Simple unit-step movement

2025-06-04

We assume that all moves are memoryless, time homogeneous, and space homogeneous or translation invariant.

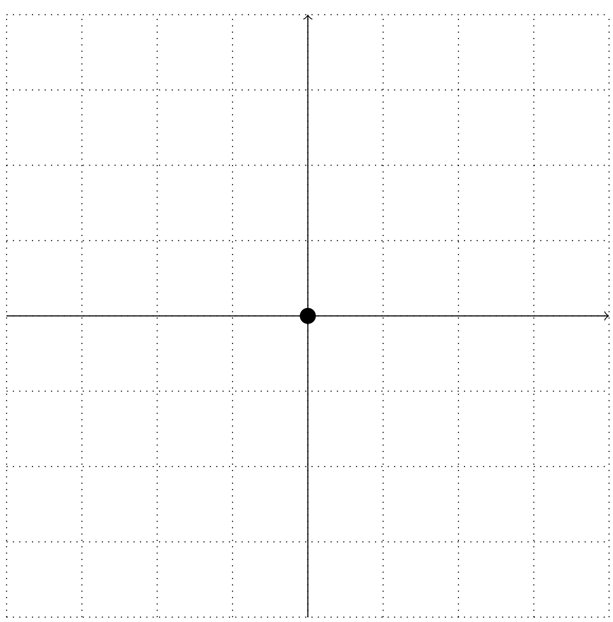
This allows us to decompose the trajectory into a set of independent displacements or offsets. The probability of each can be calculated under different movement "kernels" or models, and we can derive the total probability of the trajectory as the joint probability of these independent steps.

Here, we begin with very simple single (discrete) unit step movements in cardinal directions.

- $\mathcal{M}_1$: Unbiased, uniform random movement: every round it takes a single unit step in any of the eight possible discrete integer step directions.
- $\mathcal{M}_2$: Biased toward positive x values: e.g. 0.6 probability of taking a non-zero positive unit step in the x direction, a 0.3 probability of taking a zero-unit step in the x direction, and 0.1 probability of taking a negative step in the x direction; y direction movements are unbiased, with equal probability of positive, negative, or zero unit steps in the y direction.
- $\mathcal{M}_3$: Biased toward positive x and y values: e.g. 0.6 probability of taking a non-zero positive unit step, 0.3 probability of taking zero unit steps, and 0.1 probability of taking negative unit steps, in each of the x and y directions.

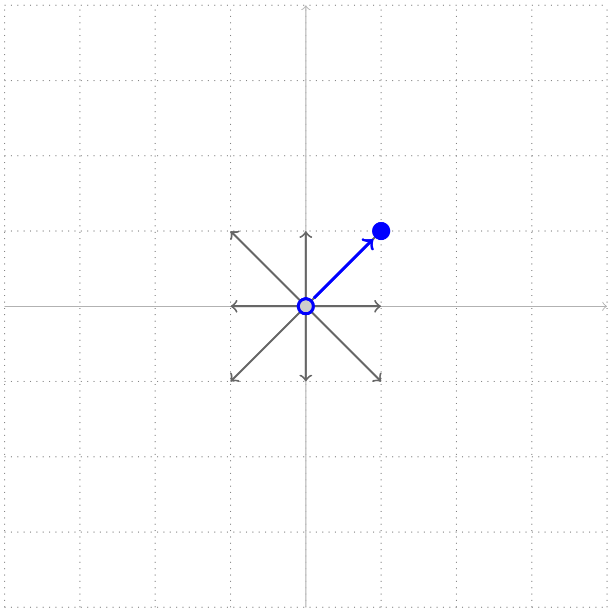
Initial state

Let the initial position of the beetle be $X_0 = (0, 0)$.



Single-step trajectory

What is the probability of a single-step trajectory, $\big[(1,1)\big]$?



Let

$$T^{n=1} = \big[(1,1)\big]$$

be a trajectory of $n = 1$ steps generated under some unknown model.

The probability of this trajectory under the uniform movement model, $\mathcal{M}_1$, is

$$\begin{aligned} \Pr(T^{n=1} \mid \mathcal{M}_1) &= \frac{1}{8} \\ &= 0.125. \end{aligned} \tag{0.1}$$

The probability of trajectory $T^{n=1}$ under the directed movement model $\mathcal{M}_2$ is

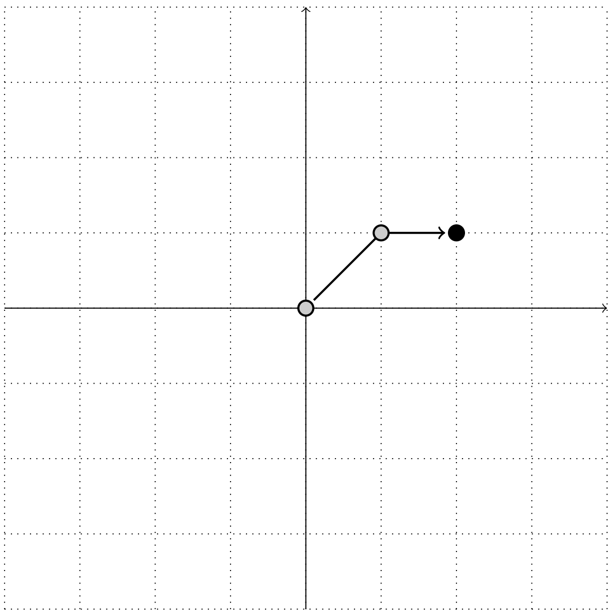
$$\begin{aligned} \Pr(T^{n=1} \mid \mathcal{M}_2) &= (0.6) \cdot \tfrac{1}{3} \\ &= 0.2. \end{aligned} \tag{0.2}$$

The probability of trajectory $T^{n=1}$ under the directed movement model $\mathcal{M}_3$ is

$$\begin{aligned} \Pr(T^{n=1} \mid \mathcal{M}_3) &= (0.6)(0.6) \\ &= 0.36. \end{aligned} \tag{0.3}$$

Two-step trajectory

Another step is observed:



What is the probability of the two-step trajectory

$$T^{n=2} = \begin{bmatrix} (1,1) \\ (2,1) \end{bmatrix} ?$$

> This will be the *joint* probability of reaching the final position in the previous single-step trajectory, *and* the probability of the new step taken from that point.

We assume the movement model $\mathcal{M}$ is **Markovian** and **translation-invariant**.

Let the displacements be: - $S_1 = X_1 - X_0 = (1,1)$ - $S_2 = X_2 - X_1 = (1,0)$

Then:

$$\Pr(T^{n=2} \mid \mathcal{M}) = \Pr(S_1 \mid \mathcal{M}) \cdot \Pr(S_2 \mid \mathcal{M})$$

Under $\mathcal{M}_1$

$$\Pr(S_1) = \frac{1}{8}, \quad \Pr(S_2) = \frac{1}{8}$$

$$\Pr(T^{n=2} \mid \mathcal{M}_1) = \frac{1}{64} = 0.015625$$

Under $\mathcal{M}_2$

$$\begin{aligned}
\Pr(S_1) &= 0.6 \cdot \tfrac{1}{3} = 0.2 \\
\Pr(S_2) &= 0.6 \cdot \tfrac{1}{3} = 0.2 \\
\Pr(T^{n=2} \mid \mathcal{M}_2) &= 0.2 \cdot 0.2 = 0.04
\end{aligned}$$

Under $\mathcal{M}_3$

$$\begin{aligned}
\Pr(S_1) &= 0.6 \cdot 0.6 = 0.36 \\
\Pr(S_2) &= 0.6 \cdot 0.3 = 0.18 \\
\Pr(T^{n=2} \mid \mathcal{M}_3) &= 0.36 \cdot 0.18 = 0.0648
\end{aligned}$$

General trajectory probability

Let a trajectory of length n be given as a sequence of positions:

$$(X_0, X_1, \dots, X_n)$$

Define the step displacements as:

$$S_i = X_i - X_{i-1}, \quad i = 1, \dots, n$$

Then the trajectory probability under a Markovian, translation-invariant model $\mathcal{M}$ is:

$$\Pr(T^n \mid \mathcal{M}) = \prod_{i=1}^{n} \Pr(S_i \mid \mathcal{M})$$

Extension to continuous or other movement models

Generalizing to continuous or alternative movement models is conceptually straightforward, but there might be issues with normalization, support, integration, identifiability?

We can think of a Gaussian movement kernel, where the displacement at time t is drawn from an isotropic multivariate Gaussian, just like in your example, with the key difference is that we are modeling the *displacement* using a Gaussian, rather than position + noise as a Gaussian, I think?