Theoretical Computer Science (M21276)

Part A/3: Regular Languages

(Oct 2-6, 2023)

Question 1. Find the language corresponding to each of the following regular expressions over the alphabet $\{a, b, c\}$.

(i) $a + b$

Answer: $\{a, b\}$

(ii) $a + bc$

Answer: $\{a, bc\}$

(iii) $ab + ba$

Answer: $\{ab, ba\}$

(iv) ab^*

Answer: $\{a, ab, abb, abbb, \dots\}$

(v) $a + b^*$

Answer: $\{a, \Lambda, b, bb, bbb, \dots, b^n, \dots\}$

(vi) $ab^* + c$

Answer: $\{c, a, ab, ab^2, \dots, ab^n, \dots\}$

(vii) $ab^* + bc^*$

Answer: $\{a, b, ab, bc, abb, bcc, \dots, ab^n, bc^n, \dots\}$

(viii) $a^*bc^* + ac$

Answer: $\{ac, b, ab, bc, abc, aabc, \dots, a^n bc^m, \dots\}$

Question 2. Find all strings in $L((a + b)^*b(a + ab)^*)$ of length less than four.
Answer: b, ab, bb, aab, abb, bbb, bab, aba, bba, baa, ba

Question 3. Find a regular expression to describe each of the following languages over the alphabet $\{a, b, c\}$:

(i) $\{a, b, c\}$

Answer: $a + b + c$

(ii) $\{aa, ab, ac\}$

Answer: $aa + ab + ac$

(iii) $\{\Lambda, bb, bbbb, bbbbbb, \dots\}$

Answer: $(bb)^*$

(iv) $\{a, b, ab, ba, abb, baa, \dots, ab^n, ba^n, \dots\}$

Answer: $ab^* + ba^*$

(v) $\{a, aaa, aaaaa, \dots, a^{2n+1}, \dots\}$

Answer: $a(aa)^*$

(vi) $\{\Lambda, a, abb, abbbb, \dots, ab^{2n}, \dots\}$

Answer: $\Lambda + a(bb)^*$

(vii) $\{\Lambda, a, b, c, aa, bb, cc, \dots, a^n, b^n, c^n, \dots\}$

Answer: $a^* + b^* + c^*$

(viii) $\{\Lambda, a, b, ca, bc, cca, bcc, \dots, c^n a, bc^n, \dots\}$

Answer: $\Lambda + c^*a + bc^*$

(ix) $\{a^n b^m, n \geq 4, m \leq 3\}$

Answer: $aaaaa^*(\Lambda + b + bb + bbb)$

(x) $\{a^n b^m, n \geq 3, m \text{ is even}\}$

Answer: $aaaa^*(bb)^*$

(xi) $\{a^n b^m, (n + m) \text{ is even }\}$

Answer: $(aa)^*(bb)^* + a(aa)^*b(bb)^*$

Question 4. Find a regular expression for each of the following languages over the alphabet $\{a, b\}$.

(i) All strings with even length.

Answer: $(aa + ab + ba + bb)^* = ((a + b)(a + b))^*$

(ii) All strings whose length is a multiple of 3.

Answer: $((a + b)(a + b)(a + b))^*$

(iii) All strings containing the substring aba

Answer: $(a + b)^*aba(a + b)^*$

(iv) All strings with an odd number of a's.

Answer: $b^*ab^*(ab^*ab^*)^*$

(v) All strings except those with the substring aa.

Answer: $(b + ab)^*(\Lambda + a)$

(vi) All strings except those with the substring aaa.

Answer: $(b + ab + aab)^*(\Lambda + a + aa)$

Question 5.[hard] Prove the following equalities of regular expressions.

(i) $b + ab^* + aa^*b + aa^*ab^* = a^*(b + ab^*)$

Answer: $b + ab^* + aa^*b + aa^*ab^* = (b + ab^*) + aa^*(b + ab^*) = (\Lambda + aa^*)(b + ab^*) = a^*(b + ab^*)$

(ii) $a^*(b + ab^*) = b + aa^*b^*$

Answer: $a^*(b + ab^*) = a^*b + a^*ab^* = (\Lambda + aa^*)b + aa^*b^* = b + aa^*b + aa^*b^* = b + aa^*(b + b^*) = b + aa^*b^*$

(iii) $a(a + b)^* + aa(a + b)^* + aaa(a + b)^* = a(a + b)^*$

Answer: It is enough to realise that $aa(a + b)^*$, $aaa(a + b)^*$ a subset of $a(a + b)^*$

Question 6.[hard] Prove that for any regular expression R holds $R^* = R^*R^*$.

Answer: Since $L(R^*) = L(R)^*$, we need to show that $L(R)^* = L(R)^*L(R)^*$.
$\Rightarrow$ Let $x \in L(R)^*$. Then $x = x\Lambda \in L(R)^*L(R)^*$. Therefore, $L(R)^* \subseteq L(R)^*L(R)^*$.
$\Leftarrow$ Suppose, $x \in L(R)^*L(R)^*$. Then $x = yz$, where $y, z \in L(R)^*$. Thus $y \in L(R)^k$ and $z \in L(R)^n$ for some k and n. Therefore, $yz \in L(R)^{k+n}$, which says that $x = yz \in L(R)^*$, so $L(R)^*L(R)^* \subseteq L(R)^*$. Thus $L(R)^* = L(R)^*L(R)^*$, so $R^* = R^*R^*$.