Theoretical Computer Science (M21276)

Part A/7: Pushdown automata

(Oct 16-20, 2023)

In each question design a pushdown automaton that accepts the given language. Write also plan and discuss whether your automaton is deterministic or non-deterministic. Can you produce a context-free grammar for each language?

Question 1. $L = \{ab^n cd^n \mid n \geq 0\}$ over the alphabet $\Sigma = \{a, b, c, d\}$.

Answer: 4 state PDA: 0 (start), 1, 2, 3 (final);
input alphabet: $\{a, b, c, d\}$;
stack alphabet: $\{X, Y, \$\}$
instructions: $(0, a, \$, nop, 1)$, $(1, b, \$, push(X), 1)$, $(1, b, X, push(X), 1)$, $(1, c, X, nop, 2)$, $(1, c, \$, nop, 2)$, $(2, d, X, pop, 2)$, $(2, \Lambda, \$, nop, 3)$

Question 2. $L = \{wcw^R \mid w \in \{a, b\}^*\}$ over the alphabet $\Sigma = \{a, b, c\}$.

Answer: Plan: Use stack to record the string w, until c is seen. Then check that following string w^R matches with what is at top of stack.
3 state PDA: 0 (start), 1, 2 (final); input alphabet: $\{a, b\}$; stack alphabet: $\{X, Y, \$\}$
instructions: $(0, a, \$, push(X), 0)$, $(0, a, X, push(X), 0)$, $(0, a, Y, push(X), 0)$, $(0, b, \$, push(Y), 0)$, $(0, b, X, push(Y), 0)$, $(0, b, Y, push(Y), 0)$, $(0, c, \$, nop, 1)$, $(0, c, X, nop, 1)$, $(0, c, Y, nop, 1)$
$(1, a, X, pop, 1)$, $(1, b, Y, pop, 1)$, $(1, \Lambda, \$, nop, 2)$

Question 3. Any string from $\{a, b\}^*$ with an odd number of b's over the alphabet $\Sigma = \{a, b\}$.

Think again about the same language: Do we really need a stack to recognise this language? Can you design a finite automaton that accepts that language?

Answer: 2 state PDA: 0 (start) and 1(final); input alphabet: $\{a, b\}$; stack alphabet: $\{X, \$\}$
transitions: $(0, a, \$, nop, 0)$, $(0, a, X, nop, 0)$, $(0, b, \$, push(X), 0)$, $(0, b, X, pop, 0)$, $(0, \Lambda, X, nop, 1)$
Grammar: terminals $\{a, b\}$, nonterminals S (init), T, rules: $S \to aS \mid bT$, $T \to aT \mid bS \mid \Lambda$

Question 4. $L = \{a^m b^m b^n a^n \mid m \geq 0, n \geq 0\}$ over the alphabet $\Sigma = \{a, b\}$.

Answer: Plan: use stack to record number of a's (by pushing X's), then match up as b's come in. When stack empty, start pushing Y's. When a's begin again, start matching them with the Y's on the stack. Accept only if they match.
4 state PDA: 0(start), 1, 2, 3(final); input alphabet: $\{a, b\}$; stack alphabet: $\{X, Y, \$\}$

Instructions: $(0, a, \{\$, X\}, push(X), 0)$, $(0, \Lambda, \$, nop, 3)$, $(0, b, X, pop, 1)$, $(0, b, \$, push(Y), 1)$, $(1, b, X, pop, 1)$, $(1, b, \{\$, Y\}, push(Y), 1)$, $(1, a, Y, pop, 2)$, $(2, a, Y, pop, 2)$, $(2, \Lambda, \$, nop, 3)$, $(1, \Lambda, \$, nop, 3)$

Question 5. $L = \{a^n b^{n+2} \mid n \geq 0\}$ over the alphabet $\Sigma = \{a, b\}$.

Answer: Keeping track of number of a's with the stack and making sure number of b's matches. However first b seen pushes rather than pops!
3 state PDA: 0 (start), 1, 2 (final); input alphabet: $\{a, b\}$; stack alphabet: $\{X, \$\}$
Instructions: $(0, a, \$, push(X), 0)$, $(0, a, X, push(X), 0)$, $(0, b, \$, push(X), 1)$, $(0, b, X, push(X), 1)$, $(1, b, X, pop, 1)$, $(1, \Lambda, \$, nop, 2)$

Question 6. Consider strings consisting entirely of left and right brackets. Such a string is called balanced if (a) reading from left to right the number of left brackets is always at least the number of right brackets, and (b) the total number of left brackets equals the total number of right brackets. For example, (()()) is balanced, but (() is not.
(i) Can you produce a context-free grammar for balanced strings?

Answer: $S \to (S) \mid SS \mid \lambda$
(ii) Can you draw a PDA which will accept only balanced brackets?

Answer: Each opening bracket (is simply pushed, each closing bracket) causes a matching is pop.

Question 7. Any string with twice as many a's as b's.

Answer: Plan: use stack to keep track of $\#b$'s – odd a's. Every b, push an Y onto the stack or pop X from the stack. Every other a, push a X onto the stack or pop an Y from the stack.
3 state PDA: 0(start), 1, 2(final), input alphabet: $\{a, b\}$; stack alphabet: $\{X, Y, \$\}$
Instructions : $(0, b, \$, push(Y), 0)$, $(0, b, Y, push(Y), 0)$, $(1, b, \$, push(Y), 1)$, $(1, b, Y, push(Y), 1)$, $(0, b, X, pop, 0)$, $(1, b, X, pop, 1)$, $(0, a, \$, push(X), 1)$, $(0, a, X, push(X), 1)$, $(0, a, Y, pop, 1)$, $(1, a, \$, nop, 0)$, $(1, a, X, nop, 0)$, $(1, a, Y, nop, 0)$, $(0, \Lambda, \$, nop, 2)$