Signal and Image Processing by Computer (236327)

Spring 2018

Homework #2

Publication: May 9th , 2018

Submission: by May 29th at 13:30

Guidelines:

- Submission is only in pairs.
- Submit your solution to the course box at Taub floor 1.
- Matlab code should be electronically submitted via the course website.
- You may answer in Hebrew or English, in a clear-handwrite or a printed form.
- Follow the FAQ at course website, as clarifications and corrections may be published only there.

הנחיות:

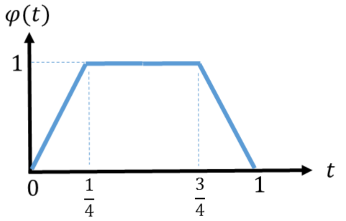
- **תאריך ושעת הגשה:** עד ה-29.5.2018 בשעה 13:30.
- ההגשה בזוגות בלבד.
- ההגשה ידנית לתא הקורס בטאוב קומה 1.
- את קוד המטלב יש להגיש אלקטרונית באתר הקורס.
- ניתן לענות בעברית או באנגלית, בכתב ברור או מודפס.
- חובה לעקוב אחרי FAQ באתר. הבהרות ותיקונים בקשר לתרגיל יפורסמו שם בלבד.

Question #1 (15 points) – Uniform Sampling

Consider the signal:

$$\varphi(t) = \begin{cases} 4t & , \text{ for } t \in \left[0, \frac{1}{4}\right) \\ 1 & , \text{ for } t \in \left[\frac{1}{4}, \frac{3}{4}\right) \\ 1 - 4\left(t - \frac{3}{4}\right) & , \text{ for } t \in \left[\frac{3}{4}, 1\right) \end{cases}$$

that is graphically described as



The signal is uniformly sampled based on N equal-length intervals, resulting in a piecewise-constant reconstruction of the form

$$\hat{\varphi}_N(t) = \varphi_i \qquad \text{for } t \in \left[\tfrac{i-1}{N}, \tfrac{i}{N}\right)$$

where $i = 1, \dots, N$ denotes the sampling interval index, and φ_i is the corresponding sample (a real scalar).

The corresponding Mean Squared Error is defined as

$$MSE_N = \int_{t=0}^{1} (\varphi(t) - \hat{\varphi}_N(t))^2 dt$$

a. For $N = 4$, what are the samples $\varphi_1, \dots, \varphi_N$ and the corresponding MSE_N ?

b. For $N = 8$, what are the samples $\varphi_1, \dots, \varphi_N$ and the corresponding MSE_N ?

Question #2 (25 points) –

Signal Discretization using a Piecewise-Linear Approximation

In class we discussed a signal sampling procedure that relies on a piecewise-constant approximation of the given signal. In this question, we extend the sampling procedure to rely on a piecewise-linear approximation of the signal.

The given signal, $\varphi(t)$, is defined for $t \in [0,1)$ as a mapping to the range of values $[\varphi_L, \varphi_H]$.

Let us consider a discretization procedure based on a uniform segmentation of the unit interval into N intervals of equal size, i.e.,

$$\Delta_i = \left[\frac{i-1}{N}, \frac{i}{N}\right) \qquad , \quad i = 1, \ldots, N.$$

The approximated signal, $\hat{\varphi}(t)$, is formed from linear approximations, each associated with an interval:

$$\text{For } t \in \left[\frac{i-1}{N}, \frac{i}{N}\right): \qquad \hat{\varphi}(t) = a_i(t - t_i) + c_i$$

where a_i and c_i are real-valued scalar constants defining the linear approximation of the i^{th} interval, and t_i is the center of the i^{th} interval.

The approximations are evaluate here for the MSE criterion.

a. Show that for a positive integer k:

$$\int_{t \in \Delta_i} (t - t_i)^k dt = \begin{cases} 0 & , k \text{ is odd} \\ \dfrac{|\Delta_i|^{k+1}}{2^k \cdot (k+1)} & , k \text{ is even} \end{cases}$$

where $|\Delta_i|$ is the size of the interval.

b. What are the **optimal coefficients** a_i and c_i that minimize the MSE of representing the entire signal using N intervals?

c. Formulate the **minimal MSE** of representing the entire signal using N intervals.

d. Compare the minimal MSE for using **piecewise-linear** approximation and the minimal MSE for using **piecewise-constant** approximation (as given in class – no need to develop it).
Which MSE is lower? Mathematically justify your answer.

Note:

- Your answers should include the full mathematical developments leading to the requested expressions.
- The obtained expressions should be relatively simple (somewhat extending the expressions obtained in class for a piecewise-constant approximation).

Question #3 (20 points) – Bit Allocation of a One-Dimensional Signal

Consider the following signal for $t \in [0,1)$:

$$\varphi(t) = A \cdot \sin(2\pi\omega t + \phi)$$

where A, ω, and ϕ are the signal's amplitude, frequency and phase parameters, respectively. We assume here that ω is an integer.

We would like to find the optimal bit-allocation for $\varphi(t)$.

a. Express the derivative-energy ($Energy(\varphi')$) and the value-range $(\varphi_H - \varphi_L)$ of $\varphi(t)$, as required for the bit-allocation optimization.

b. Find the optimal number of samples (N_t) and quantization bits (b) under the constraint of overall bit-budget B.
   i. Formulate the bit-allocation optimization problem.
   ii. Develop the problem to optimization on a single variable (i.e., b or N_t), and formulate the mathematical expression (equation) that defines optimality.

c. You are given a total bit-budget of $B = 200\ bits$. Compare (and explain) the values of the optimal bit-allocation parameters (i.e., N_t and b) of the following signals:
   i. $\varphi_1(t) = 5 \cdot \sin\left(2\pi t + \frac{2}{3}\pi\right)$.
   ii. $\varphi_2(t) = 5 \cdot \sin\left(20\pi t + \frac{2}{3}\pi\right)$.

   Here you can use Matlab for numerically solving the optimality equation from section (b.ii). Useful Matlab functions here are '*solve'* and '*lambertw*'. Note that *'solve'* may return an expression that mathematically depends on the '*lambertw*' function, so, in turn, you can get a real valued result by appropriately applying the '*lambertw*' Matlab function.

Matlab Part (40 points) –

Numerical and Practical Bit Allocation for Two-Dimensional Signals

General instructions:

- Figures should be appropriately titled.
- Your written report should be submitted with the dry part, whereas the code should be electronically submitted.

Section 1

Consider the separable 2D cosine:

$$\varphi_1(x,y) = A \cdot \cos(2\pi\omega_x x)\cos\left(2\pi\omega_y y\right) \quad \text{for} \quad (x,y) \in [0,1] \times [0,1].$$

with parameters : $A = 5000$, $\omega_x = 5$, $\omega_y = 2$.

a. **Analytically** calculate (i.e., mathematically develop formulas for derivatives and integrals) the parameters needed for bit-allocation: the value-range $\varphi_H - \varphi_L$, the horizontal-derivative energy $Energy(\varphi_x')$, and the vertical-derivative energy $Energy\left(\varphi_y'\right)$.

b. Approximate the continuous-domain signal $\varphi_1(x,y)$ in Matlab by a very high resolution digital version of the $\varphi_1(x,y)$ function.
For example, $\varphi_1(x,y)$ can be approximated in Matlab using the high-resolution 2D grid:

```
continuous_approx_delta  = 0.001;
grid = 0 : continuous_approx_delta : 1;
[x_grid,y_grid] = meshgrid(grid,grid);
```

and a matrix of high-precision signal values (e.g., use double type variables):

```
phi_xy=A*cos(2*pi*omega_x*(x_grid)).* cos(2*pi* omega_y *(y_grid));
```

Let us denote the size of this matrix as $N_y^{cont} \times N_x^{cont}$ (the 'cont' notation associates these parameters to the continuous signal approximation – and not to the bit-allocation procedure that will be applied later.)

Present the signal as an image using the imshow function (use an appropriate graylevel scaling that suits the value of A).

c. **Numerically** calculate the parameters needed for bit-allocation: the value-range $\varphi_H - \varphi_L$, the horizontal-derivative energy $Energy(\varphi_x')$, and the vertical-derivative energy $Energy\left(\varphi_y'\right)$.

The calculations should rely on the following examples for numerical approximations:

- Horizontal derivative at (x_0, y_0): $\frac{\partial}{\partial x}\varphi(x,y)\Big|_{(x,y)=(x_0,y_0)} = \frac{\varphi(x_0+\Delta_x^{cont},y_0)-\varphi(x_0,y_0)}{\Delta_x^{cont}}$ for some Δ_x^{cont}, here it can be set to the $\text{continuous\_approx\_delta}$ defined above.

- 2D integration:
$$\int_{x=0}^{1}\int_{y=0}^{1} f(x,y)dxdy = \sum_{i=0}^{N_x-1}\sum_{j=0}^{N_y-1} f\left(i\Delta_x^{cont}, j\Delta_y^{cont}\right)\Delta_x^{cont}\Delta_y^{cont}$$
where $\Delta_x^{cont} = \frac{1}{N_x^{cont}}$ and $\Delta_y^{cont} = \frac{1}{N_y^{cont}}$ for the integers N_x^{cont} and N_y^{cont} defined above.

Write the values of the numerical approximations of $\varphi_H - \varphi_L$, $Energy(\varphi_x')$, and $Energy\left(\varphi_y'\right)$.

Compare these numerical results to the analytically calculated values from subsection (a).

d. Use the numerical approximations of $\varphi_H - \varphi_L$, $Energy(\varphi_x')$, and $Energy\left(\varphi_y'\right)$ and numerically solve the bit-allocation optimization to determine $N_x^{opt,numeric}$, $N_y^{opt,numeric}$ and $b^{opt,numeric}$.
For the numerical solution you can use the Matlab function fmincon with nonlinear equality constraint, the other constraints are not needed and can be set to the empty vector [] .

Consider two bit-allocation procedures with the bit-budgets $B_{low} = 5000$ and $B_{high} = 50000$.
Write the obtained values for $N_x^{opt,numeric}$, $N_y^{opt,numeric}$ and $b^{opt,numeric}$.

e. In this subsection we will **practically** search for the best bit-allocation parameters. Recall that given a bit-budget B, setting two parameters determines the third one. For example, for some N_x and N_y the quantizer bit-cost is $b = \left\lfloor \frac{B}{N_x N_y} \right\rfloor$ (note that b should be an integer).
Accordingly, implement a searching procedure that finds the best bit-allocation parameters by practically evaluating the bit-allocation MSE for many combinations of parameters.
For example, you can implement two nested for loops that run over integer values of N_x and N_y, then in each (inner) iteration evaluate the bit-allocation MSE by applying sampling+quantization in resolution corresponding to the iteration parameters $\left(N_x, N_y, b\right)$.

Apply the practical searching procedure for two bit-budgets $B_{low} = 5 \times 10^3$ and $B_{high} = 5 \times 10^4$.

- For each of the two bit-budgets, what are the optimal $\left(N_x, N_y, b\right)$?
- Are these similar to the corresponding values ($N_x^{opt,numeric}$,$N_y^{opt,numeric}$, $b^{opt,numeric}$) from section c ? Explain in case of significant differences (for example, more than 10% for large value).
- The report should include the reconstructed images obtained in the experiments of this subsection.

Section 2

Repeat on section 1 for the separable 2D cosine:

$$\varphi_1(x, y) = A \cdot \cos(2\pi\omega_x x) \cos\left(2\pi\omega_y y\right) \quad \text{for} \quad (x, y) \in [0,1] \times [0,1].$$

however, with parameters : $A = 5000$, $\omega_x = 5$, $\omega_y = 5$.

Compare the results to those obtained for the signal of the previous section. Explain the differences.