CV201 HW5 HadadYakir BukraBarak

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1 Matrix Lie Group

1.1 Problem 1

These 3 equations below show that GL(n) is a matrix group

$$|I| = 1 \to I \in GL(n) \tag{1}$$

□

$$\begin{aligned} &A, B \in GL(n) \to |A|, |B| \neq 0 \\ &|AB| = |A||B| \neq 0 \\ &AB \in GL(n) \end{aligned} \tag{2}$$

□

$$\begin{aligned} &A \in GL(N) \to |A| \neq 0 \\ &|A| \neq 0 \to \exists\ A^{-1} \\ &|A^{-1}| \neq 0 \to A^{-1} \in GL(n) \end{aligned} \tag{3}$$

□

1.2 Problem 2

These 3 equations below show that $GL_+(n)$ is a matrix group

$$|I| = 1 > 0 \to I \in GL_+(n) \tag{4}$$

□

$$\begin{aligned} &A, B \in GL_+(n) \to |A|, |B| > 0 \\ &|AB| = |A||B| > 0 \\ &AB \in GL(n) \end{aligned} \tag{5}$$

□

$$
\begin{aligned}
&A \in GL(N) \to |A| > 0 \\
&|A| > 0 \to \exists\ A^{-1} \\
&AA^{-1} = I \to |A||A^{-1}| = 1 \\
&|A^{-1}| = |A|^{-1} \\
&|A^{-1}| > 0 \to A^{-1} \in GL_{+}(n)
\end{aligned} \tag{6}
$$

□

1.3 Problem 3

I not included in $GL_{-}(n)$, this is the first rule for group to be a matrix group. So $GL_{-}(n)$ is not a matrix group.

$$
|I| = 1 \to I \notin GL_{-}(n) \tag{7}
$$

□

1.4 Problem 4

Denote $C = AB = [a_1 a_2 \cdots a_n]^T [b_1 b_2 \cdots b_n]$ where a_i are row vectors and b_i are columns vector, so for example $c_{12} = a_1 b_2$ while $c_{21} = a_2 b_1$ which is usually not equal. This is only one case so obviously for every case is even less likely.

1.5 Problem 5

These 3 equations below show that US(n) is a matrix group

$$
I = 1I \to I \in US(n) \tag{8}
$$

□

$$
\begin{aligned}
&A, B \in US(n) \to A = aI,\ B = bI \\
&AB = abI \\
&AB \in US(n)
\end{aligned} \tag{9}
$$

□

$$
\begin{aligned}
&A \in US(N) \to A = aI \\
&denote\ \hat{A} = \frac{1}{a} I \to A\hat{A} = I \exists\ A^{-1} \to A^{-1} = \hat{A} \\
&A^{-1} = \frac{1}{a} I \to A^{-1} \in US(n)
\end{aligned} \tag{10}
$$

□

1.6 Problem 6

The matrix $0_{n\times n}$ is the zero element of $R^{n\times n}$

1.7 Problem 7

One of the axioms of linear space is that exist the zero element of the parent space, since the 0 matrix is not invertible $0 \notin$ group matrix $\rightarrow$ group matrix is no a sublinear space of $R^{n\times n}$.

2 Linear and Affine Maps

2.1 Problem 8

The condition for matrix A to be invertible is that it will be linear independent.

2.2 Problem 9

For a function f to be linear it needs to fulfill this requirement, $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$ in case of affine functions:

$$f(\alpha x_1 + \beta x_2) = a\alpha x_1 + a\beta x_2 + b \tag{11}$$

$$\alpha f(x_1) + \beta f(x_2) = a\alpha x_1 + +a\beta x_2 + \beta b \tag{12}$$

So for $f(\alpha x_1 + \beta x_2) = \alpha f(x_1) + \beta f(x_2)$, β must be $0 \rightarrow$ affine functions aren't linear functions $\square$

2.3 Problem 10

$$\begin{aligned}
&f(x) = Ax + b \\
&f(x) - b = Ax \\
&A^{-1}(f(x) - b) = A^{-1}Ax \\
&A^{-1}(f(x) - b) = x
\end{aligned} \tag{13}$$

For f to be invertible A must be invertible.

2.4 Problem 11

$$\begin{aligned}
f(x) &= A_1x + b_1 \\
g(x) &= A_2x + b_2 \\
h(x) &= f(g(x)) = A_2(A_1x + b_1) + b_2 \\
&= A_2A_1x + (A_2b_1 + b_2) \\
&= A_3x + b_3 \\
A_3 &= A_2A_1,\ A_3 \in R^{k\times n} \\
b_3 &= (A_2b_1 + b_2),\ b_3 \in R^k
\end{aligned} \tag{14}$$

A_2A_1 and A_2b_1 are o.k with sizes

2.5 Problem 12

From the previous problems f and g invertibles iff A_1, A_2 exists. h will be invertible if A_3 is invertible. Let $\hat{A}_3 = A_1^{-1}A_2^{-1}$

$$\begin{aligned} A_3\hat{A}_3 &= A_2A_1A_1^{-1}A_2^{-1} \\ &= A_2IA_2^{-1} \\ &= A_2A_2^{-1} \\ &= I \end{aligned} \tag{15}$$

$\hat{A}_3 = A_3^{-1} \rightarrow A_3$ invertible

2.6 Problem 13

Let Q define the group that look like $\begin{bmatrix} A & b \\ 0_{1\times n} & 1 \end{bmatrix}$
These 3 equations below show that Q is a matrix group

$$A = I_{n\times n},\ b = 0_{n\times 1} \rightarrow I_{n+1\times n+1} \in Q \tag{16}$$

□

$$Q_1 = \begin{bmatrix} A_1 & b_1 \\ 0_{1\times n} & 1 \end{bmatrix}, Q_2 = \begin{bmatrix} A_2 & b_2 \\ 0_{1\times n} & 1 \end{bmatrix} \in Q \tag{17}$$

In case of $Q_3 = Q_1Q_2$The first N rows I'm not interested to calculate because it can be in any form, our interest part is the $(n+1)_{th}$ row. The first n values equal to the equation below:

$$\begin{bmatrix} 0_{1\times n} \ 1 \end{bmatrix} \begin{bmatrix} \vdots \\ 0 \end{bmatrix} = 0 \tag{18}$$

And the last value $Q_{3(n+1)\times(n+1)}$

$$\begin{bmatrix} 0_{1\times n} & 1 \end{bmatrix} \begin{bmatrix} \vdots \\ 1 \end{bmatrix} = 1 \tag{19}$$

So $Q_3 = \begin{bmatrix} \ddots_{n\times n} & \vdots \\ 0_{1\times n} & 1 \end{bmatrix} \rightarrow Q_3 \in Q$
□
It given that $|A| \neq 0 \rightarrow$ A is invertible $\rightarrow$ The columns of A is linear independent. Concatenating $0_{n\times n}$ to $A \rightarrow [A\ 0_{n\times n}]^T$ still doesn't change that the columns are independent. I don't know if b is dependent with any column of A but I do can say that $[b\ 1]^T$ independent with all columns of $[A\ 0_{n\times n}]^T$ because of the

last element 1. We know from linear algebra that matrix is invertible iff the columns are independent $\rightarrow \begin{bmatrix} A_2 & b_2 \\ 0_{1\times n} & 1 \end{bmatrix}$
$\square$

2.7 Problem 14

This prove is very similar to problems 6 and 13, so I'll be short with words.
Let Q_+ defined to be that group as defined in the question and from the definition $Q_+ \subset Q$.
$I_{(n+1)\times(n+1)} \in Q$, $|I| = 1 \rightarrow I_{(n+1)\times(n+1)} \in Q_+$
$\square$
Let $A_1, A_2 \in Q_+ \rightarrow A_1A_2 \in Q$, $|A_1| > 0, |A_2| > 0 \rightarrow |A_1A_2| = |A_1||A_2| > 0 \rightarrow A_1A_2 \in Q_+$.
$\square$
Let $A_1 \in Q_+ \rightarrow A_1 \in Q$, from problem 13 A_1^{-1} exists, $|A_1| > 0 \rightarrow |A_1^{-1}| = |A_1|^{-1} > 0$
$\square$
Q_+ is a matrix group $\square$

2.8 Problem 15

Define Q_- as the group of matrices that defined in the question $|I| > 0 \rightarrow IQ_-$
Q_- is not a matrix group.

2.9 Problem 16

Let $c(t) = (1-t)f + tg \rightarrow c(t, x) = ((1-t)A_0 + tA_1)x + ((1-t)b_0 + tb_1) = A(t)x + b(t)$. This curve fulfill these requirements, $c(0) = f$ and $c(1) = g$ additionally $A(t) \neq 0, \forall t \in (0, 1)$.

3 Least Squares and Projections on a Linear Subspace

3.1 Problem 17

Part i)

Given $\boldsymbol{u_i} = P \begin{bmatrix} \boldsymbol{X_i} \\ 1 \end{bmatrix} + \varepsilon_i$

$$
\begin{aligned}
\varepsilon_i &= \boldsymbol{u_i} - P \begin{bmatrix} \boldsymbol{X_i} \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} u_i \\ v_i \end{bmatrix} - \begin{bmatrix} \theta_1 & \theta_2 & 0 & a \\ \theta_3 & b & 0 & \theta_4 \end{bmatrix} \begin{bmatrix} X_i \\ Y_i \\ Z_i \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} u_i \\ v_i \end{bmatrix} - \begin{bmatrix} \theta_1 X_i + \theta_2 Y_i + a \\ \theta_3 X_i + bY_i + \theta_4 \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} u_i - a \\ v_i - bY_i \end{bmatrix}}_{y_i} - \underbrace{\begin{bmatrix} X_i & Y_i & 0 & 0 \\ 0 & 0 & X_i & 1 \end{bmatrix}}_{H_i} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \\
H &= \begin{bmatrix} H_1 \\ \vdots \\ H_N \end{bmatrix} = \begin{bmatrix} X_1 & Y_1 & 0 & 0 \\ 0 & 0 & X_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ X_N & Y_N & 0 & 0 \\ 0 & 0 & X_N & 1 \end{bmatrix} \\
y &= \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} u_1 - a \\ v_1 - bY_1 \\ \vdots \\ u_N - a \\ v_N - bY_N \end{bmatrix} \\
\theta_{LS} &= (H^T H)^{-1} H^T y
\end{aligned} \tag{20}
$$

□

Part ii)

$$\begin{aligned}
p((u_i, v_i)_{i=1}^N|\theta) &= \frac{1}{z} exp\Bigg(-\frac{1}{2}\sum_i^N \Bigg(\begin{bmatrix} u_i \\ v_i \end{bmatrix} - P\begin{bmatrix} X_i \\ 1 \end{bmatrix} \Bigg)^T (\sigma^2 I)^{-1} \Bigg(\begin{bmatrix} u_i \\ v_i \end{bmatrix}_i - P\begin{bmatrix} X_i \\ 1 \end{bmatrix} \Bigg) \Bigg) \\
&= \frac{1}{z} exp\Bigg(-\frac{1}{2\sigma^2}\sum_i^N (y_i - H_i\theta)^T (y_i - H_i\theta) \Bigg) \\
&= \frac{1}{z} exp\Bigg(-\frac{1}{2\sigma^2}\sum_i^N ||y_i - H_i\theta|| \Bigg) \\
z &= \sum_{(u,v)} exp\Bigg(-\frac{1}{2\sigma^2}\sum_i^N ||y_i - H_i\theta|| \Bigg)
\end{aligned} \tag{21}$$

□

Part iii)

$$\begin{aligned}
\arg\max_\theta \; p((u_i, v_i)_{i=1}^N|\theta) &= \arg\max_\theta \; exp\Bigg(-\frac{1}{2\sigma^2}\sum_i^N ||y_i - H_i\theta|| \Bigg) \\
&= \arg\min_\theta \; ||y_i - H_i\theta|| \\
&= \theta_{LS}
\end{aligned} \tag{22}$$

□

3.2 Problem 18

for some non-negative constant λ. It is the sum of squares of the residuals plus a multiple of the sum of squares of the coefficients themselves.

Consider the matrix H augmented with rows corresponding to $\sqrt{\lambda}$ times the $k \times k$ identity matrix I:

$$\hat{H} = \begin{bmatrix} H \\ \sqrt{\lambda} I_{k\times k} \end{bmatrix}$$

$$\hat{y} = \begin{bmatrix} y \\ 0_{k\times 1} \end{bmatrix}$$

The matrix product in the objective function adds k additional terms of the form $(0 - \sqrt{\lambda}\theta_i)^2 = \lambda\beta_i^2$ to the original objective. Therefore:

$$(\hat{y} - \hat{H}\theta)^T(\hat{y} - \hat{H}\theta) = (y - H\theta)^T(y - H\theta) + \lambda\theta^T\theta$$

From the form of the left hand expression it is immediate that the Normal equations are

$$(\hat{H}^T\hat{H})\theta = \hat{H}^T\hat{y}$$

Because we adjoined zeros to the end of y, the right hand side is the same as H^Ty. On the left hand side λI is added to the original H^TH. Therefore the new Normal equations simplify to

$$(H^TH + \lambda I)\theta = H^Ty \rightarrow \theta = (H^TH + \lambda I)^{-1}H^Ty$$

□

3.3 Problem 19

$\hat{x}$ that give the $\min_{\hat{x}\in span(V)} ||x - \hat{x}||$ is the one that is the closet from $span(V)$ to x. From linear algebra we know that this $\hat{x} = proj_V(x)$. to calculate this we can do it this way:

$$\hat{x} = \underbrace{V}_{d\times k}\left(\underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_k \end{bmatrix}}_{k\times d}\underbrace{x}_{d\times 1}\right)$$

The multiplication inside the parentheses is for calculating the coefficients of x on V. We don't need to normalize the answer because V is orthonormal by saying that $V^TV = I$

3.4 Problem 20

Part i)
V is orthonormal $\rightarrow V^TV = I_{k\times k}$
Part ii)
Let $P = VV^T$, by looking at $P^2 = V\underbrace{V^TV}_{I}V^T = P$. Using induction we can say that $P^m = P \rightarrow P^{17} - P^{2000} = 0$
Part iii)
$rank(P) \leq k \rightarrow |P| = 0$

4 Image Warping

4.1 Computer Exercise 2

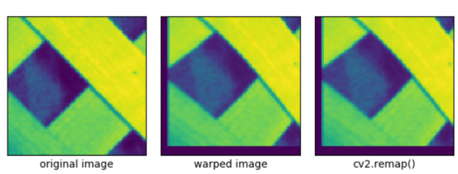

Figure 1: Computer exercise 2

4.2 Computer Exercise 3

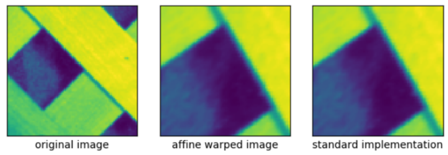

Figure 2: Computer exercise 3

5 Optical Flow

5.1 Problem 21

$$
\begin{aligned}
\arg\max_{\mu} \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) &= \arg\max_{\mu} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \\
&= \left\{e^x \ rising \ monotonic \ function\right\} \\
&= \arg\min_{\mu}\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \\
&= \arg\min_{\mu}(x-\mu)^2
\end{aligned}
\tag{23}
$$

□

5.2 Problem 22

Using sloppier notation

$$
\begin{aligned}
0 &= I_x u + I_y v + I_t \\
&= \nabla \begin{bmatrix} I_x \\ I_y \end{bmatrix} \mathbf{u} + I_t \\
&= \{\mathbf{u} = u_{normal} + u_{tangent} \\
&= \nabla_{\mathbf{x}} I u_{normal} + \nabla_{\mathbf{x}} I u_{tangent} + I_t \\
&= \{u_{tangent} \textit{ perpendicular to } \nabla_{\mathbf{x}} I\} \\
&= \nabla_{\mathbf{x}} I u_{normal} + I_t \\
\nabla_{\mathbf{x}} I u_{normal} &= -I_t \\
&= \{u_{normal} \textit{ parallel to } \nabla_{\mathbf{x}} I^T\} \\
u_{normal} &= \alpha \nabla_{\mathbf{x}} I^T \\
-I_t &= \alpha \nabla_{\mathbf{x}} I \nabla_{\mathbf{x}} I^T \\
&= \alpha ||\nabla_{\mathbf{x}} I||^2 \\
\alpha &= \frac{-I_t}{||\nabla_{\mathbf{x}} I||^2} \\
u_{normal} &= \frac{-I_t}{||\nabla_{\mathbf{x}} I||^2} \begin{bmatrix} I_x \\ I_y \end{bmatrix} \\
&\square \\
||u_{normal}|| &= \sqrt{u_{normal} u_{normal}^T} \\
&= \sqrt{\frac{I_t^2}{||\nabla_{\mathbf{x}} I||^4} \nabla_{\mathbf{x}} I \nabla_{\mathbf{x}} I^T} \\
&= \sqrt{\frac{I_t^2}{||\nabla_{\mathbf{x}} I||^4} ||\nabla_{\mathbf{x}} I||^2} \\
&= \sqrt{\frac{I_t^2}{||\nabla_{\mathbf{x}} I||^2}} \\
&= \frac{|I_t|}{||\nabla_{\mathbf{x}} I||} \\
&\square
\end{aligned}
\tag{24}
$$

5.3 Problem 23

$$\begin{bmatrix} R_x & R_y & R_t \\ G_x & G_y & G_t \\ B_x & B_y & B_t \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

There are 3 equations and 2 unknowns, so the rank of this equation system tell us how much solutions.
Rank 2 $\rightarrow$ unique solutions.
Rank 1 $\rightarrow$ infinitely solutions.
Rank 3 $\rightarrow$ no solution.

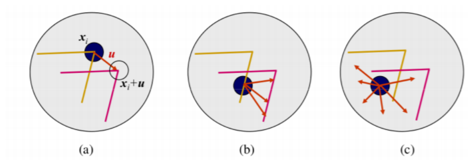

Figure 3: image (a) = rank 2, image (b) = rank 1, image (c) = rank 3

Examples of uv planes:

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{unique\ solution} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{infinitely\ solutions} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}}_{no\ solution}$$

5.4 Problem 24

$$\varepsilon(u,v) = \left\| \begin{bmatrix} R_x & R_y & R_t \\ G_x & G_y & G_t \\ B_x & B_y & B_t \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} \right\|^2$$

Yes, this cost function convex as we saw on the second least squares lecture that sum of convex functions is convex function too.

5.5 Problem 25

This equation is simply derive from the same way we got the 2D gradient constraint equation using tailor series. I'm using sloppier writing

$$I(x+u, y+v, z+h, t+1) \approx I_{\mathbf{x}} I_x u + I_y v + I_z h + I_t \tag{25}$$

$$\varepsilon = (I_x u + I_y v + I_z h + I_t)^2 \tag{26}$$

5.6 Problem 26

$$
\begin{aligned}
P_Y(y) &= p(Y < y) \\
&= p(aX + b < y) \\
&= p(X < \tfrac{y-b}{a}) \\
&= \int_{-\infty}^{\frac{y-b}{a}} \frac{1}{\sqrt{2\pi\sigma}} exp\bigg(- \frac{\big(x-\mu\big)^2}{2\sigma^2}\bigg) dx \\
&= \{y = ax + b,\ dy = adx,\ upper\ limit\ = \ y\} \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma}} exp\bigg(- \frac{\big(\frac{y-b}{a} - \mu\big)^2}{2\sigma^2}\bigg) \frac{1}{a} dy \\
&= \frac{1}{a} p_X(\tfrac{y-b}{a}) \\
&= \frac{1}{a}\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma}} exp\bigg(- \frac{\big(y-(a\mu+b)\big)^2}{2(a\sigma)^2}\bigg) dy \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi(a\sigma)}} exp\bigg(- \frac{\big(y-(a\mu+b)\big)^2}{2(a\sigma)^2}\bigg) dy \\
&= \mathcal{N}(a\mu + b, (a\sigma)^2)
\end{aligned}
\tag{27}
$$

□

5.7 Problem 27

$$
\begin{aligned}
\varepsilon &\sim \mathcal{N}(0, \sigma^2) \\
p_Y(y) &= p(Y < y) \\
&= p(\mu + \varepsilon < y) \\
&= p(\varepsilon < y - \mu) \\
&= \int_{-\infty}^{y-\mu} \frac{1}{\sqrt{2\pi\sigma}} exp\bigg(- \frac{\varepsilon^2}{2\sigma^2}\bigg) d\varepsilon \\
&= \{y = \mu + \varepsilon\} \\
&= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma}} exp\bigg(- \frac{\big(y-\mu\big)^2}{2\sigma^2}\bigg) dy \\
&\sim \mathcal{N}(\mu, \sigma^2)
\end{aligned}
\tag{28}
$$

□

5.8 Problem 28

$$-I_t = \mu + \varepsilon \tag{29}$$

From problem 27

$$-I_t \sim \mathcal{N}(\mu, \sigma^2) \tag{30}$$

□

5.9 Computer Exercise 4

5.10 Computer Exercise 4

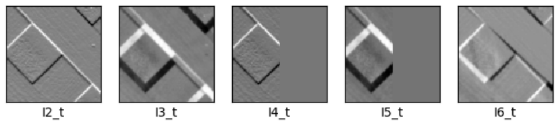

Figure 4: Computer exercise 4

5.11 Horn-and-Schunck Optical Flow

5.11.1 Problem 29

$\frac{E_{data}}{\partial u_{ij}}$ and $\frac{E_{data}}{\partial v_{ij}}$ are the same on borders and corners because it doesn't get affected by its neighbors.
On the other hand $E_{smoothness}$ does get affected by its neighbors so the formulas in these cases are:
$Borders$)

$$
\begin{aligned}
\frac{E_{smoothness}}{\partial u_{1j}} &= 2\lambda\big(3u_{1j} - (u_{2j} + u_{1j+1} + u_{1j-1})\big) \\
\frac{E_{smoothness}}{\partial v_{1j}} &= 2\lambda\big(3v_{1j} - (v_{2j} + v_{1j+1} + v_{1j-1})\big) \\
\frac{E_{smoothness}}{\partial u_{Nj}} &= 2\lambda\big(3u_{Nj} - (u_{N-1j} + u_{Nj+1} + u_{Nj-1})\big) \\
\frac{E_{smoothness}}{\partial v_{Nj}} &= 2\lambda\big(3v_{Nj} - (v_{N-1j} + v_{Nj+1} + v_{Nj-1})\big) \\
\frac{E_{smoothness}}{\partial u_{i1}} &= 2\lambda\big(3u_{i1} - (u_{i+1,1} + u_{i-1,1} + u_{i2})\big) \\
\frac{E_{smoothness}}{\partial v_{i1}} &= 2\lambda\big(3v_{i1} - (v_{i+1,1} + v_{i-1,1} + v_{i2})\big) \\
\frac{E_{smoothness}}{\partial u_{iN}} &= 2\lambda\big(3u_{iN} - (u_{i+1N} + u_{i-1N} + u_{iN-1})\big) \\
\frac{E_{smoothness}}{\partial v_{iN}} &= 2\lambda\big(3v_{iN} - (v_{i+1N} + v_{i-1N} + v_{iN-1})\big)
\end{aligned}
\tag{31}
$$

Corners)

$$\begin{aligned}
\frac{E_{smoothness}}{\partial u_{11}} &= 2\lambda\big(2u_{11} - (u_{21} + u_{12})\big)\\
\frac{E_{smoothness}}{\partial v_{11}} &= 2\lambda\big(3v_{1j} - (v_{2j} + v_{2j+1} + v_{2j-1})\big)\\
\frac{E_{smoothness}}{\partial u_{N1}} &= 2\lambda\big(2u_{N1} - (u_{N-1,1} + u_{N2})\big)\\
\frac{E_{smoothness}}{\partial v_{N1}} &= 2\lambda\big(2v_{N1} - (v_{N-1,1} + v_{N2})\big)\\
\frac{E_{smoothness}}{\partial u_{N1}} &= 2\lambda\big(2u_{i1} - (u_{N-1,1} + u_{N2})\big)\\
\frac{E_{smoothness}}{\partial v_{N1}} &= 2\lambda\big(2v_{i1} - (v_{N-1,1} + v_{N2})\big)\\
\frac{E_{smoothness}}{\partial u_{1N}} &= 2\lambda\big(2u_{1N} - (u_{2N} + u_{1N-1})\big)\\
\frac{E_{smoothness}}{\partial v_{1N}} &= 2\lambda\big(2v_{1N} - (v_{2N} + v_{1N-1})\big)
\end{aligned} \tag{32}$$

5.11.2 Problem 30

(i) E_{data} is the gradient constraint equation, so if $\boldsymbol{u}$ satisfy this equation this is the flow that minimize E_{data}
(ii) $E_{smoothness}$ get its minimum to 0 for every image that its flow is constant no matter the value, so there is no unique minimizer.

5.11.3 Problem 31

$$\begin{aligned}
f(u) &= \sum_{s} \sum_{s':s'\sim s} (u(s) - u(s'))^2\\
\frac{d}{du(s)} f(u) &= 2 \sum_{s':s'\sim s} (u(s) - u(s')) - 2 \sum_{s':s'\sim s} (u(s') - u(s))\\
&= 4 \sum_{s':s'\sim s} (u(s) - u(s'))
\end{aligned} \tag{33}$$

□
By using the proof above and the given partial derivatives this is simply place it instead of $E_smoothness$ which give us similar solution to problem 29

5.11.4 Problem 32

One based indexing gradient term is the same for every pixel include the boundaries

5.11.5 Problem 33

Every empty place its 0, $\#i$ means row number i of A
Borders)

$i = 1, \qquad i_s = j$

$\#2i_s - 1$

$$\begin{bmatrix} \cdots & \underbrace{-2\lambda}_{2i_s-3} & 0 & \underbrace{I_x^2+6\lambda}_{2i_s-1} & \underbrace{I_xI_y}_{2i_s} & \underbrace{-2\lambda}_{2i_s+1} & 0 & \cdots & \underbrace{-2\lambda}_{2i_s+2N_{cols}-1} & 0 & \cdots \end{bmatrix}$$

$\#2i_s$

$$\begin{bmatrix} \cdots & 0 & \underbrace{-2\lambda}_{2i_s-2} & \underbrace{I_xI_y}_{2i_s-1} & \underbrace{I_y^2+6\lambda}_{2i_s} & 0 & \underbrace{-2\lambda}_{2i_s+2} & \cdots & 0 & \underbrace{-2\lambda}_{2i_s+2N_{cols}} & \cdots \end{bmatrix}$$

$i = N_{rows}, \qquad i_s = (N_{rows}-1)N_{cols} + j$

$\#2i_s - 1$

$$\begin{bmatrix} \cdots & \underbrace{-2\lambda}_{2i_s-2N_{cols}-1} & 0 & \cdots & \underbrace{-2\lambda}_{2i_s-3} & 0 & \underbrace{I_x^2+6\lambda}_{2i_s-1} & \underbrace{I_xI_y}_{2i_s} & \underbrace{-2\lambda}_{2i_s+1} & 0 & \cdots \end{bmatrix}$$

$\#2i_s$

$$\begin{bmatrix} \cdots & 0 & \underbrace{-2\lambda}_{2i_s-2N_{cols}} & \cdots & 0 & \underbrace{-2\lambda}_{2i_s-2} & \underbrace{I_xI_y}_{2i_s-1} & \underbrace{I_y^2+6\lambda}_{2i_s} & 0 & \underbrace{-2\lambda}_{2i_s+2} & \cdots \end{bmatrix}$$

$j = 1, \qquad i_s = (i-1)N_{cols} + 1$

$\#2i_s - 1$

$$\begin{bmatrix} \cdots & \underbrace{-2\lambda}_{2i_s-2N_{cols}-1} & 0 & \cdots & \underbrace{I_x^2+6\lambda}_{2i_s-1} & \underbrace{I_xI_y}_{2i_s} & \underbrace{-2\lambda}_{2i_s+1} & 0 & \cdots & \underbrace{-2\lambda}_{2i_s+2N_{cols}-1} & 0 & \cdots \end{bmatrix}$$

$\#2i_s$

$$\begin{bmatrix} \cdots & 0 & \underbrace{-2\lambda}_{2i_s-2N_{cols}} & \cdots & \underbrace{I_xI_y}_{2i_s-1} & \underbrace{I_y^2+6\lambda}_{2i_s} & 0 & \underbrace{-2\lambda}_{2i_s+2} & \cdots & 0 & \underbrace{-2\lambda}_{2i_s+2N_{cols}} & \cdots \end{bmatrix}$$

$j = N_{cols}, \qquad i_s = iN_{cols}$

$\#2i_s - 1$

$$\begin{bmatrix} \cdots & \underbrace{-2\lambda}_{2i_s-2N_{cols}-1} & 0 & \cdots & \underbrace{-2\lambda}_{2i_s-3} & 0 & \underbrace{I_x^2+6\lambda}_{2i_s-1} & \underbrace{I_xI_y}_{2i_s} & \cdots & \underbrace{-2\lambda}_{2i_s+2N_{cols}-1} & 0 & \cdots \end{bmatrix}$$

$\#2i_s$

$$\begin{bmatrix} \cdots & 0 & \underbrace{-2\lambda}_{2i_s-2N_{cols}} & \cdots & 0 & \underbrace{-2\lambda}_{2i_s-2} & \underbrace{I_xI_y}_{2i_s-1} & \underbrace{I_y^2+6\lambda}_{2i_s} & \cdots & 0 & \underbrace{-2\lambda}_{2i_s+2N_{cols}} & \cdots \end{bmatrix}$$

$Corners)$

$i=1, j=1 \qquad i_s = 1$

$\#1$

$$\left[\underbrace{I_x^2+4\lambda}_{1} \quad \underbrace{I_xI_y}_{2} \quad \underbrace{-2\lambda}_{3} \quad 0 \quad \cdots \quad \underbrace{-2\lambda}_{2N_{cols}+1} \quad 0 \quad \cdots\right]$$

$\#2$

$$\left[\underbrace{I_xI_y}_{1} \quad \underbrace{I_y^2+4\lambda}_{2} \quad 0 \quad \underbrace{-2\lambda}_{4} \quad \cdots \quad 0 \quad \underbrace{-2\lambda}_{2N_{cols}+2} \quad \cdots\right]$$

$i=1, j=N_{cols} \qquad i_s = N_{cols}$

$\#2N_{cols}-1$

$$\left[\cdots \quad \underbrace{-2\lambda}_{2N_{cols}-3} \quad 0 \quad \underbrace{I_x^2+4\lambda}_{2N_{cols}-1} \quad \underbrace{I_xI_y}_{2N_{cols}} \quad \cdots \quad \underbrace{-2\lambda}_{4N_{cols}-1} \quad 0 \quad \cdots\right]$$

$\#2N_{cols}$

$$\left[\cdots \quad 0 \quad \underbrace{-2\lambda}_{2N_{cols}-2} \quad \underbrace{I_xI_y}_{2N_{cols}-1} \quad \underbrace{I_y^2+4\lambda}_{2N_{cols}} \quad 0 \quad \cdots \quad 0 \quad \underbrace{-2\lambda}_{4N_{cols}} \quad \cdots\right]$$

$i=N_{rows}, j=1 \qquad i_s = N-N_{cols}+1$

$\#2N-2N_{cols}+2$

$$\left[\cdots \quad \underbrace{-2\lambda}_{2N-4N_{cols}+1} \quad 0 \quad \underbrace{I_x^2+4\lambda}_{2N-2N_{cols}+1} \quad \underbrace{I_xI_y}_{2N-2N_{cols}} \quad \underbrace{-2\lambda}_{2N-2N_{cols}+2} \quad 0 \quad \cdots\right]$$

$\#2i_s$

$$\left[\cdots \quad 0 \quad \underbrace{-2\lambda}_{2N-4N_{cols}+2} \quad \underbrace{I_xI_y}_{2N-2N_{cols}-1} \quad \underbrace{I_y^2+4\lambda}_{2N-2N_{cols}} \quad 0 \quad \underbrace{-2\lambda}_{2N-2N_{cols}+2} \quad \cdots\right]$$

$i=N_{rows}, j=N_{cols} \qquad i_s = N$

$\#2N-1$

$$\left[\cdots \quad \underbrace{-2\lambda}_{2N-2N_{cols}-1} \quad 0 \quad \cdots \quad \underbrace{-2\lambda}_{2N-3} \quad 0 \quad \underbrace{I_x^2+4\lambda}_{2N-1} \quad \underbrace{I_xI_y}_{2N}\right]$$

$\#2N$

$$\left[\cdots \quad 0 \quad \underbrace{-2\lambda}_{2N-2N_{cols}} \quad \cdots \quad 0 \quad \underbrace{-2\lambda}_{2N-2} \quad \underbrace{I_xI_y}_{2N-1} \quad \underbrace{I_y^2+4\lambda}_{2N}\right]$$

5.11.6 Problem 34

(i)

$$\begin{aligned}\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} &= \begin{bmatrix}y-x\\ z-y\end{bmatrix}\\ A &= \begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\end{bmatrix}\end{aligned} \tag{34}$$

(*ii*)

$$
\begin{aligned}
\left\| \begin{bmatrix} y-x \\ z-y \end{bmatrix} \right\|^2 &= \begin{bmatrix} y-x & z-y \end{bmatrix} \begin{bmatrix} y-x \\ z-y \end{bmatrix} \\
&= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
&= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\
Q &= \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\end{aligned} \tag{35}
$$

(*iii*) Q is symmetric and we know that normal of vector is ≥ 0 So this matrix is at least SPSD, now we need to calculate its eigenvalue to see if its even stronger as SPD.

$$
\begin{aligned}
0 &= |Q - \lambda I| \\
&= \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ -1 & 1-\lambda \end{vmatrix} \\
&= (1-\lambda)\big((2-\lambda)(1-\lambda) - 1\big) - (1-\lambda) \\
&= (1-\lambda)\big((2-\lambda)(1-\lambda) - 2\big) \\
&= \{\lambda_1 = 1\} \\
&= (2-\lambda)(1-\lambda) - 2 \\
&= 2 - 3\lambda + \lambda^2 - 2 \\
&= \lambda(\lambda - 3) \\
&= \{\lambda_2 = 0,\ \lambda_3 = 3\}
\end{aligned} \tag{36}
$$

$\lambda_2 = 0$ which means that Q is only SPSD and not SPD as we learn from the very first lectures

5.12 Computer Exercise 5

After problem 35

5.13 Problem 35

$$
\begin{aligned}
E(u(x), v(x)) &= \sum_{i=1}^{N} g(\mathbf{x}, t)\left(\nabla_{\mathbf{x}} I(\mathbf{x}_i, t)\begin{bmatrix} u(\mathbf{x}_i) \\ v(\mathbf{x}_i) \end{bmatrix} + I_t(\mathbf{x}_i, t)\right)^2 \\
&= \sum_i \left(\sqrt{g}\Big(I_x A\theta + I_t\Big)\right)^2 \\
&= \sum_i \left(\sqrt{g}\Big(I_x A\theta + I_t\Big)\right)^T \left(\sqrt{g}\Big(I_x A\theta + I_t\Big)\right) \\
&= \sum_i \left(\underbrace{\sqrt{g} I_x A}_{H_i}\theta + \underbrace{\sqrt{g} I_t}_{-y_i}\right)\left(s\sqrt{g} I_x A\theta + \sqrt{g} I_t\right)
\end{aligned}
$$

$$
\begin{aligned}
\theta_{LS} &= \big(H^T H\big)^{-1} H^T y \\
&= \left(\big(\sqrt{g} I_x A\big)^T\right)\big(\sqrt{g} I_x A\big)\Bigg)^{-1} \cdot \big(\sqrt{g} I_x A\big)^T \cdot -\sqrt{g} I_t \\
&= A^T U_x^T \sqrt{g}^T \sqrt{g} I_x A \qquad\qquad\qquad\qquad \cdot - I_x^T g I_t \\
&= \underbrace{\big(g A^T I_x^T I_x A{-}1}_{M(x)}\big) \qquad\qquad\qquad\qquad \cdot \underbrace{-g I_x^T I_t}_{b(x)} \\
&= M^{-1}(x) b(x)
\end{aligned}
$$

$\square$

5.14 Computer Exercise 5

I calculated Horn and Schunck optical flow for 4 different lambda and these are my results, I add the wrapped image to see which lambda was the best option and because the results were similar I decided to add mse to be more precise of my results.
I saw that the difference was only about a thousandth between each lambda. except zero that some of the pictures doesn't give a really good results.

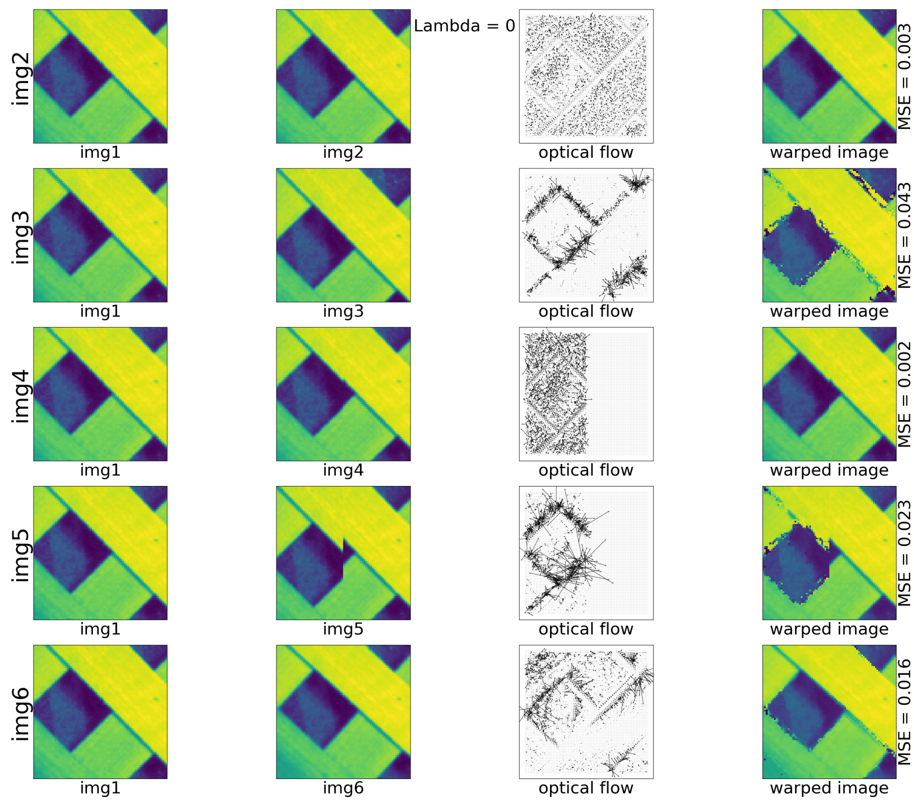

Figure 5: Lambda = 0

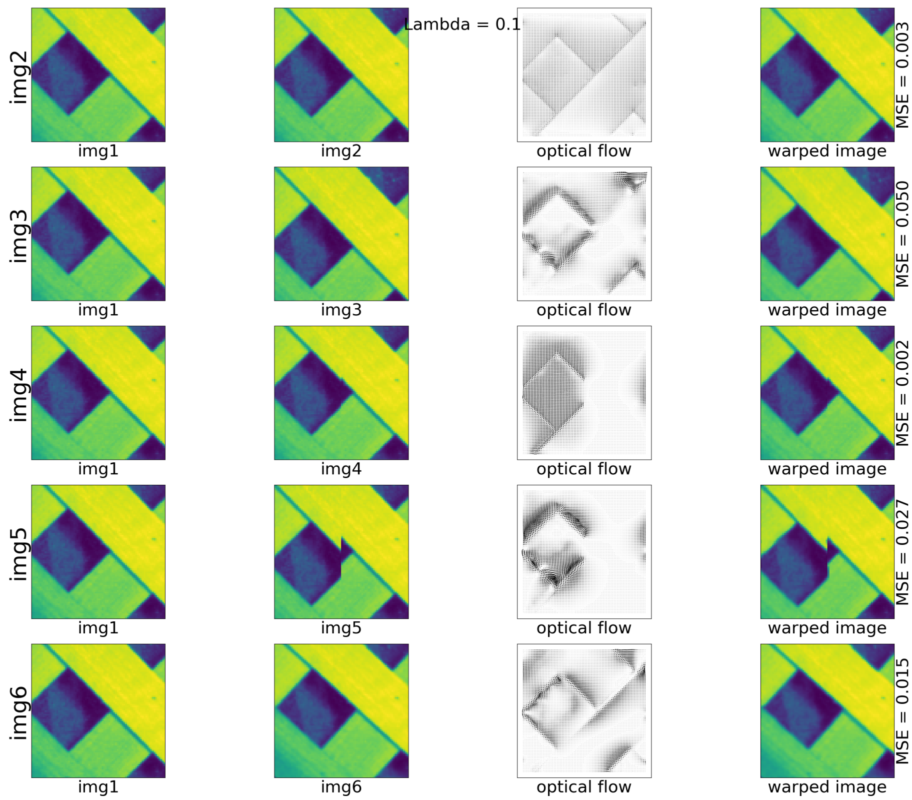

Figure 6: Lambda = 0.1

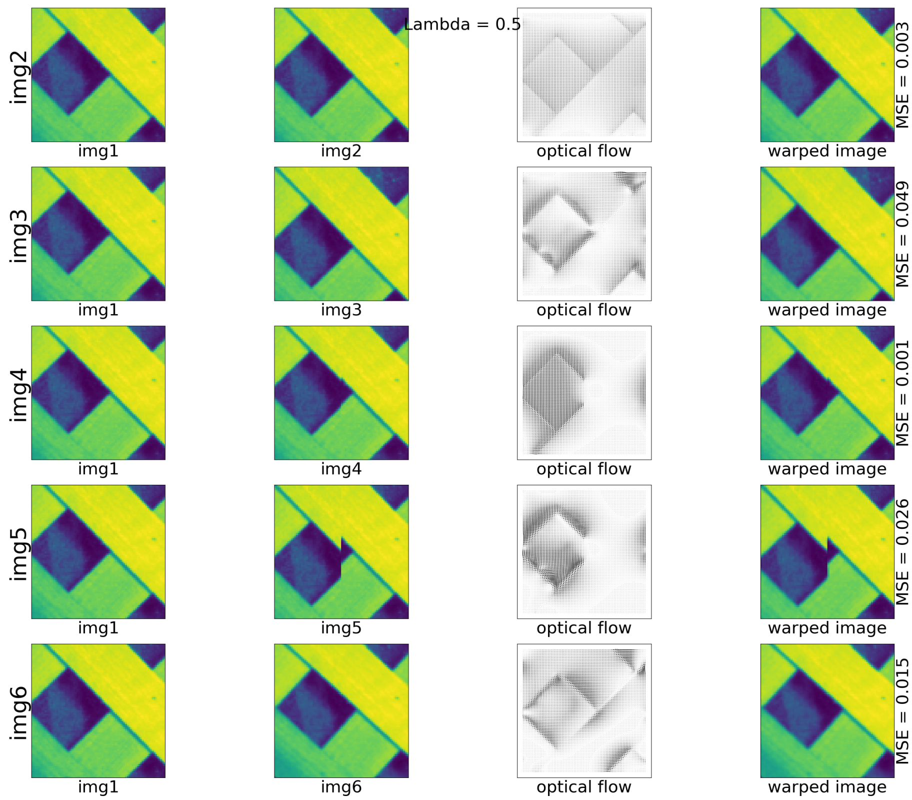

Figure 7: Lambda = 0.5

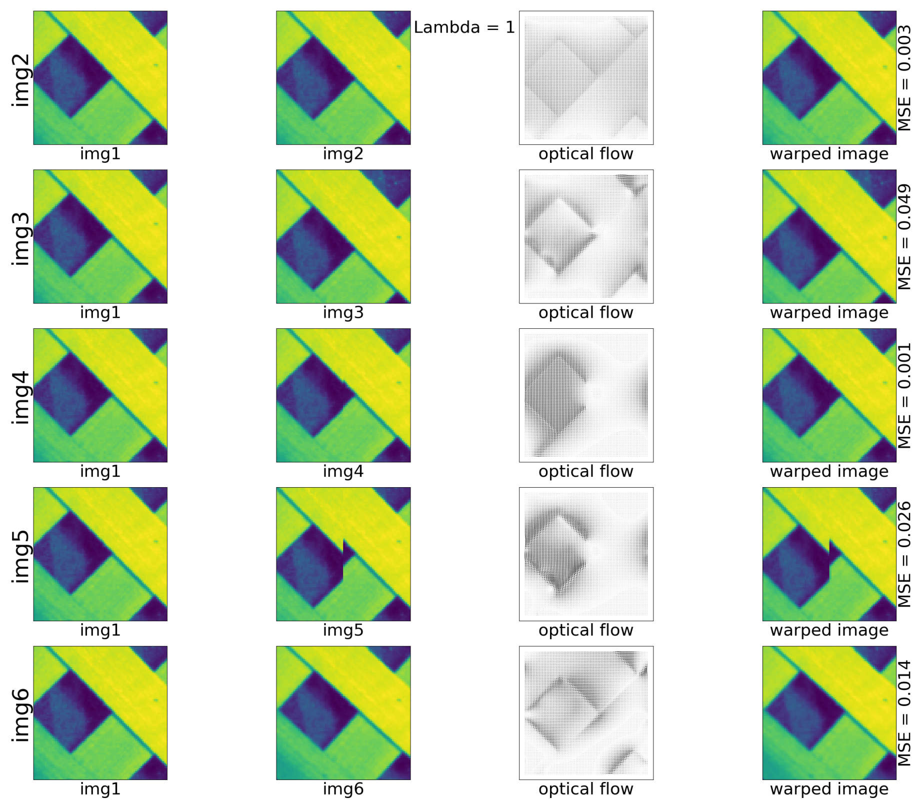

Figure 8: Lambda = 1

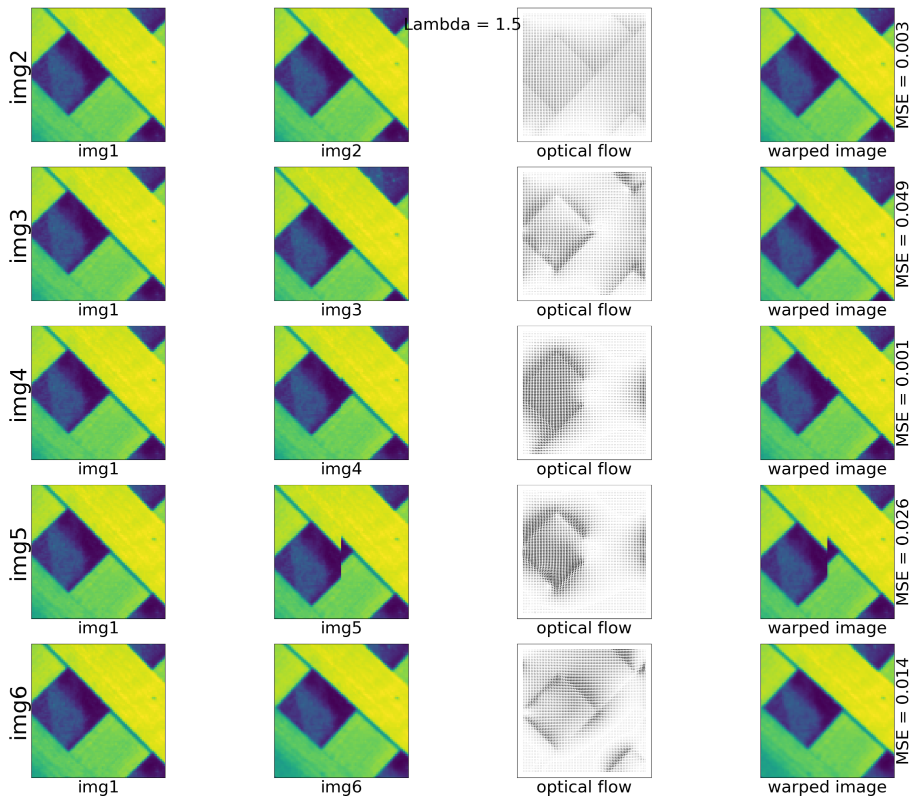

Figure 9: Lambda = 1.5