 52

23. 8.35% of what amount is 18.37?

Solution

Let x be the known number whose 8.35% is 18.37

0.0835x X=18.37

X= $\frac{18.37}{0.0835}$= $\underline{\underline{220}}$

24. A 6% tax on a pair of shoes amounts to Birr 102. What is the cost of the pair of shoes?

Solution

Let x be the price of shoes without tax. 506% tax levied on the shoe which is equal to birr 102. Thus, 0.06x=102$\Longrightarrow x = \frac{102}{0.06}$= 1700

The price of the shoes is Birr $\underline{\underline{1700.00}}$

25. If the average daily wage of a laborer increased from Birr 16.00 to Birr 21.64 in the last three years, what is the rate of ↗ ?

Solution

Rate of change= $\frac{final\ value - original\ value}{orginal\ value}$

$= \frac{21.64-16}{16} = \frac{5.64}{16} = \underline{\underline{0.3525}}$

The rate of increase in three years is $\underline{\underline{35.25}}$%

26. A radio recorder sold for Birr 210 has a markup of 25% on the selling price. What is the cost?

Solution

Markup percent= $\frac{markup}{selling\ price}$=100%

$\Rightarrow$ 25= $\frac{markup}{210}$x100%

$\Rightarrow$ markup=210.025=52.5

$\therefore$ Cost of the radio recorder is 210-52.5= $\underline{\underline{157.5.}}$

27. Ato Alula deposited Birr 5,000 in a saving account that pays 6% interest rate per year, compounded quarterly, what is the amount of interest obtained at the end of seven years? (No deposit or withdrawal was mode in the se seven years)

Solution

Given: P=3000, r=6%, m=4xn=7x4=28

i= $\frac{r}{4} = \frac{0.06}{4}$ = $\underline{\underline{0.015}}$

A_7 =p(1+i)^{28}=3000-$(1+0.015)^{28}$

=3000(1.015)^{28} ≈3000x1.517= $\underline{\underline{4551.67}}$

$\therefore$ The interest earned can be

4551.67-3000=1551.67

28. If you receive 6% interest compounded monthly, about how many years will it take for a deposit at time-0 to triple?

Solution

Let p be the amount deposited at a time

0, r=6%, m=12, i=$\frac{0.06}{12}$=0.005

3p=p$(1+0.005)^n$

$\Rightarrow$ 3=$(1.005)^n$

Take natural lagrithm both sides

ln3=ln$(1.005)^n$ $\Rightarrow$ln3=n ln(1.005)

$\Rightarrow$ n=12xt=220.27

$\Rightarrow$ t=$\frac{220.27}{12}$= $\underline{\underline{18.35}}$

If takes about 18 years to triple the deposited amount by 6% compounded interest monthly.

29. Ato Alemu makes regular deposits of Birr 230 at the end of each month for three years. What is the future value of his deposit, if interest rate per years is 9% compounded monthly? What is the amount of interest?

Solution

This is an ordinary annuity problem. Monthly deposit R=230, n= 12x3=36.

r= 9%, i=$\frac{0.09}{12}$=0.0075

An=R$\left(\frac{(1+i)^n-n}{i}\right)$

$\Rightarrow$A=230$\left(\frac{(1+0.0075)^{36}-1}{0.0075}\right)$=230$\left(\frac{(1.0075)^{36}}{0.0075}\right)$

$= 230x41.1527$ $\Rightarrow$A=9465.1247

30. At the end of each month Ato Mohammed deposits 10% of his salary in a saving insertion that pays annual interest rate of 6% for one years & then 15% for the next three years. If the salary of Ato Mohammed is Birr 1800, find the future value of his deposits at the end of the 4-years.

Solution

Let A_1=180$\left(\frac{(1+0.005)^{12}-1}{0.005}\right)$=180$\left(\frac{(1.005)^{12}-1}{0.005}\right)$

$= 180x12.3357$

$\Rightarrow A_1=2220.40$

For the three years A to Mohammed deposited 0.15x1800=270 monthly.

$A_2=270\left(\frac{(1+0.005)^{36}-1}{0.005}\right)=270\left(\frac{(1.005)^{36}-1}{0.005}\right)$

$\Rightarrow A_2$ 270x39.3361=$\underline{\underline{10620.75}}$

31. Find the future value annuity factor for a monthly annuity, assuming the term is fifteen years and the interest rate is 7.5% compounded monthly.

Solution

$S_{\frac{n}{i}}=\left(\frac{(1+i)^n-1}{i}\right) \Rightarrow S\left(\frac{180}{0.00675}\right)=\left(\frac{(1+0.00675)^{180}}{0.00675}\right)$

$\Rightarrow S\left(\frac{180}{0.00675}\right)$ =$\underline{\underline{331.11}}$

32. Find the future values of Birr 857.35 per years for 20 years at 10.5% annual interest rate.

Solution

$Fv=R\left(\frac{(1+i)^n-1}{i}\right)=857.35\left(\frac{(1+0.105)^{20}-1}{0.105}\right)$

$\Rightarrow$ Fv=$\underline{\underline{51,981.82}}$

33. Apiece of machinery costs birr 50,000 with an estimated residual value of birr 7,000 and a useful life of 8 years. It was placed in service on July 1 of the current fiscal year. Determine the accumulated depreciation and book value at the end of the following fiscal year using.

iii. The straight line method

iv. The percentage method

Solution: initial value=50,000

Useful life= 8years, residual value=7,000

iii. By straight lien method

Depreciation= $\frac{initial\ value - residual\ value}{useful\ life\ in\ years}$

= $\frac{50,000-7000}{8}$=5775

iv. Assume the machinery depreciate 10% annually. The depreciated value of the year can be: Depreciation= 0.1x50,000=5000

34. A share company needs Birr 10,000,000 to expand the existing company. From the following which option gives better economic advantage for the company?

d. A share issuing 5,000 shaves with a value of Birr 1,000 and borrowing the remaining amount from commercial banks.

e. Issuing 10,000 shares with par value Birr 1,000.

f. Borrowing Birr 10,000 from commercial banks

Solution

All the three cases have their own advantage and disadvantages. It depends on the performance of the company. If the company perform well and is able to pay loan without any difficult financing by loan is good. If the company will have a chance of loss selling a share is suitable

.