MATHEMATICS

Grade 11

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2016 EC

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Unit One

Relation & Function

Relation

Defination Relation is aset of ordered pairs

Relations is a set of whose elements are ordered pairs given two sets A & B arelation R from Set A to Set B is defined as any subject of AxB

R- is said to be relation fromset A to set B ,iff

- ✓ R$\leq A \times B$
- ✓ Domain of R $\leq A$
- ✓ Range of $R \leq B$
- A relation R on A is any subject of $A \times A$
- The set of all first elements in a relation R is called the domain of the relations R domain of R= $\{x: (x, y) \in R \; for \; same \; y \; \in B\}$
- The set of all second elements in a relations R is called the range R range R $=\{y: (x, y) \in R \; for \; same \; x \; \in A\}$

  Example

  1) In a certain school there are four sections of Grade 11 & each section is related with the number of students in the given sections shown below.

Sections	Numbers of students
Section 1	49
Section 2	51
Section 3	48
Section 4	50

a) From the above table find the relation R

R=$\{(Section\ 1,49), (Section\ 2,51), (Section\ 3,48), (Section\ 4,50)\}$

b) Find the domain and range of R – Domain of

R={$(Section\ 1), (Section\ 2), (Section\ 3), (Section\ 4)$ }

Range of R={49,51,48,50}

Example let R=$\{(x, y): y = x^2 - 2\}$ $and\ x\epsilon\{0,1,2,3,4,5\}$ then find the domain and range of R

Solution

R=$\{(0, -2), (1, -1), (2,2)(3,7), (4,14), (5,23)\}$

Domain R= $\{0,1,2,3,4,5\}$

Range of R =$\{-2, -1,2,7,14,23\}$

Representations of Relations

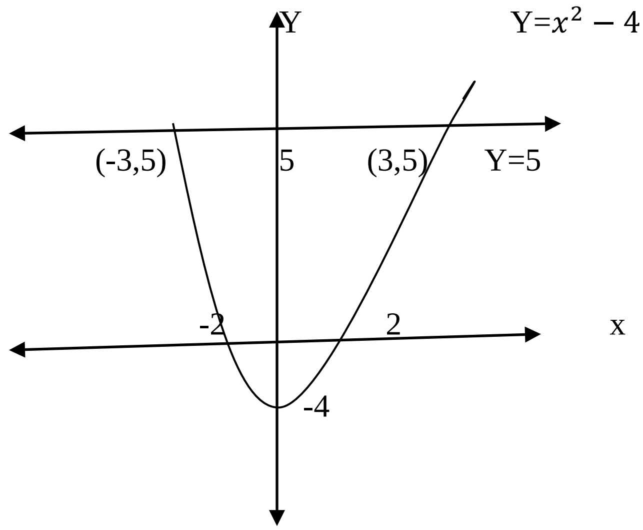
Relation can be represented by

1) A set of ordered pairs
2) A corresponding b/n domain and range
3) A Graph
4) An equation
5) An inequality or combination of these etc.

Examples: - find the domain & range of the following relations

a) R=$\{(x, y): y \geq\ x^2 - 4\ \ and\ y \leq 5\}$
b) R=$\{(x, y): x^2 + y^2 \leq 125\ and\ y \geq 2|x|\}$

Solution



To find the domain first find the intersection points of Y=$x^2 4$ $y = 5, y = y$

- $x^2 - 4 = 5$
- $x^2 - 4 - 5 = 0$
- $x^2 - 9 = 0$
- $x^2 - 3^2 = 0$
- $(x-3)(x+3) = 0$

⇔ X=3 or x=-3

Domain of the R={x: $-3 \le x \le 3$}

Range of R ={y: $-4 \le y \le 5$}

Examples: - **b** Solutions

Given R={(x, y): $x^2 + y^2 \le 125$ &$y \ge 2|x|$}

$= x^2 + y^2$=125 & y=2$|x|$

Put y=2$|x|$

=$(x^2) + (2|x|)$=125

$= x^2 + 4x^2$=125

$= 5x^2 = 125$

x^2=25

=$\sqrt{x^2}$=$\sqrt{25}$

when x= 5, y=2$|5|$

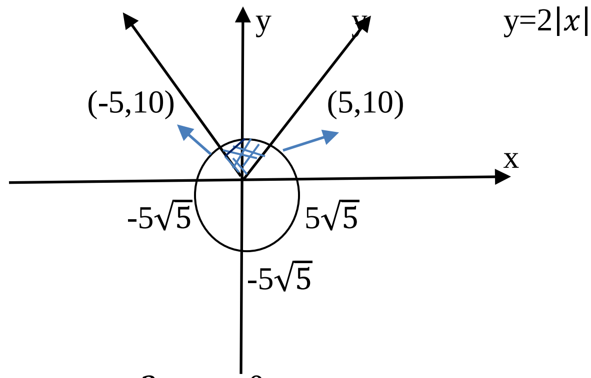
= 10

when x=5, y= 2$|-5|$

=10

But the equation$x^2 + y^2 = 125$, is equation of a circle with center at (0,0)&$radius$ $((r) = 5\sqrt{5}$

Graphically



And $2|x|$=y=$\begin{cases} 2x, x > 0 \\ -2x, x < 0 \end{cases}$

- Domain of R=$\{x: -5 \leq x \leq 5\}$ & range of R= $\{y: 0 \leq y \leq 5\sqrt{5}\}$

Examples 4 find the domain and range of R, if R=$\{(x,y): y = |x-1|, x\epsilon z \,\&|X| \leq 3\}$

Solution: $|x| \leq a \; for \; a \geq 0 \; is - a \leq x \leq a$

$|x| \leq 3 \leftrightarrow$-3$\leq x \leq 3 \,\& x\epsilon z$

x$\epsilon\{-3,-2,-1,0,1,2,3\}$

y=$|x-1|$

When, x=3, y=$|3-1| = 2$

when x=2, y= $|2-1| = 1$

When x=1, y=$|1-1| = 0$

When x=0, y=$|0-1| = 1$

When x=-1, y=$|-1-1| = 2$

When x=-2, y=$|-2-1| = 3$

When x=-3, y= $|-3-1| = 4$

Domain of R =$\{x: -3 \leq x \leq 3\} \; or \{-3,-2,-1,0,1,2,3\}$ & range of R={0,1,2,3,4}

- A relation may consist of a finite set of ordered pairs or a finite set of ordered pairs

Equalities of ordered pairs

- For an ordered pair (x,y)order is important that is $(x,y) = (y,x)$, only when x=y
- For two ordered pairs $(a,b)\&(c,d)$, (a,b)=.$iff, a = b \; and \; b = d$

Examples 1, find the values of a and b when

i) $(a+3, b-2) = (5,,1)$

ii) $\left(\frac{a}{3}+1,\ b-\frac{1}{3}\right)=\left(\frac{5}{3},\frac{2}{3}\right)$

solutions

i) $(a+3, b-2) = (5,,1) \leftrightarrow$ a+3=5 & b-2=1

A=5-3&b=1+2

A=2 &b=3

ii) $\left(\frac{a}{3}+1,\ b-\frac{1}{3}\right) = \left(\frac{5}{3},\frac{2}{3}\right)$

$\frac{a}{3}+1=\frac{5}{3}$ & b-$\frac{1}{3}$=$\frac{2}{3}$

$\frac{a}{3}=\frac{5}{3}-1$ & b=$\frac{2}{3}+\frac{1}{3}$

$\frac{a}{3}=\frac{5-3}{3}$ $\&b=\frac{2+1}{3}$

$\frac{a}{3}=\frac{2}{3}$ $\&b=\frac{3}{3}$

3*$\frac{a}{3}=\frac{2}{3}*3$ $\& b=1$

a=2 &b=1

Inverse Relations & their Graphs

Inverse Relations

Let R be a relation from seta to Set B .the inverse of R denoted by R^{-1} is a relation from set B to set A given by R^{-1} =$\{(y,x),(x,y)\epsilon R\}$ YR^{-1}x, iff, XRy

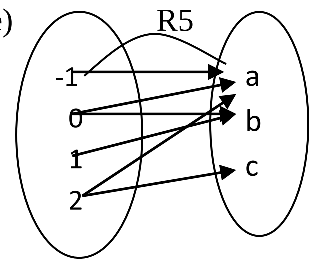
- An inverse relation is the set of ordered pairs obtained by interchanging the first and second elements of each pair in the original relations
- If R^{-1} is the inverse of a relation R then
  - ✓ Domain of R =range of R^{-1}
  - ✓ Range of R = domain of R^{-1}

Examples

1) Find the inverse of each of the following relations and determine the domain & range of R and R^{-1}
   a) R1=$\{(-3,2),(-2,3),(-1,4),(0,5),(1,6),(2,7),(3,8)\}$
   b) R2= $\{(x,y): x, y\epsilon 1R\ \&y=2x-4\}$

c) R3=$\{(x, y): y \geq x - 3 \text{ and } y \leq x + 5\}$

d) R4= $\{(x, y): y = 2 - 4^{x}\}$

e) 

f) R6 =$\left\{(x, y) y = \sqrt{1 - |x|}\right\}$

Solutions

a) R^{-1} =$\{(2, -3), (3, -2), (4, -1), (5,0), (6,1), (7,2), (8,3)\}$

Domain of R^{-1} =range of R^{-1}=$\{-3, -2, -1,0,1,2,3\}$ =$\{2,3,4,5,6,7,8\}$

b) $R2^{-1}$=$\{(x, y): x = 2y - 4\}$=$\left\{(x, y): \frac{x+4}{2} = y\right\}$, DR2 =R$R^{-1}$=D$R2^{-1}$=RR2=1R

$or\ R2^{-1}$=$\{(y, x): y = 2x - 4\}$

c) $R3^{-1}$=$\{(y, x): y \geq x - 3\ \&y \leq x + 5\}$

DR3=R$R3^{-1}$=$\{x: x\epsilon 1R\}$

RR3=D$R3^{-1}$ $= 1R$

Or =$\{(y, x): x \geq y - 3\ \&x \leq y + 5\}$

=$\{(x, y): x + 3 \geq y \& x - 5 \leq y\}$

=$\{(x, y): y \leq x + 3 \& y \geq x - 5\}$

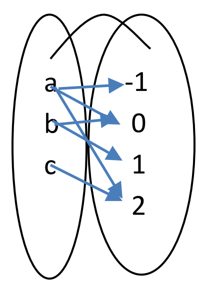
d) $R4^{-1}$=$\{(y, x): y = 2 - 4^{x}\}$

= or $R4^{-1}$=$\{(x, y): x = 2 - 4^{y}\}$

=$\{(x, y): x - 2 = -4^{y}\}$

= $\{(x, y): 2 - x = 4^{y}\}$

=$\{(x, y): \log 4\ 2 - x = y\}$

e. 

- D$R5^{-1}$=RR5 =$\{a, b, c\}$
- R$R5^{-1}$=DR5 =$\{0,1,2, -1\}$

- DR4=$\{(x: x\epsilon IR)\}$ RR4 =$\{y: y < 2\}$
- $R4^{-1}$ =$\{x: x < 2\}$ $RR4^{-1}$ =$\{y: y\epsilon IR\}$

- F) $R6^{-1}$=$\left\{(y,x): y=\sqrt{1-|x|}\right\}$

  Or $R6^{-1}$=$\{(x,y): x=\sqrt{1-|y|}\}$

  =$\{(x,y): x^2=1-|y|\}$

  =$\{(x,y): x^2-1=|y|\}$

  =$\{(x,y): 1-x^2=|y|\}$

  $R6^{-1}=\{(x,y): y=\pm(x^2-1)\}$

  $=\{(x,y): y=x^2-1 \; or y=1-\; x^2\}$

- Domain of R6=$\{x: -1\leq x\leq 1\}$
- Range of R6 = $\{y: 0\leq y\leq 1\}$
- Domain of $R6^{-1}$= $\{x: 0\leq x\leq 1\}$
- Range of $R6^{-1}$= $\{y: -1\leq y\leq 1\}$

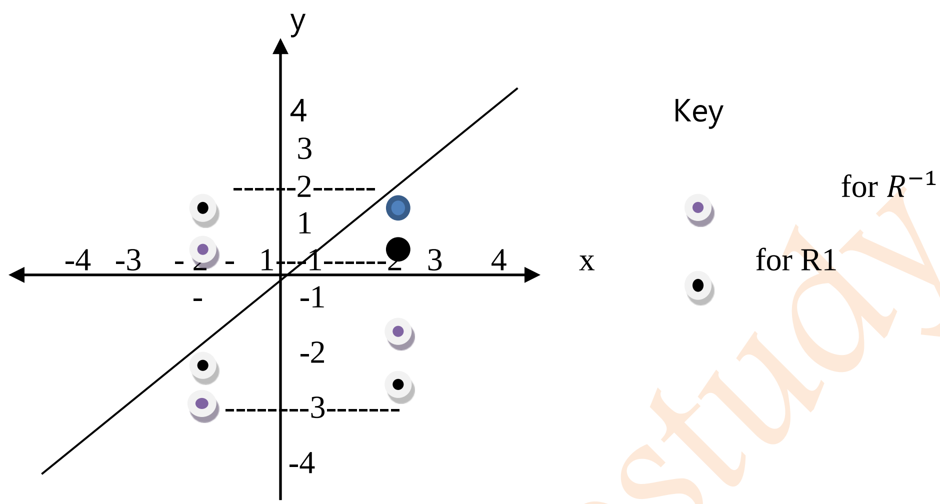
Graphs of an inverse relation is the reflection (mirror image) of the graph of the original relation along the line y=x

Examples

1) For each of the following relations
   a) Find the inverse of each relations
   b) Draw the graph of R & its inverse on the same plane.
   c) Determine the domain & range of R &R^{-1}
      i) R1 =$\{(1,2),(-3,1),(-2,-4),(2,-3)\}$
      ii) R2 =$\{(x,y): y=x+5\}$
      iii) R3 =$\{(x,y): y<x+5\}$
      iv) R4 =$\{(x,y): y<x+5 \;\&\; y\leq -x+4, y\geq -4\}$
      v) R5 =$\{(x,y): y\geq x^2-1 \;\&\; y\leq 3\}$

Solutions

1i) R^{-1} $=\{(2,1),(1,-3),(-4,-2),(-3,2)\}$

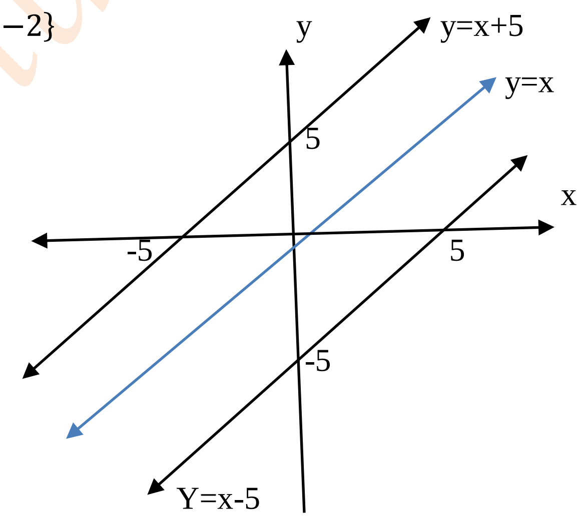


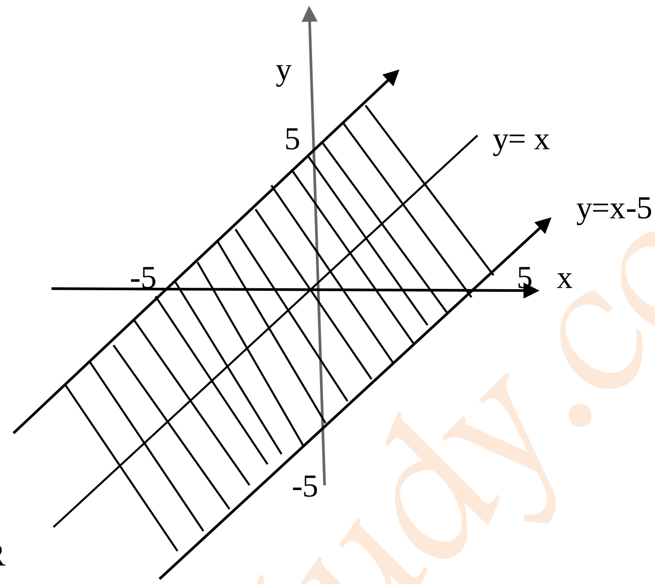
- Domain of R1 $=\{1,2,-3,-2\}$
- Range of R1 $=\{1,2,-3,-4\}$
- Domain of $R1^{-1}$$=\{1,2,-3,-4\}$
- Range of $R1^{-1}$$=\{1,2,-3,-2\}$

1ii) $R2^{-1}=\{(y,x): y=x+5\}$

Or $R2^{-1}=\{(x,y): x=y+5\}$

$=\{(x,y): x-5=y\}$





1iii) R_3^{-1} = {(y, x): y< $x+5$}

y   y=x+5

Or R_3^{-1} = (x, y) : x< $y+5$}

={(x, y): x-5 < y}

= {(x ,y) : y> x-5}

Domain of R_3 = Range of R_3 =IR

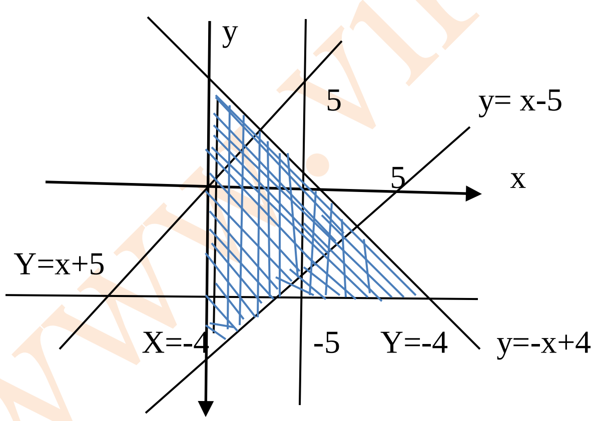
Domain of R_3^{-1} =Range of R_3^{-1} =IR

1iV). R_4^{-1} ={(y, x) : y< $x+5$,y≤ $-x+4$ & $y \geq -4$}

Or R_4^{-1} = {(x, y) : x < $y+5$, x ≤ $-y+4$ & $x \geq -4$}

= {(x, y) : x-5 < $y, 4-x \geq y$ & $x \geq -4$}

={(x, y):y> $x-5$, $y \leq 4-x$ & $x \geq -4$}

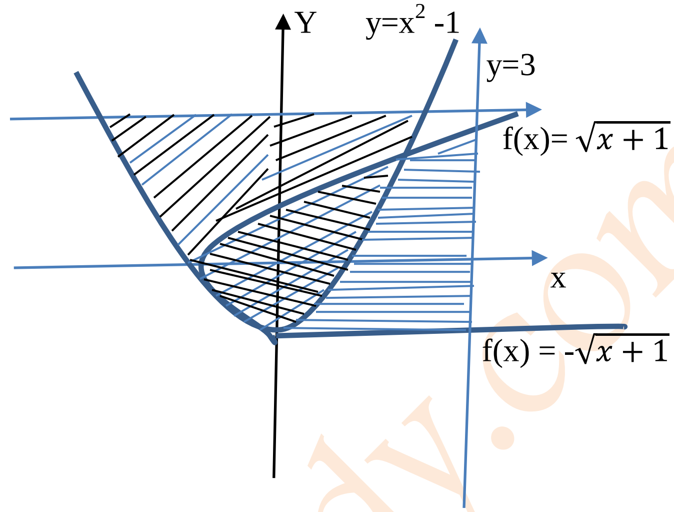


Domain of R_4 ={x: -9< $x \leq 8$}

Range of R_4 ={y:-4≤ $y < \frac{9}{2}$}

- DR_4^{-1} = {x: -4≤ $x < \frac{9}{2}$}

RR_4^{-1} ={y: -9 < $y \leq 8$}



1V). R_5^{-1} ={(y, x): y≥ x^2 -1 & y≤ 3}

Or R_5^{-1} ={(x, y) : x≥ y^2 -1 & x≤ 3}

= {(x, y): x+1≥ y^2 & x≤ 3}

={(x, y): $y^2 \leq x + 1$ & $x \leq 3$}

= {(x, y):-$\sqrt{x+1} \leq y \leq \sqrt{x+1}$ & x≤ 3}

Domain of R_S ={x:-2≤ $X \leq 2$}   Domain of R_S^{-1} = {x:-1≤ $x \leq 3$}

Range of R_5 ={y: -1≤ $y \leq 3$}   Range of R_S^{-1} ={y: -2≤ $y \leq 2$}

Types of functions

Function: - is a relation in which no two distinct ordered pairs have the same first element.

- Function is a special type of relation.
- A relation is said to be a function, if the domain is not repeated. Let f:A→ B or $A \to B$ is a function with domain A & range ⊆ B.
- Given a relation R, if (x, y) $\in R$ &$(X, Z) \in R$ implies y=z for any x, y & z, then R is a function otherwise R is not a function. Or if (x, y_1)∈ R &$(x, y_2) \in R$, then y_1=y_2.

Examples: - 1. Which one of the following relations are functions?

a). R= {(1,0), (2,0), (3,1), (1,6)} c). R={(x, y):y< $x + 2$}

b). R= {(x, y): y=x+2}   d). R= {(x, y): y is the brother of x}

e). R= {(x, y): y is a mother of x}   f). R= {(x, y): $x^2 + y^2$ =9}

g). R={(a,1) ,(a,2), (a,3) , a ϵ IR }.

N.B: - All one – to –one relations are functions.

All many –to –one relations are functions.

All one –to –many relations are not functions.

Vertical line test

- A set of points in the Cartesian plane is the graph of a functions, if no vertical line intersects the set more than once.

Even and odd functions (optional)

- The evenness or oddness of a given function is called its parity.
- A function f: A $\rightarrow B$ is said to be

  i. Odd, if, for any x $x\epsilon A$,f(-x) = -(f(x) and the domain of f is asymmetric set.

  ii. Even , if , for any x ϵA, f(-x) =f(x) and the domain of f is a symmetric set.

- The graphs of all even functions are symmetric with respect to the y- axis i.e.(x,y) lies on the graph of f (-x, y) lies on the graph off.
- The graphs of all odd functions are symmetric with respect to the origin.

  i.e. (x, y) lies on the graph of f (-x, -y) lies on the graph off.

power functions

Definition :- a power function is a function which can be written in the form of f(x) = ax^r ,where r ϵ Q & a ϵ IR.

- Power function is a function with a single term. (A function which contains only single term).

Examples: - 1). The following are examples of power functions.

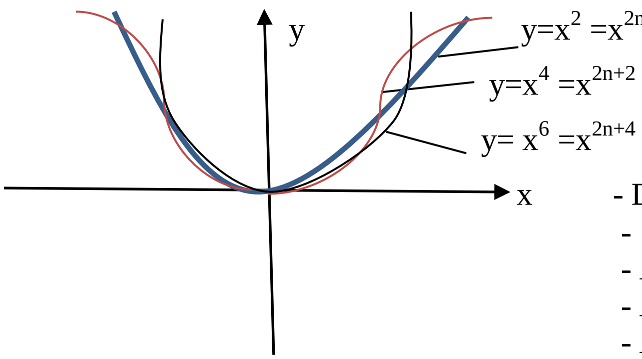
a) Area of a circle $A = \pi r^2$, where r is the radius of a circle

b) Volume of a sphere (v) =$\frac{4}{3}A = \pi r^3$, where r is the radius of a sphere.

c) y =1, x, x, x^{-3}, $x^{1/2}$, $x^{-3/2}$ etc.

Graphs & Properties of Power Functions

1). Graphs (functions) of the form $f(x) = x^n$ where n is even natural number.



for all even consecutive natural number n.

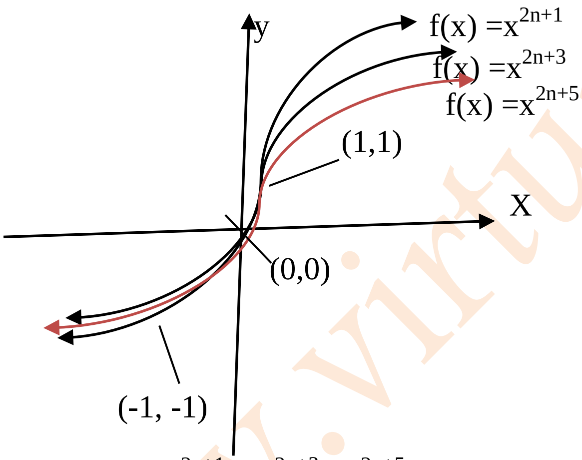
common properties

- Domain = IR
- Range $=\{y: y\geq 0\}$
- All are even functions
- All Are S.w.r.t the y-axis
- All passes through (0,0), (1,1), (-1,1)
- All are $\uparrow on[0,\infty)$
- All are $\downarrow on\ (-\infty,0]$

For $x > 1\ \&\ x < -1, x^{2n+4} > x^{2n+2} > x^{2n}$

For $0 < x < 1\ \&-1 < x < 0$, $x^{2n+4} < x^{2n+2} < x^{2n}$

2). Functions of the form $f(x) = x^n$, where n is an odd natural number.



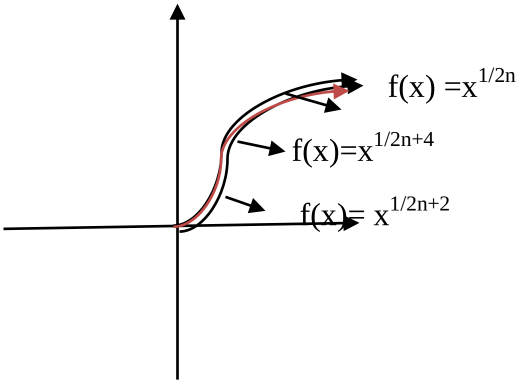
<u>**Common Properties**</u>

- Domain = IR
- Range =IR
- All are odd functions
- All are symmetric about the origin
- All passes through the points (0,0) & (-1, -1), (1,1)
- All are increasing functions

for x = 0 $x^{2n+1} = x^{2n+3} = x^{2n+5} = 0$

for $x > 1\ \&-1 < x < 0$, $x^{2n+5} > x^{2n+3} > x^{2n+1}$

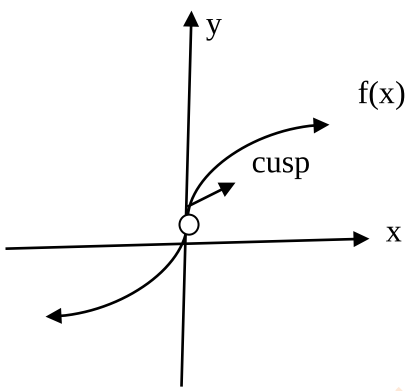
3). Functions of the form $f(x) = x^{1/n}$, n is even natural number



Common Properties

- Domain = $\{x : x \geq 0\}$
- Range = $\{y : y \geq 0\}$
- No parity
- No symmetry
- All passes through (0.0) & (1,1)
- All are increasing (i.e. $\uparrow on\ [0, \infty)$)
- For x >, $x^{1/2n} > x^{1/2n+2} > x^{1/2n+4}$
- For $0 < x < 1$, $x^{1/2n+4} < x^{1/2n+2} < x^{1/2n}$
  (For all consecutive even natural numbers.

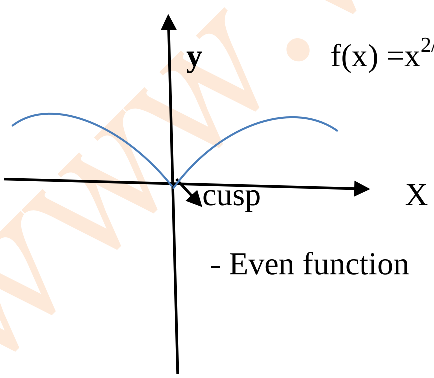
4). Functions of the from $f(x) = x^{1/n}$ n is odd.



$f(x) = x^{1/n}$ n is odd natural number Type equation here.

- Domain = IR
- Range = IR
- $\uparrow on\ (-\infty, \infty)$
- $Odd\ function$
- S.w.r.t the origin

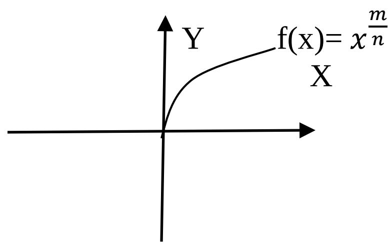
5). Functions of the form f(x) =, $x^{\frac{m}{n}}$ where m – is even natural number



$f(x) = x^{2/3} = x^{\frac{4}{5}}$ n- is odd natural number

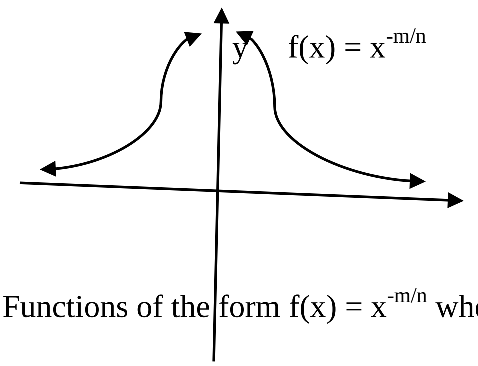
- Domain = IR
- Range = $\{y : y \geq 0\}$
- Even function
- S.w.r.t the y-axis
- $\uparrow on[0. \infty)$
- $\downarrow on(-\infty, 0]$

6). Functions of the form $f(x) = x^{\frac{m}{n}}$ where m- is even & n- is odd



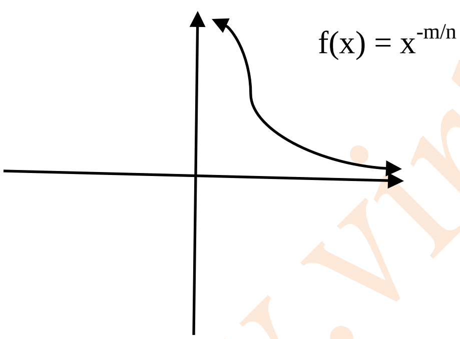
. – Domain = {x: x ≥ 0} - No parity
- Range ={y: y≥ 0 - N
-↑ $on[0.\infty)$

7). Functions of the form $f(x) = x^{-m/n}$ where m- is even & n- is odd



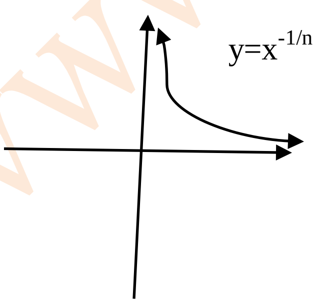
- Domain = IR {0}
- Range ={y: y> 0}
- even function
- ↑ $on(\infty, 0)$
- ↓ $on(0, \infty)$
- s.w.r.t the y-axis

8). Functions of the form $f(x) = x^{-m/n}$ where m –is odd & n- is even



- Domain ={x: x > 0}
- Range ={y: y > 0}
- No parity
- No symmetry
- -↓ $on(0, \infty)$

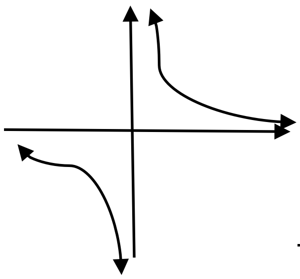
9). Functions of the form $f(x) = x^{-1/n}$ where n –is even natural number



n- is even natural number

- Domain ={x: x> 0}
- Range ={y: y> 0 }
- No parity
- No symmetry

10). Functions of the form f(x) = $x^{-1/n}$ where n –is odd natural number



- Domain = IR {0}
- Range = IR {0}
- odd function
- s.w.r.t the origin

All power functions of the form f(x) =ax^n satisfies the multiplicative property f(xy) =f(x). f(y), when a =1

Absolute value (modulus functions)

Definition:- for any real number x, the absolute value or modulus of x denoted by $|x|$ & is defined as :

$$|x|=\begin{cases} x\,, \text{if } x > 0 \\ 0, \text{if } x = 0 \\ -x. \text{if } x < 0 \end{cases}$$

Properties: -

1. $\sqrt{x2} = |x| \geq 0$
2. $|x|$ is the distance between the origin & the point corresponding to x.
3. $|x| = |-x|$
4. $|xy|= |x||y|$
5. $|x| \geq \underline{+}\,\underline{x}$
6. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, y≠ 0.
7. $|x + y| \leq |x| + |y|(\Delta inequality\,)$
8. For any x $\in IR$ & a≥ 0, if , x= $\underline{+}$ a

$\boldsymbol{Examples} : -$

1). Evaluate each of the following absolute value expressions.

a). $\left|\sqrt{2} - 1.14\right|$ b). $|\pi - 3.14|$ c). $\left|\pi - \frac{22}{7}\right|$ d). $|1 - 0.9|$

2). Find the solution set of the following absolute value equations.

a). $|x|=\frac{2}{3}$ c). $|2x + 3|$= x+6 e). $|4x + 3|$=x^2-9

b). $\left|\frac{x}{2} + \frac{x}{3}\right| = \frac{1}{2}$ d). $|x3 - 21|$=6 f). $|x| - 7$ = -2

Absolute Value Functions

Definition: - The absolute value (modulus) function is the function that can be expressed in the form of f(x) = $|x|$

Domain of f(x) = $|x|$ is IR

Range of f(x) = $|x|$ is {y: y≥ 0}

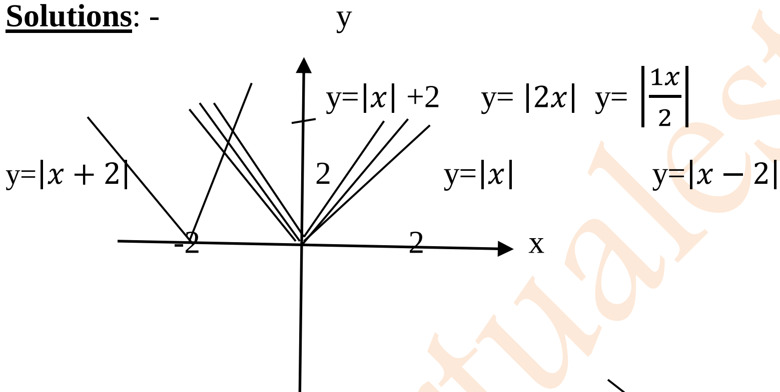
Graphs of absolute value functions

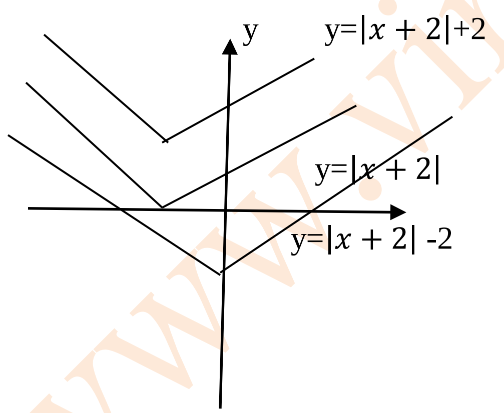
Examples: - 1. Draw the graphs of the following absolute value functions

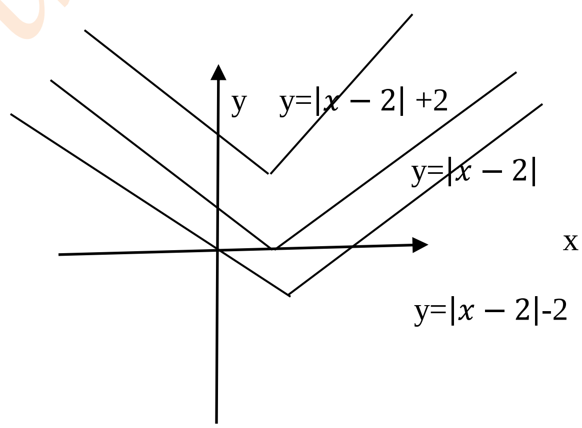
a). f(x) = $|x|$ b). f(x) = $|x|$ +2 c). f(x) = $|x|$ -2 d). f(x) =$|x+2|$ e). f(x) =$|x-2|$

f). f(x)= $|2x|$

Solutions: -







General Properties of Absolute Value Functions

Functions	Domain	range	parity	Symmetry	Increasing on	Decreasing
$f(x) = \|x\|$	IR	$y \geq 0$	even	About the y- axis	$[0,\infty)$	$(-\infty, 0]$
$f(x) = \|kx\|$ $k \neq 0$	IR	$y \geq 0$	Even	About the y- axis	$[0,\infty)$	$(-\infty, 0]$
$y = \|x + c\|$,c∈ N	IR	$y \geq 0$	Neither	About the line x= -c	$[-c,\infty)$	$(-\infty, -c]$
$y = \|x - c\|$,c∈ N	IR	$y \geq 0$	Neither	About the line x= c	$[c,\infty)$	$(-\infty, c]$
$y = \|x - c\|+d$ c, d ∈ N	IR	$y \geq d$	Neither	About the line x=c	$[c,\infty)$	$(-\infty, c]$
$y = \|x\|+c$,c ∈ N	IR	$y \geq c$	Even	About the y- axis	$[0,\infty)$	$(-\infty, 0]$
$y = \|x\|- c$,c ∈ N	IR	$y \geq -c$	Even	About the y- axis	$[0,\infty)$	$(-\infty, 0]$
$y = -\|x\|+c$,c ∈ N	IR	$y \leq c$	Even	About the y- axis	$[-\infty, 0)$	$[0,\infty)$

Signum function

- In mathematics, the word sign refers to the property of being positive or negative.
- Every real number except zero is either positive or negative, & therefore has a sign.
- The sign function or signum function is an odd mathematical function that extracts the sign of a real number.
- The signum function reads as signum x, is written as sgn(x) & defined by : f(x) =y= sgn(x)

$$\begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases} = \frac{x}{|x|} = \frac{|x|}{x},\ x \neq 0$$

$$0,\ x = 0$$

Generally, $f(x) = y = x^n \operatorname{sgn}(x) = \begin{cases} xn, for\ x > 0 \\ 0, for\ x = 0 \\ -xn, for\ x < 0 \end{cases}$

Examples: - 1). Evaluate each of the following expressions.

a). $\operatorname{sgn}(\log_{1/2} 64) = -1$ b). $\operatorname{sgn}(\sqrt{22} - 30)$ c). $\operatorname{sgn}(\pi 2 - 5) = 1$ d). $\operatorname{sgn}(x^2 - 2x + 3) = 1$ e). $\operatorname{sgn}(\sin(300^0)) = -1$

2. find the solution sets for the following equations.

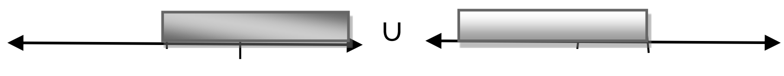
a). $\operatorname{sgn}(x^2 - x) = 1$, since for every sgn $(x > 0) = 1$

$$x^2 - x > 0$$

$x(x-1) > 0$

$x > 0 \;\&\; x - 1 > 0$ or

$x < 0 \;\&\; x - 1 < 0$



$x > 1 \cup x < 0$

$S.S = \{x: x > 1 \text{ or } x < 0\}$

b). sgn(x-7) = -1

$x-7 < 0 \quad x < 7 \quad$ S.S $\{x: x < 7\}$.

c).sgn($9-x^2$) = 1

$9 - x^2 > 0$

$9 > x^2 \implies x^2 < 9 \quad -3 < x < 3$ S.S $\{x: -3 < X < 3\}$.

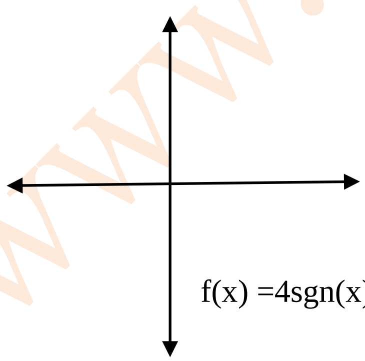
Graphs of Signum Function

Examples: - 1. Sketch the graphs of the following functions & state their basic properties.

a). f(x) = sgn(x) c). f(x) = 2+3sgn(x)

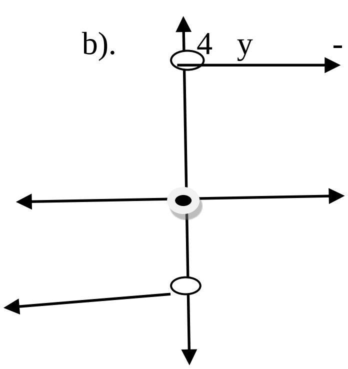
b). f(x) = 4sgn(x) d). f(x) = 3-sgn(x)

Solutions: - 1a).



- Domain = IR
- Range ={1,0,-1}
- Odd function
- S.w.r.t the origin

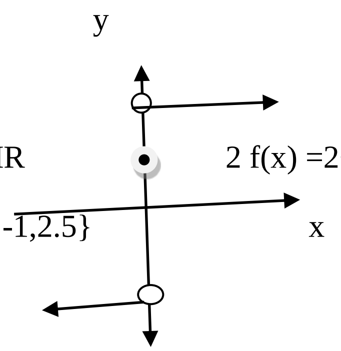
b).



- Domain =IR
- Range ={-4,4,0}
- odd function

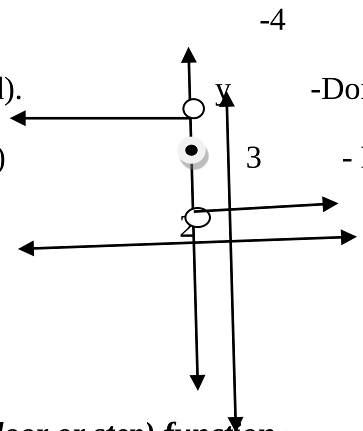
f(x) =4sgn(x) -S.w.r.t the origin

c).



- domain = IR
- Range = {-1,2.5}
- No parity
- No symmetry

d).



- Doman = IR
- Range ={2,3,4}
- No parity
- No symmetry

<u>Greatest integer (floor or step) function.</u>

Definition: - The greatest integer function denoted by $\lfloor x \rfloor$ & given by f(x) = $\lfloor x \rfloor$, is defined as the greatest integer less than or equal to x. i.e $\lfloor x \rfloor \leq x$. Or (a function f: IR→ z , $defined$ by f(x) = $\lfloor x \rfloor$. Domain = IR , Range = z

❖ The greatest integer function (GIF) rounds off the given number to the nearest integer.

Examples: - 1. Evaluate each of the following.

a). $\lfloor 2.7 \rfloor$ = 2 b). $\lfloor -2.7 \rfloor$ = -3 c). $\lfloor \pi \rfloor$= 3 d). $\lfloor -\pi \rfloor$= -4 e). $\lfloor 7 \rfloor$=7 f). $\lfloor -\sqrt{2} \rfloor$= -2

g). $\lfloor -7 \rfloor$= -7 h). $\lfloor -0.09 \rfloor$ = -1 i). $\lfloor 0.35 \rfloor$= 0 j). $\left\lfloor \frac{23}{25} \right\rfloor$ = 0 k). $\left\lfloor -\frac{23}{25} \right\rfloor$= -1

<u>Equation Involving Greatest Integers</u>

❖ For any real number x & integer a, the equation $\lfloor x \rfloor$-a =0 is called the greatest integer equation.

❖ If $\lfloor x \rfloor$ =a , if a$\leq x < a+1$, where a is an integer.

Examples: - 1. find the solution sets for each of the following equations.

a). $\lfloor x \rfloor$= -4 -4 $\leq x < -4+1$

= -4$\leq x < -3$

∴S. S= {x: -4$\leq x < -3$ }, x ∈ IR

= **<u>x ∈ [−4, −3), *X ϵ* IR</u>**

b). $\lfloor 2x+3 \rfloor$= -11

= -11$\leq 2x+3 < -11+1$

= -14$\leq 2x+3-3 < -10-3$

= $\frac{-14}{2} \leq \frac{2x}{2} < \frac{13}{2}$

= -7$\leq x < \frac{-13}{2}$

c). $\lfloor 1+\lfloor x \rfloor \rfloor$ = 5

5$\leq 1+\lfloor x \rfloor < 6$

$4 \leq \lfloor x \rfloor < 5$

Since $\lfloor x \rfloor$ is the greatest integer Less than or equal to x & $\lfloor x \rfloor$ $is\ always$ an integer

$4 \leq \lfloor x \rfloor < 5 = 4 \leq x < 5$

S.S $= \{x: 4 \leq X < 5\}$, x $\in IR$

x $\in [4,5), X \in R$

S.S = $\{x: -7 \leq x < \frac{-13}{2}\}$

$\underline{x \in [-7, \frac{-13}{2})}$, x$\in IR$.

d). $\lfloor x - 3 \rfloor = 7$

$10 \leq x < 11$

S.S = $\{x: 10 \leq x < 11, x \in IR\}$

$=$ x$\in [10,11), X \in IR$.

Properties of Greatest Integer Function

If n is integer & x ϵ IR between n & n+1, then

a). $\lfloor -n \rfloor$ = - $\lfloor n \rfloor$

b). $\lfloor x + n \rfloor$= $\lfloor x \rfloor$ +n

c). $\lfloor x \rfloor$ = x, if x $\in z$

d). $\lfloor -x \rfloor = \begin{cases} -\lfloor x \rfloor, if\ x \notin z \\ -\lfloor x \rfloor - 1, if\ x \notin z \end{cases}$

e). $\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} -1, if\ x\ x \epsilon z \\ 0, if\ x\ x \epsilon z \end{cases}$

f). $\lfloor x \rfloor$ =a, if a$\leq x < a + 1, a \in z$

g). $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$ or $\lfloor x \rfloor + \lfloor y \rfloor$ +1, y $\in IR$

Graphs of Greatest Integer Function

Examples: - 1. Draw the graphs of the following G.I.F

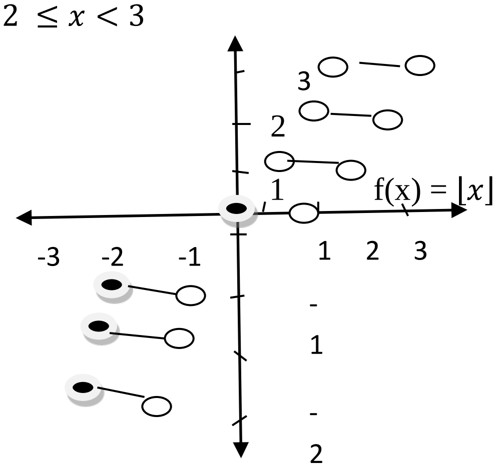
a**)**.f(x) = $\lfloor x \rfloor$, for -3 $\leq x \leq 3$

b). f(x) = 3$\lfloor x \rfloor$, for -2$\leq x < 3$

c). f(x) = $\lfloor x + 1 \rfloor, -2 \leq x < 3$

d). f(x) = $\lfloor 2x \rfloor, -2 \leq x < 3$

Solutions: -1a) $\lfloor x \rfloor = \begin{cases} -3\ , if\ -3 \leq x < -2 \\ -2, if\ -2 \leq x < -1 \\ -1, if\ -1 \leq x < 0 \\ 0, if\ 0 \leq x < 1 \\ 1, if\ 1 \leq x < 2 \\ 2, if\ 2 \leq x < 3 \\ 3, if\ 3 \leq x < 4 \end{cases}$

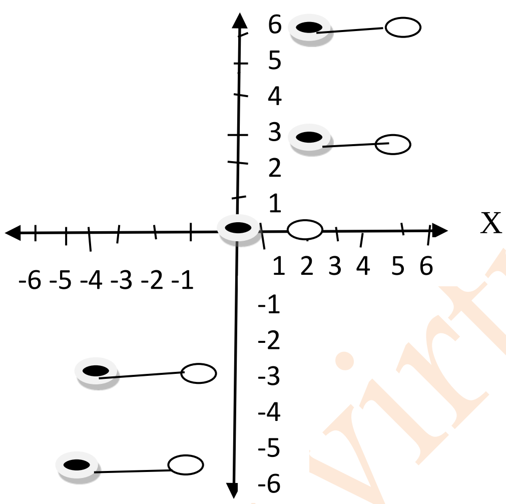


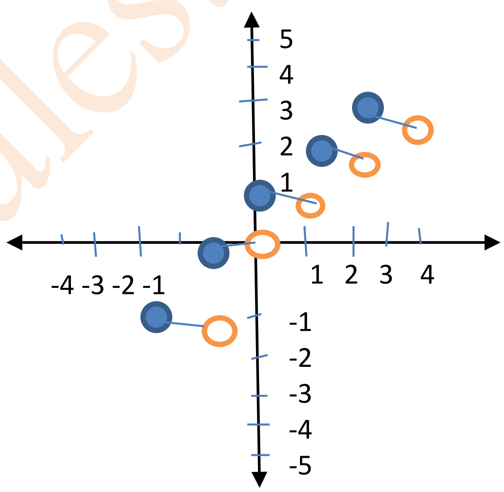
b).

x	-2≤ $x < -1$	-1≤ $x < 0$	0≤ $x < 1$	1≤ $x < 2$	2≤ $x < 3$
y=$\lfloor x \rfloor$	-6	-3	0	3	6

c). f(x) =$\lfloor x + 1 \rfloor$,-2≤ $x < 3$

x	-2≤ $x < -1$	-1≤ $x < 0$	0≤ $x < 1$	1≤ $x < 2$	2≤ $x < 3$
y=$\lfloor x + 1 \rfloor$	-1	0	1	2	3





Composition of Function

Definition: - let f: A→ B & $g: B \to C$ be two functions, then the composition og g by f, denoted by gof is the function, defined by:

gof: A→C: (gof)(x) =g(f(x)), for all x ϵ A.

- ✓ Domain of gof C Domain of f

- ✓ Domain of fog C domain of g
- ✓ gof is defined only when range of f C domain of g.
- ✓ if Range of f n domain of g = ∅ , then gof doesn't exist

Examples: - i. For each of the following functions find a). (fog)(x), (gof)(x) ,(fof)(x) ,(gog)(x) , if 1). F(x) = x+3 & g(x) = 4x+7

Solution: -1a). **(fog)(x)** =f(g(x) =f(4x+7) **(gof)(x)** =g(f(x)) = g(x+3)

=4x+7+3 = 4(x+3) +7

= **4x+10** = 4x+12+7

= **4x+19**

(fof)(x) =f(f(x)) = f(x+3) **(gog)(x)** = g(g(x)) = g(4x+7)

= x+3+3 = 4(4x+7) +7

= **3+6** = **16x+35**

2). Let f(x) = $\sqrt{x}$ & g(x) =2x-5

- **f(g(x))** =f(2x-5) = $\boldsymbol{\sqrt{2x-5}}$
- **g(f(x))** =g($\sqrt{x}$) $= \boldsymbol{2\sqrt{x}-5}$
- **f(f(x)) =f(**$\sqrt{x}$) $= \sqrt{\sqrt{x}}$
- **g(g(x))** = g(2x-5)

= 2(2x-5)-5

= 4x-10-5

= **4x-15**

3). F(x) = x^2 -1 & g(x) =$|x|$

f(g(x)) = f($|x|$) = $\boldsymbol{|x|^2-1}$

g(f(x)) =g(x^2-1) = $\boldsymbol{|x2-1|}$

f(f(x)) = f(x^2-1) = $(x^2-1)^2$ -1

= x^4-2x^2+1-1

= $\boldsymbol{x^4-2x^2}$

4). F= {(0,1),(2,4).(3,-1} & g= {(-1,2) ,(1,3),(4,5)

- f(g(x)) = f(g(-1) = f(2) =4
- f(g(1))= f(3) = -1

-f(g(4))= f(5) – doesn't exist

∴ fog= **{(-1,4),(1,-1)}**

5). f(x) =$\frac{3}{2x-1}$ & g(x) = $\frac{x-1}{x+1}$

- f(g(x))= $f(\frac{x-1}{x+1}) = \frac{3}{2\left(\frac{x-1}{x+1}\right)-1} = \frac{3}{\frac{2x-2}{x+1}-1} = \frac{3(x-1)}{x-3} = \boldsymbol{\frac{3x-3}{x-3}}$//

- g(f(x)) =$g(\frac{3}{2x-1}) = \frac{\frac{3}{2x-1}-1}{\frac{3}{2x-1}+1} = \frac{\frac{3-2x+1}{\cancel{2x-1}}}{\frac{3+2x-1}{\cancel{2x-1}}} = \frac{4-2x}{2x+2} = \boldsymbol{\frac{2-x}{x+1}}$//

-f(f(x)) = $f(\frac{3}{2x-1}) = \frac{3}{2\left(\frac{3}{2x-1}\right)-1} = \frac{3}{\frac{6-2x+1}{2x-1}} = \frac{3(2x-1)}{7-2x} = \boldsymbol{\frac{6x-3}{-2x+7}}$//

- g(g(x)) = $g(\frac{x-1}{x+1}) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{\frac{x-1-x-1}{\cancel{x+1}}}{\frac{x-1+x+1}{\cancel{x+1}}} = \frac{-2}{2x} = \boldsymbol{\frac{-1}{x}}$//

ii. For each of the following functions find the unknown functions.

a). let (fog) (x) = 44x +5 & g(x) = 2x+5., since (fog)(x) linear & g(x) is linear, f(x) must be linear.

let f(x) = ax +b, a, b $\in IR$ & $a \neq 0$.

-f(g(x)) =44x+5 a=22 & 5(22) +b =5

f(2x+5) = 44x + 5 a= 22 &110 +b =5

a(2x+5) +b = 44x+5 a= 22 & B= -105

2ax+5a+b= 44x+5 $\therefore$ f(x)= ax +b = <u>**22x-105**</u>

2a=4 & 5a+b =5

b). (fog)(x) = $3x^2$+7x+4 & g(x) = 2x+5, since (fog)(x) is quadratic (second degree) & g(x) is linear, f(x) must be quadratic (second degree)

let f(x) = ax^2+ b x +c, a, b, c $\in IR$ & $a \neq 0$.

= f(g(x)) = $3x^2+7x+4$ ==> 15+2b =7 and 25a+5b+c =4

= a(2x+5) = $3x^2+7x+4$ = 2b= -8 = $25.\frac{3}{4}+5(-4)+c=4$

= $a(2x+5)^2 + b(2x+5) + c = 3x^2+7x+4$ <u>**b= -4**</u> = $\frac{75}{4}-20+c=4$

= $a(4x^2+20x+25)+2bx+5b+c = 3x^2+7x+4$ = $\frac{75-80}{4}+c=4$

= $4ax^2+20ax+25a+2bx+5b+c = 3x^2+7x+4$ = $\frac{5}{4}+c=4$ $c=4+\frac{5}{4}$

=4a= 3, 20a+2b =7 & 25a+5b+c =4 $c=\frac{21}{4}$

$a=\frac{3}{4}$ & $20.\frac{3}{4}+2b=7$ $\therefore f(x) = ax^2+bx+c = \frac{3}{4}x - 4x + \frac{21}{4}$

c). (fog)(x) = $18x^2+21x+6$, f(x) =$2x^2+3x+1$, since fog is quadratic & f is quadratic, then g must be linear

➔ let g(x) = ax +b, a, b ϵ IR & a $\neq$ 0.   = $2a^2$=18   4ab+3a =21   **b = 1**   when a = -3

F(g(x)) = $18x^2+21x+6$ = a^2=9   when a=3   = 4(-3) (b)+3(-3) = 21

= $2(ax +b)^2$ +3(ax=b) +1 = $18x^2+21x+6$ **a= $\pm$ 3//**   4(3)(b) +3(3) =21   = -12b-9 =21

= $2(a^2x^2+2abx+b^2)$ +3ax+3b+1 =$18x^2+21x+6$ = 12b + 9 =21   = -12b= 30

$2a^2x^2+4abx+2b^2+3ax+36+1 = 18x^2+21x+6$ = 12b= 21-9   b= $\frac{-30}{12}$ = $\frac{-10}{4}$ = $\frac{-5}{2}$ //

and g(x) = ax +b = 3x+1 or -3x - $\frac{-5}{2}$

iii. for each of the following functions find a). fog, gof, fof, gog b). domain of fog, gof, fof, gog.

1). f(x) = $\sqrt{X}$ & g(x) = 2x-5

Solution: -

f(g(x)) = f(2x-5) = $\sqrt{2x-5}$,   Domain of fog= {x: x≥ $\frac{5}{2}$}

g(f(x)) = g ($\sqrt{x}$) = 2$\sqrt{x}$-5,   Domain of gof = {x: x $\geq$ 0}

f(f(x)) = f ($\sqrt{x}$) = $\sqrt{\sqrt{x}}$,   Domain of fof = {x: x $\geq$ 0}

g(g(x)) =g(2x-5) =2(2x-5) -5 = 4x-10-5 = 4x-15, Domain of gog = {x:x ϵ IR}

2). f(x) = $\sqrt{x-1}$ & g(x) = $\sqrt{2-x}$

f(g(x)) = f ($\sqrt{2-x}$) = $\sqrt{\sqrt{2-x}}$-1,   Domain of fog = {x: x≤ 1}

g(f(x)) = g ($\sqrt{x-1}$= $\sqrt{2-\sqrt{x-1}}$,   Domain of gof ={x: x≤ 5 $\cap$ $x \geq 1$}

3). f(x) = $\frac{x+2}{x-3}$ & g(x) = $\frac{x-1}{x+5}$

f(g(x)) = f ($\frac{x-1}{x+5}$) = $\frac{\frac{x-1}{x+5}+2}{\frac{x-1}{x+5}-3}$ = $\frac{x-1+2x+10}{x-1-3x-15}$ = $\frac{3x+9}{-2x-16}$,

Domain of fog ={x: x≠ -5 and $g(x) \neq 3$} = {x:x≠ -5 & $x \neq -8$}= IR{-5,-8)

$g(f(x)) = g\left(\frac{x+2}{x-3}\right) = \frac{\frac{x+2-1}{x-3}}{\frac{x+2}{x-3+5}} = \frac{x+2-x+3}{x+2+5x-15} = \frac{5}{6x-13}$, Domain of gof $=\{x: x\neq 3 \ \& \ f(x) \neq -5\}$

$= \{x: x\neq 3 \ \& \ x \neq \frac{13}{6}\} = IR \ \{3, \frac{13}{6}\}$//

Inverse Function

Definition: - The inverse of a function udo or reverse the result of the given function.

- The inverse of a given function $f:A\rightarrow B$ is denoted by f^{-1} & is defined by $f^{-1}: B\rightarrow A$.
- The given function f is said to be invertible if f is one-to –one or $f^{-1}$is a function.
- If f and g are inverse of each other i.e. if f and g are invariable functions, then.

  a). $(fog)^{-1} = g^{-1}o f^{-1}$ c). $f(g(x)) = x$ & $g(f(x)) = x$ I (which is called the identity function).

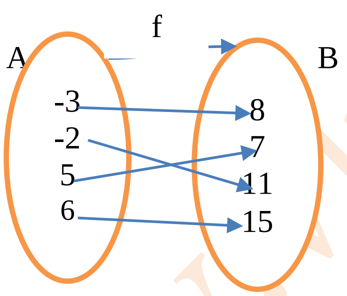
  b). $(gof)^{-1}= f^{-1} o g^{-1}$ $\forall x \in g$ & $\forall x \in f$

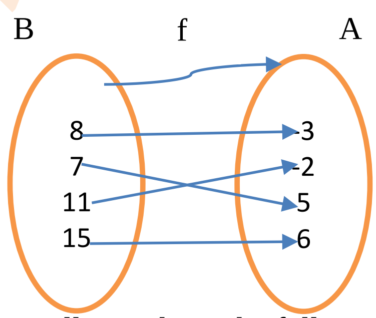
d). Domain of f= Range of g and Range of f= domain of g.

$f(x) = y => f^{-1}(y) = x$

Examples: - 1. Find the inverse of each of the following functions.

a). f= {(2,3) ,(4,5) ,(-2,7) f^{-1} = {(3,2) ,(5,4) ,(7,-2)}.

b) 



To find the inverse a given function algebraically, we have the following steps.

1. Put f(x)= y
2. Interchange x and y
3. Solve for y
4. Put the last y as $f^{-1}(x)$

c). $f(x) = 2x+3$

Solution

$= y = 2x+3$

$= x = 2y+3$

$= x-3=2y$

$= \frac{x-3}{2} = y = f^{-1}(x)$

d). $f(x) = x^2+7$

$= y= x^2+7$

$= x=y^2+7$

$= x-7 = y^2$

$y= f^{-1}(x) = \pm \sqrt{x-7}$

e). $f(x) = \frac{3x}{2x+5}$

$y = \frac{3x}{2x+5}$

$x = \frac{3y}{2y+5}$

$=2xy+5x = 3y$

$= 2xy-3y = -5x$

$y = f^{-1}(x) = \frac{-5x}{2x-3}$//

f). $f(x) = \frac{3^x - 3^{-x}}{2}$

$y = \frac{3^x - 3^{-x}}{2}$

$x = \frac{3^y - 3^{-y}}{2}$

$2x = 3^y - 3^{-y}$

$2x = 3^y - 3^{-y}$

$2x = 3^y - \frac{1}{3^y}$

$2x = \frac{3^{2y-1}}{3^y}$

$2x.3^y = 3^{2y} - 1$ let $(3^y) = k$

$2x.k = k^2 - 1$

$k^2 - 2kx - 1 = 0$

$k = \frac{2x \pm \sqrt{4x^2+4}}{2}$

$k = \frac{2x \pm \sqrt{4(x^2+4)}}{2}$

$k = \frac{2x \pm 2\sqrt{x^2+1}}{2}$

$k = x \pm \sqrt{x^2+1}$

$k = 3^y = x \pm \sqrt{x^2+1}$

$\log_3 3 = \log_3(x \pm \sqrt{x^2+1}$

$y = f^{-1}(x) = \mathbf{\log_3(x \pm \sqrt{x^2+1}}$//

- The graphs of the given function and its inverse are mirror image (reflection) of each other along the line y =x

Application of Relations & Functions

Examples: - 1. Let p+ 3q =30 be an equation involving two variables p(price) and q(quantity). Indicate the meaningful domain and range of this function, when.

a). The is considered as an independent variable.

Solution: - a) price p is an independent variable - q must be the dependent variable; we can express the dependent variable of interims of the independent variable p.

$3q = -p+30$

$Q(p) = \frac{-p}{3} + 10$, since price is non-negative $\frac{-p}{3} + 10 \geq 0$

$\frac{-p}{3} \geq -10$

$-p \geq -30$

$P \leq 30$, Domain = {p: $0 \leq p \leq 30$} Range = {q: when p= 0 & when p=30}

= **{q: 0≤ $q \leq 10$}**

b). when quantity (q) is an independent variable

3q+p =30

P(q) = -3q+30

-3q+30≥ 0

-3q≥ -30 , $q \leq 10$, Domain = **{q:0≤ $q \leq 10$}** and Range = {**p: 0≤ $p \leq 30$}**

2). A car rental company charges an initial fee, which is also called a flat fee of birr 300 & an additional birr 15 per kilometer to rent a van.

a). write a function that approximates the cost (in birr) in terms of x, the number of kilometers driven.

b). How much would an 80 km trip cost?

Solution: - 2a) fixed cost = birr 300.

Total cost (y) = variable cost x number of kms + fixed cost

y=15x+300

b). when x= 80 km, y= 15x+300

= 1200 =300

y =1500 birr

3). A firm produces an item whose production cost function is C(x) =200+60x, where x is the number of items produced.

a). if entire stock is sold at a price of each item which is birr 800, then determine the revenue function.

Solution: - The revenue function is given by R(x) =800x.

b). if the total number of items produced was 1000 and entire stock is sold at a price of each item which is birr 800, find the profit of the firm.

Solution: - The profit the firm is profit= revenue – total cost

=800 x 1000 – (200+60 x 1000)

= 800,000 -60200

= **739800 birrs**

Review exercise on unit 1

Q3). find the inverse of each of the following functions.

a). $g(x) = x^3+5$ b). $h(x) = \frac{2x+3}{4x-5}$ c). $f(x) = -2\ x\ \frac{5}{4}$

Solutions: - 3a). $g(x) = x^3+5$ b). $h(x) = \frac{2x+3}{4x-5}$ c). $f(x) = 2\ x\ \frac{5}{4}$

$y = x^3+5$ $y = \frac{2x+3}{4x-5}$ $y = -2\ x\ {}^5/_4$

$x = y^3+5$ $x = \frac{2y+3}{4y-5}$ $x = -2y\ x\ {}^5/_4$

$x-5 = y^3$ $4xy-5x= 2y+3$ ${}^{-x}/_2 = y^{5/4}$

$g^{-1}(x)= y= \sqrt[3]{\boldsymbol{x-5}}$// $4xy-2y =5x+3$ $({}^{-x}/_2)^{4/5} = f^{-1}(x)$

$y= h^{-1}(x)= \frac{\boldsymbol{5x+3}}{\boldsymbol{4x-2}}$ // $f^{-1}(x)= \sqrt[5]{(\frac{-x}{2})^4}$//

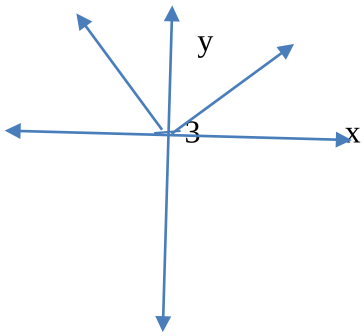
Q4). Given $f(x) = 3+2|x|$, then find

a). $f(-5) = 3+2|-5|= 3+10 =$ **13**

Q5). Given f(x) = 2-3sgn(x) , then find:

a). f(-3) = 2-3sgn(-3)

b). domain of f(x) ={x: xϵ IR) and Range of f(x) = { y: y≥ 3} = 2-3(-1)



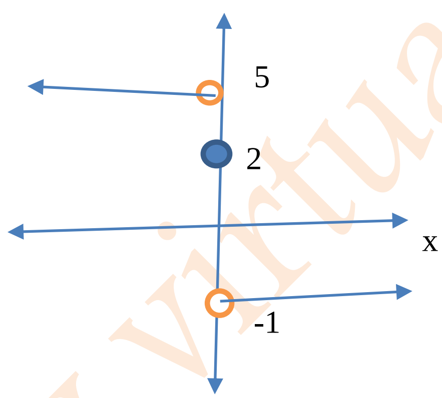
& f $(\frac{1}{2})$ = 2-3sgn $(\frac{1}{2})$

y=3+2$|x|$ = 2-3x1 = **-1**

domain of f(x) = {x: x ϵ IR}

Range of f(x) = {y: y=2, -1,5}

= {-1,2,5}



Q6). Determine if each pair of the following functions are inverse of each other or not.

b). f(x) = $\frac{2x+3}{x-1}$ & g(x) = $\frac{x+3}{2x+1}$, using composition of functions , we can check whether the given pairs are inverse each other or not.

f(g(x))= f($\frac{x+3}{2x+1}$) = $\frac{2\left(\frac{x+3}{2x+1}\right)+3}{\frac{x+3}{2x+1}-1} = \frac{2x+6+6x+3}{x+3-2x-1} = \frac{8x+9}{-x+2} \neq x$

∴ f & g are not inverse each other.

c). f(x) =2x-3 & g(x) =3-2x

= f(g(x)) = f(3-2x) = 2(3-2x)-3 = 6-4x-3 = -4x+3$\neq x$

∴f & g are not inverse each other.

a). f(x) = x-1 and g(x) = x+1

f(g(x)) = f(x+1) = x+1-1 =x

∴f & g are inverse each other.

UNIT TWO

Rational expression

Definition: -A rational expression is an expression that can be written as the quotation of two polynomial expressions $\frac{P(x)}{Q(x)}$,Where Q(x) and P(x) are polynomial &Q(x)≠0.

Examples

1. Which one of the following expressions are rational expressions?

   a). $\frac{x2+1}{\sqrt{2-x}}$ not a rational expression because $\sqrt{2-x}$ is not a polynomial expression.

b). $\frac{x-2}{\sqrt{(x2+1)2}} = \frac{x-2}{|x2+1|} = \frac{x-2}{x2+1}$ it is a rational expression because both the nominator as well as the denominator one polynomials.

c). $|x2+3|$ $=x^2+3$ it is a rational expression.

d). $\frac{\sqrt{x}2}{x2+x-2} = \frac{|x|}{x2+x-2}$ it is not a rational expression because the numerator $|x|$ is not a polynomial.

e). $\frac{x}{x}$ it is a rational expression.

f). 5 it is a rational expression.

g). $\log_{1/4} 64$ it is a rational expression.

h). $\frac{x2-x}{\sqrt{x4+4x2+4}} = \frac{x2-x}{\sqrt{(x2+2)2}} = \frac{x2-x}{x2+2}$ it is a rational expression.

Rational function

Definition :- A rational function is a function that can be expressed as the quotation of two polynomial functions f(x) $=\frac{P(X)}{Q(X)}$,Where $Q(x) \neq 0$ and P(x) & Q(x) are a polynomial functions.

Example 1:-which one of the following functions are rational functions ?

A. $f(x) = \frac{x2-1}{x+2}$ B. $f(x) = \frac{|x2+3|}{\sqrt{(x2+1)2}}$

Solutions

a). since both x^2-1 & x + 2 are polynomials the expression $\frac{x2-1}{x+2}$ is a rational expression.

b). $\frac{|x2+3\,|}{\sqrt{(x2+1\,)2}} = \frac{|x2\,+3\,|}{|x2+1\,|} = \frac{x2+3}{x2+1}$ is a rational expression & it is a rational function.

c). $\frac{|x4+1\,|}{\sqrt[3]{27\,x3}} = \frac{x4+1}{3x}$ it is a rational expression & it is a rational function.

Evaluating Rational Expression (Function)

Examples

a). if $f(x) = \frac{ax2-17}{x-2a}$,then find 'a' such that f (1)=5

Solution

Since $f(x) = \frac{ax2-17}{x-2a}$ $\Rightarrow$ f(1) = 5 $\Rightarrow$ $\frac{a-17}{1-2a} = 5$

5-10a = a- 17

-11a = -22

a = 2

b). if $f(x) = \frac{ax+b}{x-1}$, find a and b

such that f(-1)=3 & f(2) =6

Solution

f (-1)= 3 $\Rightarrow$ $\frac{-a+b}{-1-1}$ = $\Rightarrow$ $\frac{-a+b}{-2}$ = 3 $\Rightarrow$ b- a = -6

and f (2) =6 $\Rightarrow$ $\frac{2a+b}{2-1}$ = 6 2a +b =6

$\Rightarrow$ 2 $\begin{cases} -a + b = -6 \\ 2a + b = 6 \end{cases}$

$\begin{cases} 2a + b \\ -2a + 2b = -12 \end{cases}$

3b= -6

b = -2

= 2a+ b =6

2a- 2=6

2a=8

a =4

c). find the value of x which gives 2,5,6, for the function f(x) = $\frac{x}{x-1}$

Solution

c). f(x) =2 $\rightarrow \frac{x}{x-1}$ =2 $\rightarrow$ 2x- 2 =x

2x-x =2

X=2

f(x)=5 $\rightarrow \frac{x}{x-1}$ = 5 $\rightarrow$ 5x -5 =x

5x –x= 5

4x =5

X=$\frac{5}{4}$

f (x)=6 $\rightarrow \frac{x}{x-1}$ = 6 $\rightarrow$ 6x -6 =x

6x –x= 6

5x =6

X=$\frac{6}{5}$

Domain and range of rational expression /functions /

The domain of a rational expression /functions includes all real numbers except those that cause the denominator equal to zero.

To find the domain of a given rational expression /function /.

® Set the denominator equal to zero.
® Solve to find the value (s) of the variable (s) that make the denominator equal to zero.
® The domain is all real numbers except those zeros of the denominators.

Range of rational expressions /functions /

Rules

Rule 1. Range of $\frac{constant}{linear}$ $=|R|$ $\{0\}$

Equation $\frac{2}{2x-3}$, Range $=|R|$ $\{0\}$.

Range of $\frac{-5}{7x-6}$ =is $|R|$ $\{0\}$

Rule 2: - Range of $\frac{linear}{linear}$ $=\frac{ax+b}{cx+d}$, a, c $\neq 0$ is $|R|$ $\{\frac{a}{c}\}$

Examples:- range of $\frac{2x+3}{x+1}$ is $|R|$ $\{\frac{2}{1}\}$

Range of $\frac{10x-12}{8x-11}$ is $|R|$ $\{\frac{10}{8}\}$

Rule 3:-Range of $\frac{linear}{qudratic}$, $\frac{qudratic}{quadratic}$, $\frac{qudratic}{linear}$

Can be found using cross multiplication & solve the variables as quadratic in terms $\frac{x}{y}$ & check b^2 -4ac $\geq$ 0.

Examples: - 1. Find the domain & range of the following.

a). f(x) = $\frac{x2+2x-11}{2x-6}$

Solution: -

$\frac{y}{1} = \frac{x2+2x-11}{2x-6}$ == > $y(2x-6)=x^2 + 2x - 11$

$2xy-6y = x^2+2x - 11$

$X^2 - 2xy + 2x + 6y - 11 = 0$

$X^2 - 2y\,x + 2x + 6y - 11 = 0$

$X^2 - (2y-2)x + 6y - 11 = 0$

$X^2 - (2y-2)x + 6y - 11 = 0$

$a=1$, $b=-(2y-2)$, $c=6y-11$

$b^2 - 4ac \geq 0$

$(-(2y-2))^2 - 4(1)(6y-11) \geq 0$

$(2y-2)^2 - 24y + 44 \geq 0$

$4y^2 - 8y + 4 - 24y + 44 \geq 0$

$4y^2 - 32y + 48 \geq 0$

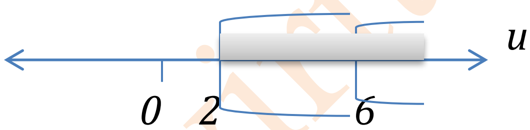
$4(y^2 - 8y + 12) \geq 0$

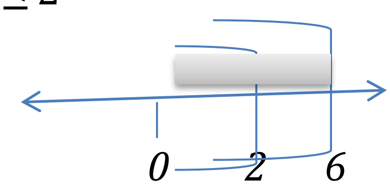
$(y-6)(y-2) \geq 0$

Case 1 **Case 2**

$y-6 \geq 0$ & $y-2 \geq 0$ or $y-6 \leq 0$ & $y-2 \leq 0$

$y \geq 6$ & $y \geq 2$ or $y \leq 6$ & $y \leq 2$





Range = $\{y : y \leq 2 \text{ or } y \geq 6\}$

b). Domain of $\frac{X2+5X+6}{X2-9}$ is $|R| \{-3,3\}$

Domain of $\frac{x2+2x-11}{2x-6}$ is $|R| \{3\}$

c). Domain of $\frac{x2-3x+4}{x2+3x+4}$ is $|R|$

Operation with Rational Expressions /Functions /

Addition, subtraction, multiplication & division of rational expression /functions follow some basic rules as addition, subtraction, multiplication & division of rational numbers.

Let P(x), S(x), Q(x) & R(x) be polynomials such that S(x) ≠ 0,Q(x) ≠ 0 R(x)≠ 0 , then.

<u>The sum</u>: $\frac{P(X)}{Q(X)} + \frac{R(X)}{S(X)} = \frac{P(X).S(X)+ R(X).Q(X)}{Q(X).\ S(X)}$ is a rational expression.

The difference: $\frac{\boldsymbol{P(X)}}{\boldsymbol{Q(X)}}$ - $\frac{\boldsymbol{R(X)}}{\boldsymbol{S(X)}} = \frac{\boldsymbol{P(X).S(X)-R(X).Q(X)}}{\boldsymbol{Q(X).S(X)}}$ is a rational expression

The product: $\frac{P(X)}{Q(X)} . \frac{R(X)}{S(X)} = \frac{P(X).R(X)}{Q(X).\ S(X)}$ = is a rational expression

<u>The quotient:</u> $\frac{P(X)}{Q(X)} \div \frac{R(X)}{S(X)} = \frac{P(X).\ S(X)}{Q(X).\ R(X)}$ is a rational expression

Examples: -

1. Perform the following operations and set their domain.

a). $\frac{2x-3}{x2+5x} + \frac{3x-5}{x2+5x} = \frac{2x-3+3x-5}{x2+5x} = \frac{5x-8}{x2+5x}$, Domain =|$R$| {0, -5}

b). $\frac{x-4}{x2-9} + \frac{x+2}{x2+11x+24} = \frac{x-4}{(x-3\ )(x+3)} + \frac{x+2}{(x-3\ )(x+8)} = \frac{(x-4)(x+8)+(x+2)(x-3)}{(x-3)(x+3)(x+8)}$

$= \frac{2x2+3x-38}{(x-3)(x+3)(x+8)}$, $x \neq \pm$ 3, -8

c). $\frac{6x-12}{x2+3x-10} - \frac{x-2}{x-3} = \frac{(6x-12\ )(x-3\ )-(x-2)(x-2)(x+5)}{(x-2)(x+3)(x-3)}$

$= \frac{\boldsymbol{x3+5x2-14\ x+16}}{\boldsymbol{(x-2)(x+3)(x-3)}}$ **,** x≠ $\pm$ 3, 2 d). $\frac{x2-x-6}{3x2-12} \div \frac{x2-3x}{2-x}$

$= \frac{(x+2)(x-3)}{3(x-2)(x+2)} \div \frac{x(x-3)}{-(x-2)}$

$= \frac{\cancel{(x+2)}\cancel{(x-3)}}{3\cancel{(x-2)}\cancel{(x+2)}} - \frac{\cancel{(x-2)}}{x\cancel{(x-3)}}$

$= \frac{(x+2)}{3x(x+2)} = \frac{\mathbf{-1}}{\mathbf{3x}}$, x≠ $\underline{+}$ 2,0

e). $\left(\frac{x3+1}{x-2} \div \frac{x2-x+1}{x-2}\right) \cdot \frac{1}{x+1}$

$= \left(\frac{\cancel{x+1}\cancel{(x2-x+1}}{\cancel{x-2}} \cdot \frac{\cancel{x-2}}{x2-\cancel{x+1}}\right) \cdot \frac{1}{\cancel{x+1}}$

= $\underline{1}$, x≠ 2, -1

Simplification of rational expressions

Definition: - a rational expression is said to be reduced to lowest terms (in simplified) form, if the numerator & denominator have no common factor except -1 & 1.

Examples: - 1. Simplify the following rational expressions & state the universe.

a). $\frac{(x-1)(x+2)}{x-3}$, Domain =$|R|$ {3} b). $\frac{x2+5x+6}{x2+3x+2} = \frac{(x+3)\cancel{(x+2)}}{(x+1)\cancel{(x+2)}} = \frac{\mathbf{x+3}}{\mathbf{x+1}}$**,** x≠ -2, -1

c). $\frac{x^3-27}{x4+3x3-27x-81} = \frac{x^3-3^3}{x^3(x+3)-27(x+3)}$

$= \frac{(x-3)(x^2+3x+9)}{(x^3-27)(x+3)}$

$= \frac{\cancel{(x-3)}\cancel{(x^2+3x+9)}}{\cancel{(x-3)}\cancel{(x^2+3x+9)}(x+3)} = \frac{\mathbf{1}}{\mathbf{x+3}}$, x≠3, -3 d). $\dfrac{x^3+x^2-4x-4}{x^5-3x^4-5x^3+15x^2+4x-12} \div \dfrac{x^3-4x^2+x+6}{x^3-2x^2+5x-24}$

$$= \frac{x^2(x+1)-4(x+1)}{x^4(x-3)-5x^2(x-3)+4(x-3)} \div \frac{x^3-4x^2+x+6}{x^3-2x^2+5x-24}$$

$$= \frac{(x^2-4)(x+1)}{(x^4-5x^2+4)(x-3)} \div \frac{(x-2)(x-3)(x+1)}{(x-3)(x^2+x+8)}$$

$$= \frac{\cancel{(x-2)}\cancel{(x+2)}\cancel{(x+1)}}{(x-1)\cancel{(x+1)}\cancel{(x-2)}\cancel{(x+2)}(x-3)} \div \frac{(x-2)(x-3)(x+1)}{(x-3)(x^2+x+8)}$$

$$= \frac{1}{(x-1)\cancel{(x-3)}} \cdot \frac{\cancel{(x-3)}(x^2+x+8)}{(x-2)(x-3)(x+1)} = \frac{\mathbf{x^2+x+8}}{\mathbf{(x-1)(x-2)(x-3)(x+1)}}, x \neq \pm 2, -1$$

e). simplify the following rational expressions.

a). $\dfrac{x-1}{x+2} + \dfrac{x+3}{x-4}$

$$= \frac{(x-1)(x-4)+(x+3)(x+2)}{(x+2)(x-4)}$$

$$= \frac{x^2-4x-x+4}{(x+2)(x-4)}$$

$$= \frac{x^2+5x+9}{(x+2)(x-4)}$$

$$= \frac{\mathbf{2x^2+13}}{\mathbf{(x+2)(x-4)}}$$

b). $\dfrac{x-1}{x+2} \div \dfrac{x+3}{x-4}$

$$= \frac{x-1}{x+2} \cdot \frac{x-4}{x+3}$$

$$= \frac{\mathbf{(x-1)(x-4)}}{\mathbf{x+2)(x+3)}}$$

Decomposition of rational expressions in to partial fractions

Fraction: - an expression of the form $\frac{P(X)}{Q(X)}$,Where P(x) & Q(x) are polynomials with Q(x)$\neq$ 0

We have two types of fractions.

a). **proper fraction**: -a fraction in which the degree of the numerator is less than the degree of the denominator.

Example: - $\frac{x+2}{x^2+2}$, $\frac{x^3+4x^2+6x+1}{x^4+5x^2+10x+10}$

b). **Improper fraction**: -a fraction in which the degree of the numerator is greater than or equal to the degree of the denominator.

Example: - $\frac{x^2+5}{x^2+5x+4}$, $\frac{x^2+4}{x+2}$, $\frac{x^2+4x+5}{x-1}$ etc…

Decomposition of rational expressions into partial fraction.

Definition: -Two Polynomials P(x) & Q(x) are equal if both polynomials have the same degree & terms of the same degree in both polynomials have equal coefficients.

Example: - 1. Let P(x)=3x^3+5x^2-7x+ c & Q(x)=ax^4+bx^3+5x^2-7x+4, find the values of constants a, b & c if P(x) =Q(x) for all x ∈ IR

P(x)=Q(x), for all x ∈ IR, if, both polynomials have the same degree with the same coefficients.

- ⇨ ax^4+bx^3+5x^2-7x+c=3x^3+5x^2-7x +4
- ⇨ ax^4+bx^3+5x^2-7x+c=0x^4+3x^3+5x^2-7x+4
- ⇨ a**=0. b=3, c=4**

examples: -1. Write the rational expression $\frac{5x^3+7x^2+5x+9}{x^2+1}$ in the form of f(x)+$\frac{r(x)}{q(x)}$, where f(x) & r(x) are polynomial expressions & the degree of r(x) is less than the degree of q(x).

Solution using long division of polynomials.

$$= x^2+1 \overline{\big)\, \cancel{5x^3} + 7x^2 \cancel{+} 5x + 9} \qquad \frac{5x^3}{x^2} = 5x$$
$$\cancel{-5}x^3 + \cancel{5}x$$

$$= x^2+1 \overline{\big)\, \cancel{7}x^2 + 9} \qquad \frac{7x^2}{x^2} = 7$$
$$-\cancel{7}x^2 + 7$$

$= 2$

⇨ r(x) = 2; $\frac{5x^3+7x^2+5x+9}{x^2+1}$ = 5x+7+$\frac{2}{x^2+1}$

⇨ f(x) = 5x+7

Decomposition of Rational Expressions into Partial Fractions

- ✓ is the process of starting with the simplified answer & taking it back a part of "decomposing" the final expression into its initial polynomial fractions?
- ✓ Is a technique used to transform algebraic expressions into sums of simpler expressions called partial fraction decomposition?
- ✓ to decompose a rational expression $\frac{P(x)}{Q(x)}$, the degree of P(x) must be less than the degree of Q(x).
- ✓ if the degree of P(x)≥ the degree of Q(x), we use long division.
- ✓ two polynomial expressions are said to be relatively prime, if their GCD=1

Theorem 2.1:-Any non-constant polynomial expression with real coefficients can be factorized as a product of linear and /or irreducible quadratic factors with the possibility of some powers.

Examples: - 1. Factorize the following polynomials, if possible.

a). x^2 +5x+6 **b).** $x^3 - 4x^2 + 7x - 6$

c). $x^3 + x^2 - 2x$ **d).** x^4+4x^2+4

Solutions

1a). $x^2 + 5x + 6$ ==> find two pairs of integers whose product is 6 & whose sum is 5. They are 2 &3

==> x^2+5x+6 and then split the middle term in to these two numbers.

$= x^2 + (3 + 2)x + 6$

$=x^2+3x+2x+6$

$= x.x+3.x+2.x+6$ ($x^2=x.x$) & (6=2x3)

$= x(x+3) +2(x+3)$ (Take out common factor X & 2 respectively).

$=(x+2)(x+3)$ This is the complete factorization of $x^2 + 5x + 6$

1b). x^3-4x^2+7x-6 using rational root test, we have

$$\begin{array}{c|cccc} 2 & 1 & -4 & 7 & -6 \\ & & 2 & -4 & 6 \\ \hline & 1 & -2 & 3 & 0 \end{array}$$

(Using synthetic division)

==>$x^3 - 4x^2 + 7x - 6 = (x-2)(x^2 - 2x + 3)$ is the complete factorization of $x^3 - 4x^2 + 7x - 6$

1c). $x^3 + x^2 - 2x$

$=x(x^2 + x - 2)$ (Take out x as a common factor)

$=x(x^2+2x-x-2)$

$=x(x(x+2)-1(x+2))$

$= x(x-1)(x+2)$

Therefore: - the complete factorization x^3+x^2-2x is **$(x)(x-1)(x+2)$**

1d). x^4+4x^2+4

It is a quadratic expression.

Let $x^2= k > 0$,

$= x^4+4x^2+4$

$= (x^2)^2 + 4(x^2) + 4$

$= k^2+4k+4$ (find two pairs of numbers, whose product is 4 & whose sum is 4

$= k^2+2k+2k+4$

=k(k+2) +2(k+2)

= (k+2) (k+2) but $k=x^2$

= $(x^2+2)(x^2+2)$ is the complete factorization of x^4+4x^2+4.

Rule sate composes a given rational expression into partial fractions.

Factors in the denominator	Corresponding term in the partial fraction
ax+b, but $(ax+b)^2$ is not a factor. $a\neq 0$ ((DN(x) < DD(x))	$\frac{a}{ax+b}$, a is constant.
$(ax+b)^k$, but $(ax+b)^{k+1}$ is not a factor. $a\neq 0$, $k\in N$	$\frac{a1}{ax+b}+\frac{a2}{(ax+2)2}$ +------- $\frac{ak}{(ax+b)k}$ a1, a2 ---------ak are constant.
ax^2+bx+c, but ax^2+bx+c is not a factor. $a\neq 0$, $b^2-4ac<0$)	$\frac{ax+b}{ax2+bx+c}$, a, b are constants.
$(ax^2+bx+c)^k$, but $(ax^2+bx+c)^{k+1}$ is not a factor. $a\neq 0$ ($b^2-4ac<0$), $k\in N$	$\frac{a1x+b1}{ax^2+bx+c}+\frac{a2x+b2}{(ax^2+bx+c)^2}$ +----+ $\frac{akx+bk}{(ax^2+bx+c)^k}$ A1, a2,…… ak & b1 ,b2 ……bk are constants

Examples: - 1. Decompose the following rational expressions into partial fractions.

a). $\frac{x+1}{x^2+4x+3}$ **b).** $\frac{2x-3}{x^2+x-2}$ **c)**. $\frac{x+2}{x^2+6x+9}$ **d).** $\frac{x-1}{x^2-4x+4}$

e). $\frac{x^2+3x+5}{(x+2)(x^2+x+1)}$ **f)**. $\frac{6x^2+5x+2}{(x^2+2x+3)^2}$ **g)**. $\frac{x^3+7x+1}{x^2+3x+2}$ **h).** $\frac{x^3}{(x+5)(x+6)}$

i). $\frac{3x-5}{(x-2)^3}$ **j).** $\frac{5x+6}{(x^2-2)(x-2)}$ k). $\frac{6x^2-14x-27}{(x+1)(x-1)^2}$ l). $\frac{x^3+2x+1}{x^2+x-2}$

m). $\frac{42-14x}{x^4-4x^3-x+4}$ **n)**. $\frac{2x^3+x^2-7x+7}{x^2+x-2}$ **o**. $\frac{-x^2+2x+2}{x(x+1)^2}$ **p).** $\frac{4x^2+3x+8}{(x+1)(x^2+2)}$

q). $\frac{x^2-1}{(x+1)(x^2+x+1)}$ r). $\frac{x^2}{(x+1)(x-2)(x+3)}$ s). $\frac{x^2-x+2}{(x+3)^3}$ t). $\frac{x^3-x+1}{x^2(x^2+2x+1)}$

u). $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)}$

Solutions: - 1a). $\frac{x+1}{x^2+4x+3} = \frac{x+1}{x^2+3x+x+3} = \frac{x+1}{(x+1)(x+3)}$ Since the denominator contains two linear factors

$\frac{a}{x+1} + \frac{b}{x+3} = \frac{x+1}{(x+1)(x+3)}$ a(x+3) +b(x+1) =x+1(multiplying both sides by the LCM of the denominator)

Let x= -3

= a (-3+3) + b (-3+1) = -3 +1

= a (0) + b (-2) = -2

= 0+-2b= -2

-2b =-2

b=1

& Let x = -1

=a (-1+3) + b (-1 +1) = -1+1

=a (2) + b (0) =0

= 2a +0 =0

or simply $\frac{x+1}{(x+1)(x+3)} = \frac{1}{x+3}$

i). $\frac{3x-5}{(x-2)^3}$ $\frac{a}{x-2}+\frac{b}{(x-2)^2} + \frac{c}{(x-2)^3} = \frac{3x-5}{(x-2)^3}$

since the denominator contains n-repeating linear factors

$a(x-2)^2$+b(x-2) +c = 3x - 5

(multiplying both sides by the LCM of the denominator)

a (x^2-4x +4) + bx-2b +c =3x-5

ax^2 -4ax +4a +bx-2b +c =3x -5

ax^2-4ax +bx +4a -2b +c =$0x^2 + 3x - 5$

==> ax^2=$0x^2$, -4a +b =3 ,4a-2b +c =-5

a=0 -4(0) +b=3

= 2a=0

=a=0

Therefore: - the partial fraction of

$$\frac{x+1}{x^2+4x+3}=\frac{a}{x+1}+\frac{b}{x+3}$$

$$=\frac{0}{x+1}+\frac{1}{x+3}$$

$$=\frac{1}{x+3}$$

b=3 & 4(0)-2(3) +c = -5

0-6+c = - 5 **c=1**

Therefore our partial fraction becomes

$$\frac{3x-5}{(x-2)^3}=\frac{a}{x-2}+\frac{b}{(x-2)^2}+\frac{c}{(x-2)^3}$$

$$\frac{0}{x-2}+\frac{3}{(x-2)^2}+\frac{1}{(x-2)^3}$$

$$\frac{3}{(x-2)^2}+\frac{1}{(x-2)^3}$$

g). $\frac{x^3+7x+1}{x^2+3x+2}$, Since the degree of the numerator is greater than the degree of the denominator, we have an improper fraction & then we use long division to change this improper fraction as a sum of polynomial & proper fraction .

$= \frac{x^3+7x+1}{x^2+3x+2} = x-3+\frac{14x+7}{x^2+3x+2}$ and decompose the proper fraction $\frac{14x+7}{x^2+3x+2}$ Into partial friction.

$$=\frac{14x+7}{x^2+2x+x+2}=\frac{14x+7}{(x+1)(x+2)}=\frac{a}{x+1}+\frac{b}{x+2}=\frac{14x+7}{(x+1)(x+2)}$$

= a(x+2) +b(x+1) =14x+7

= ax + 2a + bx + b =14x+7

= a+ b=14 & 2a +b =7

$$\begin{cases} a + b = 14 \;\ldots 1 \\ 2a + b = 7 \;\ldots 2 \end{cases}$$

Subtract equation 2 from equation 1

$= -a = 7$ $a = -7$ & $a + b = 14$

$-7 + b = 14$ $\underline{\underline{b = 21}}$

Therefore $\frac{\mathbf{x^3+7x+1}}{\mathbf{x^2+3x+2}} = x - 3 + \frac{a}{x+1} + \frac{b}{x+2}$

$$= \mathbf{x - 3 - \frac{7}{x+1} + \frac{21}{x+2}}$$

t). $\frac{x^3-x+1}{x^2(x^2+2x+1)} = \frac{x^3-x+1}{x^2(x+1)^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{(x+1)} + \frac{d}{(x+1)^2}$

$= a(x(x+1)^2 + b(x+1)^2 + c\,(x^2(x+1) + d\,(x^2) = x^3 - x + 1$

$= a(x(x^2+2x+1) + b\,(x^2+2x+1) + c\,(x^3+x^2)\,dx^2 = x^3 - x + 1$

$= a(x^3+2x^2+x) + bx^2 + 2bx + b + cx^3 + cx^2 + dx^2 = x^3 - x + 1$

$= ax^3 + 2ax^2 + ax + bx^2 + 2bx + b + cx^3 + cx^2 + dx^2 = x^3 - x + 1 + 0\,x^2$

⇨ $a + c = 1$

$2a + b + c + d = 0$ — but $2a + b + c + d = 0$

$a + 2b = -1$ — $= 2(-1) + 1 + 2 + d = 0$

$\underline{\underline{\mathbf{b = 1}}}$ $a + 2(1) = 1$ — $= -2 + 3 + d = 0$

$= a + 2 = 1$ — $= 1 + d = 0$

$\underline{\underline{\mathbf{a = -1}}}$ — $\underline{\underline{\mathbf{d = -1}}}$

and $a + c = 1$

Therefore: - $\frac{x^3-x+1}{x^2(x+1)^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$

-1 +c=1

$= \frac{-1}{x} + \frac{1}{x^2} + \frac{2}{x+1} - \frac{1}{(x+1)^2}$

c= 2

f). $\frac{6x^2+5x+2}{(x^2+2x+3)^2}$

Since the denominator x^2+2x+3 is irreducible quadratic factor, because $b^2-4ac < 0$

$$=\frac{6x^2+5x+2}{x^2+2x+3)^2} = \frac{ax+b}{x^2+2x+3} + \frac{cx+d}{x^2+2x+3)^2}$$

= (ax + b) $(x^2 + 2x + 3) + cx + d$ = $6x^2$ +5x+2

= ax^3 +$2ax^2$+3ax +bx^2+2bx +3b+cx+d =$6x^2$+5x +2

= **a = 0** 2a +b =6 **b= 6** & 3b +d =2

= 3a +2b + c =5 = 3(6) +d =2

= 3(0) +2(6) +c =5 = 18 +6 =2

12 + c =5 **d = -16**

C=-7

Therefore: - $\frac{6x^2+5x+2}{(x^2+2x+3)^2} = \frac{ax+b}{x^2+2x+3} + \frac{cx+d}{(x^2+2x+3)^2}$

$$= \frac{0x+6}{x^2+2x+3} + \frac{-7x-16}{(x^2+2x+3)^2} = \frac{6}{x^2+2x+3} + \frac{-7x-16}{(x^2+2x+3)^2}$$

$$\Rightarrow \frac{6x^2+5x+2}{(x^2+2x+3)^2} = \frac{6}{x^2+2x+3} + \frac{-7x-16}{(x^2+2x+3)^2}$$

e). $\frac{x^2+3x+5}{(x+2)(x^2+x+1)}$ Since (x+2) is a non-reducible linear factor & $x^2 + x+1$ is also non-reducible quadratic factor.

$= \frac{x^2+3x+5}{(x+2)(x^2+x+1)} = \frac{a}{x+2}+\frac{bx+c}{x^2+x+1}$

$= \frac{x^2+3x+5}{(x+2)(x^2+x+1)} = \frac{a(x^2+x+1)+(bx+c)(x+2)}{(x+2)(x^2+x+1)}$

$= a(x^2+x+1) + (bx + c)(x+2) = x^2+3x+5$

$= ax^2 +ax+a+bx^2+2bx+cx+2c = x^2+3x+5$

a+b =1 a=1-b or b=1-a

a+2b+c=3 -------------1 From 2 & 3 we have

a+2c ==5 ------------2 a+2c =5 a+2(2) =5 but b=1- a

= a+2b+c=3 -a +c =1 a+ 4 =5 b=1-1

= a+2(1-a) +c=3 3c=6 a= 1 b=0

= a+2-2a+c =3 c=2

-a+ c=1 ------------3

Therefore: - $\frac{x^2+3x+5}{(x+2)(x^2+x+1)} = \frac{a}{x+2}+\frac{bx+c}{(x^2+x+1)} = \frac{1}{x+2}+\frac{2}{x^2+x+1}$

u). $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{2x^3-4x-8}{x(x-1)(x^2+4)} = \frac{a}{x}+\frac{b}{x-1}+\frac{cx+d}{x^2+4}$

$= a(x-1)(x^2+4) + b(x(x^2+4)) + (cx+d)(x(x-1)) = 2x^3-4x-8$

$= a(x^3+4x-x^2-4) + b(x^3+4x) + (cx + d)(x^2-x) = 2x^3-4x-8$

$= ax^3-ax^2+4ax-4a+bx^3+4bx+cx^3-cx^2+dx^2-dx = 2x^34x-8$

⇨ a+ b + c =2, - a - c + d = 0, 4a +4b-d = -4 -4a =-8 a=2

a + b + c = 2 and –a- c + d=0

2+ b + c =2 4a+4b-d= -4

b + c =0 3a- c + 4b = -4

but a= 2

$3(2)- c + 4b = -4$

$6-c+4b = -4$

$\underline{\underline{4b - c = -10}}$

But from $b + c = 0$ $b = - c$

$=4c - c= -10$

$4(-c)-c = -10$

$-5c = -10$

c= 2 & $b = -c$ **b= -2** but $-a - c + d =0 - 2 - 2+ d = 0,\underline{\underline{d = 4}}$

Therefore: - $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{a}{x}+\frac{b}{x-1}+\frac{cx+d}{x^2+4} = \frac{2}{x} - \frac{2}{x-1} + \frac{2x+4}{x^2+4}$

Rational equations & rational inequalities

Rational equations

Definition: - a rational equations is an equation that can be reduced to the form $\frac{P(x)}{Q(X)} = 0,$

Where P(x) and Q(x) are polynomials, with $Q(x) \neq 0$

Extraneous solution: -a number that looks to be a solution but causes the denominator of the original equation to become zero is called an extraneous solution.Steps to solve a given rational equation.

Given a rational equation, solving the rational equation is finding all the possible numbers in the domain of the given rational expression that satisfy the given rational equation.

Step 1: -reduce the given rational equation to the form $\frac{P(X)}{Q(X)} = 0$ (without simplifying the given expression).

Step 2: - solve the equation $Q(x)= 0$

Step 3: - solve the equation P(x)= 0

The solution set is the set of all real numbers except the zeros of q(x) (the zeros of the denominator)

Examples: - 1. Solve each of the following rational equations.

a). $x - \frac{1}{2} = \frac{1}{3}x$ Solution $x-\frac{1}{2} = \frac{1}{3}x$ ==>$-\frac{1}{2} = \frac{1}{3}x - x$

$= -\frac{1}{2} = \frac{x-3x}{3}$ check: $- x-\frac{1}{2} = \frac{1}{3}x$, when $x= \frac{3}{4}$

S.S = $\{\frac{3}{4}\}$ $= -\frac{1}{2} = \frac{-2x}{3}$ $= \frac{3}{4} - \frac{1}{2} = \frac{1}{3} . \frac{3}{4}$

$= -3 = -4x$ $= \frac{3-2}{4} = \frac{1}{4}$

$x = \frac{3}{4}$ $= \frac{1}{4} = \frac{1}{4}$ True

b). $2- \frac{X}{X+1} = \frac{3}{x+1}$ **Solution** $\frac{2(x+1)-x}{x+1} = \frac{3}{x+1} = \frac{2x+2-x}{x+1} = \frac{3}{x+1}$

$= \frac{x+2}{x+1} - \frac{3}{x+1} = 0$

$= \frac{x+2-3}{x+1} = 0$ $= \frac{x-1}{x+1} = 0 = x-1 = 0$ <u>x=1</u> s. s= {1}

c). $\frac{2}{x^2-9} = \frac{x}{x+3}$ d). $\frac{x^2-5x+6}{x^2-2} = 0$

$\frac{2}{x^2-9} - \frac{x}{x+3} = 0$ $= \frac{x^2-3x-2x+6}{x^2-2} = 0$

$\frac{2(x+3)-x(x-3)}{(x-3)(x+3)} = 0$

$= \frac{2x+6-x^2+3x}{(x-3)(x+3)} = 0$

$= -x^2+5x+6 = 0$

$= -x^2+5x+6 = 0$

$= -x^2+6x-x+6 = 0$

$= -x(-x-6)-1(x-6) = 0$

$= (-x-1)(x-6) = 0$

$-x-1 = 0$ or $x-6 = 0$

$-x = 1$ or $x = 6$

$X = -1$ or $x = 6$

<u>S.S = {-1,6}</u>

$= \frac{x(x-3)-2(x-3)}{x^2-2} = 0$

$= (x-2)(x-3) = 0$

$= x-2 = 0$ or $x-3 = 0$

$=$ <u>$x = 2$ or $x = 3$</u>

S. S= {2,3}

e). $\frac{x}{x-1} = \frac{3}{x+1}$

$= \frac{x}{x-1} - \frac{3}{x+1} = 0$

$= x^2+x-3x+3 = 0$

$= x^2-2x+3 = 0$

$b^2-4ac = (-2)^2 - 4(1)(3)$

$= 4-12 = -8 < 0$

Therefore: - The equation has no solution S.S $= \emptyset$ or {}

f). $\frac{-3x}{x+1} + \frac{6}{x} = \frac{3}{x+1}$

$\frac{-3x}{x+1} + \frac{6}{x} - \frac{3}{x+1} = 0$

$= \frac{-3x^2+6(x+1)-3x}{x(x+1)} = 0$

$= -3x^2+6x+6-3x = 0$

$= -3x^2+3x+6 = 0$

g). $\frac{x}{x-1} - \frac{x+1}{x^2-x} + \frac{1}{x} = 1$

$\frac{x}{x-1} - \frac{x+1}{x(x-1)} + \frac{1}{x} - 1 = 0$

$x^2-(x+1)+1(x-1)-x(x-1) = 0$

$x^2-(x+1)+x-1-x^2+x = 0$

$\cancel{x^2}-\cancel{x}-1+\cancel{x}-1-\cancel{x^2}+x = 0$

$= -x^2+x+2 = 0$ $\quad\quad$ $-2 + x = 0$

$= -x^2+2x-x+2 = 0$ $\quad\quad$ $\underline{\underline{x = 2}}$

$= -x(x-2) - 1(x-2) = 0$ $\quad\quad$ **S.S = {2}**

$= (-x-1)(x-2) = 0$

$= -x - 1 = 0$ or $x - 2 = 0$

$x = -1$ or $x = 2$ but $-1 \notin$ the domain of the given expression.

-1 is an extraneous solution Therefore **S.S = {2}**

h). $\frac{10}{x(x+1)} + 4\,\frac{(2-x^2)}{x+1} + \frac{5x^3-5x^2}{x^2} = \frac{1+10x}{x^2}$

$= \frac{10x+4x^2\left(2-x^2\right)+(5x^3-5x^2+1\,(x+1)}{x^2(x+1)} = \frac{1+10x}{x^2}$

$= \frac{10x+8\cancel{x^2}-4x^4+5x^4\cancel{-5x^3}+x+\cancel{5x^3}-\cancel{5x^2}+1}{x^2(x+1)} = \frac{1+10x}{x^2}$

$= x^4+3x^2+11x+1 = (1+10x)(x+1)$ $\quad\quad$ $\Longrightarrow x^4-7x^2 = 0$

$= x^4+3x^2+11x+1 = x+1+10x^2+10x$ $\quad\quad$ $\Longrightarrow x^2(x^2-7) = 0$

$= x^4+3x^2-10x^2+11x-10x-x+1-1 = 0$ $\quad\quad$ $\Longrightarrow x^2 = 0$ or $x^2-7 = 0$ $x = 0$ or $x = \underline{+}\sqrt{7}$

But 0 is an extraneous solution S.S = $\{-\sqrt{7}, \sqrt{7}\}$

Rational inequalities

Definition 2.7:- is an inequality that can be reduced to the form $\frac{P(x)}{Q(x)} \le 0$, $\frac{P(x)}{Q(x)} < 0$, $\frac{P(x)}{Q(x)} \ge 0$, *or* $\frac{P(x)}{Q(x)} > 0$, Where P(x) & Q(x) are polynomial expressions & $Q(x) \neq 0$.

Steps in solving rational inequalities.

Step 1: -determine the domain of rational inequality.

Step 2: -Write the given inequality to the form $\frac{P(x)}{Q(x)} > 0$, $\frac{P(x)}{Q(x)} \geq 0$, $\frac{P(x)}{Q(x)} < 0$ or $\frac{P(x)}{Q(x)} \leq 0$.

Step 3: - find the zeros of both P(x) & Q(x).

Step 4: - divide the number line into different intervals.

Step 5: - find the interval that satisfies the given inequality.

Examples: - 1. Which of the following are rational inequalities?

a). $1 > \frac{2}{3x}$ b). $2^x < \frac{2}{x+5}$ c. $\frac{1}{x} - \frac{2}{x^2} \leq \frac{5}{x^2+1}$

Solutions: a and c are rational inequalities & b is not.

2. Solve the inequality: $\frac{x+1}{x-1} > 0$

Solutions: - domain $=|R|\ \{1\}$ using sign chart we can find the solution sets.

	-1	1	
	$x < -1$	$-1 < x < 1$	$x > 1$
x+1	----------------------	0 + + + + +	+ + + +
x-1	------------------------	------------------------	0 + + + +
$\frac{x+1}{x-1}$	------------------	0-------------------	B + + + +

Since $\frac{x+1}{x-1} > 0$ S.S =x {x: x > 1 or x < -1}

i.e. S. S= $\underline{x \in (-\infty, -1) \cup (1, \infty)}$

3. Solve each of the following inequalities.

a). $\frac{1}{3} + \frac{2}{x^2} < \frac{5}{3x}$ c). $\frac{(x-2)^2}{4-x} \leq 1$.

b). $\frac{2x^2-5x+2}{x+1}$

d). $\frac{x-1}{x+4} < \frac{x-2}{x+3}$

2Solution

3a). $\frac{1}{3}+\frac{2}{x^2} - \frac{5}{3x} < 0$

$= \frac{x^2+6-5x}{3x^2} < 0$

$= \frac{x^2-3x-2x+6}{3x^2} < 0$

$= \frac{x(x-3)-2(x-3)}{3x^2} < 0$

$= \frac{(x-2)(x-3)}{3x^2} < 0$

	x < 0 (0)	0 < x < 2 (2)	2 < x < 3 (3)	x > 3
$3x^2$	--------0	+	+	+
x - 2	----------	----------0	+	+
x - 3				
$\frac{(x-2)(x-3)}{3x^2}$	--------B	++	0----------0	+

S.S = {x: x <0 or 2 < x < 3} = $(-\infty, 0)u(2,3)$

3b). $\frac{2x^2-5x+2}{x+1} \geq 0.$

$= \frac{2x^2-4x-x+2}{x+1} \geq 0$

$= \frac{2x(x-2)-1(x-2)}{x+1} \geq 0.$

$= \frac{(2x-1)(x-2)}{x+1} \geq 0.$

Using sign chart method

	$x < -1$	$-1 < x < \frac{1}{2}$	$\frac{1}{2} < x < 2$	$x > 2$
	-1		$\frac{1}{2}$	2
x + 1	------------	0 +	+	+
2x-1	------------	------------	0 +	+
X-2	------------	------------	------------	0 +
(2x-1) (x-2)	+ + + + +	+ + + + +0	---------------0	+
$\frac{(2x-1)(x-2)}{x+1}$	------------	+ + + 0	----------------0	+

S.S = {x:-1 < x < $\frac{1}{2}$ or x >2}

= **x∈ $(-1,\frac{1}{2})$ u (2,∞)**

3c). $\frac{(x-2)^2}{4-x} \leq 1$

$= \frac{(x-2)^2}{4-x} - 1 \leq 0$

$= \frac{(x-2)^2-1(4-x)}{4-x} \leq 0$

$= \frac{x^2-\cancel{4x}+4-4}{4-x}$

$= \frac{x^2-3x}{4-x} \leq 0$

$= \frac{x(x-3)}{4-x} \leq 0$ **S.S={x:0 ≤ x ≤ 3 or x >4 }**

= **x ∈ [0, 3] u (4,∞)**

	$x < 0$	$0 < x < 3$	$3 < x < 4$	$x > 4$
	0	3	4	
x	- - - - - -0	+ + + +	+ + + + +	+ + +
x - 3	- - - - -	- - - - - - - -0	+ + + +	+ + + +
4 – x	+ +	+ + +	+ + + 0	- - -
$\frac{x(x-3)}{4-x}$	+ + + + 0	- - - - - 0	+ + + B	- - -

Rational functions & their Graphs

Rational function

Definition :- a function that can be written as the ratio of two polynomial function f(x) = $\frac{P(X)}{Q(X)}$,

Where P(x) & q(x) are polynomial functions & q(x) ≠ 0.

Examples:- 1. Which one of the follwing are rationals ?

a). f(x)= $\frac{x^2+6x+9}{x^2+4x+1}$ **b).** f(x) = $\frac{3x-5}{(x+1)^{3/2}}$

c). $f(x) = 3x^2 - 4x + 5$ **d).** $f(x) = 4$

solutiouns all except c are rational functions .

N.B:- The domain of a rational function is all real number except the zeros of the denomniator of the given rational functuion .

Graphs of rational functions

Terms related to graphs of rational functions

1. Asymphotote :- an asymphtot is a line that the graph of a given function approaches but never touches .

   Types of asymptotes

   We have three types of asymptotes for this level.

   a). ***Horzontal asymptotes (H,A)***

- Is a horizontal line that is not part of a graph of function but guides if for l-values "for " to the right and /or "for" to the left.
- The graph of a given function may cross the H.A.
- A Horzontal asymptote is aspecial case of oblique /slant asymptote .
- The line y=f(x) =b, is a horzontal asymptote of the graph of the given function f, if $y \longrightarrow b$,as $x \longrightarrow \infty$ or $x \longrightarrow -\infty$.

Rules of Horizontal asymptote

Let $f(x) = \frac{N(X)}{D(X)} = \frac{anx^n + an-1x^{n-1} + ---- + a1x + a0}{bmx^m + bm-1x^{m-1} + ---- + b1x + b0}$.

Where N(x) &D(x) are polynomials & an$\neq$ 0 & bm$\neq$ 0, n $\in$W be rational functions & let degree N(x) =n & degree of D(x) =m and f(x) is in lowest /in simplest / form

Rule 1: - if n < m degree of the nominator < degree of denominator), then the line y= 0(x-axis) is a horizontal asymptote

Rule 2: - if n=m (degree of the nominator = degree of the denominator), then the line $y = \frac{an}{bm}$ is a horizontal asymptote.

Rule 3: - if n > m (degree of the nominator > degree of the denominator), then the graphs no horizontal asymptote.

N.B: - The graph of a given rational function cannot have both H.A & O.A at the same time.

The graph of a given rational function may have both H.A & V.A and V.A.

b). ***Vertical asymptote (V.A)***

- The zeros of the denominator of a given rational function after simplifying /cancelling the common factor) is called a vertical asymptote (V.A)
- The line x =a is a V.A for the graph of a function f, if x-a is not a common factor for the numerator & the denominator (x =a is not a common root of the numerator & the denominator).(or if N(a) = 0 & D(a) ≠ 0 or if N(a) ≠ 0 & D(a) ≠ 0). If $f(x) \longrightarrow -\infty$

As $x \longrightarrow a$ from either the left or the right. if N(a) =0 & D(a) = 0 the graph of a given function has a hole at x = a.

Meaning

- ✓ x approaches a form the right.
- ✓ X approaches a form the left.
- ✓ f(x) is increasing without bound.
- ✓ f(x) is decreasing without bound.

A graph can have an infinite V.As.

The graph will never touch the V.A.

c). ***oblique asymptote /slant asymptote***

- When a linear asymptote is not parallel to the x-axis or the y –axis it is called an oblique asymptote or if n=m+1 (degree of numerator = degree of denominator +1) we have an O.A.

N.B: - if n > m+1, we have no H.A or O.A.

Examples: -1. Find the V. A, H. or O. As for each of the following functions (if any) a). f(x) = $\frac{1}{x-2}$. , Since degree of N(x) < degree of D(x) we have a H.A y= 0 (x-axis) & the nominator & denominator have no common factor x-2 =0 x= 2 is a V.A

i.e. N (2) ≠ 0 N (2) = 1 ≠ 0 & D (2) =0 our V.A is **x =2** Since we have a V.A, we have no O.A.

b). f(x) = $\frac{x+1}{x^2+3x+2}$

$= \frac{x+1}{x^2+2x+x+2}$

$= \frac{x+1}{x(x+2)+1(x+2)}$

$= \frac{x+1}{(x+1)(x+2)}$

$= \frac{1}{x+2}$

⇨ f(x) has a hole at x= -1

⇨ f(x) has a V.A at x = -2

⇨ f(x) has a H.A at y =0 (x-axis)

⇨ f(x) has no O.A

c). f(x) =$\frac{3x^2+1}{x^2+4x+2}$

= 3- $\frac{12X-11}{X^2+4X+4}$

==>f(x) = y = 3 is an O.A

= x = -2 is a V.A

= f(x) has no H.A

d). f(x) = $\frac{x^3+3x+1}{x^2+1}$

= f(x) has no V.A since $x^2+1 \neq 0$

= f(x) has not an O.A at y = x

=f(x) has no H.A.

e). f(x) = $\frac{x^3+x^2+x+1}{x+2}$

= f(x) has no H.A.

=f(x) has no O.A.

=f(x) has V.A at x = -2

f). f(x) = $\frac{x^4+2x^3-3x+1}{x-2}$ since DN(x) > DD(x)+1

= f(x) has no O.A.

= f(x) has no H.A.

since the denominator & the nominator have no common factor.

2. Domain & Range

- The domain of a given rational function is all real number except the zeros of the denominators of the given rational functions.
- The range of a given rational function is the values of y-a after rearranging.

3. Intercepts (x and y)

- To find the x- intercept (s) put y=0 & solve for x.

❖ To find the y- intercept put x = 0 & solve for y.

4. Parity (even, odd, neither

❖ The graph of an even function is S.W.R.T the y –axis.

❖ The graph of an odd function is S.W.R.T the origin.

5. Symmetry (about the y-axis, the origin or neither)

6. Values of the function near the V. As

Graphs of rational functions

Example: - 1. For each of the following rational functions.

a). find the domain.

b). find the intercepts (in there are)

c). find all the asymptotes (if any)

d). sketch the graphs.

1. $f(x) = \frac{1}{x}$ 2. $f(x) = \frac{1}{x^2}$ 3. $f(x) = \frac{x+1}{x^2+5x+6}$

4. $f(x) = \frac{1}{x^n}$, when n is even natural number & when n is odd natural number

5). $f(x) = \frac{1}{(x-a)^n}$, when n is even natural number when n is odd natural number.

Solution

1. f(x) = $\frac{1}{x}$

$\Rightarrow$ Domain = $|R|$ {0}

$\Rightarrow$ X-Intercept(s) no x- intercepts.

$\Rightarrow$ Y-intercept no y- intercept

$\Rightarrow$ V.A x =0 (y-axis) Since DN(x) < DD(x), we have a H.A at y =0 (x-axis)

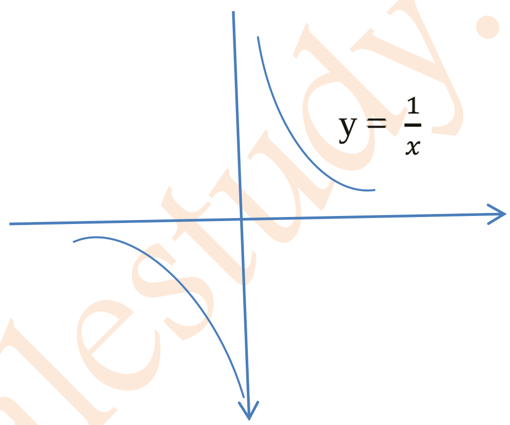
$\Rightarrow$ No O.A.

$\Rightarrow$ As $x \to 0^+$, $y \to \infty$

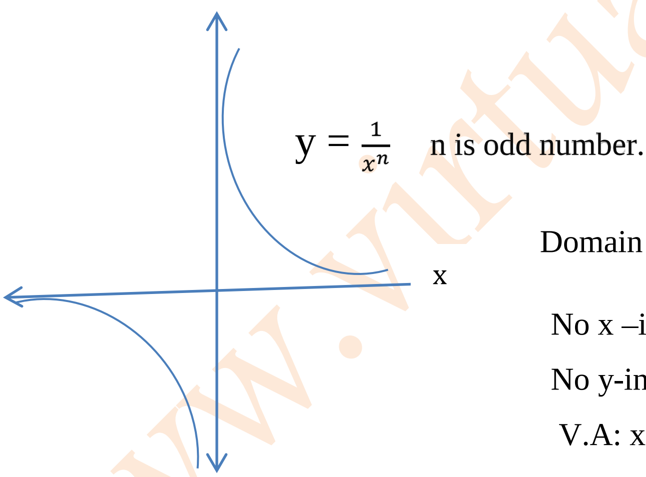
$\Rightarrow$ As $x \to 0^-$, $y \to -\infty$

$\Rightarrow$ As $x \to \infty, y \to 0$

$\Rightarrow$ As $x \to -\infty, y \to 0$



2. f(x) = $\frac{1}{x^n}$ n is odd number.



Domain = $|R|\{0\}$

No x –intercept

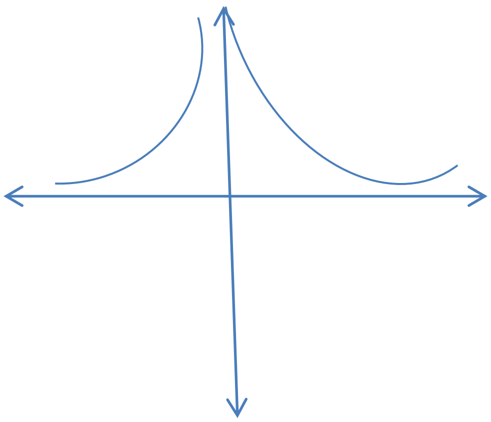
No y-intercept

V.A: x= 0 (y-axis)

H.A: y =0 (x-axis)

No O.A.

3. f(x) = $\frac{1}{x^n}$, n is even natural number.



→ Domain = $|R|\{0\}$

→ No x –intercept

→ No y –intercept

→ V.A: x = 0

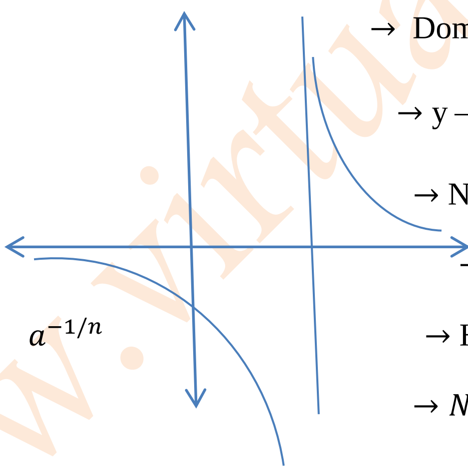
→ H.A: y=0

→ No O.A.

↑ On (-∞,)

↓ on (0, ∞)

4. f(x) = $\frac{1}{(x-a)^n}$ a ≥ 0 n- is odd natural number.



→ Domain =$|R| \in \{0\}$

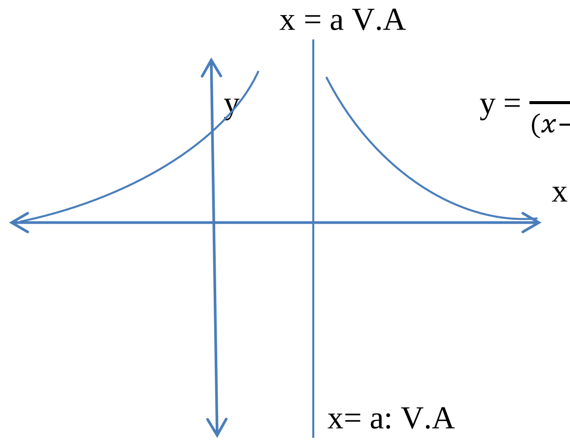
→ y –intercept at - $\frac{1}{an}$ or (0, - $\frac{1}{an}$)

→ No x- intercept

→ V.A: x=a

→ H.A: y = 0

→ *No* O.A.



$y = \frac{1}{(x-a)^n}$ n, n is even natural number.

→ Domain = $|R| \in \{a\}$
→ y- Intercept at $(0,a^{1/n})$
→ No x- intercept
→ $H.A: y = 0$
→ $V.A$: x = a
→ No O.A.

5. $f(x) = \frac{x+1}{x^2+5x+6}$ y- intercept (x= 0) at $(0,\frac{1}{6})$

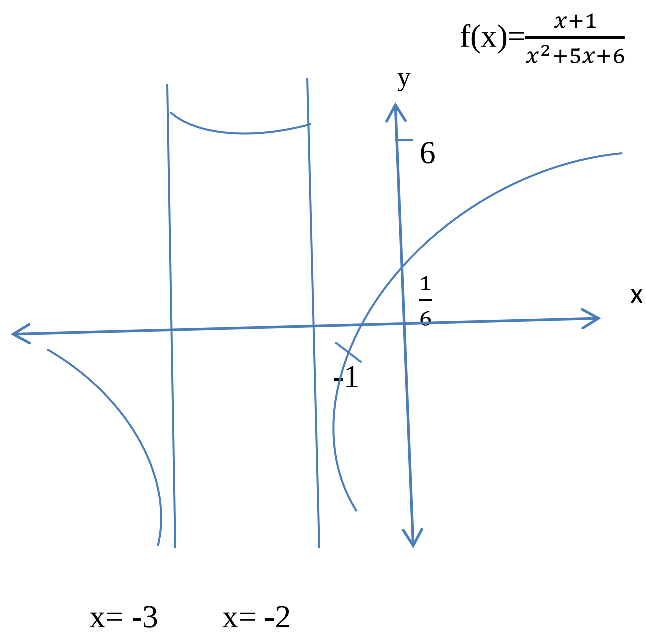
$= \frac{x+1}{x^2+3x+2x+6}$ x-intercept (y= 0) at (-1,0)

$= \frac{x+1}{(x+3)(x+2)}$ V.A: x = -3 & x = -2

H.A: y = 0

Using sign chart

	x < -3	-3 < x < -2	-2 < x < -1	x > -1
x+3	- - - - - - - 0	+ + + +	+ + + +	+ +
x+2	- - - - - - -	- - - - - 0	+ + +	+ + +
x+1	- - - - -	- -	- - - 0	+ + +
$\frac{x+1}{(x+2)(x+3)}$	- - - - B	+ + + B	- - - - 0	+ + +



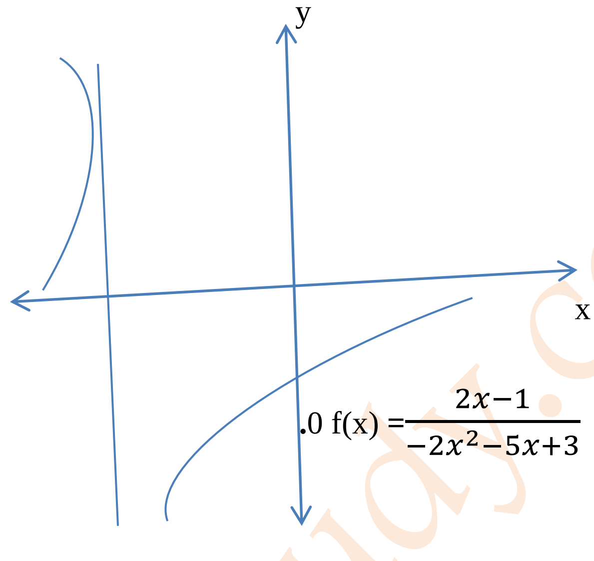
6. f(x)= $\frac{2x-1}{-2x^2-5x+3}$

$= \frac{2x-1}{-2x^2-6x+x+3}$

$= \frac{2x-1}{-2x(x+3)+1(x+3)}$

$= \frac{\cancel{2x-1}}{-(\cancel{2x-1})(x+3)}$

$= \frac{\mathbf{-1}}{\boldsymbol{x+3}}$, $\mathbf{x} \neq \frac{\mathbf{1}}{\mathbf{2}}$



Domain = $\mathbb{R}\setminus\{\frac{1}{2},-3\}$

f(x) has a V.A at x =-3

f(x) has a hole at x= $\frac{1}{2}$

f(x) has a H.A at y =0

7. f(x) = $\frac{1}{x^2+x-12} = \frac{1}{x^2+4x-3x-12} = \frac{1}{(x+4)(x-3)}$

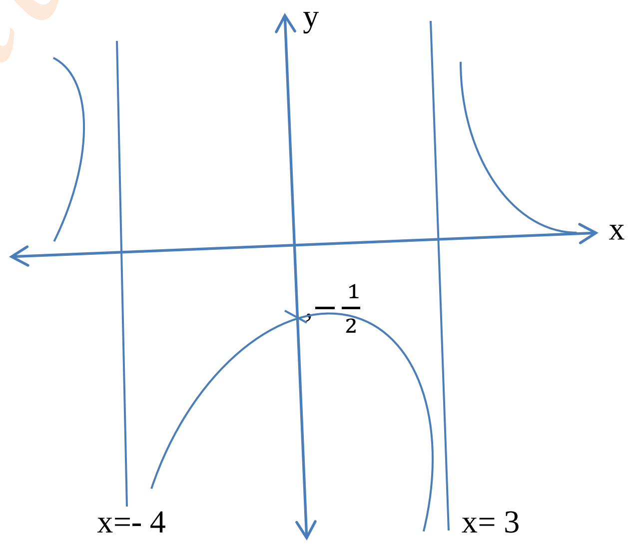
⇨ Domain = $\mathbb{R}\setminus\{-4,3\}$

⇨ f(x) has V. As at x =-4 & x =3

⇨ f(x) has a H.A at y= 0

⇨ No - x- intercept.

⇨ y – intercept at $(0,-\frac{1}{2}$



Applications on rational functions

- Rational formulas can be useful tools for representing real life situations & for finding answers to real problems.
- Equations representing direct, inverse & joint variation are examples of rational formulas that can model many real-life situations.

Steps in solving word problems.

1. Understanding the problem.
2. Setting up the equation(s) /or expressions using variables
3. Solving the given equation(s) for the given variables.
4. Interpreting the results.

Examples: - 1. A sample of 80g of **a** particular ice cream contains 15 g of fat & 70g of carbohydrate.

a). how much fat does 350g of the same ice cream contain?

b). how much carbohydrate does 350 g of the same ice cream contain?

Solution

a). if the amount of fat in 350g is x g, then $\frac{x}{350} = \frac{15}{80}$ solving for x gives us,

x= $\frac{15}{80} \times 350$ = **65.675 g**

b). if the amount of carbohydrate in 350 g is y g, then $\frac{y}{350} = \frac{70}{80}$

==> y= $\frac{70}{80} \times 350$ = **87.5 g**

2. Zeru can complete a certain job in 5 days & Tulu can complete the same job in 3 days. How long will it take them to complete the job. If they work together?

Solution

If an individual complete a given task in x days, he/she perform $\frac{1}{x}$ of the total.

Let zeru works in x days to complete the given task & Tulu works in y days to complete the same task, then they work together in $\frac{1}{x} + \frac{1}{y} = \frac{1}{Total}$

$\frac{1}{5} + \frac{1}{3} = \frac{1}{T}$

$\frac{3+5}{15} = \frac{1}{T}$ ==> $\frac{15}{8}$ = Total work

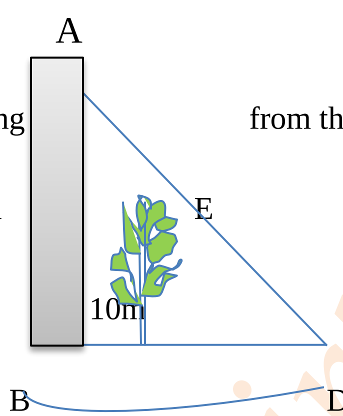
Total work (They work together) in **<u>1.875</u>** days.

3. In one morning, if the shadow of a building of 20 meters long is 15 meters long, how long is the shadow of a tree of 10 meters long?

   <u>Solution</u>

Building

20m



from the figure $\triangle$ ABD N $\triangle$ ECD(AA)

$\frac{AB}{EC} = \frac{AD}{ED} = \frac{BD}{CD}$

$= \frac{20m}{10m} = \frac{15m}{x}$

20m x = 150m .10 m

Shadow building 15 m

x= $\frac{15\cancel{0}m}{2\cancel{0}}$ = x= 7.5 m long the length of the shadow of a tree.

3). let the ratio of boys to girls in a certain high school is 4:5. if the total number of students in the school is 1260, find the number of boys & the number of girls.

<u>Solution</u>: Let the number of boys be x & the number of girls be y.

<u>x</u>: y 4:5 ==>$\frac{x}{y} = \frac{4}{5}$ 5x =4y

x=$\frac{4}{5}$ and x+ y =1260

⇨ $\frac{4}{5}y$+y =1260

⇨

$\frac{4y+5y}{5}$ =1260 x+ y =1260

$\frac{9y}{5}$ =1260 x+700 =1260

y= 1260. $\frac{5}{9}$ x= 1260-700

y= 700 is the number of girls. & x= 560 - is the number of boys

Unit three

Matrices

Introduction: - systems of linear equations occur in many areas such as in Geometry, Engineering, Business, Economics, Biology etc. solving such systems of linear equation is one of the most important applications of mathematics to other fields.

The concepts of matrix

Matrix

Definition: **- *A*** matrix is a rectangular array or table of numbers symbols or expressions, arranged in rows & columns, which is used to represent a mathematical objects or property of such an object.

The number may be real or complex it may be represented as:

$$A=\begin{pmatrix} a11\ a12\ a13\ a14\ a15---a1n \\ a2\ 1a\ 22a\ \ 23--------a2n \\ a31--------------- \\ a41--------------- \\ - \\ - \\ - \\ am1\ \ am2\ \ am3--------amn \end{pmatrix} m \times n$$

- The number m is called the number of rows of a given matrix.
- The number n is called the number of columns of the given matrix or as A=(aij)m x n , $1\leq i \leq m$ and $1\leq j \leq n$, m,nϵN. A matrix with m rows & n columns is called as m x n matrix with order or size or dimension m x n. where aij represents the element at the intersection of i^{th} row and j^{th} column.
- Matrices are usually denoted by capital letters, like A, B, C etc... and entries of matrices by small letters, like a, b, c etc.
- Matrix has no value.

The number of elements in a matrix (m x n). (matrix with order m x n) is m.n

Examples: -

1. If a matrix has 12 elements, then the possible orders will be:
   a. 12 x 1 ,1 x 12 ,6 x 2, 2 x 6 ,4 x 3, 3 x 4
2. For each of the following matrices find the size and determine its elements.

a). $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$: B. $=\begin{pmatrix} 1 \\ 0 \\ 29 \end{pmatrix}$ C= $\begin{pmatrix} -2 & 10 & 8 \\ 3 & -1 & 6 \\ 0 & 5 & 0 \\ 3 & 4 & -5 \end{pmatrix}$

a11= 1	b11=1	c11= -2	c31 =0
a12 =2	b21=0	c12 =10	c32=5
a21 =3	b31=29	c13 =8	c33=0
a22 =4	size:3x1	c21 =3	c41 =3
Size :2x2 or 2		c22 =-1	c42= 4
		c23 = 6	c43=-5

3.if $A=(aij)_{2x3}$ & $a_{ij} =$ j-i, determine matrix A

Solution:

a_{11}=1-1=0 a_{21}=1-2=-1

a_{12}=2-1=1 a_{22}=2-2=0

a_{13}=3-1=2 a_{23}=3-2=1

⇨ $A=\begin{bmatrix} a11 & a21 & a23 \\ a21 & a22 & a23 \end{bmatrix}_{2x3}=\begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$

4. Let $A=\begin{bmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 5 & -1 \\ 2 & 2 & 7 & 10 \end{bmatrix}$ then determine each of the following entries (if it exist).

a). a_{11}=1 c). a_{32}=2

b). a_{23} =5 d). a_{41}=does not exist

Types of Matrices

Let $A= (a_{ij})_{m \times n}$ be an m x n matrix, m, n ∈ N.

1. Rectangular Matrix: -a matrix in which the number of rows is not equal to the number of columns.

i.e m ≠ n (number of rows & number of columns)

Example: - A $=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}_{2X3}$

2. Square Matrix: -a matrix in which the number of rows is equal to the number of columns. i.e m=n (number of rows =number of columns)

Example: - Let B$=\begin{pmatrix} 2 & 3 & -7 \\ 4 & 6 & 10 \\ 11 & 9 & 1 \end{pmatrix}_{3x3}$

3. Diagonal matrix: a square matrix whose non-diagonal elements are zero except possible along the main diagonal (top left to bottom right).

If aij=0,for all i≠j & aij may may not be zero.

Example: - a). $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ b). $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ c). $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{5x5}$

d). $(1)_{1x1}$ or $(2)_{1x1}$ $(3)_{1x1}$ or $(4)_{1x1}$ all are square matrices .

Trace of Matrix: - The diagonal elements of a given matrix denoted by (a_{11} a_{22} a_{33} ....a_{nn})

Example Let A$=\begin{pmatrix} 1 & 2 & 1 & 0 \\ 3 & 4 & 5 & -1 \\ 2 & 2 & 7 & 10 \\ 8 & 9 & -4 & 7 \end{pmatrix}$ tr (A)=1+4+7+7 =19

4. Scalar matrix: - A square matrix whose all non –diagonal elements are zero & diagonal elements are equal.

If m =n & aij =0,$\forall i \neq j$ & aij=k, $\forall i$

Examples:- A$=\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}_{3x3}$ B$=\begin{pmatrix} k & 0 & 0 - - - 0 \\ 0 & k & 0 - - - 0 \\ - & - & - \; - \; - \\ - & - & - \; - \; - \\ 0 & 0 & - \; - \; - \; k \end{pmatrix}_{n\,x\,n}$ k$\neq 0$ or $k = 0$

5. **Unit matrix (Identify) matrix:** a square matrix is matrix whose all non-diagonal elements are zero & diagonal elements equal to one.

Examples:- Let A= $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{4x4}$ B).= $\begin{pmatrix} 1 & 0 & 0 - - - 0 \\ 0 & 1 & 0 & 0 - -0 \\ 0 & 0 & 1 & 0 - -0 \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}_{n \times n}$

All main diagonal elements are 1 & others are zero.

6. **Zero or null matrix**

A matrix whose all elements are zero.

A zero matrix may or may not be a square matrix i.e zero matrix can be square or rectangular matrix.

Examples :- $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{3 \times 3}$ B= $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}_{3 X 2}$ C) = $\begin{pmatrix} 0 & 0 & 0 - & - 0 \\ 0 & 0 & 0 & 0 - 0 \\ - & - & - & - - \\ - & - & - & - & - \\ 0 & 0 & - & - & -0 \end{pmatrix}_{n \times n}$

7. **Row matrix**: -a matrix having only one row & more than one column is said to be a row matrix or row vector.

Examples: - $(a11 \quad a12 \quad a13 \quad a14 - - - a1n)$ 1 x n

A matrix A (an m x n) matrix is said to be row matrix if m =1

Example: - A= (1 10 20 -5 -12 79)$_{1 \times 7}$

8. **Column matrix: -** A Matrix having only one column & more than one row is said to be a column matrix or column vector. An m x n matrix is said to be column matrix if n =1

Examples: - $\begin{pmatrix} a11 \\ a21 \\ a31 \\ - \\ - \\ am1 \end{pmatrix}_{m \times 1}$ **example: -** A= $\begin{pmatrix} 1 \\ 0 \\ 7 \\ 8 \end{pmatrix}_{4 X 1}$

9. **Singular Matrix**: - A square matrix A is said to be singular, if det(A) or $|A|$=0.

Example: - Let A=$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, $|A|=\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$= 1x 4 - 2 x 2 = 0

10. **triangular matrix: -** a triangular matrix is a square matrix in which elements below and /or above the diagonal are all zeros. We have mainly two types of triangular matrices.

1. Upper triangular matrix

Let A=(aij)nxn be a square matrix of order n.

a). if aij =0,for all I >j ,then A is called an upper triangular matrix

A=$\begin{pmatrix} a11 & a12 & a13 - - - a1n \\ 0 & a22 & a23 - - - a2n \\ - - - - - - - - \\ 0 \; 0 \; 0 \; 0 \; 0 \; 0 \; 0 & & ann \end{pmatrix}$ $n \; x \; n$ or A matrix A is said to be an upper triangular matrix if all the elements above the main diagonal & the main diagonal are all zero.

b). A is said to be strictly upper triangular matrix if aij=0 for all i≥ j

i.e A=$\begin{pmatrix} 0 \; a12 - - - \; a1(n-1) \; a1n \\ 0 \;\; 0 - - - - a2(n-1) \; a2n \\ - - - - - - - - - - \\ - - - - - - - - - - - \\ - - - - - - - - - - - - \\ 0 \; 0 - - - - - - - - 0 \end{pmatrix}$nx n

2. ***Lower Triangular Matrix***: - A Matrix A is said to be a lower triangular matrix if all the elements below the main diagonal & the main diagonal are non –zero & elements above the main diagonal are all zero. i.e aij=0, for all i < j.

$$\begin{pmatrix} a11 \;\; 0 \;\; 0 \;\; - - - - - - \;\; 0 \;\; 0 \\ a21 \;\; a22 \;\; 0 - - - - - - \;\; 0 \;\; 0 \\ - - - - - - - - - - - - - - - - \\ - - - - - - - - - - - - - - - - \\ - - - - - - - - - - - - - - - - \\ an1 \;\; an2 - - - - an(n-1) \; ann \end{pmatrix}$$ n x n

A is said to be strictly lower triangular matrix if aij = 0, for all i≤ j.

i.e $\begin{pmatrix} 0 & 0 & 0 & --- & 0 & 0 \\ a21 & 0 & 0 & --- & 0 & 0 \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ - & - & - & - & - & - \\ an1 & an2 & --- & & an(n-1) & 0 \end{pmatrix}$ n x n

11). ***Symmetric Matrix***: - a square matrix A is said to be symmetric matrix if A= A^t

If A= (aij) n x n, then A^t = (aij) n x n, all aij =aji

12***). Skew symmetric matrix***: - a square matrix A is said to be skew – symmetric matrix if A= -A^t. if A=(aij) n x n , then A^t =(aji)n x n , aij = -aji

aii = -aii   2aii= 0   aii = 0, hence all diagonal elements of a skew symmetric matrix are zero.

13). ***Equal matrix***: - two matrices A and B are said to be equal if

1. they have the same order.

2. their corresponding elements are equal.

Examples: - 1. Identify the types of matrices for each of the following matrices.

a). A =$\begin{pmatrix} 0 & 1 & -4 \\ -1 & 0 & 7 \\ 4 & -7 & 0 \end{pmatrix}$ 3 x 3   – it is a square matrix

- it is a skew symmetric matrix

b). B = $\begin{pmatrix} 3 & 2 & 11 & 5 \\ 2 & 9 & -1 & 6 \\ 1 & 1 & -1\ 0 & 7 \\ 5 & 6 & 7 & 9 \end{pmatrix}$ 4 x 4   - it is a square matrix

- it is a symmetric matrix

c). $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

- it is a square matrix
- it is a triangular matrix
- it is an identity matrix
- it is a scalar matrix
- it is a unit matrix
- It is a diagonal matrix

d). if D=$\begin{pmatrix} 1 & a & 2 \\ 0 & 1 & 0 \\ b & c & 5 \end{pmatrix}$ 3 x 3 find the value of a, b, c so that D is a symmetric matrix.

Solution: - D is a symmetric matrix, if D= D^t

=$\begin{pmatrix} 1 & a & 2 \\ 0 & 1 & 0 \\ b & c & 5 \end{pmatrix}$ = $\begin{pmatrix} 1 & 0 & b \\ a & 1 & c \\ 2 & 0 & 5 \end{pmatrix}$

a = 0, b=2 , c=0

e). Let A=$\begin{pmatrix} 1 & a & 2 \\ 0 & 1 & 0 \\ b & c & 5 \end{pmatrix}$ find a, b, c such that A is an upper triangular matrix.

Solution: -Ais an upper triangular, iff, aij =0 for all I > j

f). find the value of the the unknown in each of the following if A=B

i. A=$\begin{pmatrix} 1 & 2 \\ -3 & a \end{pmatrix}$ & B= $\begin{pmatrix} 1 & b \\ -3 & 4 \end{pmatrix}$ ==> a =4 ,b=2

ii. A =$\begin{pmatrix} a+b & 5 & e \\ 3 & -1 & 4 \\ 0 & 1 & 9 \end{pmatrix}$ & B=$\begin{pmatrix} 1 & 5 & 7 \\ a-b & -1 & 4 \\ d & 1 & c \end{pmatrix}$ if A=B $\begin{pmatrix} a+b & 5 & e \\ 3 & -1 & 4 \\ 0 & 1 & 9 \end{pmatrix}$ = $\begin{pmatrix} 1 & 5 & 7 \\ a-b & -1 & 4 \\ d & 1 & c \end{pmatrix}$

⇨ $\left.\begin{cases} a+b=1 \\ a-b=3 \end{cases}\right\}$ 2a =4

a=2 , d=0 , e= 7 , c= 9 & b= -1

iii. A=$\begin{pmatrix} a+b & 2 & 3 \\ a-b & 0 & -1 \\ 4 & 5 & 6 \end{pmatrix}$ & B=$\begin{pmatrix} 2 & 2 & C \\ 4 & 0 & -1 \\ 4 & d & 6 \end{pmatrix}$ if A=B $\begin{pmatrix} a+b & 2 & 3 \\ a-b & 0 & -1 \\ 4 & 5 & 6 \end{pmatrix}$ = $\begin{pmatrix} 2 & 2 & C \\ 4 & 0 & -1 \\ 4 & d & 6 \end{pmatrix}$

Operations on matrices

a). **Additions & Subtraction of Matrices**

Let A= (a i j) m x n & B= (bi j) m x n be two matrices with m ,n ε N then

a). The sum of A & B, denoted by A+B is an m x n matrix obtained by adding the corresponding entries of A & B.

i.e A+B =(aij) m x n +(bij) m x n = (aij+bij) m xn

b). the difference (subtraction of B from A), denoted by A-B =A+(-B) is an m x n matrix obtained by subtracting the entries of B from the corresponding entries of A. i.e A-B= (aij –bij) m x n .

examples: - 1. Find A+B, A-B, (A+B) +C, A+(B+C) for the following matrices.

a). A= $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, B= $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$, C= $\begin{pmatrix} 5 & 7 \\ 10 & -4 \end{pmatrix}$

b). A= $\begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix}$ & B= $\begin{pmatrix} 3 & 1 & 4 \\ 4 & 0 & -1 \\ 5 & 2 & -2 \end{pmatrix}$ C= $\begin{pmatrix} -5 & 7 & 0 \\ 6 & 8 & 9 \\ 11 & 20 & -17 \end{pmatrix}$

Solutions: -

a). A+B = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} 2 X 2$ + $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ 2 X 2 = $\begin{pmatrix} 1+0 & 2+(-1) \\ 3+1 & 4+2 \end{pmatrix}$ 2 X 2 = $\begin{pmatrix} 1 & 1 \\ 4 & 6 \end{pmatrix}$ 2 X 2 //

A-B = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} 2 X 2$ - $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} 2X2$ = $\begin{pmatrix} 1-0 & 2-(-1) \\ 3-1 & 4-2 \end{pmatrix}$ = $\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}$ 2X2 //

(A+B) +C = $\begin{pmatrix} 1 & 1 \\ 4 & 6 \end{pmatrix}$ + $\begin{pmatrix} 5 & 7 \\ 10 & -4 \end{pmatrix}$ = $\begin{pmatrix} 1+5 & 1+7 \\ 4+10 & 6+(-4) \end{pmatrix}$ = $\begin{pmatrix} 6 & 8 \\ 14 & 2 \end{pmatrix}$ 2X2//

A+(B+C) = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ + $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$ + $\begin{pmatrix} 5 & 7 \\ 10 & -4 \end{pmatrix}$ = $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ + $\begin{pmatrix} 0+5 & -1+7 \\ 1+10 & 2+(-4) \end{pmatrix}$

= $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ + $\begin{pmatrix} 5 & 6 \\ 11 & -2 \end{pmatrix}$ = $\begin{pmatrix} 6 & 8 \\ 14 & 2 \end{pmatrix}$ //

b). A+B = $\begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix}$ + $\begin{pmatrix} 3 & 1 & 4 \\ 4 & 0 & -1 \\ 5 & 2 & -2 \end{pmatrix}$ = $\begin{pmatrix} 4 & 3 & 7 \\ 2 & 0 & 0 \\ 9 & 7 & 4 \end{pmatrix}$ 3X3 //

A-B = $\begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix}$ - $\begin{pmatrix} 3 & 1 & 4 \\ 4 & 0 & -1 \\ 5 & 2 & -2 \end{pmatrix}$ = $\begin{pmatrix} -2 & 1 & -1 \\ -6 & 0 & 2 \\ -1 & 3 & 8 \end{pmatrix}$ 3X3//

(A+B)+C= $\begin{pmatrix} 4 & 3 & 7 \\ 2 & 0 & 0 \\ 9 & 7 & 4 \end{pmatrix} + \begin{pmatrix} -5 & 7 & 0 \\ 6 & 8 & 9 \\ 11 & 20 & -17 \end{pmatrix} = \begin{pmatrix} -1 & 10 & 7 \\ 8 & 8 & 9 \\ 20 & 27 & -13 \end{pmatrix}_{3X3}$//

A+(B+C) = $\begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 4 \\ 4 & 0 & -1 \\ 5 & 2 & -2 \end{pmatrix} + \begin{matrix} -5 & 7 & 0 \\ 6 & 8 & 9 \\ 11 & 20 & -17 \end{matrix} = \begin{pmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} -2 & 8 & 4 \\ 10 & 8 & 8 \\ 16 & 22 & -19 \end{pmatrix}$

= $\begin{pmatrix} -1 & 10 & 7 \\ 8 & 8 & 9 \\ 20 & 27 & -13 \end{pmatrix}_{3X3}$//

b). scalar multiplication of matrices

let A= (aij) $_{mxn}$ be a matrix x in IR & k ε IR .The scalar multiple of A by k, denoted by KA, is

the mxn matrix defined by : KA =K$\begin{pmatrix} a11 & a12 - - - a1n \\ a21 & a22 - - a2n \\ - - - - - - - \\ - - - - - - - \\ - - - - - - - \\ am1 & am2 - - amn \end{pmatrix}_{mxn}$

= $\begin{pmatrix} ka11 & ka12 - - - ka1n \\ ka21 & ka22 - - ka2n \\ - - - - - - - \\ - - - - - - - \\ - - - - - - - \\ kam1 & kam2 - - kamn \end{pmatrix}_{mxn}$

Examples: - 1. Let A= $\begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & 3 \end{pmatrix}$, then find

a). 3A, -$\frac{2}{3}$A, -5A ,2A+3B, A-$\frac{1}{2}$B, $\frac{-1}{3}$B, if B= $\begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 4 \\ 5 & 1 & 6 \end{pmatrix}$

Solution: -

1a). 3A= 3$\begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 3x2 & 3x1 & 3x1 \\ 3x(-2) & 3x(-1) & 3x3 \end{pmatrix} = \begin{pmatrix} 6 & 3 & 3 \\ -6 & -3 & 9 \end{pmatrix}$

Note: -The given matrix A with order mxn kA have the same dimension.

$-\frac{2}{3}A = -\frac{2}{3}\begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{-4}{3} & \frac{-2}{3} & \frac{-2}{3} \\ \frac{4}{3} & \frac{2}{3} & -2 \end{pmatrix}$//

$-5A = -5\begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & 3 \end{pmatrix} = \begin{pmatrix} -10 & -5 & -5 \\ 10 & 5 & -15 \end{pmatrix}$//

$2A+3B = 2\begin{pmatrix} 2 & 1 & 1 \\ -2 & -1 & 3 \end{pmatrix} + 3\begin{pmatrix} 0 & 1 & 2 \\ 1 & -1 & 4 \\ 5 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ -4 & -2 & 6 \end{pmatrix} + \begin{pmatrix} 0 & 3 & 6 \\ 3 & -3 & 12 \\ 15 & 3 & 18 \end{pmatrix}_{3X3}$

But A& B must have the same dimension 2A+3B can't be added.

Find matrix C such that A+2C =B, if 3A =2B if $A=\begin{pmatrix} 2 & 3 & 4 \\ 5 & -9 & 10 \\ 7 & 11 & -6 \end{pmatrix}$ & $B=\begin{pmatrix} 0 & 1 & 2 \\ 1 & -3 & 4 \\ 7 & 5 & 6 \end{pmatrix}$

A+2C=B 2C=B-A

2C = B-A

$C = \frac{B-A}{2}$, but $A=\frac{2}{3}B$ $C=\frac{B}{2}-\frac{A}{2}$ & $A=\frac{2}{3}B$ $C=\frac{B}{2}-\frac{\frac{2}{3}(B)}{2}$ $C=\frac{B}{2}-\frac{2}{6}B$ $=\frac{3B-2B}{6}=\frac{B}{6}$

$C = \frac{1}{6}\begin{pmatrix} 0 & 1 & 2 \\ 1 & -3 & 4 \\ 7 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{2} & \frac{2}{3} \\ \frac{7}{6} & \frac{5}{6} & 1 \end{pmatrix}$

C). **Matrix Multiplication**

Definition: - Two Matrices A & B are said to be conformable for the product AB (for every order of A&B). if the number of columns in A is equal to the number of rows in B.

If A=(aij) m x p & B=(bij) p x n be matrices in IR or in ∞ ,then the product of A & B denoted by AB is the m x n matrix AB=(cij)m x n. Where cij =ai1b1j +ai2b2j + ------+a i-p b p j, for i=1----m & j =1 -----n. i.e the (i j) th entry of the product AB is the sum of the product of the corresponding entries of the i th row of A with j th column of B.

Examples: - 1. Determine AB for each of the following matrices.

a). A= $(2 \;\; 3)_{1x2}$ & B= $\begin{pmatrix}4\\5\end{pmatrix}_{2\,x3}$

Solution: - since number of columns of A=number of rows of B.

AB is defined.

$AB = (2 \;\; 3)\begin{pmatrix}4\\5\end{pmatrix} = (2X4 + 3X5) = (8 + 15) = (23)_{1x1}$

b). A= $\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}_{2X2}$ & B= $\begin{pmatrix}6 & 8\\9 & 10\end{pmatrix}_{2X2}$

$$= AB = \begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}\begin{pmatrix}6 & 8\\9 & 10\end{pmatrix} = \begin{pmatrix}\begin{pmatrix}2 & 3\\4 & 5\end{pmatrix}\begin{pmatrix}6\\9\end{pmatrix} & (2\;3)\begin{pmatrix}8\\10\end{pmatrix}\\(4\;5)\begin{pmatrix}6\\9\end{pmatrix} & (4\;5)\begin{pmatrix}8\\10\end{pmatrix}\end{pmatrix} = \begin{pmatrix}39 & 46\\69 & 82\end{pmatrix}_{2X2} = AB$$

c). A= $\begin{pmatrix}1 & 3\\2 & 7\end{pmatrix}_{2X2}$ & B= $\begin{pmatrix}0 & 1 & -2\\4 & 9 & 3\end{pmatrix}_{2X3}$

$$AB = \begin{pmatrix}1 & 3\\2 & 7\end{pmatrix}\begin{pmatrix}0 & 1 & -2\\4 & 9 & 3\end{pmatrix} = \begin{pmatrix}(1\;3)\begin{pmatrix}0\\4\end{pmatrix} & (1\;3)\begin{pmatrix}1\\9\end{pmatrix} & (1\;3)\begin{pmatrix}-2\\13\end{pmatrix}\\(2\;7)\begin{pmatrix}0\\4\end{pmatrix} & (2\;7)\begin{pmatrix}1\\9\end{pmatrix} & (2\;7)\begin{pmatrix}-2\\3\end{pmatrix}\end{pmatrix} = \begin{pmatrix}12 & 28 & 7\\28 & 65 & 17\end{pmatrix}_{2X3}$$

Properties of Scalar Multiplication Of Matrices

Let m, n be positive integers, A & B be m x n matrices & r, s ε IR. Then

a). r(A+B) =Ra + Rb c). (rs)A =r(sA)

b). (r+s) A= rA + SA d). 1.A =A

e). if A and B are mx n matrices, then.

i. (-1) A = -A & A+(-A) =0 m x n = -A+A

-A is additive inverse of A

ii. A-B =A+(-1) B =A-B

Properties of Multiplication Of Matrices

Let A, B & C be matrices in IR & k ε IR. Assume that the indicated operations on matrices are defined then a). A(B+C) =AB+AC d). k(AB) =(KA)B

b). (A+B) C=AC+BC e). if 0 is the zero matrix of appropriate size then 0.A = 0 =A.0

c). A(BC)=(AB)C f). AB $\neq$ BA

Examples: -Let A be a 2 x 3 matrix & B be an m x n matrix. find m and n so that:

a). AB is defined b). BA is defined c). both AB & BA are defined.

Solutions: - a). AB is defined ==> numbers of A = numbers of B

- ⇨ A_{2X3} $B_{m\ x\ n}$ ↔m=3 and can be any natural number.
- ⇨ b). BA is defined $B_{m\ x\ n}$ $A_{2\ X\ 3}$ ↔n=2 and m can be any natural number.
- ⇨ c). both AB & BA defined m= 3 & n= 2

Transpose of A Matrix and Its Properties

Definition: - let A=(aij) mxn be an mx n matrix. then the matrix B= (bij) n xm Where bij =aij for 1≤j ≤n & 1≤ $i \leq m$ is called the transpose of A & denoted by A^t

i.e if A=$\begin{pmatrix} a11 & a12 & --- & a1n \\ a21 & a22 & --- & a2n \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ am1 & am2 & --- & amn \end{pmatrix}_{mx\ n}$, then

A^t=$\begin{pmatrix} a11 & a21 & --- & am1 \\ a12 & a22 & --- & am2 \\ - & - & - & - \\ - & - & - & - \\ - & - & - & - \\ a1n & a2n & --- & amn \end{pmatrix}_{n\ xm}$

The transpose of a matrix A denoted by at & is obtained by interchanging the rows & columns of A.

Properties of Transpose Of A Matrix

Let A and B be matrices such that the given operations are defined & k ε IR. then.

a). $(A \pm B)^t = A^t \pm B^t$ c). $(AB)^t = B^t. At$

b). $(kA)^t = kA^t$ d). $(A^t)^t = A$

Examples: - 1. Let A $=\begin{pmatrix}1 & 3\\4 & 5\end{pmatrix}$ & B= $\begin{pmatrix}0 & 1\\3 & 5\end{pmatrix}$, then find

a). $A^t = \begin{pmatrix}1 & 3\\4 & 5\end{pmatrix}^t = \begin{pmatrix}1 & 4\\3 & 5\end{pmatrix}$ c). $(A+B)^t = A^t + B^t = \begin{pmatrix}1 & 4\\3 & 5\end{pmatrix} + \begin{pmatrix}0 & 2\\1 & 3\end{pmatrix} = \begin{pmatrix}1 & 6\\4 & 8\end{pmatrix}$//

b). $B^t = \begin{pmatrix}0 & 1\\3 & 5\end{pmatrix}^t = \begin{pmatrix}0 & 2\\1 & 3\end{pmatrix}$ d). $(AB)^t = B^t. A^t = \begin{pmatrix}0 & 2\\1 & 3\end{pmatrix} \begin{pmatrix}1 & 4\\3 & 5\end{pmatrix} = \begin{pmatrix}6 & 10\\10 & 19\end{pmatrix}$//

e). $(B^t)^t = B = \left(\begin{pmatrix}0 & 1\\2 & 3\end{pmatrix}^t\right)^t = \begin{pmatrix}0 & 2\\1 & 3\end{pmatrix}^t = \begin{pmatrix}0 & 1\\2 & 3\end{pmatrix}$//

f). $(4AB)^t = 4B^tA^t = 4\begin{pmatrix}0 & 2\\1 & 3\end{pmatrix}\begin{pmatrix}1 & 4\\3 & 5\end{pmatrix} = \begin{pmatrix}0 & 8\\4 & 12\end{pmatrix}\begin{pmatrix}1 & 4\\3 & 5\end{pmatrix} = 4\begin{pmatrix}6 & 10\\10 & 19\end{pmatrix} = \begin{pmatrix}24 & 40\\40 & 76\end{pmatrix}$ //

Elementary row operations of matrices

Definition: - let m, n ε N & Amxn matrix with entries in IR. An elementary row operation on A is any one of the following operations.

1. Swapping: - interchanging two rows or columns. i.e Ri $\longleftrightarrow$ Rj or Ci $\longleftrightarrow$ Cj
2. Scaling: - multiplying a row or column of A by a non-zero constant.
   Multiplying the i^{th} row or j^{th} column by a non-zero scalar k denoted by Ri $\longleftrightarrow$ kRi or Cj =kCj.
3. Pivoting: - adding anon-zero constant multiple of one row of A on to another row or one column of A on to another column of A.
   Adding k times, the i^{th} row of A on to j^{th} row or the j^{th} row of B or the j^{th} column of A on to j^{th} column of A denoted by
   Rj$\leftrightarrow Rj + kRi \text{ or } Cj \leftrightarrow Cj + kCi$

Examples: - 1. Let A=$\begin{pmatrix}4 & 2\\1 & 3\end{pmatrix}$ apply each of the following elementary row operations one after the other on A & obtain a new matrix.

a) R1↔R2 = $\begin{pmatrix}4 & 2\\1 & 3\end{pmatrix} \rightarrow \begin{pmatrix}1 & 3\\4 & 2\end{pmatrix}$//

b) R2→ $R2 - 4R1$ = $\begin{pmatrix}1 & 3\\4 & 2\end{pmatrix} \longrightarrow \begin{pmatrix}1 & 3\\0 & -10\end{pmatrix}$//

c) R2 → $-\frac{1}{10}$R2 = $\begin{pmatrix}1 & 3\\0 & \frac{-10}{-10}\end{pmatrix}\begin{pmatrix}1 & 3\\0 & 1\end{pmatrix}$= //

Thus, $\begin{pmatrix}1 & 3\\0 & 1\end{pmatrix}$ is an upper triangular matrix.

Row Echelon form (REF)of a matrix.

Definition: - an m x n matrix A is said to be in REE if,

a) All rows consisting of (with no leading entries) entirely zeros, if any, are at the bottom of the matrix.

b) The leading entry (the first non-zero entry) in each row after the first is to the right of the leading entry in the previous row.

Examples: - 1. Determine if each of the following matrices are REE or not.

a) A=$\begin{pmatrix}1 & 2 & 2\\0 & 1 & 1\\0 & 0 & 1\end{pmatrix}$, B = $\begin{pmatrix}1 & -1 & 2\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{pmatrix}$ C= $\begin{pmatrix}1 & 1 & -1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 3\\0 & 0 & 0 & 1\end{pmatrix}$ D= $\begin{pmatrix}1 & 1 & 0 & 4\\0 & 1 & 0 & 0\\0 & 0 & 1 & 3\\0 & 0 & 1 & 1\end{pmatrix}$

E=$\begin{pmatrix}1 & 2 & 1 & -2\\0 & 2 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}$

Solutions: -

Matrices A, B & E are REE.

Matrices C and D are not in REE.

Reduced row Echelon form (RREF) of matrix

Definition: - a matrix A is said to be in RREE if A is in REE & the leading element in each non-zero row is 1 & it is the only non-zero number in its column.

Example: - **A**=$\begin{pmatrix}1&0&2&5\\0&1&2&6\\0&0&0&0\end{pmatrix}$ I3= $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ D= $\begin{pmatrix}1&0&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}$

A, D.I3 are RREF.

Systems of Linear Equations with Two or Three Variables

Definition: - an equation of the form a1x1, +a2x2+-----+anxn=b Where a1, a2, a3, --- an,b are real numbers ,is called a linear equation in the variables x1,x2 ---,xn. The variables x1, x2, x3, --- xn are also called unknowns.

A linear equation does not involve product of variables quotients of variables or roots of variables & all the variables that occur in the equation are only to power one.

Examples: -1. Which one of the following is linear equation & which are not?

a). $3x^2+5y=7$ b).$x-\sqrt{y}+4z=0$ c). xy-2z+w =1 d). 2x-y =5 e). 5x+6y+8z=15

Solutions: - All except d & e are not linear equation

Note: - The set of all possible solutions of a given linear equation is called the solution set or the general solution set of the given linear equation.

Augmented matrix of a system of linear equations

Definition: - let aij, bi εIR, for i=1,2----m & j=1,2 ----n a finite set of linear equations.

a11x1+a12x2+----+a1nxn =b1

a21x1+a22x2+-----+a2nxn =b2

am1x1+am2x2+------+am n x n =b m is called a system of m linear equations in n variables x1, x2, x3 -----x n if A=(aij) mx n X= $\begin{pmatrix} x1 \\ x2 \\ - \\ - \\ xn \end{pmatrix}$, B= $\begin{pmatrix} b1 \\ b2 \\ - \\ - \\ bm \end{pmatrix}$, then AX=B

A is called the coefficient matrix of the system.

A/B= $\begin{pmatrix} a11 & a12 & ---a1n & b1 \\ a21 & a22 & ---a2n & b2 \\ - & --- & --- & --- \\ - & --- & --- & --- \\ am1 & am2 & ---amn & bm \end{pmatrix}$ obtained by adjoining B to A is called the augmented the system .

Examples: - 1. The following system of linear equations, find the coefficient matrix & the augmented matrix.

a). $\begin{cases} x+y+z=8 \\ x+2y=5 \\ y+3z=9 \end{cases}$ b). $\begin{cases} x+y+z=1 \\ 2x-y-z=0 \\ 3x+5y-9z=-1 \end{cases}$ c). $\begin{cases} 2x+y=5 \\ x-3y=7 \end{cases}$

Solution: - 1a). Coefficient matrix

A= $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ & the augmented matrix (A/B) is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 8 \\ 1 & 2 & 0 & 5 \\ 0 & 1 & 3 & 9 \end{array}\right)$

b). A= $\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 3 & 5 & -9 \end{pmatrix}$ & A/B= $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & -1 & 0 \\ 3 & 5 & -9 & -1 \end{array}\right)$

c). A= $\begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$ & A/B = $\left(\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -3 & 7 \end{array}\right)$

Solutions Set of a System of Linear Equations

Definition: - consider the system AX=B of m linear equations in n variables .an order n-tuple of numbers.

$C = (c1, c2, c3----, cn)^t$ is a solution to the system AX=B if AC=B & the set of all solutions of AX=B is called the solution set of the given system.

Examples: -1. Solve the following system of linear equations.

a). $\begin{cases} x+2y=1 \\ y+3z=0 \end{cases}$ y=-3z & x+2(-3z) =1 x-6z =1 x= 6z+1

∴S. S {(x, y, z)} = {(6z+1, -3z, z): z ε IR}. if we let z=0, then x=1 y=0

{(1 ,0,0)} is a particular solution to the given system.

- ❖ A system of linear equations that has at least one solution is called a **consistent system.**
- ❖ A system that does not have a solution is called an **inconsistent system**.
- ❖ Consider a general system a1x+b1y=c1(a1 or b1≠ 0)

  a2x+b2y=c2(a2 or b2≠0)

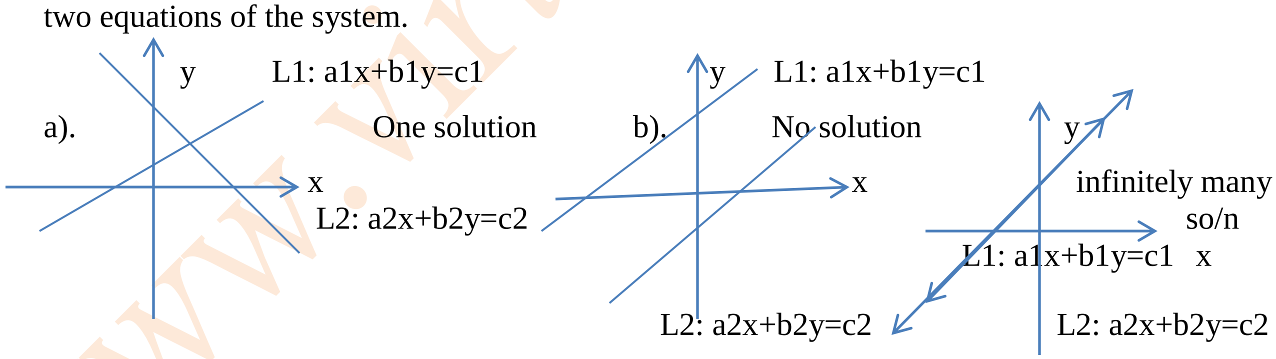
Let L1: a1x+b1y=c1

L2: a2x+b2y=c2

L1 & L2 intersect exactly at one point and this point of intersection is the only solution of the system.

L1 & L2 are two parallel lines & they do not intersect. The system has no solution.

L1 & L2 coincide (or identical).in this case, all the ordered pairs of numbers satisfying one of the two equations of the system.



- ❖ A given system of linear equations has either one so/n, not so/n or infinitely many solutions.
- ❖ A consistent system has either only one solution or infinitely many solutions.

Examples: -

a) $\begin{cases} x - y=1 \\ x + y=5 \end{cases}$ - The system has a unique solution.

b). $\begin{cases} x + y=1 \\ 2x+2y =4 \end{cases}$ - The system has no solution.

c). $\begin{cases} x -2y= 1 \\ -2x+4y=-2 \end{cases}$ - The system has an infinite number of solutions.

Homogenous systems of linear equations

Definition: - consider a system of linear equations.

$a_{11}x_1+a_{12}x_2+\text{---}+a_{1n}x_n =b_1$

$a_{21}x_1+a_{22}x_2+ \text{----}+a_{2n}x_n =b_2$

$a_{m1}x_1+a_{m2}x_2+ \text{----}+a_{mn}x_n =b_m$

if $b_1=b_2 = \text{--------} = b_m=0$, then the system is called a homogenous, otherwise, it is called anon-homogenous system.

- ❖ A homogenous system is a system of the form $AX=0$
- ❖ The given system has always a solution, namely, $x_1=x_2 = \text{-------} =x_n =0$. This solution is called the trivial solution of the homogenous system.
- ❖ A solution $(x_1, x_2, \text{-----}, x_n)^t$ of the given system for which $x_i \neq 0$, for some $i=1 ,2\text{-----}n$, is called a non-trivial solution of the given homogenous system.
- ❖ Two systems of linear equations over IR, with the same number of unknowns are said to be equivalent if every solution of one system is a solution of the other system.
- ❖ Consider a system $AX=B$

a). if the matrices (A/B) & (C/D) are row equivalent augmented matrices of two systems, then the systems $AX=B$ & $CX=D$ have the same solutions.

b). The Method of solving the system $AX=B$ by reducing (A/B) into row Echelon form (REF) is called Gaussian Elimination Method.

c). The method of solving the system $AX=B$ by reducing (A/B) in to Reduced row Echelon form (RREF) is called Gauss-Jordan Reduction Method.

examples: - 1. Solve the following system of linear equations using Gaussian Elimination Method.

a). $\begin{cases} x + y + z = 1 \\ x - y - z = 5 \\ x + y - z + 3 \end{cases}$ **Solutions**: -

the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 5 \\ 1 & 1 & -1 & 3 \end{array}\right)$$

Reduce this matrix row Echelon form using appropriate elementary row operations.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 5 \\ 1 & 1 & -1 & 3 \end{array}\right) \xrightarrow[R3 \to R3 - R1]{R2 \to R2 - R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right)$$

And the last matrix is in row Echelon form Thus, the systems with augmented matrices $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 5 \\ 1 & 1 & -1 & 3 \end{array}\right)$ and $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -2 & 4 \\ 0 & 0 & -2 & 2 \end{array}\right)$ are equivalent systems & hence they have the same solution set .

-2z =2 = z= -1

-2y – 2z =4 ==>-2y+2 = 4 y = -1 and x +y +z =1 =>x -1 =1 x-2 =1 **x=3**

S.S =**{(3,-1,-1)}.**

b). $\begin{cases} x + y + z = 3 \\ x - 2y + 3z = 1 \\ 2x + y - z = 2 \end{cases}$

The coefficient matrix is A=$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 2 & 1 & -1 \end{pmatrix}$ and the augmented matrix is A/B=$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -2 & 3 & 1 \\ 2 & 1 & -1 & 2 \end{array}\right)$

$$\xrightarrow{R2 \to R2 - R1} \quad \xrightarrow{R3 \to R3 - 2R1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & 2 & -2 \\ 0 & -1 & -3 & -4 \end{array}\right)$$

11z=10 z= $\frac{10}{11}$ & -3y+2z = -2

-3y + $\frac{20}{11}$ = -2 and x+ y +z =3

-3y = -2 - $\frac{20}{11}$ x + $\frac{14}{11}$ + $\frac{10}{11}$ = 3

-3y = - $\frac{42}{11}$ x= 3-$\frac{24}{11}$ x = $\frac{33-24}{11}$ =$\frac{9}{11}$

y = - $\frac{42}{33}$ = $\frac{14}{11}$ **S. S= {($\frac{9}{11}$, $\frac{14}{11}$, $\frac{10}{11}$)}**

c). $\begin{cases} x-y-z=0, \\ 2x+y+2z=3 \end{cases}$ A/B $=\left(\begin{matrix}1 & -1 & -1\\ 2 & 1 & 2\end{matrix}\middle|\begin{matrix}0\\3\end{matrix}\right)$ R2 $\rightarrow$ R2-2R1 $\left(\begin{matrix}1 & -1 & -1\\ 0 & 3 & 4\end{matrix}\middle|\begin{matrix}0\\3\end{matrix}\right)$

R2 $\rightarrow$ 3R1+R2 $\left(\begin{matrix}1 & -1 & -1\\ 0 & 3 & 4\end{matrix}\middle|\begin{matrix}0\\3\end{matrix}\right) \approx \left(\begin{matrix}1 & -1 & -1\\ 0 & 0 & 1\end{matrix}\middle|\begin{matrix}0\\3\end{matrix}\right)$

z=3,x-y-z =0

x-y-3 =0

x-y =3   x= y+3 but 3y+4z =3

z=$\frac{3-3y}{4}$ and   x- y –z =0

x –y - $\frac{3}{4}$ + $\frac{3}{4}$ y =0

4x-4y-3+3y =0

4x=y+3

X= $\mathbf{\frac{y+3}{4}}$//

We can express in terms of variable y.

S. S = {(x, y, z)} = {($\frac{y+3}{4}$,y,$\frac{3-3y}{4}$)}, y ε IR but the solution can be {(0, -3,3)}

Inverse of a square matrix

Definition: - The inverse of a square matrix A, denoted by A^{-1} , is the square matrix so that the product of A & A^{-1} is the identity matrix. i.e $AA^{-1}=A^{-1}A$ =I, the identity matrix that results will be the same size as the matrix A. or

- let A be a square matrix of order n. then A is said to be invertible (or has an inverse), if there exists a square matrix B of order n such that AB=In=BA.
- If such matrix B exists, then it is called an inverse of A.
- If square matrix A is invertible, then its inverse is unique.

Examples: - 1. Let A=$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$,find A^{-1}

Let A^{-1} =$\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ ==>$AA^{-1}=I_2$ $\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}$ $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ = $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

=$\begin{pmatrix}a & b\\ 2c & 2d\end{pmatrix}$ = $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ ==> a=1, b=0 2c =0 c=0 2d=1 d= $\frac{1}{2}$

$A^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

b). $A=\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$, let $A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $AA^{-1} = I_2$

$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+2c & b+2d \\ 3a+5c & 3b+5d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

a+2c =1 ,3a+5c =0, b+2d =0 ,3b+5d =1

a= 1-2c & b= -2d

5c+3(1-2c) =0 3(-2d) +5d =1

5c+3-6c=0 -6d+5d =1

-c+3=0 -d=1

c=3 a= -5 **d=-1** **b=2**

$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$

Properties of Inverse of Matrices

Let A, B be invertible matrices of the same size. then

a). A^{-1} is invertible & $(A^{-1})^{-1} = A$

b). At is invertible & $(A^t)^{-1} = (A^{-1})^t$

c). AB is invertible & $(AB)^{-1} = B^{-1}A^{-1}$

d). An invertible matrix is also called non-singular matrix.

We can find the inverse of a given square matrix using augmented matrix.

Examples: - 1. Using elementary row operations determine the invariability of each of the following matrices.

a). $A=\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

Solutions

A with $I_2 = \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right)$ $R_2 \to R_2 - R_1$ $\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$

$R_1 \to R_1 - R_2$ $\left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right)$ $A^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

b). $B=\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ B with I_3 B/I_3 $\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$$R_1 \rightarrow R_1 - R_2 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_1 \rightarrow R_1 + R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$R_2 \rightarrow R_2 - 2R_3 \quad \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \quad B^{-1} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

c). $C = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$ $\quad C/I_3 \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ $\quad R_1 \rightarrow R_1 - 2R_2 \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$R_1 \rightarrow R_1 + R_3 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$ $\quad R_2 \rightarrow R_2 - 2R_3 \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$

$$C^{-1} = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Applications

Examples: - 1. In a Δ, the smallest angle measures 10^0 more than half of the largest angle. The middle angle measures 12^0 more than the smallest angle. Find the measure of each angel of the Δ.

Solution: -

Let x be the measurement of the smallest angle.

Let y be the measurement of the middle angle.

Let z be the measurement of the largest angle of the Δ.

$x + y + z = 180^0$(sum of interior angles of any Δ gives 180^0)

$x = \frac{1}{2}z + 10 \qquad x + y + z = 180^0$

$y = 12 + x \qquad x - \frac{1}{2}z = 10$ & the augmented matrix is $\left(\begin{array}{c|ccc} 1 & 1 & 1 & 180 \\ 1 & 0 & -\frac{1}{2} & 10 \\ -1 & 1 & 0 & 12 \end{array}\right)$

$-x + y = 12$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 1 & 0 & -0.5 & 10 \\ -1 & 1 & 0 & 12 \end{array}\right) \xrightarrow[R_3 \rightarrow R_3 + R_1]{R_2 \rightarrow R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & -1 & -1.5 & -170 \\ 0 & 2 & 1 & 192 \end{array}\right)$$

$R_3 \rightarrow R_3+2R_2$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & -1 & -1.5 & -170 \\ 0 & 0 & -2 & -148 \end{array}\right)$ -2z =-148 z = 174

-y - $\frac{3}{2}$z = -170

- y - $\frac{3}{2}$.74 = -170 ==> -y =-170+111 y=170-111 y=59^0 & x=47^0

$\therefore The$ 3 angles are 47^0,59^0 & $74^{0.}$

***Examples* 2 : -** if a 30% salt solution is to be mixed with 20% salt to form a mixture of a 25% salt solution of 5000 liters how much of each is needed?

***solution* : -** let x be the amount of 30% salt solution needed & y be the amount of 20% salt solution needed. The total amount of the mixture must be 500 liters x+ y =500 and the amount of salt in the result is 25% of 500 liters 0.25 x 500=125

0.3 x +0.2y =125 $\left(\begin{array}{cc|c} 1 & 1 & 500 \\ 0.3 & 0.2 & 125 \end{array}\right)$

X + y =500 $R_1 \rightarrow R_2 + - 0.3R_1$ $\left(\begin{array}{cc|c} 1 & 0 & 500 \\ 0 & -0.1 & -25 \end{array}\right)$ -0.1y = -25 **<u>y= 250</u>**

and x + y =500 x =500 -250 = **<u>250</u>**

UNIT FOUR

DETERMINANTS AND THEIR PROPERTIES

The Determinant of a Matrix: - is a scalar number, obtained from the elements of a matrix by specified operations, which is the characteristics of the matrix. The determinants are defined only for square matrices.

- It is denoted by det(A) or $|A|$ for a square matrix A.
- Determinant of a matrix with

  1). Order 1: A=(a) = $|A|$= det(A)= $|a|$= a

  2). Order 2: A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$= $|A|$ = $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ = ad-bc or $\begin{pmatrix} a11 & a12 \\ a21 & a22 \end{pmatrix}$ ==>$|A|$ = $\begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix}$

  = a22a11-a21a12

Examples: -1. Determine the determinant of each of the following matrices.

a). A=$\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$ b). B=$\begin{pmatrix} 5 & 2 \\ 3 & 1 \end{pmatrix}$ c). $\begin{pmatrix} -5 & -7 \\ 6 & 8 \end{pmatrix}$

S*olutions*: - 1a) $|A|$ = $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$ = 4 x 1 -1 x 2 = 4-2 = **2**

b) $|B|$= $\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix}$ = 5 x 1 -3 x2 = 5-6 = **-1**

c). $|c|$=$\begin{vmatrix} -5 & -7 \\ 6 & 8 \end{vmatrix}$ = -5 x 8 – (6 x-7) = -40 +42 = **2**

2). If det(A) =0 & A=$\begin{pmatrix} 1 & x \\ 2 & x+1 \end{pmatrix}$ = 0 x+1-2x =0 –x+1 = 0, x=1

Minors & Cofactor of Elements of Matrices

- The minor or Mij of the element aij in each matrix is the determinant of order (n-1 x n-1) Obtained by deleting the ith row & jth column of Am x n. let n≥ 2, n ε N, A=(aij) nxn be a square matrix of order n.

Let $A=\begin{pmatrix} a11 & a12 & a13 --- & a1n \\ a21 & a22 & ------ & a2n \\ & & - & \\ & & - & \\ an1 & an2 & ---- & ann \end{pmatrix}$ n x n let (i,j), for $1\leq i \leq n$ & $1\leq j \leq n$

be an ordered pair of positive integers.

Cross out the i^{th} row and j^{th} column of A & obtain an (n-1) x (n-1) matrix and this matrix is denoted by Aij i.e

$$Aij = \begin{pmatrix} a11 & a12 & --- a1j --- & a1n \\ a21 & a22 & --- a2j --- & a2n \\ & & - & \\ & & - & \\ & & - & \\ ai1 & ai2 & --- aij --- & ain \\ an1 & an2 & --- anj --- & ann \end{pmatrix}$$ then det(Aij) is called the minor of the entry aij of A. denoted by Mij.

Example :- let $A=\begin{pmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{pmatrix}$ $|A|=\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$ M11 $=\begin{vmatrix} a22 & a23 \\ a32 & a33 \end{vmatrix}$ M12= $\begin{vmatrix} a21 & a23 \\ a31 & a33 \end{vmatrix}$

M13= $\begin{vmatrix} a21 & a22 \\ a31 & a32 \end{vmatrix}$ M21= $\begin{vmatrix} a12 & a13 \\ a32 & a33 \end{vmatrix}$ M22 = $\begin{vmatrix} a11 & a13 \\ a31 & a33 \end{vmatrix}$ M23 = $\begin{vmatrix} a11 & a22 \\ a31 & a32 \end{vmatrix}$ M31 = $\begin{vmatrix} a12 & a13 \\ a22 & a33 \end{vmatrix}$

M33 = $\begin{vmatrix} a11 & a12 \\ a21 & a22 \end{vmatrix}$

- The scalars c i j = $(-1)^{i+j}$ Mij are called the cofactor of the element a i j.

  Examples: - 1. Determine the minors & cofactor of all the entries of the following matrices.

  a). $A=\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & -1 & 5 \end{pmatrix}$ b). $B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

  <u>Solutions</u>: - M11 = $\begin{vmatrix} 1 & 0 \\ -1 & 5 \end{vmatrix}$ = 5, C11= 5

M12 = $\begin{vmatrix} 0 & 0 \\ 3 & 5 \end{vmatrix}$ = 0, C12 = 0 M13 = $\begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix}$ = -3, C13 =-3 M21 = $\begin{vmatrix} 2 & 1 \\ -1 & 5 \end{vmatrix}$ = C21 = -11

M22 = $\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix}$ = 5-3 = 2, C22 = 2 M23= $\begin{vmatrix} 1 & 2 \\ 3 & -1 \end{vmatrix}$ =-1-6 = -7, C23 =7 M31 = $\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix}$ = -1, C31 = -1

M32 = $\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}$ = 0, C32= 0 M33= $\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$ =1, C33= 1

b). M11= $\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$ = 45-48 = -3, C11= -3 M12 = $\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix}$ = 36- 42 = -6 C12 = 6

M13= $\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$ = 32-35 = -3 , C13 = -3 M21= $\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix}$ = 16-24 = -8 , C21 =8 M22 = $\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix}$ =9-21= -12, C22 = -12 M23= $\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$ = 8-14 = -6 , C23 = 6 M31= $\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$ = 12-15 = -3 , C31 = -3

M32 = $\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$ = 6-12 = -6, C32 =6 M33 = $\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$ = 5-8 = -3, C33 = -3

Determinants of Matrices of Order 3

Definition: - let A =(Aij) $_{3x3}$ be a square matrix of order 3, then $|A|$ is the number, defined by det(A) = A11c11+a12c12+a13c13 & this sum is called the cofactor expansion of the determinant along the first row.

Examples: - 1. Find the determinant of a). A= $\begin{pmatrix} 2 & 7 & 9 \\ 3 & 5 & 6 \\ 7 & 4 & 1 \end{pmatrix}$ $|A|$ = $\begin{vmatrix} 2 & 7 & 9 \\ 3 & 5 & 6 \\ 7 & 4 & 1 \end{vmatrix}$= a11c11+a12c12+a13c13

=2(-1)$^{1+1}\begin{vmatrix} 5 & 6 \\ 4 & 1 \end{vmatrix}$ +7(-1)$^{1+2}$ $\begin{vmatrix} 3 & 6 \\ 7 & 1 \end{vmatrix}$ + 9(-1)$^{1+3}$ $\begin{vmatrix} 3 & 5 \\ 7 & 4 \end{vmatrix}$ = 2(-19) -7(-39) +9(-23) = ***28***

N.B: - The determinant of a given matrix A can be expressed as a cofactor expansion along any row of A. i.e $|A|$ = ai1ci1 + ai2ci2 +ai3ci3 for each I = 1,2,3.

Checkerboard

$\begin{pmatrix} + & - \\ - & + \end{pmatrix}$ & $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

For order 2 for order 3

Examples: - 1. find the determinant of

a). A= $\begin{pmatrix} + & - & + \\ 2 & 1 & 0 \\ 3 & -2 & 2 \\ 1 & 0 & 2 \end{pmatrix}$ $|A|$= 2 $\begin{vmatrix} -2 & 2 \\ 0 & 2 \end{vmatrix}$ -1 $\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$ + 0$\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix}$

= 2(-4) -1(6-2) +0(0+2) = -8-4 = **-12**

properties of determinants

1. The determinant of a triangular matrix is the product of the main diagonal elements.

Example: - A=$\begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$, $|A|$ = 2 x -1 x1 = **-2**

2. The determinant of an identity matrix is 1 i.e A= $\begin{pmatrix} 1 & 0 & - & - & - & 0 & 0 \\ 0 & 1 & - & - & - & 0 & 0 \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ 0 & 0 & - & - & - & 0 & 1 \end{pmatrix}$, $|A|$ = 1

3. If A is a square matrix of order n& k ϵ IR k $\neq 0$, then A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, if $|A|$= -2, then

a)$\begin{vmatrix} 3a & 3b \\ 3c & 3d \end{vmatrix}$ = $3^2|A|$ = 9 x -2 = -18 b). $\begin{vmatrix} \frac{d}{2} & \frac{-b}{2} \\ \frac{-c}{2} & \frac{a}{2} \end{vmatrix}$ = $\frac{d}{2} . \frac{a}{2}$ $\frac{-(-b)}{2}$ $(-\frac{c}{2})$ = $\frac{1}{4}$(ad- bc)

= $\frac{1}{4}$(-2) = - $\mathbf{\frac{1}{2}}$// **ii.** If A= $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ & $|A|$ = 2, then a). $\begin{vmatrix} 4a & 4b & 4c \\ 4d & 4e & 4f \\ 4g & 4h & 4i \end{vmatrix}$ = $4^3|A|$ = 64 x 2 = **128**

4. $|AB|$= $|A|$. $|B|$ (6) $|A^n|$= $|A|^n$, n $\in N$
5. $|A|$= $|A^t|$ **Exampl**e: - Let $|A|$= 4 & $|B|$== 5 , & A_{3X3} , B_{3X3}, then find

a). $|AB|$= $|A|$. $|B|$= 4 x 5 = 20 b). $|3A|$=$3^3|A|$= 27 x 4 = 108

c). $|2AB^t|$=$2^3|A|$. $|B|$= 8 x 4 x 5 = 160 d). $|A^5|$= $|A|^5$ = 4^5 = 1028

6. $|A^{-1}|$=$\frac{1}{|A|}$
7. If two rows or two columns of a determinant are interchanged, then the sign of the determinant is changed.

Example:- let A = $\begin{pmatrix} a1 & b1 & c1 \\ a2 & b2 & c2 \\ a3 & b3 & c3 \end{pmatrix}$, if $|A|$= k & B = $\begin{pmatrix} a2 & b2 & c2 \\ a1 & b1 & c1 \\ a3 & b3 & c3 \end{pmatrix}$, then $|B|$= - k

8. If every element of a row or column of a determinant is zero, then the value of the determinant is zero.
9. If every element of a row or column of a determinant is multiplied by the same non-zero constant k, then the value of the determinate is multiplied by that constant.
10. If two rows or columns of a determinant are identical, then the value of the 12. $|adjA|$ = $|A|^{n-1}$, n is order of A determinant is zero.

Inverse of a square matrix

Ad joint of A Square Matrix

Definition :- Let A= (aij) n x n be a square matrix of order n & cij= $(-1)^{i+j}|Aij|$ be the cofactor of aij, for $1\leq i \leq n$ & $1 \leq j \leq n$. Then the matrix $(cij)^t_{n \times n}$ is called adjoint of A & it is denoted by Adj(A) for any 2 x 2 square matrix A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ adj A= $\begin{pmatrix} d & -b \\ -c & d \end{pmatrix}$.

Example: - 1. Find the Adjoint of the following matrices.

a). A= $\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$ b). A= $\begin{pmatrix} 1 & 2 & 3 \\ 6 & 1 & 9 \\ 4 & 0 & 5 \end{pmatrix}$

Solution: - 1a). adj A= $\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$ b). C11 =5 C12 = $(-1)^{1+2}\begin{vmatrix} 6 & 9 \\ 4 & 5 \end{vmatrix}$ = (-1) (30-36) = (-1) (-6) = **6**

C13 = $\begin{vmatrix} 6 & 1 \\ 4 & 0 \end{vmatrix}$ = **-4** C21= $-\begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix}$ = **-10** C22= $\begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix}$ = 5-12 = **-7** C23= $-\begin{vmatrix} 1 & 2 \\ 4 & 0 \end{vmatrix}$ = -(-8) = **8**

C31= $\begin{vmatrix} 2 & 3 \\ 1 & 9 \end{vmatrix}$ = 18-3 = **15** C32= $-\begin{vmatrix} 1 & 3 \\ 6 & 9 \end{vmatrix}$ = -(9-18) = **9** C33= $-\begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix}$ = 1-12 = **-11**

Adj A = $(cij)^t$ adj A= $\begin{pmatrix} c11 & c12 & c13 \\ c21 & c22 & c23 \\ c31 & c32 & c33 \end{pmatrix}^t = \begin{pmatrix} 5 & 6 & -4 \\ -10 & -7 & 8 \\ 15 & 9 & -11 \end{pmatrix}^t$ adj A = $\begin{pmatrix} 5 & -10 & 15 \\ 6 & -7 & 9 \\ -4 & 8 & -11 \end{pmatrix}$

Theorem 4.6: - Let A be a square matrix of order n, then A(adj(A)) = adj(A). A = $|A|$ in, if $|A| \neq 0$, then A is invertible & its inverse is given by $A^{-1} = \frac{adjA}{|A|}$.

Examples: - 1. Find the inverse of the following matrices.

a) A= $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, b). A = $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 4 \\ 5 & 1 & 5 \end{pmatrix}$

Solution: - a). adj A= $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$, $|A|$ = 3-2 = 1

$A^{-1} = \frac{adj\ A}{|A|} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{3} & \mathbf{-1} \\ \mathbf{-2} & \mathbf{1} \end{pmatrix}$//

b). C11= 1 C21= -9 C31=7

C12 = 5 C22 =0 C32= -1

C13= -2 C23=9 C33= -5

Adj A= $\begin{pmatrix} c11 & c12 & c13 \\ c21 & c22 & c23 \\ c31 & c32 & c33 \end{pmatrix}^t = \begin{pmatrix} 1 & 5 & -2 \\ -9 & 0 & 9 \\ 7 & -1 & -5 \end{pmatrix}^t = \begin{pmatrix} 1 & -9 & 7 \\ 5 & 0 & -1 \\ -2 & 9 & -5 \end{pmatrix}$ and

$|A| = \begin{vmatrix} + & - & + \\ 1 & 2 & 1 \\ 3 & 1 & 4 \\ 5 & 1 & 5 \end{vmatrix} = 1\begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} - 2\begin{vmatrix} 3 & 4 \\ 5 & 5 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 5 & 1 \end{vmatrix}$ = 5-4-2(15-20 +1(3-5) = 1+10 -2 = **9**

$A^{-1} = \frac{adjA}{|A|} = \frac{\begin{pmatrix} 1 & -9 & 7 \\ 5 & 0 & -1 \\ -2 & 9 & -5 \end{pmatrix}}{9}$ $A^{-1} = \begin{pmatrix} \frac{1}{9} & -1 & \frac{7}{9} \\ \frac{5}{9} & 0 & \frac{-1}{9} \\ \frac{-2}{9} & 1 & \frac{-5}{9} \end{pmatrix}$

$N.B:-$ adj (AB) = (adjB) (adj A)

Solution of System of Linear Equations Using Cramer's Rule (Unique Solution)

Theorem 4.7: - (Cramer's rule for two variable)

Given the system ax+ by = e in two cx +dy = f variables x and y, if ad- bc ≠0, then the given system has a unique solution & it is given by x= $\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{ad-bc}$, y= $\frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{ad-bc}$ i.e x= $\frac{DX}{D}$, y= $\frac{Dy}{D}$, where D- the determinant of the coefficient matrix & Dx = $\begin{vmatrix} e & b \\ f & d \end{vmatrix}$, Dy = $\begin{vmatrix} a & e \\ c & f \end{vmatrix}$

Example: - solve the following using Cramer's rule. (if possible)

a). $\begin{cases} x + y = 2 \\ 2x + 3y = 4 \end{cases}$ b). $\begin{cases} 2x + y = 9 \\ 3x - 4y = 2 \end{cases}$

A = $\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $|A|$ = 3-2 = 1 Dx = $\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix}$ = 6-4 = 2 Dy = $\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix}$ = 4-4 = 0 x = $\frac{Dx}{D} = \frac{2}{1}$ =2

& y= $\frac{Dy}{D} = \frac{0}{1}$ = 0 ∴ S.S = {(2,0)} b). $|A| = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix}$ = -8-3 = **-11**

Dx = $\begin{vmatrix} 9 & 1 \\ 2 & -4 \end{vmatrix}$ = -36 -2 = **-38** Dy = $\begin{vmatrix} 2 & 9 \\ 3 & 2 \end{vmatrix}$ = 4-27 = **-23** x = $\frac{Dx}{D} = \frac{-38}{-11} = \frac{38}{11}$, Dy = $\frac{-23}{-11} = \frac{23}{11}$

S. S= {($\frac{38}{11}$, $\frac{23}{11}$)}.

Theorem 4.7 (Cramer's rule for three variables)

Given a11x+a12y +a13z = b1

a21x+a22y +a23z = b2

a31x+a32y+a33z = b3, in three variables x, y & z if

$\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix} \neq 0$, then the given system has a unique solution given by x = $\frac{Dx}{D}$, y =$\frac{Dy}{D}$, Z =$\frac{Dz}{D}$, Where D = $\begin{vmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{vmatrix}$ Dx = $\begin{vmatrix} b1 & a12 & a13 \\ b2 & a22 & a23 \\ b3 & a32 & a33 \end{vmatrix}$, Dy = $\begin{vmatrix} a11 & b1 & a13 \\ a21 & b2 & a23 \\ a31 & b3 & a33 \end{vmatrix}$ Dz= $\begin{vmatrix} a11 & a12 & b1 \\ a21 & a22 & b2 \\ a31 & a32 & b3 \end{vmatrix}$

Examples: - 1. Use Cramer's rule, if possible, & solve each of the following systems of linear equations.

a). $\begin{cases} x + y + z = 3 \\ -2x + 2y + 2z = 5 \\ x + y + 2z = 4 \end{cases}$ b). $\begin{cases} x + y - z = 1 \\ 2x + z = 4 \\ y - z = 5 \end{cases}$

Solutions: - a). D = $\begin{vmatrix} + & - & + \\ -2 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix}$ D = $\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}-1\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix}$+$\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix}$ D= (4-2) + -1(-4-2) +1(-2-2) =2+6-4 = 8-4 = $\underline{\underline{4}}$

Dx = $\begin{vmatrix} + & - & + \\ 3 & 1 & 1 \\ 5 & 2 & 2 \\ 4 & 1 & 2 \end{vmatrix}$= 3$\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix}$ -1 $\begin{vmatrix} 5 & 2 \\ 4 & 2 \end{vmatrix}$ + 1$\begin{vmatrix} 5 & 2 \\ 4 & 1 \end{vmatrix}$ = 3(4-2) -1(10-8) +1(5-8) = 3(2) -1(2) = (-3) = 6-2-3 = $\underline{\underline{1}}$

Dy= $\begin{vmatrix} + & - & + \\ 1 & 3 & 1 \\ -2 & 5 & 2 \\ 1 & 4 & 2 \end{vmatrix}$= 1$\begin{vmatrix} 5 & 2 \\ 4 & 1 \end{vmatrix}$ -3$\begin{vmatrix} -2 & 2 \\ 1 & 2 \end{vmatrix}$+1$\begin{vmatrix} -2 & 5 \\ 1 & 4 \end{vmatrix}$ = 1(10-8)-3(-4-2) +1(-8-5) = 2+18-13=$\underline{\underline{7}}$

$$Dz = \begin{vmatrix} + & - & + \\ 1 & 1 & 3 \\ -2 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix} = 1\begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} - 1\begin{vmatrix} -2 & 5 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = 8\text{-}5\text{-}1(\text{-}8\text{-}5) + 3(\text{-}2\text{-}2) = \underline{\mathbf{4}}$$

$$x = \frac{Dx}{D},\ y = \frac{Dy}{D},\ z = \frac{Dz}{D}$$

$$x = \frac{1}{4},\ y = \frac{7}{4},\ z = \frac{4}{4} = 1$$

S. S= $\mathbf{\{(= \frac{1}{4}, = \frac{7}{4}, 1)\}}$//

<u>Systems of linear equations with no solution or infinitely many solutions</u>

Given a system AX =B of n- linear equations in n- variables, $n\epsilon N$ if $|A| = 0$, we cannot apply Cramer's rule to solve the system.

If $|A|= 0$, there are two possibilities

- ✓ The system has no solution.
- ✓ The system has infinitely many solutions.

  Such types of systems of equations can be solved using augmented matrix by applying a series of elementary operations.

  <u>Applications: -</u>

  1. ***<u>Polynomial interpolation</u>***

     <u>Examples:</u> - 1. Consider the data points (0,1), (1,2) & (2,22). Find an interpolating polynomial p(x) of degree at most two & estimate the value of p (3).

     Let p(x) = a+ b x +cx^2

     P (0) = 1 =a= 1, p (1) = a+ b+ c =2

     P (2) = 22 ==>a+2b+4c= 22

     a=1

     a+ b+ c =2

     a+2b+4c =22, this gives us a system of linear equations. the coefficient matrix is.

A= $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$ & $|A|$= 2≠ 0 a= $\frac{Da}{D}$, b = $\frac{Db}{D}$, c = $\frac{Dc}{D}$

a= $\frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 22 & 2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}} = \underline{\underline{\mathbf{1}}}$ b = $\frac{\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 22 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}} = \frac{\mathbf{-17}}{\mathbf{2}}$// c = $\frac{\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 22 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{vmatrix}} = \frac{19}{2} = \underline{\underline{\mathbf{9.5}}}$

2). *Area of a* Δ *in xy – plane*

Determinant can be used to find the area of a triangle whose three vertices are in the x y – plane.

Let A (x1, y1) , B(x2,y2), c(x3, y3) be the vertices of a given Δ in the x y- plane , then

$A(\Delta ABC) = \frac{1}{2}\begin{vmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{vmatrix}$, if $\begin{vmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{vmatrix} > 0$ & $A(\Delta ABC) = -\frac{1}{2}\begin{vmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{vmatrix}$, if $\begin{vmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{vmatrix} < 0$.

Examples: - 1. Find the area of the triangle with vertices.

a). A (-2,0), (0,2), (2,0)

Solution: - $A(\Delta ABC) = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ but $\begin{vmatrix} + & - & + \\ 2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ = -2 $\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$ + 0 $\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}$+1$\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$ = -2(2) +0-4 =-4-4 = $\underline{\underline{\mathbf{-8}}}$ since $|A| < 0$ $A(\Delta ABC) = \frac{-1}{2}|A| = \frac{-8}{-2} = \underline{\underline{\mathbf{4}}}$

Test for Collinear Points In The XY- Plane

Three points in XY- plane are said to be collinear if all the three points lie on the same plane.

Let points A(x1, y1), B(x2,y2) & C(x3,y3) be points are collinear if $\begin{vmatrix} x1 & y1 & 1 \\ x2 & y2 & 1 \\ x3 & y3 & 1 \end{vmatrix} = 0$

Example: - Determine if the points A (1,1), B (2,2), C (3,3) are collinear or not.

Solution: - $\begin{vmatrix} + & - & + \\ 1 & 1 & 1 \\ 2 & 2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$ = 1$\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$ -1 $\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$ +1$\begin{vmatrix} 2 & 2 \\ 3 & 3 \end{vmatrix}$ = 2-3-(2-3) +6-6 = -1+1+0 = $\underline{\underline{\mathbf{0}}}$

∴ A, B & C are Collinear Points.

Unit five

Vectors

Revision on vectors & scalars

Physical quantity

- Physical quantity is the measurable & quantitative physical property that carries unique information with it. Based on the dependency of direction, physical quantities can be classified in two categories.
  1. Scalar
  2. Vector

1. **Scalar**: - The quantity, which has only magnitude & no direction, is termed as a scalar quantity.

   Example: - mass, density, volume etc.
2. **Vector**: - The physical quantity, which comprise of both magnitude & direction, is termed as a vector.

   Example: - velocity, force, displacement, acceleration, weight, torque etc.

- Vector quantity can be one, two or three dimensions.
- Vector quantity changes with the change in their direction or magnitude or both.
- Vectors can be resolved in any direction.

Representation of a vector

- A vector is represented by:

a). boldface letters as a, b v, u etc.

b). a letter with an arrow directed to the right above the letter as:

$\underset{a}{\rightarrow}, \underset{b}{\rightarrow}, \underset{v}{\rightarrow}, \underset{u}{\rightarrow}$ etc.

c). A directed line segmented as $\underset{AB}{\rightarrow}$

The arrow indicates the direction of the vector & the length represents the magnitude of the vector.

- Let V be a vector in the plane with initial point p (x1, y1) & terminal point Q (x2, y2), then.

1. Its coordinate form is (x2-x1, y2-y1)
2. Its column form is $\begin{pmatrix} x2 - x1 \\ y2 - y1 \end{pmatrix}$
3. Its norm (magnitude) is given by $|V|$ = $\sqrt{(x2 - x1)^2 + (y2 - y1)^2}$

<u>Examples</u>: - 1. Let V= $P\underset{Q}{\rightarrow}$ with p (1,2) & Q (6,7), find

a) Its coordinate form:

= (x2-x1, y2-y1) = (6-1, 7-2) = **<u>(5,5)</u>**

b). Its column form $\begin{pmatrix} x2 - x1 \\ y2 - y1 \end{pmatrix}$ = $\begin{pmatrix} 6 - 1 \\ 7 - 2 \end{pmatrix}$= $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$

c). Its norm, $|V|$ = $\sqrt{(X2 - X1)^2 + (y2 - y1)^2}$

= $\sqrt{(6 - 1)^2 + \ (7 - 2)^2}$

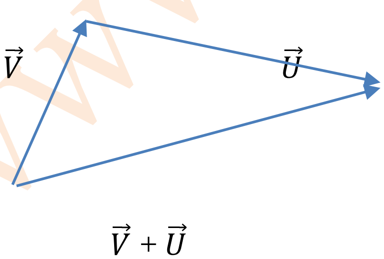
= $\sqrt{5^2 + 5^2}$= $\sqrt{50}$

= $\sqrt[5]{\mathbf{2}}$//

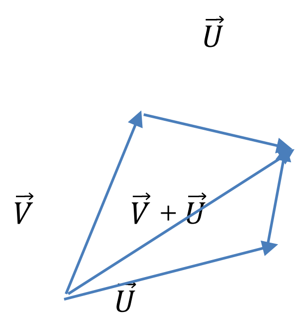
<u>Operations on vectors</u>Vectors can be added using triangle rule and parallelogram.

Let $\vec{V}$ & $\vec{U}$ be two vectors, then

a). $\vec{V}$

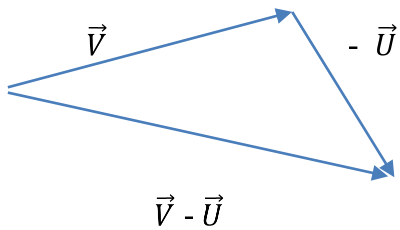


Triangle law



parallelogram law

Subtraction of vectors



Multiplication of vectors by scalars

Let $\vec{V}$ be a given vector.

- The product of V & a scalars k is the vector that is k times as larges $\vec{V}$
- If k $>, k\vec{V}$ has the same direction as $\vec{V}$
- If k$<$ 0, k$\vec{V}$ has opposite direction to $\vec{V}$.
- K $\vec{V}$ is a scalar multiple of $\vec{V}$

Examples: - 1. If v= (5, -1), w= (2, -5) & u= (3,4), then find a). v+ u, u-v, (u +v) +w, (u-v)-w

b). 5v+3u, -3u+7v, $\frac{-5}{2}$ w + $\frac{2}{3}\vec{V}$

Solutions: - a). v+ u = (5, -1) + (3,4) = (5+3, -1+4) = **(8,3)**//

u-v= (3,4)- (5, -1) = (3-5,4-(-1) = **(-2,5)**//

(u +v) + w **=** (8,3) + (2, -5) = (8+2,3+(-5) = **(10, -2)**//

(u-v)-w = ((3,4) -(5, -1))- (2,-5) = ((3-5,4-(-1))-(2,-5) = (-2,5) –(2,-5)= **(-4,10)**//

b). 5(5, -1) +3(3,4) = (25, -5) +(9,12) = **(34,7)**//

-3u+7v= (-9, -12) +(35, -7) **= (26, -19)**//

$\frac{-5w}{2} + \frac{2v}{3}$ = (-5,$\frac{25}{2}$)+($\frac{10}{3}$,$\frac{-2}{3}$)= **($\frac{-5}{3}$,$\frac{71}{6}$)**//

Unit vectors

- A vector whose magnitude is one is called a unit vector.
- The standard unit vectors are i = (1,0) & j= (0,1)
- Any vector can be express as a linear combination of unit vector i & j

  Let $\vec{v}$ = (a, b)

  $\vec{v}$ = (a, b) = a (1, 0) + b (0,1) = ai + bj

  $\vec{v}$ = (2,3) = 2(1,0) +3(0,1) = 2i + 3j

  $\vec{v}$ = (-4,8) = -4(1,0) + 8(0,1) = -4i+8j

<u>unit vector in the direction of a vector in the direction of a vector $\vec{v}$</u>

if $\vec{v}$ is a non-zero vector in the plane, then the vector $\vec{U}$ = $\vec{V}$ has length 1 & the $|\vec{V}|$ same direction as $\vec{u}$

Example: - find unit vector $\vec{U}\ \vec{v} - \vec{u} = \frac{\vec{v} - \vec{U}}{|\vec{v} - \vec{u}|} = \frac{(2,3)}{\sqrt{2^2+3^2}} = \frac{(2,3)}{\sqrt{13}} = \mathbf{(\frac{2}{\sqrt{13}}, \frac{3)}{\sqrt{13}}}$//

i. In the direction of $\vec{V}$

ii. In the direction opposite $\vec{V}$

iii. In the direction of $\vec{V}$ + $\vec{U}$

iv. In the direction of $\vec{V}$ - $\vec{U}$ if

a). $\vec{v}$ = (3,6), $\vec{U}$ = (1,3).

<u>Solutions</u>: - ai). $\vec{U} = \frac{\vec{v}}{|v|}$, $\quad \vec{v} = \sqrt{3^2 + 6^2} = \sqrt{25} = 5$

$\vec{U} = \frac{(3,6)}{5} = (\frac{3}{5}, \frac{6}{5}$

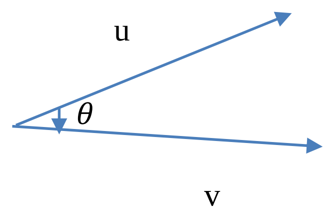
$\vec{U} = \frac{(-3,-6)}{5} = (\frac{-3}{5}, \frac{-6}{5})$

$\vec{U}$ v + u = $\frac{\vec{v} + \vec{U}}{|\vec{v} + \vec{U}|} = \frac{(4,9)}{\sqrt{97}} = \mathbf{\frac{4}{\sqrt{97}}, \frac{9}{\sqrt{97}}}$//

scalar (inner or dot)product

Theorem: - if u & v are vectors & θ is the angle between u & v , then the dot product of u & v , denoted by u. v is given by

u. v = $|u|\ |v| \cos\theta$



if θ is a cute then u. v > 0

if θ is obtuse, then $\vec{u}\ .\ \vec{v} < 0$

if θ is $\frac{\pi}{2}$, then $\vec{u}\ .\ \vec{v} = 0$ ($\vec{u}\ \lrcorner\ \vec{v}$)

if $\theta = 0^0$, then $\vec{u}\ //\ \vec{v}$.

If $\theta = \pi$, then $\vec{u}\ //\ \vec{v}$ but opposite in direction.

Example: - 1. Find the dot product of the vectors u & v when.

a). u= (2,0) & v= (4,0)

b). u= (0,3) & v= ($2\sqrt{3}$, 2)

Solutions: - a). u. v = (2,0), (4,0) = 2.4 + 0.0 = **8** b). u. v = (0,3). ($2\sqrt{3}$,2) = 0. $2\sqrt{3}$ + 3.2 = **6**

Properties of dot product

If $\vec{u}, \vec{v}, \vec{w}$ be vectors & k ∈ IR.

- ✓ $\vec{u}\ .\ \vec{v} = \vec{v}\ .\ \vec{u}$ -> $\vec{u}.(\ \vec{v} + \vec{w}\ ) = \vec{u}\ .\ \vec{v} + \vec{u}\ .\ \vec{w}$ -> $|\vec{u}^2| = \vec{u}\ .\ \vec{u} = |\vec{u}|.|\vec{u}|$
- ✓ k ($\vec{u}\ .\ \vec{v}$) =k$\vec{u}\ .\ \vec{v} = \vec{u}\ .\ k\vec{v}$ ->$|\vec{u} + \vec{v}| = (\vec{u} + \vec{v}).\ (\vec{u} + \vec{v}) = |\vec{u}^2| + 2\vec{u}.\vec{v} + |\vec{v}^2|$
- ✓ $|\vec{u} - \vec{v}| = |\vec{u} - \vec{v}|.\ |\vec{u} - \vec{v}| = |\vec{u}|^2 - 2\vec{u}\ .\ \vec{v} + |\vec{v}|^2$
- ✓ $if\ \vec{u}$

(u1, u2) & $\vec{v}$ = (v1, v2) are vectors, then u. v = u1v1+ u2v2

Examples: - 1. Let $|u|$ = 3, $|v|$= 6, $\theta = \frac{\pi}{3}$, then find a).$|3u - 2v|^2$= (3u-2v) (3u-2v)

=$9u^2$-12u.v + $4v^2$ = 9. 3^2-12$|u||v|$cos$\frac{\pi}{3}$ +4.6^2

= 81-12.3.6.$\frac{1}{2}$ + 144 since u.v = $|\vec{u}|.|\vec{v}|cos\theta$ = **117**

$|3u - 2v|$ = $\sqrt{\mathbf{117}}$ //

b). Find the cosine of the angle between 3u-2v & u let θ be the angle between (3u-2v) & u.

then (3u-2v). u = $|3u - 2v|.|u|$ cosθ

3. 3^2-2 x 6 x 3 x $\frac{1}{2}$ = $\sqrt{117}$. 3 cosθ

= 9 = 3$\sqrt{117}$cosθ

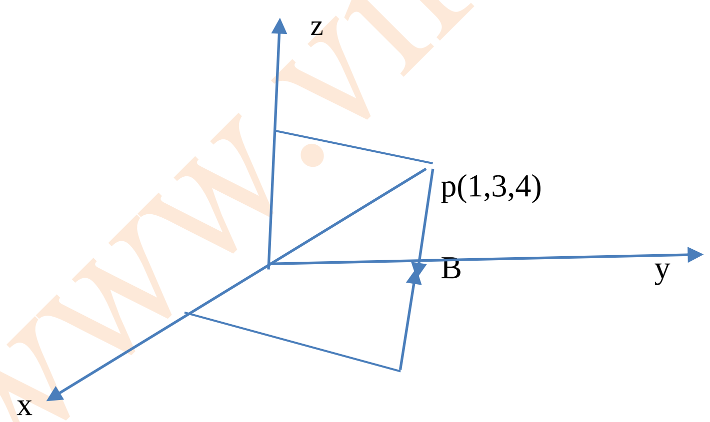
Cos θ = $\frac{\mathbf{3}}{\sqrt{\mathbf{117}}}$//

Vectors in three-dimensional space

Coordinate of a point in space

It is common to have the x & the y-axis on a horizontal plane & the z-axis vertical or perpendicular to the plane containing the x & the y- axis at the point.

The direction of the z axis is determined by the right-hand rule.



Vectors in Space

- it is represented by the directed line segment from p (x1, y1, z1) to Q (x2, y2, z2), then the component of V is $(P\vec{Q})$
  (x2-x1, y2-y1)
- its magnitude is $|v|= \sqrt{(x2-x1)^2+(y2-y1)^2+(z2-z1)^2}$

Example: - Let the initial point of v= (4,2, -1) & its terminal point is (7,1,5), then find $|v|$.

$\vec{V}$ = (7-4,1-2,5+1) = <u>(3, -1, 6).</u>

$\overrightarrow{|V|} = \sqrt{3^2+(-1)^2+6^2} = \sqrt{9+1+36} = \sqrt{\mathbf{46}}$//

Cross Product

Definition: - Given two non- zero vectors.

U= u1i+u2j+u3k & v= v1i+v2j+v3k, then the cross product of u & v is the vector

u x v = (u2v3- u3v2) i –(u1v3-u3v1) j + (u1v2- u2v1) k

Examples: - 1. If u= i+3j+4k & v = (2i+7j-5k), then find the cross product of u x v.

Solution: -

u x v = $\begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}$i - $\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}$ j + $\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$k

= (-15-28) I – (-5-8) j +(7-6) k

= -43i+13j +k

N.B: - i x j = k, j x k = i, k x i =j, j x i = -k, k x j = -i & i x k= -j.

Let u =$\frac{1}{2}$ i +3j +k & v= 2i -$\frac{3}{4}$j -2k, then find

a). u x v, $|u\ xv|$

<u>Solution</u>: - u x v = $\begin{vmatrix} + & - & + \\ i & j & k \\ \frac{1}{2} & 3 & 1 \\ 2 & \frac{-3}{4} & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ \frac{-3}{4} & -2 \end{vmatrix}$i - $\begin{vmatrix} \frac{1}{2} & 1 \\ 2 & -2 \end{vmatrix}$j + $\begin{vmatrix} \frac{1}{2} & 3 \\ 2 & \frac{-3}{k} \end{vmatrix}$k

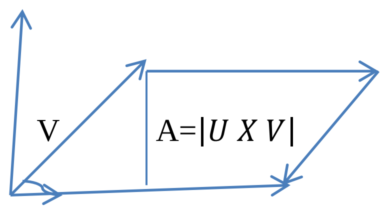
$=((3x-2)-(\frac{3}{4}x1))\,i$ $\sqrt{(\frac{-21}{4})^2+3^2+(\frac{-51}{8})^2}$ = **<u>8.79</u>**

$-((\frac{1}{2}x-2)-(2x1))\,j$

$+((\frac{1}{2}x-\frac{3}{4})-(2\,x3))\,k$

$=\frac{21}{4}\,i+3j-(\frac{51}{8})^2$ = **<u>8.79</u>**

- u x v is perpendicular to both u and v.
- Area of a parallelogram is given by A=$|u\ x\ v|$.



$|u\ x\ v|=|u|\ |v|\ \sin\theta$

Application of scalars & cross product

<u>Vectors & Lines</u>

Let R(x ,y) be any point on L. then the position vector $\underset{(r)}{\rightarrow}$ from the origin to any point on line L is obtained by

$\underset{ORO}{\longrightarrow}+\underset{ROR}{\longrightarrow}=\underset{OR}{\rightarrow}$

$=\underset{ro}{\rightarrow}+t\underset{v}{\rightarrow}=\underset{r}{\rightarrow}$

$\underset{r}{\rightarrow}=\underset{ro}{\rightarrow}+t\underset{v}{\rightarrow}$,t is scalar.

$\underset{v}{\rightarrow}$ is a direction vector of the line?

1. if $\underset{v}{\rightarrow}$& RO (x0, y0) are given, then the vector equation of the line L is.

   $\underset{r}{\rightarrow}=\underset{r0}{\rightarrow}+tv$, t$\in R$ & $\underset{V}{\rightarrow}\neq 0$ ... 1

2. (x0, y0) + t (p, q)

   = (x0+tp, yo + tq)

   X= x0+ tp & y= y0+tq ...2

3. If $p\neq 0$,$q\neq 0$, then t=$\frac{x-x0}{p}=\frac{y-y0}{q}$

   $\therefore\ \underset{r}{\rightarrow}$ r0 + t$\underset{v}{\rightarrow}$ is called vector equation of a line L.

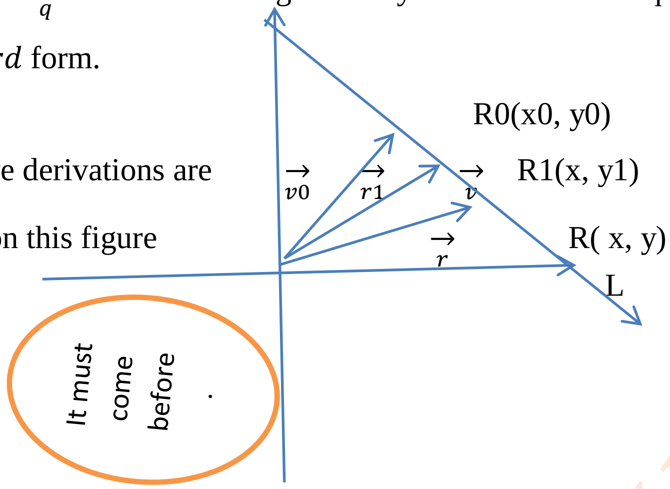
$X = x0 + tp$ & $Y = y0 + tq$ is called parametric equation of a line L.

& $\frac{x-x0}{p} = \frac{y-y0}{q}$ is called rectangular or symmetric form of equation of a line L.

standard form.

The above derivations are

Based on this figure



Examples: -1. Consider the line passes through the point (1 ,4) & (2,2), then.

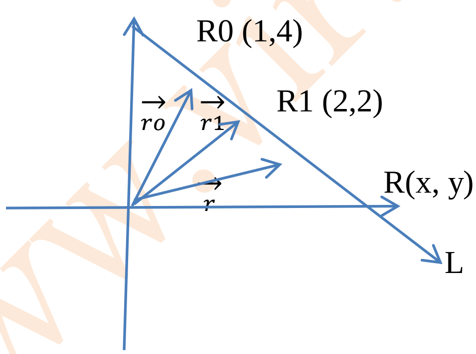
a). find the vector equation of a line.

b). find the parametric equations of a line.

c). find the standard equations a line.

<u>Solution: -</u>

Let R0= (1, 4) & R1 = (2,2)



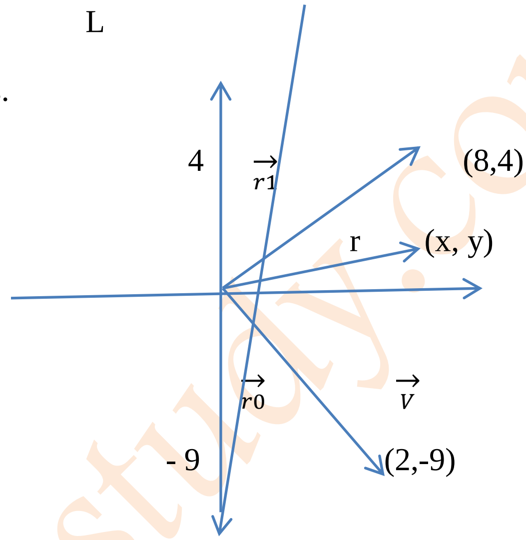
The vector equation of the line is b) from the vector equation in (a)

(x, y) = R0 + t$\overrightarrow{R1\ R0}$ x = 1 +t, y=4-2t

= (1, 4) + t ((2,2) – (1,4)) x-1=t & $\frac{y-4}{-2}$ =t , t ε IR

= (1, 4) +t (1, -2) $x-1 = \frac{y-4}{-2}$ 2x+y-6 =0 is its standard form

∴(x, y) = (1,4) + t(1-4)



2. Consider the line defined by the following points.

a). (2, -9) & (8,4)

b). (7,3) & (4, -3)

c). (-1, -1) & (3, 10) ,then fond

i. The vector equation of the line

ii. The parametric equations of the line.

iii. The standard form of equation of the line.

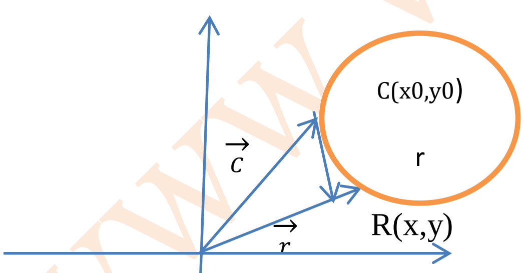
a). (x,y) = (2,-9) + t(8,4) –(2,-9)

= (2, -9) +t (6,13) is vector equation of a line from (a) 2+6t =x & 13t-9 =y is parametric form.
&from (b) t= $\frac{x-2}{6}$, t= $\frac{y+9}{13}$ $\frac{x-2}{6} = \frac{y+9}{13}$ 13x-26=6y+54

13x-6y-80=0 is its standard form

Vectors & circles

Consider a circle with the center.



c(x0, y0) radius r >0 & R(x,y) in the plane.

and $\left|\overrightarrow{r-c}\right|$ = r, where $\vec{r}$ & $\vec{c}$ are position vectors of R(x ,y) & c(x0 ,y0) respectively . squaring both sides, we

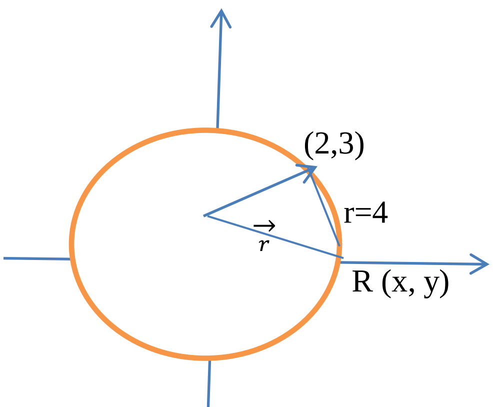
$\left|\vec{r}-\vec{c}\right|^2 = r^2$ x^2+y^2 +Ax +By + c =0, where

$(\vec{r}-\vec{c})\ (\vec{r}-\vec{c}) = r^2$ A= -2x0, B = -2y0

$\underset{r}{\rightarrow} . \underset{r}{\rightarrow} - 2.\underset{r}{\rightarrow}.\underset{c}{\rightarrow} + \underset{c}{\rightarrow} . \underset{c}{\rightarrow} = r^2$ $\quad$ $c = x0^2 + y0^2 - r^2$

$(x - x0)^2 + (y - y0)^2 = r^2$

Examples: - 1. Find an equation of the circle centered at c (2,3) & radius 4.



Let $\underset{r}{\rightarrow}$ & $\underset{c}{\rightarrow}$ be the position vectors of R (x, y) & c (x0, y0) respectively then.

$\underset{r}{\rightarrow} . \underset{r}{\rightarrow} - 2.\underset{r}{\rightarrow}.\underset{c}{\rightarrow} + \underset{c}{\rightarrow} . \underset{c}{\rightarrow} = r^2$

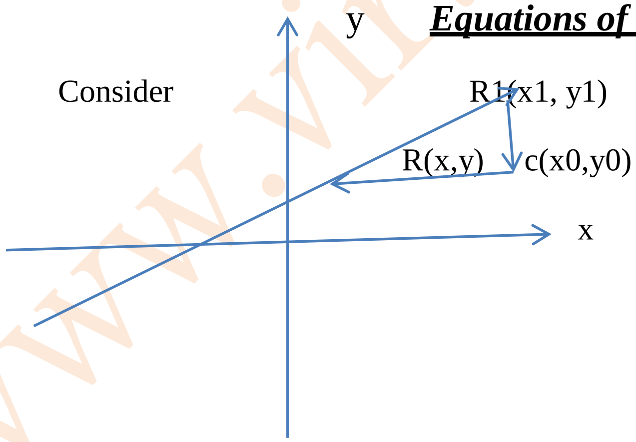
(x, y). (x, y) -2(x, y). (2.3) +(2,3) (2,3) $=r^2$

$x^2+y^2 -2(2x+3y) +4+9 =16$

$\underline{y^2+x^2-4x-6y-3 =0}$

Equations of tangent lines & a circle

Consider



Let R1(x1, y1) be a point of tangency to the circle $(x1-x0)^2+(y1-y0) = r^2$, r>0

If R is an arbitrary point on L, then.

$\overrightarrow{R1R}\cdot\overrightarrow{CR1}=0$, since R1c $\perp$ R1R

(x- x1, y-y1) (x1-x0, y1-y0) =0

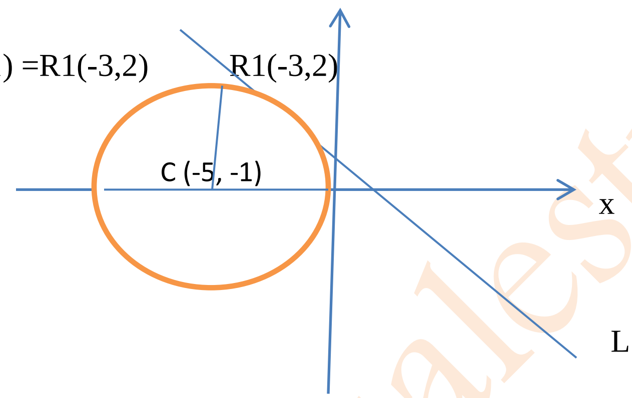
(x-x1) (x1-x0) +(y-y1) (y1-y0) =0

<u>**(x-x0) (x1-x0) + (y-y0) (y1-y0) = r^2**</u> is the equation of a line tangent to the circle at (x1,y1) & center at (x0 ,y0)

If c(x0, y0) =(0,0), then x1x+y1y = r^2

Examples: - 1. Find the equation of the tangent line to the circle $x^2+y^2+10x+2y+13=0$ at the point (-3,2).

Let R (x1, y1) =R1(-3,2)



From $x^2+y^2+10x+2y+13=0$

C (-5, -1), r=$\sqrt{13}$

Slope of CR1= $\frac{\Delta y}{\Delta x} = \frac{2+1}{-3+5} = \frac{3}{2}$, Since R1c $\perp$ R1R

R1c. R1R=0

(-5+3, -1-2). (x+3, y-2) =0

= (-2, -3). (x+3, y-2) =0

-2(x+3) + -3(y-2) =0

- 2x -3y=0 is the equation a tangent line to the circle at (-3,2) & center (-5, -1)

UNIT: - 6

TRANSFORMATION OF THE PLANE

Rigid motion

A rigid motion is a motion which preserves distance. Rigid motion carries any plane figure to a congruent plane figure.

Translation

Definition: - a translation is a transformation that occurs when every point of a figure is moved from one location to another location along the same direction through the same distance.

If point p is translated to point p', then the vector $p\underset{p'}{\rightarrow}$ is called the translation vector.

Let T=(a,b) be a translation vector , then

a). The origin is translated to(a ,b) i.e (0,0) $\rightarrow$ (a,b)

b). The image of the point p (x, y) under the translation vector T will be the point p' (x + a , y + b)

Example: - let T= (2,3) be the translation vector, then find the image of a). A (2,1), B (-1,2), C (-5,1) through the translation vector T.

Solution: -

1. To translate a (2,1) under T (2,3) it must move 2 units right & then 3 units up which is the point (4,4) i.e A (2,1) T (2,3) (4,4)
   B (-1,2) T (2,3) (1,5)
   C (-5,1) T (2,3) (-3,4)
2. If a translation T takes the origin to (-2,2), then find the image of the points P (3,5) & Q (-1,4)
   Solution: -The translation vector T is (-2,2)
   The image of P (3,5) T (-2,2) P' (1,7)
   The image of Q (-1,4) T (-2,2) Q' (-3,6)

3. if a translation T takes the point (-2,3) to the point (1,2), then what are the images of the following lines.

a). l:3x-y+4=0

b). l: 4y+2x+1 =0

Solution: -

T=$\underset{AB}{\longrightarrow}$ = (1-(-2) ,2-3) = (3-1)

The point P' = (x, y') x' =x+3, y'=y-1

X= x'-3, y=y'+1

A translation map lines on to parallel lines L': 3(x'-3) -(y'+1) +4=0

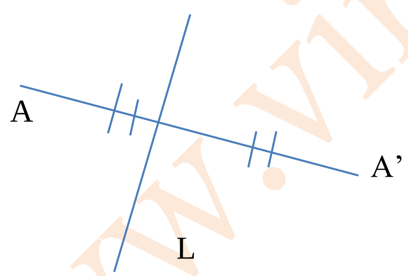
= 3x'-9-y'-1+4=0

= 3x'-y'-6=0

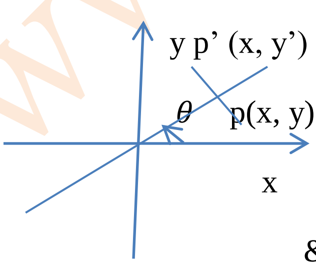
∴ L: $\underline{\underline{\text{4y+2x-1=0}}}$

1. Reflection

Definition :-A reflection about a line L is a transformation of the plane on to itself which carries point A of the plane in to the point A' of the plane such that L is the perpendicular bisector of $\underset{AA}{\rightarrow}$'. The line L is called the line of reflection or the axis of reflection. the reflection of a point A about the line L, is denoted by M(A); i.e. A'= M(A)



A Reflection in the line y= mx.



1. When θ=$0^{0,\text{ you}}$ will have reflection in the x-axis (y=0) thus (x, y) is mapped to (x,-y)
2. When θ=π/2, we will have reflection on in the y-axis (x=0) & (x, y) is mapped to (-x, y).

4. When $\theta = \pi/4$,we will have reflection about the line y=x & hence (x, y) is mapped to (y, x).

5. When $\theta = 3\pi/4$, we will have reflection about the line y=-x & (x, y) is mapped to (-y, -x)

Examples: - 1. Find a reflection of point p (4,5) over the following.

a) x- axis b) y-axis c). The line y=x d). the line y= -x

Solutions: -

a) About the x- axis any point (a, b) is reflected about the x-axis, it becomes (x,-y)

(4,5) $\rightarrow$ (4,-5)

b). about the y-axis, (4,5) $\rightarrow$ (-4,5)

c). about the line y=x (4,5) $\rightarrow$ (5,4)

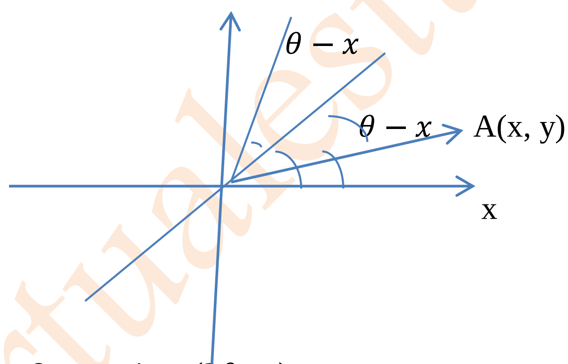
d). about the line y=-x (4,5) $\rightarrow$ (-5,-4)

let L be a line

Passing through the

Origin & making an angle

θ with the + ve x-axis



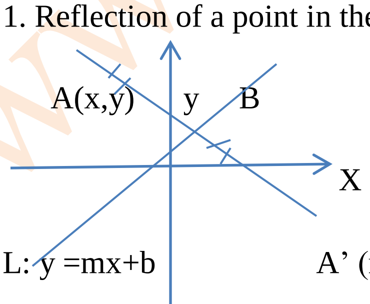
coordinates of A are x= r cos x & y= r sin x (2θ –x)

x' = x cos 2θ + y sin 2θ

y' = x sin 2θ – ycos 2θ, where θ is the angle of inclination of L: y=mx

B). Reflection in the line y= mx + b

1. Reflection of a point in the plane



To find the image of any point A(x,y) about a line L.

1. Find the slope of L A' (x', y')

2. Find the slope of $\overrightarrow{AA'}$

3.Find the intersection point of L & $\underset{AA'}{\longrightarrow}$

4. Using B as middle point of $\underset{AA'}{\longrightarrow}$, find A’

Slope of $\underset{AA'}{\longrightarrow}$ is -1/m

2. Reflection of a line in the line

- If a line $\underset{AA'}{\longrightarrow}$ is ⅃ the axis of reflection L, then $\underset{AA'}{\longrightarrow}$ is own image .
- If $\underset{AA'}{\longrightarrow}$ is a line / / to the line of reflection L, to find the image of $\underset{AA'}{\longrightarrow}$, when reflected about L.

  a). choose any point B on $\underset{AA'}{\longrightarrow}$

  b). find the image of B M(B) =B’

  C). Find the equation $\underset{AA'}{\longrightarrow}$ which is the line passing through B’ with slope equal to the slope of L.

3. Reflection of A Circle In The Slope

- If the center of a circle is on the line of reflection L, then the image of the circle is itself.
- If the center of a circle o has image o’ when reflected about a line L, then the image of a circle has center o’ & radius the same as the original circle.

Examples: - 1. The image of point A (4,9) in a reflection is A’ (5,1) . Find the line of reflections equation.

Solution: -

$\underset{AA'}{\longrightarrow}$ ⅃ L

Slope of $\underset{AA'}{\longrightarrow}$ $= \frac{1-9}{5-4} = \frac{-8}{1} = -8$

Equation of a line passing through $\underset{AA'}{\longrightarrow}$ L: y-9 = -8(x-4)

y= -8x+32+9

y= -8x+41

slope of L X slope $\underset{AA'}{\longrightarrow}$ = -1

m_2.-8 = -1

m_2 =1/8 but the line passes through the midpoint of $\underset{AA'}{\longrightarrow}$.

midpoint of $\underset{AA'}{\longrightarrow}$ $=(\frac{4+5}{2},\frac{9+1}{2}) = (\frac{9}{2},5)$

∴ The equation of the line of reflection.

L: $y-5=\frac{1}{8}(x-\frac{9}{2})$ $y=\frac{1}{8}x-\frac{9}{16}+5$ $y=\frac{1}{8}x-\frac{9+80}{16}$ $y=\frac{2x+71}{16}$ ***$\underline{\underline{16y -2x -71 = 0}}$***

2.Find the equation of the image of the circle $x^2+y^2+16x-24y+183 = 0$ when it is reflected about the line $4x+7y+13 =0$?

Solution: -

To find the image of a given circle, simply find the image of the center of the given circle. from $x^2+y^2+16x-24y+183 =0$

$(x^2+16x) +(y^2-24y) = -183$

$(x+8)^2+(y-12)^2 = -183+64+144$

$(x+8)^2+(y-12)^2 = 25$

C (-8,12) r= 5, reflect (-8,12)

Along the line y=

The equation of the new circle will be: $(x+16)^2 +(y+2)^2 =25$, it becomes

$\underline{\mathbf{x^2+32x+y^2+4y+235 = 0}}$

Rotation

Definition:- a rotation R of θ about a point 0, is called center of rotation & θ is called an amount of rotation .

- If $\theta >0^0$, the rotation is clockwise rotation.

A Rotation When the Center of Rotation Is About the Origin

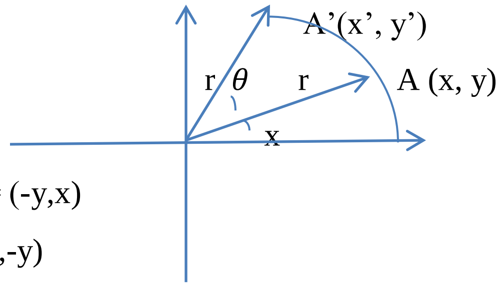
- Rotation is a rigid motion.
- Rotation when the center of rotation is about the origin.

Theorem

Let R be a rotation through an angle θ about the origin. if R (x, y) = (x', y'), then

x' = x cos A – y sin θ

y' =x sin $\theta + y\cos\ \theta$.



$\Rightarrow$ if $\theta = \pi/2$, (x', y') = (-y,x)

$\Rightarrow$ if $\theta = \pi$, R(x, y) =(-x,-y)

$\Rightarrow$ if $\theta = \frac{3}{2}\pi$, R (x, y) =(y ,-x).

4.If θ =2nπ ,nϵ z

R is the identity transformation.

5. every circle with center at the center of rotation is fixed.

B). Rotation when the center of rotation is the point (a,b).

Rotation about the point (a,b) can be divided into the following three transformation .

1. Translation by T=(-a,-b)

   $(x_1', y_1') = (x-a ,y-b)$.

   The center of rotation is moved to the origin.

2. Rotation through an angle θ about the origin.

   $x_2' = x_1 \cos\theta - y_1' \sin\theta$

   $y_{2'} = x_1 \sin\theta + y' \cos\theta$

3. Translation T=(a,b)

   $X_{3'} = a + x_{2'} = a + x_{1'} \cos\theta - y_{1'} \sin\theta$

   $= a + (x-a)\cos\theta - (y-b)\cos\theta$

Examples:-1. Find the image of the point A(3,4) when it is rotated through $\theta = \pi$ about (6,5)

Solution :- x' =a +(x-a) cos θ –(y-b)sin θ

y' = b+(x-a) sin θ +(y-b)cos θ where (x,y) =(3,4) & (a,b) = (6,5) , $\theta = \pi$

$x' = 6+(3-6)\cos\pi - (4-5)\sin\pi$

$= 6+(-3)(-1) - (-1)(0) = \underline{\mathbf{9}}$

$y' = 5+(3-6)\sin\pi + (4-5)\cos\pi$

$= 5+(-3)(0) + (-1)(-1)$

$= \underline{\mathbf{6}}$ &

the image of(3,4) when it rotated about $\theta = \pi$ about (6,5) is **<u>(9,6)</u>**

2.Determine the image of the line L: y=3x+1 after a rotation of $\theta = \pi/2$ about the point (2,2).

<u>Solution</u>:- Let $(x_1,y_1)=(1,4)$ - any point on L. & $(x_2,y_2)= (-2,-5)$ – point on L. and then

$x' = a+(x-a)\cos\theta - (y-b)\sin\theta$

$y' = b+(x-a)\sin\theta + (y-b)'\cos\theta.$

Where $(x,y) = (1,4)$ or $(x_2,y_2) = (-2,-5)$ thus $p(x_1,y_1) = (a+(x_1-a)\cos\theta - (y-b)\sin\theta$

$=, b+(x_1-a)\sin\theta + (y-b)\cos\theta$

==>p(1,4) =(0,1) & p(-2,-5) =(9,-2) & the image of the line L containing (0,1) & (9,-2) is

L': $y = \frac{-1}{3}x + 1.$

Unit -7

Statistics

Definition: -we can define statistics into senses.

i. ***In the sense of plural:*** - statistics are the raw data (body of numerical facts & figures or list of numbers.

Examples: - statistics of birth statistics of death

ii. ***In the sense of singular:*** - it is the subject that deals with collection, classifying, summarizing, organizing, analyzing, interpretation & presentation of numerical data.

Data:-They are measurements or observations .

Data can be classified as

a). **Quantitative: -** data that can be numerically described or measurable quantity or the amount of something.

Example: - observation regarding, weight, height ,distance ,pressure , speed , income & life length of an atom etc…

b). **Qualitative data: -** data that cannot be measurable or cannot be expressed in numbers but can be categorized.

Example:- Honesty – sex – efficiency

Intelligent –poverty – color – beauty – religion etc…

Qualitative data can be :-

1. Discrete variable
2. Continues variable.

1. Discrete variable

❖ Is one which takes only whole number values.

Example:- number of students in a class .

Number of chairs in a room etc…

2. Continues variable.

❖ Is one which takes all real values between two given real values. They can take fractional or decimal values.

Example:- height of students in a class . – size of a shirt

- Lifetime of an electric bulb. –price of a kilo of sugar

Raw: - recorded information in its original collected form, which is not organized numerically

Frequency:- The number of observations in a particular class.

Cumulative frequency: -The cumulative of a particular class in the sum of all frequencies up to this class.

Cumulative frequency distribution: - is a form of frequency distribution that represents frequency distribution.

Class interval:- is the numerical width of any class in a particular distribution .

Class limit :- is the maximum value & the minimum value the class interval may contain.

Class width:- is the difference between the upper & lower class boundary of any class (category 0.

Introduction To Grouped Data

Definition: -

Grouped discrete data: - is discrete data that has been grouped into categories.

Grouped frequency distribution.

- Is the organization of raw data in table form using classes and frequencies for the purpose of summarizing a large sample of data .
- Steps for constructing grouped frequency distribution.
  1. Determine the number of classes required (usually between 5 & 20)
  2. Approximate the class width i.e C. W=$\frac{L.V-S.V}{number\ of\ class\ required}$

Example:- 1. The following data shows the test results of 20 students in mathematics:

1 2 3 3 4 5 5 5 6 6

6 6 6 7 8 8 9 10 10 10

a). constructing a grouped frequency distribution with 5 classes.

b). what is the frequency of the 4^{th} classes?

c). what is the frequency of the 2^{nd} class?

d). what is the upper –class limit and lower-class limit of the 3^{rd} class?

solution: - C.W $= \frac{M.V - M.V}{K} = \frac{10-1}{5} = \frac{9}{5} = 1.8 \approx 2$

a).

Class	Frequency	Class frequency	Class boundary
1 -2	2	2	0.5-2.5
3-4	3	5	2.5-4.5
5-6	8	13	4.5-6.5
7-8	3	16	6.5-8.5
9-10	4	20	8.5-10.5
Total	20		

b). The frequency of the 4^{th}class is 3.

c). The frequency of the 2^{nd}class is 3.

d). upper class limit of the 3^{rd}class is 6. And the lower-class limit of the 3^{rd}class is 5.

Graphical Representation of Grouped Data

- The histogram, and frequency polygon graph are the most applied graphical representation for continues data.
- Steps for constructing histogram, and frequency polygon.

  1). Draw & lable the x-axis & y-axis.

  2). Choose a suitable scale for the frequencies or cumulative frequencies & label it on the y-axis.

  3). Represent the class boundaries for the frequency histogram or the mid points for the frequency polygon on the x-axis.

  4). Plot the points.

  5). Draw the bars or lines to connect the points.

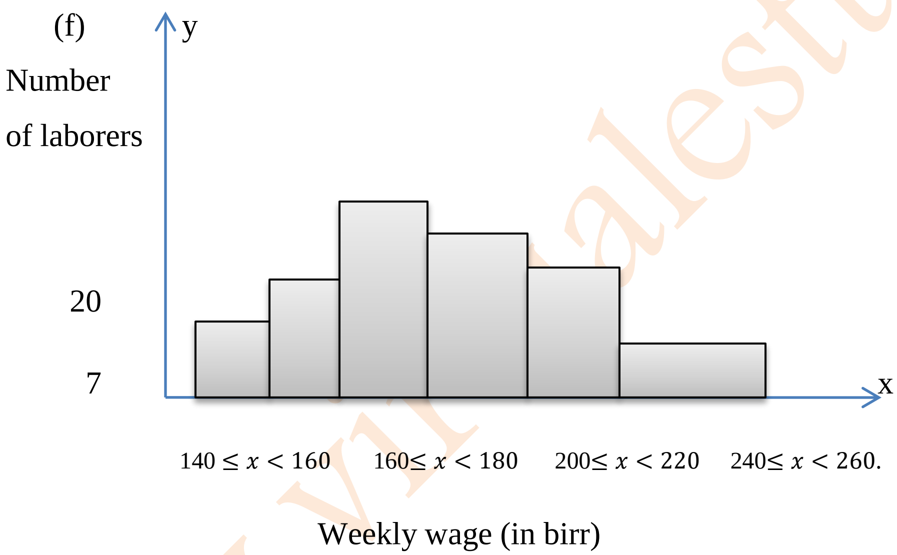
Frequency polygon

- Is a graph constructed by using lines to join the mid-points of each interval, or bin.
- The frequency is placed along the vertical axis & the classes mid-points are placed along the horizontal axis.

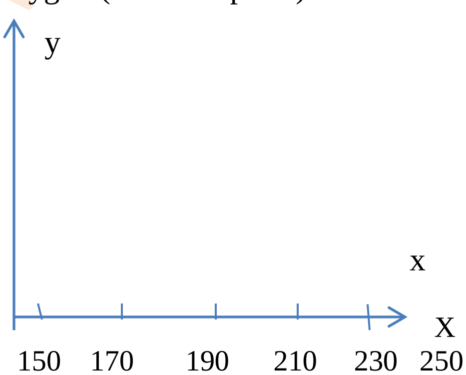
Examples: -1. Construct histogram, & frequency polygons for the following data.

Let x be the weekly wage.

Weekly wage (in birr)	Number of labors (f)
$140 \leq x < 160$	7
$160 \leq x < 180$	20
$180 \leq x < 200$	33
$200 \leq x < 220$	25
$220 \leq x < 240$	11
$240 \leq x < 260$	4



iii.Frequency polygon (use mid- point)



Measures of central tendency and their interpretation

⇒ The most commonly used measures of central tendency are: -

- ✓ Mean (Arithmetic mean)
- ✓ Median
- ✓ Mode
- ✓ Quintiles (quartiles, declines, percentiles)

1.The mean ($\overline{x}$)

Definition :- The mean x of a set of data ,denoted by $\bar{x}$ is equal to the sum of the data items divided by the number of items contained in the data set .

If x_1 ,x_2 ,x_3 ,… ,x_n are set of data ,then

$\bar{x} = \frac{x1,+x2+x3+ -- +xn}{n} = \sum_{i=1}^{n} fixi \ \sum_{i=1}^{n} fi$ - mean for discrete data.

Mean for Grouped Data

$$\bar{x} = f_1 \ (xc1 + f2xc2 + --- + fnxcn)/(f1 + f2 + --- + fn) = \frac{\sum_{i=1}^{n} fixi}{\sum_{i=1}^{n} fi}$$ Type equation here.

Combined mean

⇒ If $\bar{x}1$ is the mean of n_1, observations.

⇒ if $\overline{x2}$ is the mean of n_2 observation

$\bar{x}$ k is the mean of n_k observation, then the mean of all observations in all groups is called combined mean & it is given by :

$\bar{x}C = \frac{n1\bar{x}1 \ + n2\bar{x}2 + --- + nk\bar{x}k}{n1+n2+n3+ --- +nk}$

Examples: − 1. Find the meaning of the following distributions.

a). 5,7,8,20,15,8,7,8,20,8

b).

x	4	6	8	10	11
f	5	3	7	7	4

c).

Marks	f	xci
0-9	12	4.5
10-19	18	14.5
20-29	27	24.5
30-39	20	34.5
40-49	17	44.5

2.one group of 8 students has a mean average score of 67 in a test. A second group of 17 students has a mean average of 81 in the same test. what is the mean averaging all 25 students?

Solutions: -

1a). $\bar{x} = \frac{5+7+8+20+15+8+7+8+20+8}{10}$

$= \frac{106}{10} = \underline{\underline{10.6}}$

b). $\bar{x} = \frac{4x5+6x3+8x7+10x7+11x4}{26}$

$= \frac{20+18+56+70+44}{26}$

$= \frac{208}{26} = \mathbf{8}$

c). $\bar{x} = \frac{4.5x12+14.5x18+24.5x27+34.5x20+44.5x17}{94}$

$= \frac{54+261+661.5+690+756.5}{94}$

$\bar{x} = \underline{\underline{25.78}}$

2.Median $(\widetilde{X})$

® The median divides the distribution into two equal parts, when the data is arranged in either increasing or decreasing order of magnitude.

1). $\widetilde{X}$ for raw data and discrete data

® To find the $\widetilde{X}$ for row data and discrete data first put the data in ascending or descending order of magnitude. for these type data the position of the median is given by: $\tilde{x}$= x(n/2 +0.5), where n – is total number of observations

2. $\widetilde{X}$ for grouped data

$\widetilde{X}$ = BL+($\frac{\frac{n}{2}-c+b}{fc}$) xi where, BL –lower class boundary of the median class. n – total number of observations.

$(n/2)^{th}$ – the position of the median class.

Fc –frequency of the median class

Cfb – cumulative frequency just before the median class

i- Class width

Example: - 1. Find the $\tilde{X}$ for the following data sets.

a). 9,7,9,11,14,7,8,8,6,7,3,5,7

b).

x	4	6	8	10	12	14	16
f	2	4	5	3	2	4	1

c).

class	Frequency	Cumulative frequency	Cumulative boundaries
1-5	3	3	0.5-5.5
6-10	8	11	5.5-10.5
11-15	12	23	10.5-15.5
16-20	6	29	15.5-20.5
21-25	5	34	20.5-25.5
26-30	4	38	25.5-30.5
31-35	2	40	30.5-35.5

Solutions: -

a). first put the data in ascending or descending order

i.e 3,5,6,7,7,7,7,8,8,9,9,11,14

$= x\,(\frac{13}{2}+0.5) = x(6.5+0.5) = x7$ the median is the 7^{th} observation in either direction , which is 7

i.e $\underline{\underline{\tilde{X} = 7}}$

b). $x(\frac{21}{2}+0.5) = x10.5 + 0.5 = x11$ the median is the 11^{th}observation which is **8.**

c). $(\frac{n}{2})^{th}$ median class $(\frac{40}{2})^{th} = 20^{th}$ the median is found in the 3^{rd} class

BL =10.5, I =5

Fc =12

Cfb = 11

$\therefore \tilde{X} = BL + (\frac{\frac{n}{2}-cfb}{fc})\, xi$

$= 10.5 + \left(\frac{20-11}{12}\right) x\ 5$

$= 10.5 + \frac{9}{12} .5$

$= 10.5 + \frac{45}{12}$

$= 10.5+3.75$

$\tilde{X} = 14.25$

3.Quintile's (Quartiles, decals & percentiles)

a). Quartiles: - A Quartile divides a set of data in to three equal parts .

There are 3 quartiles - These are first quartile (lower quartile) ,meddle quartile(median) & 3rd quartile (upper quartile) .

- The difference between the upper quartile & the lower quartile is called the interquartile range (IQR)

  i.e . IQR $=Q_3-Q_1$

  b). **Decals :-** divide a set of data into ten equal parts .There are nine decals , namely $D_1, D_2, D_3, \ldots D_9$.

  C). Percentile :- divides a set of data in to 100 equal parts . There are 99 percentiles , namely $P_1, P_2, P3, \ldots P_{99}$.

  Percentiles are mostly used for the ranking system.

Quartiles , decals & percentiles for raw data & discrete data

$Q_K = (\frac{K(n+1)}{4})^{th}$ item , if n, is odd $= (\frac{\frac{kn}{4}+\frac{kn+1}{4}}{2})^{th}$ item if n- is even.

$DK = (\frac{k(n+1)}{10})^{th}$ item if n- is odd

$= \left(\frac{\frac{kn}{10}+\frac{kn}{10}+1}{2}\right)^{th}$ item if n- is even .

$Pk = \left(\frac{k(n+1)}{100}\right)^{th}$ item if n – is odd

$= \left(\frac{\frac{kn}{100}+\frac{kn}{100}+1}{2}\right)^{th}$ if n – is even .

Quartiles ,decals & percentiles for grouped data

® The kth quartile for a grouped frequency distribution is :

® $Qk(k^{th}$ quartile) = BL + $\left(\frac{\frac{Kn}{4}-cfb}{fc}\right)$ xi where k =1,2,3

BL - Lower class boundary of the k^{th} quartile class .

n- Total number of observations

cfb – Cumulative frequency before the k^{th} quartile class.

fc-. Frequency of the k th quartiles class

i- The class size.

The k^{th} decal for G.F.D is:

$Dk(k^{th}$ decal) = BL + $(\frac{\frac{kn}{10}-cfb}{fc})$ xi where , k = 1,2,3 - - - 9

BL- lower class boundary of the k^{th} decal class.

$\left(\frac{kn}{10}\right)^{th}$ –The position (class) of the k^{th} decal

$fc -$ Frequency of the k^{th} decal class

$cfb -$ Cumulative frequency before the k^{th} decal class.

$i -$ The class size.

$n -$ Total number of observation

The k^{t} percentile for grouped frequency distribution is given by:-

$Pk(k^{th}$ percentile) = BL +($\left(\frac{\frac{kn}{100}-cfb}{fc}\right)$ xi where BL- lower class boundary of the k^{th} percentile class.

n- Total number of observations

cfb- Cumulative frequency before the k^{th} percentile class.

$\left(\frac{kn}{100}\right)^{th}$ – The location of the k^{th} percentile

i – class size.

fc-Frequency of the k^{th} percentile

Examples :- 1. For each of the following data sets find . Q_2 , Q_3,D_4,D_8 ,P_{12} ,P_{24} ,P_{87}

a).78, 68 ,19 ,35, 46, 58 ,35, 35 ,31, 10 48 ,78.

b).

x	10	14	15	17	19	20	26
f	12	18	20	2	4	4	1

c).

Age	f
5-14	4
15-24	12
25-34	10
35-44	7
45-54	2

<u>Solutions</u> :- a). first arrange the data 10 ,19 ,28 ,31 , 35, 35, 46, 48, 58, 68, 78 here n =12

$Q_2 = \left(\frac{\frac{kn}{4}+\frac{kn}{4}+1}{2}\right)^{th}$ item

$= \left(\frac{\frac{(2x12)}{4}+\frac{2x12}{4}+1}{2}\right)^{th}$

$= \frac{6th + 7th}{2}$ Q_2 which is found between the 6^{th} & 7^{th} observations.

$Q_2 = \frac{6th + 7th}{2}$

$Q_2 = \frac{35 + 35}{2} = 35$

$Q_3 = \left(\frac{\frac{kn}{4} + \frac{kn}{4} + 1}{2}\right)^{th}$

$= \left(\frac{\frac{3X12}{4} + \frac{3X12}{4} + 1}{2}\right)^{th}$

$= \left(\frac{9\,th + 10\,th}{2}\right)^{th}$

$= \left(\frac{48 + 58}{2}\right) = \left(\frac{106}{2}\right) = \underline{\underline{53}}$

$P_{12} = \left(\frac{\left(\frac{12x2}{100}\right) + \frac{12x2}{100} + 1}{2}\right)^{th}$

$P_{12} = \underline{\underline{18.46}}$

$P_{24} = \left(\frac{\frac{24x12}{100} + \frac{24x12}{100}}{2} + 1\right)^{th}$

$= \left(\frac{(2.88) + 3.88}{2}\right)^{th}$

$= \frac{2nd + 0.88(3rd - 2nd) + 3rd + 0.88(4th - 3rd)}{2}$

$= \frac{19 + 0.88(9) + 28 + 0.88x3)}{2}$

$= \frac{26.92 + 30.64)}{2} = \underline{\underline{28.78}}$

$p_{87} = \left(\frac{\frac{87x12}{100} + \frac{87x12}{100} + 1}{2}\right)^{th}$

$p_{87} = \underline{\underline{67.4}}$

$D_4 = \left(\frac{\frac{4x12}{10} + \frac{4x12}{10} + 1}{2}\right)^{th}$ item

$= \left(\frac{4.8 + 5.8}{2}\right)^{th}$

$= \frac{4th + (5th - 4th)0.8 + 5th + 0.8(6th - 5th)}{2}$

$= \frac{31 + 0.8\,x4 + 35 + 0.8\,x\,0}{2}$

$= \frac{\mathbf{31 + 3.2 + 35}}{\mathbf{2}}$

$D_4 = \frac{34.2 + 35}{2} = \frac{69.2}{2} = \underline{\underline{34.6}}$

b).

Q_2	Q_3	D_4	D_8	P_{12}	P_{24}	P_{87}
15	15	14	15	10	14	19

c).

Age	Frequency	Cumulative frequency	Cumulative boundaries	Class size
5-14	4	4	4.5 -14.5	10
15-24	12	16	14.5-24.5	10
25-34	10	26	24.5-34.5	10
35-44	7	33	34.5-44.5	10
45-54	2	35	44.5-54.5	10

$Q_2 = BL + (\frac{\frac{kn}{4}}{fc} - cfb)xi$ first find Q_2 class

® $\left(\frac{2X35}{4}\right)^{th} = \underline{(17.5)^{th}}$ which is found in the 3rd class.

$BL = 24.5$

$i=10$

$cfb = 16$

$fc = 10$

$\frac{kn}{4} = \underline{17.5}$

$Q_2 = 24.5 + (\frac{17.5-16}{10})X10$

$= 24.5 + 1.5$

$= \underline{26}$

$Q_3 = BL + \left(\frac{\frac{Kn}{4} - cfb}{fc}\right)xi$

but Q_3 class is $\left(\frac{3x35}{4}\right)^{th}$

$= (26.25)^{th}$ which is found in the 4th class

$BL = 34.5$ $i=10$ $cfb = 26$ $fc = 7$

$Q_3 = 34.5 + (\frac{26.25-26}{7})X10$

$= 34.5 + \frac{0.25X10}{7}$

$= \underline{34.86}$

$BL = 14.5$

$cfb = 4$

$i = 10$

$fc = 12$

$D_4 = 14.5 + (\frac{14-4}{12})x10$

$D_4 = BL + (\frac{\frac{Kn}{10} - cfb}{fc})xi$

$= (\frac{kn}{10})^{th} = (\frac{4x35}{10})^{th} = \mathbf{14^{th}}$

14th which is found in the 2nd class .

$p_{87} = BL + (\frac{\frac{Kn}{100} - cfb}{fc})xi$ but $(\frac{kn}{100})^{th} = (\frac{87 x 35}{100})^{th} = \underline{(30.45)^{th}}$

which is found in the 4th class.

$BL = 34.5$, $fc = 7$

$cfb = 26$ $i=10$

$p_{87} = 34.5 + (\frac{30.45-26}{7})x10$

$=14.5 + \frac{10x10}{12})$ p_{87} = ***46.57***

= ***22.83***

4.The mode ($\hat{x}$)

Definition :- The mode is the value that occurs most often in data .

Mode for Grouped Data

($\hat{x}$) = BL+($\frac{d1}{d1+d2}$) xi where

BL – lower class boundary of the modal class

d_1 diffrence between the frequency of the modal class & the frequency of the pre-modal class

d_2 – difference between the frequency of the modal class & the frequency of the next class.

i- Class size

Examples: -1. Find the mode for each the following sets of data.

a). 18,14 ,15,10,11,3,10,12,10

b).

x	7	12	15	17	19	23
f	6	4	6	5	6	5

c). The following is a distribution of the number of the number of drivers violated traffic – safety in Addis Ababa. Find the mode of the distribution.

Violated traffic safety	Number of drivers	Cumulative boundaries
0-3	6	0.5-3.5
4-7	10	3.5-7.5
8-11	20	7.5-11.5
12-15	22	11.5-15.5
16-19	6	15.5-19.5
20-23	2	19.5-23.5

Solutions :- 1a). The mode ($\hat{x}$) =10 which occurs most frequently.

b). The mode ($\hat{x}$) = 7, 15, 19, they have the same highest frequency

c). first identify the modal class the modal class is the 4^{th} class because it contains the highest frequency. $\hat{x}$ = BL+($\frac{d1}{d1+d2}$) xi

BL =11.5 $= 11.5+(\frac{2}{2+16})$ x4

d1=22-20 =2 $= 11.5+\frac{2}{18}$ x4

i= 4 $\hat{x}=$ ***11.94***

Real Life Applications of Statistics

⇒ There are numerous applications of statistics in different professions. for instance , statistics can be applied in clinical trial & design , corporate sectors , whether forecasting , sports & financial markets etc….

Unit -8

Probability

Definition: - The chance that a given event will occur.

❖ The measure of uncertainty involved in the happening of event so that definite value may be assigned to it.

Basic Terminologies on Probability

1. ***Experiment***: - a planned operation carried out under controlled conditions

❖ An activity (measurement or observation) that generates well-defined outcomes (result).

❖ We have two types of experiments: -

a). **Deterministic**: - is one whose outcome may be predicted with certainty beforehand.

Examples: - Combining hydrogen & oxygen or adding two numbers such as 2 & 3.

b) Random Experiment: - an experiment in which all possible outcomes are known & the exact outcome cannot be predicted in advance.

❖ An experiment whose outcome cannot be predicted with certainty.

Examples: -

✓ Tossing a fair coin.
✓ Rolling an unbiased die.
✓ Drawing a card from a pack of well-shuffled cards.
✓ Picking up a ball from a bag of balls of different colures. etc…

2. ***Sample space (probability space) or possibility set***
   - Is the set of all possible outcomes of a random experiment? sample space denoted the letter S'.
   - There are three ways to represent a sample space: -
     i. By listing all the possible outcomes
     ii. By creating a tree diagram
     iii. By creating a Venn-diagram
3. ***Outcome (sample point)***:- is any result obtained in an experiment .
4. ***Event :-***any subset of a sample space.

- The collection of all or some of the possible out-comes

Examples:- 1. If a fair die is rolled once the possible results are either 1, 2, 3, 4, 5, or 6, then

a). Give the sample space

b). Give the event of obtaining odd numbers.

c). Give the event of obtaining even numbers.

d). Give the event of obtaining number seven.

Solutions :- 1a). The sample space (S) is {1,2,3,4,5,6}

b). The event of obtaining odd numbers is {1,3,5}

c). The event of obtaining even numbers is {2,4,6}

d).The event of obtaining number seven is ∅ which is an impossible event.

2. In tossing a fair coin the sample space is {H.T}.

3. In throwing two well-balanced dice , the sample space is

S={(1,1) ,(1,2) ,(1,3), (1,4) ,(1,5) ,(1,6),

(2,1) ,(2,2) ,(2,3) ,(2,4) ,(2,5) ,(2,6),

(3,1) ,(3,2),(3,3) ,(3,4) ,(3,5),(3,6),

(4,1),(4,2),(4,3),(4,4) ,(4,5),(4,6),

(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),

(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}.

4.From a group of 3 boys & 2 girls, we select two children. what would be the sample space for this experiment?

S={B_1B_2,B_1B_3,B_1G_1, B_1G_2,B_2B_3,B_2G_1,B_2G_2,B_3G_1,B_3G_2,G_1G_2}

Techniques of counting

- There are three types of techniques of counting.

1. ***Fundamental principles of counting:-***In some counting problems we can find out the answer without actually counting.

There are two types:-

a). Multiplication principle:-If one selection can be made in "m" different ways , & following this another selection can be made in "n" different ways , then both selections , one after the other can be made in the given order m x n different ways . the same principle can be generalized to three or more selections occurring in succession.

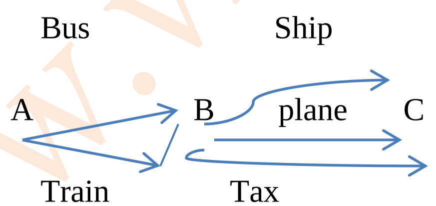
b). Addition principle:- Let E_1 and E_2 be two mutually exclusive events ,then if E_1, can occur in "m" ways and another event E_2 can occur in "n" ways ,then the number of ways in which either E_1 or E_2 can occur in (m + n) ways .

Examples :-1. How many two digit even number are that greater than 58?

Solution :- 4 x 5 = **20** different two digit even numbers. In the unit place there are 0, 2, 4, 6, 8 options & in the tens place we have 6, 7, 8, 9 options (since the number is >58).

2).If Dawit wants to travel from city A to B , then city c, then in how many different ways can be travel if the means of transportation from city A to B , are bus or train & from city B to C the means of transportations are ship or plane or taxi.

Solutions :-



We have 2x3 =6 different ways

3).Suppose that a man has 5 coats ,10 shirts and 8 different trousers . In how many different ways can a man dress .

Solution :- 5 x 10 x 8 = 400 different ways

4).4 red and 6 green marbles are placed in a bag. How many marbles are there to choose from?

Solutions:- since the red & green marbles cannot be chosen together or at the same , so there are 4+6 =10 marbles to choose from .

5). Aster will draw one card from a standard deck of playing cards. How many ways can she choose?

a). an even number b). a king or queen? c). heart,a diamond or a club ?

d). a king or a black?

Solution:- There are 4 of each card (4Aces ,4 kings ,4 queens , etc…) & there are 4 suits (clubs hearts ,diamonds & spades)

From 52 cards → 26 red → 13 spades, 13 clubs

From 52 cards → 26 black → 13 hearts, 13 diamonds

From those cards 20 are even numbers.

a). 20 even number = 20 ways (2,4,6,8,10) x 4 = 20

b). 4 kings +4 queens = 8 ways (K or Q)

C). 13 hearts or 13 diamonds or 13 clubs

= 13+13+13 =39 different ways

d). 4 kings or a black there are 2 black kings in 2d black cards 2 are found both 4 kings & 26 blacks.

4 kings +26 black - 2 black kings = **28** different ways.

Permutations & combinations

1. **Factorial:-**

® Factorial of a number denoted by n! is the product of all positive integers less than or equal to n.

$n! = n(n-1)(n-2)(n-3) - - - x\ 3x2x1\ \&\ 0! = 1,\ 1! = 1$

2. **Permutations**:- is the number of arrangements of objects with attention given to the order of arrangements
   - ❖ The number of permutation of a set of "n" objects taken all together is denoted by p(n,n) = n pn= n !

❖ The number of permutation of "n" objects taking 'r' at a time where $0 \leq r \leq n$ is denoted by p(n , r) = $^{n}p_{r}$ = $p^{n}{}_{r}$ = pn_1r & is given by p(n,r) = $\frac{n!}{(n-r)!}$.

❖ The number of arrangements of 'n' objects in which n_1 are like, n_2 are alike, - - - , n_r a like objects or rth type is $\frac{n!}{n1!x\, n2!xn3!x- - -x\, nr!}$, where $n_1+n_2+n_{3+}$ - - - $n_r = n$ & $0 < r \leq n$.

Examples:- 1. How many different permutations can be made from the letters in the word 'MATHEMATICS '? In how many of these permutations.

a) Do all the vowels occur together.

b) Do the word start with I.

c) Begins with the two A.

d) Begins with E and ends with C.

<u>**Solution :-**</u> In the Mathematics , we have 11 letters and we have 2A's , 2M's, 2T's = $\frac{11!}{2!2!2!}$

$= \frac{11\,x10\,x9\,x8\,x7x6x5x4x2!}{2\,x1\,x2x1\,x2!}$

= 990 x 56 x30 = <u>**1,663,200**</u>

Do all the vowels occur together i.e. MAAIETHMTCS?

Counts as one letter

$= \frac{8!}{2!}\,x\,\frac{4!}{2!} = \frac{8x7x6x5x4x3x2!}{2!}\,x\,\frac{4x3x2!}{2!}$

= 56x30x12x12

= 56x30x144

= 56 x 4320

=<u>**241,920**</u>

b). Do the word starts with I . we have $\frac{10!}{2!2!2!}$ I – is fixed.

c). begins with the two A, we have: $\frac{9!}{2!2!}$

d). begins with E & ends with C, we have $\frac{9!}{2!2!2!}$

2). How many permutations can be made from the word MISSISSIPPI?

Solutions: - We have $\frac{11!}{4!4!2!}$ permutations

3). Find the value of the following.

a). $6P_2$ b). $\frac{6!}{2!}$

Solutions: - 3a). $6P_2 = \frac{6!}{(6-2)!} = \frac{6X5X\cancel{4!}}{\cancel{4!}} = \underline{\underline{\mathbf{30}}}$

3b). $\frac{6!}{2!} \neq 3! \Longrightarrow \frac{6!}{2!} = \frac{6x5x4x3x\cancel{2!}}{\cancel{2!}} =$ 30 x12 =**360**

4).If $np_4 = 12^n p_2$, find n .

$\frac{n!}{(n-4)!} = 12(\frac{n!}{(n-2)!})$ $n^2 - 5n + 6 - 12 = 0$

$12n!(n-4)! = n!(n-2)!$ $n^2 - 5n - 6 = 0$

$= 12(n-4)! = (n-2)(n-3)(n-4)!$ $n^2 - 6n + n - 6 = 0$

$12 = (n-2)(n-3)$ $n(n-6) + 1(n-6) = 0$ $(n+1)(n-6) = 0$

$12 = n^2 - 3n - 2n + 6$ n=-1 or n-6 =0 n= -1 or n=6 but n is non-negative **n=6**

5). In how many ways can 4 different biology books & 3 different physics books be arranged on a shelf if :

a). There are no restrictions?

Solutions: - we have 4 biology books +3 physics books = 7 books . using fundamental counting principles: - 7! = 7x 6 x 5 x 4 x 3 x 2 x1 = 5040 or $7P_7$ = **5040** ways.

b). The biology books are on the left & the physics books are on the right?

Solution: - for biology books: 4! = 4 x 3 x 2 x 1

For physics books: 3!= 3x2x1

Total: 4x 3 x 2 x 1 | 3 x 2 x 1

4! | 3! = 4 x 3 x 2 x 1 x 3 x 2 x 1 = **144**.

c). The BIOLOGY books are kept together?

Solution: - If biology books are kept together, we count 4 biology as one unit .

_x_ _x_ _x_ _x - - - -

4 biology book as one unit

4! x 4! = **576**

d). The physics books kept together? we have: 5x 4 x 3 x 2 x 1
5! x 3!

Bi Bi Bi Bio phys

5! x 3! as one unit

=**720**

Circular permutation

- Circular permutation is the total number of ways in which "n" distinct objects can be arranged around a fixed circle.
  Examples; - In how many ways can 5 boys & 5 girls be seated at around table so that no two girls may be together?
  Solution: - leaving one seat vacant between two boys, 5 boys may be seated in 4! Ways. then at reaming 5 seats, 5 girls can sit in 5! Ways. Hence, the required number = 4! x 5!

Combinations

- The way of selecting the objects or numbers from a group of objects or collection, in such a way that the order of objects does not matter.
- Combinations are used for things of similar kind.
- The number of combinations of 'n' objects taking 'r' of them at a time , denoted by c(n , r), $\binom{n}{r}$, $^{n}r^{c}$ =cr^{n} & defined by :
  C (n, r) = $\frac{n!}{(n-r)!r!}$ $0\leq r \leq n.$
  N.B:C(n, r) = $\frac{npr}{r!}$

Problems of permutations (we use permutation of the following type)

1). Problems based on arrangement (arrangements of books on a shelf)

2). Problems based on standing in a line.

3). Problems based on seated in arrow (at around table)

4). Problems based on digits (numbers from a given digits)

5). Problems based on arrangements letters of a word.

6). Problems based on rank of a word (in a dictionary)

We use combination of the following type

1. Problems based on selections or choose.
2. Problems based on groups or committee.
3. Problems based on geometry.

Examples:-1. Compute c(8,6) = $\frac{8!}{(8-6)!6!} = \frac{8x7x\cancel{6!}}{2!x\cancel{6!}} = \underline{\underline{28}}$

2.A committee of 7 students has to be formed from 10 boys & 10 girls. In how many ways can this be done when the committee consists of

a). 3 girls & 4 boys? c). all girls?

b). all boys? d). at least 4 boys?

Solutions:- we have $\frac{10\ BOYS}{3}$& $\frac{10\ Girls}{4}$ we from a committee of 6 members

a). $10C_3$ x $^{10}C_4$ = $\frac{10\,!}{7!3!}$ x $\frac{10!}{6!4!}$ = $\frac{10\ X\ 9\ X\ 8\ X\ 7\ X\ 6!}{\cancel{6!}X\ 4\ X\ 3\ X\ 2\ X\ 1}$ = 120 x 210

= **25200**

b). $\frac{10\ BOYS}{7}$, $\frac{10Girs}{0}$

10_{C_7} x 10_{C_0} = $\frac{10\ X\ 9\ X\ 8}{3X7}$ X 1 = **120**

C). 10 boys , 10 girls

0 7

$10C_0$ x $10C_7$ = 1 x $\frac{10\ x\ 9\ x\ 8}{3x2}$ =**120**

e) 10 boys 10 girls

4 $\longrightarrow$ 3 $10C_4$ x $10C_3$ + $10C_5$ x $10C_2$ + $10C_6$ x $10C_1$ +$10C_7$ x $10C_0$

$5 \rightarrow 2$
$6 \rightarrow 1$
$7 \rightarrow 0$

= **37760**

e). 10 boys   10 girls

$3 \leftarrow 4$
$4 \leftarrow 3$
$5 \leftarrow 2$
$6 \leftarrow 1$
$7 \leftarrow 0$

$10C_3 \times 10C_4 + 10C_4 \times 10C_3 + 10C_5 \times 10C_2 + 10C_6 \times 10C_1 + 10C_7 \times 10C_0$

= **63,960**

3). If C(n,3) =c(n,4) , find n.

$$\frac{n!}{(n-3)!3!} = \frac{n!}{(n-4)!4!}$$

$(n-4)!4! = (n-3)!3!$

$(n-4)!\ 4 \times 3! = (n-3)\ (n-4)!\ 3!$

$4 = n-3$

$n = 7$

Binomial Theorem

Pascal's triangle

Definition :- Is a triangular arrangement of numbers that gives the coefficients in the expansion of any binomial expression , such as $(a+b)^n$, for n=0 ,1,- - - , n.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 –
- -

Binomial theorem

® For any positive integer n, the binomial expression of

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots \binom{n}{n}b^n \quad \text{with } \binom{n}{0} = C_0, \binom{n}{1} = C_1, \binom{n}{2} = C_2$$

$C_0, C_1, C_2, \cdots, C_n$ are coefficients.

® The number of terms in the expansion of $(a+b)^n$ is n+1

Examples :- 1. Find the coefficient of a^3b^2 in the expansion of $(a + b)^5$.

Solution :- The general term of the binomial expression $(a+ b)^n$ is $\binom{n}{r}a^{n-r}b^r$

$\binom{5}{r}a^{5-r}b^r$ $a^{5-r}b^r = a^3b^2$ 5-r =3 & r= 2 **r=2**

$==> \binom{5}{2}a^3b^2$

$\left(\frac{5!}{(5-2)!2!}\right)a^3b^2 = \frac{5x4x\cancel{3!}}{\cancel{3!}2!} = \frac{20}{2} = 10$

∴ The coefficient of a^3b^2 in the expansion of $(a + b)^5$ is **10**

2.Find the coefficient of the term containing a^4b^2 in the expansion of $(2a+b)^6$

$\binom{6}{r}a^{n-r}b^r = \binom{6}{r}(2a)^{6-r}b^r = a^4b^2$ 2^6-r$\binom{6}{r}$

$\mathbf{a^{6-r}b^r = a^4b^2}$ $a^{6-r}b^r = a^4b^2$ 6-r =4 & r =2 **r= 2**

$\binom{6}{2}2^{6-2} = \frac{6!}{(6-2)!2!} \times 2^4 = \frac{6x5x\cancel{4!}}{\cancel{4!}x2x1}$ x16 = 15 x 16 = **240** which is the coefficient of a^4b^2 in the expansion of $(2a+b)^6$.

3).Find the term independent of x in the expansion of a) $(2x^3 - \frac{1}{x})^{12}$

Solution :- $\binom{n}{r}a^{n-r}b^r = \binom{12}{r}(2x^3)^{12-r}(\frac{-1}{x})^r$

$= \binom{12}{r}2^{12-r}(x^3)^{12-r}(-\frac{1}{x})^r$

$= \binom{12}{r}2^{12-r}x^{36-3r} - x^{-r}$

To be independent of x 36-3r+-r =0

36-4r =0

36= 4r r = **9**

$\binom{12}{9}(2^{12-9})(x^{3})^{12-9} - (x)^{-9}$

$= -\binom{12}{9}(2^3).x^{36-27}.x^{-9}$

$= -\binom{12}{9}(8)(x^9.x^{-9})$

$= -\binom{12}{9}(8)(1)$

$= \left(\frac{12!}{9!3!} x\, 8\right) = \left(\frac{12\, x11x10)(-1)}{3x2}\right)$

=(220x8) (-1) = **-1760** is the constant term .

Types of events

1). Simple (Elementary) Event

- An event containing only one outcome.

2.Compound Events

- An event that contains more than one outcome

3.Impossible Event

- An event that cannot happen.

4.Certain or Sure Event

- One that contains the whole sample space.

4. Occurrence or Non-Occurrence of One Event

- An event is said to occur if the outcome is associated to the events sample space. otherwise, it is non-occurrence event.

6.Algebra of Events

- a) complementary of an event E, denoted by E' (not E_1) consists of all events in the sample space that are not in E_1 $E_1 \cup E_1' =$ sample space

7.Exhaustive Events

- A set of events is said to be exhaustive events if the performance of the experiment always results in the occurrence of at least one of them. Thus, if a set of events $E_1, E_2, \ldots E_n$ are subset of a sample space S, they are said to be exhaustive, if $E_1 \cup E_2 \cup E_3 \cup - - - \cup E_n = S$

8. Mutually Exclusive Events

- Two events E_1 & E_2 are said to be mutually exclusive events if the occurrence of any one of them excludes the occurrence of the other event. Thus, they cannot occur simultaneously.

9. Exhaustive & Mutually Exclusive Events

- If S is a sample space associated with a random experiment & if $E_1, E_2 \ldots E_n$ are subset of S such that.

  a).$E_1 \cap E_2 = \emptyset$, for $i \neq j$

  b). $E_1 \cup E_2 \cup \ldots \cup En = S$, then the collection of the events $E_1, E_2, \ldots, En$ forms a mutually exclusive & exhaustive set of events.

10. Independent Events

- Events are said to be independent, if the occurrence or non-occurrence of one does not affect the occurrence or non-occurrence of the other.

11. Dependent Events

- Events are said to be dependent, if the occurrence or non-occurrence of one event affects the occurrence or non-occurrence of the other.

Examples :- 1. If a die is thrown, then S= {1,2,3,4,5,6}. Let Me be the event of getting an even number , then E={2,4,6} when you throw the die , if the outcome is 4, as $4 \in E$,the outcome is 3,then as $3 \notin E$, you say that E has not occurred (not E) Here E'={1,3,5}

$E \cup E' = \{2,4,6\} \cup \{1,3,5\} = S$ or E'= S-E ={1,2,3,4,5,6} –{2,4,6} = **{1,3,5}**

Here $E \cap E' = \emptyset$ E and E' are mutually exclusive and exhaustive events.

2.When two dice thrown, consider the following events

E_1 = getting a prime number **a).** E_1 = {(2,3) ,(2,5) ,(3,3) ,(3,5) ,(5,2) (5,3) ,(5,5)}.

E_2 = getting an even number. **b).** E_2 = {(2,2),(2,4) ,(2,6) ,(4,2) ,(4,4) ,(4,6) ,(6,2) ,(6,4) ,(6,6)}

a). List E_1 & E_2 b) Are E_1 and E_2 exhaustive events? No because $E_1 \cup E_2 \neq S$

S ={(1,1),(1,2),(1,3),(1,4),(1,5)(1,6), c) Are E_1 and E_2 mutually exclusive events? No because

(2,1) ,(2,2),(2,3) ,(2,4) ,(2,5) ,(2,6), $E_1 \cap E_2 \neq \emptyset$.

(3,1) ,(3,2) ,(3,3) ,(3,4) ,(3,5) ,(3,6),

(4,1) ,(4,2) ,(4,3) ,(4,4) ,(4,5) ,(4,6) ,

(5,1) ,(5,2) ,(5,3) ,(5,4) ,(5,5) ,(5,6) ,

(6,1) ,(6,2) ,(6,3) ,(6,4) ,(6,5) ,(6,6)}.

3.You draw a card from a pack of 52 playing cards .

a). How many possible events are there: **Ans :-52**

b). is the event that you draw 9 of spade a simple or a compound event? **Ans** :-**a simple event** .

c).is the event that you draw any card of spade as simple or compound event.? **Ans**:- **a compound event .**

d). How do you call the event of drawing the card of heart? Ans :-Impossible Event

Probability of an Event

- Probability can be reported as a fraction, decimal ,percentage or ratio.

- The greater the probability , the more likely the event is to occur.
- The smaller the probability , the less likely the event is to occur.
- Probability is a real number in the closed interval [0,1].
- A probability of 0 means that an event will never occur (impossible event).
- A probability of 1 means that an event will always occur (certain event).
- Probability can be measured in three different approaches (assignment of probability).

1. *The classical (Mathematical) approach*

- In the classical approach to probability , if all the outcomes of random experiment are equally likely & mutually exclusive , then the probability of an event E is :

P(E) = $\frac{number\ of\ outcomes\ favoring\ E}{number\ of\ all\ possible\ outcomes} = \frac{n(E)}{n(s)}$

- Outcomes of a random experiment are said to be equally likely , when each element has equal chance of being chosen.

2. *Empirical approach*

- This is based on the relative frequency of an event when the experiment is repeated a large number of times.

Examples :- 1.

x	1	2	3	-1	-2	-3
f						

P(E)= $\frac{Frequency\ of\ E}{Total\ number\ of\ observation} = \frac{E}{N}$

P(negative numbers) = $\frac{frequency\ \ negative\ numbres}{N}$

$\frac{10}{20} = \frac{\mathbf{1}}{\mathbf{2}}$//

3. The axiomatic approach

- Based on the following conditions. Let S be a sample space ..

i). 0≤ $P(E) \leq 1$ ii). P(S) =1

® if E_1 and E_2 are mutually exclusive events , then $P(E_1 \cup E_2) = P(E_1) + P(E_2)$.

® If $E \cup E' = S$, then $P(E \cup E') = P(S) = 1$, & $P(E') = 1 - P(E)$.

Examples :- 1. If a die is rolled once, find the probability of getting an even number.

Solution:- Our sample space is S= {1,2,3,4,5,6}

E= {2,4,6}, n(S) =6, n(E) =3

P(even no) = $\frac{n(Even\ number}{Total\ outcome(sample\ space)} = \frac{3}{6} = \frac{\mathbf{1}}{\mathbf{2}}$//

2). If a die is rolled twice , then find the following probabilities .

a). The probability of getting the sum 2.

b). The probability of the sum greater than 6.

c). The probability of getting the sum greater than 9 or an odd .

Solutions :- S={(1,1) ,(1,2) ,(1,3),(1,4) ,(1,5),(1,6),
(2,1),(2,2),(2,3) ,(2,4),(2,5),(2,6),
(3,1) ,(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}

a). E=getting the sum 2 = {(1,1)} ,n(E) =1

P(getting the sum 2) = $\frac{n(E)}{n(S)} = \frac{\mathbf{1}}{\mathbf{36}}$//

b). E = getting the sum greater than 6.

={(1,6),(2,5),(2,6),(3,4),(3,5),(3,6),(5,2),
(4,3),(4,4) ,(4,5),(4,6),(5,3),(5,4),
(6,4) ,(5,5),(5,6),(6,1)(6,2),(6,3),(6,5),(6,6)}.

n(E) = 21 P(E)= $\frac{n(E)}{n(S)} = \frac{21}{36} = \frac{\mathbf{7}}{\mathbf{12}}$ //

c).E= getting the sum greater than 9 or odd.

E={(4,6),(5,5),(5,6) ,(6,4) ,(6,5) ,(6,6),(1,2) ,(1,4), (1,6) ,(2,1) ,(2,3),(2,5),(3,2),(3,4), (3,6), (4,1) ,(4,3) ,(4,5) ,(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)}.

n(E) =24 P(E) = $\frac{n(E)}{n(s)} = \frac{24}{36} = \frac{4}{6} = \frac{\mathbf{2}}{\mathbf{3}}$//

3. There are 4 blue balls & 3 red balls in a bag. you pick 3 balls at random . find the probability of getting 2 blue balls & 1 red ball.

Solution: Total number of outcomes = $7C_3 = \frac{7!}{(7-3)!3!} = \frac{7X6X5X4!}{(4)!3X2X1} = \underline{\underline{35}}$

For red balls , there are $3c_1$ possible outcomes & for 2 blue balls ,there $4c_2$ possible outcomes , since picking up blue balls & picking up red balls are independent , the number of possible outcomes of picking 2 blue balls &1 red ball is :-

$4C_2 \text{ x } 3C_1 = \frac{4!}{2!2!} \text{ X } \frac{3!}{2!1!} = 6x3 = \underline{\underline{18}}$

P(2 blue or 1 red) = $\frac{4C2 \text{ x } 3C1}{7C3} = \frac{\mathbf{18}}{\mathbf{35}}$//

4. Three cars are chosen at random from a certain car station containing 8 defective & 12 non-detective cars . what is the probability that .

   a). all are defective b). all are non-defective

   c). two cars are defective & the other is non-detective .

   Solutions:- 4a). all are defective only if : the number of ways of choosing 3 defectives & zero non-defective is $8C_3 \text{ x } 12C_0 = \frac{8!}{5!3!} \text{ x } \frac{12!}{12!0!} = \frac{8X7X6X5!}{5!3X2} \text{ X } \frac{12!}{12!X1} = \underline{\underline{56}}$

But total number of possible outcomes 3 cars out of 20 cars (8 defective +12 non-defective =20)

P(all are defective) = $\frac{n(defective)}{Total} = \frac{8C3\ X12C0}{20C3} = \frac{56}{1140} = \frac{\mathbf{14}}{\mathbf{285}}$ //

b).all are non-defective means there is no defective cars.

$8c_0 \text{ x } 12c_3 = \underline{\underline{220}}$

P(all are non-defective) = $\frac{8C0\ X12C3}{20C3} = \frac{220}{1140} = \frac{22}{114} = \frac{\mathbf{11}}{\mathbf{57}}$//

c). two are defectives & the other are non-defective

two out of 8 & 1 out of 12

$\frac{8c2\ x\ 12c1}{20C3} =$ P(two are defective & one non-defective)

$= \frac{\frac{8!}{6!2!} x \frac{12!}{11!x1!}}{1140} = \frac{8x7x\cancel{6}!}{\cancel{6}!x2x1} \text{ X } \frac{12x\cancel{11}!}{\cancel{11}!x1} = \frac{28x12}{1140} = \frac{336}{1140} = \frac{168}{570} = \frac{84}{285} = \frac{\mathbf{28}}{\mathbf{95}}$ //

5. In a sample of 50 people ,21 had type 0 blood ,22 had type A blood ,5had type B blood , and 2 had type AB blood ,then find the probability of :

a). A person has type O blood

b). A person has type A or type B blood.

c). A person has neither type A or type O blood .

d). A person does not have type AB blood .

Solutions :- 5a). P(type O blood) = $\frac{f0\ blood}{total\ blood\ type} = \mathbf{\frac{21}{50}}$ //

b). P(A or B) = p(A) +P(B) (since A & B are mutually exclusive)

$$= \frac{22}{50} + \frac{5}{50} = \mathbf{\frac{27}{50}} //$$

c). P(neither A nor O) = P(B or AB) = $\frac{5}{50} + \frac{2}{50} = \mathbf{\frac{7}{50}}$ //

d). P(not AB) = 1-p(AB) = 1- $\frac{2}{50} = \frac{48}{50} = \mathbf{\frac{24}{25}}$//

6. If one card is drawn from a deck, find the probability of picking

a). king b) Not king c). king & queen d). king or queen e). a6 or a spade .

Total number of cards =52

In a standard or deck cards , we have 52 cards

26 red 13 diamonds

26 black 13 hearts

13 spades

Clubs(flowers)

n(king) =4

a). P(king) = $\frac{n(king)}{total} = \frac{4}{52} = \mathbf{\frac{1}{13}}$//

b). P(not king) = 1- P(king) = 1- $\frac{1}{13} = \mathbf{\frac{12}{13}}$//

c). a card cannot be a king & a queen at the same time.

P(king & queen) $= \frac{n(\emptyset)}{52} = \frac{0}{52} = \underline{\underline{\mathbf{0}}}$

d). P(king or queen) = P(a king) + P(a queen) $= \frac{1}{13} + \frac{1}{13} = \frac{\mathbf{2}}{\mathbf{13}}$//

e). P(a 6 or a spade) = P(6) +P(spade) –P(6 & spade) = $\frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{\mathbf{4}}{\mathbf{13}}$//

7. which of the following cannot be valid assignments of probabilities for outcomes of sample space .

$S=\{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$.

	a_1	a_2	a_3		a_4	a_5	a_6	a_7
a	0.2	0.001	0.09		0.03	0.01	0.008	0.3
b	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$		$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
c	0.1	0.2	0.3		0.4	0.5	0.6	0.7
d	-0.7	0.007	0.3		0.4	-0.2	0.1	0.3
e	$\frac{1}{13}$	$\frac{2}{13}$	$\frac{3}{13}$		$\frac{4}{13}$	$\frac{5}{13}$	$\frac{6}{13}$	$\frac{7}{13}$

<u>**Solutions**</u> :-

- The sum of probability of the given outcome must be 1.
- The probability of an event cannot be negative.
- $0 \leq P(E) \leq 1$.
- The probability of the given outcome not greater than 1.

a). $P(a_1) + P(a_2) + P(a_3) + P(a_4) + P(a_5) + P(a_6) + P(a_7) = 1$

but $P(a_1) + P(a_2) + P(a_3) + P(a_4) + P(a_5) + P(a_6) + P(a_7) = 0.639$

∴a is not valid

b). not valid because the sum of probability of the given outcome does not exceed 1.

c). $P(a_1) + P(a_2) + P(a_3) + P(a_4) + P(a_5) + P(a_6) + P(a_7) = 1$ not valid, because the sum of the probabilities is 2.8

d). The probability of the given event is non-negative

d). Not valid

odds in favour of & odds a gainst an event

Definition :-if m and n are probabilities of the occurrence & non-occurrence of an event respectively , then the ratio m:n is called the odds in favor of the event & the ratio n:m is called the odds against the event .

N.B: odds in favor of an event E(ratio of success to failures)

$$= \frac{P(E)}{P(E')} = \frac{P(success)}{P(failuress)}$$

Odds against $= \frac{P(E)}{P(E')} = \frac{P(success)}{P(failuress)}$

an event

Examples :- 1. If a race horse runs 100 races & wins 25 times & loses the other 75 times , what are the probability of winning & the odds of the horse winning ?

Solution :-

P(w) = $\frac{25}{100}$ = **0.25** & P(losing) = $\frac{75}{100}$ = **0.75** odds in favor = 0.25:0.75 $\frac{25}{75}$ = **0.333**

2.The odds against certain events are 6:8 , find the probability of its occurrence .

Solution :- E= $\frac{8}{8+6} = \frac{8}{14} = \frac{4}{7}$//

The rules of probability (rules on probability)

a). **Addition Rule** :- for any three events E_1,E_2 & E_3

1. $P(E_1 \cup E_2) = p(E_1) + P(E_2) - P(E_1 \cap E_2)$
2. $P(E_1 \cup E_2 \cup E_3) = p(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3)$
   $- P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$
3. $P(E) + P(E^C) = 1$
4. $P(E_1 - E_2) = P(E_1 \cap E_2^C) = P(E_1) - P(E_1 \cap E_2)$.
5. $P(E_1 \cup E_2) + P(E_1 \cup E_2)^C = 1$, E_1 and E_2 are in the sample space .

Examples :- 1. Two dice are rolled .find the probability of getting

a) sum of 8,9 or 10

c) a sum greater than 9 or a sum 12.

b) doubles or a sum 7.

Solutions :- S = ={(1,1) ,(1,2) ,(1,3),(1,4) ,(1,5),(1,6),
(2,1),(2,2),(2,3) ,(2,4),(2,5),(2,6),
(3,1) ,(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1),(4,2),(4,3),(4,4),(4,5),(4,6),
(5,1),(5,2),(5,3),(5,4),(5,5),(5,6),
(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)}.

a) a sum of 8 ,or 9 or 10

E = {(2,6) ,(3,5) ,(3,6) ,(4,4) ,(4,5) ,(4,6) ,(5,3) ,(5,4) ,(5,5) ,(6,2) (6,3) ,(6,4)}.

$P(E)=\frac{n(E)}{36} = \frac{12}{36} = \frac{\mathbf{1}}{\mathbf{3}}//$ or =P(sum 8 or 9 or 10)

= P(sum 8) + P(sum 9) + P(sum10)

$= \frac{5}{36} + \frac{4}{36} + \frac{3}{36} = \frac{12}{36} = \frac{\mathbf{1}}{\mathbf{3}}//$

b) Doubles or a sum of 7

E_1 =getting doubles = {(1,1) ,(2,2) ,(3,3) ,(4,4) ,(5,5) (6,6) }.

$n(E_1) = 6$

E_2 = getting a sum 7 ={(1,6), (2,5) ,(3,4) ,(4,3) ,(5,2) ,(6,1)}

$n(E_2) = 6$

E_1 and E_2 are mutually exclusive events.

$P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{\mathbf{1}}{\mathbf{3}}//$

c). E_1= getting a sum greater than 9 = {(4,6) ,(5,5) ,(5,6) ,(6,4) ,(6,5) ,(6,6) }.

E_2 =getting a sum of 12 ={(6,6) }.

$P(E_1 \text{ or } E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$

$= \frac{6}{36} + \frac{1}{6} - \frac{1}{6} = \frac{\mathbf{1}}{\mathbf{6}}//$

Multiplication Rule

- It is based on the concepts of dependence or independence of events.

Examples :- 1. A jar contains 4 black & 3 white balls. You draw two balls one after the other with replacement (the second is drawn after the first is replaced). Find the probability that the first ball is black & the second ball is also black.

Solution:- Let E_1 be the first ball black

Let E_2 be the second ball is black.

$P(E_1$ & then $E_2) = P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{4}{7} \times \frac{4}{7} = \frac{16}{49}$//

(Since the selection is with replacement , E_1 & E_2 are independent

Conditional Probability

❖ When occurrence of one event depends on the occurrence of another event , we say the second event is conditioned by the first event .

❖ The conditional probability of an event E_2 in relationship to an event E_1 is the probability that event E_2 occurs after event E_1 has already occurred. The notation for conditional probability is $P(E_2/E_1)$, it means the probability that event E_2 occurs given that E_1 has already occurred .

❖ Let E_1 and E_2 be any two events , the probability that both events occur denoted by $P(E_1$ and $E_2) = P(E_1E_2) = P(E_1 \cap E_2)$ & is given by : $P(E_1 \cap E_2) = P(E_1) \times P(E_2/E_1), P(E_1) \neq 0$

$= P(E_2) \times P(E_1/E_2), P(E_2) \neq 0$

(If E_1 & E_2 are dependent)

If E_1 & E_2 are independent $P(E_1/E_2 = P(E_1)$ & $P(E_2/E_1) = P(E_2)$

$P(E_1 \cap E_2) = P(E_1 \times P(E_2)$

Example :- 1. A jar contains 3 red , 5 Green , 2 blue & 6 yellow marbles 3 marbles are drawn one after the other . find the probability of getting a green marble on the first draw , a yellow marble on the second draw & a red marble on the third draw , if

a). each marble is drawn , but there is replaced back before the next draw . (with replacement)

b). the marbles are drawn without replacement .

Solutions :- Let E_1= getting green ball , in the first draw

E_2 = getting yellow ball , in the second draw

E_3 =getting red ball, in the third draw.

a) The marbles are replaced after each draw.

$P(E_1\ E_2\ E_3) = P(E_1) \times P(E_2) \times P(E_3)$ (They are independent)

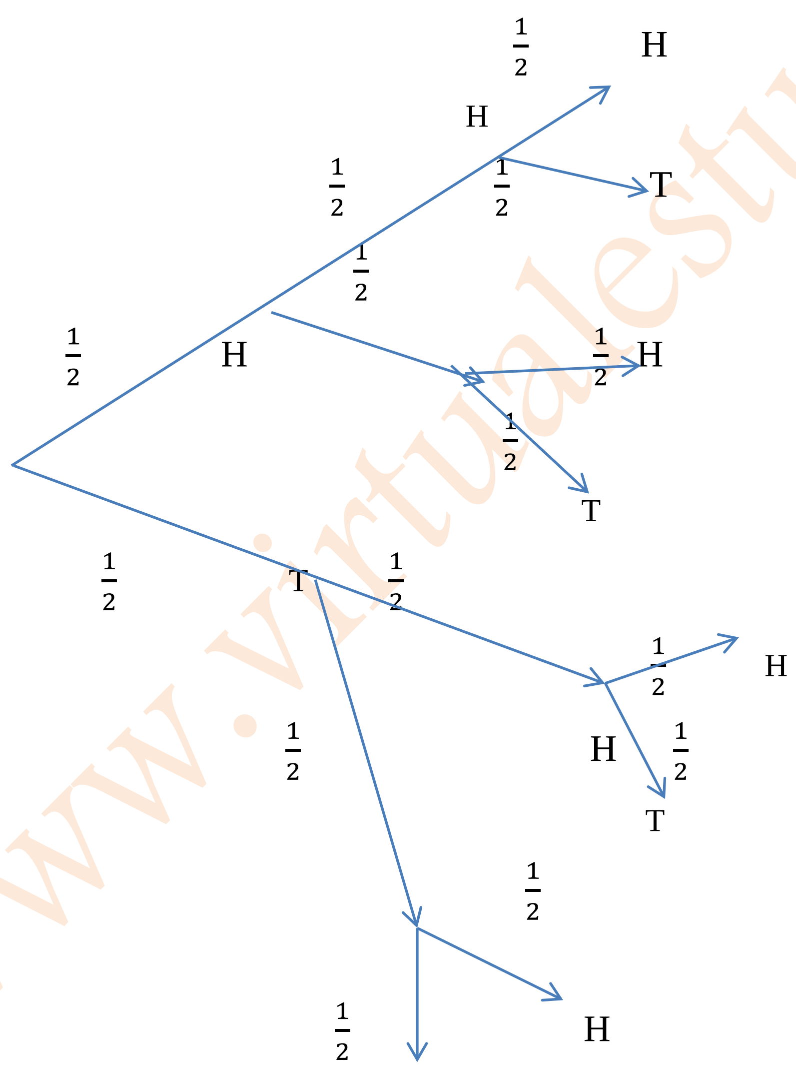
$$= \frac{5}{16} \text{ X } \frac{6}{16} \text{ X } \frac{3}{16} = \frac{45}{2048}$$

b) Without replacement (they are dependent)

$P(E_1 \cap E_2 \cap E_3) = P(E_1) \text{ x } P(E_2)(E_1) \text{ x } P(E_3)(E_1 \cap E_2) = \frac{5}{16} \text{ X } \frac{6}{15} \text{ X } \frac{3}{14} = \mathbf{\frac{3}{112}}$//

Sequential Events (Using tree diagram)

Examples: - A fair coin is tossed three times. find the probability that all outcomes will be heads. **Solution**: - we have 2^3 =8 possible outcomes using tree diagram.



Sequential events	Probability of sequential events
HHH	$\frac{1}{8}$
HHT	$\frac{1}{8}$
HTH	$\frac{1}{8}$
HTT	$\frac{1}{8}$
THH	$\frac{1}{8}$
THT	$\frac{1}{8}$
TTH	$\frac{1}{8}$
TTT	$\frac{1}{8}$

P(all outcomes are head) = $\frac{\mathbf{1}}{\mathbf{8}}$//

Real –Life Applications of Probability

⇒ Probability theory is widely used in the area of studies such as statistics, finance insurance policy, traffic signals ,medical decisions & weather forecasting etc…

Examples :- 1. A traffic light at a certain road crossing starts green at 6:30 hours & continues to be green till 6:32 hours & again turns green at 6:36 hours & continues green till 6:38 hours . this cycle is repeated throughout the day. If a person's arrival time at this crossing is random & uniform over the interval 18:20 to 18:35 hours , then find the probability that he has to wait the signal.

Solution :- In every 6 minutes , the light remains green for 2 minutes & red for 4 minutes . so , in the interval 18:20 hours to 18:35 hours i.e 15 minutes ,the light will remain green for $\frac{2}{16}$x15 = 5 minutes & red for 10 minutes .

The probability that he has to wait at the signal $= \frac{10}{15} = \frac{2}{3} =$ **67%**

2.If the probability that a person lives in an industrialized country of the word is $\frac{1}{5}$. Find the probability that a person does not live in an industrialized country.

Solution: - P(E)= $\frac{1}{5}$ & P(E')=? P(E') = 1- P(E) = 1- $\frac{1}{5}$ $=$ $\frac{4}{5}$//.