Logic background

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<Propositional Logic>

The Basics of Propositional Logic: Syntax

- **Basic syntax** defined as follows:

$$F := P_1 \mid (F_1 \wedge F_2) \mid (F_1 \vee F_2) \mid (F_1 \rightarrow F_2) \mid (F_1 \leftrightarrow F_2) \mid \neg F \mid \top \mid \bot.$$

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Example well-formed formulas:

$$(P_1 \wedge \neg P_2) \rightarrow (P_1 \vee \neg P_3)$$

$$((P_1 \vee \neg P_1) \wedge P_2)$$

$$((P_1 \rightarrow \neg P_1) \wedge (P_2 \leftrightarrow \neg P_2))$$

$$((((P_1 \rightarrow P_2) \rightarrow P_2) \rightarrow \neg P_1) \rightarrow \neg P_3)$$

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From atomic propositions (*atoms*) P, formulas F are created by repeated application of logical connectives: $\wedge, \vee, \rightarrow, \neg, \leftrightarrow$

Not well-formed formulas:

$$(P_1 \wedge \wedge \neg P_2) \rightarrow \rightarrow (P_1 \vee \neg P_3)$$

$$\wedge \wedge \neg P_1$$

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Semantics: propositions are *bivalent* (take values **true**, **false**), **truth assignment** $v : \{\mathtt{atoms}\} \to \{\mathbf{true}, \mathbf{false}\}$.

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1. $\overline{v}(F) = v(F)$ if F is an atom
2. $\overline{v}(\neg F) \quad = \begin{cases} \textbf{true} & \text{If } \overline{v}(F) = \textbf{false} \\ \textbf{false} & \text{otherwise} \end{cases}$
3. $\overline{v}(F_1 \wedge F_2) = \begin{cases} \textbf{true} & \text{if } \overline{v}(F_1) = \textbf{true} \text{ and } \overline{v}(F_2) = \textbf{true} \\ \textbf{false} & \text{otherwise} \end{cases}$
4. $\overline{v}(F_1 \vee F_2) = \begin{cases} \textbf{true} & \text{if } \overline{v}(F_1) = \textbf{true} \text{ or } \overline{v}(F_2) = \textbf{true} \text{ (or both)} \\ \textbf{false} & \text{otherwise} \end{cases}$
5. $\overline{v}(F_1 \to F_2) = \begin{cases} \textbf{false} & \text{if } \overline{v}(F_1) = \textbf{true} \text{ and } \overline{v}(F_2) = \textbf{false} \\ \textbf{true} & \text{otherwise} \end{cases}$
6. $\overline{v}(F_1 \leftrightarrow F_2) = \begin{cases} \textbf{true} & \text{if } \overline{v}(F_1) = \overline{v}(F_2) \\ \textbf{false} & \text{otherwise} \end{cases}$

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$$
\begin{aligned}
\overline{v}_{\prime}(\neg P_1) &= \textbf{true} \\
\overline{v}_{\prime}(\neg P_1 \vee P_2) &= \textbf{true} \\
\overline{v}_{\prime}(P_2 \wedge P_1) &= \textbf{false} \\
\overline{v}_{\prime}(P_1 \rightarrow P_2) &= \textbf{true}.
\end{aligned}
$$

The Basics of Propositional Logic: Semantics

Example: Given the truth assignment $v_{\prime}(P_1) = \textbf{false}$, $v_{\prime}(P_2) = \textbf{true}$

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\overline{v}_{\prime}(\neg P_1) &= \textbf{true} \\
\overline{v}_{\prime}(\neg P_1 \vee P_2) &= \textbf{true} \\
\overline{v}_{\prime}(P_2 \wedge P_1) &= \textbf{false} \\
\overline{v}_{\prime}(P_1 \rightarrow P_2) &= \textbf{true}.
\end{aligned}$$

Visualizing semantics using **truth tables**

	atoms		*complex formulas*				
	P_1	P_2	$\neg P_1$	$(P_1 \wedge P_2)$	$(P_1 \vee P_2)$	$(P_1 \rightarrow P_2)$	$(P_1 \leftrightarrow P_2)$
w_1	**true**	**true**	**false**	**true**	**true**	**true**	**true**
w_2	**true**	**false**	**false**	**false**	**true**	**false**	**false**
w_3	**false**	**true**	**false**	**false**	**true**	**true**	**false**
w_4	**false**	**false**	**true**	**false**	**false**	**true**	**true**

The Basics of Propositional Logic: Logical Equivalences

- **Truth tables** useful for convincing ourselves of certain *logic equivalences* ($\equiv$), for example::

$$P_1 \rightarrow P_2 \equiv \neg P_1 \vee P_2$$
$$P_1 \leftrightarrow P_2 \equiv (P_1 \rightarrow P_2) \wedge (P_2 \rightarrow P_1)$$

	P_1	P_2	$(\neg P_1 \vee P_2)$	$P_1 \rightarrow P_2$
w_1	**true**	**true**	**true**	**true**
w_2	**true**	**false**	**false**	**false**
w_3	**false**	**true**	**true**	**true**
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w_4	**false**	**false**	**true**	**true**

The Basics of Propositional Logic: Logical Equivalences

- Standard **algebraic rules**

	$\neg\neg F \equiv F$	Negation
$\neg(F_1 \wedge F_2) \equiv (\neg F_1 \vee \neg F_2)$	$\neg(F_1 \vee F_2) \equiv (\neg F_1 \wedge \neg F_2)$	De Morgan
$(F \wedge F) \equiv F$	$(F \vee F) \equiv F$	Indempotency
$(F \wedge \top) \equiv \top$	$(F \vee \bot) \equiv F$	Absorption
$(F_1 \wedge F_2) \equiv (F_2 \wedge F_1)$	$(F_1 \vee F_2) \equiv (F_2 \wedge F_1)$	Commutativity
$(F_1 \wedge (F_1 \wedge F_3)) \equiv ((F_1 \wedge F_1 \wedge F_3)$	$(F_1 \vee (F_1 \vee F_3)) \equiv ((F_1 \vee F_1 \vee F_3)$	Associativity
$(F_1 \wedge (F_1 \vee F_3)) \equiv ((F_1 \wedge F_2) \vee (F_1 \wedge F_3))$	$(F_1 \vee (F_1 \wedge F_3)) \equiv ((F_1 \vee F_2) \wedge (F_1 \vee F_3))$	Distributivity

Basic Concepts in Propositional Inference

- **Satisfiability**: The set $\Gamma = \{F_1, ..., F_n\}$ is said to be **satisfiable iff** there is an assignment to all variables in Γ such that $\overline{v}(F_j) =$ **true** for all $F_j \in \Gamma$

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satisfying assignment:

$$v(P_1) = \mathbf{true}, v(P_2) = \mathbf{true}, v(P_3) = \mathbf{true}, v(P_4) = \mathbf{false}$$

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Validity: F is **valid** (or a *tautology*) if it is true in all assignments

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- **Entailment**: Any formula α is a **logical consequence** of (or *logically entailed by*) a set of formulas $\Gamma = \{F_1, F_2, ...\}$

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Example entailments:

$$(P_1 \wedge P_2) \models P_1$$
$$(P_1 \rightarrow P_2) \models ((P_1 \wedge P_3) \rightarrow P_2)$$
$$((P_1 \wedge P_2) \rightarrow P_3) \rightarrow P_4) \models ((P_1 \rightarrow P_3) \rightarrow P_4)$$

Relating these different semantic concepts

Theorem : Relating Propositional Entailment, Validity and Satisfiability

For any $\Gamma = F_1, ..., F_m$ and formula α, the entailment relation $\Gamma \models \alpha$ is equivalent to the following statements:

1. The formula $(F_1 \wedge ..., \wedge F_m) \rightarrow \alpha$ is **valid**, or a **tautology**;
2. The formula $(F_1 \wedge ... \wedge F_m \wedge \neg\alpha)$ is **unsatisfiable**.

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1. The formula $(F_1 \wedge ..., \wedge F_m) \rightarrow \alpha$ is **valid**, or a **tautology**;
2. The formula $(F_1 \wedge ... \wedge F_m \wedge \neg \alpha)$ is **unsatisfiable**.

Important take-away: All problems are reducible to satisfiability testing.

Using propositional solvers

```
from z3 import * ## to install: pip install z3-solver

solver = Solver()
P,Q,R,S = Bools("P Q R S")

### Γ = [(((P∧Q)→R)→S)], add as assertion to solver
Gamma = [ Implies(Implies(And(P,Q),R),S) ]
solver.add(Gamma)

## check if satisfiable
solver.check() ## => "sat",
solver.model() ## => [S = True, R = False, ...]

## query, α = ((P ∧ Q)→S)
alpha = Implies(Implies(P,Q),S)

## add negated query, creating assertion Γ ∧ ¬α
solver.add(Not(alpha))
solver.check() # => unsat, i.e., Γ ⊨ α
```

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  Challenge: Efficient inference, *(for us)* making our logic amendable to gradient-based learning, differentiable.

Towards Algorithms for SAT: Normal Forms

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  Example: $\neg(P_1 \rightarrow P_2)$

$$
\begin{aligned}
\neg(P_1 \rightarrow P_2) &\equiv \neg(\neg P_1 \vee P_2) && \text{Conversion of } \rightarrow \text{ to } \vee \\
&\equiv (\neg\neg P_1 \wedge \neg P_2) && \text{De-morgan's rule} \\
&\equiv (P_1 \wedge \neg P_2) && \text{negation rule}
\end{aligned}
$$

Towards Algorithms for SAT: Normal Forms

Conjunctive Normal Form (**CNF**) has the following form:

$$\bigwedge_{j=1} \left(\bigvee_{i=1}^{k_j} l_i^j \right),$$

or consists of a <u>conjunction</u> of **clauses**, or *a disjunction of **literals** (an atom or its negation, e.g., P_1 or $\neg P_1$.)*

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A formula in **Disjunctive Normal Form** (**CNF**) has the following form:

$$\bigvee_{j=1} \left(\bigwedge_{i=1}^{k_j} l_i^j \right)$$

or consists of a disjunction of **terms**, or *a conjunction of literals*

Why normal forms matter

Example: $P_1 \rightarrow (P_2 \wedge P_3)$

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$$\textbf{CNF:}\ \underbrace{(\neg P_1 \vee P_2)}_{\text{clause } C_1} \wedge \underbrace{(\neg P_1 \vee P_3)}_{\text{clause } C_2}$$

$$\textbf{DNF:}\ \underbrace{\neg P_1}_{\text{term } T_1} \vee \underbrace{(P_1 \wedge P_2)}_{\text{term } T_2}$$

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Any formula has an equivalent CNF and DNF formula. **Satisfiability** is easy for DNF, but tautology is hard (*vice versa for CNF*).

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A more complicated case for variables $P_0^1, \ldots, P_0^n$ *and* $P_1^1, \ldots, P_1^n$*:*

$$(P_0^1 \vee P_1^1) \wedge (P_0^2 \vee P_1^2) \wedge \ldots \wedge (P_0^n \vee P_1^n)$$

the corresponding DNF:

$$\begin{aligned}
&(P_0^1 \wedge P_0^1 \wedge \ldots \wedge P_0^{n-1} \wedge P_0^n) \vee \\
&(P_0^1 \wedge P_0^1 \wedge \ldots \wedge P_0^{n-1} \wedge P_1^n) \vee \\
&\qquad\qquad\qquad\qquad \ldots \\
&(P_1^1 \wedge P_1^2 \wedge \ldots \wedge P_1^{n-1} \wedge P_0^n) \vee \\
&(P_1^1 \wedge P_1^2 \wedge \ldots \wedge P_1^{n-1} \wedge P_1^n) \vee
\end{aligned}$$

Converting representations in Sympy

```python
from sympy import *
from sympy.logic.boolalg import to_dnf

bool_vars = "P01,P02,P03,P04,P11,P12,P13,P14"
### defining variables
P01,P02,P03,P04,P11,P12,P13,P14 = symbols(bool_vars)

### cnf formula
cnf_formula = And(
            Or(P01,P11),Or(P02,P12),
            Or(P03,P13),Or(P04,P14)
)

### conversion to DNF
to_dnf(cnf_formula)
```

Towards Algorithms for SAT: Conditioning

- **Set based representations** of CNF formulas, Δ, e.g., will represent the formulas

$$(\neg P_1 \vee P_2) \wedge (\neg P_1 \vee P_3)$$

as $\Delta = \{\{\neg P_1, P_2\}, \{\neg P_1, P_3\}\}$.

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Conditioning to a literal l:

$$\Delta|l = \left\{ C - \{\neg l\} \mid C \in \Delta, l \notin C \right\}.$$

e.g., $\Delta|P_1 = \{\{P_2\}, \{P_3\}\}$

Towards Algorithms for SAT: Splitting and DPLL

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Theorem : Splitting Rule

Given a CNF formula Δ and a variable l, Δ is satisfiable if and only if $\Delta|l$ is satisfiable or $\Delta|\neg l$ is satisfiable.

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input : A CNF formula Δ
output: Satisfiability of Δ

```
Function DPLL(Δ):
    if {} ∈ Δ then
        return unsat                                  // empty clause
    else if Δ = {} then
        return sat                                    // No more clauses
    select literal l from Δ                 // Branching rule ;
    return DPLL(Δ|l) ∨ DPLL(Δ|¬l)              // Splitting Rule ;
```

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- $\Delta = \{\{P_1, P_2\}, \{P_1, \neg P_2, \neg P_3\}, \{\neg P_1, P_2, \neg P_3\}\}$, DPLL($\Delta$)

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        return unsat                                  // empty clause
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        return sat                                    // No more clauses
    select literal l from Δ                           // Branching rule ;
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- $\Delta = \{\{P_1, P_2\}, \{P_1, \neg P_2, \neg P_3\}, \{\neg P_1, P_2, \neg P_3\}\}$, $\mathtt{DPLL}(\Delta)$
  select P_1: $\Delta | P_1 = \{\{\neg P_1, P_2, \neg P_3\}\}$ **call** $\mathtt{DPLL}(\Delta | P_1)$

Towards Algorithms for SAT: Splitting and DPLL

input : A CNF formula Δ
output: Satisfiability of Δ

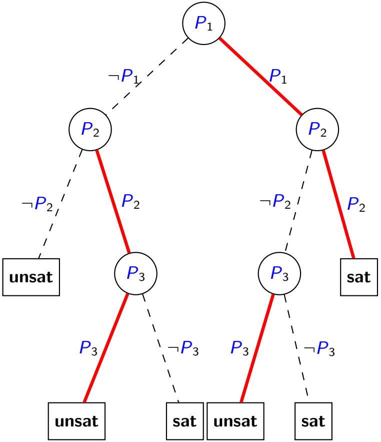
Function `DPLL`(Δ):
  if $\{\} \in \Delta$ **then**
    return **unsat** `// empty clause`
  else if $\Delta = \{\}$ **then**
    return sat `// No more clauses`
  select literal l from Δ `// Branching rule ;`
  return `DPLL`$(\Delta|l) \vee$ `DPLL`$(\Delta|\neg l)$ `// Splitting Rule ;`

Example: $(P_1 \vee P_2) \wedge (P_1 \vee \neg P_2 \vee \neg P_3) \wedge (\neg P_1 \vee P_2 \vee \neg P_3)$

- $\Delta = \{\{P_1, P_2\}, \{P_1, \neg P_2, \neg P_3\}, \{\neg P_1, P_2, \neg P_3\}\}$, `DPLL`$(\Delta)$
  select P_1: $\Delta|P_1 = \{\{\neg P_1, P_2, \neg P_3\}\}$ **call** `DPLL`$(\Delta|P_1)$
    select P_2 : $\Delta|P_2 = \{\}$ **call** `DPLL`$(\Delta|P_2)$ *(returns **sat**)*

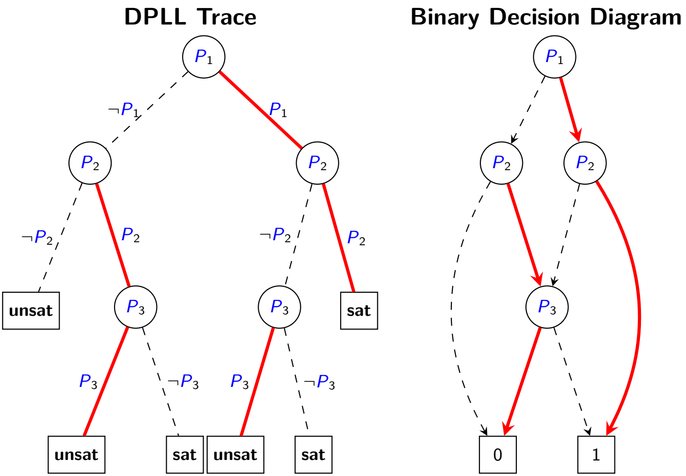
Visualizing DPLL

$$(P_1 \lor P_2) \land (P_1 \lor \neg P_2 \lor \neg P_3) \land (\neg P_1 \lor P_2 \lor \neg P_3)$$



- With some extra work we can store a trace of the search, *exhaustive DPLL* (Huang and Darwiche, 2007; Oztok and Darwiche, 2018).

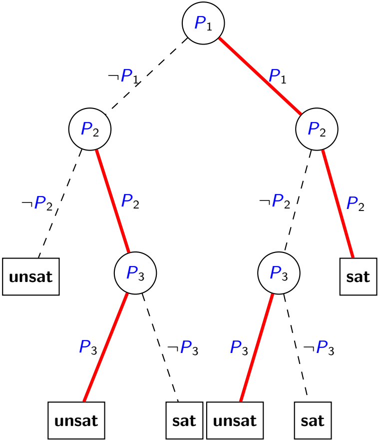
Visualizing DPLL



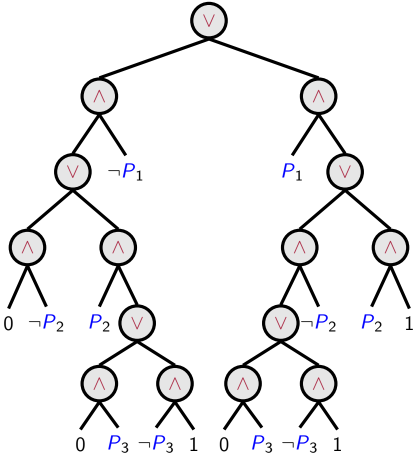
- A little more work, can get (ordered) BDDs (Bryant, 1986).

Visualizing DPLL

DPLL Trace



Negation Normal Form Circuit



- Or equivalent NNF circuit representation (Darwiche and Marquis, 2002).

Interim Summary

- SAT is at the foundations of much of automated reasoning, classical algorithm is **DPLL** (Davis and Putnam, 1960; Davis et al., 1962).

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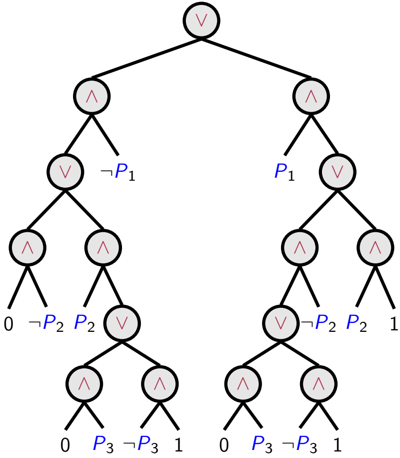
- SAT is at the foundations of much of automated reasoning, classical algorithm is **DPLL** (Davis and Putnam, 1960; Davis et al., 1962).
- SAT algorithms have played another role (Huang and Darwiche, 2005)), *top-down compilers* that transform logic into structures that:

  Facilitate tractable inference, i.e., polytime queries or transformations;

  Are amendable to gradient-based learning, differentiable.

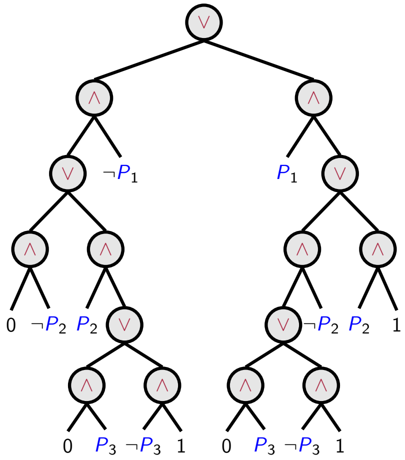
Knowledge Compilation

Negation Normal Form (NNF) Circuits



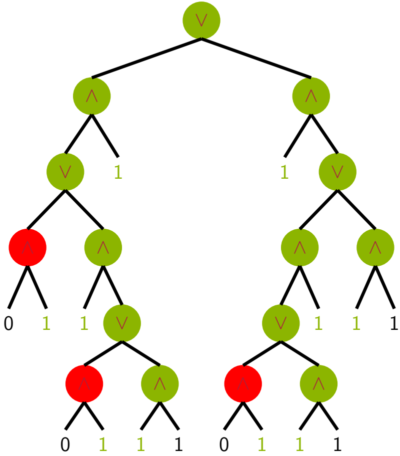
- **NNF**: A rooted DAG where, *leaf nodes* are labels with **true**, **false** (0,1), P or $\neg P$ (from set of *atoms*), internal nodes labeled with $\wedge, \vee$.

Why?



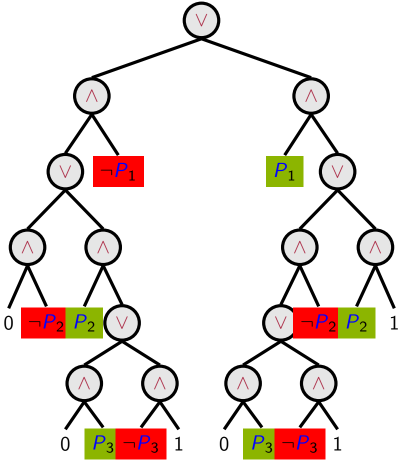
- **Decomposable NNF** (Darwiche, 2001a): A type of NNF that has some interesting properties.

Why?



- Certain *queries* can be answered in linear time, e.g., **SAT** (*make literal nodes 1 and evaluate*).

Why?



- Entailment, e.g., $\Delta \models (\neg P_1 \vee \neg P_2 \vee \neg P_3)$ *(Negate query, assign literal values accordingly in DAG, evaluate)*

Tractable Classes of Boolean Circuits and Their Properties

Odd-parity function, assigns **true** when odd number of variables are true, **false** otherwise. E.g., for variables P_1, P_2, P_3, P_4 in DNF form

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$$
\begin{aligned}
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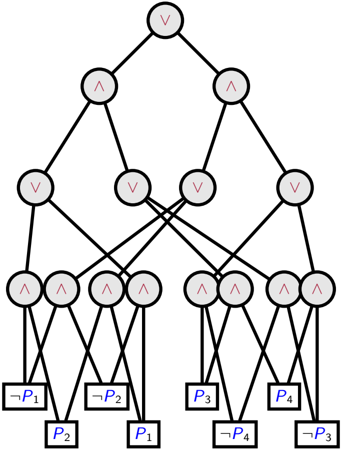
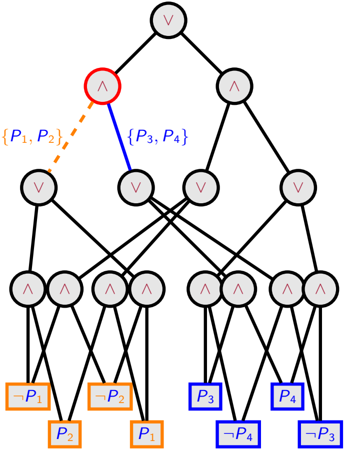
\end{aligned}
$$

How many satisfying assignments (or models) does this have?

Tractable Classes of Boolean Circuits and Their Properties

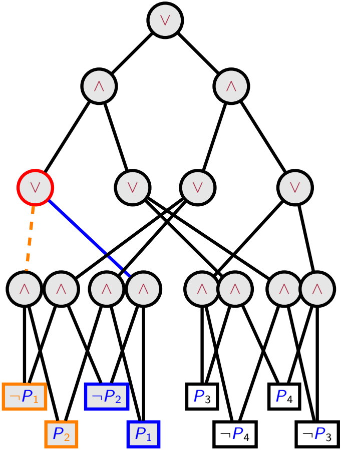


Tractable Classes of Boolean Circuits and Their Properties



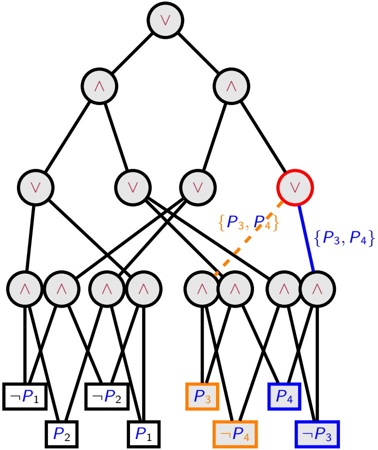
Decomposability (DNNF): Conjunctions (And-gates) do not share variables.

Tractable Classes of Boolean Circuits and Their Properties



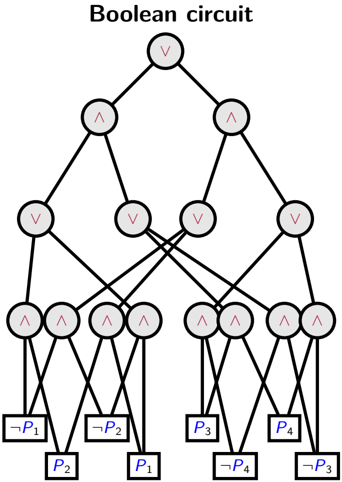
Determinism: Disjunctions (Or-gates) have at most one true input, d-DNNF circuits (Darwiche, 2001b).

Tractable Classes of Boolean Circuits and Their Properties



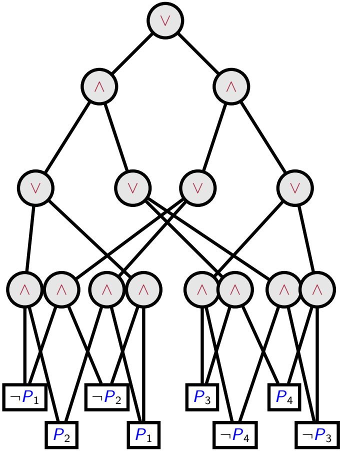
Smoothness: Disjunctions (Or-gates) mention the same variables.

Example: Model Counting

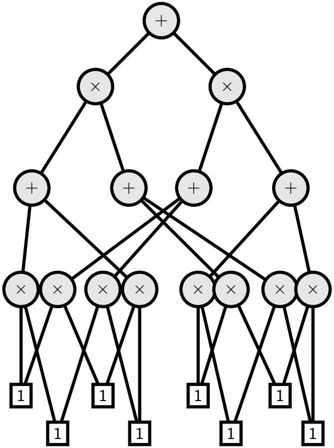


Example: Model Counting

Boolean circuit

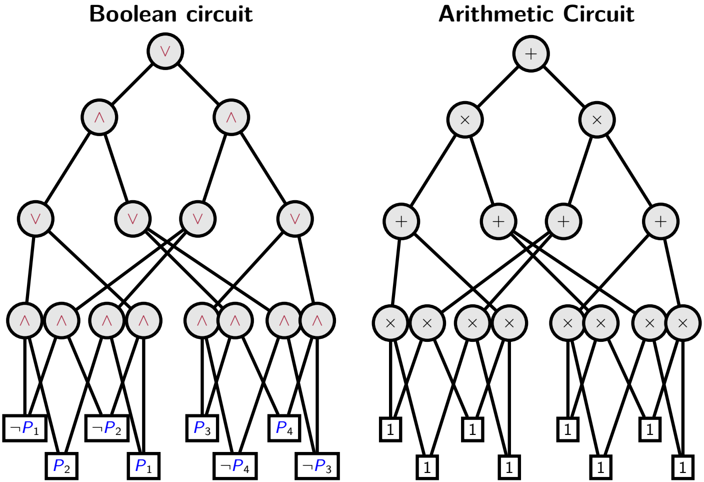


Arithmetic Circuit



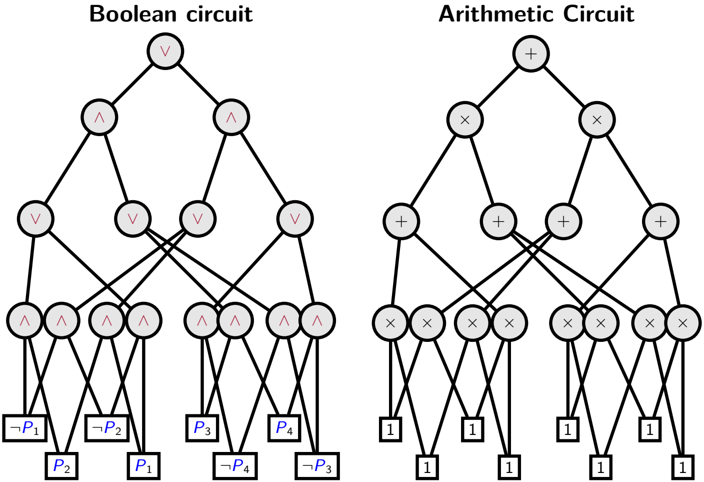
Model Counting: Change $\vee, \wedge$ to $+, \times$, assign 1s, evaluate.

Example: Model Counting



Weighted Model Counting: is useful for tractable probabilistic reasoning (Chavira and Darwiche, 2008), supports differentiable computation.

Example: Model Counting



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Compilation Software

```
from pysdd.sdd import SddManager

sdd = SddManager(var_count=3)
p1, p2, p3 = sdd.vars
parity_1 = (p1 & -p2 & -p3) | (-p1 & p2 & -p3) \
        | (-p1 & -p2 & p3)
parity_3 =  (p1 & p2 & p3)
parity = sdd.disjoin(parity_1,parity_3)

count = parity.wmc(log_mode=False)
print(f"model count: {count.propagate()}")
## 4.0
```

- More conditions can be imposed, we will use **Sentential Decision Diagrams** (SDD) later (Darwiche, 2011), *bottom-up compilation*.

Credits and more reading

Many examples taken/adapted from the work cited throughout, see also Darwiche (2022), many relevant lectures from his group at UCLA: `https://www.youtube.com/@UCLA.Reasoning`.

- **Logic background** follows Davis et al. (1994). Other books consulted: Kroening and Strichman (2016); Enderton (2001); Raedt et al. (2016)
- More on **Knowledge Compilation**: (Darwiche and Marquis, 2002; Marquis, 2008)

Thank you.

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