Computational methods for wave modeling

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Summary

Introduction

Things to consider when modeling:

- Governing differential equation(s)
- Initial conditions
- Boundary conditions (geometry of the domain)

Example: acoustic wave equation in one dimension

$u(x, t)$: wave field; x is the space variable, and t is the time variable

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

c here denotes the (phase) velocity of the wave; consider the plane wave solution $u(x, t) = e^{i(\xi x - \omega t)}$

- Initial condition: $u(x, 0) = e^{i\xi x},\ u_t(x, 0) = -i\omega e^{i\xi x}$
- Periodic boundary condition: $u(x, t) = u(x + L, t)$

Logic Based Methods

- Finite Difference (FD)- simple, robust, easy to implement
- Finite Volume (FV)- works well with discontinuities, and shock
- Finite Element (FE)- works well with complicated geometry

Example: FD method for 1D acoustic wave

$$\frac{\partial u}{\partial t} \approx \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\swarrow \qquad\qquad\qquad \searrow$$

$$\frac{u(x,t+\Delta t) - 2u(x,t) + u(x,t-\Delta t)}{(\Delta t)^2} = c^2 \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}$$

[2]

We implement this method here

Data Driven methods

Scientific Machine Learning methods [1]

- Linear regression
- Support vector machine
- Neural networks

Example: PINN for 2nd order ODE

Consider the Initial value problem

$$t^2y'' + 3ty' + 4y = t,\ y(1) = 0,\ y'(1) = 1,\ t \in [1, 3]$$

We set up a sequential neural network, and feed the neural network the two initial conditions. We ask the neural network to find the solution that minimizes the residual enforced by the differential equation:

$$r = t^2y'' + 3ty' + 4y - t$$

The example is implemented here

Reference I

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