Project 3: MECH 6371 Computational Thermal and Fluid Science

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1. Introduction

The problem given is a 1 – D PDE of the inviscid Burger's equation with the following initial conditions:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u(x,0) = \begin{cases} 1 & for\ \ 0 < x < 0.2 \\ 0 & for\ \ 0.2 < x < 1 \end{cases}$$

Where $u(x,t)$ is the value of the function at the spatio-temporal coordinate (x,t). For simplicity, the value $u(\ x = 0.2,\ t = 0\ )$ has been initialized as 1 because of the ambiguity in the problem statement. This equation is to be solved using four different methods:

1. Upwind 1st order
2. Central difference
3. Lax method
4. Lax – Wendroff

Required:

1. Comparison of the results for different schemes
2. Effect of stability parameters on the results
3. Numerical dispersion and dissipation effects on the schemes

2. Methodology

Each scheme follows a different discretization method which is explained as follows:

1. **1st order Upwind scheme:**
   This scheme discretizes the PDE using the nonlinear second term as the partial derivative of $(u^2/2)$ as follows:

$$\frac{U_i^{\,n+1} - U_i^{\,n}}{\Delta t} + \frac{1}{2}\frac{U^2{}_{i+1}{}^{n} - U^2{}_{i}{}^{n}}{\Delta x} = 0$$

The computation is conducted for all nodes except the boundary nodes where the initial condition is enforced for all timesteps. The stability requirement of this scheme is as follows:

$$\left|u_{max}\frac{\Delta t}{\Delta x}\right| \leq 1$$

Where u_{max} is the maximum value of u(x,t) at the previous timestep. This means either the computation needs to be designed with adaptive timestep where the Δt would change for every timestep according to the stability requirement, or an apriori u(x,t) range needs to be known. It is however possible to consider a conservative estimate of $\frac{\Delta t}{\Delta x}$ being less than 0.5, or in other words, an assumption of u(x,t) always being bounded within the range of [0 , 2]. And hence for this scheme, a conservative stability criterion of the following was used:

$$\frac{\Delta t}{\Delta x} \leq 0.5$$

2. **Central difference:**
   This scheme discretizes the spatial derivative nonlinear term with central difference discretization as follows:

$$\frac{U_i^{n+1} - U_i^{n}}{\Delta t} + \frac{1}{2}\frac{{U^2}_{i+1}^{\ \ n} - {U^2}_{i-1}^{\ \ n}}{2\,\Delta x} = 0$$

   The stability analysis of the scheme concludes that this scheme is unconditionally unstable.

3. **Lax method:**
   This is the same as the central difference, but with the transient term at nth timestep also discretized as the average of neighboring spatial points. This scheme hence discretizes the PDE as follows:

$$U_i^{n+1} = \frac{U_{i+1}^{\ \ n} + U_{i-1}^{\ \ n}}{2} - \left(\frac{\Delta t}{\Delta x}\right)\frac{1}{2}\left(\frac{{U^2}_{i+1}^{\ \ n} - {U^2}_{i-1}^{\ \ n}}{2}\right)$$

   The stability requirement of this scheme is as follows:

$$\left|u_{max}\frac{\Delta t}{\Delta x}\right| \leq 1$$

   Similar to the upwind scheme stability criteria, the conservative estimate of the CFL value was considered as follows:

$$\frac{\Delta t}{\Delta x} \leq 0.5$$

4. **Lax – Wendroff:**
   This is a modification to the Lax method where there is an addition of the 2nd order terms which makes this the only 2nd order method for the purpose of this project. The scheme is discretized as follows:

$$U_i^{n+1} = U_i^n - \left(\frac{\Delta t}{\Delta x}\right)\frac{1}{2}\left(\frac{{U^2}_{i+1}^{\ \ n} - {U^2}_{i-1}^{\ \ n}}{2}\right) + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\right)^2\left(\left(\frac{{U_{i+1}}^n + {U_i}^n}{2}\right)\left(\frac{{U^2}_{i+1}^{\ \ n} - {U^2}_i^{\ n}}{2}\right) - \left(\frac{{U_i}^n + {U_{i-1}}^n}{2}\right)\left(\frac{{U^2}_i^{\ n} - {U^2}_{i-1}^{\ \ n}}{2}\right)\right)$$

The stability criteria of this scheme are the same as the Lax method and similar approach has been implemented here.

3. Results

3.1 Comparison of the schemes:

Before comparing the results, it is important to note that the Central difference scheme is unconditionally unstable and hence the evolution of the wave is shown in Fig. 1 for $\Delta t = 0.0025\ seconds$ and $\Delta x = 0.005\ units$. It is however possible to solve the Central difference method using an implicit scheme, but it is assumed this solution is not a requirement for this project.

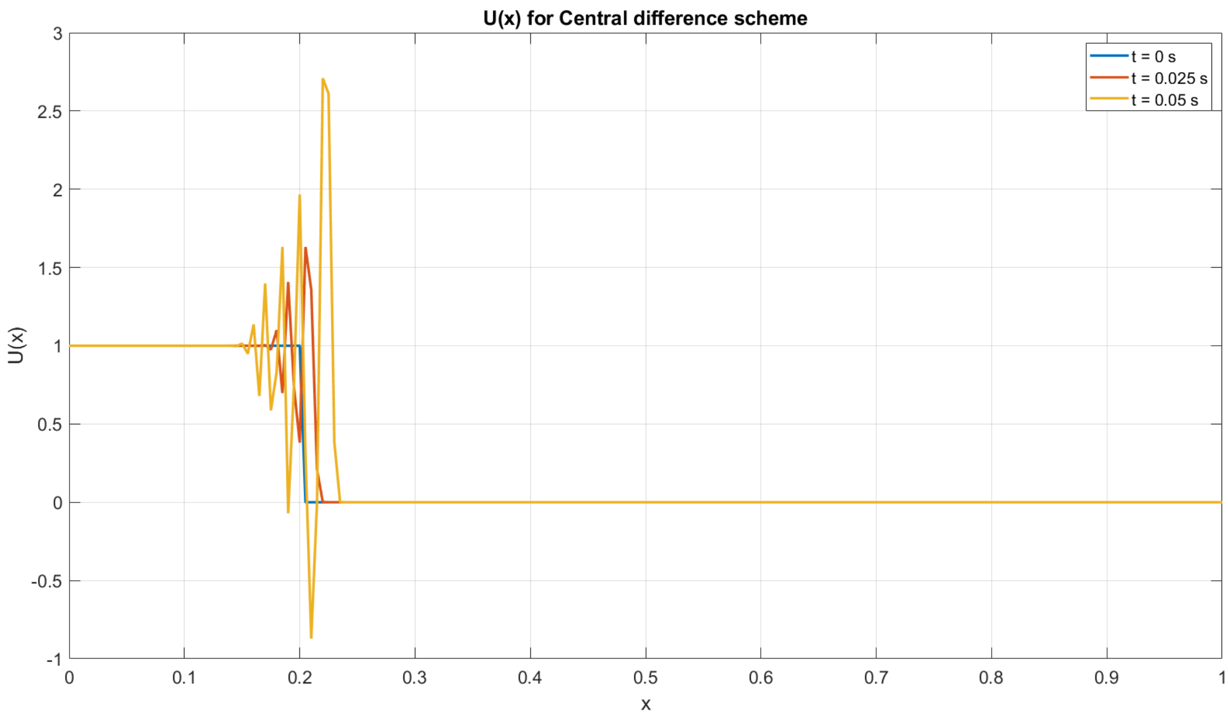
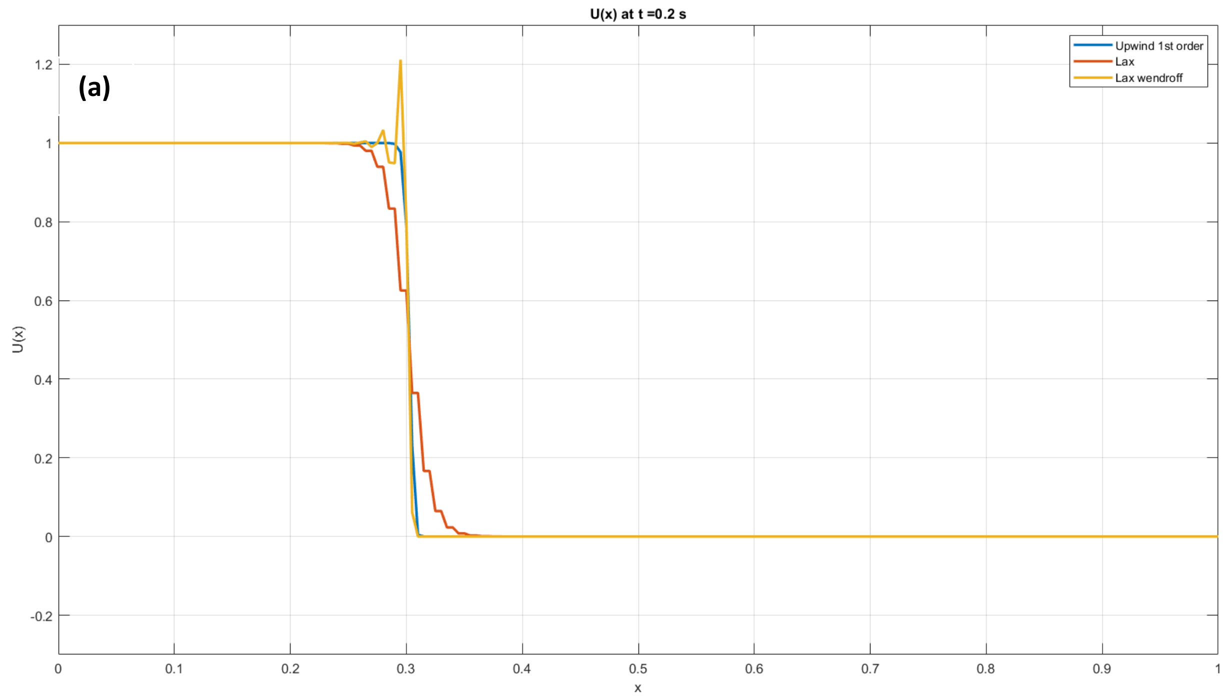
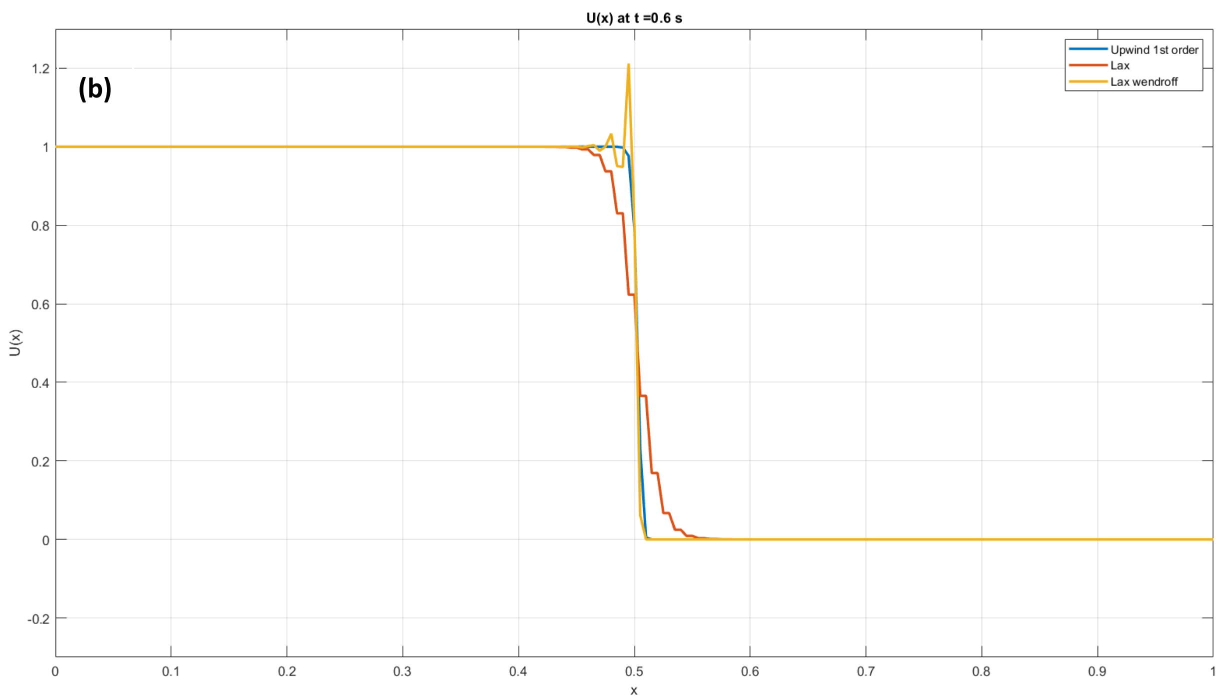


Fig. 1: Unstable evolution of u(x,t) for central difference scheme

Hence the rest of the report will contain the comparison of only the rest three schemes – Upwind, Lax and Lax – Wendroff methods.

For comparison purpose, the values of Δt and Δx are chosen to be the same for all three schemes – $\Delta t = 0.0025\ seconds$ and $\Delta x = 0.005\ units$, which makes the ratio $\frac{\Delta t}{\Delta x} = 0.5$. The evolution for all three schemes for 4 different timesteps is shown in Fig. 2 (a) – (d).


U(x) at t =0.2 s
(a)
Upwind 1st order
Lax
Lax wendroff
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
U(x)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x


U(x) at t =0.6 s
(b)
Upwind 1st order
Lax
Lax wendroff
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
U(x)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

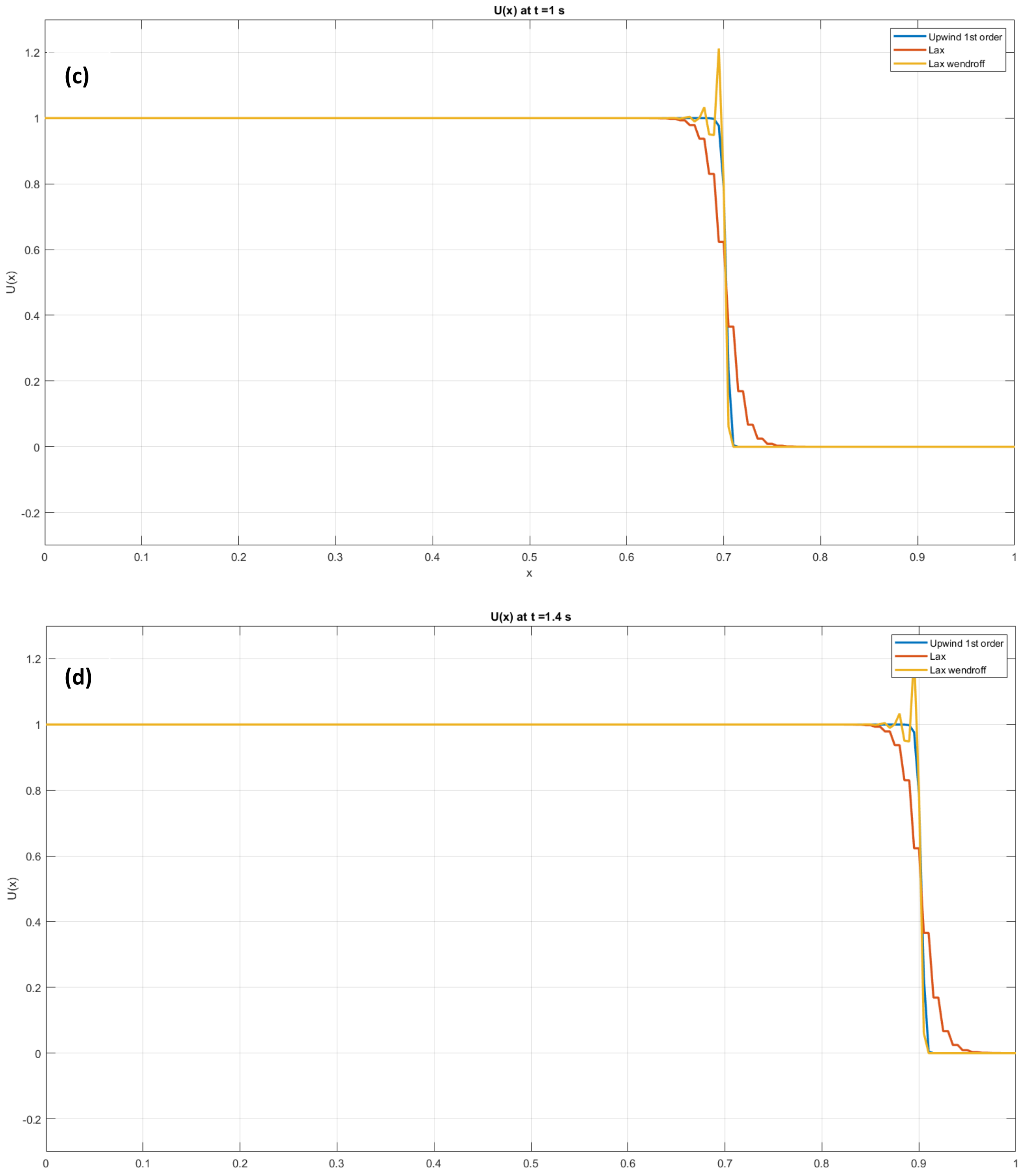


Fig. 2: $u(x,t)$ profiles for all 3 schemes for (a) $\Delta t = 0.2\ s$ (b) $\Delta t = 0.6\ s$ (c) $\Delta t = 1\ s$ (d) $\Delta t = 1.4\ s$
(Stability parameter $= \frac{\Delta t}{\Delta x} = 0.5$)

It is noted that the wave curve progresses from left to right and ends on the other end of the domain at approx. t = 1.6 s. The smoothest curve is the upwind scheme curve which is the blue curve in the Fig. 2. There is nothing special about the upwind scheme as it is the simplest finite difference scheme for the hyperbolic

PDE. This scheme contains the lowest dissipation error which is noted in the least amount of smoothing of the curve near the discontinuity. The Lax curve while correctly predicts the moving discontinuity, due to the dissipative nature of the scheme, there are several intervals close to the discontinuity where there are step-like shapes noted. This is due to the dissipation error from the presence of 2^{nd} order derivative term in the modified equation of the Lax method. However, for Lax – Wendroff method which is a 2^{nd} order method, there are odd derivative terms in the modified equation which makes the dispersion effects more prominent in this scheme. This is noted in the sharp peaks in Fig. 2 for Lax – Wendroff curves. An important point to note is that these peaks are shaped only on the left side of the discontinuity and not on the right side. This is because a dispersive scheme like Lax – Wendroff is primarily asymmetric because of the direction of the wave from left to right.

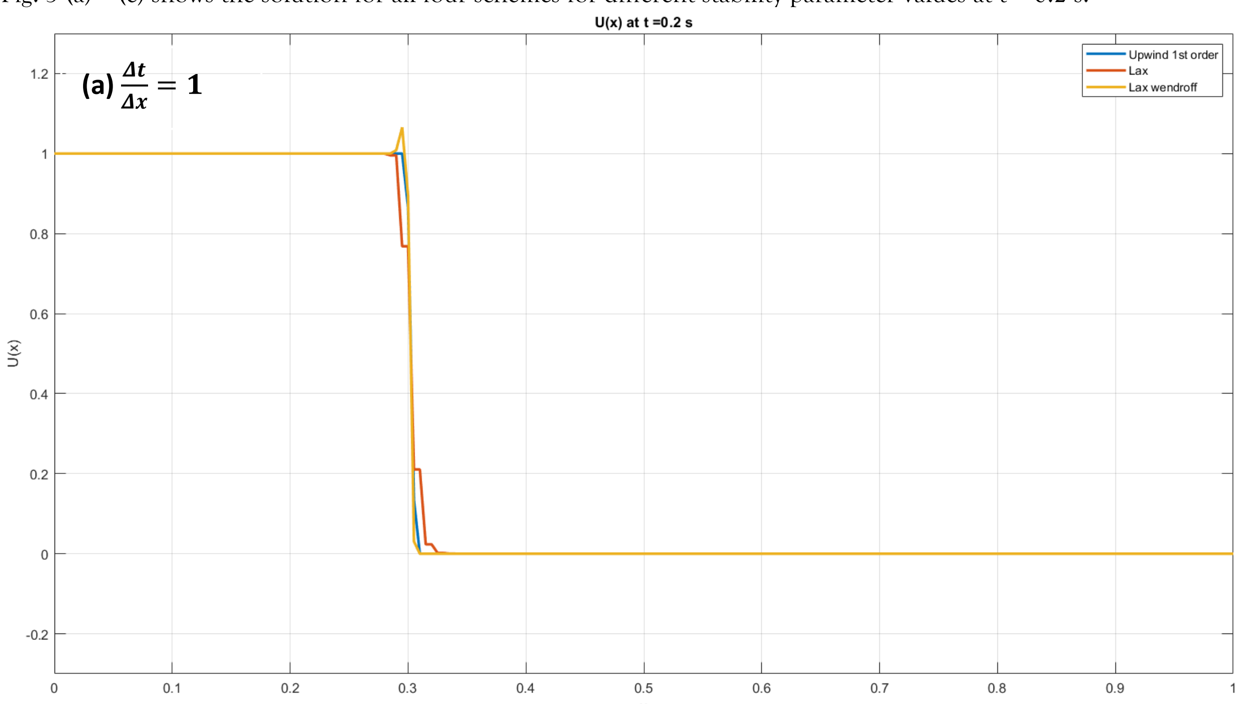
3.2 Effect of stability parameter, dispersion and dissipation:

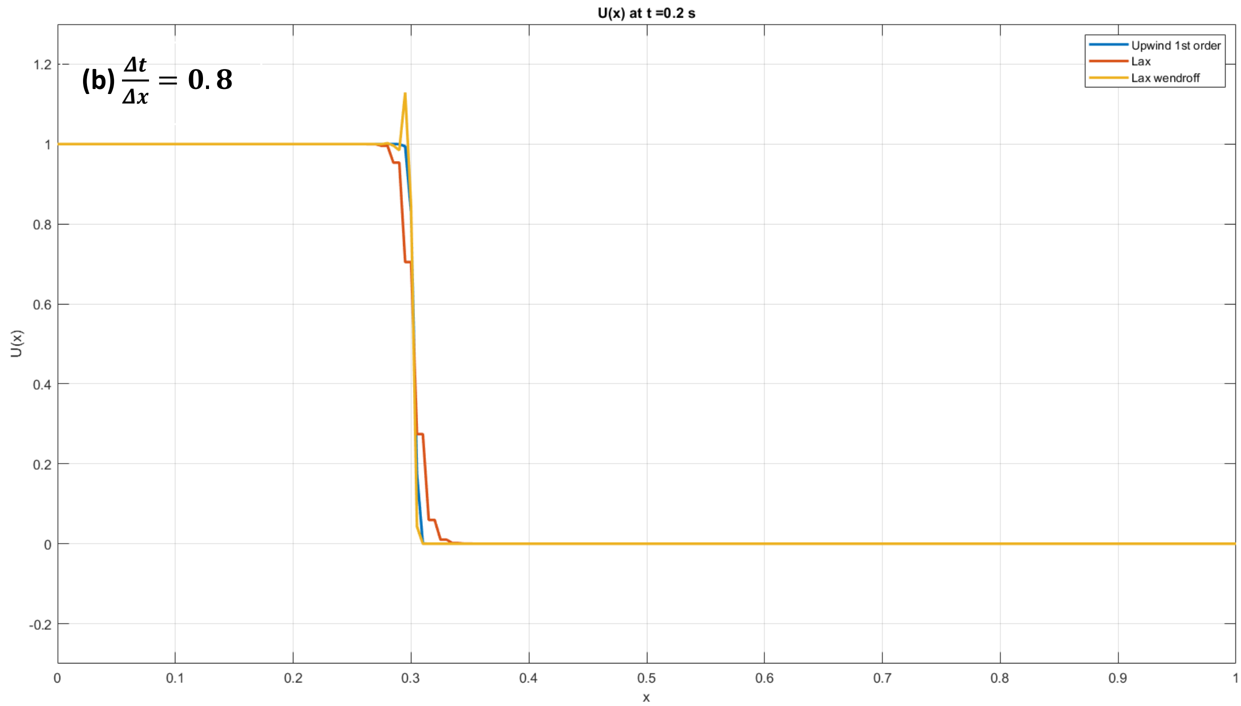
The effect of stability parameter $\frac{\Delta t}{\Delta x}$ is investigated for every scheme at a particular timestep, t = 0.6 s. Five different stability parameter cases are considered for this study. In order to see any notable differences due to the stability parameter, only the Δt value is varied, i.e. the mesh size is kept constant for all stability parameters as shown Table 1.

Δt (s)	**Δx** (m)	$\frac{\Delta t}{\Delta x}$
0.005	0.005	1
0.004	0.005	0.8
0.0025	0.005	0.5
0.002	0.005	0.4
0.001	0.005	0.2

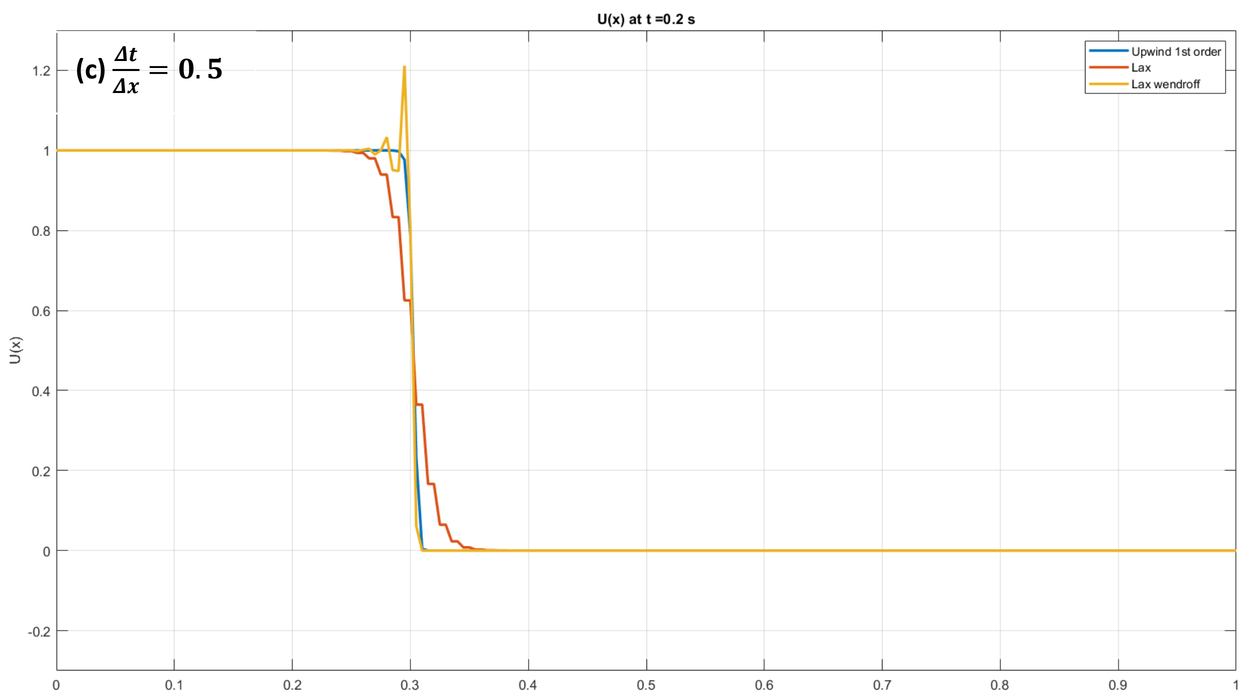
Table 3: Δt and Δx variation for the stability parameter cases

Fig. 3 (a) – (e) shows the solution for all four schemes for different stability parameter values at t = 0.2 s.





U(x) at t =0.2 s
(b) $\frac{\Delta t}{\Delta x} = 0.8$
Upwind 1st order
Lax
Lax wendroff
U(x)
x



U(x) at t =0.2 s
(c) $\frac{\Delta t}{\Delta x} = 0.5$
Upwind 1st order
Lax
Lax wendroff
U(x)
x

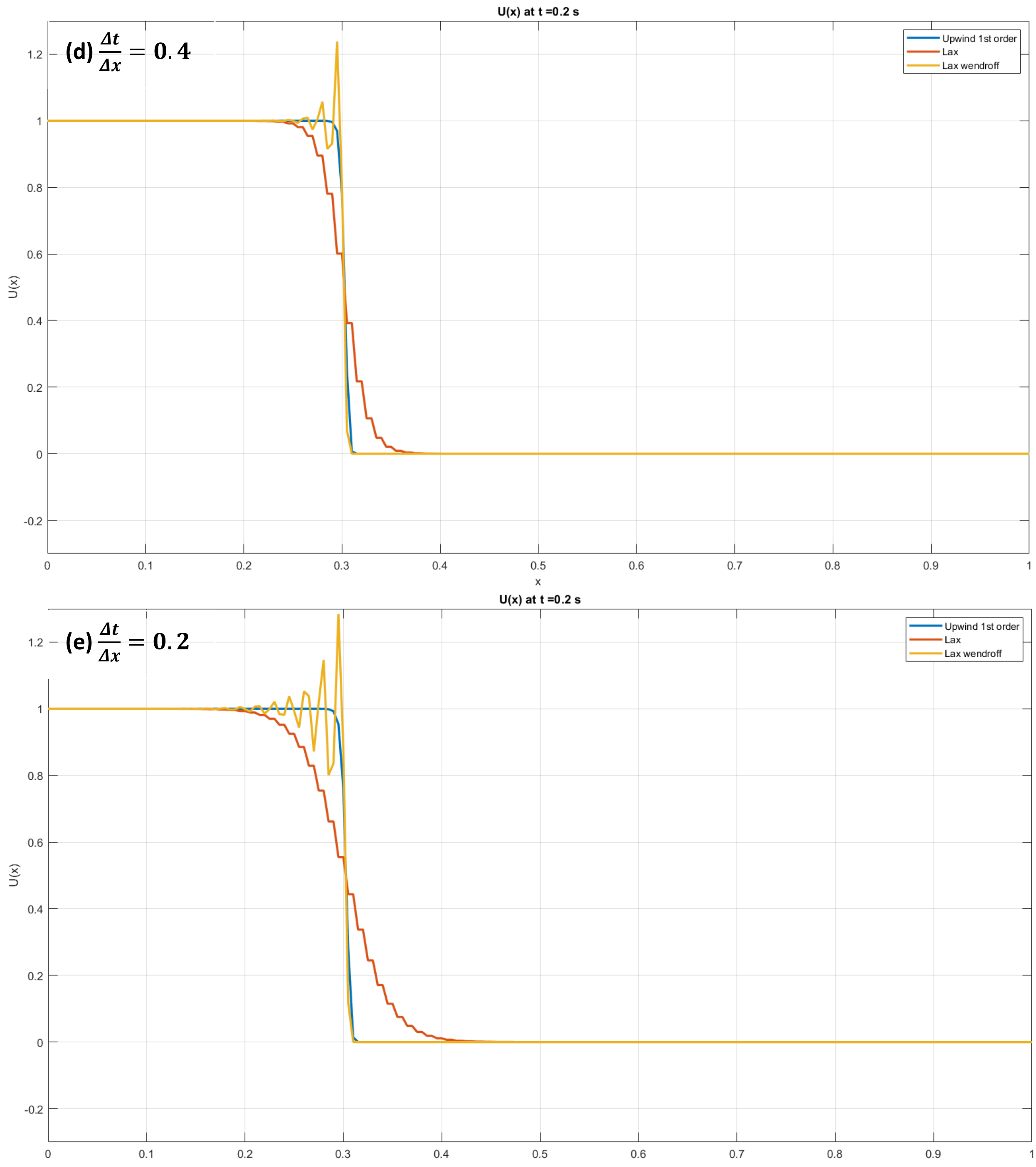


Fig. 3: $u(x,t)$ profiles for all 3 schemes for $\frac{\Delta t}{\Delta x}$ = (a) 1 (b) 0.8 (c) 0.5 (d) 0.4 and (e) 0.2

From Fig. 3 above, it is noted that the gradient of the Upwind curve gets smoother with decrease in the stability parameter. This is because the numerically added dispersion term of the modified equation gets larger with decrease in the stability parameter from the value of 1. Hence the upwind solution with $\frac{\Delta t}{\Delta x} = 1$ is the closest to the exact solution. Similarly, for the Lax method the curve gets more distorted with a decrease in the stability parameter. As mentioned before, the *steps* in the solution near the discontinuity are due to the dispersive term. For the Lax – Wendroff scheme, again the solution has least oscillations for the maximum stability parameter – 3 (a). As the stability parameter increases, the oscillations start to increase dramatically near the discontinuity. In this case, the dissipative term in the modified equation depends inversely on the stability parameter. It is thus important to note that simply decreasing the stability parameter for this scheme is not always favorable as the dissipative error gets influenced by this decrement. The asymmetry is due to the movement of the wave in the right direction.

It is important to note that these effects are independent of the spatial resolution of the solution. The number of points for all stability parameter cases are kept constant. But Fig. 4 shows the solution for the same stability parameter as in Fig. 3(e), i.e. $\frac{\Delta t}{\Delta x} = 0.2$, but with a lower $\Delta x = 0.0025$. It is noted that all the curves get compressed, or closer to the discontinuity, however, the number of oscillations is the same for the Lax – Wendroff method. This shows that an increase in the spatial resolution leads to a geometrically *closer* solution to the exact solution.

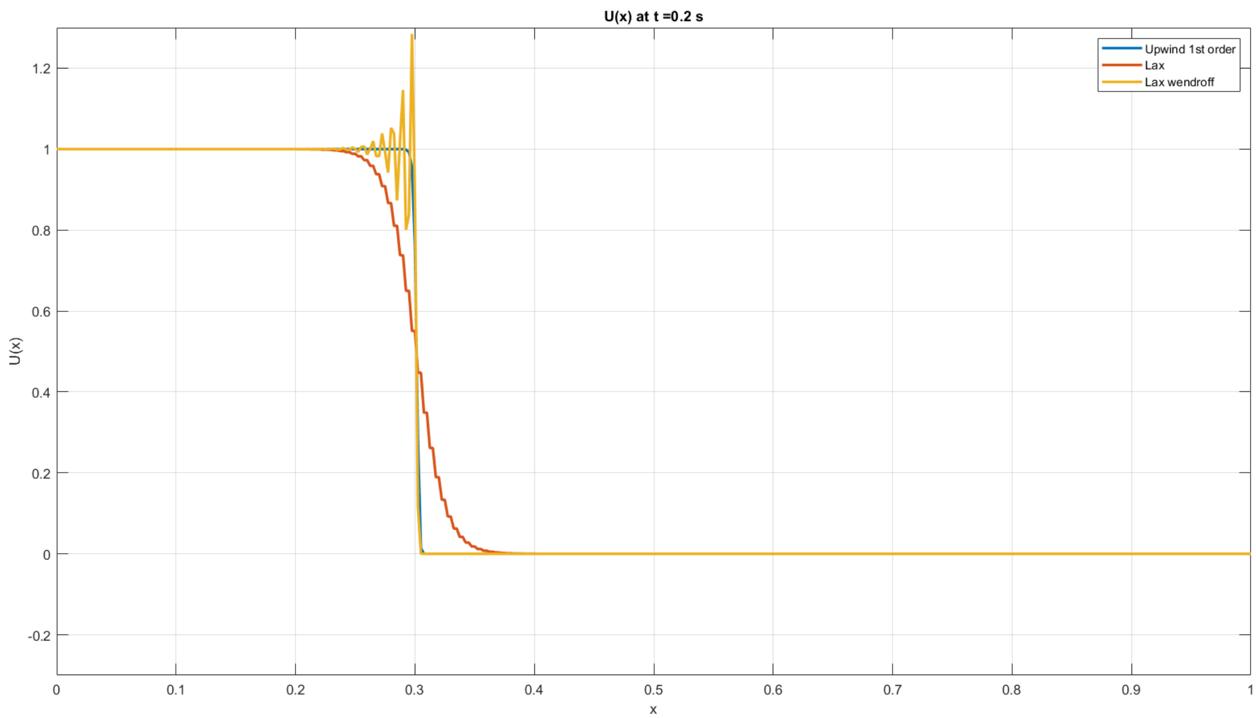


Fig. 4: $u(x,t)$ profiles for all 3 schemes for $\frac{\Delta t}{\Delta x} = 0.2$ and $\Delta x = 0.0025$

4. Conclusions

This project consisted of formulating and implementing a numerical code for a 1D PDE of the Burger's equation which is a non-linear hyperbolic PDE with prescribed initial and boundary conditions. The problem algorithm was developed using MATLAB with four different finite difference schemes namely Upwind 1st order, Central difference, Lax scheme and Lax – Wendroff scheme. To investigate the effects of the stability parameter, five different cases were considered by varying $\frac{\Delta t}{\Delta x}$.

It was noted that the Central difference scheme which was an explicit scheme was unconditionally unstable. The rest of the schemes are all conditionally stable with the stability criteria of $\frac{\Delta t}{\Delta x} < 1$. The upwind 1st order solves the exact equation for $\frac{\Delta t}{\Delta x} = 1$, whereas for the rest of the stability values, it induces dissipation in the solution and this effect is also noted for the Lax scheme. For the Lax – Wendroff method, the dispersive effects were prominent in the solution. For all three schemes, it was noted that a decrease in the stability parameter substantially deviates the solution from the exact solution. Finally, it was noted that an increase in the spatial resolution leads to a geometric compactness for the same stability parameter values.

It was thus possible to visualize the evolution of the wave using Burger's hyperbolic PDE using four different finite difference schemes out of which one was unconditionally unstable. Numerical dispersion and dissipation effects were noted in all the solutions and their variation with respect to the stability parameter was also studied.

The following MATLAB code was developed for this project:

```
clc
clear
%% Initialization
nx=201;
dx=1/(nx-1);
stability=0.5; % dt/dx
dt=stability*dx;
tmax=5;
nt=tmax/dt;
u=zeros(nx,nt+1); % For upwind
u1=zeros(nx,nt+1); % For central difference
u2=zeros(nx,nt+1); % For Lax
u3=zeros(nx,nt+1); % For Lax Wendroff
%% Initial condition - Discontinuity
u(1:1+0.2/dx,1:end)=1;
u(2+0.2/dx:end,1:end)=0;
u1(1:1+0.2/dx,1:end)=1;
u1(2+0.2/dx:end,1:end)=0;
u2(1:1+0.2/dx,1:end)=1;
u2(2+0.2/dx:end,1:end)=0;
u3(1:1+0.2/dx,1:end)=1;
u3(2+0.2/dx:end,1:end)=0;
%% FD schemes

    for ntt=2:1:nt+1
        for x=2:1:nx-1
            u(x,ntt)=u(x,ntt-1)-(dt/dx)*0.5*((u(x,ntt-1))^2-u(x-1,ntt-1)^2);
% Upwind 1st order

            u1(x,ntt)=u1(x,ntt-1)-(dt/dx)*0.25*((u1(x+1,ntt-1))^2-u1(x-1,ntt-
1)^2);  % Central difference

            u2(x,ntt)=0.5*(u2(x+1,ntt-1)+u2(x-1,ntt-1))-
(dt/dx)*0.25*((u2(x+1,ntt-1))^2-(u2(x-1,ntt-1))^2); % Lax
```

```
            u3(x,ntt)=u3(x,ntt-1)-((dt/dx)*0.25*((u3(x+1,ntt-1))^2-u3(x-
1,ntt-1)^2))...
                +0.125*((dt/dx)^2)*(((u3(x+1,ntt-1)+u3(x,ntt-
1))*((u3(x+1,ntt-1))^2-(u3(x,ntt-1))^2))  ...
                - ((u3(x,ntt-1)+u3(x-1,ntt-1))*((u3(x,ntt-1))^2-(u3(x-1,ntt-
1))^2))); % Lax Wendroff

        end
    end


%% POST PROCESSING

for ntt=1:0.2/dt:nt+1
    figure;set(gcf, 'Position', get(0, 'Screensize'));
    plot(0:dx:1,u(:,ntt),'LineWidth',2);grid on;hold on;ylim([-0.3 1.3]);
    % plot(0:dx:1,u1(:,ntt),'LineWidth',2); Central diffeerence, UNSTABLE

plot(0:dx:1,u2(:,ntt),'LineWidth',2);plot(0:dx:1,u3(:,ntt),'LineWidth',2);
    xlabel('x');ylabel('U(x)');
    title(['U(x) at t =',num2str((ntt-1)*dt),' s']);
    legend('Upwind 1st order','Lax','Lax wendroff');
    w = waitforbuttonpress;
    close
end

%% POST PROCESSING - CENTRAL DIFFERENCE, UNSTABLE
    figure;set(gcf, 'Position', get(0, 'Screensize'));
    plot(0:dx:1,u1(:,1),'LineWidth',2);grid on;hold on;
    plot(0:dx:1,u1(:,11),'LineWidth',2);
    plot(0:dx:1,u1(:,21),'LineWidth',2);
    xlabel('x');ylabel('U(x)');
    title(['U(x)']);
    legend('t = 0 s','t = 0.025 s','t = 0.05 s');
```