Signal and Image Processing by Computer (236327)

Spring 2018

Homework #4

Publication: June 15th , 2018

Submission: by July 1st at 23:59

Guidelines:

- Submission is only in pairs.
- Submit your solution to the course box at Taub floor 1.
- Matlab code should be electronically submitted via the course website.
- You may answer in Hebrew or English, in a clear-handwrite or a printed form.
- Follow the FAQ at course website, as clarifications and corrections may be published only there.

הנחיות:

- **תאריך ושעת הגשה:** עד ה-1.7.2018 בשעה 23:59.
- ההגשה בזוגות בלבד.
- ההגשה ידנית לתא הקורס בטאוב קומה 1.
- את קוד המטלב יש להגיש אלקטרונית באתר הקורס.
- ניתן לענות בעברית או באנגלית, בכתב ברור או מודפס.
- חובה לעקוב אחרי FAQ באתר. הבהרות ותיקונים בקשר לתרגיל יפורסמו שם בלבד.

Question #1 (22 Points)

Consider a scalar quantizer $Q(x)$ that is defined for any $x \in \mathbb{R}$.

The quantizer has K representation values, $\{r_i\}_{i=1}^{K}$, and corresponding decision levels $\{d_i\}_{i=1}^{K-1}$ such that the quantization function is defined as

$$Q(x) = \begin{cases} r_1 & ,\text{for} \quad x < d_1 \\ r_i & ,\text{for} \quad d_{i-1} \le x < d_i \qquad i = 2, \dots, K-1 \\ r_K & ,\text{for} \quad x \ge d_{K-1} \end{cases}$$

Note that the quantizer defined in this question is not necessarily optimized in any particular way.

The discrete signal $\overline{\boldsymbol{\varphi}}^{\boldsymbol{in}} \triangleq \begin{bmatrix} \varphi_1^{in} \\ \varphi_2^{in} \\ \vdots \\ \varphi_N^{in} \end{bmatrix}$ is an N-length real-valued column vector.

The system $\mathcal{H}\{\cdot\}$ does scalar quantization on the input values.

The system's output for the input signal $\overline{\boldsymbol{\varphi}}^{\boldsymbol{in}}$ is $\overline{\boldsymbol{\varphi}}^{\boldsymbol{out}} \triangleq \begin{bmatrix} \varphi_1^{out} \\ \varphi_2^{out} \\ \vdots \\ \varphi_N^{out} \end{bmatrix}$ obeying the relation

$$\overline{\boldsymbol{\varphi}}^{\boldsymbol{out}} = \mathcal{H}\left\{\overline{\boldsymbol{\varphi}}^{\boldsymbol{in}}\right\}$$

where the components of the output signal are defined via

$$\varphi_k^{out} \triangleq Q\left(\varphi_k^{in}\right) \quad \text{for} \quad k = 1, \dots, N$$

a. Is $\mathcal{H}$ a linear system? Prove.

b. Is $\mathcal{H}$ a shift-invariant system? Prove.

Question #2 (12 points) – Commutativity of Discrete Linear Shift Invariant Operators

Show that two LSI operators, denoted by the matrices $\mathbf{H}_{\mathrm{LSI}}^{(1)}$ and $\mathbf{H}_{\mathrm{LSI}}^{(2)}$, commute, i.e., prove that $\mathbf{H}_{\mathrm{LSI}}^{(1)}\mathbf{H}_{\mathrm{LSI}}^{(2)} = \mathbf{H}_{\mathrm{LSI}}^{(2)}\mathbf{H}_{\mathrm{LSI}}^{(1)}$.

Question #3 (10 points)

A unitary transform $\mathbf{U}$ is applied on the random vector $\mathbf{x}$, resulting in $\mathbf{y} = \mathbf{U}^T\mathbf{x}$.

The autocorrelation matrices of $\mathbf{x}$ and $\mathbf{y}$ are denoted as $\mathbf{R}_x$ and $\mathbf{R}_y$, respectively.

Prove that $Trace\left[\mathbf{R}_y\right] = Trace[\mathbf{R}_x]$.

Question #4 (28 points) – The Discrete Cosine Transform

The $N \times N$ DCT matrix, $\mathbf{U}$, is comprised from the elements defined as

$$u_{n,k} = \begin{cases} \dfrac{1}{\sqrt{N}} & \text{for } k=1 \quad , \ 1 \le n \le N \\ \sqrt{\dfrac{2}{N}} \cos\left(\dfrac{\pi(2n-1)(k-1)}{2N}\right) & \text{for } 2 \le k \le N \ , \ 1 \le n \le N \end{cases}$$

where $u_{n,k}$ is the value of the matrix component at the k^{th} column and n^{th} row.

a. Prove that the DCT matrix $\mathbf{U}$ is unitary.

Hints:
You may assist trigonometric identities such as

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$
$$2\cos\theta\cos\varphi = \cos(\theta-\varphi) + \cos(\theta+\varphi)$$

and that the sum of the roots of unity of order N (or their nonzero powers, $p \neq 0$) is zero, i.e., for $p \neq 0$:

$$\sum_{n=1}^{N} W_N^{n \cdot p} = 0$$

where $W_N \triangleq e^{-\frac{i2\pi}{N}}$.

b. Is the DCT matrix the real part of the DFT matrix? Explain.

c. Consider a class of discrete random signals that is described by zero mean and the autocorrelation matrix:

$$R = \begin{bmatrix} 1-\alpha & -\alpha & 0 \\ -\alpha & 1 & -\alpha \\ 0 & -\alpha & 1-\alpha \end{bmatrix}$$

where $0 < \alpha < 1$ is some scalar value.

What is the PCA for this class? i.e., what is the unitary matrix that diagonalizes ?

d. Is the PCA matrix from subsection c related to the 3×3 DCT matrix?

Question #5 (28 Points)

A class of one-dimensional discrete signals is defined as follows.

A signal is the column vector of N samples, where $\boldsymbol{N}$ **is an even number**, of the form:

$$\overline{\varphi} = [M, \quad \cdots \quad M, \quad \underbrace{M+L,}_{K^{th}\text{ component}} \quad M, \quad \cdots \quad M, \quad \underbrace{M+L,}_{\left(K+\frac{N}{2}\right)^{th}\text{ component}} \quad M, \quad \cdots \quad M]^T$$

i.e., all the vector components have the value M except for the components in the K and the $\left(K+\frac{N}{2}\right)$ coordinates that have the value $M+L$.

Note that $\boldsymbol{K} \in \left\{\mathbf{1}, \ldots, \frac{\boldsymbol{N}}{\mathbf{2}}\right\}$.

The vector components are indexed starting at 1, i.e., the vector can be generally formulated as

$$\overline{\varphi} = [\varphi_1, \ldots, \varphi_N]^T$$

M, L, and K are **independent random variables**:

K is a uniform random variable over the integers $\left\{1, \ldots, \frac{N}{2}\right\}$.

M obeys $E\{M\} = 0$ and $E\{M^2\} = c$

L obeys $E\{L\} = 0$ and $E\{L^2\} = \frac{N}{2}(1-c)$

where $0 < c < 1$ is a (deterministic) constant.

a. Calculate the autocorrelation matrix of $\overline{\varphi}$, denoted as $\mathbf{R}_{\overline{\varphi}}$, and show it is circulant.
Note that the next subsections do not depend on your answer here.

b. Explain how the eigenvalues of $\mathbf{R}_{\overline{\varphi}}$ can be computed in a way that is simpler than an explicit diagonalization\eigendecomposition procedure.

c. Consider two general and **independent** random vectors $\overline{\varphi}^{(1)}$ and $\overline{\varphi}^{(2)}$, having the same size and **zero-mean**.
The autocorrelation matrix of $\overline{\varphi}^{(1)}$ is denoted as $\mathbf{R}^{(1)}$, and the autocorrelation matrix of $\overline{\varphi}^{(2)}$ is denoted as $\mathbf{R}^{(2)}$.
An additional random vector is defined as their sum:
$$\overline{\varphi}^{sum} = \overline{\varphi}^{(1)} + \overline{\varphi}^{(2)}$$
Express the autocorrelation matrix of $\overline{\varphi}^{sum}$, denoted as $\mathbf{R}^{sum}$, in terms of $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$. Show the mathematical developments leading to your result.

d. The random vector $\overline{\varphi}$ is deteriorated by a linear degradation operator $\mathbf{H}$ and then an additive noise vector $\overline{\mathrm{n}}$, resulting in the degraded signal
$$\overline{\varphi}^{deg} = \mathbf{H}\overline{\varphi} + \overline{\mathrm{n}}$$
The noise $\overline{\mathrm{n}}$ and the signal $\overline{\varphi}$ are independent.
The signal autocorrelation matrix is denoted as $\mathbf{R}_{\overline{\varphi}}$, and the noise-vector autocorrelation matrix is denoted as $\mathbf{R}_{\mathrm{n}}$.

- What is the autocorrelation matrix of the degraded signal $\overline{\varphi}^{deg}$?
- Formulate the Wiener filter appropriate to the above problem.

Optional Matlab Part (10 Bonus points) – Random Signals and Denoising using the Wiener Filter

Important: In order to get the bonus points of this section, you are required to submit a solution to all the sections below. For example, this means that if you will submit a solution only to section (a) you will not get any bonus points.

General instructions:

- Figures should be appropriately titled.
- Your written report should be submitted with the dry part, whereas the code should be electronically submitted.

Consider again the class of one-dimensional discrete signals is defined above in Question #5 and note the specific definition given here below in red:

A signal is the column vector of N samples, where ***N* is an even number**, of the form:

$$\overline{\varphi} = [M, \quad \cdots \quad M, \quad \underbrace{M+L,}_{K^{th}\text{ component}} \quad M, \quad \cdots \quad M, \quad \underbrace{M+L,}_{\left(K+\frac{N}{2}\right)^{th}\text{ component}} \quad M, \quad \cdots \quad M]^T$$

i.e., all the vector components have the value M except for the components in the K and the $\left(K+\frac{N}{2}\right)$ coordinates that have the value $M+L$. **Note that $\boldsymbol{K} \in \left\{\mathbf{1}, \dots, \frac{\boldsymbol{N}}{\mathbf{2}}\right\}$.**

The vector components are indexed starting at 1, i.e., the vector can be generally formulated as $\overline{\varphi} = [\varphi_1, \dots, \varphi_N]^T$.

M, L, and K are **independent random variables**:

K is a uniform random variable over the integers $\left\{1, \dots, \frac{N}{2}\right\}$.

M obeys $E\{M\} = 0$ and $E\{M^2\} = c$

L obeys $E\{L\} = 0$ and $E\{L^2\} = \frac{N}{2}(1-c)$

where $0 < c < 1$ is a (deterministic) constant.

Consider signals of length $N = 64$ samples.
Moreover, the random variables M and L have here Gaussian distributions that obey the second-order statistics of the class for a parameter $= 0.8$.

a. Produce a large (and sufficient) amount of realizations of the class defined above (note that each signal realization relies on realizations of K,M and L).
   Use these realizations to calculate the empirical approximation of the mean signal and the autocorrelation matrix of the class.

   - Present the empirical mean of the class (using the 'plot' command), and show the empirically estimated autocorrelation matrix (using the 'imagesc' command).

- How well do your empirical results approximate the analytical results obtained in Question #1?
- What is the number of realizations needed for obtaining a good empirical approximation of the second-order statistics?

b. Signals of the above class are deteriorated by an additive white Gaussian noise:

$$\overline{\varphi}_{noisy} = \overline{\varphi} + \overline{\mathrm{n}}$$

where

$\overline{\mathrm{n}}$ is an additive noise vector, considered as a realization of an N-length random vector, having i.i.d components (with variance σ_n^2) that follow

$$\bar{\mu}_{\mathrm{n}} \triangleq E\overline{\mathrm{n}} = \overline{0} \quad \text{and} \quad \mathbf{R}_{\mathrm{n}} \triangleq E\overline{\mathrm{n}}\overline{\mathrm{n}}^T = \sigma_n^2 \mathbf{I}$$

$\overline{\varphi}_{noisy}$ is the given degraded signal (a N-length column vector).

Consider a noise variance of $\sigma_n^2 = 1$.

- Construct in Matlab the Wiener filter for denoising the above defined noisy signals.
  Show the filter matrix using the 'imagesc' command and explain the filter structure.
- Produce a large (and sufficient) amount of realizations of the class defined above, each should be deteriorated by a (different) realization of the noise vector defined above.
  Note that the clean signal should be kept for the purpose of computing the MSE of the denoised signal. However, the denoising task should be applied with respect to the noisy signal only.
- Use the Wiener filter constructed above to denoise the realizations of the noisy signals.
  - Plot several examples of denoised signals with respect to their clean and the noisy versions.
  - For each denoised realization, compute the MSE with respect to the clean version of the same realization.
    Average the MSE for all the realizations and write this number. This is the **empirical** approximation of the **expected** MSE of denoising using the Wiener filter.

c. Repeat subsection (b) for a noise variance of $\sigma_n^2 = 5$. Explain the differences in the obtained results.