Tree Power Notes

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ABSTRACT.

Key words:

Introduction

Thermoelectric devices has been used in several projects [1], add Orlando's ref. Such devices convert energy from temperature difference to electricity, and power up small hardwares. In Washington state, we harvest energy from tree trunks, and power up devices that collect information of pollution in the nearby area.

In this work, we consider the numerical modeling of temperature distribution in tree trunks in the Washington area. Our result should provide reference for polution device design.

Literature review:

1 Assumptions

1. We assume all trees we consider are roughly the same. That means same height, same radius
2. With the previous assumption, we place all devices at the same height (therefore eliminating the z variable in the heat equation), and same depth

2 To do (Summer)

1. run experiments in Python3, see similar results from Potter and Andersen.
   - study source terms
   - improve 2D heat code with CN-method, and ADT method.
2. find appropriate parameters for testing (ask hardware group)
3. ways to validate computation (numerical analysis methods)
4. fill in documentation with theory, data, numertical methodology, and numerical experiments.

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3 PDE model of temperature within a tree trunk

Heat transfers witinin the tree trunk vertically, radially, and might differ at different azimuth angle. To fully model the temperature distribution within a tree trunk, we need to study the heat equation in three dimensions.

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \text{ source term} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{k}{r}\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \text{ source term} \quad \text{heat3d} \quad (1)$$

However, for our application, we can greatly simplify the equation with reasonable assumptions. As our devices would be installed in forests with trees of similar age and growth, it is reasonable to assume that we install all devices at the same height- this eliminates the dependence on the height (z- variable). We therefore obtain the following standard heat equation for temperature distribution in two dimensional cylindrical coordinates:

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + \text{ source term} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \phi}\left(\frac{k}{r}\frac{\partial T}{\partial \phi}\right) + \text{ source term} \quad \text{heat2d} \quad (2)$$

We add details of source terms for the diffusion equation to obtain

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) - \frac{1}{\Delta r}[H + (1-\alpha)(S_{dir} + S_{dif}) + (IR_{in} - IR_{out})] \quad (3)$$

In the following, we explain the source terms:

1. Free convective heat loss/gain happens when the tree surface temp is different from the ambient temp. Forced convection happens when there is wind. We denote the heat loss/gain by H, and
$$H = h(T_{sfc} - T_{air}), \quad (4)$$
with h (W/(m^2K)) being the convective heat transfer coefficient. Here
$$h = h_{free} - h_{forced}, \quad (5)$$
more details in 3.

2. $S_{dir} + S_{dif}$ represents direct solar radiation, plus diffusion solar radiation.

3. $IR_{in} - IR_{out}$ represents wave radiation from and to the tree. More details see [3].

We modify above source terms to fit the parameters for trees in Washington state.

3.1 Parameters to collect and verify. Hardware group

Notations:

1. T: temperature (K)

2. ρ: density (kg/m^3)

3. c: specific heat (J/(kgK))

4. t: time (s)

5. k: thermal conductivity (W/(mK))
6. r: distance from center of the tree (m)
7. ϕ: azimuth angle, measured clockwise with south being $\phi = 0$ (radian/degree), [3] assumes $\partial k/\partial\phi = 0$
8. α: albedo of the surface

3.2 Furthur simplification to one dimensional equation

Although we will be working mostly with the above two dimensional heat equation, [3] provides justification for further reducing the above problem to one dimensional case. As seen in Fig. 2 of [3], biggest temperature difference occurs in south and west asepcts. In our hardware application, we seek to install our devices in the aspects with the highest temperature differences. Based on the location of Washington state and data from hardware group, we install all devices in the south of the tree trunks. This furthur reduces the heat equation to solve to a one dimensional problem

$$\rho c \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \text{ source term} = k\frac{1}{r}\frac{\partial T}{\partial r} + k\frac{\partial^2 T}{\partial r^2} + \text{ source term.} \tag{6}$$

heat1d

In the following, we discuss the numerical simulation of both two and one dimensional heat equations with several numerical schemes.

4 Finite Difference schemes for the heat equation

We discuss Finite Difference (FD) schemes for both two and one dimensional heat equation, as the FD method is simple to implement, and robust.

4.1 FD schemes for the two dimensional heat equation

A detailed discussion on axi-symmetric heat equation is in [7]. We model the heat equation to depend on the azimuth angle ϕ, as trees in Washington state experience different sunlight and therefore growth in different direction. pg.251, section 3.5.6

For simplicity of discussion, we assume $\rho,\ c, k$ are all constants. This is a reasonable assumption, as the trees should not have parameters that change drastically within the trunk. Then the constant coefficient equation (2) simplifies to

$$\frac{\rho c}{k}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2} \tag{7}$$

simple_heat

With centered FD discretization in space, and forward in time for example, we have

- $\dfrac{\partial T}{\partial t} \approx \dfrac{T^1_{i,j} - T^0_{i,j}}{\Delta t}$
- $\dfrac{\partial T}{\partial r} \approx \dfrac{T^0_{i+1,j} - T^0_{i-1,j}}{2\Delta r}$
- $\dfrac{\partial^2 T}{\partial r^2} \approx \dfrac{T^0_{i-1,j} + T^0_{i+1,j} - 2T^0_{i,j}}{(\Delta r)^2}$
- $\dfrac{\partial^2 T}{\partial \phi^2} \approx \dfrac{T^0_{i,j-1} + T^0_{i,j+1} - 2T^0_{i,j}}{(\Delta \phi)^2}$

Here $T^n_{i,j} \approx T(r_i, \phi_j, t_n)$ denotes the numerical solution at the point (r_i, ϕ_j, t_n). In general, we compute the heat equation iteratively in time with the general time forwarding scheme:

$$\frac{T^1_{i,j} - T^0_{i,j}}{\Delta t} = D\ T_{i,j}, \tag{8}$$

where the right hand side is a discretization in space, with D being the difference operator. Depending on D, the above *continuous differential equation* can be discretized in the following two ways:

4.1.1 Forward in time, centered difference Method (FTCS)

The FTCS is the most straightforward method, where we directly apply the above listed discretization to the heat equation.

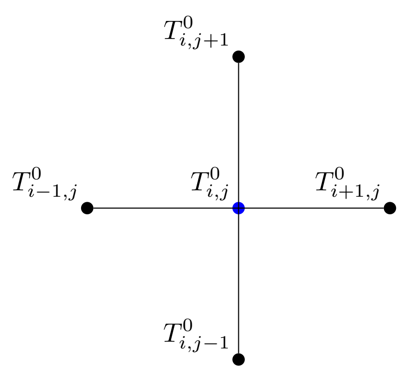
$$\begin{aligned} T^1_{i,j} =& \frac{k\Delta t}{\rho c}\Bigg[\underbrace{\frac{1}{(\Delta r)^2}\left(T^0_{i-1,j} + T^0_{i+1,j} - 2T^0_{i,j}\right)}_{\frac{\partial^2 T}{\partial r^2}} + \underbrace{\frac{1}{2r_i\Delta r}\left(T^0_{i+1,j} - T^0_{i-1,j}\right)}_{\frac{1}{r}\frac{\partial T}{\partial r}} \\ &+ \underbrace{\frac{1}{r_i^2(\Delta\phi)^2}\left(T^0_{i,j-1} + T^0_{i,j+1} - 2T^0_{i,j}\right)}_{\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}}\Bigg] + T^0_{i,j}, \end{aligned} \tag{9}$$

with $k_1 = \dfrac{k\Delta t}{\rho c(\Delta r)^2}$, $k_2 = \dfrac{k\Delta t}{2\rho c r_i \Delta r}$, and $k_3 = \dfrac{k\Delta t}{\rho c r_i^2(\Delta\phi)^2}$, above equation can be implemented as

$$T^1_{i,j} = k_1\left(T^0_{i-1,j} + T^0_{i+1,j} - 2T^0_{i,j}\right) + k_2\left(T^0_{i+1,j} - T^0_{i-1,j}\right) + k_3\left(T^0_{i,j-1} + T^0_{i,j+1} - 2T^0_{i,j}\right) + T^0_{i,j}, \tag{10}$$

where we obtain $T^1_{i,j}$ each time step with T values in several locations from previous step. This system of equations can be solved directly. See implementation (that needs modification) in `heatPolar.py`. Here we can denote the general scheme to be

$$\frac{T^1_{i,j} - T^0_{i,j}}{\Delta t} = D\ T^0_{i,j}. \tag{11}$$



Space information at each time step.

Imagine one more axis normal to the paper, pointing outwards above the blue dot. That would be $T^1_{i,j}$.

4.1.2 Crank-Nicolson Method (CNM)

We also solve the heat equation with the Crank-Nicolson method, as it is easy to modify given the FTCS method. To put simply, the general scheme is modified by averaging the space information of the current step and the next step:

$$\frac{T^1_{i,j} - T^0_{i,j}}{\Delta t} = \frac{1}{2} D\ T^0_{i,j} + \frac{1}{2} D\ T^1_{i,j}. \tag{12}$$

The specific scheme is

$$\begin{aligned} &T^1_{i,j} - 0.5k_1\left(T^1_{i-1,j} + T^1_{i+1,j} - 2T^1_{i,j}\right) - 0.5k_2\left(T^1_{i+1,j} - T^1_{i-1,j}\right) - 0.5k_3\left(T^1_{i,j-1} + T^1_{i,j+1} - 2T^1_{i,j}\right) \\ &= 0.5k_1\left(T^0_{i-1,j} + T^0_{i+1,j} - 2T^0_{i,j}\right) + 0.5k_2\left(T^0_{i+1,j} - T^0_{i-1,j}\right) + 0.5k_3\left(T^0_{i,j-1} + T^0_{i,j+1} - 2T^0_{i,j}\right) + T^0_{i,j}, \end{aligned} \tag{13}$$

here $k_1,\ k_2,\ k_3$ are previously defined.

4.2 FD schemes for the one dimensional heat equation

Discussion with Orlando: should we use $\rho c\frac{\partial T}{\partial t} = k\frac{1}{r}\frac{\partial T}{\partial r} + k\frac{\partial^2 T}{\partial r^2}$ or $\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial r^2}$

For the one dimensional heat equation, we apply centered-difference and Crank-Nicholson methods.

centered difference:

$$(U^{n+1}_i - U^n_i)/k = 1/h^2(U^n_{i-1} - 2U^n_i + U^n_{i+1})$$

CN:

$$(U^{n+1}_i - U^n_i)/k = 1/2(D^2U^n_i + D^2U^{n+1}_i) = 1/2h^2(U^n_{i-1} - 2U^n_i + U^n_{i+1} + U^{n+1}_{i-1} - 2U^{n+1}_i + U^{n+1}_{i+1})$$

rewrite to be ($r = k/2h^2$)

$$-rU^{n+1}_{i-1} + (1+2r)U^{n+1}_i - rU^{n+1}_{i+1} = rU^n_{i-1} + (1-2r)U^n_i + rU^n_{i+1}$$

4.3 Discretization of the heat equation in 1D polar form

The full IBVP we seek to solve is the following

$$\begin{cases} \rho c\dfrac{\partial T}{\partial t} = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(kr\dfrac{\partial T}{\partial r}\right) + \dfrac{1}{\Delta r}g_s(t) \\ \dfrac{\partial T(r,t)}{\partial r}\bigg|_{r=0} = 0,\ T(r,t)\bigg|_{r=R} = g_1(t) \\ T(r,0) = \text{temperature in tree trunk at 7 am} \end{cases} \tag{14}$$

Let $T^n_i \approx T(r_i, t_n)$ denote the numerical solution at the point (r_i, t_n), Δt the time step, and Δr grid size. With second order central difference in space

$$\frac{\partial T}{\partial r} \approx \frac{T_{i+1} - T_{i-1}}{2\Delta r} \tag{15}$$

$$\frac{\partial^2 T}{\partial r^2} \approx \frac{T_{i-1} - 2T_i + T_{i+1}}{(\Delta r)^2} \tag{16}$$

and first order forward in time

$$\frac{\partial T}{\partial t} \approx \frac{T^1 - T^0}{\Delta t}, \tag{17}$$

we obtain the Crank-Nicolson method by averaging the above spatial differences in time:

$$\begin{aligned}(T_i^{n+1} - T_i^n)/\Delta t &= 1/(4ar_i\Delta r)(T_{i+1}^n - T_{i-1}^n + T_{i+1}^{n+1} - T_{i-1}^{n+1}) \\ &+ 1/(2a(\Delta r)^2)(T_{i-1}^n - 2T_i^n + T_{i+1}^n + T_{i-1}^{n+1} - 2T_i^{n+1} + T_{i+1}^{n+1}),\end{aligned} \tag{18}$$

1dCNscheme

where $a = \dfrac{\rho c}{k}$.

We implement boundary conditions in the numerical scheme. For the boundary at the center of the tree, we follow the same no-flux Neumann condition in **??**. We use the second order central difference approximation at $r = 0$:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} \approx \frac{T_2 - T_0}{2\Delta r} = 0. \tag{19}$$

Equation (4.3) gives a representation for the ghost cell value T_0 in terms of T_2. When implementing, we replace T_0 by T_2 in 1dCNscheme when $i = 1$. It is implemented in the first row of the matrix below.

For the outer boundary condition at the surface of the tree, we implement the source term from the previous section as a Dirichlet condition

$$T(r,t)\Big|_{surface} = g_1(t). \tag{20}$$

We again introduce a ghost cell at T_{M+1} in 1dCNscheme, with $T_M^1 = g_1(t)$, and $T_{M+1}^1 = g_1(t + \Delta t)$. It is inplemented the last row of the following matrix equation.

The source term $g_s(t)$ is only present at the boundary. To discretize the source term, we have the following vector **??**

$$G_s(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_s(t) \end{pmatrix} \tag{21}$$

To match with the continuous problem, we multiply $G_s(t)$ by $\dfrac{\Delta t}{ak\Delta r}$.

Together, we solve the following system to compute the heat distribution in the tree trunk:

$$\begin{pmatrix} 1+\beta & -\beta & & 0 \\ \alpha_2 & 1+\beta & -\gamma_2 & \\ \ddots & \ddots & \ddots & \\ & \alpha_{m-1} & 1+\beta & -\gamma_{m-1} \\ 0 & & \alpha_m & 1+\beta \end{pmatrix} \begin{pmatrix} T_1^1 \\ T_2^1 \\ \vdots \\ T_{m-1}^1 \\ T_m^1 \end{pmatrix} =$$

$$\begin{pmatrix} 1-\beta & \beta & & 0 \\ -\alpha_2 & 1-\beta & \gamma_2 & \\ \ddots & \ddots & \ddots & \\ & -\alpha_{m-1} & 1-\beta & \gamma_{m-1} \\ 0 & & -\alpha_m & 1-\beta \end{pmatrix} \begin{pmatrix} T_1^0 \\ T_2^0 \\ \vdots \\ T_{m-1}^0 \\ T_m^0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ rhs(M) \end{pmatrix} \tag{22}$$

where

$$rhs(M) = \gamma_m(g_1(t) + g_1(t+\Delta t)) + \frac{\Delta t}{2ak\Delta r}(g_s(t) + g_s(t+\Delta t)),$$

$\alpha_i = \dfrac{\Delta t}{4ar_i\Delta t} - \dfrac{\Delta t}{2a(\Delta r)^2}$, $\beta = \dfrac{\Delta t}{a(\Delta r)^2}$, and $\gamma_i = \dfrac{\Delta t}{4ar_i\Delta t} + \dfrac{\Delta t}{2a(\Delta r)^2}$.

If the boundary condition is Neumann

$$\left.\frac{\partial T}{\partial r}\right|_{r=\text{ surface}} = g_1(t),$$

than the last row of above matrix equation should change to

$$\begin{pmatrix} \vdots & \vdots & \vdots \\ 0 & -\beta & 1+\beta) \end{pmatrix} \begin{pmatrix} \vdots \\ T_m^1 \end{pmatrix} = \tag{23}$$

$$\begin{pmatrix} \vdots & \vdots & \vdots \\ 0 & \beta & 1+\beta) \end{pmatrix} \begin{pmatrix} \vdots \\ T_m^0 \end{pmatrix} + \begin{pmatrix} \vdots \\ rhs(M) \end{pmatrix} \tag{24}$$

where

$$rhs(M) = 2\Delta r\gamma_m(g_1(t) + g_1(t+\Delta t)) + \frac{\Delta t}{2ak\Delta r}(g_s(t) + g_s(t+\Delta t)),$$

5 Implementation in Python and numerical results

Record numerical results here

With the FTCS method, we need to worry about numerical stability. By the von Neumann stability analysis, we write $T(r,\phi,t_n) = e^{ikr+il\phi}$, and $T(r,\phi,t_{n+1}) = Ge^{ikr+il\phi}$. We force $G \le 1$. Plug $T(r,\phi,t_n) = T_{i,j}^0$ and $T(r,\phi,t_{n+1}) = T_{i,j}^1$ into the Finite Difference equation, we get (we will temporarily denote $c = \dfrac{k}{\rho c}$)

$$G = c\Delta t\left[\frac{1}{(\Delta r)^2}(e^{-ik\Delta r} + e^{ik\Delta r} - 2) + \frac{1}{2r_i\Delta r}(e^{ik\Delta r} - e^{-ik\Delta r}) + \frac{1}{(r_i\Delta\phi)^2}(e^{-il\Delta\phi} + e^{il\Delta\phi} - 2)\right] + 1 \tag{25}$$

As $r_i \ge \Delta r$, we replace all r_i with Δr, and we get

$$G \le c\Delta t\left[\frac{1}{(\Delta r)^2}(2\cos(k\Delta r) - 2) + \frac{1}{2(\Delta r)^2}2i\sin(k\Delta r) + \frac{1}{(\Delta r\Delta\phi)^2}(2\cos(l\Delta\phi) - 2)\right] + 1 \tag{26}$$

We request $|G| \le 1$, and it comes down to the following inequality:

$$\left[1 - \frac{c\Delta t}{(\Delta r)^2}4\sin^2(k\Delta r/2) - \frac{c\Delta t}{(\Delta r\Delta\phi)^2}4\sin^2(l\Delta\phi/2)\right]^2 + \left[\frac{c\Delta t}{(\Delta r)^2}\sin(k\Delta r)\right]^2 \le 1 \tag{27}$$

Worst case is when $k\Delta r = l\Delta\phi = \pi$, and we get the stability requirement:

$$\left(1 - \frac{c\Delta t}{(\Delta r)^2}4 - \frac{c\Delta t}{(\Delta r\Delta\phi)^2}4\right)^2 \le 1 \tag{28}$$

For vectorization, we rewrite the above equation as

$$T_{i,j}^1 = (k_1 - k_2)T_{i-1,j}^0 + (k_1 + k_2)T_{i+1,j}^0 + (1 - 2k_1 - 2k_3)T_{i,j}^0 + +k_3T_{i,j-1}^0 + k_3T_{i,j+1}^0 \tag{29}$$

Read Chapter 9 of Randy LeVeque.

Reference BAD FORMAT. FOR convience only

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