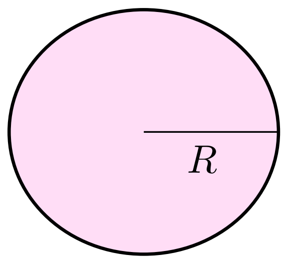
Dirichlet & Heat Problems in Polar Coordinates

Section 13.1

1 Steady State Temperature in a circular Plate

Consider the problem

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \quad x^2 + y^2 < R^2 \tag{1.1}$$
$$u(x,y) = f(x,y), \quad x^2 + y^2 = R^2,$$
$$u(x,y) \text{ bounded on } x^2 + y^2 \le R^2. \tag{1.2}$$



It turns out that we cannot solve this problem using separation of variables as it is written. But, as you will see, if we change coordinates to polar coordinates then separation of variables works fine.

To this end recall that polar coordinates are given by

$$x = r\cos(\theta), \quad y = r\sin(\theta)$$

or

$$r^2 = x^2 + y^2, \quad \theta = \tan^{-1}(y/x).$$

So we need to translate

$$u_{xx}(x,y) + u_{yy}(x,y) = 0$$

into the variables r and θ. First we use $r^2 = x^2 + y^2$ and implicit differentiation to compute

$$2rr_x = 2x \ \Rightarrow \ r_x = \frac{x}{r}, \quad 2rr_y = 2y \ \Rightarrow \ r_y = \frac{y}{r}$$

so we have

$$r_x = \frac{x}{r} = \cos(\theta) \ \text{ and } \ r_y = \frac{y}{r} = \sin(\theta). \tag{1.3}$$

Similarly, differentiating $y = r\sin(\theta)$ with respect to x and using (1.3) we have

$$0 = r_x\sin(\theta) + r\cos(\theta)\theta_x \tag{1.4}$$
$$= \cos(\theta)\sin(\theta) + r\cos(\theta)\theta_x \tag{1.5}$$

which implies

$$\theta_x = -\frac{\sin(\theta)}{r}. \tag{1.6}$$

Differentiating $x = r\cos(\theta)$ with respect to y and using (1.3) we have

$$\begin{aligned} 0 &= r_y \cos(\theta) - r\sin(\theta)\theta_y && (1.7) \\ &= \cos(\theta)\sin(\theta) - r\sin(\theta)\theta_y && (1.8) \end{aligned}$$

and we obtain

$$\theta_y = \frac{\cos(\theta)}{r}. \tag{1.9}$$

Next, using the chain rule and (1.3) we compute

$$u_x = u_r r_x + u_\theta \theta_x = u_r \cos(\theta) - u_\theta \frac{\sin(\theta)}{r} \tag{1.10}$$

and

$$u_y = u_r r_y + u_\theta \theta_y u_r \sin(\theta) + u_\theta \frac{\cos(\theta)}{r}. \tag{1.11}$$

Using the formulas (1.10) and (1.11) we now compute the second derivatives:

$$\begin{aligned} u_{xx} &= (u_{rr} r_x + u_{r\theta}\theta_x)\cos(\theta) - (u_{\theta r} r_x + u_{\theta\theta}\theta_x)\,\frac{\sin(\theta)}{r} \\ &\quad + u_r(\cos(\theta))_x - u_\theta \left(\frac{\sin(\theta)}{r}\right)_x \\ &= \left(u_{rr}\cos(\theta) + u_{r\theta}\left(-\frac{\sin(\theta)}{r}\right)\right)\cos(\theta) \\ &\quad - \left(u_{\theta r}\cos(\theta) + u_{\theta\theta}\left(-\frac{\sin(\theta)}{r}\right)\right)\frac{\sin(\theta)}{r} \\ &\quad + u_r(-\sin(\theta))\theta_x - u_\theta\left(\frac{(r\cos(\theta)\theta_x - \sin(\theta) r_x)}{r^2}\right) \\ &= u_{rr}\cos^2(\theta) - 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\sin^2(\theta)}{r^2}\right) \\ &\quad + u_r\frac{\sin^2(\theta)}{r} - u_\theta\left(\frac{-\cos(\theta)\sin(\theta) - \sin(\theta)\cos(\theta)}{r^2}\right). \end{aligned}$$

Finally we arrive at

$$\begin{aligned} u_{xx} &= u_{rr}\cos^2(\theta) - 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\sin^2(\theta)}{r^2}\right) \\ &\quad + u_r\frac{\sin^2(\theta)}{r} + u_\theta\left(\frac{(2\sin(\theta)\cos(\theta))}{r^2}\right) \end{aligned} \tag{1.12}$$

Exactly the same type of calculation which begins with

$$u_y = u_r r_y + u_\theta \theta_y = u_r \sin(\theta) + u_\theta\left(\frac{\cos(\theta)}{r}\right)$$

leads to

$$\begin{aligned} u_{yy} &= u_{rr}\sin^2(\theta) + 2u_{r\theta}\left(\frac{\sin(\theta)\cos(\theta)}{r}\right) + u_{\theta\theta}\left(\frac{\cos^2(\theta)}{r^2}\right) \\ &+ u_r\frac{\cos^2(\theta)}{r} - u_\theta\left(\frac{(2\sin(\theta)\cos(\theta))}{r^2}\right) \end{aligned} \tag{1.13}$$

Now adding (1.12) and (1.13) leads to

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}. \tag{1.14}$$

With this we can rewrite the problem (1.1) as

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < R,\ 0 \le \theta \le 2\pi \tag{1.15}$$

$$u(R,\theta) = f(\theta), \quad 0 \le \theta \le 2\pi$$

$$u(r,\theta) \text{ bounded.} \tag{1.16}$$

Remark 1.1. We could just as easily have considered the following problem.

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 < r < R,\ -\pi \le \theta \le \pi \tag{1.17}$$

$$u(R,\theta) = g(\theta), \quad -\pi \le \theta \le \pi$$

$$u(r,\theta) \text{ bounded.} \tag{1.18}$$

In other words we could write the problem on any 2π interval. The only difference is that the function $f(x)$ and $g(x)$ would be different since we are assuming that the given function is 2π periodic. So if we intend to use the interval $-\pi < \theta < \pi$ instead of $0 < \theta < 2\pi$ and we intend to use the same periodic function then we need to make some adjustments. For example, if

$$f(\theta) = \begin{cases} f_1(\theta) & ,0 < \theta < \pi \\ f_2(\theta) & ,\pi < \theta < 2\pi \end{cases} \quad \Rightarrow \quad g(\theta) = \begin{cases} f_2(\theta - 2\pi) & ,-\pi < \theta < 0 \\ f_1(\theta) & ,0 < \theta < \pi \end{cases}.$$

As a specific example consider

$$f(\theta) = \begin{cases} \theta & ,0 < \theta < \pi \\ 2\pi - \theta & ,\pi < \theta < 2\pi \end{cases} \quad \Rightarrow \quad g(\theta) = \begin{cases} -\theta & ,-\pi < \theta < 0 \\ \theta & ,\pi < \theta < 2\pi \end{cases}.$$

The advantage of the interval $[-\pi, \pi]$ is that in the integrations you are integrating over a symmetric interval about zero, so you can use even and odd to your advantage.

Separation of variables proceeds as follows. We seek simple solutions to (1.15) in the form

$$u = \varphi(\theta)\psi(r).$$

Substituting this into the equation in (1.15) we have

$$(\varphi(\theta)\psi(r))_{rr} + \frac{1}{r}\left(\varphi(\theta)\psi(r)\right)_r + \frac{1}{r^2}\left(\varphi(\theta)\psi(r)\right)_{\theta\theta} = 0$$

$$\varphi(\theta)\psi''(r) + \frac{1}{r}\varphi(\theta)\psi'(r) + \frac{1}{r^2}\varphi''(\theta)\psi(r) = 0.$$

Next we divide by $\varphi(\theta)\psi(r)$ and multiply by r^2 to obtain

$$\frac{r^2(\psi'' + (1/r)\psi')}{\psi} = -\frac{\varphi''}{\varphi}$$

which as usual in separation of variables must equal a constant λ.

Recall the solutions for so-called Euler-Cauchy equations:

$$r^2\psi'' + ar\psi' + b\psi = 0$$

Consider the change of variables $s = \ln(r)$ or $r = e^s$. By the chain rule

$$\frac{d\psi}{dr} = \frac{d\psi}{ds}\frac{ds}{dr} = \frac{1}{r}\frac{d\psi}{ds}$$

and

$$\frac{d^2\psi}{dr^2} = \frac{d}{ds}\left(\frac{d\psi}{dr}\frac{1}{r}\right) = \frac{1}{r^2}\frac{d\psi^2}{ds^2} - \frac{1}{r^2}\frac{dR}{ds}$$

So the equation becomes

$$r^2\left(\frac{1}{r^2}\frac{d\psi^2}{ds^2} - \frac{1}{r^2}\frac{d\psi}{ds}\right) + ar\frac{1}{r}\frac{d\psi}{ds} + b\psi = 0$$

which simplifies to

$$\frac{d\psi^2}{ds^2} + (a-1)\frac{d\psi}{ds} + b\psi = 0.$$

This is a constant coefficient equation and we recall from ODEs that there are three possibilities for the solutions depending on the roots of the characteristic equation. In the present case we have $a = 1$ and $b = \lambda$.

Thus we obtain the pair of BVPs for ODEs:

$$\varphi''(\theta) + \lambda\varphi(\theta) = 0, \quad \varphi(-\pi) = \varphi(\pi), \quad \varphi'(-\pi) = \varphi'(\pi)$$

For $\lambda = 0$ we have $\varphi(\theta) = 1$. For nonzero λ we have $\lambda = \mu^2$ and the general solution is $\varphi(\theta) = A\cos(\mu\theta) + B\sin(\mu\theta)$.

The first boundary condition implies $B\sin(\mu\pi) = B\sin(-\mu\pi)$ or $\sin(\mu\pi) = 0$. This implies $\mu = n$ an integer. The second boundary condition implies $-\mu A\sin(\mu\pi) = -\mu A\sin(-\mu\pi)$ which again gives $\mu = n$ so that $\lambda = n^2$ and both A and B are arbitrary. Thus we have $\lambda = n^2$ and

$$\varphi(\theta) = A_m\cos(n\theta) + B_m\sin(n\theta)$$

With $\lambda = n^2$ with $n = 0, 1, 2, \cdots$ as above

$$r^2\psi''(r) + r\psi'(r) - n^2\psi(r) = 0$$

for which we seek solutions in the form $\psi(r) = r^c$. When $n = 0$ we get

$$r^2\psi''(r) + r\psi'(r) = 0$$

which is an Euler-Cauchy problem with general solution $\psi(r) = a + b\ln(r)$.

But in order for the solution to be bounded we need $b = 0$ so ψ is an arbitrary constant, say $\psi = 1$.

For $m \neq 0$ we have again an Euler-Cauchy problem with general solution

$$\psi = ar^n + br^{-n}.$$

Once again, for ψ bounded we need $b = 0$, so we take

$$\psi = r^n.$$

Combining these results we seek our general solution in the form

$$u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] \tag{1.19}$$

In the case of (1.15) where the boundary function is given on $[0, 2\pi]$

$$f(\theta) = u(R, \theta) = a_0 + \sum_{n=1}^{\infty} R^m \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

This is a general Fourier series and we have

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi R^m} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta$$

$$b_n = \frac{1}{\pi R^m} \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta$$

In the case of (1.17) where the boundary function is given on $[-\pi, \pi]$ (see Remark 1.1)

$$g(\theta) = u(R, \theta) = a_0 + \sum_{n=1}^{\infty} R^m \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

This is a general Fourier series and we have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi R^m} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta)\, d\theta$$
$$b_n = \frac{1}{\pi R^m} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta)\, d\theta$$

Example 1.1.

$$\begin{aligned} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0, \quad 0 < r < 1,\ -\pi \le \theta \le \pi \\ u(1,\theta) &= \cos^2(\theta), \\ u(r,\theta) &\text{ bounded.} \end{aligned}$$

Using the trig identity

$$\cos^2(\theta) = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$$

and our usual orthogonality conditions the solution is given by

$$u(r,\theta) = \frac{1}{2} + \frac{1}{2} r^2 \cos(2\theta)$$

Example 1.2.

$$\begin{aligned} u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0, \quad 0 < r < 1,\ -\pi \le \theta \le \pi \\ u(1,\theta) &= \sin(3\theta), \\ u(r,\theta) &\text{ bounded.} \end{aligned}$$

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]$$

Now we need

$$\sin(3\theta) = u(1,\theta) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos(n\theta) + b_n \sin(n\theta)\right].$$

But we can argue (using our knowledge of orthogonality) that the solution is given by

$$u(r,\theta) = r^3 \sin(3\theta).$$

Notice that this solution can be transformed back into rectangular coordinates but it would be a mess.

Example 1.3 (Integral Formula for Dirichlet Problem in a Disk)**.** We recall that the Dirichlet problem for for circular disk can be written in polar coordinates with $0 \le r \le R$, $-\pi \le \theta \le \pi$ as

$$\begin{aligned} \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} &= 0 \\ u(R,\theta) &= f(\theta). \end{aligned}$$

As we have seen, we can obtain the solution u in the form

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty} (r)^n \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right).$$

We need

$$f(\theta) = u(R,\theta) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\theta) + b_n \sin(n\theta)\right),$$

and

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\cos(n\theta)\, d\theta,$$

$$b_n = \frac{1}{\pi R^n}\int_0^{2\pi} f(\theta)\sin(n\theta)\, d\theta.$$

Lemma 1.1. *The series solution can thus be written as*

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left(R^2 - r^2\right) f(\alpha)}{R^2 + r^2 - 2rR\cos(\theta - \alpha)}\, d\alpha.$$

Proof. To prove this result we need only insert the formulas for a_n and b_n into the infinite sum representation for the solution, interchange the sum and integral and sum a resulting geometric series:

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, d\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \int_0^{2\pi} f(\alpha)\left[\cos(n\alpha)\cos(n\theta) + \sin(n\alpha)\sin(n\theta)\right] d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\, d\alpha + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \int_0^{2\pi} f(\alpha)\, \cos[n(\theta-\alpha)]\, d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \cos[n(\theta-\alpha)]\right] d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\left[1 + \sum_{n=1}^{\infty}\left(\frac{r}{R}\right)^n \left\{e^{in(\theta-\alpha)} + e^{-in(\theta-\alpha)}\right\}\right] d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\left\{1 + \frac{re^{i(\theta-\alpha)}}{R - re^{i(\theta-\alpha)}} + \frac{re^{-i(\theta-\alpha)}}{R - re^{-i(\theta-\alpha)}}\right\} d\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)}{(R^2 + r^2 - 2rR\cos(\theta-\alpha))} f(\alpha)\, d\alpha.
\end{aligned}$$

$\square$

2 Heat Equation in a Disk

Next we consider the corresponding heat equation in a two dimensional wedge of a circular plate. So we write the heat equation with the Laplace operator in polar coordinates.

Example 2.1.

$$u_t = k\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}\right), \quad 0 < r < 1,\ 0 \le \theta \le \pi/2 \tag{2.1}$$

$$\begin{aligned}
&u(1,\theta,t) = 0, \quad 0 \le \theta \le \pi/2 \\
&u(r,0,t) = 0, \quad u_\theta(r,\pi/2,t) = 0, \quad 0 < r < 1, \\
&u(r,\theta,t) \text{ bounded}, \\
&u(r,\theta,0) = f(r,\theta) = (r - r^3)\sin(\theta).
\end{aligned}$$

Separation of variables proceeds as follows. We seek simple solutions to (2.1) in the form

$$u = T(t)\Theta(\theta)R(r).$$

Substituting this into the equation in (2.1) we have

$$\frac{T'(t)}{kT(t)} = \frac{\left(\Theta(\theta)R''(r) + \frac{1}{r}\Theta(\theta)R'(r) + \frac{1}{r^2}\Theta''(\theta)R(r)\right)}{\Theta(\theta)R(r)} = -\lambda^2.$$

But from the right hand side we can further see that there must be another constant which we denote by μ^2

$$\frac{r^2\left(R''(r) + \frac{1}{r}R'(r) + \lambda^2 R(r)\right)}{R(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = \mu^2.$$

This leads to the following equations

$$\begin{gathered}
T'(t) + k\lambda^2 T(t) = 0 \\
r^2 R'' + rR' + (\lambda^2 r^2 - \mu^2)R = 0 \\
\Theta'' + \mu^2\Theta = 0
\end{gathered}$$

with boundary conditions

$$\begin{gathered}
|R(0)| < \infty, \quad R(1) = 0 \\
\Theta(0) = 0, \quad \Theta'(\pi/2) = 0.
\end{gathered}$$

First we consider the equation for Θ:

$$\Theta'' + \mu^2\Theta = 0, \quad \Theta(0) = 0, \quad \Theta'(\pi/2) = 0$$

which gives

$$\mu_m = (2m-1), \quad T_m(t) = \Theta_m(\theta) = \frac{2}{\sqrt{\pi}}\sin((2m-1)\theta).$$

The eigenfunction $\Theta_m(\theta)$ satisfy the orthogonality relations

$$\int_0^{\pi/2} \Theta_m(\theta)\Theta_k(\theta)\, d\theta = \delta_{m,k}.$$

Then for each m we have

$$r^2R'' + rR' + (\lambda^2 r^2 - (2m-1)^2)R = 0, \quad |R(0)| < \infty, \quad R(1) = 0. \tag{2.2}$$

For each m the differential equation in (2.2) is a Bessel equation which has a regular singular point at $r = 0$. The solutions are called Bessel functions of the first kind of order μ. The problem (2.2) (i.e., the equation and boundary conditions) is a singular Sturm-Liouville problem and the associated eigenvalues and eigenfunctions

$$\lambda_{m,n}, \quad R_{m,n}, \quad m,n = 0,1,2,\cdots.$$

The eigenfunctions are orthogonal with the integral

$$\int_0^1 R_{m,n}(r)R_{m,p}(r)\, r\, dr = \delta_{n,p}.$$

We will use standard notation for the normalized eigenfunctions

$$R_{m,n}(r) = \kappa_{m,n}\mathrm{BesselJ}((2m-1), \lambda_{m,n}r)$$

where

$$\kappa_{m,n} = \left(\int \mathrm{BesselJ}^2((2m-1), \lambda_{m,n}r)\, r\, dr\right)^{-1}.$$

The eigenvalues λ are the zeros of the Bessel function obtained from the boundary condition $R(1) = 0$ by

$$\mathrm{BesselJ}((2m-1), \lambda_{m,n}) = 0. \tag{2.3}$$

It is well known that for each m this equation has infinitely many solutions $\lambda_{m,n}$ which tend to infinity with n.

The solution is given by

$$u(r,\theta,t) = \frac{2}{\sqrt{\pi}}\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\kappa_{m,n}C_{m,n}e^{-k\lambda_{m,n}^2 t}\mathrm{BesselJ}((2m-1), \lambda_{m,n}r)\sin((2m-1)\theta)\right).$$

For this example the special form of the initial condition $f(r,\theta) = (r - r^3)\sin(\theta)$ together with the orthogonality relations in θ show that the solution reduces to

$$u(r,\theta,t) = \frac{2}{\sqrt{\pi}}\left(\sum_{n=1}^{\infty}\kappa_{1,n}C_{1,n}e^{-k\lambda_{1,n}^2 t}\mathrm{BesselJ}(1, \lambda_{1,n}r)\right)\sin(\theta).$$

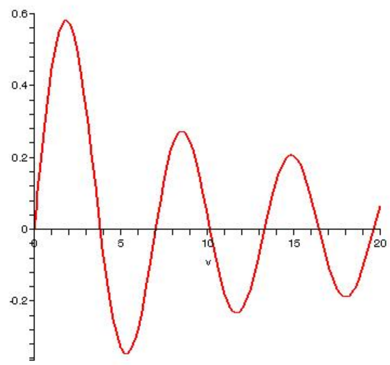
Also, in this case we can express $\kappa_{1,n}$ by

$$\kappa_{1,n} = \frac{\sqrt{2}}{\text{BesselJ}(0, \lambda_{1,n})}.$$

Thus we only need the eigenvalues $\lambda_{1,n}$ for $n = 1, 2, \cdots$. We cannot solve for this values exacly but we can solve numerically and, for example, we find

$$\lambda_{1,1} = 3.831705970, \quad \lambda_{1,2} = 7.015586670, \quad \lambda_{1,2} = 10.17346814$$

and a plot of $y = \text{BesselJ}(1, \xi)$ for ξ from 0 to 20 is given below



We obtain the corresponding Fourier coefficients $C_{1,n}$ from

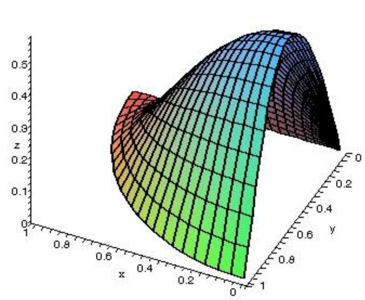
$$C_{1,n} = \frac{\kappa_{1,n} 2}{\sqrt{\pi}} \int_0^{\pi/2} \int_0^1 (r - r^3)\text{BesselJ}(1, \lambda_{1,n} r)\, dr\, d\theta$$

from which, using Maple, we obtain

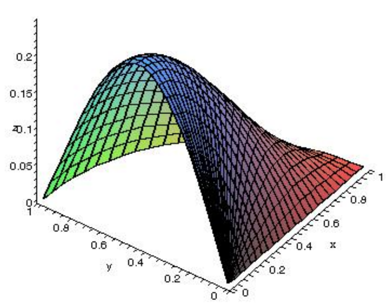
$$C_{1,n} = \frac{-16\sqrt{2}}{\lambda_{1,n}^3 \sqrt{\pi}}$$

and

$$u(r, \theta, t) = \frac{-64}{\pi} \sum_{n=1}^{\infty} \left(\frac{e^{-k\lambda_{1,n}^2 t}\text{BesselJ}(1, \lambda_{1,n} r)\sin(\theta)}{\lambda_{1,n}^3 \text{BesselJ}(0, \lambda_{1,n})} \right).$$



$u(r, \theta, 0)$



$u(r, \theta, 3)$