Signal and Image Processing by Computer (236327)

Spring 2018

Homework #3

Publication: May 29th , 2018

Submission: by June 13th at 13:30

Guidelines:

- Submission is only in pairs.
- Submit your solution to the course box at Taub floor 1.
- Matlab code should be electronically submitted via the course website.
- You may answer in Hebrew or English, in a clear-handwrite or a printed form.
- Follow the FAQ at course website, as clarifications and corrections may be published only there.

הנחיות:

- **תאריך ושעת הגשה:** עד ה-13.6.2018 בשעה 13:30.
- ההגשה בזוגות בלבד.
- ההגשה ידנית לתא הקורס בטאוב קומה 1.
- את קוד המטלב יש להגיש אלקטרונית באתר הקורס.
- ניתן לענות בעברית או באנגלית, בכתב ברור או מודפס.
- חובה לעקוב אחרי FAQ באתר. הבהרות ותיקונים בקשר לתרגיל יפורסמו שם בלבד.

Question #1 (30 points)

The 4×4 **Walsh-Hadamard** matrix is defined as

$$\mathbf{W}_4 = \frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

and its columns are used to form a set of 4 orthonormal functions, $\left\{\psi_j^W(t)\right\}_{j=1}^4$, defined for t in the continuous interval $[0,1)$.

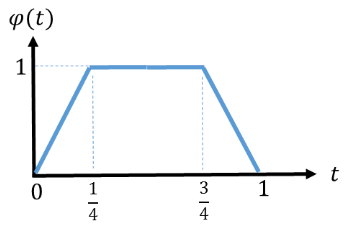
The procedures considered in this question address the minimization of the approximation MSE.

a. (5 Points) Show the set of orthonormal Walsh-Hadamard functions $\left\{\psi_j^W(t)\right\}_{j=1}^4$.
All the functions should be presented using graphs with explicit notation of relevant values on the two axes.

b. (10 Points) Consider the signal:

$$\varphi(t) = \begin{cases} 4t & , \quad \text{for } t \in \left[0, \frac{1}{4}\right) \\ 1 & , \quad \text{for } t \in \left[\frac{1}{4}, \frac{3}{4}\right) \\ 1 - 4\left(t - \frac{3}{4}\right) & , \quad \text{for } t \in \left[\frac{3}{4}, 1\right) \end{cases}$$

that is graphically described as



What are the **coefficients** in the representation of $\varphi(t)$ using the set of **4 Walsh-Hadamard** functions $\left\{\psi_j^W(t)\right\}_{j=1}^4$?
Provide the numerical values of the coefficients.
Note that some of the calculations can be simplified by correctly using the form of the signal and the orthonormal functions.

c. (5 Points) What is the **best 2-term approximation** of $\varphi(t)$ based on the set of the Walsh-Hadamard functions $\left\{\psi_j^W(t)\right\}_{j=1}^4$?
Recall that the best K-term approximation is made without considering the order of orthonormal functions in the set.

d. (5 Points) The 4×4 **Haar** matrix is

$$\mathbf{H}_4 = \frac{1}{2}\begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

and its columns are used to form a set of 4 orthonormal functions, $\left\{\psi_j^H(t)\right\}_{j=1}^4$, defined for t in the continuous interval $[0,1)$.

Show the set of orthonormal **Haar** functions $\left\{\psi_j^H(t)\right\}_{j=1}^4$.
All the functions should be presented using graphs with explicit notation of relevant values on the two axes.

e. (5 Points) Consider the best 2-term approximation of $\varphi(t)$ using the set of 4 **Haar** functions.
Which set of 4 orthonormal functions, the Walsh-Hadamard set or the Haar set, provides a better 2-term approximation (in the sense of minimal MSE)?
You may justify your answer to this subsection by a good explanation or a calculation.

Question #2 (20 points)

Let us consider a real-valued unitary matrix $\mathbf{U}$ of size $N \times N$, formed from N column-vectors $\{\mathbf{u}_i\}_{i=1}^{N}$. The direct-transform of some signal-vector $\mathbf{x}$ is considered here as $\mathbf{U}^T\mathbf{x}$ (note the transpose).

An orthonormal set of N continuous functions can be constructed as:

$$\psi_i(t) = \sum_{k=1}^{N} u_i^{(k)} \cdot \psi_k^s(t) \quad , i = 1, \dots, N$$

where,

$u_i^{(k)}$ is the k^{th} element of the i^{th} column-vector $\mathbf{u}_i$

$\psi_k^s(t)$ is the k^{th} function of the standard family (for N samples), defined as

$$\psi_i^s(t) = \begin{cases} \sqrt{N} & for \quad t \in \left[\frac{i-1}{N}, \frac{i}{N}\right] \\ 0 & otherwise \end{cases}$$

Prove that the orthonormality of the columns, $\{\mathbf{u}_i\}_{i=1}^{N}$, leads to orthonormality of the functions $\{\psi_i(t)\}_{i=1}^{N}$.

Question #3 (15 points) – The Discrete Fourier Transform

Calculate the DFT (of order N) of the discrete one-dimensional signal with values defined for $n = 0, \dots, N-1$ via

$$x[n] = \begin{cases} 1 & \text{for } n = 0, T, \dots, (c-1)T \\ 0 & \text{otherwise} \end{cases}$$

where $N = cT$ for some positive integer c.

Matlab Part (35 points) – Image Restoration using Filtering in The DFT Domain

General instructions:

- Figures should be appropriately titled.
- Your written report should be submitted with the dry part, whereas the code should be electronically submitted.

a. Select a grayscale image of size 512×512 pixels and create a deteriorated version of it by adding a constant to the graylevel values for all the pixels belonging to columns that their index is an integer multiplication of 16. Present the original image and the deteriorated one (that should have damages in the form of periodically placed vertical-lines).

b. Considering the j^{th} line, what is the DFT of the interference? You may assist your solution to Question #3 above.
In which DFT components the interference has a strictly positive values?

c. Restore the image from its corrupted version by applying Notch filtering in the DFT domain for each row.
Compare the MSE of the deteriorated image to the MSE of the restored image.
Note that in this question you should implement the DFT by yourselves (and not by using Matlab's fft function).