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Numerical Simulation by FDM of Unsteady Heat Transfer in Cylindrical Coordinates

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Abstract. In this paper the heat transfer problem in transient and cylindrical coordinates will be solved by the Crank-Nicolson method in conjunction the Finite Difference Method. To validate the formulation will study the numerical efficiency by comparisons of numerical results compared with two exact solutions.

Introduction

The objective of this paper is to solve a heat transfer problem in transient and described in cylindrical coordinates in the following form [1-2],

$$\frac{\partial T}{\partial t} + V_r \frac{\partial T}{\partial r} + V_z \frac{\partial T}{\partial z} = \alpha \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] + \frac{\dot{q}}{\rho C_p} \tag{1}$$

where $T = T(z,r,t)$ is the temperature in K, r and z are the cylindrical coordinates in the direction of the radius and cylinder lenght, respectivily, t is the time (in s), V_r is the velocity in the r, V_z is the velocity in the z direction, ρ is the specific mass in kg/m^3, C_p is the specific heat in $kJ/kg.K$, $\dot{q}$ is the energy generation in K/s and α is the thermal difusivity in m^2/s.

To solve the Eq. (1) will be used methods of Crank-Nicolson for time discretization and the Difference Central (order 2) for spatial discretization. The efficiency of this formulation is analyzed at the last instant of time comparing the numerical solution with the exact solution by L_∞ norm.

Numerical Formulation

Initially adjust the Eq. (1) as follows,

$$\frac{\partial T}{\partial t} + V_r \frac{\partial T}{\partial r} + V_z \frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\dot{q}}{\rho C_p} \quad \Rightarrow$$

$$\frac{\partial T}{\partial t} + V_r \frac{\partial T}{\partial r} + V_z \frac{\partial T}{\partial z} = \alpha\frac{\partial^2 T}{\partial r^2} + \frac{\alpha}{r}\frac{\partial T}{\partial r} + \alpha\frac{\partial^2 T}{\partial z^2} + \frac{\dot{q}}{\rho C_p} \quad \Rightarrow$$

$$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial r^2} + \frac{\alpha}{r}\frac{\partial T}{\partial r} + \alpha\frac{\partial^2 T}{\partial z^2} - V_r \frac{\partial T}{\partial r} - V_z \frac{\partial T}{\partial z} + \frac{\dot{q}}{\rho C_p} \tag{2}$$

Then apply the Crank-Nicolson method to discretize in the time the Eq. (2),

$$\left(\frac{T^{n+1}-T^n}{\Delta t}\right) = \frac{1}{2}\left(\alpha\frac{\partial^2 T}{\partial r^2} + \frac{\alpha}{r}\frac{\partial T}{\partial r} + \alpha\frac{\partial^2 T}{\partial z^2} - V_r \frac{\partial T}{\partial r} - V_z \frac{\partial T}{\partial z} + \frac{\dot{q}}{\rho C_p}\right)^{n+1} + \frac{1}{2}\left(\alpha\frac{\partial^2 T}{\partial r^2} + \frac{\alpha}{r}\frac{\partial T}{\partial r} + \alpha\frac{\partial^2 T}{\partial z^2} - V_r \frac{\partial T}{\partial r} - V_z \frac{\partial T}{\partial z} + \frac{\dot{q}}{\rho C_p}\right)^{n} \quad \Rightarrow$$

$$\frac{T^{n+1}}{\Delta t} - \frac{1}{2}\left(\alpha\frac{\partial^2 T}{\partial r^2} + \frac{\alpha}{r}\frac{\partial T}{\partial r} + \alpha\frac{\partial^2 T}{\partial z^2} - V_r \frac{\partial T}{\partial r} - V_z \frac{\partial T}{\partial z} + \frac{\dot{q}}{\rho C_p}\right)^{n+1} = \frac{1}{2}\left(\alpha\frac{\partial^2 T}{\partial r^2} + \frac{\alpha}{r}\frac{\partial T}{\partial r} + \alpha\frac{\partial^2 T}{\partial z^2} - V_r \frac{\partial T}{\partial r} - V_z \frac{\partial T}{\partial z} + \frac{\dot{q}}{\rho C_p}\right)^{n} + \frac{T^n}{\Delta t} \quad \Rightarrow$$

$$\left(\frac{T}{\Delta t}-\frac{\alpha}{2}\frac{\partial^2 T}{\partial r^2}-\frac{\alpha}{2r}\frac{\partial T}{\partial r}-\frac{\alpha}{2}\frac{\partial^2 T}{\partial z^2}+\frac{V_r}{2}\frac{\partial T}{\partial r}+\frac{V_z}{2}\frac{\partial T}{\partial z}\right)^{n+1}-\left(\frac{\dot{q}}{2\rho C_p}\right)^{n+1}=\left(\frac{\alpha}{2}\frac{\partial^2 T}{\partial r^2}+\frac{\alpha}{2r}\frac{\partial T}{\partial r}+\frac{\alpha}{2}\frac{\partial^2 T}{\partial z^2}-\frac{V_r}{2}\frac{\partial T}{\partial r}-\frac{V_z}{2}\frac{\partial T}{\partial z}+\frac{\dot{q}}{2\rho C_p}+\frac{T}{\Delta t}\right)^{n}\quad\Rightarrow$$

$$\left(\frac{T}{\Delta t}-\frac{\alpha}{2}\frac{\partial^2 T}{\partial r^2}-\frac{\alpha}{2r}\frac{\partial T}{\partial r}-\frac{\alpha}{2}\frac{\partial^2 T}{\partial z^2}+\frac{V_r}{2}\frac{\partial T}{\partial r}+\frac{V_z}{2}\frac{\partial T}{\partial z}\right)^{n+1}=\boldsymbol{A} \tag{3}$$

where A is defined by the expression

$$\boldsymbol{A}=\left(\frac{\alpha}{2}\frac{\partial^2 T}{\partial r^2}+\frac{\alpha}{2r}\frac{\partial T}{\partial r}+\frac{\alpha}{2}\frac{\partial^2 T}{\partial z^2}-\frac{V_r}{2}\frac{\partial T}{\partial r}-\frac{V_z}{2}\frac{\partial T}{\partial z}+\frac{\dot{q}}{2\rho C_p}+\frac{T}{\Delta t}\right)^{n}+\left(\frac{\dot{q}}{2\rho C_p}\right)^{n+1}$$

To discretize the partial derivatives of the first order will be used the following central finite differences of second order [3-6],

$$\frac{\partial T}{\partial z}=\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta z} \tag{4}$$

$$\frac{\partial T}{\partial r}=\frac{T_{i,j+1}-T_{i,j-1}}{2\Delta r} \tag{5}$$

Now, to discretize the partial derivatives of the second order will be used the following central finite differences of second order [3-6],

$$\frac{\partial^2 T}{\partial z^2}=\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta z^2} \tag{6}$$

$$\frac{\partial^2 T}{\partial r^2}=\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta r^2} \tag{7}$$

Thus, substituting Eqs. (4-7) in Eq. (3) it has,

$$\left[\frac{T_{i,j}}{\Delta t}-\frac{\alpha}{2}\left(\frac{T_{i,j+1}-2T_{i,j}+T_{i,j-1}}{\Delta r^2}\right)-\frac{\alpha}{2r}\left(\frac{T_{i,j+1}-T_{i,j-1}}{2\Delta r}\right)-\frac{\alpha}{2}\left(\frac{T_{i+1,j}-2T_{i,j}+T_{i-1,j}}{\Delta z^2}\right)+\frac{V_r}{2}\left(\frac{T_{i,j+1}-T_{i,j-1}}{2\Delta r}\right)+\frac{V_z}{2}\left(\frac{T_{i+1,j}-T_{i-1,j}}{2\Delta z}\right)\right]^{n+1}=\boldsymbol{A}\quad\Rightarrow$$

$$\left[\left(-\frac{\alpha}{2\Delta r^2}+\frac{\alpha}{4r\Delta r}-\frac{V_r}{4\Delta r}\right)\boldsymbol{T}_{i,j-1}+\left(-\frac{\alpha}{2\Delta z^2}-\frac{V_z}{4\Delta z}\right)\boldsymbol{T}_{i-1,j}+\left(\frac{1}{\Delta t}+\frac{\alpha}{\Delta r^2}+\frac{\alpha}{\Delta z^2}\right)\boldsymbol{T}_{i,j}+\left(-\frac{\alpha}{2\Delta z^2}+\frac{V_z}{4\Delta z}\right)\boldsymbol{T}_{i+1,j}+\left(-\frac{\alpha}{2\Delta r^2}-\frac{\alpha}{4r\Delta r}+\frac{V_r}{4\Delta r}\right)\boldsymbol{T}_{i,j+1}\right]^{n+1}=\boldsymbol{A} \tag{8}$$

Numerical Applications

Through the Eq. (8) a linear system is generated which is solved by the method Gauss-Seidel built in Fortran language, and thus calculated the temperature values in all nodes of computational mesh built in the predetermined domain.

Application 1. For this first application is considered $\alpha=\rho=C_p=1$, $V_z=V_r=1$ and an exact solution proposed is given in the form, $T(z,r,t)=e^{z+r+t}$, being that,

$$q=\rho C_p e^{r+z+t}\left(1+V_r+V_z-\frac{\alpha}{r}-2\alpha\right)$$

By varying the values of $h=\Delta z=\Delta r$ and Δt will study the influence of temporal and spatial refinements by comparisons of results of the numerical solution and the exact solution as shown in Table 1.

Table 1. L_∞ norm for the application 1.

Δt	**h = Δz = Δr**				
	1/10	**1/20**	**1/30**	**1/40**	**1/50**
1/10	2.14E-03	9.21E-04	6.84E-04	6.08E-04	5.75E-04
1/20	1.85E-03	5.97E-04	3.38E-04	2.43E-04	1.98E-04
1/30	1.79E-03	5.44E-04	2.80E-04	1.83E-04	1.36E-04
1/40	1.78E-03	5.25E-04	2.61E-04	1.63E-04	1.15E-04
1/50	1.77E-03	5.17E-04	2.52E-04	1.53E-04	1.06E-04

It is noted from Table 1 that for a fixed Δt and varying h or for a fixed h and varying the Δt value the numerical accuracy of the numerical formulation is always improving.

Application 2. For this application α, ρ and C_p values is equal the 1, $V_z = V_r = 1$ and exact solution proposed is $T(z,r,t) = \sin(2(r+z+t))$ with,

$$q = \rho C_p \left[2\cos(2(r+z+t))\left(1 + V_r + V_z - \frac{\alpha}{r}\right) + 8\alpha sen(2(r+z+t))\right]$$

It is noted in this application (see Table 2), which uses an exact solution from a sine function that the numeric results achieve a precision order more than in the previous application where exact solution proposed makes use of an exponential function. It is also important to note that with the refinement both in space and in time continuously improve the numerical accuracy.

Table 2. L_∞ norm for the application 2.

Δt	**Δz = Δr**				
	1/10	**1/20**	**1/30**	**1/40**	**1/50**
1/10	1.35E-03	6.69E-04	5.31E-04	4.86E-04	4.68E-04
1/20	1.15E-03	4.23E-04	2.53E-04	1.88E-04	1.58E-04
1/30	1.12E-03	3.80E-04	2.06E-04	1.39E-04	1.06E-04
1/40	1.11E-03	3.65E-04	1.89E-04	1.23E-04	8.94E-05
1/50	1.10E-03	3.58E-04	1.83E-04	1.16E-04	8.18E-05

Conclusion

The formulation proposed in this study proved to be easy to implement and excellent numerical accuracy. Even in the case of a thermal problem evaluated in cylindrical coordinates, the expression constructed and presented in Eq. (2) secured the already expected efficiency of the Central Difference Method.

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