Signal and Image Processing by Computer (236327)

Spring 2018

Homework #1

Publication: April 16th , 2018

Submission: by May 2nd at 13:30

Guidelines:

- Submission is only in pairs.
- Submit your solution to the course box at Taub floor 1.
- Matlab code should be electronically submitted via the course website.
- You may answer in Hebrew or English, in a clear-handwrite or a printed form.
- Follow the FAQ at course website, as clarifications and corrections may be published only there.

הנחיות:

- **תאריך ושעת הגשה:** עד ה-2.5.2018 בשעה 13:30.
- ההגשה בזוגות בלבד.
- ההגשה ידנית לתא הקורס בטאוב קומה 1.
- את קוד המטלב יש להגיש אלקטרונית באתר הקורס.
- ניתן לענות בעברית או באנגלית, בכתב ברור או מודפס.
- חובה לעקוב אחרי FAQ באתר. הבהרות ותיקונים בקשר לתרגיל יפורסמו שם בלבד.

Question #1 (25 points) –

Optimal Quantization for Minimum Expected Absolute Deviation

In this question we consider the design of a scalar quantizer for the criterion of minimum expected absolute-deviation.

The input x is a realization of the random variable X that follows the probability density function $p(x)$. The PDF $p(x)$ is assumed to be positive only for $x \in [\varphi_L, \varphi_H]$.

The expected absolute-deviation is defined as

$$E\varepsilon_Q^1 \triangleq \int_{\varphi_L}^{\varphi_H} |x - Q(x)|\, p(x)dx$$

The considered quantizer uses b bits for representing values, providing $K = 2^b$ representation levels. The representation levels are denoted as $\{r_i\}_{i=1}^{J}$ and the decision levels are $\{d_i\}_{i=0}^{K}$.

a. Formulate the **optimization problem** for designing a b-bit quantizer that minimizes the expected absolute-deviation for a given input PDF $p(x)$.

b. Develop the conditions for optimal **representation** levels **<u>given</u>** the **decision** levels. Which mathematical operation do the optimality conditions resemble?

c. Develop the conditions for optimal **decision** levels **<u>given</u>** the **representation** levels.

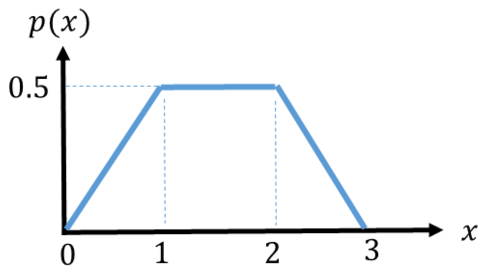
d. Formulate the **Max-Lloyd procedure** for designing a b-bit quantizer that minimizes the expected absolute-deviation for a given input PDF $p(x)$.

Question #2 (25 points)

Consider a random variable that corresponds to the following probability density function:

$$p(x) = \begin{cases} \frac{x}{2} & , \text{ for } x \in [0,1) \\ \frac{1}{2} & , \text{ for } x \in [1,2) \\ \frac{1}{2} - \frac{1}{2}(x-2) & , \text{ for } x \in [2,3) \\ 0 & , \text{ otherwise} \end{cases}$$

that is graphically described as



We consider here the design of a quantizer having 3 representation values: r_1, r_2, r_3.

The corresponding decision levels are: $d_0 = 0,\ d_1,\ d_2, d_3 = 3$.

The optimizations in this question are in the sense of **minimum expected squared error.**

Consider the Max-Lloyd optimization for the initialization of the decision levels to:

$$d_0^{(0)} = 0, \quad d_1^{(0)} = 1, \quad d_2^{(0)} = 2,\ d_3^{(0)} = 3$$

a. What are the optimal representation values,

$$r_1^{(1)}, r_2^{(1)}, r_3^{(1)}$$

calculated in the first iteration of the Max-Lloyd procedure ?

b. What are the optimal decision levels,

$$d_0^{(1)}, d_1^{(1)}, d_2^{(1)}, d_3^{(1)}$$

computed in the first iteration of the Max-Lloyd procedure ?

Question #3 (15 points) – The Miraculous Expressions of Optimal Quantization in the Sense of Minimum Expected Squared-Error

In this question we consider the design of a scalar quantizer for the criterion of minimum expected squared-error.

The quantizer consists of K representation values, $\{r_i\}_{i=1}^{K}$, and $K+1$ decision levels $\{d_i\}_{i=0}^{K}$. Accordingly, the mapping function is $Q(x) = r_i$ for $x \in [d_{i-1}, d_i)$.

The input x is a realization of the random variable X that follows the probability density function $p_X(x)$.

The expected squared-error for a quantizer of K representation values is

$$E\varepsilon_Q^2 \triangleq \sum_{i=1}^{K} \int_{d_{i-1}}^{d_i} (x - r_i)^2 \, p_X(x) dx$$

a. Let us define $Y \triangleq Q(X)$.
   Explain why Y is a discrete random variable, and write its probability mass function (PMF), denoted as $P_Y(y)$, using the PDF $P_X(x)$.

b. Prove (in detail) that the **optimal** expected squared-error holds

$$\sum_{i=1}^{K} \int_{d_{i-1}^{opt}}^{d_i^{opt}} \left(x - r_i^{opt}\right)^2 p_X(x) dx = \mathrm{Var}(X) - \mathrm{Var}(Y)$$

   where $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$ are the variances of X and Y, respectively.

   In particular, consider the case where X has a non-zero mean.

Matlab Part (35 points)

Instructions:

- In this part use a grayscale image (of 256 graylevels) and of a size of at least 256x256 pixels.
  You can transform RGB-color image to a grayscale image using the matlab function rgb2gray.
- Along this part show the image in the same size and dynamic range (pass [0,255] to the imshow function).
- Figures should be appropriately titled.
- Your written report should be submitted with the dry part, whereas the code should be electronically submitted.

1. We would like to estimate the probability-density-function of the graylevels in the image using the image histogram. This can be calculated using the Matlab function imhist (Note that imhist is sensitive to the graylevel range, e.g. [0,1], [0,255], etc.). If the histogram seems too uniform, please pick another image with a non-uniform distribution.

2. Apply uniform quantization on the image using b bits per pixel.
   a. Show the MSE as a function of the bit-budget b for $b = 1, \dots, 8$.
   b. Plot the decision and representation levels for representative b values.

3. Implement the Max-Lloyd quantizer.
   a. Show the MSE as a function of the bit-budget b (for $b = 1, \dots, 8$).
   b. Plot the decision and representation levels for representative b values.
   c. Compare the results to those of the uniform quantizer. Explain the differences.