A Study of Numerical Methods and Sensitivity Analysis of Option Pricing

A project submitted to University of Dhaka in partial fulfillment of the requirements for the degree of Bachelor of Science in Applied Mathematics

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Abstract

In this project, we discuss the Numerical Methods, namely, Binomial tree model, Trinomial tree model and Monte Carlo simulation for valuing European options. We also use Monte Carlo simulation for valuing Asian options and measure the sensitivity of European options to parameters on which the value of the underlying instrument depends by using Finite Difference Method. We compare the numerical values of European options with the exact values obtained by the renowned Black Scholes Model. Moreover, the effects of dividend payments on option pricing are also considered. All the results have been shown numerically as well as graphically with the help of Python 3.12.2 and MATLAB R2020a.

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Abbreviations

BSM	:	Black Scholes Merton Model
BT	:	Binomial Tree Model
TT	:	Trinomial Tree Model
MC	:	Monte Carlo Simulation
PDE	:	Partial Differential Equation
SDE	:	Stochastic Differential Equation
FDM	:	Finite Difference Method
GBM	:	Geometric Brownian Motion

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CHAPTER 1

Introduction

Financial markets have been known for playing a crucial role in supporting the global economy by providing a structured platform for trading various financial instruments. These markets facilitate capital flow, enabling businesses to grow, economies to develop, and individuals to meet their financial objectives. Among the many instruments traded, derivatives stand out for their flexibility in managing risk and optimizing investment strategies. Derivatives are financial agreements that change in value based on the price changes of other assets, like stocks, indexes, or interest rates. Options are a kind of derivative that allow the buyer to choose whether to buy or sell a specific asset at a set price by a certain date. Understanding options involves grasping different types of options, their payoff structures, and the distinctions between American, European, and Asian options. Pricing an option requires the use of mathematical tools that are complex and sophisticated in nature. Important ideas like Brownian motion and stochastic processes are crucial for understanding how asset prices move randomly. Factors that influence valuation of an option are price of the asset the option is based on, the market volatility, time to expiration or maturity time, interest rates, and dividends. The Greeks, which include: Delta, Gamma, Theta, Vega, and Rho, provide estimates of sensitivity, offering insights into how these variables affect option pricing. Advanced pricing techniques like the binomial and trinomial tree methods provide discrete-time frameworks for option valuation. These models build possible price paths for the underlying asset step by step, offering a structured approach to determining prices of options. Monte Carlo simulation is a well known statistical procedure, employing random sampling to model and price options, particularly in complex cases. The Black Scholes Merton Model revolutionized option valuation by providing a closed-form solution for European options using continuous-time stochastic processes. More exotic options, such as Asian options, use partial differential equations to account for the complexities of their payoff structures. The formula for the European Fixed Strike Asian Call Option provides a specific solution for this option type. This project aims to explore the diverse world of financial markets and derivatives, focusing specifically on options and their sensitivities. **Chapter 2** covers fundamental concepts, including derivatives, types of markets, contracts, traders, options, payoffs, key definitions, stochastic procedures, Geometric Brownian motion, along with Greek Letters. **Chapter 3** discusses the Binomial Tree model, Trinomial Tree model, Monte Carlo simulation, the Black Scholes Merton Model for valuing European options, and the relevant Partial Differential Equation formulation for Asian call options. **Chapter 4** focuses on the valuation of the described methods, both numerically and graphically. **Chapter 5** presents the conclusions drawn from the various methods discussed in the project.

CHAPTER 2

Preliminaries

2.1 Introduction

Many people start their investment journey with a savings account. Banks or similar institutions offer interest, a portion of your deposited money, in exchange for holding it for a while. Interest can be calculated in various ways. This chapter will explore some popular methods and highlight the key differences between them. It's also a chance to revisit exponential functions and geometric series, which are relevant to understanding interest calculations. [4].

2.2 Derivatives

Derivatives have gained significant importance in the financial sector. Futures contracts and options are widely traded over numerous global exchanges. Derivates market has surpassed the stock market in size when evaluated based on underlying assets. The total value of assets tied to active derivatives contracts is multiple times larger than the global gross domestic product. This chapter will explore the utilization of derivatives for hedging, arbitrage, and speculation. [4]

Definition 2.2.1

"A **derivative** can be defined as a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables. A stock option, for example, ves can be dependent on almost any variable, from the price of hogs to the amount of snow falling at a certain ski resort. We can also say, a derivative is a financial security with a value that is reliant upon or derived from, an under lying asset or group of assets."

A derivative is a contract between two parties, with its value based on changes in the price of an underlying asset. Most derivatives are traded off-exchange and are primarily used by institutions for risk management or speculation on price movements of the underlying asset. In contrast, exchange-traded derivatives like futures and stock options are standardized, helping to minimize the risks associated with over-the-counter derivatives. [4]

2.3 Exchange-Traded Markets

Exchange-Traded Markets are centralized platforms where all transactions are routed through an intermediary. In simpler terms, they serve as a bridge connecting buyers and sellers, ensuring that every trade goes through a regulated process.

Definition 2.3.1

"A derivatives exchange is a marketplace where standardized contracts, predefined by the exchange, are traded by individuals. "

Definition 2.3.2

"An exchange-traded derivative refers to a financial contract that is listed and traded on a regulated exchange. "

Exchange-traded derivatives have gained popularity due to their numerous advantages over over-the-counter (OTC) derivatives, including standardization, enhanced liquidity, and the reduction of default risk. These derivatives provide a more transparent and regulated trading environment, making them one of the preferable choices for a lot of the market participants. Additionally, they can be effectively used for hedging purposes or for speculative trading across a broad range of financial devices, such as commodities, currencies, equities, and even interest rates. [4]

2.4 Over-the-Counter Markets

Definition 2.4.1

“ An over-the-counter **(OTC)** market is a decentralized market in which market participants trade stocks, commodities, currencies, or other instruments directly between two parties and without a central exchange or broker. ”

OTC markets are virtual, with no physical trading locations. Instead, all trading takes place electronically. OTC markets are mainly used for trading bonds, currencies, derivatives, and structured financial products.
All kind of derivatives may not be traded over an exchange. In fact, the over-the-counter (OTC) market now surpasses the exchange-traded market in terms of total trading volume. OTC markets are virtual, with transactions conducted electronically, and they carry a higher degree of counterparty risk compared to exchange-traded derivatives. **Counterparty risk** refers to the probability that one party involved in an exchange may fail to meet its contractual obligations. This type of risk can arise in credit, investment, and trading transactions. [4]

2.5 Forward Contracts

Definition 2.5.1

“ A **forward contract** is an agreement to buy or sell an asset at a certain future time for a certain price. In contrast, a **spot contrast** is an agreement that is made for an asset to bought or sold at the present. ”

A forward contract, mostly known as a relatively **simple derivative**. [4]
A forward contract is an agreement between two parties to buy or sell an asset at a specific price on a future date. One party agrees to buy (long position), while the other agrees to sell (short position). These contracts are typically traded outside

of a formal exchange, often between financial institutions or with their clients. [4] The payoff or profit assuming a long position in the forward contract for a single unit of the asset is given as

$$f(S,T) = S_T - K$$

where K is the delivery or strike price and S_T is the price of the asset at maturity of the contract. This is because the holder of the contract must to buy an asset worth S_T for K. Likewise, the payoff received with a short position in a single unit of the asset is

$$g(S,T) = K - S_T$$

The payoffs seen here can have either sign. Since there's no upfront cost to start a forward contract, any profit or loss you make is the total gain or loss from the contract. [4]

2.6 Futures Contracts

Definition 2.6.1

" A **futures contract** is a legal agreement to buy or sell a particular commodity asset, or security at a predetermined price at a specified time in the future. "

Different from forward contracts, futures contracts are traded on a central exchange. This allows the exchange to set standard features for the contracts, making them easier to buy and sell. [4]

Characteristics of futures contract

1. Futures contracts are financial agreements that require the buyer to buy or the seller to sell a specific asset at a set value and time sometime in future.
2. Futures contracts give investors the ability to bet whether the value of some asset will go up or down. They can do this by buying or selling a futures contract on that asset, and they can use leverage to increase their potential gains or losses.
3. Futures contracts may also be used for protection against losses from value fluctuations in an underlying asset. This is known as hedging. [4]

2.7 Trading strategies

Derivatives markets have achieved remarkable success, primarily because they attract a wide variety of traders and offer substantial liquidity. As a result, when an investor seeks to enter into a contract, it is typically easy to find a counterparty willing to take the opposite position.

A **trader** is an individual who engages in the buying and selling of financial assets in any financial market, either for himself or on behalf of another person or institution. The main difference between a trader and an investor is the duration for which the person holds the asset. [4]

We will discuss three types of traders in this chapter: **hedgers**, **speculators** and **arbitrageurs**.

2.7.1 Hedgers

A **hedge** is an investment aimed at minimizing the risk of unfavorable price fluctuations in an asset. Typically, it involves taking an opposite position in a related security. **Hedgers** are exposed to the risk of price changes in an asset and use derivatives to mitigate or eliminate this risk.

2.7.2 Speculators

Speculators aim to profit from anticipated price movements of an asset. They use derivatives to gain additional leverage. For example, an investor expecting a particular stock to increase in value may buy shares of that company. If their prediction is correct, they earn a profit; otherwise, they do not. This is an example of speculation. While hedgers seek to avoid the risk of unfavorable price changes, speculators actively take positions in the market, betting that the asset's price will either rise or fall.

2.7.3 Arbitrageurs

Arbitrageurs operate to capitalize on price differences between two markets. For instance, if they notice a misalignment between the futures price and the cash price of an asset, they will take opposite positions in both markets to secure a profit. They form a third key group of participants in futures, forward, and options markets. Arbitrage involves making risk-free profits by simultaneously executing transactions across multiple markets. [4]

2.8 Options

An option is a contract issued by a seller that gives the buyer the right, but not the obligation, to purchase (in the case of a **call option**) or sell (in the case of a **put option**) a specific asset at a predetermined price (**Strike price** / **Exercise price** K) at a future date. In exchange for granting this right, the seller receives a payment known as the **premium** from the buyer. [4]

Options are traded in either of the two places: exchanges and over-the-counter markets. Options have two basic forms: **call** and **put** options.

Call Option:

Definition 2.8.1

“ A call option gives the holder the right to buy the underlying stock from a seller by a certain date for a certain price or strike price K. ”

Put Option:

Definition 2.8.2

“ A put option gives the holder the right to sell the underlying stock to the writer by a certain date for a certain price or strike price K . ”

In this definition, the price that the bet is done upon is called the strike price, and the certain time that is fixed is usually called the time to maturity. K is reserved for the strike price, S is for the price of the underlying stock. The cost of some call option will be denoted by C and the cost of a put by P.
Option Holder is the one who is able to make a decision, to exercise it or not,to buy or not, to sell or not, and they pay the option seller a fee for this freedom.
Option Writer is the one who has a decision forced upon them at some future time, so they receive some in return.
Premium is the amount that is paid initially for the option. [4]

European Call Option:

A European call option allows the bearer the ability to buy from a writer an underlying asset for a strike price K on the completion of maturity time T.

European Put Option:

A European put option allows the bearer the ability to sell to a writer an underlying asset for a strike price K on the completion of maturity time T.

American Call Option:

An American call option allows the bearer the ability to buy from a writer an underlying asset for a strike price K any time before the specified maturity time T completes. The value of this option, as denoted by C, is given by"

$$C(S,T) = \max_{0<t\leq\tau\leq T}(S_T - K, 0)$$

"

American Put Option:

An American put option allows the bearer the ability to sell to a writer an underlying asset given a strike price K any time before the specified maturity time T completes. The value of this option, as denoted by P, is given by [4]"

$$P(S,T) = \max_{0<t\leq\tau\leq T}(K - S_T, 0)$$

"

2.9 Option Payoff

Definition 2.9.1

" Payoff Diagram is a graph of the value of the option position at maturity $t = T$ as a function of the underlying stock price S. "

- Price of call when $t = T$:"

$$C(S,T) = \max(S_T - K, 0) = \begin{cases} 0 & \text{if } S_T < K \\ S_T - K & \text{if } S_T > K \end{cases}$$

"

- Price of put when $t = T$:“

$$P(S,T) = \max(K - S_T, 0) = \begin{cases} K - S_T & \text{if } S_T < K \\ 0 & \text{if } S_T > K \end{cases}$$

”

Remark 2.9.1

we are easily understand from the diagram If a trader thinks that the stock price is on the rise, he can make money by buying a call option without buying the stock. If a trader thinks the stock price is on the decline, he can make money by buying put options. [4]

Example 2.9.1. *Suppose, a six-month call option of the European type on Bond Price share having a strike price of $K = 50$ and time to maturity, $T = 0.30$. If someone enters into this contract, he has the right but not the obligation to buy one share valued $K = 50$ after a six months maturity period. Whether they exercise it or not depends on the stock price of the asset after maturity T reaches completion:*

- *If the stock price is above \$50, let \$60, he can buy this share for \$50 and sell it immediately for \$60, making a profit of \$10.*
- *If the stock price is below \$50, there is no financial sense to buy it. The option is worthless.*

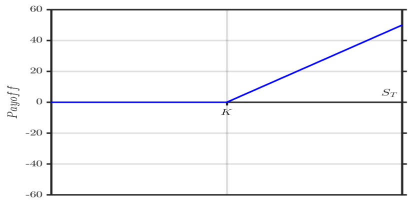
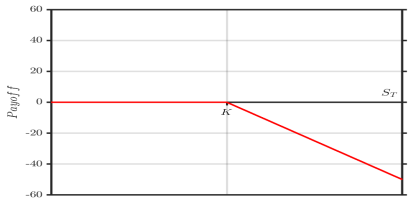
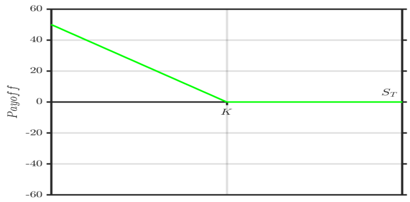
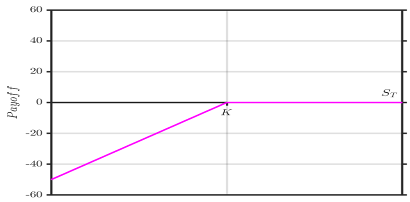
2.9.1 Option Positions

“There are four types of option positions:

1. A **long** position in a **call** option
2. A **long** position in a **put** option
3. A **short** position in a **call** option
4. A **short** position in a **put** option

” For these four combinations, the payoffs and the payoff functions are shown here : [4]

“

Option Positions	**Payoff Functions**	**Graph**
Long Call	$C_{\text{Long}} = \max(S_T - K, 0) = \begin{cases} 0 & \text{if } S_T < K \\ S_T - K & \text{if } S_T \geq K \end{cases}$	
Short Call	$C_{\text{Short}} = -\max(S_T - K, 0) = \begin{cases} 0 & \text{if } S_T < K \\ K - S_T & \text{if } S_T \geq K \end{cases}$	
Long Put	$P_{\text{Long}} = \max(K - S_T, 0) = \begin{cases} 0 & \text{if } S_T > K \\ K - S_T & \text{if } S_T \leq K \end{cases}$	
Short Put	$P_{\text{Short}} = -\max(K - S_T, 0) = \begin{cases} 0 & \text{if } S_T > K \\ S_T - K & \text{if } S_T \leq K \end{cases}$	

”

2.9.2 Maintaining a Long Position vs a Short Position

Investors hold “long” security positions when they expect the stock to increase in value in the future. The reverse of a “long” position is a “short” position. A "short" position typically involves selling a stock that the investor does not own. Short sellers believe the stock’s price will decline.

For example, an investor with 50 shares of Beximco stock in their portfolio is considered long 50 shares. This means the investor has fully paid for these shares. In the case of stocks, a long position indicates ownership of the stock after purchasing shares. In contrast, an investor in a short position owes stock to another party but has not yet purchased it. Short sellers expect the stock's price to fall. For options, holding or buying a call or put option is a long position, giving the investor the right to buy or sell at a specific price. Conversely, selling or writing a call or put option is a short position, obligating the writer to either buy from or sell to the long position holder. [4]

2.10 Arbitrage Opportunity

Definition 2.10.1

" A portfolio strategy $(\bar{\xi}_t)_{t=1,\ldots,N}$ constitutes an arbitrage opportunity for option value (V) if all three following conditions are satisfied :

1. $V_0 \leq 0$ (start from 0 even with a debt)
2. $V_N \geq 0$ (finish with a non-negative amount)
3. $P(V_N > 0) > 0$ (a profit is made with non-zero probability)

"

[10]

Arbitrage occurs when you can simultaneously buy and sell the same or a similar product or asset at different prices, leading to a risk-free profit. According to economic theory, arbitrage opportunities should not exist because if markets are efficient, such chances to profit would be eliminated. [5]

2.11 Stochastic Process

Definition 2.11.1

" If a random variable X is dependent on time, so that it is defined instances of time $t_1, t_2, \ldots, t_n$, then $X(t)$ is called a stochastic process. "

A stochastic procedure can also be defined as $\{X(t, \omega) : t \in T\}$ to reflect that it is a function of two variables, $t \in T$ along with $\omega \in \Omega$. [4, 11]
We provide some real life examples of stochastic procedures for better intuition:

- **Stock Prices**: The movement of stock prices in financial markets is the most appropriate application of a stochastic procedure, often modeled using Brownian motion or geometric Brownian motion.
- **Weather Patterns**: Weather systems, including temperature, precipitation, and wind speed, are stochastic processes due to their inherent randomness and complexity.
- **Radioactive Decay**: The decay of radioactive atoms occurs randomly over time, following a Poisson process.
- **Queueing Systems**: In telecommunications and computer networks, the arrival of data packets and their processing times are modeled as stochastic processes to optimize performance and manage congestion.
- **Population Dynamics**: The growth and fluctuations of populations in ecology, such as the number of individuals in a species, can be described using stochastic models to account for random birth and death events.
- **Thermal Noise in Electrical Circuits**: In electronics, thermal noise, better known as "Johnson-Nyquist noise", is a stochastic process that arise from randomly distributed motion of electrons present in conductors.
- **Brownian Motion**: The movement of particles suspended in a fluid is apparently random due to large amount of intermolecular forces, and is known as Brownian motion. This is widely studied in chemistry and physics.
- **Genetic Drift**: In biology, genetic drift refers to random changes in allele frequencies within a population, a stochastic process that can significantly impact evolutionary dynamics.

Definition 2.11.2

“ The standard Wiener process $W(t)$ is a Gaussian process such that

- $W(t)$ has independent increments:
  i.e. if $u \leq v \leq s \leq t$, then $W(t) - W(s)$ and $W(u) - W(v)$ are independent.

- $W(s+t) - W(s) \sim \mathcal{N}(0, t)$ and $W(0) = 0$.

”

[4]

2.12 Geometric Brownian Motion

Definition 2.12.1

“ A geometric Brownian motion is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also known as a Wiener process) with drift. It is also known as an exponential Brownian motion. ”

A stochastic procedure $S(t)$ is defined to follow a geometric Brownian motion when it satisfies the associated differential equation, which has the form “

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

”in this, μ is known as the coefficient of drift, σ the coefficient of diffusion while dB_t is an incremental Brownian motion or a Wiener process. [?] We can find an exact solution by integrating this, which would result in the following formula

$$S_t = S_0 \exp\left(\mu t - \frac{\sigma^2}{2} t\right) + \sigma B_t$$

2.13 Factors that affect Pricing of Options

There are seveal factors that affect the price of a stock option, but we discuss the main **six**: "

1. The initial stock price, S_0, that is the price the underlying stock had at the time of the purchase of the option.

2. The strike price, K, the agreed price that was fixed by the writer of the option.

3. The time of maturity, T, the point upon which (European) or up until which (American) the option is to be exercised.

4. The volatility of the stock price, σ, best understood as the uncertainty in the movement of the stock price.

5. The risk-free interest rate, r, represents the interest an investor is expected to get from an absolutely risk-free investment over a specified period of time.

6. Dividends offered, D, the distribution of the company's earnings with its shareholders, to be determined by the directing board of the company.

" The table below shows how option prices are affected by changes in various factors, assuming all other factors remain constant. A '+' means that an increase in the factor either raises or leaves the option price unchanged. A '−' means that an increase in the factor either lowers or leaves the option price unchanged. A '?' indicates that the effect of a rise in said factor on the price of option is uncertain. [4]

""

Variable	**European Call**	**European Put**	**American Call**	**American Put**
Stock Price	+	−	+	−
Strike Price	−	+	−	+
Time to Expiration	?	?	?	?
Volatility	+	+	+	+
Risk-free Rate	+	−	+	−
Dividends	−	+	−	+

Table 2.1: Table showcasing the relation between the various factors that affect the pricing of options.

Understanding these dynamics of option pricing involves recognizing how different factors interact with the underlying stock and its movement.

2.13.1 Stock price, S_0

Call options rise in value as stock prices increase since the bearer of the option gains the ability to purchase the stock at some set price (called strike price). Conversely, when the stock price is low compare to stock price cause price of the option to decline. Put options, however, gain value as stock prices decrease because the option bearer can sell the stock at a higher value than its market price.

2.13.2 Strike price, K

The relationship between the strike price and the option price is inverted. A lower strike price raises the value of call options, offering greater profit potential if the stock price exceeds it. Conversely, a higher strike price reduces the value of call options. In the case of put options, a higher strike price boosts their value, while a lower strike price diminishes it.

2.13.3 Maturity time, T

The difference between long-life and short-life options is crucial for understanding their relative values. Long-life options include all the exercise opportunities of short-life options plus additional ones, making them inherently more valuable. The longer an option's lifespan, the more chances it has to benefit from favorable stock price movements. As a result, a long-life option must always be worth at least as much as a short-life option, given its wider range of potential outcomes.

2.13.4 Volatility, σ

Volatility is a key factor when valuing options. High volatility means a greater probability of significant stock price movements, which would raise the potential for profit for both call and put option holders. For call options, increased volatility means a higher chance of the stock price exceeding the strike price, which results in a high profit. Put options benefit from volatility as well, as larger stock price declines can lead to higher profits for put holders.

2.13.5 Interest rate, r

Interest rates influence option pricing through their impact on stock returns and present value calculations. When interest rates rise, the expected return required by investors from holding stocks also increases. This causes the value of call option to grow, since the expected returns from owning the stock become more attractive. Conversely, put option values decrease because the "present value of future cash flows", which the put bearer may receive if said option is exercised, diminishes with higher interest rates.

2.13.6 Dividends, D

Dividends affect option prices, particularly on the date of the ex-dividend when the stock price typically declines equal to the amount paid out by the dividend. This reduction in stock price negatively impacts call option values since the potential for stock price appreciation is diminished. However, put option values benefit from dividends, as the reduction in stock price aligns with the put holder's objective of profiting from falling stock prices.

2.13.7 Put-Call Parity

Definition 2.13.1

" If S denotes the stock price of a non-dividend paying stock, the value of a European call C with a certain exercise price K and exercise date T can be deduced from the value of a European put P with the same exercise price and exercise date and vice-versa by the following relation. This relationship between the underlying asset and its options is called Put-Call Parity.

$$P + S = C + K\exp\{-r\,(T-t)\}$$

"

When dividends exist, we can modify the equation as follows “

$$P + S = C + D + K\exp\{-r\,(T - t)\}$$

” This relationship only holds for European options. To derive relations for American options, we will use “

$$S - K \leq C - P \leq S - K\exp\{-r\,(T - t)\}$$

”and for a dividend paying version, we can use “

$$S - D - K \leq C - P \leq S - K\exp\{-r\,(T - t)\}$$

”

2.14 The Greeks

Definition 2.14.1

“[Greek Letters] The Greek Letters or just Greeks are risk sensitivities representing the price sensitivity to changes in underlying parameters on which the value of the financial instrument depends. ”

The Greeks are vital tools in risk management, allowing for the rebalancing of a financial portfolio to achieve the desired exposure. In finance, there is often an interest in hedging a financial position, and one approach is to build a risk-neutral portfolio concerning a specific predefined quantity. The Greeks can evaluate component risk independently, making them essential for constructing a risk-neutral portfolio. An option's price is influenced by the forces of supply and demand in a free and open market. Option traders must consider the interrelationship of various factors in an option-pricing model, such as stock price, time, volatility, interest rates, dividends, and strike price. This multidimensional perspective on asset pricing is distinctive to option traders. [19, 20]

2.14.1 Delta

Definition 2.14.2

"[Delta] The Delta Δ of an option is the rate of change of an option's value relative to a change in the price of the underlying security. "

"It is the slope of the curve that relates the option price to the underlying asset price.

Suppose that the delta of a call option on a stock is 0.6. This means that when the stock price changes by a small amount, the option price changes by about 60% of that amount. Mathematically we can write

$$\Delta = \frac{\partial C}{\partial S} \tag{2.1}$$

We know from Black Scholes Model that the exact explicit solution for the European call option is

$$C(S,t) = SN(d_+) - Ke^{-r(T-t)}N(d_-) \tag{2.2}$$

where $N(.)$ is the cumulative distribution function for a standard normal random variable, given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \tag{2.3}$$

Here

$$d_+ = \frac{\log(S/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{2.4}$$

and

$$d_- = \frac{\log(S/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_+ - \sigma\sqrt{T-t} \tag{2.5}$$

It is easy to see that

$$\frac{\partial d_+}{\partial S} = \frac{1}{\sigma\sqrt{T-t}} \times \frac{1}{\frac{S}{K}} \times \frac{1}{K} = \frac{1}{S\sigma\sqrt{T-t}} = \frac{\partial d_-}{\partial S} \tag{2.6}$$

and

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \tag{2.7}$$

Then

$$\begin{aligned}
N'(d_+) &= N'(d_- + \sigma\sqrt{T-t}) \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(d_- + \sigma\sqrt{T-t})^2} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[d_-^2 + 2d_-\sigma\sqrt{T-t} + \sigma^2(T-t)]} \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_-^2} \times e^{-\frac{1}{2}[2d_-\sigma\sqrt{T-t} + \sigma^2(T-t)]} \\
&= N'(d_-) \times e^{-[\log\frac{S}{K} + r(T-t)]} \\
\therefore SN'(d_+) &= N'(d_-) \times Ke^{-r(T-t)}
\end{aligned} \tag{2.8}$$

Now for a European call option in the long position

$$\begin{aligned}
\Delta_{call} &= \frac{\partial C}{\partial S} \\
&= \frac{\partial}{\partial S}[SN(d_+) - Ke^{-r(T-t)}N(d_-)] \\
&= N(d_+) + SN'(d_+)\frac{\partial d_+}{\partial S} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial S} \\
&= N(d_+) + SN'(d_+)\frac{\partial d_+}{\partial S}N(d_+) - SN'(d_+)\frac{\partial d_+}{\partial S} \\
&= N(d_+)
\end{aligned} \tag{2.9}$$

So for a European call option in the short position

$$\Delta_{call} = -N(d_+) \tag{2.10}$$

For a European put option we know

$$P(S,t) = Ke^{-r(T-t)}N(-d_-) - SN(-d_+) \tag{2.11}$$

Now for a European put option in the long position

$$\begin{aligned}
\Delta_{put} &= \frac{\partial P}{\partial S} \\
&= \frac{\partial}{\partial S}[Ke^{-r(T-t)}N(-d_-) - SN(-d_+)] \\
&= \frac{\partial}{\partial S}[Ke^{-r(T-t)}(1 - N(d_-)) - S(1 - N(d_+))] \\
&= N(d_+) - 1 + SN'(d_+)\frac{\partial d_+}{\partial S} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial S} \\
&= N(d_+) - 1 + SN'(d_+)\frac{\partial d_+}{\partial S}N(d_+) - SN'(d_+)\frac{\partial d_+}{\partial S} \\
&= N(d_+) - 1
\end{aligned} \tag{2.12}$$

Similarly, for a European put option in the short position

$$\Delta_{put} = -(N(d_+) - 1) \tag{2.13}$$

"

2.14.2 Gamma

Definition 2.14.3

"[Gamma] Gamma (Γ) is the rate of change of an option's Δ given a change in the price of the underlying security. It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 C}{\partial S^2} \tag{2.14}$$

"

"Then for European call option

$$\begin{aligned}
\Gamma_{call} &= \frac{\partial \Delta_{call}}{\partial S} \\
&= \frac{\partial N(d_+)}{\partial S} \\
&= N'(d_+) \times \frac{\partial d_+}{\partial S} \\
&= \frac{N'(d_+)}{S\sigma\sqrt{T-t}}
\end{aligned} \tag{2.15}$$

and for European put option

$$\begin{aligned}
\Gamma_{put} &= \frac{\partial \Delta_{put}}{\partial S} \\
&= \frac{\partial}{\partial S}[N(d_+) - 1] \\
&= N'(d_+) \times \frac{\partial d_+}{\partial S} \\
&= \frac{N'(d_+)}{S\sigma\sqrt{T-t}} \\
\therefore \Gamma_{call} &= \Gamma_{put}
\end{aligned} \tag{2.16}$$

"

2.14.3 Theta

"The decline in the value of an option because of the passage of time is called time decay, or erosion. Incremental measurements of time decay are represented by the Greek letter theta Θ. "

Definition 2.14.4

"[Theta] Theta is the rate of change in an option's price given a unit change in the time to expiration.

$$\Theta = \frac{\partial C}{\partial t} \tag{2.17}$$

"

"We know

$$d_- = d_+ - \sigma\sqrt{T-t} \Rightarrow d_+ - d_- = \sigma\sqrt{T-t} \Rightarrow \frac{\partial d_+}{\partial t} - \frac{\partial d_-}{\partial t} = \frac{-\sigma}{2\sqrt{T-t}} \tag{2.18}$$

By using the Black Scholes Model for the European call option , we get

$$\begin{aligned}
\Theta_{call} &= \frac{\partial}{\partial t}[SN(d_+) - Ke^{-r(T-t)}N(d_-)] \\
&= SN'(d_+)\frac{\partial d_+}{\partial t} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial t} - rKe^{-r(T-t)}N(d_-) \\
&= SN'(d_+)(\frac{\partial d_+}{\partial t} - \frac{\partial d_-}{\partial t}) - rKe^{-r(T-t)}N(d_-) \\
&= \frac{-\sigma SN'(d_+)}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_-)
\end{aligned} \tag{2.19}$$

Similarly for European put option

$$\begin{aligned}
\Theta_{put} &= \frac{\partial}{\partial t}[Ke^{-r(T-t)}N(-d_-) - SN(-d_+)] \\
&= \frac{\partial}{\partial t}[Ke^{-r(T-t)}(1 - N(d_-)) - S(1 - N(d_+))] \\
&= rKe^{-r(T-t)}N(-d_-) - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial t} + SN'(d_+)\frac{\partial d_+}{\partial t} \\
&= \frac{-\sigma SN'(d_+)}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_-)
\end{aligned} \tag{2.20}$$

"

2.14.4 Vega

"The Black Scholes Merton model assumes that the volatility of the asset underlying an option is constant.But in practice the volatility of an asset changes over time. As a result, the value of an option is liable to change because of movements in volatility as well as because of changes in the asset price and the passage of time. "

Definition 2.14.5

"[Vega] The vega of an option, ν , is the rate of change in its value with respect to the volatility of the underlying asset.

$$\nu = \frac{\partial C}{\partial \sigma} \tag{2.21}$$

"

"When vega is highly positive or highly negative, there is a high sensitivity to changes in volatility. If the vega of an option position is close to zero, volatility changes have very little effect on the value of the position.
We know

$$d_- = d_+ - \sigma\sqrt{T-t} \Rightarrow d_+ - d_- = \sigma\sqrt{T-t} \Rightarrow \frac{\partial d_+}{\partial \sigma} - \frac{\partial d_-}{\partial \sigma} = \sqrt{T-t} \tag{2.22}$$

By using the Black Scholes Model for the European call option , we get

$$\begin{aligned} \nu_{call} &= \frac{\partial}{\partial \sigma}[SN(d_+) - Ke^{-r(T-t)}N(d_-)] \\ &= SN'(d_+)\frac{\partial d_+}{\partial \sigma} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial \sigma} \\ &= SN'(d_+)(\frac{\partial d_+}{\partial \sigma} - \frac{\partial d_-}{\partial \sigma}) \\ &= SN'(d_+)\sqrt{T-t} \end{aligned} \tag{2.23}$$

Similarly for European put option

$$\begin{aligned} \nu_{put} &= \frac{\partial}{\partial \sigma}[Ke^{-r(T-t)}N(-d_-) - SN(-d_+)] \\ &= \frac{\partial}{\partial \sigma}[Ke^{-r(T-t)}(1 - N(d_-)) - S(1 - N(d_+)] \\ &= SN'(d_+)\frac{\partial d_+}{\partial \sigma} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial \sigma} \\ &= SN'(d_+)(\frac{\partial d_+}{\partial \sigma} - \frac{\partial d_-}{\partial \sigma}) \\ &= SN'(d_+)\sqrt{T-t} \end{aligned} \tag{2.24}$$

$$\therefore \nu_{call} = \nu_{put}$$

"

2.14.5 Rho

Definition 2.14.6

"[Rho] Rho is the option's sensitivity to small changes in the risk-free interest rate. The rho of an option is the rate of change of its price C with respect to the interest rate r .

$$\rho = \frac{\partial C}{\partial r} \tag{2.25}$$

"

"In practice r is usually set equal to the risk-free rate for a maturity equal to the option's maturity. This means that a trader has exposure to movements in the whole term structure when the options in the trader's portfolio have different maturities. We know

$$d_- = d_+ - \sigma\sqrt{T-t} \Rightarrow d_+ - d_- = \sigma\sqrt{T-t} \Rightarrow \frac{\partial d_+}{\partial r} - \frac{\partial d_-}{\partial r} = 0 \tag{2.26}$$

By using the Black Scholes Model for the European call option , we get

$$\begin{aligned}
\rho_{call} &= \frac{\partial}{\partial r}[SN(d_+) - Ke^{-r(T-t)}N(d_-)] \\
&= SN'(d_+)\frac{\partial d_+}{\partial r} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial r} + K(T-t)e^{-r(T-t)}N(d_-) \\
&= SN'(d_+)(\frac{\partial d_+}{\partial r} - \frac{\partial d_-}{\partial r}) + K(T-t)e^{-r(T-t)}N(d_-) \\
&= K(T-t)e^{-r(T-t)}N(d_-)
\end{aligned} \tag{2.27}$$

Similarly for European put option

$$\begin{aligned}
\rho_{put} &= \frac{\partial}{\partial r}[Ke^{-r(T-t)}N(-d_-) - SN(-d_+)] \\
&= \frac{\partial}{\partial r}[Ke^{-r(T-t)}(1 - N(d_-)) - S(1 - N(d_+))] \\
&= -K(T-t)e^{-r(T-t)}N(-d_-) + SN'(d_+)\frac{\partial d_+}{\partial r} - Ke^{-r(T-t)}N'(d_-)\frac{\partial d_-}{\partial r} \\
&= SN'(d_+)(\frac{\partial d_+}{\partial r} - \frac{\partial d_-}{\partial r}) - K(T-t)e^{-r(T-t)}N(-d_-) \\
&= -K(T-t)e^{-r(T-t)}N(-d_-)
\end{aligned} \tag{2.28}$$

”

2.15 Asian Option

Definition 2.15.1

“[Asian Option] The European or American calls and puts are sometimes called vanilla or even plain vanilla options. Their payoffs depend only on the final value of the underlying asset. Options whose payoffs depend on the path of the underlying asset are called path-dependent or exotic . [12] ”

Asian options are a fundamental type of exotic option. Unlike standard options, the strike price or settlement price of Asian options is based on an average of observations rather than a single observation. As a result, Asian options have lower volatility, making them cheaper compared to European options. They are typically traded on currencies and commodities with low trading volumes. Asian options were first introduced in 1987 when Banker's Trust's Tokyo office used them to price average options on crude oil contracts, which is why they are called "Asian" options. [13]

2.15.1 Mathematical Representation of Asian Options

“There are two types of Asian options: the Average Price Option (fixed strike), where the strike price is set in advance, and the payoff is calculated using the average price of the underlying asset; and the Average Strike Option (floating strike), where the strike price is determined by averaging the price of the underlying asset over the option's duration. The payoff profiles of these options are clearly defined. [9]

Average-price call:

$$\max\{0, average[S_T] - K\}$$

Average-price put:

$$\max \{0, K - average[S_T]\}$$

Average-strike call:

$$\max \{0, S_T - average[K_t]\}$$

Average-strike put:

$$\max \{0, average[K_t] - S_T\}$$

There are many ways to specify the average of the stock prices. Some examples are :
Discrete arithmetic average

$$A = \frac{1}{n} \sum_{i=1}^{n} S_{t_i} \tag{2.29}$$

Discrete geometric average

$$G = (\prod_{i=1}^{n} S_{t_i})^{\frac{1}{n}} \tag{2.30}$$

Continuous arithmetic average

$$A = \frac{1}{T-t} \int_t^T S_u du \tag{2.31}$$

Continuous geometric average

$$G = \exp(\frac{1}{T-t} \int_t^T \ln S_u du) \tag{2.32}$$

There are n prices measured at the time points

$$0 \leq t_1 < t_2 < ... < t_n \leq T$$

Usually, these time points are equidistant, e.g. weekly averaging over 1 year. "

2.16 Sensitivity Analysis

Sensitivity analysis is a method that is used to assess how changes in underlying independent variables affect the growth or decay of the over-arching dependent variables. This method helps establish how the input and output variables of a model are related to one another. Sensitivity analysis is vital for decision-making as it reveals how uncertainties or changes in input parameters might affect the final results.

In this context, the main six factors influence valuation of options. A model is thus required, as it enables traders to predict, assess, and understand how various forces impact an option's value. Since all factors except the strike price (K) are dynamic, it is essential for option traders to understand how these factors interact. These factors are independent of each other, thus they can change simultaneously, resulting massive changes that cannot easily be pinpointed to a single factor. In subsequent chapters, we will explore the Greek letters, which help quantify the sensitivities of these variables in relation to one another.

2.17 Numerical Methods

Numerical methods are techniques used to approximate mathematical processes when analytical solutions are not feasible. These methods aim to provide an approximate solution or, at the very least, define the range within which the solution lies. In finance, key themes include time, uncertainty, and information. Probability and statistics are essential tools for dealing with basic forms of uncertainty. When it comes to fairly pricing financial instruments like derivatives, numerical methods become necessary because arbitrage arguments often result in pricing equations that take the form of partial differential equations (PDEs). Although some PDEs can be solved analytically to produce closed-form pricing formulas, numerical methods are crucial for approximating these formulas. Essentially, there are three primary numerical approaches to pricing derivatives:

1. Solving the PDE using finite difference schemes
2. Using a randomized sampling simulation (Monte Carlo Simulations)
3. n-step trees, such as binomial and trinomial trees

In the coming chapters we will discuss each in detail.

CHAPTER 3

Option Pricing Methods

3.1 Introduction

Numerical methods are techniques used to approximate mathematical processes when analytical solutions are not possible. These methods provide approximate solutions or, at the very least, help identify the range within which a solution exists. In finance, the key themes are time, uncertainty, and information. Probability and statistics are essential tools to address the most basic forms of uncertainty. To fairly price a financial instrument, such as a derivative, numerical methods are often required because arbitrage arguments lead to pricing equations in the form of partial differential equations. While some of these equations can be solved analytically to produce a closed-form pricing formula, numerical approaches are crucial for approximating such formulas. There are three primary numerical methods for pricing derivatives: “

1. Solution to a partial differential equation, e.g., by finite difference approximations
2. Monte Carlo Simulation
3. Binomial or Trinomial Lattices

” Accurately pricing options is crucial for both trading and risk management. This chapter delves into several advanced techniques for option pricing, each with its unique strengths and applications.

3.2 Binomial tree

A binomial tree is a graphical representation used in finance to model the potential fluctuations in the value of the underlying asset over time. Constructed by branching out at each time step, each branch represents a possible price change. The tree is utilized in option pricing and risk management to find the price of options at different points of time. By considering all possible price paths, the binomial tree provides a method to visualize and analyze potential investment outcomes. It is similar to a decision tree where each node has two branches, analogous to decisions leading to two possible outcomes. Just as a tree branches into multiple paths, a binomial tree represents various possible price movements or states in a financial model. [1, 4, 5]

The concept of a binomial tree originated in mathematics, specifically in the study of probability and finance. This model was first proposed by William Sharpe in 1978 [3] and formalized by Cox, Ross, and Rubinstein in 1979 [1] and by Rendleman and Bartter in the same year. [2]

3.2.1 No-Arbitrage Argument

Definition 3.2.1

An arbitrage portfolio is a portfolio that guarantees a profit by trading in the market with probability 1.

An arbitrage portfolio is thus a deterministic money-making mechanism, and its existence indicates a significant mispricing in the market [6]. The term "no arbitrage argument" refers to the principle in finance and economics that in an efficient market, there should be no opportunity for risk-free profits. It implies that if there were a way to make a profit without any risk, market forces would quickly eliminate that opportunity. [4]

3.2.2 Risk-Neutral Valuation

Risk-neutral valuation is the technique used in finance to calculate the price of an investment or financial device while assuming that investors are ignorant of risk. The approach values assets based on their expected return, ignoring the risk associated with them. By using this technique, investors can determine the fair value of options, derivatives, and other financial products without considering the specific risk preferences of market participants.

A risk-neutral universe has two main characteristics that make the pricing of derivatives easier: "

1. The expected return on a stock (or any other investment) is the risk-free rate.

2. The discount rate used for the expected payoff on an option (or any other instrument) is the risk-free rate.

" Risk-neutral valuation has its limitations. Assuming that investors are risk takers will not always reflect reality. Additionally, this method may not be suitable for assets with non-linear payoffs or complex structures, limiting its applicability in certain scenarios. However, risk-neutral valuation simplifies complex financial calculations by assuming a risk-free rate of return, making it easier to compare different investment opportunities. It also provides a straightforward method for pricing financial derivatives. [4]

3.2.3 Binomial Option Pricing Model

The continuous random walk that is called the Brownian motion, is approximated using similar but small steps, with the following characteristics: "

1. The asset price S changes only at discrete times δt, $2\delta t$, $3\delta t$, ..., up to $M\delta t = T$, the maturity date of the derivative security. δt denotes the small but finite time-step between movements in the asset price.

2. If the asset price is S^m at time $m\delta t$, then at time $(m+1)\delta t$ it may take one of only two possible values, $uS^m > S^m$ or $dS^m < S^m$. Thus, during a single time-step, the asset price may move from S up to uS or down to dS.

3. The probability, p, of S moving up to uS is known (as is the probability $1-p$ of S moving down to dS). [5]

"

3.2.4 One Step Binomial Model

Consider unit stock with price S_0 and an option on the stock with a present price of f. Assume that the option matures at time T and that the option can increase from S_0 to a higher value, uS_0, where $u > 1$, or decrease from S_0 to a lower value, dS_0, where $d < 1$. For the rising movement of the stock price, the equivalent price of option is f_u, and for the down movement, the corresponding option price is f_d. Considering a long position in Δ amount of shares and a short position in unit option is, we consider a riskless portfolio, implying it has the same probability of rising as it has for declining, implying [4]

$$S_0u\Delta - f_u = S_0d\Delta - f_d \tag{3.1}$$

Then the value of Δ for a riskless portfolio is

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d} \tag{3.2}$$

Since this is riskless, opportunities for arbitrage cannot be present. Thus, the portfolio earns the risk-free interest rate, denoted by r. If the current price of the portfolio is "$(S_0u\Delta - f_u)e^{-rT}$" and the cost of setting up the portfolio is "$S_0\Delta - f$", then the no-arbitrage argument gives "

$$S_0\Delta - f = (S_0u\Delta - f_u)e^{-rT}$$

or

$$f = S_0\Delta(1 - ue^{-rT}) + f_ue^{-rT}$$

Substituting from equation (3.2) for Δ, we obtain

$$f = S_0\left(\frac{f_u - f_d}{S_0u - S_0d}\right)(1 - ue^{-rT}) + f_ue^{-rT}$$

or

$$f = \frac{f_u(1 - de^{-rT}) + f_d(ue^{-rT} - 1)}{u - d}$$

or

$$f = e^{-rT}[pf_u + (1 - p)f_d] \tag{3.3}$$

where

$$p = \frac{e^{rT} - d}{u - d} \tag{3.4}$$

” Here, this value of p is considered the probability the price rises, because for a specific value of p, the portfolio sees average, risk-free rate of growth. Hence, the probability of a decline in price is “$1 - p$.” “” [4]

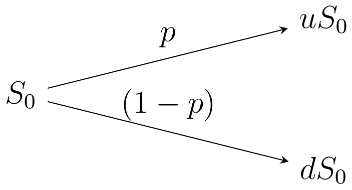


Figure 3.1: One Step Binomial Model

3.2.5 Two Step Binomial Model

For two time steps, divide the maturity date T into two intervals of equal length Δt. For each step, the stock price either rises up to u or declines to d of original value. [4]
“”

The price of the portfolio can be obtained from the value of a one-step model after each time step. By iterating this process backward through each node, it is possible to calculate the current option price.

3.2.6 General Model

“The general model for binomial option pricing is derived from the two-step model. Suppose that the maturity date T is divided into N time steps, each of length

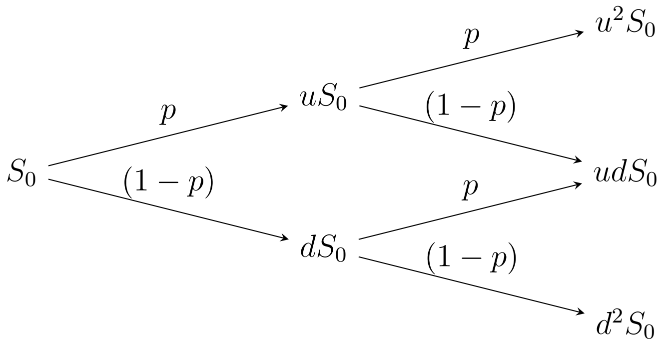


Figure 3.2: Two Step Binomial Model

$\Delta t = \frac{T}{N}$. During each time step, the stock price either moves up to u times or down to d times. For a call option, the value at the final step can be obtained by

$$C_T = \max(S_T - K, 0)$$

Here, S_T denotes the final stock price at time T and K the strike price. For any i, the price is

$$C_i = e^{-r\Delta t}[pC_u + (1-p)C_d]$$

This result can be recursively solved backward for all nodes of the tree to find the price at the initial node C_0. [5]"

3.3 Trinomial tree

"The trinomial tree method, introduced by Phelim Boyle in 1986, extends the binomial options pricing model [7]. This method divides the timeframe into intervals of Δt, and assumes that the stock price can either increase, decrease, or remain unchanged [4]. "

3.3.1 Trinomial Option Pricing Model

Initially, let us assume the price of the stock is S_0 with a probability p_u that it will increase by a factor of u to become S_0u in the next interval Δt. Similarly, there's a probability p_d that the price will decrease by a factor of d to become

S_0d. There's also a probability p_m that the stock will remain the same, making the value S_0m. After interval Δt at the rate r is, the expected return is:"

$$S_0e^{r\Delta t} = p_uS_0u + p_mS_0m + p_dS_0d$$

Simplified, this equation becomes:

$$e^{r\Delta t} = p_uu + p_mm + p_dd \tag{3.5}$$

From the variance, we derive:

$$u^2p_u + m^2p_m + d^2p_d - e^{2r\Delta t} = \sigma^2\Delta t \tag{3.6}$$

And by considering the property of probabilities:

$$p_u + p_m + p_d = 1 \tag{3.7}$$

Solving these three equations, the unknown probabilities are found to be:

$$p_u = \frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}$$

$$p_d = \frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}$$

$$p_m = 1 - p_u - p_d$$

Where:

$$u = e^{\sigma\sqrt{2\Delta t}}, \quad d = e^{-\sigma\sqrt{2\Delta t}}, \quad m = 1$$

[4]

" "Procedures for pricing options using a trinomial tree is closely related to that of a binomial tree. If the stock price tree is constructed and the payoff from option at maturity time T is calculated, for a call option it is $\max(S_T - K, 0)$, and for a put option it is $\max(K - S_T, 0)$. The following backward induction algorithm is then applied, where i represents the time position and j represents the space position:

$$C_{i,j} = e^{-r\Delta t}\left[p_uC_{i+1,j+1} + p_mC_{i+1,j} + p_dC_{i+1,j+1}\right]$$

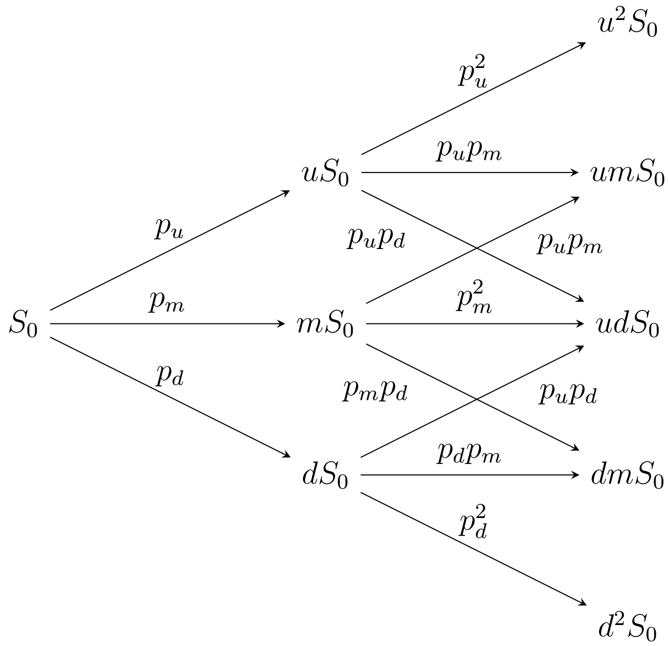


Figure 3.3: Trinomial Model Representation of Stock Price Movements

If the option is American-style and can be exercised before the expiry date T with the same strike price, the backward recursion is as follows: For a call option:

$$C_{i,j} = \max\left(S_{i,j} - K, e^{-r\Delta t}\left[p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j+1}\right]\right)$$

For a put option:

$$C_{i,j} = \max\left(K - S_{i,j}, e^{-r\Delta t}\left[p_u C_{i+1,j+1} + p_m C_{i+1,j} + p_d C_{i+1,j+1}\right]\right)$$

[8]"

3.3.2 Comparison: Binomial vs Trinomial Trees

The trinomial model, with its additional degree of freedom, requires more computational effort than the binomial model. However, a trinomial tree can be built in order to recombine and achieve better measures of accuracy as a binomial tree with lesser time steps [9]. The third choice makes the trinomial model more realistic, as it accounts for the possibility of the price of the underlying asset remaining unchanged over a period, say a month or a year. For vanilla options, as we increase number of steps, the convergence occurs rapidly, making the binomial model preferable because of its simpler working principle. For exotic options, the trinomial model often provides more stability and accuracy, regardless of step size.

3.4 Monte Carlo Simulation for Pricing Options

Monte Carlo simulations are essentially a techniques for evaluating integrals. Although it may not always be the most efficient or quickest method for solving a particular problem, it is straightforward to implement and can be applied to a diverse array of issues. These methods depend on repeated random sampling to acquire numerical results. The core idea is to leverage randomness to address problems that could be deterministic in nature. [13] While Monte Carlo methods can differ, they generally adhere to a specific pattern:

1. Define the domain of all possible inputs
2. Sample random results from this domain
3. Keep generating random samples to calculate the probabilities for each outcome
4. Aggregate the results

The working procedure of a Monte Carlo process that aims to price derivatives, which could be simple or complex in nature, is to repeatedly sample a finite but large number of walks or paths, then accumulate the derivative payoffs that come from each path, dividing by the total number of paths to get an average. [15]

3.4.1 MC Simulation for Pricing Options

When applied to option valuation, Monte Carlo simulation utilizes the risk-neutral valuation principle. As it samples multiple paths, it estimates the payoff that is expected in a risk-neutral environment, by using the risk-free rate as discounting factor, the payoff is discounted to present. This is used for contracts or derivatives reliant on a single market variable that yields some payoff after a maturity time. “

3.4.2 Algorithm

Assuming that interest rates are constant, we can value the derivative as follows:

1. Sample a random path for S in a risk-neutral world.
2. Calculate the payoff from the derivative.
3. Repeat steps 1 and 2 to get many sample values of the payoff from the derivative in a risk-neutral world.
4. Calculate the mean of the sample payoffs to get an estimate of the expected payoff in a risk-neutral world.
5. Discount this expected payoff at the risk-free rate to get an estimate of the value of the derivative.

" We can create a random path using Geometric Brownian Motion. These generated samples are then utilized to calculate the statistical price of the option. Each simulation produces a path for the underlying asset, allowing us to use a discrete time approach to compute the arithmetic average for each sample. Subsequently, we can apply the law of large numbers, which asserts that with a sufficient number of samples, the mean of all sampled averages will converge to the statistical average of the underlying asset's path. [4]
For a non dividend paying stock following **GBM**, it is know that "

$$S_t = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right),$$

" where B_t is a Brownian motion, and its increments are distributed as a normal distribution which has a zero mean and the variance the size of time step, which means "

$$B_t \sim N(0, t),$$

" and "

$$\log\left(\frac{S_t}{S_0}\right) \sim N\left(\left(r - \frac{1}{2}\sigma^2\right) t, t\sigma^2\right).$$

" To obtain the price, we set "

$$S_t(i) = S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2\right) t + \sigma\sqrt{t}Z(i)\right)$$

” here “$Z(i) \sim N(0,1), i = 1,2,3,\ldots,n$”, considered to be independent. Using the law of large numbers stated “

$$M = \frac{1}{n}\sum_{i=1}^{n} S_t(i) \to E\left[S_t\right], \text{ as } n \to \infty.$$

”

Variance for the estimator must be found as well, which is found as M

$$\mathbb{V}\text{ar}(M) = \mathbb{V}\text{ar}\left(\frac{1}{n}\sum_{i=1}^{n} S_t(i)\right) = \frac{1}{n^2}\mathbb{V}\text{ar}\left(\sum_{i=1}^{n} S_t(i)\right) = \frac{\mathbb{V}\text{ar}\left(S_t\right)}{n}$$

and so we notice that, as n goes to infinity, that is, as we sample infinite times, M goes to zero, which shows the characteristics of Monte Carlo simulation. [14, 13]

3.4.3 Pseudocode

Here is the relevant Monte Carlo pseudocode for European option :

set $sum = 0$
for $i = 1$ to n
generate $S(T)$
set $sum = sum + \max(0, S_T - K)$
end
set $\widehat{C^K} = \exp(-rT)sum/n$

3.4.4 Advantages

The primary strengths of Monte Carlo simulation can be found in its versatility; it is used reliably if the payoff is influenced by the entire path of the variable S, not just its final value. For instance, it can also work with average values of S from time 0 to time T. Payoffs are realized at various points within the derivative's lifespan, not only at the end. This method can accommodate any stochastic process for S and can be extended to cases where the derivative's payoff depends on multiple underlying market variables. However, the disadvantages include its high computational demands and difficulty in handling early exercise opportunities. [4]

3.5 Black Scholes Model

"In the early 1970s, Fischer Black, Myron Scholes, and Robert Merton achieved a major breakthrough in the pricing of European stock options. This was the development of what has become known as the Black–Scholes–Merton (or Black–Scholes) model. The model has had a huge influence on the way that traders price and hedge derivatives. In 1997, the importance of the model was recognized when Robert Merton and Myron Scholes were awarded the Nobel prize for economics. Sadly, Fischer Black died in 1995; otherwise he too would undoubtedly have been one of the recipients of this prize. [4, 16]
By adjusting the proportion of the underlying asset and option continuously in a portfolio, Black and Scholes demonstrated that investors can create a riskless hedging portfolio where the risk exposure associated with the stochastic asset price is eliminated. In an efficient market with no riskless arbitrage opportunity, a riskless portfolio must earn an expected rate of return equal to the riskless interest rate. [17] "

3.5.1 Riskless Hedging Principle

"We illustrate how to use the riskless hedging principle to derive the governing partial differential equation for the price of a European call option. In their seminal paper (1973), Black and Scholes made the following assumptions on the financial market. "

(i) Continuous placement of trades within given time

(ii) Riskless interest rate, r is known to be constant over the entire lifespan

(iii) Non-dividend paying asset

(iv) Transaction costs, taxes and other costs are neglected

(v) Assets can be divided perfectly

(vi) Proceeds are permitted to be used fully, no backlash for short-selling.

(vii) Arbitrage opportunities do not exist

"The stochastic process of the asset price S_t is assumed to follow the Geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{3.8}$$

where μ is the expected rate of return, σ is the volatility and W_t is the standard Brownian process.Now, we need a new tool, namely, Itô's Lemma which determines how a function of a stochastic variable varies with changes of the independent variable.

3.5.2 Itô's Lemma

We wish to find the estimates given here by using theory of Brownian motion with the increment dW over some infinitely small time interval dt. We can now state Itô's lemma: Suppose that the value of a variable X follows the Itô's process [4]:

$$\begin{aligned} dX(t) &= a(X,t)dt + b(X,t)dW \\ X(0) &= X \end{aligned} \tag{3.9}$$

" Assuming that $u(x,t)$ is a smooth function of two independent variables x and t. Then

$$du = \left[\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}b(x,t)^2\right]dt + b(x,t)\frac{\partial u}{\partial x}dW \tag{3.10}$$

The proof of this lemma is based on the Taylor expansion

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial t}dt + \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial t}dtdx + \frac{\partial^2 u}{\partial t^2}dt^2\right] + \dots \tag{3.11}$$

We let $dx = dX$, place it for dX in the stochastic differential equation and collect terms of order $\sqrt{dt}$ We take $dW^2 = dt$ and that $dWdt$ is negligible. The dominant term is $b(x,t)\frac{\partial u}{\partial x}dW$ since dW behaves like $\sqrt{dt}$. If we apply Itô's lemma to $u = lns$ where

$$dS(t) = \mu S dt + \sigma S dW \tag{3.12}$$

Then (with $X = S, a(S,t) = \mu S, b(S,t) = \sigma S$) we find

$$\frac{\partial u}{\partial s} = \frac{1}{s}, \quad \frac{\partial f^2}{\partial s^2} = -\frac{1}{s^2}, \quad \frac{\partial u}{\partial t} = 0$$

so that

$$du = \left(\mu - \frac{\sigma^2}{2}dt\right) + \sigma dW \tag{3.13}$$

This is the equation for Brownian motion which has the solution

$$\begin{aligned} u(t) - u(t_0) &= \left(\mu - \tfrac{\sigma^2}{2}\left(t - t_0\right) + \sigma\left(W(t) - W\left(t_0\right)\right)\right. \\ \Rightarrow u(t) &= u\left(t_0\right) + \left(\mu - \tfrac{\sigma^2}{2}\left(t - t_0\right) + \sigma\left(W(t) - W\left(t_0\right)\right)\right. \end{aligned} \tag{3.14}$$

so that

$$S(t) = S\left(t_0\right) e^{\left(\mu - \frac{\sigma^2}{2}(t - t_0) + \sigma(W(t) - W(t_0))\right)} \tag{3.15}$$

Let V be the value of a put or call written on an underlying asset with value S_t at time t. We assume that V depends on the two independent variables S and t, where S itself moves randomly. Then according to Itô's lemma, V changes over the infinitesimal time interval dt according to

	dt	dW
dt	0	0
dW	0	dt

$$\begin{aligned} dV &= \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\left[\frac{\partial^2 V}{\partial S^2}dS^2 + 2\frac{\partial^2 V}{\partial S \partial t}dtdS + \frac{\partial^2 V}{\partial t^2}dt^2\right] \\ &= \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\left[(\mu S dt + \sigma S dW)^2\right] + \frac{\partial^2 V}{\partial S \partial t}dt(\mu S dt + \sigma S dW) \\ &= \frac{\partial V}{\partial S}[\mu S dt + \sigma S dW] + \frac{\partial V}{\partial t}dt + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 dt \\ &= \left(\frac{\partial V}{\partial t}dt + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S \frac{\partial V}{\partial S}dW \end{aligned}$$

So, the Itô's lemma for two variables S and t be:

$$\boxed{dV = \left(\frac{\partial V}{\partial t}dt + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW} \tag{3.16}$$

"

3.5.3 Black Scholes PDE

"Let us now assume that we have a portfolio consisting of a long position on one option of value V and short position on Δ shares of the underlying asset where Δ is as yet undetermined.The value of the portfolio at any time t is

$$\Pi = V(S,t) - \Delta S \tag{3.17}$$

Over the time interval dt the gain in the value of the portfolio is

$$d\Pi = dV(S,t) - \Delta dS \tag{3.18}$$

i.e.

$$d\Pi = \left(\frac{\partial V}{\partial t}dt + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW - \Delta(\mu S dt + \sigma S dW) \tag{3.19}$$

We now observe that if $\Delta = \frac{\partial V}{\partial S}$ then the stochastic terms cancel so that the gain is deterministic.

$$d\Pi = \left(\frac{\partial V}{\partial t}dt + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW - \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW) \tag{3.20}$$

i.e.

$$d\Pi = \left(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt \tag{3.21}$$

If the gain in the value of Π is deterministic, then it cannot be more or less than the gain in the value of the portfolio were it invested at the risk free interest rate r according to the no arbitrage principle. It follows that

$$\frac{d\Pi}{\Pi} = rdt$$
$$\Rightarrow d\Pi = \Pi rdt$$
$$\Rightarrow d\Pi = \left(V - \frac{\partial V}{\partial S}S\right) rdt$$

Using value of $d\Pi$ from 3.21 we find that

$$\frac{\partial V}{\partial t}dt + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = \left(V - \frac{\partial V}{\partial S}S\right) r \tag{3.22}$$

Hence,

$$\frac{\partial V}{\partial t} + \frac{1}{2}a^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{3.23}$$

This is the famous Black Scholes Partial Differential Equation for the value of an option." [4]

3.5.4 Assumptions of BSM

Some assumptions were made for simplification of the derivation of **BSM** equation:

(i) The geometric Brownian motion $dS = \mu Sdt + \sigma SdW$ describes the value of asset

(ii) Selling or buying options are instantenous and can be done at any time, since δ can take on any value over time

(iii) The derivative $\frac{\partial V}{\partial S}$ is assumed to be a smooth function of S; thus the number of shares in Π can be made fractional. This implies that the portfolio can consist of fractions of all kind.

(iv) The option is European in nature, that is, can only be exercised at maturity.

(v) The riskless rates and the market volatility are known throughout the entire time and are only temporal, not assumed to be stochastic.

(vi) There is no obstruction or bias from buying or selling of any type of option.

(vii) Short-selling incurs no penalty.

"The resulting equation is a mathematical model for the value of an option. It holds for all options depending on S and t as long as S is modeled by the equation of geometric Brownian motion $dS = \mu S dt + \sigma S dW$. [4, 5] "

3.5.5 European Option Pricing Using BSM

"When r and σ are constants, the exact explicit solution for the European call option with strike price K ,maturity time T and zero dividend yield is [4]

$$C(S,t) = SN(d_1) - Ke^{-r(T-t)}N(d_2) \tag{3.24}$$

where $N(.)$ is the cumulative distribution function for a standard normal random variable, given by

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy \tag{3.25}$$

Here

$$d_1 = \frac{\log(S/K) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{3.26}$$

and

$$d_2 = \frac{\log(S/K) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{3.27}$$

The solution for a put option is given by

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1) \tag{3.28}$$

The Δ for a European call is

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) \tag{3.29}$$

The Δ for a European put is

$$\Delta = \frac{\partial P}{\partial S} = N(d_1) - 1 \tag{3.30}$$

When r and σ are constants, the exact explicit solution for the European call option with strike price K ,maturity time T and dividend yield q is [4]

$$C(S,t) = Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{3.31}$$

Here

$$d_1 = \frac{\log(S/K) + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \tag{3.32}$$

and

$$d_2 = \frac{\log(S/K) + (r - q - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{3.33}$$

The solution for a put option is given by

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) \tag{3.34}$$

"

3.5.6 Initial & Boundary Condition

Since the Black-Scholes equation involves only the first derivative with respect to t, but both the first and second derivatives with respect to the asset value S, we require one boundary condition concerning time (specifically, the value at the right boundary or the maturity time $t = T$) and two boundary conditions concerning S (stemming from the behavior of the option at $S = 0$ and as $S \to \infty$). Considering a call option, the boundary condition at maturity is $C(S,T) = \max(S - K, 0)$ for all S. When the underlying asset becomes worthless, option is priced at zero. Therefore, $C(0,t) = 0$ for all t. Conversely, if S becomes very large, the option is sure to be exercised, which implies that the exercise price K can become negligible S. As a result, the option's value will closely match the value of the underlying asset itself.

The boundary conditions could therefore be found as "

$$\frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0, \quad 0 \le S < \infty, 0 \le t \le T \tag{3.35}$$

$$\begin{aligned} C(S,T) &= \max(S-K,0) & \quad 0 \le S < \infty \\ C(0,t) &= 0 & \quad 0 \le t \le T \\ C(S,t) &= S & \quad \text{as } S \to \infty \end{aligned} \tag{3.36}$$

"

Similarly for a put option from the "Put-Call Parity" Formula the expiry time condition is "$P(S,T) = max(K - S, 0)$" for all S. In case of the underlying option becoming worthless, the value of the option would be the adjusted strike price using with the risk free interest rate. It is trivial that as S becomes grows, the option deteriorates to nothing.

The boundary conditions could therefore be found as "

$$\frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} - rP = 0, \quad 0 \leq S < \infty, 0 \leq t \leq T \tag{3.37}$$

$$\begin{aligned} P(S,T) &= \max(K-S,0) & 0 \leq S < \infty \\ P(0,t) &= Ke^{-r(T-t)} & 0 \leq t \leq T \\ P(S,t) &= 0 & \text{as } S \to \infty \end{aligned} \tag{3.38}$$

"

3.6 PDE Formulation of Asian Option

In this section, we derive the differential equation that governs the price of an Asian option using the Black-Scholes method. The price function $V(S, A, \tau)$ depends on the time to expiry τ and the two state variables: asset price S and the average asset value A. When constructing an option pricing model through the partial differential equation approach, it is convenient to omit the subscript t in the state variables.

Suppose we define the average price of the asset as "

$$A = \int_0^t f(S,u)du \tag{3.39}$$

where $f(S,t)$ is chosen according to requirements. For example, $f(S,t) = \frac{1}{t}S$ corresponds to continuous arithmetic average, $f(S,t) = \exp\left(\frac{1}{n}\sum_{i=1}^{n} \delta\left(t - t_i\right) \ln S\right)$ corresponds to discrete geometric average, etc. Suppose $f(S,t)$ is a continuous time function, then by the mean value theorem

$$\begin{aligned} dA = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} f(S,u)du = \lim_{\Delta t \to 0} f\left(S, u^*\right) dt = f(S,t)dt, \\ where \quad t < u^* < t + \Delta t, \end{aligned} \tag{3.40}$$

"

"So dA is not stochastic, rather it is deterministic. Hence, a riskless hedge for the Asian option requires only eliminating the asset-induced risk, so the exposure on the Asian option can be hedged by holding an appropriate number of units of the underlying asset.

Consider a portfolio that contains one unit of the Asian option and $-\Delta$ units of the underlying asset. We then choose Δ such that the stochastic components associated with the option and the underlying asset cancel each other out. Assume the asset price dynamics to be given by

$$\frac{dS}{S} = \mu dt + \sigma dZ \tag{3.41}$$

where Z is the standard Brownian process, q is the dividend yield on the asset, μ and σ are the expected rate of return and volatility of the asset price, respectively. Let $V(S, A, t)$ denote the value of the Asian option and let Π denote the value of the above portfolio. The portfolio value is given by

$$\Pi = V(S, A, t) - \Delta S, \tag{3.42}$$

and assuming Δ to be kept instantaneously "frozen," the differential of Π is found to be

$$d\Pi = \frac{\partial V}{\partial t}dt + f(S,t)\frac{\partial V}{\partial A}dt + \frac{\partial V}{\partial S}dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}dt - \Delta dS - \Delta qSdt. \tag{3.43}$$

The last term in the above equation corresponds to the contribution of the dividend dollar amount from the asset to the portfolio value. As usual, we choose $\Delta = \frac{\partial V}{\partial S}$ so that the stochastic terms containing dS cancel. The absence of arbitrage dictates

$$d\Pi = r\Pi dt, \tag{3.44}$$

where r is the riskless interest rate.

Putting the above results together, we obtain the following governing differential equation for $V(S, A, t)$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial A} - rV = 0. \tag{3.45}$$

The equation is a degenerate diffusion equation since it contains diffusion term corresponding to S only but not A. The auxiliary conditions in the pricing model depend on the specific details of the Asian option contract. [17] ”

3.6.1 Continuously Monitored Geometric Averaging Options

"Here we derive analytic price formulas for the European Asian options whose terminal payoff depends on the continuously monitored geometric averaging of the underlying asset price. We take time zero to be the initiation time of the averaging period, t is the current time and T denotes the expiration time. We define the continuously monitored geometric averaging of the asset price S_u over the time period $[0, t]$ by

$$G_t = \exp\left(\frac{1}{t}\int_0^t \ln S_u du\right) \tag{3.46}$$

The terminal payoff of the fixed strike call option and floating strike call option are, respectively,

$$\begin{aligned} c_{fix}\left(S_T, G_T, T; K\right) &= \max\left(G_T - K, 0\right) \\ c_{f\ell}\left(S_T, G_T, T\right) &= \max\left(S_T - G_T, 0\right), \end{aligned} \tag{3.47}$$

where K is the fixed strike price. We illustrate how to use the risk neutral valuation approach to derive the price formula of the European fixed strike Asian call option. [17] "

3.6.2 European Fixed Strike Asian Call Option

"We assume the existence of a risk neutral pricing measure Q under which discounted asset prices are martingales, implying the absence of arbitrage. Under the measure Q, the asset price dynamics follows

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dZ_t, \tag{3.48}$$

where Z_t is Q-Brownian and q is the constant dividend yield of the underlying asset. For $0 < t < T$, the solution of the above stochastic differential equation is given by

$$\ln S_u = \ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)(u - t) + \sigma\left(Z_u - Z_t\right). \tag{3.49}$$

Substituting the above relation into 3.46 and performing the integration, we obtain

$$
\begin{aligned}
\ln G_T =& \frac{t}{T}\ln G_t + \frac{1}{T}\left[(T-t)\ln S_t + \left(r-q-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2}\right] \\
&+\frac{\sigma}{T}\int_t^T (Z_u - Z_t)\,du.
\end{aligned}
\tag{3.50}
$$

The stochastic term $\frac{\sigma}{T}\int_t^T (Z_u - Z_t)\,du$ can be shown to be Gaussian with zero mean and variance $\frac{\sigma^2}{T^2}\frac{(T-b)^3}{3}$. By the risk neutral valuation principle, the value of the European fixed strike Asian call option is given by

$$
c_{fix}(S_t, G_t, t) = e^{-r(T-t)} E_Q\left[\max(G_T - K, 0)\right], \tag{3.51}
$$

where the expectation is taken under Q conditional on the filtration generated by the Q-Brownian process. We assume the current time t to be within the averaging period. By defining

$$
\bar{\mu} = \left(r-q-\frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2T} \text{ and } \bar{\sigma} = \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}, \tag{3.52}
$$

G_T can be written as

$$
G_T = G_t^{t/T} S_t^{(T-t)/T} \exp(\bar{\mu} + \bar{\sigma}\hat{Z}). \tag{3.53}
$$

where $\hat{Z}$ is the standard normal random variable. Our usual expectation calculations with call payoff:

$$
\begin{aligned}
&E_Q[\max(F\exp(\bar{\mu}+\bar{\sigma}\hat{Z}) - K, 0] \\
&= Fe^{\bar{\mu}+\bar{\sigma}^2/2} N\left(\frac{\ln\frac{F}{K}+\bar{\mu}+\bar{\sigma}^2}{\bar{\sigma}}\right) - KN\left(\frac{\ln\frac{F}{K}+\bar{\mu}}{\bar{\sigma}}\right)
\end{aligned}
\tag{3.54}
$$

We then deduce that [17]

$$
c_{fix}(S_t, G_t, t) = e^{-r(T-t)}\left[G_t^{t/T} S_t^{(T-t)/T} e^{\bar{\mu}+\bar{\sigma}^2/2} N(d_1) - KN(d_2)\right] \tag{3.55}
$$

where

$$
\begin{aligned}
d_2 &= \left(\frac{t}{T}\ln G_t + \frac{T-t}{T}\ln S_t + \bar{\mu} - \ln K\right)/\bar{\sigma}, \\
d_1 &= d_2 + \bar{\sigma}
\end{aligned}
\tag{3.56}
$$

”

CHAPTER 4

Result & Discussion

In the previous chapters, we have discussed European options, the Greek letters , fundamental concepts of Asian options, four methods of option pricing ; Binomial Tree Method, Trinomial Tree Method, Monte Carlo Simulation and the Black Scholes Model. In this chapter we will apply these methods to European option with dividends and without dividends . We will also calculate the Greeks for the European options. Furthermore , we will calculate the value of Asian Options and compare with the European option results.

4.1 European Option Pricing

In order to produce results using numerical methods described in the previous chapters, we will use two datasets, the first of which considers no dividend yield for the underlying stock and the second one considers a dividend yield.

Dataset 1:

"We use the daily data of Tesla (NASDAQ. TSLA) from $2015 - 01$ to $2017 - 07$ and find the historical volatility 23%, we consider the initial stock price \$21.5, risk free interest rate 1% , strike price \$18 and 100 days maturity times, we consider the 252 trading days and zero dividend yield here. [8]

"**Dataset 2 (with dividend):**

"We use the daily data of Tesla (NASDAQ. TSLA) from $2015 - 01$ to $2017 - 07$ and find the historical volatility 23%, we consider the initial stock price \$21.5, risk free interest rate 1% , strike price \$18 and 100 days maturity times, we consider the 252 trading days and 0.1% dividend yield here. [8] "

“

European Option Pricing					
Variable	**Value**	**BTM**	**TTM**	**BSM**	**MCS**
Strike Price	15	6.5594	6.4731	6.9990	6.5644
	17	4.6172	4.6092	4.7713	4.6245
	18	3.7048	3.6489	3.6606	3.7138
	22	1.1054	1.0729	0.7335	1.0579
	24	0.4703	0.4248	0.6701	0.4439
	26	0.1213	0.0747	0.0344	0.1606
Maturity Time	25	3.5178	3.4390	3.5010	3.5210
	75	3.6133	3.6009	3.5964	3.6361
	100	3.7048	3.6824	3.7492	3.7138
	175	3.9966	3.9721	3.9742	3.9592
	250	4.2346	4.1558	4.2187	4.1970
	500	4.8545	4.8320	4.9526	4.8940
Volatility	15%	3.5900	3.5241	3.8459	3.5906
	23%	3.7048	3.6375	3.7784	3.7138
	30%	3.9396	3.9117	3.6178	3.9030
	35%	4.1084	4.0872	4.0807	4.0683
	40%	4.2777	4.2101	3.9533	4.2504
	45%	4.4471	4.3940	4.7293	4.4445
Interest Rate	1%	3.7048	3.6160	3.5699	3.7138
	2%	3.7686	3.7444	3.5477	3.7766
	3%	3.8325	3.8304	3.7515	3.8395
	4%	3.8963	3.8048	4.1313	3.9025
	5%	3.9602	3.9446	3.7925	3.9656
	6%	4.0241	3.9258	4.2018	4.0287
Number of Tree Steps	5	3.7048	3.6104	-	-
	10	3.7018	3.6054		
	50	3.7144	3.6149		
	100	3.7130	3.7091		
	500	3.7139	3.6294		
	1000	3.7139	3.6974		

Table 4.1: Summary of the effect on the price of a European call option without dividends. Values are generated by changing one variable while keeping all the others fixed for each case.

”

“

European Option Pricing with Dividend					
Variable	**Value**	**BTM**	**TTM**	**BSM**	**MCS**
Strike Price	15	6.5535	6.5144	6.5935	6.5585
	17	4.6108	4.6096	4.7786	4.6182
	18	3.6984	3.6501	3.5351	3.7075
	22	1.1018	1.0530	1.3667	1.0543
	24	0.4683	0.4216	0.2274	0.4419
	26	0.1207	0.0775	0.2136	0.1597
Maturity Time	50	3.5161	3.5021	3.4847	3.5192
	75	3.6083	3.5105	3.5327	3.6312
	100	3.6984	3.6277	3.6347	3.7075
	175	3.9867	3.9426	3.9064	3.9492
	250	4.2212	4.1719	4.1643	4.1835
	500	4.8307	4.7513	4.9027	4.8704
Volatility	15%	3.5831	3.5217	3.6967	3.5837
	23%	3.6984	3.6232	3.6139	3.7075
	30%	3.9339	3.9221	3.7169	3.8973
	35%	4.1031	4.0909	3.7991	4.0629
	40%	4.2725	4.2527	4.3131	4.2452
	45%	4.4421	4.3922	4.5716	4.4395
Number of Tree Steps	5	3.6984	3.6627	3.6977	3.7075
	10	3.6954	3.6816	3.6954	3.7075
	50	3.7081	3.6341	3.7081	3.7075
	100	3.7067	3.6496	3.7067	3.7075
	500	3.7076	3.6255	3.7076	3.7075
	1000	3.7076	3.7051	3.7076	3.7075
Interest Rate	1%	3.6984	3.6888	3.8748	3.7075
	2%	3.7622	3.6805	3.6324	3.7703
	3%	3.8261	3.8217	3.7086	3.8332
	4%	3.8899	3.8559	4.0771	3.8962
	5%	3.9538	3.9265	3.9838	3.9592
	6%	4.0177	4.0116	4.1372	4.0224
Dividend Yield	0.1%	3.6984	3.7034	3.7075	3.7075
	0.2%	3.6920	3.6970	3.7028	3.7013
	0.3%	3.6856	3.6906	3.6861	3.6950
	0.4%	3.6793	3.6843	3.6968	3.6887
	0.5%	3.6729	3.6779	3.6770	3.6825
	0.6%	3.6665	3.6715	3.6855	3.6762

Table 4.2: Summary of the effect on the price of a European call option with dividends. Values are generated by changing one variable while keeping all the others fixed for each case.

”

4.1.1 Visualizing option prices (Without dividend)

Below are the graphs of the result found from the sensitivity analysis using **Dataset 1**.

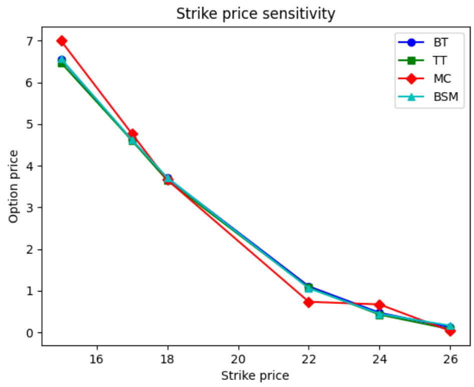


Figure 4.1: Call Price vs Strike Price

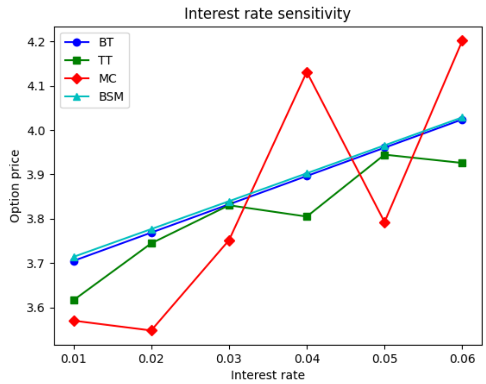


Figure 4.2: Call Price vs Interest

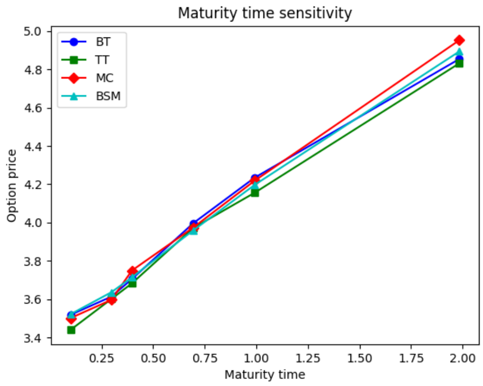


Figure 4.3: Call Price vs Maturity

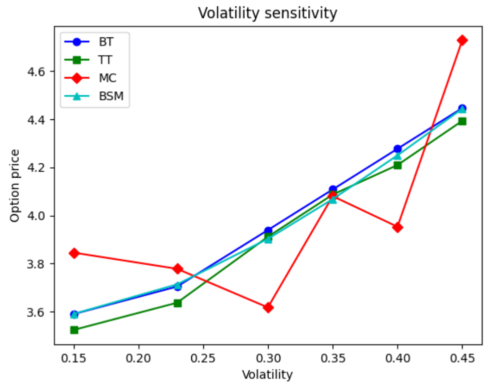


Figure 4.4: Call Price vs Volatility

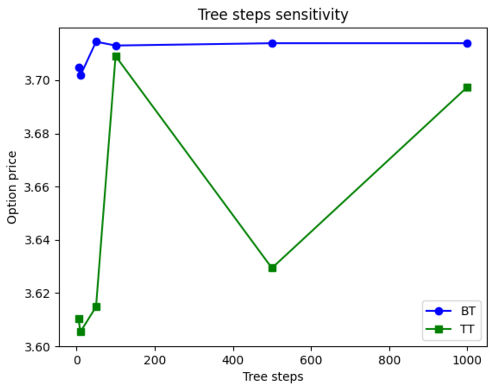


Figure 4.5: Call Price vs TreeSteps

4.1.2 Visualizing option prices (With dividend)

Below are the graphs of the result found from the sensitivity analysis using **Dataset 2**.

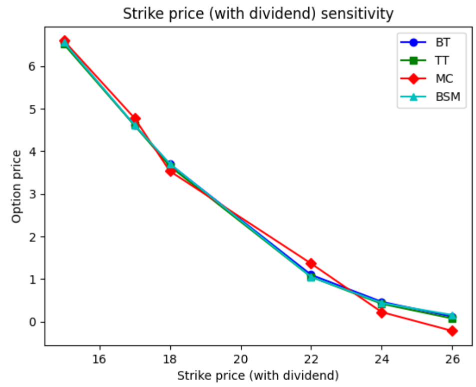


Figure 4.6: Call Price vs Strike Price

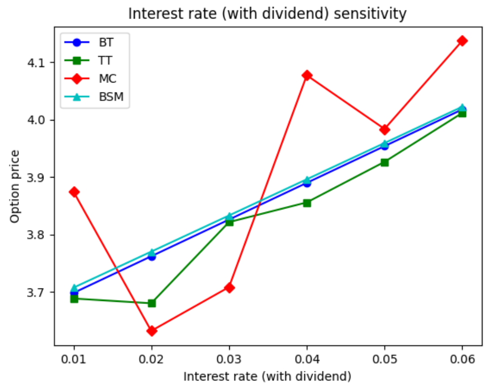


Figure 4.7: Call Price vs Interest

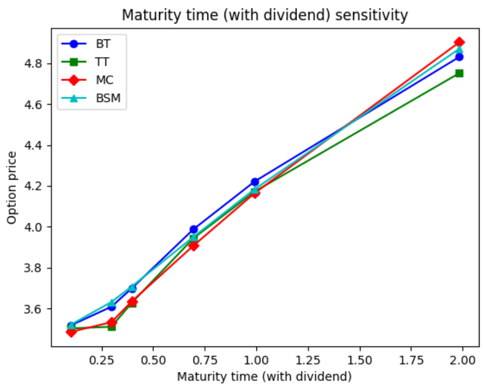


Figure 4.8: Call Price vs Maturity

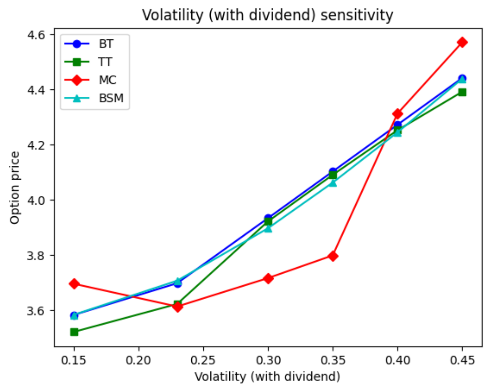


Figure 4.9: Call Price vs Volatility

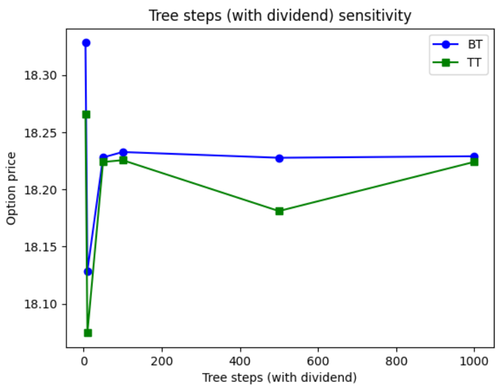


Figure 4.10: Call Price vs TreeSteps

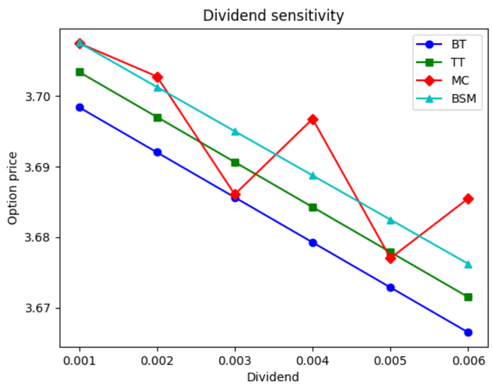


Figure 4.11: Call Price vs Dividends

4.2 Evaluating The Greeks

4.2.1 Delta Δ

For numerical approach we can use finite difference method

$$\Delta = \frac{\partial C}{\partial S} = \frac{C(S+\delta S) - C(S-\delta S)}{2\delta S} \tag{4.1}$$

where δS is a very small change of stock price . We use equation 4.1 to find the Δ by using the lattice tree and Monte Carlo methods we discussed in the previous chapters .
“We use the daily data of Tesla (NASDAQ. TSLA) from $2015-01$ to $2017-07$ and find the historical volatility 23%, we consider the initial stock price \$21.5, risk free interest rate 1% , strike price \$18 and 100 days maturity times, we consider the 252 trading days, zero dividend yield and let’s consider $\delta S = 1$ here. [8] ” “

Evaluating Delta for European Option					
Delta	**Stock Price**	**BTM**	**TTM**	**BSM**	**MCS**
$\Delta = \frac{\partial C}{\partial S}$	10	0.0000	-0.0014	0.0086	0.0001
	21	0.8825	0.8785	0.9120	0.9034
	30	1.0000	0.9951	1.0055	0.9998
	45	1.0000	1.0072	0.9991	1.0000
	57	1.0000	1.0016	0.9958	1.0000
	70	1.0000	1.0090	0.9994	1.0000

Table 4.3: Value of Delta of a European call option for various stock prices.

”

4.2.2 Gamma Γ

Using finite difference we can write“

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{C(S+\delta S) - 2C(S) + C(S-\delta S)}{(\delta S)^2} \tag{4.2}$$

” We derive the numerical approximations for Γ using the same set of information we used in the previous table .Let us consider $\delta S = 1$ here. “

Evaluating Gamma for European Option					
Gamma	**Stock Price**	**BTM**	**TTM**	**BSM**	**MCS**
$\Gamma = \frac{\partial^2 C}{\partial S^2}$	10	0.1000	0.0933	-0.0075	0.0002
	21	0.1332	0.1246	0.0537	0.0538
	30	0.0333	0.0170	0.0097	0.0001
	45	0.0222	0.0089	0.0007	0.0000
	57	0.0175	0.0088	0.0037	0.0000
	70	0.0143	-0.0035	0.0018	0.0000

Table 4.4: Value of Gamma of a European call option for different stock prices.

"

4.2.3 Theta Θ

By using finite difference method we may write "

$$\Theta = \frac{\partial C}{\partial t} = \frac{C(T+\delta t) - C(T-\delta t)}{2\delta t} \tag{4.3}$$

" We derive the numerical approximations for Θ using the same set of information we used in the previous table. Let us consider $\delta t = 15$ days here. "

Evaluating Theta for European Option					
Theta	**Maturity Time**	**BTM**	**TTM**	**BSM**	**MCS**
$\Theta = \frac{\partial C}{\partial t}$	50	0.3148	0.3144	0.3285	0.3239
	75	0.6161	0.6120	0.7421	0.7442
	100	1.1294	1.1264	0.8111	0.8056
	175	0.8710	0.8626	0.8240	0.8196
	250	0.7404	0.7386	0.7847	0.7759
	500	0.5431	0.5344	0.6459	0.6387

Table 4.5: Value of Theta of a European call option for different Maturity Times.

"

4.2.4 Vega ν

By using finite difference method we may write

$$\nu = \frac{\partial C}{\partial \sigma} = \frac{C(T+\delta\sigma) - C(T-\delta\sigma)}{2\delta\sigma} \tag{4.4}$$

We derive the numerical approximations for ν using the same set of information we used in the previous table. Let us consider $\delta\sigma = 5\%$ here. "

Evaluating Vega for European Option					
Vega	**Volatility**	**BTM**	**TTM**	**BSM**	**MCS**
$\nu = \frac{\partial C}{\partial \sigma}$	15%	0.5937	0.5863	0.8187	0.8184
	23%	2.5801	2.5719	2.1959	2.2037
	30%	3.3687	3.3655	3.0602	3.0669
	35%	3.3810	3.3803	3.4730	3.4736
	40%	3.3863	3.3800	3.7578	3.7618
	45%	3.3874	3.3808	3.9757	3.9690

Table 4.6: Value of Vega of a European call option for different Volatility.

"

4.2.5 Rho ρ

By using finite difference method we may write

$$\rho = \frac{\partial C}{\partial \sigma} = \frac{C(r+\delta r) - C(r-\delta r)}{2\delta r} \tag{4.5}$$

We derive the numerical approximations for ρ using the same set of information we used in the previous table .Let us consider $\delta r = 0.5\%$ here.

Evaluating Rho for European Option					
Rho	**Interest rate**	**BTM**	**TTM**	**BSM**	**MCS**
$\nu = \frac{\partial C}{\partial \sigma}$	1%	6.3808	6.3723	6.2661	6.2697
	2%	6.3841	6.3796	6.2791	6.2828
	3%	6.3864	6.3814	6.3032	6.2944
	4%	6.3879	6.3812	6.3130	6.3047
	5%	6.3885	6.3872	6.3159	6.3136
	6%	6.3883	6.3815	6.3194	6.3211

Table 4.7: Value of Vega of a European call option for different Volatility.

4.2.6 Visualizing the Greeks

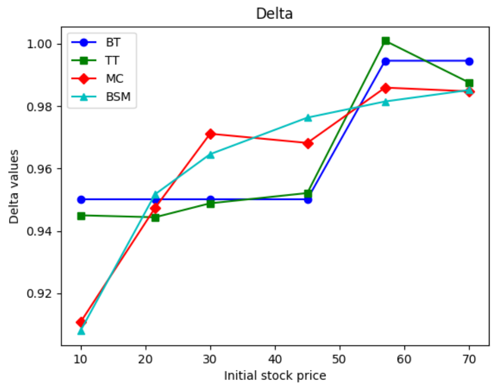


Figure 4.12: Delta vs Stock Price

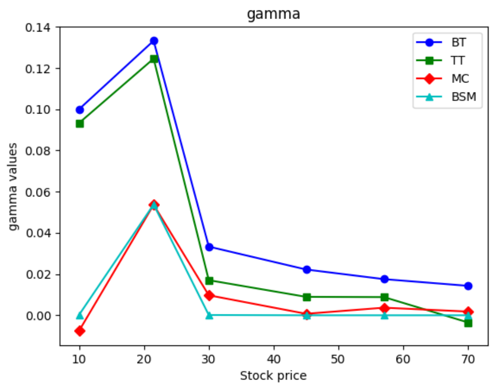


Figure 4.13: Gamma vs Stock Price

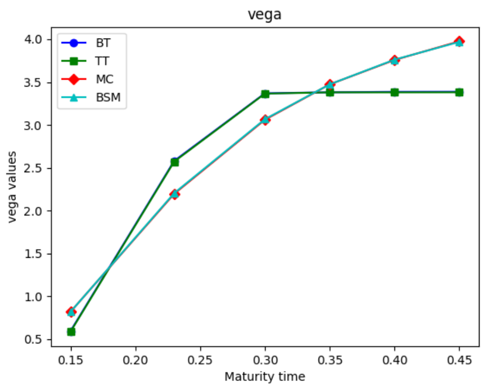


Figure 4.14: Vega vs Volatility

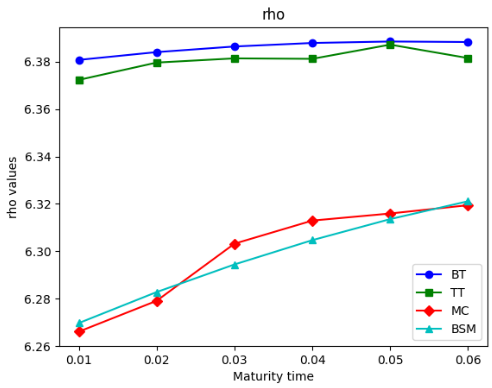


Figure 4.15: Rho vs Interest Rate

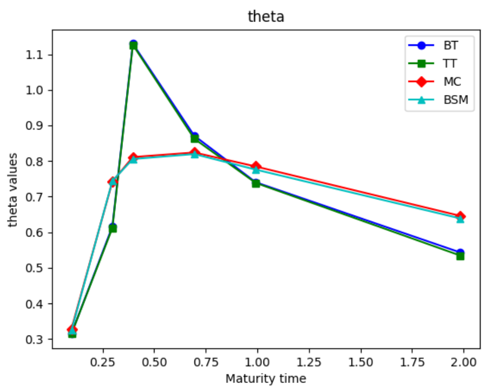


Figure 4.16: Theta vs Maturity Time

From the graphs, we can infer various information about the greeks.

- The value of **Delta** increases with the stock. This implies that as the stock price increases, price of the call option should increase along with it.
- The value of **Gamma** decreases with the stock. This implies that the Delta of the call option will decrease significantly.
- The value of **Vega** increases with the market volaitility. This implies that more volatile conditions of the market will drive the price of the call option upwards.
- The value of **Theta** decrease with time to maturity. This implies that more time to expire for a call option in the case of European variant will cause the call options price to fall.
- The value of **Rho** increases with risk-free interest rate. This implies that the value of the call option increases with increase in risk-free interest rate.

4.3 Asian Option Pricing

We will now determine the call option value of Asian option using the method of the arithmetic average. We will analyze the Asian option prices using two prominent method: Monte Carlo simulatio and the Black Scholes model. These will then be compared with the respective European option price.

Comparison of Asian option with European Option					
Variable	**Value**	**BSM European**	**MC European**	**BSM Asian**	**MCS Asian**
Strike Price	15	6.5644	6.4783	6.4743	6.5045
	17	4.6245	4.7053	4.4834	4.5147
	18	3.7138	3.6385	3.4960	3.5552
	22	1.0579	1.1144	0.5008	0.5046
	24	0.4439	0.4284	0.0834	0.0922
	26	0.1606	0.2420	0.0078	0.0102
Maturity Time	25	0.1522	0.1470	0.0066	0.0075
	75	0.8165	0.7403	0.1369	0.1695
	100	1.1359	1.1350	0.2285	0.2723
	175	1.9971	1.9962	0.5105	0.6785
	250	2.7480	2.6556	0.7708	1.0087
	500	4.7865	4.7680	1.4546	2.0106
Volatility	15%	3.5906	3.5646	3.4863	3.5290
	23%	3.7138	3.6921	3.4960	3.5384
	30%	3.9030	3.8203	3.5325	3.5777
	35%	4.0683	3.9715	3.5783	3.5803
	40%	4.2504	4.2428	3.6388	3.6883
	45%	4.4445	4.4101	3.7117	3.7313
Interest Rate	3.7138	3.6627	3.4960	3.5502	
	3.7766	3.8373	3.4822	3.5452	
	3.8395	3.7677	3.4683	3.5939	
	3.9025	3.9755	3.4545	3.6235	
	3.9656	3.9603	3.4407	3.6373	
	4.0287	4.0880	3.4270	3.6774	

Table 4.8: Comparison of Asian option and European Option without dividends. Values are generated by changing one variable while keeping all the others fixed for each case.

4.3.1 Visualizing Asian and European option pricing

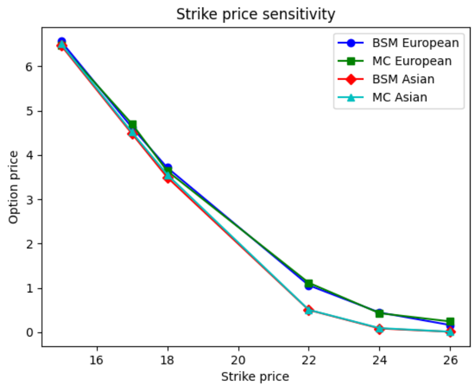


Figure 4.17: Call Price vs Strike Price

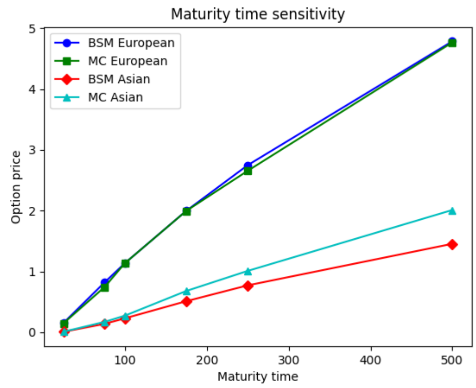


Figure 4.18: Call Price vs Maturity Time

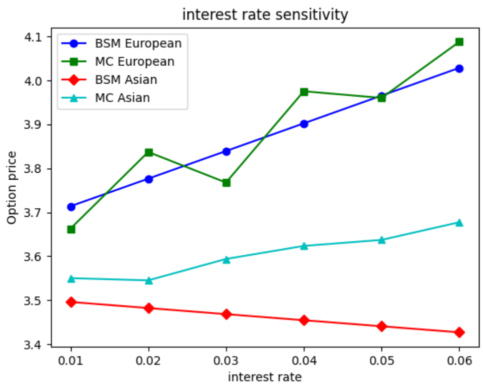


Figure 4.19: Call Price vs Interest Rate

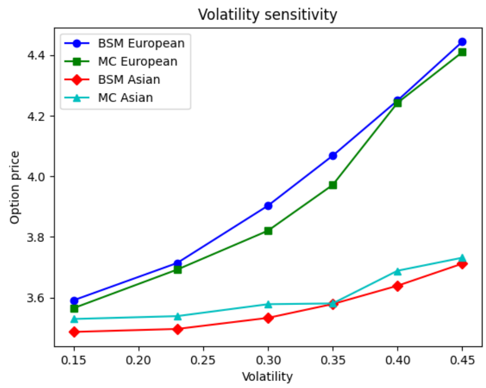


Figure 4.20: Call Price vs Volatility

CHAPTER 5

Conclusion

In this project, we have covered several topics related to derivatives, with a focus on options, particularly European and Asian options. Our primary objective was to value these options. For the European option, we employed the Binomial and Trinomial Tree Method, Monte Carlo simulation, and the Black Scholes Model. Table 4.1 shows the numerical values for the European call option without Dividend Yield, while Table 4.2 shows the values for the European call option with Dividend Yield. Table 4.2 demonstrates that as the dividend rate increases, the call option value decreases. Table 4.1 illustrates that the price of the European call rises with an increase in Maturity Time, Volatility, and the risk-free Interest Rate, and declines as the Strike Price increases. Tables 4.3, 4.4, 4.5, 4.6, 4.7 describe the Greeks: Delta Δ, Gamma Γ, Theta Θ, Vega ν, and Rho ρ respectively. These tables reveal that for a European call option, Δ increases as the stock price rises, Γ decreases with increasing stock price, Θ grows with a longer maturity time T, ν increases as volatility σ rises, and ρ grows with the risk-free Interest Rate r. Table 4.8 provides the results for a Fixed Strike Asian Call Option without Dividend Yield using Monte Carlo Simulation and the numerical solution 3.55 of its PDE, comparing these with the European call option without Dividend Yield using Monte Carlo Simulation and the Black Scholes exact solution 3.24. We have implemented the discussed procedures in Python for both numerical results and graphical representations. Finally, we conclude that this work serves as a valuable guide for further study in pricing other types of options.

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Python Codes

European Option Pricing without Dividend

```python
# Functions for option pricing


def binomial_option_price(option_type, S0, K, r, sigma, T, steps):
    dt = T / steps  # Time step
    u = np.exp(sigma * np.sqrt(dt))  # Up factor
    d = 1 / u  # Down factor
    q = (np.exp(r * dt) - d) / (u - d)  # Risk-neutral probability
    discount = np.exp(-r * dt)  # Discount factor

    # Generate the possible stock prices at maturity
    stock_prices = np.asarray([S0 * (u ** i) * (d ** (steps - i)) for
     i in range(steps + 1)])

    # Calculate the option values at maturity
    if option_type == 'call':
        option_values = np.maximum(stock_prices - K, 0)
    elif option_type == 'put':
        option_values = np.maximum(K - stock_prices, 0)
    else:
        raise ValueError("Invalid option type. Use 'call' or 'put'.")

    # Step backward through the tree
    for i in range(steps - 1, -1, -1):
        option_values = discount * (q * option_values[1:i+2] + (1 - q
    ) * option_values[0:i+1])

    return option_values[0]

def trinomial_option_price(option_type, S0, K, r, sigma, T, steps):
    dt = T / steps  # Time step
    dx = sigma * np.sqrt(2 * dt)  # Stock price change factor

    u = np.exp(dx)  # Up factor
    d = 1 / u  # Down factor

```

```python
    # Risk-neutral probabilities
    p_up = ((np.exp(r * dt / 2) - np.exp(-dx)) / (np.exp(dx) - np.exp(-dx))) ** 2
    p_down = ((np.exp(dx) - np.exp(r * dt / 2)) / (np.exp(dx) - np.exp(-dx))) ** 2
    p_mid = 1 - p_up - p_down  # Probability of no movement

    discount = np.exp(-r * dt)  # Discount factor

    # Initialize stock prices at maturity
    stock_prices = np.asarray([S0 * (u ** (i - steps)) for i in range(2 * steps + 1)])

    # Calculate option values at maturity
    if option_type == 'call':
        option_values = np.maximum(stock_prices - K, 0)
    elif option_type == 'put':
        option_values = np.maximum(K - stock_prices, 0)
    else:
        raise ValueError("Invalid option type. Use 'call' or 'put'.")

    # Step backward through the tree
    for i in range(steps, 0, -1):
        new_option_values = []
        for j in range(1, len(option_values) - 1):  # Calculate the value for each node
            value = (p_up * option_values[j + 1] +
                     p_mid * option_values[j] +
                     p_down * option_values[j - 1]) * discount
            new_option_values.append(value)
        option_values = np.array(new_option_values)  # Update option_values for the next step

    return option_values[0]  # Final option price at the root node

def monte_carlo_option_price(option_type, S0, K, r, sigma, T, simulations=100000):
    # Simulate end-of-period stock prices
    dt = T
    Z = np.random.standard_normal(simulations)  # Random normal variables for stock path
    ST = S0 * np.exp((r - 0.5 * sigma ** 2) * dt + sigma * np.sqrt(dt) * Z)
```

```python

    # Calculate option payoffs
    if option_type == 'call':
        payoff = np.maximum(ST - K, 0)
    elif option_type == 'put':
        payoff = np.maximum(K - ST, 0)
    else:
        raise ValueError("Invalid option type. Use 'call' or 'put'.")

    # Discount payoffs to present value and take the average
    option_price = np.exp(-r * T) * np.mean(payoff)

    return option_price

def black_scholes_merton_option_price(option_type, S0, K, r, sigma, T
    ):
    d1 = (np.log(S0 / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.
    sqrt(T))
    d2 = d1 - sigma * np.sqrt(T)

    if option_type == 'call':
        option_price = S0 * norm.cdf(d1) - K * np.exp(-r * T) * norm.
    cdf(d2)
    elif option_type == 'put':
        option_price = K * np.exp(-r * T) * norm.cdf(-d2) - S0 * norm
    .cdf(-d1)
    else:
        raise ValueError("Invalid option type. Use 'call' or 'put'.")

    return option_price



# Main Code


from lib.tools import *
import pandas as pd

S0 = 21.5
K = 18
r = 0.01
vol = 0.23
```

```
T = 100/252
steps = 5

#list for sensitivity
K_l = [15, 17, 18, 22, 24, 26]
r_l = [0.01, 0.02, 0.03, 0.04, 0.05, 0.06]
vol_l = [0.15, 0.23, 0.3, 0.35, 0.4, 0.45]
T_l = [25/252, 75/252, 100/252, 175/252, 250/252, 500/252]
steps_l = [5, 10, 50, 100, 500, 1000]

ck = []
for K in K_l:
ck1 = binomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
ck2 = trinomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
ck3 = monte_carlo_option_price(S,K,T,r,vol,q,tree_steps)
ck4 = black_scholes_merton(S,K,T,r,vol,q)
ck.append([ck1,ck2,ck3,ck4])

K = K_l[0]
#T = T_l[0]
r = r_l[0]
q = q_l[0]
vol = vol_l[0]
tree_steps = tree_steps_l[0]

ct = []

for T in T_l:
ct1 = binomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
ct2 = trinomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
ct3 = monte_carlo_option_price(S,K,T,r,vol,q,tree_steps)
ct4 = black_scholes_merton(S,K,T,r,vol,q)
ct.append([ct1,ct2,ct3,ct4])

K = K_l[0]
T = T_l[0]
q = q_l[0]
#r = r_l[0]
vol = vol_l[0]
tree_steps = tree_steps_l[0]

cr = []

```

```
for r in r_l:
cr1 = binomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
cr2 = trinomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
cr3 = monte_carlo_option_price(S,K,T,r,vol,q,tree_steps)
cr4 = black_scholes_merton(S,K,T,r,vol,q)
cr.append([cr1,cr2,cr3,cr4])

K = K_l[0]
T = T_l[0]
r = r_l[0]
q = q_l[0]
#vol = vol_l[0]
tree_steps = tree_steps_l[0]

cv = []

for vol in vol_l:
cv1 = binomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
cv2 = trinomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
cv3 = monte_carlo_option_price(S,K,T,r,vol,q,tree_steps)
cv4 = black_scholes_merton(S,K,T,r,vol,q)
cv.append([cv1,cv2,cv3,cv4])

K = K_l[0]
T = T_l[0]
r = r_l[0]
vol = vol_l[0]
q = q_l[0]
#tree_steps = tree_steps_l[0]

cts = []

for tree_steps in tree_steps_l:
cts1 = binomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
cts2 = trinomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
cts3 = monte_carlo_option_price(S,K,T,r,vol,q,tree_steps)
cts4 = black_scholes_merton(S,K,T,r,vol,q)
cts.append([cts1,cts2,cts3,cts4])

K = K_l[0]
T = T_l[0]
r = r_l[0]
vol = vol_l[0]
```

```
#q = q_l[0]
tree_steps = tree_steps_l[0]

cqs = []

for q in q_l:
cqs1 = binomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
cqs2 = trinomial_tree_option_price(S,K,T,r,vol,q,tree_steps)
cqs3 = monte_carlo_option_price(S,K,T,r,vol,q,tree_steps)
cqs4 = black_scholes_merton(S,K,T,r,vol,q)
cqs.append([cqs1,cqs2,cqs3,cqs4])


print("Base case: S0 = "+str(S)+", K = "+str(K)+", r = "+str(r)+", vol = "+str(vol)+", steps = "+str(tree_steps))
print("With K ="+str(K_l)+"\n")
data = {"K" : K_l,
  "BM" : [item[0] for item in ck],
  "TM" : [item[1] for item in ck],
  "MC" : [item[2] for item in ck],
  "BS" : [item[3] for item in ck]}
df = pd.DataFrame(data)
df.to_csv('K_variable.csv', index=False)
print("With T ="+str(T_l)+"\n")
data = {"T" : T_l,
  "BM" : [item[0] for item in ct],
  "TM" : [item[1] for item in ct],
  "MC" : [item[2] for item in ct],
  "BS" : [item[3] for item in ct]}
df = pd.DataFrame(data)
df.to_csv('T_variable.csv', index=False)
print("With vol ="+str(vol_l)+"\n")
data = {"vol" : vol_l,
  "BM" : [item[0] for item in cv],
  "TM" : [item[1] for item in cv],
  "MC" : [item[2] for item in cv],
  "BS" : [item[3] for item in cv]}
df = pd.DataFrame(data)
df.to_csv('vol_variable.csv', index=False)
print("With r ="+str(r_l)+"\n")
data = {"r" : r_l,
  "BM" : [item[0] for item in cr],
  "TM" : [item[1] for item in cr],
```

```
  "MC" : [item[2] for item in cr],
  "BS" : [item[3] for item in cr]}
df = pd.DataFrame(data)
df.to_csv('r_variable.csv', index=False)
print("With TreeSteps ="+str(tree_steps_l)+"\n")
data = {"TreeSteps" : tree_steps_l,
  "BM" : [item[0] for item in cts],
  "TM" : [item[1] for item in cts],
  "MC" : [item[2] for item in cts],
  "BS" : [item[3] for item in cts]}
df = pd.DataFrame(data)
df.to_csv('TreeSteps_variable.csv', index=False)

print("With Dividends ="+str(q_l)+"\n")
data = {"TreeSteps" : q_l,
  "BM" : [item[0] for item in cqs],
  "TM" : [item[1] for item in cqs],
  "MC" : [item[2] for item in cqs],
  "BS" : [item[3] for item in cqs]}
df = pd.DataFrame(data)
df.to_csv('Dividends_variable.csv', index=False)
print("All data printed")


# Main Code


from lib.tools import *
import pandas as pd

S = 21.5
K = 18
r = 0.01
vol = 0.23
T = 100/252
steps = 5

#list for sensitivity
K_l = [15, 17, 18, 22, 24, 26]
r_l = [0.01, 0.02, 0.03, 0.04, 0.05, 0.06]
vol_l = [0.15, 0.23, 0.3, 0.35, 0.4, 0.45]
T_l = [25/252, 75/252, 100/252, 175/252, 250/252, 500/252]
steps_l = [5, 10, 50, 100, 500, 1000]
```

```

#S = S_l[0]
K = K_l[0]
T = T_l[0]
r = r_l[0]
vol = vol_l[0]
tree_steps = tree_steps_l[0]
ds=5

ck = []
for S in S_l:
ck1 = (binomial_tree_option_price(S+ds,K,T,r,vol,tree_steps)-
    binomial_tree_option_price(S-ds,K,T,r,vol,tree_steps))/(2*ds)
ck2 = (trinomial_tree_option_price(S+ds,K,T,r,vol,tree_steps)-
    trinomial_tree_option_price(S-ds,K,T,r,vol,tree_steps))/(2*ds)
ck3 = (monte_carlo_option_price(S+ds,K,T,r,vol,tree_steps)-
    monte_carlo_option_price(S-ds,K,T,r,vol,tree_steps))/(2*ds)
ck4 = black_scholes_merton_delta(S,K,T,r,vol)
ck.append([ck1,ck2,ck3,ck4])

print("With S ="+str(S_l)+"\n")
data = {"S" : S_l,
   "BM" : [item[0] for item in ck],
   "TM" : [item[1] for item in ck],
   "MC" : [item[2] for item in ck],
   "BS" : [item[3] for item in ck]}
df = pd.DataFrame(data)
df.to_csv('S_variable_delta.csv', index=False)

#S = S_l[0]
K = K_l[0]
T = T_l[0]
r = r_l[0]
vol = vol_l[0]
tree_steps = tree_steps_l[0]
ds=30

ck = []
for S in S_l:
ck1 = (binomial_tree_option_price(S+ds,K,T,r,vol,tree_steps)-2*
    binomial_tree_option_price(S,K,T,r,vol,tree_steps)+
    binomial_tree_option_price(S-ds,K,T,r,vol,tree_steps))/(ds**2)
ck2 = (trinomial_tree_option_price(S+ds,K,T,r,vol,tree_steps)-2*
```

```
    trinomial_tree_option_price(S,K,T,r,vol,tree_steps)+
    trinomial_tree_option_price(S-ds,K,T,r,vol,tree_steps))/(ds**2)
ck3 = (monte_carlo_option_price(S+ds,K,T,r,vol,tree_steps)-2*
    monte_carlo_option_price(S,K,T,r,vol,tree_steps)+
    monte_carlo_option_price(S-ds,K,T,r,vol,tree_steps))/(ds**2)
ck4 = black_scholes_merton_gamma(S,K,T,r,vol)
ck.append([ck1,ck2,ck3,ck4])

print("With S ="+str(S_l)+"\n")
data = {"S" : S_l,
  "BM" : [item[0] for item in ck],
  "TM" : [item[1] for item in ck],
  "MC" : [item[2] for item in ck],
  "BS" : [item[3] for item in ck]}
df = pd.DataFrame(data)
df.to_csv('S_variable_gamma.csv', index=False)

S = S_l[0]
K = K_l[0]
T = T_l[0]
#r = r_l[0]
vol = vol_l[0]
tree_steps = tree_steps_l[0]
dr=0.001

ck = []
for r in r_l:
ck1 = (binomial_tree_option_price(S,K,T,r+dr,vol,tree_steps)-
    binomial_tree_option_price(S,K,T,r-dr,vol,tree_steps))/(2*dr)
ck2 = (trinomial_tree_option_price(S,K,T,r+dr,vol,tree_steps)-
    trinomial_tree_option_price(S,K,T,r-dr,vol,tree_steps))/(2*dr)
ck3 = (monte_carlo_option_price(S,K,T,r+dr*80,vol,tree_steps)-
    monte_carlo_option_price(S,K,T,r-dr*80,vol,tree_steps))/(2*dr*80)
ck4 = black_scholes_merton_rho(S,K,T,r,vol)
ck.append([ck1,ck2,ck3,ck4])

print("With r ="+str(r_l)+"\n")
data = {"r" : r_l,
  "BM" : [item[0] for item in ck],
  "TM" : [item[1] for item in ck],
  "MC" : [item[2] for item in ck],
  "BS" : [item[3] for item in ck]}
df = pd.DataFrame(data)
```

```
df.to_csv('r_variable_rho.csv', index=False)

S = S_l[0]
K = K_l[0]
#T = T_l[0]
r = r_l[0]
vol = vol_l[0]
tree_steps = tree_steps_l[0]
dt=21

ck = []
for T in T_l:
ck1 = (binomial_tree_option_price(S,K,T+dt,r,vol,tree_steps)-
    binomial_tree_option_price(S,K,T-dt,r,vol,tree_steps))/(-2*dt
    /252)
ck2 = (trinomial_tree_option_price(S,K,T+dt,r,vol,tree_steps)-
    trinomial_tree_option_price(S,K,T-dt,r,vol,tree_steps))/(-2*dt
    /252)
ck3 = (monte_carlo_option_price(S,K,T+dt,r,vol,tree_steps)-
    monte_carlo_option_price(S,K,T-dt,r,vol,tree_steps))/(-2*dt/252)
ck4 = black_scholes_merton_theta(S,K,T,r,vol)
ck.append([ck1,ck2,ck3,ck4])

print("With T ="+str(T_l)+"\n")
data = {"T" : T_l,
  "BM" : [item[0] for item in ck],
  "TM" : [item[1] for item in ck],
  "MC" : [item[2] for item in ck],
  "BS" : [item[3] for item in ck]}
df = pd.DataFrame(data)
df.to_csv('T_variable_theta.csv', index=False)

S = S_l[0]
K = K_l[0]
T = T_l[0]
r = r_l[0]
#vol = vol_l[0]
tree_steps = tree_steps_l[0]
dv=0.05

ck = []
for vol in vol_l:
ck1 = (binomial_tree_option_price(S,K,T,r,vol+dv,tree_steps)-
```

```
binomial_tree_option_price(S,K,T,r,vol-dv,tree_steps))/(2*dv)
ck2 = (trinomial_tree_option_price(S,K,T,r,vol+dv,tree_steps)-trinomial_tree_option_price(S,K,T,r,vol-dv,tree_steps))/(2*dv)
ck3 = (monte_carlo_option_price(S,K,T,r,vol+dv,tree_steps)-monte_carlo_option_price(S,K,T,r,vol-dv,tree_steps))/(2*dv)
ck4 = black_scholes_merton_vega(S,K,T,r,vol)
ck.append([ck1,ck2,ck3,ck4])

print("With V ="+str(vol_l)+"\n")
data = {"V" : vol_l,
  "BM" : [item[0] for item in ck],
  "TM" : [item[1] for item in ck],
  "MC" : [item[2] for item in ck],
  "BS" : [item[3] for item in ck]}
df = pd.DataFrame(data)
df.to_csv('V_variable_vega.csv', index=False)

print("All data printed")


#Functions


import numpy as np
from scipy.stats import norm

#if T is in day-lengths, divide by 252 before passing the value to the function

def binomial_tree_option_price(S, K, T, r, sigma, n, option_type='call'):
T=T/252
delta_t = T / n
u = np.exp(sigma * np.sqrt(delta_t))
d = 1 / u
p = (np.exp(r * delta_t) - d) / (u - d)

stock_prices = np.zeros((n+1, n+1))
option_values = np.zeros((n+1, n+1))

for i in range(n+1):
for j in range(i+1):
stock_prices[j, i] = S * (u ** (i - j)) * (d ** j)
```

```
if option_type == 'call':
option_values[:, -1] = np.maximum(0, stock_prices[:, -1] - K)
elif option_type == 'put':
option_values[:, -1] = np.maximum(0, K - stock_prices[:, -1])

for i in range(n-1, -1, -1):
for j in range(i+1):
option_values[j, i] = np.exp(-r * delta_t) * (p * option_values[j, i+1] + (1 - p) * option_values[j+1, i+1])

option_price = option_values[0, 0]
return option_price

def trinomial_tree_option_price(S, K, T, r, sigma, n, option_type='call'):
T=T/252
delta_t = T / n
u = np.exp(sigma * np.sqrt(2 * delta_t))
p_u = ((np.exp(r * delta_t/2) - np.exp(-sigma * np.sqrt(delta_t/2))) /
(np.exp(sigma * np.sqrt(delta_t/2)) - np.exp(-sigma * np.sqrt(delta_t/2))))**2
p_d = ((np.exp(sigma * np.sqrt(delta_t/2)) - np.exp(r * delta_t/2)) /
(np.exp(sigma * np.sqrt(delta_t/2)) - np.exp(-sigma * np.sqrt(delta_t/2))))**2
p_m = 1 - p_u - p_d
stock_prices = np.zeros((2*n+1, n+1))
option_values = np.zeros((2*n+1, n+1))
stock_prices[n,0] = S
for j in range(1,n+1):
stock_prices[n-j,j] = S*u**j
for i in range(n-j+1,n+j+1):
stock_prices[i,j] = stock_prices[i-1,j]/u
if option_type == 'call':
option_values[:, -1] = np.maximum(0, stock_prices[:, -1] - K)
elif option_type == 'put':
option_values[:, -1] = np.maximum(0, K - stock_prices[:, -1])
for j in range(n-1, -1, -1):
for i in range(-j, j+1):
option_values[i+n, j] = np.exp(-r * delta_t) * (p_u * option_values[i+n-1, j+1] +
p_d * option_values[i+n+1, j+1] +
```

```
p_m * option_values[i+n, j+1])
option_price = option_values[n, 0]
return option_price

def monte_carlo_option_price(S, K, T, r, sigma,tree_steps,option_type='call', num_simulations=10000):
num_steps = tree_steps*1000
T=T/252
dt=T/num_steps
S_T = np.zeros(num_simulations)

for i in range(num_simulations):
rand = np.random.normal(0, 1, num_steps)

stock_path = S * np.exp(np.cumsum((r - 0.5 * sigma ** 2) * dt + sigma * np.sqrt(dt) * rand))
S_T[i] = stock_path[-1]

if option_type == 'call':
payoff = np.maximum(0, S_T - K)
elif option_type == 'put':
payoff = np.maximum(0, K - S_T)
else:
raise ValueError("Invalid option type. Use 'call' or 'put'.")

option_price = np.exp(-r * T) * np.mean(payoff)
return option_price
''' stock_path = S * np.exp((r - 0.5 * sigma ** 2) * T + sigma * np.sqrt(T) * rand) '''


def black_scholes_merton_delta(S, K, T, r, sigma, option_type='call'):
T=T/252
d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
d2 = d1 - sigma * np.sqrt(T)

if option_type == 'call':
option_price = norm.cdf(d1)
elif option_type == 'put':
option_price = norm.cdf(d1)-1
else:
```

```
raise ValueError("Invalid option type. Use 'call' or 'put'.")

return option_price

def black_scholes_merton_gamma(S, K, T, r, sigma, option_type='call'):
T=T/252
d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
d2 = d1 - sigma * np.sqrt(T)

if option_type == 'call':
option_price = norm.pdf(d1)/(S*sigma* np.sqrt(T))
elif option_type == 'put':
option_price = norm.pdf(d1)/(S*sigma* np.sqrt(T))
else:
raise ValueError("Invalid option type. Use 'call' or 'put'.")

return option_price


def black_scholes_merton_theta(S, K, T, r, sigma, option_type='call'):
T=T/252
d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
d2 = d1 - sigma * np.sqrt(T)

if option_type == 'call':
option_price = -sigma*S*norm.pdf(d1)/(2*np.sqrt(T))-r*K* np.exp(-r*T) * norm.cdf(d2)
elif option_type == 'put':
option_price = -sigma*S*norm.pdf(d1)/(2*np.sqrt(T))+r*K* np.exp(-r*T) * norm.cdf(-d2)
else:
raise ValueError("Invalid option type. Use 'call' or 'put'.")

return option_price

def black_scholes_merton_vega(S, K, T, r, sigma, option_type='call'):
T=T/252
d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
```

```
d2 = d1 - sigma * np.sqrt(T)

if option_type == 'call':
option_price = S*norm.pdf(d1)*np.sqrt(T)
elif option_type == 'put':
option_price = S*norm.pdf(d1)*np.sqrt(T)
else:
raise ValueError("Invalid option type. Use 'call' or 'put'.")

return option_price

def black_scholes_merton_rho(S, K, T, r, sigma, option_type='call'):
T=T/252
d1 = (np.log(S / K) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
d2 = d1 - sigma * np.sqrt(T)

if option_type == 'call':
option_price = K*norm.cdf(d2)*T* np.exp(-r*T)
elif option_type == 'put':
option_price = -K*norm.cdf(-d2)*T* np.exp(-r*T)
else:
raise ValueError("Invalid option type. Use 'call' or 'put'.")

return option_price
```

Asian Option Pricing

```
#Main Code
from lib.tools import *
import pandas as pd

S0 = 21.5
K = 18
r = 0.01
vol = 0.23
T = 100/252
steps = 5

#list for sensitivity
K_l = [15, 17, 18, 22, 24, 26]
```

```
r_l = [0.01, 0.02, 0.03, 0.04, 0.05, 0.06]
vol_l = [0.15, 0.23, 0.3, 0.35, 0.4, 0.45]
T_l = [25/252, 75/252, 100/252, 175/252, 250/252, 500/252]
steps_l = [5, 10, 50, 100, 500, 1000]

#K = K_l[0]
T = T_l[0]
r = r_l[0]

vol = vol_l[0]

ck = []
for K in K_l:
ck4 = black_scholes_merton(S,K,T,r,vol)
ck.append([ck4])

K = K_l[0]
#T = T_l[0]
r = r_l[0]

vol = vol_l[0]

ct = []

for T in T_l:

ct4 = black_scholes_merton(S,K,T,r,vol)
ct.append([ct4])

K = K_l[0]
T = T_l[0]
#r = r_l[0]
vol = vol_l[0]

cr = []

for r in r_l:
cr4 = black_scholes_merton(S,K,T,r,vol)
cr.append([cr4])

K = K_l[0]
T = T_l[0]
r = r_l[0]
```

```
#vol = vol_l[0]

cv = []

for vol in vol_l:
cv4 = black_scholes_merton(S,K,T,r,vol)
cv.append([cv4])

print("Base case: S0 = "+str(S)+", T = "+str(T)+", K = "+str(K)+", r
    = "+str(r)+", vol = "+str(vol))
print("With K ="+str(K_l)+"\n")
data = {"K" : K_l,
  "BS" : [item[0] for item in ck]}
df = pd.DataFrame(data)
df.to_csv('bs_K_variable.csv', index=False)
print("With T ="+str(T_l)+"\n")
data = {"T" : T_l,
  "BS" : [item[0] for item in ct]}
df = pd.DataFrame(data)
df.to_csv('bs_T_variable.csv', index=False)
print("With vol ="+str(vol_l)+"\n")
data = {"vol" : vol_l,
  "BS" : [item[0] for item in cv]}
df = pd.DataFrame(data)
df.to_csv('bs_vol_variable.csv', index=False)
print("With r ="+str(r_l)+"\n")
data = {"r" : r_l,
  "BS" : [item[0] for item in cr]}
df = pd.DataFrame(data)
df.to_csv('bs_r_variable.csv', index=False)

print("All data printed")


#Functions

import numpy as np
from scipy.stats import norm

#if T is in day-lengths, divide by 252 before passing the value to
    the function

def black_scholes_merton(S, K, T, r, sigma, option_type='call'):
```

```
T=T/252
mu=(r-0.5*sigma**2)*0.5*T
sigmabar=sigma*np.sqrt(T/3)
d2 = (np.log(S / K) + mu) / (sigmabar)
d1 = d2 + sigmabar

if option_type == 'call':
option_price = np.exp(-r * T)*(S*np.exp(mu+0.5*sigmabar**2)* norm.cdf(d1)- K*norm.cdf(d2))
elif option_type == 'put':
option_price = np.exp(-r * T)*(K*norm.cdf(d2)-S*np.exp(mu+0.5*sigmabar**2)* norm.cdf(d1))
else:
raise ValueError("Invalid option type. Use 'call' or 'put'.")

return option_price
```

Python Code

Monte Carlo Simulation of Asian option

```

  def black_scholes_asian_option(option_type, S0, K, r, sigma, T):
    # Adjusted volatility and risk-free rate for the geometric average
    sigma_adj = sigma / math.sqrt(3)
    r_adj = 0.5 * (r - 0.5 * sigma ** 2 + sigma_adj ** 2)

    # Calculate d1 and d2
    d1 = (math.log(S0 / K) + (r_adj + 0.5 * sigma_adj ** 2) * T) / (sigma_adj * math.sqrt(T))
    d2 = d1 - sigma_adj * math.sqrt(T)

    if option_type == "call":
        price = math.exp(-r * T) * (S0 * norm.cdf(d1) - K * norm.cdf(d2))
    elif option_type == "put":
        price = math.exp(-r * T) * (K * norm.cdf(-d2) - S0 * norm.cdf(-d1))
    else:
        raise ValueError("Invalid option type. Use 'call' or 'put'.")

    return price

def monte_carlo_asian_option(option_type, S0, K, r, sigma, T, n_simulations=10000, n_steps=100):
    dt = T / n_steps  # Time increment
    discount_factor = np.exp(-r * T)  # Discount factor for present value

    # Array to store the payoff for each simulation
    payoffs = np.zeros(n_simulations)

    # Run Monte Carlo simulations
    for i in range(n_simulations):
        # Simulate a path of stock prices
        path = np.zeros(n_steps + 1)
```

```python
        path[0] = S0

        for t in range(1, n_steps + 1):
            z = np.random.normal(0, 1)  # Generate a standard normal random variable
            path[t] = path[t - 1] * np.exp((r - 0.5 * sigma ** 2) * dt + sigma * np.sqrt(dt) * z)

        # Calculate the average stock price across the path
        average_price = np.mean(path)

        # Calculate the payoff for this simulation
        if option_type == "call":
            payoffs[i] = max(average_price - K, 0)
        elif option_type == "put":
            payoffs[i] = max(K - average_price, 0)
        else:
            raise ValueError("Invalid option type. Use 'call' or 'put'.")

    # Calculate the average payoff and discount it back to present value
    option_price = discount_factor * np.mean(payoffs)

    return option_price
```