①

1.1)
(a) $E[t] = \frac{1}{2}(3 + E(T)) + \frac{1}{2}\left(\frac{1}{3}\cdot 2 + \frac{2}{3}\cdot(5 + E(T))\right)$

$$E(T) = \frac{3}{2} + \frac{E(T)}{2} + \frac{1}{2}\left(\frac{2}{3} + \frac{10}{3} + \frac{2}{3}E(T)\right)$$

$$= \frac{3}{2} + \frac{E(T)}{2} + \frac{1}{3} + \frac{5}{3} + \frac{E(T)}{3}$$

$$\frac{E(T)}{2} - \frac{E(T)}{3} = \frac{3}{2} + 2 = \frac{7}{2}$$

$$\frac{E(T)}{6} = \frac{7}{2} \quad \text{so } E(T) = 21$$

(b) $1_l = \begin{cases} 1 & \text{if goes right and back to initial position} \\ 0 & \text{o/w} \end{cases}$

$1_x = \begin{cases} 1 & \text{if goes left and exits} \\ 0 & \text{o/w} \end{cases}$

$1_r = \begin{cases} 1 & \text{if goes left and back to initial position} \\ 0 & \text{o/w} \end{cases}$

$$T = (3 + T)1_l + 2\,1_x + (5 + T)1_r$$

$$T^2 = (3+T)^2 1_l + 4\,1_x + (5+T)^2 1_r$$

$$E(T^2) = E((3+T)^2)\,P(L) + 4P(X)$$
$$+ E((5+T)^2)\,P(r)$$

$$E(T^2) = E(9+6T+T^2)\,P(L) + 4P(X)$$
$$+ E(25+10T+T^2)\,P(r)$$

$$E(T^2) = \left[9 + 6E(T) + E(T^2)\right] P(L)$$
$$+ 4P(X) + \left[25 + 10\,E(T) + E(T^2)\right] P(r)$$

$$E(T^2) = 9P(L) + 126\,P(L) + E(T^2)\,P(L)$$
$$+ 4P(X) + 25P(r) + 210P(r) + E(T^2)\,P(r)$$

$$E(T^2)\,(1 - P(L) - P(r)) = 135\,P(L) + 4P(X) + 235\,P(r)$$

$$E(T^2) = \frac{135\,P(L) + 4P(X) + 235\,P(r)}{1 - P(L) - P(r)}$$

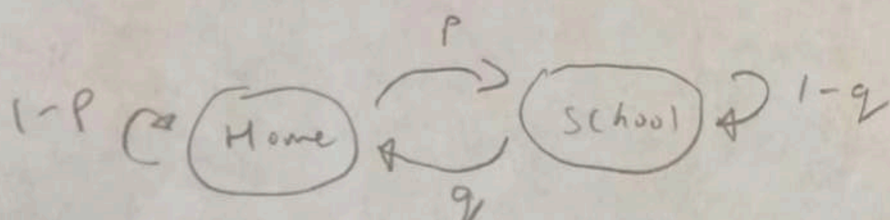
$$E(T^2) = \frac{405\,P(X) + 4P(X) + 470\,P(X)}{1 - 5P(X)} = \frac{879\,P(X)}{1 - 5P(X)}$$

$$= \frac{879\left(\frac{1}{6}\right)}{1 - \frac{5}{6}} = \frac{879\left(\frac{1}{6}\right)}{\left(\frac{1}{6}\right)} = 879$$

1.1)(b) $\mathrm{Var}(T) = E(T^2) - (E(T))^2$

$= 879 - 21^2 = 438$

1.2) X_n : student's location either in school or home for n^{th} hour.



T_h = amount of time until the student goes home given that he starts in school

T_s : amount of time until the student goes to school given that he starts at home

$(1+E(T_h))^2$

$\mathrm{Var}(T) = E(T_h^2) - (E(T_h))^2$

$E[T_h] = q\,E\left[T_h \mid \begin{matrix}\text{first step}\\ h \to \text{home}\end{matrix}\right] + (1-q)E\left[T_h \mid \begin{matrix}\text{first step}\\ \text{stay at school}\end{matrix}\right]$

$E[T_h] = q + (1-q)(1+E[T_h]) = q + 1 + E[T_h] \cancel{-q} - qE[T_h]$

$0 = 1 - qE[T_h] \rightarrow E[T_h] = \frac{1}{q}$

$E[T_h^2] = q1^2 + (1-q)E((T_h+1)^2) = q + (1-q)E[T_h^2 + 2T_h + 1]$

$= \cancel{q} + E[T_h^2] + 2E[T_h] + 1 - qE[T_h^2] - 2qE[T_h] \cancel{-q}$

$E[T_h^2] = 1 + (1-q)E[T_h^2] + (2-2q)E[T_h]$

$qE[T_h^2] = 1 + \frac{2-2q}{q}$ Thus, $qE[T_h^2] = 1 + \frac{2}{q} - 2 = \frac{2}{q} - 1$

$$q\,E[T_n^2] = \frac{2-q}{q} \rightarrow E[T_n^2] = \frac{2-q}{q^2}$$

$$Var(T) = \frac{2-q}{q^2} - \frac{1}{q^2} = \frac{1-q}{q^2} = \frac{1}{q^2} - \frac{1}{q}$$

1.3) (a)

$$E(T) = \frac{1}{3}E\left[T \mid \substack{\text{first step} \\ \text{to sleep}}\right] + \frac{2}{3}E\left[T \mid \substack{\text{first step} \\ \text{to coffee}}\right]$$

$$= \frac{1}{3}\cdot 1 + \frac{2}{3}\left(1 + E(T_c)\right)$$

T_c - no. of hrs until she goes home to sleep starting from coffee

$$E(T_c) = \frac{1}{3}E\left[T_c \mid \substack{\text{first step} \\ \text{to work}}\right] + \frac{2}{3}E\left[T_c \mid \substack{\text{first step} \\ \text{to coffee}}\right]$$

$$= \frac{1}{3}\left(1 + E(T)\right) + \frac{2}{3}\left(1 + E(T_c)\right)$$

$$E(T) = 1 + \frac{2E(T_c)}{3}$$

$$E(T_c) = \frac{1}{3} + \frac{E(T)}{3} + \frac{2}{3} + \frac{2E(T_c)}{3}$$

$$\frac{E(T_c)}{3} = 1 + \frac{E(T)}{3}$$

$$E(T) = 1 + 2\left(1 + \frac{E(T)}{3}\right) = 1 + 2 + \frac{2E(T)}{3}$$

$$= 3 + \frac{2E(T)}{3} \quad \text{so} \quad \frac{E(T)}{3} = 3 \rightarrow E(T) = 9$$

$$E(T_c) = 12$$

1.3) (b) $Var(T) = E(T^2) - (E(T))^2$ ⑤

$E(T^2) = \frac{1}{3} \cdot 1^2 + \frac{2}{3} E((1+T_c)^2)$

$= \frac{1}{3} + \frac{2}{3} E(1 + 2T_c + T_c^2)$

$= \frac{1}{3} + \frac{2}{3} + \frac{4}{3} E(T_c) + \frac{2}{3} E(T_c^2) = 1 + 16 + \frac{2}{3} E(T_c^2)$

$= 17 + \frac{2}{3} E(T_c^2)$

$E(T_c^2) = \frac{1}{3} E((1+T)^2) + \frac{2}{3} E((1+T_c)^2)$

$= \frac{1}{3} E(1 + 2T + T^2) + \frac{2}{3} E(1 + 2T_c + T_c^2)$

$= \frac{1}{3} + \frac{2}{3} E(T) + \frac{E(T^2)}{3} + \frac{2}{3} + \frac{4}{3} E(T_c) + \frac{2}{3} E(T_c^2)$

$= 1 + \frac{2}{3}(9) + \frac{E(T^2)}{3} + \frac{4}{3}(12) + \frac{2}{3} E(T_c^2)$

$= 1 + 6 + \frac{E(T^2)}{3} + 16 + \frac{2}{3} E(T_c^2)$

$\frac{E(T_c^2)}{3} = 23 + \frac{E(T^2)}{3} \rightarrow E(T_c^2) = 69 + E(T^2)$

$E(T^2) = 17 + \frac{2}{3}(69 + E(T^2)) = 17 + 46 + \frac{2}{3} E(T^2)$

$\frac{E(T^2)}{3} = 63 \rightarrow E(T^2) = 189$

$Var(T) = 189 - 81 = 108$

1.4) (a) Because N depends on the X_i value

(b)

$$E[S] = E\left[\sum_{i=1}^{N} X_i\right] = \left(E\left[\sum_{i=2}^{N} X_i \mid X_1 = 1\right] + 1\right) P(X_1 = 1)$$

$$+ \left(E\left[\sum_{i=2}^{N} X_i \mid X_1 = 2\right] + 2\right) P(X_1 = 2)$$

$$+ \left(\underbrace{E\left[\sum_{i=2}^{N} X_i \mid X_1 = 3\right]}_{0} + 3\right) P(X_1 = 3)$$

$$E(S) = (E(S) + 1)\, P(X_1 = 1) + (E(S) + 2)\, P(X_1 = 2)$$
$$+ 3P(X_1 = 3) = \frac{E(S)}{3} + \frac{1}{3} + \frac{E(S)}{3} + \frac{2}{3}$$
$$+ 1 = 2 + \frac{2E(S)}{3} \rightarrow \frac{E(S)}{3} = 2 \rightarrow E(S) = 6$$

From Theorem 5.14, $N \sim Geo\left(\frac{1}{3}\right)$ $E(N) = 3$

$E(X) = \frac{1}{3} + \frac{2}{3} + \frac{3}{3} = 2$ $\quad E(S) = 6$ so it is the same.

1.4)

(c)

$$Var(S) = E(S^2) - (E(S))^2$$

$$E(S^2) = \frac{E(S+1)^2}{3} + \frac{E(S+2)^2}{3} + \frac{3^2}{3}$$

$$= \frac{E(S^2+2S+1)}{3} + \frac{E(S^2+4S+4)}{3} + 3$$

$$= \frac{E(S^2)}{3} + \frac{2}{3}E(S) + \frac{1}{3} + \frac{E(S^2)}{3} + \frac{4E(S)}{3} + \frac{4}{3} + 3$$

$$E(S^2) = \frac{2}{3}E(S^2) + 2E(S) + \frac{5}{3} + \frac{9}{3}$$

$$\frac{1}{3}E(S^2) = 2 \cdot 6 + \frac{14}{3} = 12 + \frac{14}{3}$$

$$E(S^2) = 36 + 14 = 50$$

$$Var(S) = 50 - 36 = 14$$

From Theorem 5.14, $Var(X) = E(X^2) - (E(X))^2$

$$E(X^2) = \frac{1^2}{3} + \frac{2^2}{3} + \frac{3^2}{3} = \frac{1}{3} + \frac{4}{3} + 3 = \frac{5}{3} + \frac{9}{3} = \frac{14}{3}$$

$$Var(N) = \frac{1-\frac{1}{3}}{\frac{1}{9}} = \frac{2}{3} \times 9 = 6$$

$$Var(X) = \frac{14}{3} - 4 = \frac{14}{3} - \frac{12}{3} = \frac{2}{3}$$

$$Var(S) = 3\left(\frac{2}{3}\right) + 6(4) = 2 + 24 = 26$$

2.1) (a) $\hat{Y}(z) = \hat{N}(\hat{X}(z))$ $\hat{X}(z) = \dfrac{zq}{1-z(1-q)}$

$$\hat{Y}(z) = \hat{N}\left(\frac{zq}{1-z(1-q)}\right)$$

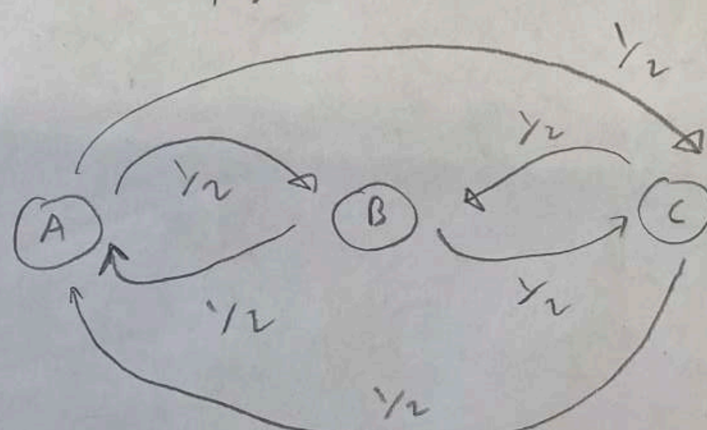
$$= \frac{\left(\frac{zq}{1-z(1-q)}\right)p}{1 - \left(\frac{zq}{1-z(1q)}\right)(1-p)}$$

$$= \frac{pqz}{(1-(1-q)z)\left(1 - \frac{(1-p)qz}{1-(1-q)z}\right)}$$

$$= \frac{pqz}{z(pq-1)+1} = \frac{zpq}{1-z(1-pq)}$$ $Y \sim \text{Geo}(pq)$

So, Y follows a geometric distribution with parameter pq.

2.2) (a)



$R_{A,A}$ = return from A to A. X_i = state at time i

2.27
a)

$$E(R_{A,A}) = E(R_{A,A} \mid X_1 = B)\, P_{A,B} + E(R_{A,A} \mid X_1 = C)\, P_{A,C}$$

$$= (1 + E(R_{B,A}))\tfrac{1}{2} + (1 + E(R_{C,A}))\cdot\tfrac{1}{2}$$

$$E(R_{B,A}) = (1 + E(R_{C,A}))\, P_{B,C} + P_{B,A}$$

$$= (1 + E(R_{C,A}))\tfrac{1}{2} + \tfrac{1}{2}$$

$$E(R_{C,A}) = (1 + E(R_{B,A}))\, P_{C,B} + P_{C,A}$$

$$= (1 + E(R_{B,A}))\tfrac{1}{2} + \tfrac{1}{2}$$

Let $x = E(R_{A,A})$, $y = E(R_{B,A})$, $z = E(R_{C,A})$

$$x = (1 + y)\tfrac{1}{2} + (1 + z)\tfrac{1}{2}$$

$$x = 1 + \frac{y}{2} + \frac{z}{2}$$

$$y = (1 + z)\tfrac{1}{2} + \tfrac{1}{2} = 1 + \frac{z}{2}$$

$$z = (1 + y)\tfrac{1}{2} + \tfrac{1}{2} = 1 + \frac{y}{2}$$

$x = 3 = E(R_{A,A})$

$y = 2$

$z = 2$

2.27) b) $\hat{R}_{A,A}(z) = E(z^{R_{A,A}})$

$= E(z^{R_{A,A}} \mid X_1 = B) p_{A,B} + E(z^{R_{A,A}} \mid X_1 = C) p_{A,C}$

$= E(z^{1+R_{B,A}}) p_{A,B} + E(z^{1+R_{C,A}}) p_{A,C}$

$= \frac{1}{2} \cdot E(z^{1+R_{B,A}}) + \frac{1}{2} E(z^{1+R_{C,A}})$

$= \frac{z}{2} E(z^{R_{B,A}}) + \frac{z}{2} E(z^{R_{C,A}})$

$\hat{R}_{B,A}(z) = E(z^{R_{B,A}})$

$= E(z^{1+R_{C,A}}) \frac{1}{2} + \frac{z}{2} = \frac{z}{2} E(z^{R_{C,A}}) + \frac{z}{2}$

$\hat{R}_{C,A}(z) = E(z^{R_{C,A}})$

$= E(z^{1+R_{B,A}}) \frac{1}{2} + \frac{z}{2} = \frac{z}{2} E(z^{R_{B,A}}) + \frac{z}{2}$

$E(z^{R_{B,A}}) = \frac{z}{2}\left(\frac{z}{2} E(z^{R_{B,A}}) + \frac{z}{2}\right) + \frac{z}{2}$

$E(z^{R_{B,A}}) = \frac{z^2}{4} E(z^{R_{B,A}}) + \frac{z^2}{4} + \frac{z}{2}$

$E(z^{R_{B,A}}) = \dfrac{\frac{z^2}{4} + \frac{z}{2}}{1 - z^2/4}$

$$E\left(z^{K_{C,A}}\right) = \frac{z}{2}\left(\frac{\frac{z^2}{4} + \frac{z}{2}}{1 - \frac{z^2}{4}} + 1\right)$$

$$E\left(z^{K_{A,A}}\right) = \frac{z}{2}\left(\frac{z^2/4 + z/2}{1 - \frac{z^2}{4}} + \frac{z}{2}\left(\frac{\frac{z^2}{4} + \frac{z}{2}}{1 - \frac{z^2}{4}} + 1\right)\right)$$

After simplification

$$E\left(z^{K_{A,A}}\right) = \frac{z^2}{2 - z}$$

(12)

2.2) (c) $n \to \infty$ case

$$P = \begin{vmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{vmatrix} \qquad \pi = \begin{vmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{vmatrix}$$

$$0 \cdot \pi_1 + \frac{1}{2} \cdot \pi_2 + \frac{1}{2} \cdot \pi_3 = \pi_1$$

$$\frac{1}{2} \cdot \pi_1 + 0 \cdot \pi_2 + \frac{1}{2} \cdot \pi_3 = \pi_2$$

$$\frac{1}{2} \pi_1 + \frac{1}{2} \pi_2 + 0 \cdot \pi_3 = \pi_3$$

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

general n case

$$\pi_{1,n} = 0 \cdot \pi_{1,n-1} + \frac{1}{2} \pi_{2,n-1} + \frac{1}{2} \pi_{3,n-1}$$

$$= \frac{1}{2}(1 - \pi_{1,n-1})$$

$$(-2)^n \pi_{1,n} = (-2)^n \cdot \frac{1}{2}(1 - \pi_{1,n-1})$$

$$= -(-2)^{n-1} + (-2)^{n-1} \pi_{1,n-1}$$

$$f_n = (-2)^n \pi_{1,n}$$

$$f_n = -(-2)^{n-1} + f_{n-1}$$

$$f_1 = (-2) \pi_{1,1} = -2$$

$$f_n = (-2) - ((-2) + \dots + (-2)^{n-1})$$

$$f_n = -2 - \frac{(-2)(1 - (-2)^{n-1})}{1 - (-2)} \qquad \pi_{1,n} = \frac{-4 + (-2)^n}{3(-2)^n}$$