

Keynes Meets Markowitz: The Trade-off Between Familiarity and Diversification

January 2011



Phelim Boyle,
Wilfrid Laurier University

Lorenzo Garlappi,
University of British Columbia

Raman Uppal,
EDHEC Business School

Tan Wang,
University of British Columbia

Abstract

We develop a model of portfolio choice that nests the views of Keynes—who advocates concentration in a few familiar assets—and Markowitz—who advocates diversification across assets. We rely on the concepts of ambiguity and ambiguity aversion to formalize the idea of an investor's "familiarity" toward assets. The model shows that when an investor is equally ambiguous about all assets, then the optimal portfolio corresponds to Markowitz's fully-diversified portfolio. In contrast, when an investor exhibits different degrees of familiarity across assets, the optimal portfolio depends on (i) the relative degree of ambiguity across assets, and (ii) the standard deviation of the estimate of expected return on each asset. If the standard deviation of the expected return estimate and the difference between the ambiguity about familiar and unfamiliar assets are low, then the optimal portfolio is composed of a mix of both familiar and unfamiliar assets; moreover, an increase in correlation between assets causes an investor to increase concentration in the assets with which they are familiar (*flight to familiarity*). Alternatively, if the standard deviation of the expected return estimate and the difference in the ambiguity of familiar and unfamiliar assets are high, then the optimal portfolio contains only the familiar asset(s) as Keynes would have advocated. In the extreme case in which the ambiguity about all assets and the standard deviation of the estimated mean are high, then no risky asset is held (*non-participation*). The model also has empirically testable implications for comparative statics with respect to idiosyncratic and systematic risk: in response to a change in idiosyncratic risk, the Keynesian portfolio always exhibits a bigger change than the Markowitz portfolio, while the opposite is true for a change in systematic risk.

Keywords: Investment, portfolio choice, ambiguity, robust control, underdiversification.

JEL Classification: G11, G12, G23, D81

We gratefully acknowledge financial support from Inquire-UK and Inquire-Europe; however, this article represents the views of the authors and not of Inquire. We would also like to thank the following for comments: Ron Anderson, Suleyman Basak, Bruno Biais, Victor DeMiguel, David Feldman, Francisco Gomes, Juan-Pedro Gomez, Phillip Illeditsch, Dietmar Leisen, Sujoy Mukerji, Vasant Naik, Anthony Neuberger, Lorian Pelizzon, Valery Polkovnichenko, Stijn Van Nieuwerburgh, Dimitri Vayanos, Laura Veldkamp, Johan Walden, Hongjun Yan, Xunyu Zhou, and participants at seminars given at EDHEC Business School, London Business School, McGill University, Northwestern University, Oxford University, Tilburg University, University of Toronto, University of Venice, Wilfrid Laurier University, the Oxford-Man conference on Robust Techniques in Quantitative Finance, the joint conference of INQUIRE-UK and INQUIRE-Europe, the Nordic Finance Network (NFN) Research Workshop, the meetings of the European Finance Association, and the meetings of the American Finance Association.

EDHEC is one of the top five business schools in France. Its reputation is built on the high quality of its faculty and the privileged relationship with professionals that the school has cultivated since its establishment in 1906. EDHEC Business School has decided to draw on its extensive knowledge of the professional environment and has therefore focused its research on themes that satisfy the needs of professionals.

EDHEC pursues an active research policy in the field of finance. EDHEC-Risk Institute carries out numerous research programmes in the areas of asset allocation and risk management in both the traditional and alternative investment universes.

1. Introduction

John Maynard Keynes and Harry Markowitz personify two contrasting schools of thought regarding the process of portfolio selection. On the one hand, Keynes advocates allocating wealth in the few stocks about which one feels most favorably:

The right method in investment is to put fairly large sums into enterprises which one thinks one knows something about [. . .]. It is a mistake to think that one limits one's risk by spreading too much between enterprises about which one knows little and has no reason for special confidence. [. . .] One's knowledge and experience are definitely limited and there are seldom more than two or three enterprises at any given time in which I personally feel myself entitled to put full confidence.¹

Keynes is not alone in his view on portfolio selection. Loeb (1950), for example, advocates: "Once you obtain confidence, diversification is undesirable; diversification [is] an admission of not knowing what to do and an effort to strike an average." Moreover, the Keynesian view is far from old-fashioned: Warren Buffett, in his letter to the Berkshire Hathaway shareholders in 1991, espouses the same view and supports it by citing the above quote from Keynes.²

On the other hand, Markowitz (1952, p. 77), championing the concept of diversification, argues that it is inefficient to put a large holding in just a few stocks, and that an investor should instead diversify across a large number of stocks, as the following quote effectively summarizes:

Diversification is both observed and sensible; a rule of behavior which does not imply the superiority of diversification must be rejected both as a hypothesis and as a maxim.

Even though Markowitz's idea of diversification has been accepted as one of the most fundamental tenets of modern financial economics, a vast body of empirical evidence that we review below suggests that investors do not hold diversified portfolios but rather invest heavily in only a few assets, as suggested by Keynes. However, because of the lack of an analytical characterization, the academic literature has so far paid relatively little attention to Keynes's view on portfolio selection, preferring instead the more extensively developed analytical framework supporting Markowitz's idea of diversification. Our objective in this paper is to fill this gap by developing a model that allows us to assess, both qualitatively and quantitatively, the different trade-offs advocated by Keynes and Markowitz.³

In particular, our goal is to understand the implications of Keynes's view for the portfolios selected by individual investors, and to answer the following questions. Under what circumstances should investors hold only the assets they "feel confident about" (*familiar assets*), and when should they hold only the broadly diversified (and possibly *unfamiliar*) market portfolio? When they hold both, how should their portfolio be allocated between the familiar asset(s) and the market portfolio? How does this allocation depend on the volatility and correlation of asset returns? When the number of assets available for investment is large, are the benefits of investing in the familiar assets overwhelmed by the benefits from diversification?

We model the lack of familiarity via the concepts of *ambiguity* (or uncertainty) in the sense of Knight (1921) and ambiguity *aversion*.⁴ A framework based on ambiguity aversion is appealing to model individual investment decisions. Experimental evidence shows that ambiguity aversion is particularly strong in cases where people feel that their competence in assessing the relevant probabilities is low and in comparative situations (Heath and Tversky (1991) and Fox and Tversky (1995)). Recent experimental evidence in Ahn, Choi, Gale, and Kariv (2010) and Bossaerts,

1 - From Keynes's letter to F. C. Scott, 15 August 1934, Keynes (1983).

2 - Warren Buffett's letter is available at <http://www.berkshirehathaway.com/letters/1991.html>.

3 - The view of Keynes has been making a comeback also in the field of macroeconomics in the face of the recent financial crisis; see, for example, the article titled "The other-worldly philosophers" in the July 16, 2009 issue of *The Economist*.

4 - LeRoy and Singell (1987) claim that even the distinction between risk and uncertainty (ambiguity) is made by Keynes (1921) and that Knight (1921) did not intend such a distinction.

Ghirardato, Guarnaschelli, and Zame (2010) also shows that investors are averse to ambiguity and that maxmin preferences are a good way to model this behavior.

In our model, the investor exhibits different degrees of ambiguity about the distributions of asset returns and considers "familiar" the asset(s) with the lowest level of ambiguity. Specifically, we build on the portfolio selection framework of an ambiguity averse investor developed in Garlappi, Uppal, and Wang (2007). The framework we develop has three attractive features: (i) it is simple and only a mild departure from the well understood Markowitz (1959) model; (ii) it has a solid axiomatic foundation; and (iii) it is capable of capturing in a parsimonious way several aspects of the observed evidence on household portfolios. Our framework can also be thought of as providing an analytic characterization of the concept of familiarity introduced by Huberman (2001). Note that while we use the same modeling framework as Garlappi, Uppal, and Wang (2007), there are several important differences. First, the focus of our work is to provide a positive model in which one can compare the Markowitz and Keynesian views of optimal portfolio selection, while the focus of Garlappi, Uppal, and Wang is to provide a normative model of portfolio selection. Second, our paper develops several testable predictions that are not contained in Garlappi, Uppal, and Wang; in particular, the predictions regarding the effect on the optimal portfolio of a change in idiosyncratic risk, systematic risk, and correlations are new.

To isolate the effect of ambiguity on portfolio selection, we consider an economy with N identical assets that differ only in the degree of uncertainty the investor exhibits toward the expected returns of each asset. We operationalize the concept of ambiguity as the size of the confidence interval each investor has for the statistical estimate of the mean of each asset return. This characterization of ambiguity, that dates back to Bewley (1988), allows us to "quantify" ambiguity as the size of the confidence interval for the estimate of an asset's expected return.

The structure of the optimal portfolio that emerges from our model depends on two quantities: (i) the standard deviation of the estimate of the expected return, and (ii) the relative degree of ambiguity across different assets. Specifically, we show that when an investor is equally ambiguous about all assets, then the optimal portfolio corresponds to Markowitz's fully diversified portfolio. In contrast, when the investor exhibits different degrees of ambiguity across assets, then the structure of the optimal portfolio depends on the *relative* degree of ambiguity and on the standard deviation of the estimate of expected return. If the standard deviation is low and the difference between the ambiguity levels of the familiar and unfamiliar assets is small, then the optimal portfolio is composed of a mix of both familiar and unfamiliar assets. Alternatively, if the standard deviation of the expected return estimate is high and the difference between the ambiguity levels of the familiar and unfamiliar assets is large, then the optimal portfolio contains *only* the familiar asset(s) as Keynes would have advocated. In the extreme case in which the ambiguity levels for all assets is high and the standard deviation of the estimated mean is also high, then the investor optimally decides not to hold any risky asset (*non-participation*).⁵

Several interesting testable implications emerge from the above structure of the optimal portfolio. First, in all cases other than in the limiting case in which the optimal portfolio corresponds to Markowitz's fully diversified portfolio, the optimal portfolio will be exposed to idiosyncratic risk. Second, when the optimal portfolio contains both familiar and unfamiliar assets, the optimal holding of the familiar asset *increases* with the correlation across assets, a manifestation of the "light to familiarity" phenomenon, which is an increase in the holdings of familiar assets during a financial crisis.⁶ The reason is that when correlations increase, the benefit from diversification from holding the unfamiliar asset together with the familiar asset decreases, and therefore the investor holds relatively more of the familiar asset. Thus, in our model we obtain a light to familiarity uniquely from the ambiguity-aversion channel, which distinguishes our mechanism from liquidity-based theories in, for instance, Brunnermeier and Pedersen (2009) and Vayanos (2004).

5 - Easley and O'Hara (2009) show how in such circumstances regulation can be effective; Cao, Wang, and Zhang (2005) show that some agents may not participate in the stock market even in equilibrium.
6 - Several papers have documented that correlations across asset returns increase during a crisis; see, for instance, Longin and Solnik (1995), Scholes (2000), Lo (2001), and Poon, Rockinger, and Tawn (2004).

Third, a change in idiosyncratic risk induces more rebalancing in the ambiguity-averse portfolio than in the Markowitz portfolio, while a change in systematic risk has the opposite effect.

Our quantitative assessment of the model reveals that reasonable differences in familiarity across assets can generate portfolios that are biased toward a few assets. For example, using data calibrated to the U.S. stock market, we find that an investor who uses a 1.0 standard-deviation bound on the estimate of the equity premium for the stock market and a 0.75 standard-deviation bound on the estimate of the expected excess return on the familiar stock will invest 20% in the familiar asset and 80% in the market. On the other hand, an investor who uses a 1.5 standard-deviation bound on the estimate of the equity premium will hold only the familiar asset and not invest at all in the market portfolio; thus, the model can explain why an investor might find it optimal to hold only two or three stocks. Furthermore, an investor who uses a 2.0 standard-deviation bound on the estimate of the equity premium for all assets will find it optimal not to participate in the stock market at all.

It is, of course, not surprising that familiarity toward some assets leads to a bias in portfolio holdings toward those assets. However, the contribution of this paper is: First, to provide a unified framework that nests the views of Keynes and Markowitz and to propose an empirically motivated measure of ambiguity that originates from classical statistics. Second, to explore the implications of the familiarity bias for (i) the riskiness of an investor's portfolio; (ii) the sensitivity of the portfolio to variations in asset-return correlations and the ensuing flight to familiarity effect; and (iii) the opposite effects of systematic and idiosyncratic volatility on the change in the optimal weights observed in the Keynes and Markowitz portfolios. Third, to show quantitatively that, for reasonable parameter values, the implications of the model are consistent with several empirical observations such as the low number of stocks held by investors, the "own-company stock," "homecountry equity," and "limited-stock-market-participation" puzzles.

We now describe the literature related to our work. Since the early work of Epstein and Wang (1994), many papers have relied on the concept of ambiguity (or Knightian uncertainty) to model an individual investor's attitude towards uncertain scenarios and explore its implications for asset prices (Dow and Werlang (1992), Cao, Wang, and Zhang (2005)); portfolio choice (Uppal and Wang (2003), Garlappi, Uppal, and Wang (2007)), regulation (Easley and O'Hara (2009)), liquidity (Easley and O'Hara (2010) and Routledge and Zin (2009)) and information processing (Epstein and Schneider (2007) and Illeditsch (2009)). Empirical evidence for the relation between expected returns and ambiguity is provided by Anderson, Ghysels, and Juergens (2009). Although some of the modeling features of our paper are shared with these papers, the main difference of our work is the focus on the trade-off between diversification and familiarity.⁷

Typically, the few assets that investors do hold are ones with which they are "familiar." Huberman (2001) introduces the idea that people invest in familiar assets and provides evidence of this in a multitude of contexts. Massa and Simonov (2006) also find that investors tilt their portfolios away from the market portfolio and toward stocks that are geographically and professionally close to the investor, resulting in a portfolio biased toward familiar stocks. French and Poterba (1990) and Cooper and Kaplanis (1994) document that investors bias their portfolios toward "home equity" rather than diversifying internationally; Grinblatt and Keloharju (2001) and Massa and Simonov (2006) find this bias to be present among Finnish and Swedish investors, respectively; and Feng and Seasholes (2004) find that Chinese investors not only overweight local companies but also companies that are traded on a local exchange. Coval and Moskowitz (1999) show that the bias toward familiar assets is not just in the international portfolios of small investors, but also U.S. investment managers exhibit a bias toward companies that are geographically close to the managers.⁸ A good summary of this literature is provided in Vissing-Jørgensen (2003, Section 4.2).

7 - Early evidence on the lack of diversification is provided by Blume and Friend (1975): using data on income-tax returns, they find that most investors hold only one or two stocks. Polkovnichenko (2005), using data from SCF, finds that of the households that hold individual stocks directly, the median number of stocks held was two from 1983 until 2001, when it increased to three, and that poor diversification is often attributable to investments in employer stock, which is a significant part of equity portfolios. Barber and Odean (2000) and Goetzman and Kumar (2008) report similar findings of underdiversification based on data for individual investors at a U.S. brokerage. Calvet, Campbell, and Sodini (2007), based on detailed government records covering the entire Swedish population, find that some households are poorly diversified and bear significant idiosyncratic risk. A comprehensive summary of this aspect of household finance is provided in Campbell (2006, Section 3).

8 - Sarkissian and Schill (2004) find that the familiarity bias is not just in investment decisions but also in financing decisions: the decision to cross-list equity in a particular foreign market is driven by how familiar that market is rather than diversification considerations.

The literature has also proposed several other theories to rationalize the tendency of investors to hold only a small number of stocks in their portfolios. These include: the presence of fixed transaction costs (Brennan, 1975), awareness about only a subset of the available stocks (Merton, 1987), investors' preference for skewness (Kraus and Litzenberger, 1976; Conine and Tamarkin, 1981; Harvey and Siddique, 2000; Mitton and Vorkink, 2007; Barberis and Huang, 2008), employees' loyalty toward the company where they work (Cohen, 2009), rank-dependent preferences (Polkovnichenko, 2005), the desire of investors to construct portfolios in "layers" in order to obtain downside protection and upside potential (Shefrin and Statman, 2000), the desire for portfolio insurance in the presence of margin and short-sale constraints (Liu, 2009), and overconfidence (Odean, 1999).

Recently, information advantage has also been offered as a potential explanation for underdiversification in portfolio choice. Van Nieuwerburgh and Veldkamp (2010) develop a model of information *acquisition* and portfolio choice which is capable of explaining portfolio concentration by exploiting the increasing return to scale from specializing in one asset. They use this model in Van Nieuwerburgh and Veldkamp (2006) to study the bias toward investing in own-company stock, and they extend this to a two-country general equilibrium setting in Van Nieuwerburgh and Veldkamp (2009) to study the "home-bias" puzzle. While appealing, an information-based explanation cannot explain entirely the observed bias toward familiar stocks, such as the "own-company" and "home-bias" puzzles. First, this explanation will hold true only if the investor believes that the expected return on the familiar asset (for example, own-company stock or domestic equity) will always be higher than that on other assets; but information about the familiar asset can also be negative, in which case the information-based explanation would recommend a reduced position in the familiar asset; and, in the case of sufficiently negative information about the familiar asset, the information-based explanation would recommend a short position. Second, our model can generate a portfolio with a small number of assets, while in the information-based explanation there is always a component of the Markowitz portfolio. Third, our model can explain non-participation. Finally, an information-based explanation would indicate that investors who hold portfolios that are biased toward a few assets outperform diversified portfolios, but the empirical evidence on this is mixed—for example, while Ivkovic and Weisbenner (2005) and Massa and Simonov (2006) find some support for this hypothesis, Calvet, Campbell, and Sodini (2007) and Goetzman and Kumar (2008) do not.

The remainder of this paper is organized as follows. In Section 2, we develop the theoretical framework that allows us to study the tradeoff among return, risk, and familiarity. In Section 3, we derive the optimal portfolio weights for an investor who exhibits familiarity toward some assets, and in Section 4 we study the implications of these weights for portfolio variance and changes in portfolio weights with respect to systematic and unsystematic volatility. Our conclusions are presented in Section 5. Our main results are highlighted in propositions, and the proofs of all propositions are relegated to the Appendix.

2. A Model that nests the Views of Keynes and Markowitz

In this section, we introduce a model of portfolio choice that incorporates an investor's ambiguity about the true distribution of each asset's return. We first describe our assumptions about the statistical properties of asset returns. We then formulate the portfolio problem of an investor in the classical setting of Markowitz (1959) and finally generalize it to the case of an investor who is averse to ambiguity.

We have made a conscious decision to consider a static, discrete-time model where all assets have the same expected return, volatility, and correlation with other assets, with the only difference across assets being the degree of ambiguity an investor has about their returns.⁹ Our choice of a static model set in discrete-time is dictated by the desire for simplicity; our choice of identical

9 - The issue of portfolio choice in a dynamic setting when the agent is averse to ambiguity has already been addressed in the literature. The case of dynamic portfolio choice with only a single risky asset in a robust-control setting is addressed in Maenhout (2004); the case of multiple risky assets in a dynamic setting in which the investor is averse to ambiguity is considered in Chen and Epstein (2002), Epstein and Miao (2003), and Uppal and Wang (2003).

asset returns is driven by the desire to focus on the key characteristic of our framework: differences in familiarity (or ambiguity) across assets. It is however straightforward to extend the model to a setting where each asset has distinct first and second moments.

2.1 Asset returns

We consider a static economy with N identical risky assets and one risk-free asset. The return on each risky asset, in excess of the risk-free rate, is

$$r_i = r_S + u_i \quad i = 1, \dots, N, \quad (1)$$

in which r_S represents the *systematic* component of individual returns and u_i is a zero-mean random variable capturing the *unsystematic* (or *idiosyncratic*) component of individual returns that is not correlated to r_S . We denote by μ and σ_S the expected return and volatility, respectively, of the systematic component r_S . For simplicity, we assume that the unsystematic components of any two assets are uncorrelated, that is, $\text{Cov}(u_i, u_j) = 0$ for all $i \neq j$. We also assume that the volatility σ_U of u_i to be identical across assets. Observe that the common total variance of each asset is given by $\sigma^2 \equiv \sigma_S^2 + \sigma_U^2$, and the common correlation across any two assets $i \neq j$ is given by $\rho = \sigma_S^2 / (\sigma_S^2 + \sigma_U^2) = \sigma_S^2 / \sigma^2 > 0$. Given these assumptions, the N -dimensional vector of expected excess returns is given by $\mu = \mu \mathbf{1}_N$, where $\mathbf{1}_N$ is a $N \times 1$ vector of ones. The $N \times N$ variance-covariance matrix is

$$\Sigma = \begin{bmatrix} \sigma_S^2 + \sigma_U^2 & \sigma_S^2 & \cdots & \sigma_S^2 \\ \sigma_S^2 & \sigma_S^2 + \sigma_U^2 & \cdots & \sigma_S^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_S^2 & \sigma_S^2 & \cdots & \sigma_S^2 + \sigma_U^2 \end{bmatrix} = \begin{bmatrix} \sigma^2 & \rho\sigma^2 & \cdots & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 & \cdots & \rho\sigma^2 \\ \vdots & \vdots & \ddots & \vdots \\ \rho\sigma^2 & \rho\sigma^2 & \cdots & \sigma^2 \end{bmatrix}, \quad (2)$$

This simplified specification implies that if risky assets are to be held in equilibrium then the equity premium μ must be positive.

2.2 The Markowitz mean-variance portfolio problem

The investor's problem is to choose a vector of portfolio weights, $\pi = (\pi_1, \pi_2, \dots, \pi_N)^\top$, for the available N risky assets. According to the classical mean-variance model (Markowitz (1952, 1959), and Sharpe (1970)), the optimal portfolio of a risk averse investor is given by the solution to the following optimization problem,

$$\max_{\pi} \pi^\top \mu - \frac{\gamma}{2} \pi^\top \Sigma \pi, \quad (3)$$

where γ denotes the investor's risk-aversion parameter. If the investor knows the true parameters μ and Σ , the solution to this problem is

$$\pi = \frac{1}{\gamma} \Sigma^{-1} \mu = \frac{1}{\gamma} \left(\frac{\mu}{\sigma^2 (1 + \rho(N-1))} \right) \mathbf{1}_N, \quad (4)$$

where the last equality follows from imposing the properties of returns described in Section 2.1.

The *relative* weights in the portfolio of only-risky assets, that is, the fraction of the risky-asset portfolio invested in each asset, is

$$\omega_i = \frac{\pi_i}{\pi^\top \mathbf{1}_N} = \frac{1}{N}, \quad i = 1, \dots, N. \quad (5)$$

The variance of this portfolio of only-risky assets is given by

$$\omega^\top \Sigma \omega = \frac{1}{N} \sigma_U^2 + \sigma_S^2, \quad (6)$$

so that when the number of assets is large, the variance of the portfolio depends only on *systematic* risk, σ_S^2 , and not on *idiosyncratic* risk, σ_U^2 :

$$\lim_{N \rightarrow \infty} \omega^\top \Sigma \omega = \sigma_S^2. \quad (7)$$

2.3 The Keynesian portfolio problem

A fundamental assumption of the standard mean-variance portfolio selection problem formulated in (3) is that the investor knows the true expected returns and covariance matrix. In practice, however, the investor has to estimate these quantities. The estimation of expected returns is a notoriously difficult task, in comparison to the estimation of second moments.¹⁰ For simplicity we therefore assume that the investor has perfect knowledge of σ , the volatility of asset returns, and ρ , the correlation between asset returns, but is *ambiguous* about the true expected return.¹¹

An intuitive way of modeling this ambiguity is to rely on results from classical statistics and consider the confidence interval for the estimator of the expected returns. Let $\hat{\mu}$ be the common estimated value of the mean return of an asset n obtained by using a return time series of length T , and

$$\sigma_{\hat{\mu}} \equiv \sqrt{\frac{\sigma^2}{T}} = \sqrt{\frac{\sigma_S^2 + \sigma_U^2}{T}}, \quad (8)$$

the standard deviation of the estimate $\hat{\mu}$.¹² Therefore, we can define the following *confidence interval* for the expected return μ of asset n :

$$\alpha_n\text{-confidence interval} = \left\{ \mu : \frac{(\mu - \hat{\mu})^2}{\sigma_{\hat{\mu}}^2} \leq \alpha_n^2 \right\}, \quad (9)$$

where n is the critical value determining the size, or level, of the confidence interval. The above expression suggests that the critical level n can be directly interpreted as a measure of the amount of ambiguity about the estimate of the expected return. A larger value of α_n will result in a larger confidence interval, and hence, a larger set of possible distributions to which the true returns may belong.¹³ Moreover, if excess asset returns are Normally distributed and σ is known, the quantity $(\hat{\mu} - \mu)/\sigma_{\hat{\mu}}$ has a standard Normal distribution.

If historical data is all the information an investor has, then the above confidence interval, which is based on historical data, is a good description of the ambiguity about the returns on that asset. If, in addition, the investor has further knowledge about a particular asset, say, through working for that company, then this familiarity would typically lead to a reduction in the ambiguity about this asset. Mathematically, this can be represented by a smaller α_n in (9). A smaller n is the result of familiarity *above and beyond* the information contained in historical data on returns. In the rest of our analysis, we will refer to *familiar* asset(s) as those for which the investor exhibits the lowest amount of ambiguity about the estimated expected return.

To capture both the presence of ambiguity and the aversion to it, we rely on the work of Gilboa and Schmeidler (1989), using the approach developed in Garlappi, Uppal, and Wang (2007). We introduce two new components into the standard mean-variance portfolio selection problem in (3). First, we model the *presence* of ambiguity by imposing the confidence interval (9) as an *additional constraint* on the mean-variance optimization program. Second, to account for *aversion* to ambiguity, we impose that the investor chooses his portfolio by *minimizing* over the set of expected returns he considers plausible according to (9). Thus, the extended mean-variance model takes the following form:

$$\max_{\pi} \min_{\mu} \pi^T \mu - \frac{\gamma}{2} \pi^T \Sigma \pi, \quad (10)$$

subject to

$$\frac{(\mu - \hat{\mu})^2}{\sigma_{\hat{\mu}}^2} \leq \alpha_n^2, \quad n = 1, \dots, N, \quad (11)$$

10 - Merton (1980) describes the theoretical reason why estimating expected returns is much more difficult than estimating second moments; empirical evidence for the resulting poor performance of the Markowitz portfolio is provided in Frost and Savarino (1986, 1988), Michaud (1989), Best and Grauer (1991), Litterman (2003), and DeMiguel, Garlappi, and Uppal (2009).

11 - In the literature, parameter uncertainty and estimation risk are also used to describe the problem the investor faces. Parameter uncertainty is often dealt with using a Bayesian approach in which the unknown parameters are treated as random variables that are "integrated out" while maximizing utility in order to find optimal portfolios. A Bayesian investor is neutral to ambiguity, since he is capable of aggregating the uncertainty about the parameters via a subjective prior. We are interested in the case of an investor who cannot form a unique prior on the uncertain parameters and is averse to this ambiguity. We adopt this model in contrast to the Bayesian approach because our model allows us to obtain zero holdings in some assets, a result that cannot be obtained in a Bayesian model.

12 - We consider the case where the length of time over which the mean return is estimated is the same for all assets. It is straightforward to extend the model to allow for different lengths of time over which the mean returns for different assets are estimated.

13 - Bewley (1988) originally formulated the argument that confidence intervals can be interpreted also as a measure of the level of ambiguity associated with the estimated parameters. For another recent paper that uses Bewley's characterization of Knightian uncertainty, see Easley and O'Hara (2010).

where α_n refers to the level of ambiguity associated with asset n . In the preferences described by (10) and (11), α_n is a constant that reflects both the investor's ambiguity and his aversion to ambiguity. Specifically, the parameter α_n should be understood as the product of ambiguity aversion (common across assets) and ambiguity (asset-specific). Because the ambiguity aversion parameter of the investor is not observationally separable from the level of ambiguity in our model, we choose a parsimonious characterization of preferences by normalizing the degree of ambiguity aversion to one.¹⁴

Observe that in the extreme case in which $\alpha_n = 0$ for all n , the optimal portfolio in (10) reduces to the mean-variance portfolio problem in (3) in which the estimated means, $\hat{\mu}$, are used as values for the expected returns, μ .

3. The Optimal Portfolio Weights in the Keynesian Model

In this section, we solve for the optimal portfolio weights of an investor who is familiar with one or more assets, and then explain how the optimal portfolio can be interpreted in terms of the views of both Markowitz and Keynes. The implications of these portfolio weights are examined in the next section.

In Section 3.1, we consider the problem where there are N risky assets, and the investor is more familiar with one of these assets. In Section 3.2, we consider the case where the number of risky assets goes to infinity. Finally, in Section 3.3, we consider the case of N risky assets, which are divided into M classes, with the investor having a different level of familiarity for each of the M asset classes.

3.1 The case with N risky assets and familiarity about only one asset

Let us first consider the limiting case with *no ambiguity* about asset returns, $\alpha_n \rightarrow 0$, and so

$$\hat{\mu} = \mu, \text{ for all risky assets: (12)}$$

In this case, "Keynes meets Markowitz" and the optimal portfolio is given by (4). These portfolio weights have the well-known properties that investment in the risky assets increases with expected return, $\hat{\mu}$, and decreases with risk aversion, γ , volatility of returns, σ , and the correlation between the returns on the two assets, ρ . And for the special case where the assets are uncorrelated, $\rho = 0$, the expression for the weights reduces to:

$$\pi|_{\rho=0} = \frac{1}{\gamma} \frac{1}{\sigma^2} \hat{\mu}. \quad (13)$$

On the other hand, an investor who is averse to ambiguity and has maxmin preferences acts conservatively (or pessimistically), so in his choice of μ_n he will use its lowest possible estimate within the confidence interval when deciding to go long an asset. Doing the inner minimization for the problem in (10)-(11), we see that the investor will use the following estimates for μ_n :

$$\mu_n = \hat{\mu} - \text{sign}(\pi_n) \sigma_{\hat{\mu}} \alpha_n, \quad n = 1, 2, \dots, N. \quad (14)$$

This equation implies that if the weight of Asset n , π_n , were positive, then the estimated mean return, $\hat{\mu}$, should be reduced downward by the product of σ_n , the ambiguity about Asset n , and $\sigma_{\hat{\mu}}$, the standard deviation of the estimate $\hat{\mu}$. This, of course, would reduce the magnitude of the position in Asset n . In our simplified setting, because the excess return on all the risky assets is positive, the investor will never choose to short any of the risky assets. In a more general setting, where all assets do not have the same return distribution, it is possible that the investor would like to be long some assets and short other assets. In that case, for assets that the investor would

14 - Klibano, Marinacci, and Mukerji (2005) develop an alternative model of decision-making under uncertainty where ambiguity aversion is potentially separable from ambiguity; but, this model requires the decision maker to specify subjective probabilities over the priors. Moreover, in the model developed by Klibano, Marinacci, and Mukerji, one obtains smooth demand functions, in contrast to the kinked demand functions generated by the maxmin specification that we use. Hayashi and Miao (2010) and Ju and Miao (2010) extend the recursive smooth ambiguity model of Klibano, Marinacci, and Mukerji to allow for a three-way separation among risk aversion, ambiguity aversion, and intertemporal substitution; Chen, Ju, and Miao (2009) apply this model to the dynamic asset allocation problem in the presence of predictability.

have shorted, the investor will use the *highest* possible estimate of μ_n within the confidence interval when deciding to go short that asset, thereby again *reducing* the magnitude of the position that is taken. Of course, there is also the possibility that if $\hat{\mu} > 0$, then, after subtracting $\sigma_{\hat{\mu}} \alpha_n$, the resulting portfolio weight becomes negative, or that if $\hat{\mu} < 0$, then after adding $\sigma_{\hat{\mu}} \alpha_n$ the resulting weight is positive. In this case, the ambiguity is so large that the investor is neither willing to take a long position nor a short position, and so the optimal portfolio is $\pi_n = 0$.

Substituting (14) back into the problem in (10) gives

$$\max_{\pi} \left\{ \pi^T \hat{\mu} - \sum_{n=1}^N \text{sign}(\pi_n) \pi_n \sigma_{\hat{\mu}} \alpha_n - \frac{\gamma}{2} \pi^T \Sigma \pi \right\}. \quad (15)$$

The optimal portfolio in (15) is given by the following proposition where we assume, without loss of generality, that the investor is relatively more familiar with the first asset, which we label as "Asset F." We use α_F to denote the level of ambiguity for the mean of the return distribution of Asset F, and α_U to denote the level of ambiguity for the remaining $N - 1$ "unfamiliar" assets in the economy, with $\alpha_F \leq \alpha_U$.¹⁵ Because the remaining $N - 1$ assets are identical in every respect, the optimal portfolio weights in these assets must be identical. We denote by π_F the portfolio holding in the familiar asset and by π_U the $(N - 1)$ -dimensional vector of holdings in the remaining unfamiliar assets. The following proposition characterizes the optimal portfolio choice of an ambiguity averse investor.

Proposition 1 *Let α_F be the level of ambiguity for the familiar asset and α_U the level of ambiguity common across the remaining $N - 1$ assets, with $\alpha_F \leq \alpha_U$. Then the optimal portfolio weights, (π_F, π_U) , of an investor who is averse to ambiguity are given by the following three cases:*

Case I. If $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} > \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$:

$$\pi_F = \frac{1}{\gamma \sigma^2 (1 + \rho(N - 1))} \left(\hat{\mu} - \sigma_{\hat{\mu}} \left(\alpha_F - (N - 1) \frac{\rho}{1 - \rho} (\alpha_U - \alpha_F) \right) \right) > 0, \quad (16)$$

$$\pi_U = \frac{1}{\gamma \sigma^2 (1 + \rho(N - 1))} \left(\hat{\mu} - \sigma_{\hat{\mu}} \left(\alpha_F + \frac{1}{1 - \rho} (\alpha_U - \alpha_F) \right) \right) \cdot \mathbf{1}_{N-1} > \mathbf{0}_{N-1}. \quad (17)$$

Case II. If $\alpha_F < \frac{\hat{\mu}}{\sigma_{\hat{\mu}}} < \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$:

$$\pi_F = \frac{1}{\gamma \sigma^2} (\hat{\mu} - \sigma_{\hat{\mu}} \alpha_F) > 0, \quad (18)$$

$$\pi_U = \mathbf{0}_{N-1}. \quad (19)$$

Case III. If $0 < \frac{\hat{\mu}}{\sigma_{\hat{\mu}}} \leq \alpha_F$:

$$\pi_F = 0, \quad (20)$$

$$\pi_U = \mathbf{0}_{N-1}. \quad (21)$$

To provide intuition for the result in the proposition we start by studying the simpler situation in which the correlation between the assets is zero, $\rho = 0$. In Case I above, for the case of zero correlation:

$$\pi|_{\rho=0} = \begin{bmatrix} \pi_F \\ \pi_U \end{bmatrix} = \frac{1}{\gamma \sigma^2} \begin{bmatrix} \hat{\mu} - \sigma_{\hat{\mu}} \alpha_F \\ (\hat{\mu} - \sigma_{\hat{\mu}} \alpha_U) \times \mathbf{1}_{N-1} \end{bmatrix}. \quad (22)$$

Comparing the expressions in (13) and (22), we see that in the presence of ambiguity about a particular risky asset n , the investment in that risky asset is reduced. Moreover, the investor will

¹⁵ - As we show in Proposition 3 below, it is possible to generalize the model to more than two categories of assets with different degrees of ambiguity for each category.

hold asset n only if $\hat{\mu}/\sigma_{\hat{\mu}} > \sigma_n$. Conversely, if the above inequality is not satisfied, that is, the ambiguity α_n about asset n is too large, then the investor will not hold asset n at all, even though holding this asset would allow the investor to diversify the portfolio. Clearly, as the level of ambiguity α_n increases, the ratio $\hat{\mu}/\sigma_{\hat{\mu}}$ that is required so that the investor is willing to hold the asset also increases; that is, either the mean return needs to increase or the standard deviation of the mean return needs to decrease.

When the assets are correlated, we see from the expressions for the optimal portfolio weights in (16) and (17) that the investor will hold more of Asset F than the unfamiliar assets if ambiguity about Asset F is less than that for the unfamiliar assets: $\alpha_F < \alpha_U$; that is, the relative investment in these two asset classes depends on the relative ambiguity regarding their return distributions. Therefore, the portfolio is diversified across the two assets but has a *bias* toward the familiar asset. Only when the level of ambiguity in both assets is the same, $\alpha_F = \alpha_U$, is the proportion of wealth allocated to each asset the same.

Comparing the portfolio weights in (16) and (17) with those for the Markowitz portfolio in (4), we also see that the investor will hold the unfamiliar assets only if the ratio $\hat{\mu}/\sigma_{\hat{\mu}}$ for these assets is sufficiently large to offset the effect of its own ambiguity and the effect that comes from Asset F 's correlation with the unfamiliar assets:

$$\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} > \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F) = \alpha_U + \frac{\rho}{1-\rho}(\alpha_U - \alpha_F). \quad (23)$$

The quantity on the right-hand side of the above expression can be thought of as the "correlation-adjusted" ambiguity threshold to induce holding in the unfamiliar assets. This threshold accounts for the fact that, due to positive correlation, holding Asset F already entails partially holding the unfamiliar assets. In the limit as $\rho \rightarrow 1$, the right-hand side of (23) goes to infinity, implying that the investor will not hold unfamiliar assets at all. The intuition for this result is that Asset F , the more familiar asset, has a higher ambiguity-adjusted return than the relatively unfamiliar assets (because $\alpha_F < \alpha_U$), and as these two asset classes become increasingly correlated, the diversification benefit from holding unfamiliar assets diminishes. This is a prediction of the model that is consistent with empirical evidence. For instance, during a financial crisis there is an increase in the correlations between returns on different assets, and at such times we often observe investors pulling out of foreign assets and going into more familiar domestic assets.

Proposition 1 allows us to formalize the contrasting views of Keynes and Markowitz on portfolio choice. In Case I, the investor holds both the familiar asset (Asset F) and the unfamiliar assets (all assets other than Asset F). But, because $\alpha_F < \alpha_U$, the weight of Asset F in the portfolio exceeds that of each of the other assets. So, the investor holds familiar assets, as advocated by Keynes, but balances this investment by holding also a portfolio of all the other assets, as advocated by Markowitz.¹⁶ This portfolio is biased toward familiar assets, and the relative investment between the familiar asset and the unfamiliar assets depends on the relative ambiguity regarding the return distributions of these two asset classes, α_F and α_U .

Case II of Proposition 1 corresponds to the setting where the investor is relatively familiar with a particular asset (that is, $\hat{\mu}/\sigma_{\hat{\mu}} > \alpha_F$) and sufficiently unfamiliar with all the other assets (that is, $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} \leq \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$). In this case, the investor follows the advice of Keynes to the extreme: "The right method in investment is to put fairly large sums into enterprises which one thinks one knows something about. [...] It is a mistake to think that one limits one's risk by spreading too much between enterprises about which one knows little and has no reason for special confidence." Thus, the investor holds only the familiar risky asset, $\pi_F > 0$, with zero in the unfamiliar risky assets, $\pi_U = 0_{N-1}$.

16 - Jeremy Siegel, in an article titled "Ben Bernanke's Favorite Stock," published on the web site Yahoo!Finance, reported that Bernanke held only one individual stock, Altria Group (formerly Phillip Morris); Chris Isidore in the article "Bernanke's Bucks," published on CNNMoney.com, reports that an overwhelming majority of Bernanke's holdings are in his TIAA-CREF account. Greenspan, on the other hand, has most of his wealth in Treasury bonds and bond funds, with only a small amount invested in individual stocks: besides the investment in General Electric stock held in his wife's 401(k) account, Greenspan's largest stock holdings are in Abbott Laboratories, Kimberly Clark, Anheuser Busch, and H.J. Heinz.

In Case III of Proposition 1, the ambiguity about the expected return of the familiar asset is large (that is, $\hat{\mu}/\sigma_{\hat{\mu}} \leq \alpha_F$), and the ambiguity about the other assets is even larger; consequently, $\pi_F = 0$ and also $\pi_U = 0_{N-1}$. That is, if ambiguity is sufficiently high for all risky assets, the investor will put his entire wealth in the risk-free asset and will not invest at all in any of the risky assets, which corresponds to complete non-participation in the stock market, as has been documented empirically (see, for instance, Vissing-Jorgensen (2003) and Campbell (2006)).

To get a sense of which of the three cases will obtain for different parameter values, we undertake the following quantitative exercise. We assume that the investor uses 100 years of annual data ($T = 100$) to estimate the excess return (equity premium) μ on each stock.¹⁷ We take the estimated value $\hat{\mu}$ of the equity premium to be equal to 6% p.a (Mehra and Prescott (1985); Welch (2000)). We specify the volatility of individual stock returns, σ , to be 30%.¹⁸ This implies that the standard deviation of the estimate of mean returns is $\sigma_{\hat{\mu}} = \frac{\sigma}{\sqrt{T}} = \frac{0.30}{\sqrt{100}}$ so that $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} = 2.0$. We assume that the correlation between stocks, ρ , is equal to 25%, which allows us to compute $\alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$, given values of α_F and α_U . From the conditions for the three cases that are given in the statement of Proposition 1, we see that if $\alpha = 0$ and $\alpha_U = 1$, then the relevant case is Case I; if $\alpha_F = 0$ and $\alpha_U = 1.5$, then the relevant case is Case II; similarly, if $\alpha_F = 1.5$, and $\alpha_F = 2$, then the relevant case is Case II; but, if $\alpha_F > 2$, then Case III is the relevant case. Keeping all else fixed, a decrease in $\hat{\mu}$ (or an increase in $\sigma_{\hat{\mu}}$) will increase the likelihood of Case III.

Notice that because $\sigma_{\hat{\mu}} = \sigma/\sqrt{T}$, as $T \rightarrow \infty$, $\sigma_{\hat{\mu}} \rightarrow 0$, and so the optimal portfolio weights from Proposition 1 converge to the Markowitz portfolio weights, as can be verified from equations (16) and (17). This is intuitive: with an infinite amount of data, there is no ambiguity.¹⁹

In the following corollary, we derive the expression for the *relative* weights in the familiar risky asset, $\omega_F = \frac{\pi_F}{\pi_F + \pi_U \mathbf{1}_{N-1}}$, and in the other $(N - 1)$ assets, $\omega_U = \frac{\pi_U}{\pi_F + \pi_U \mathbf{1}_{N-1}}$.

Corollary 1 *The relative weights (ω_F ; ω_U), of an investor who is averse to ambiguity are given by the following three cases:*

Case I. If $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} > \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$

$$\omega_F = \frac{1}{N} + \sigma_{\hat{\mu}} \frac{(\alpha_U - \alpha_F) \left(1 - \frac{1}{N} + \frac{\rho}{1-\rho}(N-1)\right)}{N \left(\hat{\mu} - \sigma_{\hat{\mu}} \alpha_F - \sigma_{\hat{\mu}} (\alpha_U - \alpha_F) \left(1 - \frac{1}{N}\right)\right)} > \frac{1}{N}, \quad (24)$$

$$\omega_U = \left(\frac{1}{N} - \sigma_{\hat{\mu}} \frac{(\alpha_U - \alpha_F) \left(\frac{1}{1-\rho} - \left(1 - \frac{1}{N}\right)\right)}{N \left(\hat{\mu} - \sigma_{\hat{\mu}} \alpha_F - \sigma_{\hat{\mu}} (\alpha_U - \alpha_F) \left(1 - \frac{1}{N}\right)\right)} \right) \cdot \mathbf{1}_{N-1} < \frac{1}{N} \cdot \mathbf{1}_{N-1}. \quad (25)$$

Case II. If $\alpha_F < \frac{\hat{\mu}}{\sigma_{\hat{\mu}}} \leq \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$

$$w_F = 1 \quad (26)$$

$$w_U = 0_{N-1} \quad (27)$$

Case III. If $0 < \frac{\hat{\mu}}{\sigma_{\hat{\mu}}} \leq \alpha_F$

$$w_F = 0 \quad (28)$$

$$w_U = 0_{N-1} \quad (29)$$

As expected, relative to the benchmark Markowitz portfolio, the portfolio that combines the views of Keynes and Markowitz (Case I) overweights the familiar asset $w_F > 1/N$ and underweights the unfamiliar assets, that is, each element of w_U is less than $1/N$.

17 - The assumption that we have 100 years of data is a conservative one because the effect of ambiguity decreases with T ; in reality, one would be fortunate to find data for even 50 years that was generated by a stationary distribution.

18 - This value for individual firm volatility is in line with the estimates of Chan, Karceski, and Lakonishok (1999) who find that the average volatility of large firms is 28.3%, the average volatility of the average firm is 34.3%, and the average volatility of small firms is 46.6%.

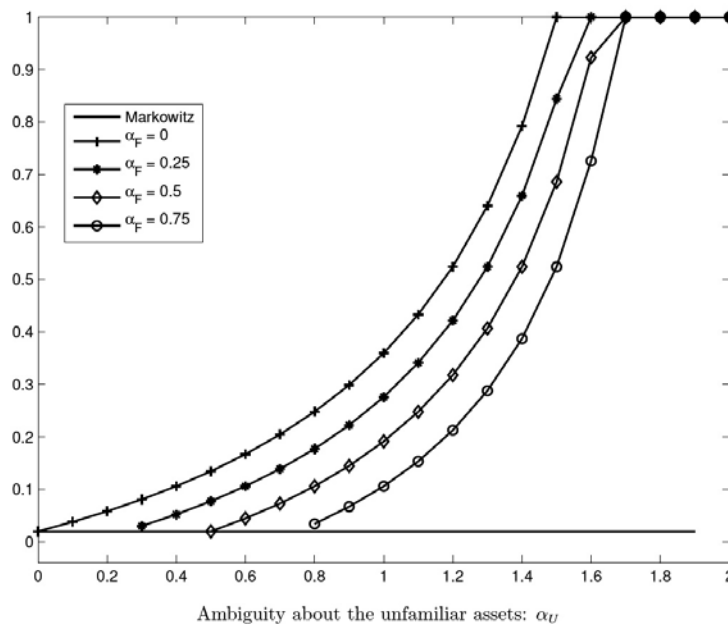
19 - Thus, another testable implication of the model is that as a longer time series of data becomes available, we should see investors moving from Keynesian to Markowitz portfolios.

To assess the magnitude of the result presented in the proposition above, we undertake the following quantitative exercise. We assume that the number of stocks available to the investor is $N = 50$. Just as before, we assume that the investor uses 100 years of annual data ($T = 100$) to estimate the excess return (equity premium) μ on each stock, the estimated value $\hat{\mu}$ of the equity premium is 6% p.a, the volatility of individual stock returns, σ , is 30% so that the standard deviation of the estimate of mean returns is $\sigma_{\hat{\mu}} = \frac{\sigma}{\sqrt{T}} = \frac{0.30}{\sqrt{100}} = 0.03$. As before, we assume that the correlation between stocks, ρ , is equal to 25%, which then implies that the volatility of the stock market is $\sqrt{\rho \times \sigma^2} = 15\%$ and that the market Sharpe ratio is 40%. We also assume that the investor's risk aversion is $\gamma = 2$.

Our results are presented in Figure 1. The horizontal axis of the plot represents the level of ambiguity about the unfamiliar assets, α_U .²⁰ The vertical axis of the plot displays w_F , the *relative* investment in the familiar asset, as a proportion of the total investment in the $N = 50$ risky assets. The horizontal solid line labeled "Markowitz" shows the investment in the familiar asset when there is no ambiguity about any of the N assets. Each of the curved lines corresponds to a different level of ambiguity for the familiar asset, $\alpha_F = \{-0.0, 0.25, 0.5, 0.75\}$.

The figure shows that there is substantial investment in the familiar stock even for low values of α_U . For instance, the leftmost curve for $\alpha_F = 0$ shows that even for $\alpha_U = 0.75$, the investment in the familiar stock exceeds 20%; for $\alpha_U = 1.0$, the investment in the familiar stock exceeds 30%; and, for $\alpha_U = 1.5$, the investment in the familiar stock is 100%. Looking at the rightmost curve for $\alpha_F = 0.75$, we see that if $\alpha_U = 1.5$, then the relative investment in the familiar stock exceeds 50%, and if $\alpha_U = 1.75$, then the relative investment in the familiar stock is 100%. The trade-offs advocated by Keynes and Markowitz are particularly stark in the context of the decision to invest in own-company stock; that is, the empirical observation that if a firm's own stock is one of the assets available for investment in 401(k) defined-contribution pension plans, many employees invest a significant fraction of their discretionary contributions in their own company's stock. For instance, Benartzi (2001), Mitchell and Utkus (2002), and Meulbroek (2005) find that about 25%–30% of the discretionary contributions of employees are invested in stocks of their own company. From the experiment above, we observe that the model can generate substantial holdings of the familiar stock that is consistent with empirical observations.

Figure 1: Relative investment in the familiar asset This figure plots the relative investment in the familiar asset, w_F , as a function of the ambiguity about the unfamiliar asset, α_U . Each curve in the figure corresponds to a different level of ambiguity about the familiar asset, $\alpha_F = \{-0.00, 0.25, 0.50, 0.75\}$. The horizontal solid line plots the relative investment in the familiar asset under the Markowitz model, where there is no ambiguity, and therefore the relative investment in the familiar asset is $1/50 = 0.02$.



20 - To better appreciate the meaning of a particular value of α_U , it may help to relate its magnitude to the size of the corresponding confidence interval. For example, under the assumption of normally distributed asset returns, a value for α_U of 0.25 corresponds to a 19% confidence interval, $\alpha_U = 0.50$ to a 38% confidence interval, $\alpha_U = 1.00$ to a 68% confidence interval, $\alpha_U = 1.64$ to a 90% confidence interval, $\alpha_U = 1.96$ to a 95% confidence interval, and $\alpha_U = 2.56$ to a 99% confidence interval.

3.2 The limiting case with an infinite number of risky assets

To gain further insight on the dichotomy between the views of Keynes and Markowitz on portfolio allocation, we analyze the optimal portfolio in an economy where the number of risky assets, N , goes to infinity. As the number of assets increases, the benefits of diversification increase. As the next proposition shows, these benefits of diversification are not sufficient to overcome the desire to hold a portfolio biased toward familiar assets. We limit our analysis to the case in which the investor optimally is holding both the familiar and unfamiliar asset (Case I of Proposition 1), because for Cases II and III the portfolio weights do not depend on N ; and so, even when $N \rightarrow \infty$, the expressions for the portfolio weights are the same as those in (18) and (20), respectively, while the relative weights are the same as those in (26) and (28), respectively.

Proposition 2 Assume the investor is less ambiguous about Asset F than about the rest of the assets, that is, $\alpha_F < \alpha_U$, and that $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} > \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$ (Case I of Proposition 1). Then, as $N \rightarrow \infty$:

1. The optimal investment of an ambiguity averse investor in the familiar asset converges to:

$$\lim_{N \rightarrow \infty} \pi_F = \frac{1}{\gamma \sigma^2 (1-\rho)} \sigma_{\hat{\mu}} (\alpha_U - \alpha_F) > 0; \quad (30)$$

2. The investment in each of the $N - 1$ assets with which the investor is not familiar goes to

$$\lim_{N \rightarrow \infty} \pi_U = \mathbf{0}_{N-1}; \quad (31)$$

3. The total investment in the $N - 1$ assets with which the investor is not familiar converges to:

$$\lim_{N \rightarrow \infty} \pi_U^\top \mathbf{1}_{N-1} = \frac{1}{\gamma \sigma^2 \rho} \left(\hat{\mu} - \sigma_{\hat{\mu}} \left(\alpha_F + \frac{1}{1-\rho} (\alpha_U - \alpha_F) \right) \right); \quad (32)$$

4. The relative portfolio weight ω_F converges to:

$$\lim_{N \rightarrow \infty} \omega_F = \frac{\sigma_{\hat{\mu}} \frac{\rho}{1-\rho} (\alpha_U - \alpha_F)}{\hat{\mu} - \sigma_{\hat{\mu}} \alpha_U} > 0. \quad (33)$$

Equation (30) in the above proposition shows that, as N approaches infinity, the weight for the more familiar asset approaches a positive constant that depends on the difference in ambiguity about this asset and all the other assets. In contrast, Equation (31) shows that the *individual* weights for each of the other less-familiar assets approach zero as the number of assets increases, while the *total* weight in these $(N - 1)$ unfamiliar assets approaches the quantity given in Equation (32).

In summary, familiarity about Asset F implies that the holding in this asset does *not* decrease to zero even as N tends to infinity, while the gains from diversification imply that an investor should hold only an infinitesimal amount in each of the remaining unfamiliar assets. As we will show further on in Section 4.3, this result will have implications for the effect of a change in idiosyncratic risk on the optimal portfolio weights.

3.3 The case with N risky assets and varying familiarity across M asset classes

In the above analysis, we considered the case where the N assets can be divided into only two classes, with Asset F in the first class and the remaining $N - 1$ assets in the second class. The characterization of the optimal portfolio weights provided in Proposition 1 takes advantage of this particular feature. However, an investor may not always group assets into just two classes. A case of particular interest for its empirical relevance is that documented by Huberman and Jiang (2006) who, using a large dataset of individual portfolio choice in retirement accounts, find that investors tend to choose three to four funds out of the ones offered which, in their study, range from 4 to 59. This evidence, combined with the finding that the percentage of retirement account wealth invested in company stock is around thirty percent,²¹ seems to suggest that investors do not view all assets other than their own-company stock as being the same in every respect.

Intuitively, the portfolio problem in which an investor categorizes assets into three (or more) separate classes with different degrees of ambiguity is a simple generalization of the case of two separate asset classes analyzed above. In the next proposition, we characterize the optimal portfolio for an arbitrary number of asset classes about which the investor has varying degrees of familiarity.

Proposition 3 Assume that the set of available N assets can be categorized into $M \leq N$ mutually exclusive asset classes, with each category containing N_m assets, $m = 1, \dots, M$ and $N_1 + \dots + N_M = N$. Each asset class is characterized by degree of ambiguity $\alpha_1 < \alpha_2 < \dots < \alpha_M$. Denote by π_m the portfolio weight in each of the assets belonging to the class $m = 1, \dots, M$, and the quantity

$$s(m) = \alpha_m + \frac{\rho}{1-\rho} \sum_{j=1}^{m-1} N_j(\alpha_m - \alpha_j) > 0, \quad m = \{1, \dots, M\}. \quad (34)$$

Then, the optimal portfolio is characterized as follows.

Case I: If $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} > s(M)$, then:

$$\pi_m = \frac{1}{\gamma\sigma^2(1+\rho(N-1))} \left(\hat{\mu} - \sigma_{\hat{\mu}} \left(\alpha_m - \frac{\rho}{1-\rho} \sum_{j \neq m}^M N_j(\alpha_j - \alpha_m) \right) \right) > 0, \quad m = \{1, \dots, M\}. \quad (35)$$

Case II: If $s(m) < \frac{\hat{\mu}}{\sigma_{\hat{\mu}}} \leq s(m+1)$, $m = \{1, \dots, M-1\}$, then:

$$\begin{aligned} \pi_i &= \frac{1}{\gamma\sigma^2 \left(1 + \rho \left(\sum_{j=1}^m N_j - 1 \right) \right)} \\ &\times \left(\hat{\mu} - \sigma_{\hat{\mu}} \left(\alpha_i - \frac{\rho}{1-\rho} \sum_{j \neq i}^m N_j(\alpha_j - \alpha_i) \right) \right) > 0, \quad i = \{1, \dots, m\}; \end{aligned} \quad (36)$$

$$\pi_j = 0, \quad j = \{m+1, \dots, M\}. \quad (37)$$

Case III: If $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} \leq s(1)$, then:

$$\pi_m = 0, \quad m = \{1, \dots, M\}. \quad (38)$$

The above proposition is a natural generalization of Proposition 1 to the case of M asset classes with different levels of ambiguity. It can be verified that by setting $M = 2$, $N_1 = 1$ and $N_2 = N - 1$, one recovers the optimal portfolio derived in Proposition 1. The general structure of the solution suggests that the portfolio weights are larger in the assets with smaller ambiguity, that is, $\pi_1 \geq \pi_2 \dots \geq \pi_M$. In contrast, when the level of ambiguity is sufficiently large in a particular asset class, the optimal decision is not to hold assets in that class, as Cases II and III of the above proposition show.

We now examine whether our model can generate the pattern of asset allocation documented by Huberman and Jiang (2006) for reasonable parameter values. In order to analyze this, we divide the class of assets *other* than the familiar own-company stock into two separate sub-classes. We take the first sub-class to be composed of three identical assets, to capture the identical allocation across the chosen funds documented by Huberman and Jiang (2006). The remaining $N-3-1$ assets are treated as distinct from own-company stock and from the funds, but otherwise identical and ex-ante indistinguishable from each other, and we assume that the total number of assets offered is $N = 50$. Using the same parameter values as before, we now determine under what values for α_1 , α_2 , and α_3 we obtain portfolio weights that are similar to the observed "30–70–0" asset allocation in own-company stocks, funds, and the rest of the assets, respectively.

In Table 1, we report the relative portfolio weights for the case with three asset classes: own-company stock, ω_1 , three funds (with aggregate weight $\omega_2^\top \mathbf{1}_3$), and the remaining 46 ($= 50 - 3 - 1$) assets (with aggregate weight in these three assets relative to the total investment in the 50 risky assets, $\omega_3^\top \mathbf{1}_{46}$). For these three asset classes, we consider values of $\alpha_1 = -0.00, 0.25, 0.50, 0.75\}$, $\alpha_2 = -0.75, 0.80, 0.85, 0.90\}$, and $\alpha_3 = -0.90, 1.10, 1.25, 1.50\}$. From Panel A of the table, in which $\alpha_1 = 0$, we see that when $\alpha_3 = 1.25$ or larger, there is no investment at all in the third asset class, and the investor chooses to invest in only the first asset (own-company stock) and the second asset (funds), holding about 50% in both the own-company stock and in the three funds. In Panels C and D, where α_1 is 0.50 and 0.75 respectively, we see that for several combinations of α_2 and α_3 , we obtain an investment of about 30% in own-company stock, 70% in the three funds, and zero in the remaining assets.

Overall, the findings suggest that in a standard asset allocation model, small changes in expected returns that fall well within condence bounds used in traditional statistical inference can generate the documented asset-allocation holdings of pension-plan participants, with roughly one-third of their holdings invested in their own-company stock, the rest invested in a few mutual funds, and zero invested in other assets offered by the pension plan.

Table 1: Relative portfolio weights when there are three asset classes

This table gives the relative investment in own-company stock, ω_1 , the sum of the investment in three other assets, $\omega_2^\top \mathbf{1}_3$, and the investment in the remaining 46 ($= 50 - 3 - 1$) assets, $\omega_3^\top \mathbf{1}_{46}$, as a function of the ambiguity about the assets in these three asset classes, α_1 , α_2 , and α_3 . Each panel in the table corresponds to a different level of ambiguity about the familiar asset, $\alpha_1 = -0.00, 0.25, 0.50, 0.75\}$. Each row corresponds to a different level of $\alpha_2 = -0.75, 0.80, 0.85, 0.90\}$. The four sets of columns correspond to $\alpha_3 = -0.90, 1.10, 1.25, 1.50\}$. All other parameter values are as described earlier in the text.

α_2	$\alpha_3 = 0.90$			$\alpha_3 = 1.10$			$\alpha_3 = 1.25$			$\alpha_3 = 1.50$		
	ω_1	$\omega_2^\top \mathbf{1}_3$	$\omega_3^\top \mathbf{1}_{46}$	ω_1	$\omega_2^\top \mathbf{1}_3$	$\omega_3^\top \mathbf{1}_{46}$	ω_1	$\omega_2^\top \mathbf{1}_3$	$\omega_3^\top \mathbf{1}_{46}$	ω_1	$\omega_2^\top \mathbf{1}_3$	$\omega_3^\top \mathbf{1}_{46}$
<i>Panel A: $\alpha_1 = 0.00$</i>												
0.75	0.29	0.18	0.53	0.42	0.41	0.18	0.48	0.52	0.00	0.48	0.52	0.00
0.80	0.30	0.13	0.57	0.42	0.35	0.23	0.50	0.50	0.00	0.50	0.50	0.00
0.85	0.30	0.09	0.62	0.42	0.30	0.28	0.52	0.48	0.00	0.52	0.48	0.00
0.90	0.30	0.04	0.66	0.42	0.25	0.33	0.55	0.45	0.00	0.55	0.45	0.00
<i>Panel B: $\alpha_1 = 0.25$</i>												
0.75	0.22	0.18	0.60	0.33	0.41	0.26	0.41	0.59	0.00	0.41	0.59	0.00
0.80	0.22	0.14	0.64	0.33	0.36	0.31	0.43	0.57	0.00	0.43	0.57	0.00
0.85	0.22	0.09	0.69	0.33	0.31	0.36	0.45	0.54	0.02	0.45	0.55	0.00
0.90	0.22	0.05	0.73	0.33	0.26	0.41	0.45	0.47	0.08	0.48	0.52	0.00
<i>Panel C: $\alpha_1 = 0.50$</i>												
0.75	0.14	0.19	0.67	0.23	0.42	0.35	0.33	0.67	0.00	0.33	0.67	0.00
0.80	0.14	0.14	0.72	0.24	0.37	0.40	0.34	0.61	0.06	0.35	0.65	0.00
0.85	0.14	0.10	0.76	0.24	0.31	0.45	0.34	0.54	0.12	0.37	0.63	0.00
0.90	0.15	0.05	0.80	0.24	0.26	0.50	0.34	0.48	0.18	0.40	0.60	0.00
<i>Panel D: $\alpha_1 = 0.75$</i>												
0.75	0.06	0.19	0.74	0.14	0.43	0.43	0.23	0.68	0.10	0.25	0.75	0.00
0.80	0.06	0.15	0.79	0.14	0.38	0.48	0.23	0.62	0.16	0.27	0.73	0.00
0.85	0.07	0.10	0.83	0.15	0.32	0.53	0.23	0.55	0.22	0.29	0.71	0.00
0.90	0.07	0.06	0.88	0.15	0.27	0.58	0.23	0.49	0.27	0.31	0.69	0.00

4. Implications for Portfolio Risk and Portfolio Sensitivity to Risk

In this section, we study the implications of the optimal portfolio derived above for portfolio risk, asset demand, and the sensitivity of the portfolio weights to idiosyncratic and systematic risk.

4.1 Implications for portfolio risk

The fact that, in the presence of different degrees of ambiguity across assets, the optimal portfolio exhibits a tilt toward the more familiar asset classes has a natural consequence on the riskiness of such a portfolio, relative to the benchmark Markowitz portfolio. The next proposition formalizes this relation in an economy with an infinite number of assets. We limit our focus on the implications for the optimal portfolios in Cases I and II of Proposition 1, because in the non-participation situation of Case III, the portfolio variance is trivially equal to zero.

Proposition 4 Assume the investor is less ambiguous about Asset F than about other assets, that is, $\alpha_F < \alpha_U$. As $N \rightarrow \infty$, the variance $\omega^\top \Sigma \omega$ of the risky part of the investor's portfolio is:

Case I: If $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} > \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$

$$\lim_{N \rightarrow \infty} \omega^\top \Sigma \omega = \sigma_S^2 + \frac{\rho^2 \sigma^2 \sigma_{\hat{\mu}}^2 (\alpha_U - \alpha_F)^2}{(1-\rho)(\hat{\mu} - \sigma_{\hat{\mu}} \alpha_U)^2}. \quad (39)$$

Case II: If $\alpha_F < \frac{\hat{\mu}}{\sigma_{\hat{\mu}}} \leq \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$,

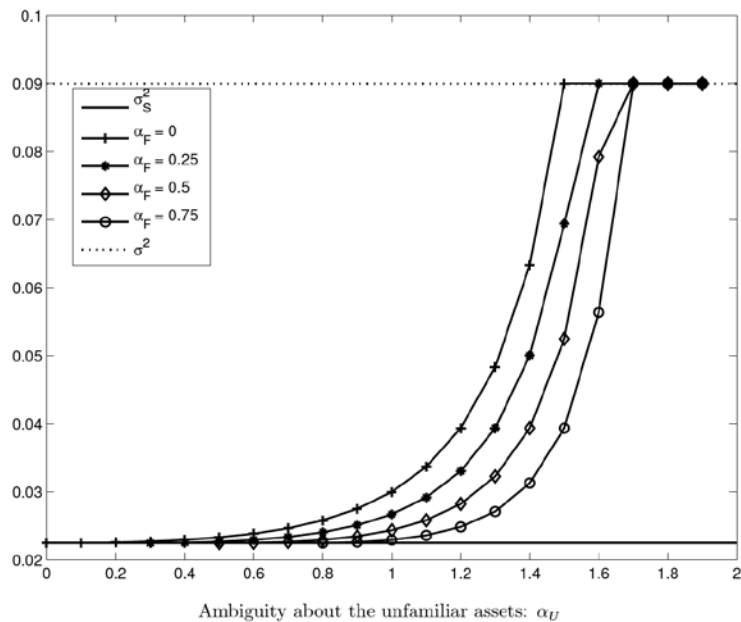
$$\lim_{N \rightarrow \infty} \omega^\top \Sigma \omega = \sigma_S^2 + (1-\rho)\sigma^2 = \sigma^2 = \sigma_S^2 + \sigma_U^2. \quad (40)$$

Recall from Equation (7) that the quantity $\sigma_S^2 = \rho\sigma^2$ in the proposition above represents systematic risk and is equal to the variance of the Markowitz portfolio. Therefore, from Proposition 4 we can conclude that in both Cases I and II, the portfolio variance is greater than what it would be in the case of the Markowitz portfolio. This is a direct consequence of the fact that in both these cases the investor's portfolio is less diversified than the Markowitz portfolio. The additional risk in Case I increases with the difference between the familiarity of the first asset relative to the other assets $\alpha_U - \alpha_F$ and it also increases in total risk, σ^2 . Case II is intuitive because the optimal portfolio ω contains only the familiar asset, and so the portfolio risk is equal to the total risk of holding a single asset, $\sigma^2 = \sigma_S^2 + \sigma_U^2$.

In Figure 2, we illustrate Proposition 4 using the same parameter values as before for the case where $N \rightarrow \infty$. From the proposition we know that the variance $\omega^\top \Sigma \omega$ of the risky part of the portfolio of an ambiguity-averse investor is larger than the variance of the benchmark Markowitz portfolio, σ_S^2 . This is because the ambiguity-averse portfolio is more concentrated in the familiar asset than the Markowitz portfolio. Each curve in the figure corresponds to a different level of ambiguity α_F , while we vary α_U along the horizontal axis. From expression (39), we can see that portfolio variance is larger the greater the spread between α_F and α_U , that is, the more familiar is Asset F, relative to the remaining assets. This follows intuitively from the fact that the more familiar an asset is, the more biased the portfolio is toward that asset.

Figure 2: Riskiness of the Keynesian portfolio

This figure plots the variance of the investor's portfolio, $\omega^\top \Sigma \omega$, as a function of the ambiguity about the unfamiliar asset, α_U . Each curve in the figure corresponds to a different level of ambiguity about the familiar asset, $\alpha_F = \{-0.00, 0.25, 0.5, 0.75\}$. The horizontal solid line plots the variance of the portfolio under the Markowitz model, where there is no ambiguity, and, therefore, the portfolio risk is $\sigma_S^2 = \rho\sigma^2$. The upper bound is the risk of holding only a single asset, $\sigma^2 = \sigma_S^2 + \sigma_U^2$.



In Figure 2, this can be seen by observing that the portfolios with lower values of α_F correspond to higher variance, for any given value of α_U . Furthermore, keeping α_F constant, the variance of a portfolio increases in the degree of ambiguity in the unfamiliar assets. This again follows from expression (39) and is a consequence of an increase in concentration as the spread between ambiguity about the two classes of assets increases.

The lower bound of all the lines in Figure 2 is σ_S^2 , the variance of the Markowitz portfolio, while the dotted upper bound is 2, the variance of an individual asset. The portfolio that reflects the view of Keynes and Markowitz (Case I) corresponds to values of α_F and α_U such that $\hat{\mu}/\sigma_{\hat{\mu}} > \alpha_F + 1/(1-\rho)(\alpha_U - \alpha_F)$, or equivalently, for $\alpha_U \in [\alpha_F, \hat{\mu}/\sigma_{\hat{\mu}}(1-\rho) + \alpha_F]$. For values of $\alpha_U > \hat{\mu}/\sigma_{\hat{\mu}}(1-\rho) + \alpha_F$, the investor holds only the familiar Asset F (Case II) and hence the variance of his portfolio will be σ^2 . The kink in Figure 2 corresponds to the boundary between Case I and Case II.

4.2 Effect of correlation on the demand for risky assets

The analytical expression for the optimal portfolio from Proposition 1 allows us to study the comparative statics with respect to some of the key parameters of the model. In this section we focus our attention on the correlation coefficient ρ , while in the next subsection we analyze the effects of idiosyncratic volatility and systematic volatility.

There is a substantial literature documenting that during a financial crisis, there is an increase in correlations between the returns of risky assets (see, for example, Longin and Solnik (1995), Scholes (2000), Lo (2001), and Poon, Rockinger, and Tawn (2004)). We would like to understand the implications of this for the portfolio weights in our model. The next proposition shows the comparative static result that an increase in correlation leads to an increase in the holding of the familiar asset.

Proposition 5 *Let ω_F be the relative weight in the familiar Asset F for the optimal portfolio in Case I of Proposition 1, where the investor biases her portfolio toward Asset F but also invests in all the other risky assets. Then:*

$$\frac{\partial \omega_F}{\partial \rho} = \frac{(N-1)(\alpha_U - \alpha_F)\sigma_{\hat{\mu}}}{(1-\rho)^2 (N(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_U) + \sigma_{\hat{\mu}}(\alpha_U - \alpha_F))} > 0, \quad (41)$$

and in the limit:

$$\lim_{N \rightarrow \infty} \frac{\partial \omega_F}{\partial \rho} = \frac{(\alpha_U - \alpha_F)\sigma_{\hat{\mu}}}{(1-\rho)^2 (\hat{\mu} - \sigma_{\hat{\mu}}\alpha_U)} > 0. \quad (42)$$

To understand the implication of this result, note first that for the Markowitz portfolio in which the relative weights are equal to $1/N$, the derivative of these weights with respect to ρ is zero. The above proposition shows that if there are differences in the ambiguity about expected returns of assets, then when correlations increase, there is *flight to familiarity*; that is, the investor increases the relative weight in the familiar asset. From (42), we also see that the magnitude of this effect increases with the difference in ambiguity between the familiar asset and the other assets and with the correlation between the assets. The intuition for this result is simple. Given that all assets have the same distribution of returns, an increase in ρ means that the unfamiliar assets become progressively less useful for diversification; that is, the unfamiliar assets tend to behave more like the familiar asset; in addition, their expected returns are estimated less precisely. An ambiguity averse investor will therefore invest relatively less in the unfamiliar assets. The implication of this result is interesting: the greater the difference in ambiguity across assets, the more concentration in the familiar asset one should expect in times of a financial crisis.

Figure 3: Flight to familiarity

This figure plots the derivative $\partial\omega_F/\partial\rho$, which gives the change in the relative weight of the familiar asset, ω_F , as one changes the correlation, ρ . This derivative is plotted as a function of the ambiguity about the unfamiliar asset, α_U . Each curve in the figure corresponds to a different level of ambiguity about the familiar asset, $\alpha_F = \{-0.00, 0.25, 0.50, 0.75\}$. The horizontal solid line plots the derivative for the Markowitz portfolio, which is 0, because if there is no ambiguity then the relative portfolio weight is $1/N$, which is insensitive to a change in ρ .

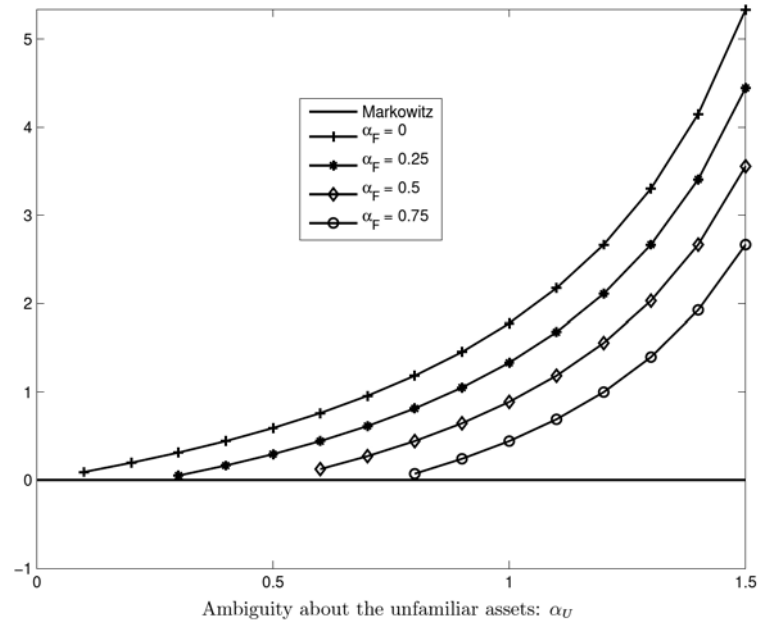


Figure 3 plots the derivative $\partial\omega_F/\partial\rho$ of the relative holdings of Asset F with respect to correlation, using the same parameter values as before. The figure displays the derivative $\partial\omega_F/\partial\rho$ in the region of α_U that gives rise to the portfolio combining the views of Keynes and Markowitz, that is, Case I of Proposition 1. From the figure we see that this derivative is (i) increasing in the degree of ambiguity α_U about the unfamiliar asset, keeping α_F constant, and (ii) decreasing in the degree of ambiguity α_F about the familiar asset, keeping α_U constant. Naturally, the more familiar the investor is with Asset F , the stronger will be the flight to familiarity effect. Note nally that in the Markowitz case, relative portfolio weights are a constant, $1/N$, and hence, their derivative with respect to ρ is zero.

Table 2: Effect of correlation on portfolio weights

This table gives the relative investment in the familiar asset, ω_F , and in the remaining $N - 1$ assets, ω_U , as a function of the ambiguity levels α_F and α_U . Each panel in the table corresponds to a different level of ambiguity about the familiar asset, $\alpha_F = \{-0.00, 0.25, 0.50, 0.75\}$. Each row corresponds to a different level of $\alpha_U = \{0.75; 0.80; 0.85; 0.90\}$. The four sets of columns correspond to $\rho = \{-0.25, 0.50, 0.70, 0.90\}$. All other parameter values are as described earlier in the text.

α_U	$\rho = 0.25$		$\rho = 0.50$		$\rho = 0.70$		$\rho = 0.90$	
	ω_F	ω_U	ω_F	ω_U	ω_F	ω_U	ω_F	ω_U
Panel A: $\alpha_F = 0.00$								
0.75	0.23	0.77	0.61	0.39	1.00	0.00	1.00	0.00
0.80	0.25	0.75	0.68	0.32	1.00	0.00	1.00	0.00
0.85	0.27	0.73	0.75	0.25	1.00	0.00	1.00	0.00
0.90	0.30	0.70	0.82	0.18	1.00	0.00	1.00	0.00
Panel B: $\alpha_F = 0.25$								
0.75	0.16	0.84	0.42	0.58	0.94	0.06	1.00	0.00
0.80	0.18	0.82	0.47	0.53	1.00	0.00	1.00	0.00
0.85	0.20	0.80	0.54	0.46	1.00	0.00	1.00	0.00
0.90	0.22	0.78	0.60	0.40	1.00	0.00	1.00	0.00
Panel C: $\alpha_F = 0.50$								
0.75	0.09	0.91	0.22	0.78	0.48	0.52	1.00	0.00
0.80	0.11	0.89	0.27	0.73	0.59	0.41	1.00	0.00
0.85	0.12	0.88	0.32	0.68	0.72	0.28	1.00	0.00
0.90	0.15	0.85	0.38	0.62	0.85	0.15	1.00	0.00
Panel D: $\alpha_F = 0.75$								
0.75	0.02	0.98	0.02	0.98	0.02	0.98	0.02	0.98
0.80	0.03	0.97	0.06	0.94	0.12	0.88	0.39	0.61
0.85	0.05	0.95	0.11	0.89	0.22	0.78	0.79	0.21
0.90	0.07	0.93	0.16	0.84	0.33	0.67	1.00	0.00

The magnitude of the effect of a change in correlation can be evaluated by the portfolio weights reported in Table 2. In each panel we consider a different level of ambiguity in the familiar asset α_F ; each row represents a different level of ambiguity α_U in the unfamiliar asset, and each pair of columns represents a different level of correlation, ρ . From the table, we see that an increase in correlation from $\rho = 50\%$ to $\rho = 70\%$ roughly doubles the holding of the familiar asset. The larger the spread between α_F and α_U (Panels A and B), the stronger is the flight to familiarity effect.

4.3 Sensitivity of portfolio weights in the Markowitz and Keynes models

When an investor is risk averse, an increase in the risk of an asset obviously leads to a decrease in the holding of that asset. An interesting extension of this insight is to distinguish between the effect of systematic and idiosyncratic risk. From traditional asset pricing theory we know that the relative weights in a well-diversified portfolio will not be affected by idiosyncratic risk because, by construction, a well-diversified portfolio contains *no* idiosyncratic risk. In our framework, however, optimal portfolios may exhibit concentration in familiar assets.

In order to study differences in the sensitivity of the portfolio strategies advocated by Keynes and Markowitz, we analyze the effect on the change in the portfolio weights of a change in systematic risk, σ_S , and idiosyncratic risk, σ_U . We focus our analysis on the case in which the views of both Keynes and Markowitz are reflected in the portfolio weights, that is, Case I of Proposition 1, and for simplicity, we confine ourselves to the case of an economy with an infinite number of assets N .

The following proposition derives the sensitivity of portfolio weights to changes in unsystematic and systematic risk.

Proposition 6 *Let π_F and π_U be the portfolios in Case I of Proposition 1. Then, as $N \rightarrow \infty$:*

1. *The effect of unsystematic risk on the total risky asset holdings where the investor exhibits familiarity toward Asset F is given by:*

$$\lim_{N \rightarrow \infty} \left(\frac{\partial \pi_F}{\partial \sigma_U} + \frac{\partial}{\partial \sigma_U} \left(\pi_U^\top \mathbf{1}_{N-1} \right) \right) = - \frac{\alpha_U \sigma_U}{\gamma \sqrt{T} \sigma_S^2 \sqrt{\sigma_U^2 + \sigma_S^2}} < 0 \quad (43)$$

while for the Markowitz portfolio it is:

$$\lim_{N \rightarrow \infty} \frac{\partial}{\partial \sigma_U} \left(\pi^\top \mathbf{1}_N \right) = 0. \quad (44)$$

2. *The effect of systematic risk on the total risky asset holdings where the investor exhibits familiarity toward Asset F is given by:*

$$\lim_{N \rightarrow \infty} \left(\frac{\partial \pi_F}{\partial \sigma_S} + \frac{\partial}{\partial \sigma_S} \left(\pi_U^\top \mathbf{1}_{N-1} \right) \right) = - \frac{2\hat{\mu}}{\gamma \sigma_S^3} + \frac{(\sigma_S^2 + 2\sigma_U^2) \alpha_U}{\gamma \sqrt{T} \sigma_S^3 \sqrt{\sigma_U^2 + \sigma_S^2}} < 0, \quad (45)$$

while for the Markowitz portfolio it is:

$$\lim_{N \rightarrow \infty} \frac{\partial}{\partial \sigma_S} \left(\pi^\top \mathbf{1}_N \right) = - \frac{2\hat{\mu}}{\gamma \sigma_S^3} < 0. \quad (46)$$

The above proposition shows that the effects of changes in unsystematic risk and systematic risk are very different in the Keynesian and Markowitz models. A non-zero value for the derivative means that, as a consequence of a change in U and S, the investor will revise his portfolio. We take the absolute value of these derivatives as a simple measure of the sensitivity of the portfolio with respect to idiosyncratic and systematic risk. Under this convention, see that the absolute value of the expression in Equation (43) is greater than that in (44), indicating that in response to a change in idiosyncratic risk the Keynesian portfolio always exhibits a *larger* change than the

Markowitz portfolio. In contrast, the absolute value of the expression in Equation (45) is smaller than that in (46), indicating that in response to a change in systematic risk the Keynesian portfolio exhibits a *smaller* change than the Markowitz portfolio.

5. Conclusion

Even though Markowitz's portfolio theory dictates that an investor should invest in only a single fund of risky assets, which in equilibrium is the market portfolio, there is substantial evidence that rather than holding just the market portfolio or a well-diversified portfolio, investors hold a substantial amount in just a few assets, often those with which they are familiar. In this paper we reconcile this apparent contradiction between Markowitz's theory and the empirical evidence by introducing Keynes's view of investment into an otherwise standard Markowitz model of portfolio selection.

Our model incorporates the view of Keynes by introducing ambiguity about the true distribution of asset returns and investors' aversion to this ambiguity, into the standard Markowitz portfolio-selection setting. The main feature of our model is that it allows investors to distinguish their ambiguity about one asset class relative to others. We show analytically that the model has the following implications, which are consistent with the stylized empirical observations. One, in the presence of ambiguity about returns on the other assets, the investor holds a disproportionately large amount of the familiar asset (relative to the Markowitz model), but continues to invest in the market portfolio. Two, the proportion of wealth allocated to the familiar asset increases with an increase in correlations between assets, implying a *flight to familiarity* effect. Three, investors who are familiar with particular assets *and* sufficiently ambiguous about all other assets hold only the familiar assets, as Keynes would have advocated, and thus the median number of stocks held by such investors is quite small. Finally, investors who are sufficiently ambiguous about all risky assets do not participate at all in the equity market (*non-participation*).

The analysis in this paper suggests that perhaps we should not ignore Keynes's view of how to make investment decisions. We hope that our work will encourage researchers to explore other implications of the Keynesian view of portfolio selection, and, in particular, to further develop theories of portfolio choice and asset pricing in which investors hold only a subset of the universe of available investable assets.

A. Appendix: Proofs of the Propositions

Proof of Proposition 1

The solution to the inner minimization problem of (10)–(11) is

$$\mu_n = \hat{\mu} - \text{sign}(\pi_n) \sigma_{\hat{\mu}} \alpha_n, \quad n = 1, \dots, N, \quad (\text{A1})$$

which, when substituted back into the original problem, gives

$$\max_{\pi} \left\{ \pi^T \hat{\mu} - \sum_{n=1}^N \text{sign}(\pi_n) \pi_n \sigma_{\hat{\mu}} \alpha_n - \frac{\gamma}{2} \pi^T \Sigma \pi \right\}. \quad (\text{A2})$$

Given our specification of the N stock return processes, it must be the case that $\pi_2 = \dots = \pi_N$. We will denote this common value by $\tilde{\pi}_U$. From the first order condition, if the optimal weights $\pi \neq 0$, they must satisfy

$$\pi = \begin{bmatrix} \pi_F \\ \pi_U \\ \vdots \\ \tilde{\pi}_U \end{bmatrix} = \frac{1}{\gamma} \Sigma^{-1} \begin{bmatrix} \hat{\mu} - \text{sign}(\pi_F) \sigma_{\hat{\mu}} \alpha_F \\ \hat{\mu} - \text{sign}(\tilde{\pi}_U) \sigma_{\hat{\mu}} \alpha_U \\ \vdots \\ \hat{\mu} - \text{sign}(\tilde{\pi}_U) \sigma_{\hat{\mu}} \alpha_U \end{bmatrix}. \quad (\text{A3})$$

Because all risky assets have the same expected return, $\hat{\mu}$, the above N -dimensional condition simplifies to the following two-dimensional one:

$$\begin{bmatrix} \pi_F \\ \tilde{\pi}_U \end{bmatrix} = \frac{1}{\gamma} \tilde{\Sigma}^{-1} \begin{bmatrix} \hat{\mu} - \text{sign}(\pi_F) \sigma_{\hat{\mu}} \alpha_F \\ (N-1)(\hat{\mu} - \text{sign}(\tilde{\pi}_U) \sigma_{\hat{\mu}} \alpha_U) \end{bmatrix}, \quad (\text{A4})$$

where

$$\tilde{\Sigma} \equiv \begin{bmatrix} \sigma^2 & (N-1)\rho\sigma^2 \\ (N-1)\rho\sigma^2 & (N-1)[1 + (N-2)\rho]\sigma^2 \end{bmatrix}. \quad (\text{A5})$$

To solve (A4) we need to consider several possible cases depending on the signs of π_F and π_U .

Case I. Suppose $\pi_F > 0$ and $\tilde{\pi}_U > 0$, then, from (A4)

$$\begin{aligned} \begin{bmatrix} \pi_F \\ \tilde{\pi}_U \end{bmatrix} &= \frac{1}{\gamma} \tilde{\Sigma}^{-1} \begin{bmatrix} \hat{\mu} - \sigma_{\hat{\mu}} \alpha_F \\ (N-1)(\hat{\mu} - \sigma_{\hat{\mu}} \alpha_U) \end{bmatrix} \\ &= \frac{1}{\gamma \sigma^2 (1-\rho)(1+(N-1)\rho)} \begin{bmatrix} \hat{\mu}(1-\rho) - \sigma_{\hat{\mu}}((1+(N-2)\rho)\alpha_F - (N-1)\rho\alpha_U) \\ \hat{\mu}(1-\rho) - \sigma_{\hat{\mu}}(\alpha_U - \rho\alpha_F) \end{bmatrix}. \end{aligned} \quad (\text{A6})$$

Because $\alpha_F < \alpha_U$, $\alpha_U - \rho\alpha_F > \alpha_F - \rho\alpha_U > (1+(N-2)\rho)\alpha_F - (N-1)\rho\alpha_U = \alpha_F - \rho\alpha_U + (N-2)\rho(\alpha_F - \alpha_U)$, and hence $\pi_F > \tilde{\pi}_U$ in (A6). Moreover, $\pi_F > 0$ and $\tilde{\pi}_U > 0$ if and only if $\tilde{\pi}_U > 0$, which is true if $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} > \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$. This proves Case I.

Cases II and III. Suppose that $0 \leq \frac{\hat{\mu}}{\sigma_{\hat{\mu}}} \leq \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F)$. We start by first proving three intermediate results: (i) it is impossible to have $\pi_U < 0$ and $\pi_F \geq 0$; (ii) it is impossible to have $\pi_U > 0$ and $\pi_F < 0$; (iii) it is impossible to have $\pi_F < 0$ and $\tilde{\pi}_U < 0$.

(i) Suppose that $(\pi_F, \tilde{\pi}_U)$ is the optimal portfolio weight vector with $\tilde{\pi}_U < 0$. Then there exists a v such that $\hat{\mu} - \sigma_{\hat{\mu}} v \in [\hat{\mu} - \sigma_{\hat{\mu}} \alpha_F, \hat{\mu} + \sigma_{\hat{\mu}} \alpha_F]$ and

$$\begin{aligned} \begin{bmatrix} \pi_F \\ \tilde{\pi}_U \end{bmatrix} &= \frac{1}{\gamma} \tilde{\Sigma}^{-1} \begin{bmatrix} \hat{\mu} - \sigma_{\hat{\mu}} v \\ (N-1)(\hat{\mu} + \sigma_{\hat{\mu}} \alpha_U) \end{bmatrix} \\ &= \frac{1}{\gamma \sigma^2 (1-\rho)(1+(N-1)\rho)} \begin{bmatrix} \hat{\mu}(1-\rho) - \sigma_{\hat{\mu}}(v(1+(N-2)\rho) + (N-1)\rho\alpha_U) \\ \hat{\mu}(1-\rho) + \sigma_{\hat{\mu}}(\alpha_U + \rho v) \end{bmatrix}. \end{aligned}$$

However, $\hat{\mu}(1-\rho) + \sigma_{\hat{\mu}}(\alpha_U + \rho v) \geq \hat{\mu}(1-\rho) + \sigma_{\hat{\mu}}(\alpha_U - \rho\alpha_F) \geq 0$. This condition would imply $\pi_U > 0$, a contradiction.

(ii) Suppose next that $\pi_F < 0$, $\pi_U > 0$. Then

$$\begin{bmatrix} \pi_F \\ \tilde{\pi}_U \end{bmatrix} = \frac{1}{\gamma \sigma^2 (1-\rho)(1+(N-1)\rho)} \begin{bmatrix} \hat{\mu}(1-\rho) + \sigma_{\hat{\mu}}(\alpha_F(1+(N-2)\rho) + (N-1)\rho\alpha_U) \\ \hat{\mu}(1-\rho) - \sigma_{\hat{\mu}}(\alpha_U + \rho\alpha_F) \end{bmatrix}.$$

Because $\hat{\mu}(1-\rho) + \sigma_{\hat{\mu}}(\alpha_F(1+(N-2)\rho) + (N-1)\rho\alpha_U) \geq \hat{\mu}(1-\rho) + \sigma_{\hat{\mu}}(\alpha_U - \rho\alpha_F) \geq 0$, we have a contradiction.

(iii) Suppose finally that $\pi_F < 0$, $\pi_U < 0$. Then, from (A4)

$$\begin{aligned} \begin{bmatrix} \pi_F \\ \tilde{\pi}_U \end{bmatrix} &= \frac{1}{\gamma} \tilde{\Sigma}^{-1} \begin{bmatrix} \hat{\mu} + \sigma_{\hat{\mu}} \alpha_F \\ (N-1)(\hat{\mu} + \sigma_{\hat{\mu}} \alpha_U) \end{bmatrix} \\ &= \frac{1}{\gamma \sigma^2 (1-\rho)(1+(N-1)\rho)} \begin{bmatrix} \hat{\mu}(1-\rho) + \sigma_{\hat{\mu}}((1+(N-2)\rho)\alpha_F + (N-1)\rho\alpha_U) \\ \hat{\mu}(1-\rho) + \sigma_{\hat{\mu}}(\alpha_U + \rho\alpha_F) \end{bmatrix}. \end{aligned} \quad (\text{A7})$$

Because $\hat{\mu} > 0$, $\pi_F > 0$ and $\tilde{\pi}_U > 0$ in (A7), we have a contradiction.

We can now prove Cases II and III. From the intermediate results derived in (i), (ii), and (iii) above, the only possibilities are either (a) $\tilde{\pi}_U = 0$ and $\pi_F \geq 0$, or (b) $\tilde{\pi}_U > 0$ and $\pi_F = 0$. In (a), the optimal π_F is the quantity x solving the utility maximization problem

$$\max_x \min_{\mu \in \Gamma_F} \mu x - \frac{\gamma}{2} x^2 \sigma^2 \quad \text{where} \quad \Gamma_F = \left\{ \mu : \frac{(\mu - \hat{\mu})^2}{\sigma_{\hat{\mu}}} \leq \alpha_F \right\}. \quad (\text{A8})$$

And, in (b), the optimal $\tilde{\pi}_U$ is the quantity x solving

$$\max_x \min_{\mu \in \Gamma_U} \mu x - \frac{\gamma}{2} x^2 \sigma^2 (1 + (N-2)\rho) \quad \text{where} \quad \Gamma_U = \left\{ \mu : \frac{(\mu - \hat{\mu})^2}{\sigma_{\hat{\mu}}} \leq \alpha_U \right\}. \quad (\text{A9})$$

Because $\alpha_F < \alpha_U$, $\Gamma_F \subset \Gamma_U$. Moreover, $\sigma^2(1 + (N-2)\rho) > \sigma^2$, and hence

$$\max_{\pi} \min_{\mu \in \Gamma_F} \mu x - \frac{\gamma}{2} x^2 \sigma^2 > \max_x \min_{\mu \in \Gamma_U} \mu x - \frac{\gamma}{2} x^2 \sigma^2 (1 + (N-2)\rho). \quad (\text{A10})$$

Thus, (b) cannot be optimal. Hence the optimal portfolio weight must correspond to (a). Because $\hat{\mu} > 0$, $\pi_F < 0$ cannot be a solution of (A8). This leaves only two cases, corresponding to Cases II and III in the proposition. In Case II, $\pi_F > 0$, which happens if $\hat{\mu} - \sigma_{\hat{\mu}} \alpha_F > 0$, or, equivalently $\alpha_F < \frac{\hat{\mu}}{\sigma_{\hat{\mu}}}$. In this case, the optimal portfolio is

$$\pi_F = \frac{1}{\gamma \sigma^2} (\hat{\mu} - \sigma_{\hat{\mu}} \alpha_F) \quad (\text{A11})$$

$$\tilde{\pi}_U = 0. \quad (\text{A12})$$

In Case III, $0 \leq \pi_F$, and so the optimal portfolio is

$$\pi_F = 0 \quad (\text{A13})$$

$$\tilde{\pi}_U = 0 \quad (\text{A14})$$

This completes the proof of the proposition.

Proof of Corollary 1

Cases II and III are immediate from the definition of relative weights. Case I follows from Proposition 1, by defining:

$$\omega_F = \frac{\pi_F}{\pi_F + \pi_U^T \mathbf{1}_{N-1}} \quad (\text{A15})$$

$$\omega_U = \frac{\pi_U}{\pi_F + \pi_U^T \mathbf{1}_{N-1}}. \quad (\text{A16})$$

Note that the quantity $\hat{\mu} - \sigma_{\hat{\mu}} \alpha_F - (\alpha_U - \alpha_F) \sigma_{\hat{\mu}} (1 - \frac{1}{N}) > 0$, because by the condition that defines Case I, $\frac{\hat{\mu}}{\sigma_{\hat{\mu}}} > \alpha_F + \frac{1}{1-\rho}(\alpha_U - \alpha_F) > \alpha_F + (\alpha_U - \alpha_F) > \alpha_F + (1 - \frac{1}{N})(\alpha_U - \alpha_F)$.

Proof of Proposition 2

The limits in (30) and (32) follow immediately from the portfolio weights in (16) and (17).

Proof of Proposition 3

The solution to the inner minimization problem of the maxmin problem in (10) and (11) is the same as in (A1), which, when substituted back into the original problem, gives the problem in (A2). Note that assets in each subclass will have identical portfolio weight. The case of one asset class is trivial. The optimal portfolio must satisfy

$$\max_{\pi} \pi (\hat{\mu} - \text{sign}(\pi) \sigma_{\hat{\mu}} \alpha_F) - \frac{\gamma}{2} \pi^2 \sigma^2. \quad (\text{A17})$$

The solution π to this problem is

$$\pi = \begin{cases} \frac{1}{\gamma\sigma^2} N_1(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_1) & \text{if } \hat{\mu} > \sigma_{\hat{\mu}}\alpha_1 \\ 0 & \text{if } \hat{\mu} \leq \sigma_{\hat{\mu}}\alpha_1 \end{cases}.$$

We now prove the proposition by induction. That is, assuming that the claim of the proposition is true for the case of N assets with $M - 1$ exclusive asset classes with $M - 2$, we show that the claim is also true for the case of N assets with M exclusive asset classes.

Let $\pi_{-1,2,\dots,m}$ denote the $m \times 1$ vector of portfolio weights in asset classes 1, 2, ..., m , where $m = 1, \dots, M$. In general, the maxmin problem has the solution of the form:

$$\pi_{\{1,\dots,M\}} = \frac{1}{\gamma} \tilde{\Sigma}_M^{-1} \begin{bmatrix} N_1(\hat{\mu} - \sigma_{\hat{\mu}}v_1) \\ N_2(\hat{\mu} - \sigma_{\hat{\mu}}v_2) \\ \vdots \\ N_M(\hat{\mu} - \sigma_{\hat{\mu}}v_M) \end{bmatrix}, \quad (\text{A18})$$

where

$$\tilde{\Sigma}_M = \sigma^2 \begin{bmatrix} N_1(1 + (N_1 - 1)\rho) & N_1N_2\rho & \dots & N_1N_M\rho \\ N_2N_1\rho & N_2(1 + (N_2 - 1)\rho) & \dots & N_2N_M\rho \\ \vdots & \vdots & \ddots & \vdots \\ N_MN_1\rho & N_MN_2\rho & \dots & N_M(1 + (N_M - 1)\rho) \end{bmatrix}, \quad (\text{A19})$$

with appropriate $v_i \in [-\alpha_i, \alpha_i]$, $i = 1, \dots, M$, to be determined. To solve for $\pi_{-1,\dots,M}$, rewrite (A18) as

$$\gamma \tilde{\Sigma}_M \pi_{\{1,\dots,M\}} = \begin{bmatrix} N_1(\hat{\mu} - \sigma_{\hat{\mu}}v_1) \\ N_2(\hat{\mu} - \sigma_{\hat{\mu}}v_2) \\ \vdots \\ N_M(\hat{\mu} - \sigma_{\hat{\mu}}v_M) \end{bmatrix}. \quad (\text{A20})$$

Using the expression for $\tilde{\Sigma}_M$ in (A19), we have

$$\begin{aligned} \gamma\sigma^2 \left((1 - \rho)\pi_1 + \rho \sum_{j=1}^M N_j\pi_j \right) &= \hat{\mu} - \sigma_{\hat{\mu}}v_1 \\ \gamma\sigma^2 \left((1 - \rho)\pi_2 + \rho \sum_{j=1}^M N_j\pi_j \right) &= \hat{\mu} - \sigma_{\hat{\mu}}v_2 \\ &\vdots \\ \gamma\sigma^2 \left((1 - \rho)\pi_M + \rho \sum_{j=1}^M N_j\pi_j \right) &= \hat{\mu} - \sigma_{\hat{\mu}}v_M. \end{aligned}$$

Subtraction yields

$$\pi_i - \pi_1 = \frac{1}{\gamma(1 - \rho)\sigma^2} \sigma_{\hat{\mu}}(v_1 - v_i) \quad (\text{A21})$$

Thus

$$\pi_i = \pi_1 + \frac{1}{\gamma(1 - \rho)\sigma^2} \sigma_{\hat{\mu}}(v_1 - v_i)$$

Substituting this expression for π_i back into the first equation of the system of equations (A20) and solving the resulting equations yields an expression for π_1 . Then it can be verified that the component π_i of $\pi_{1,\dots,M}$ can be expressed as

$$\pi_i = \frac{\hat{\mu}(1-\rho) - \sigma_{\hat{\mu}} \left((1-\rho)\alpha_i + \rho \sum_{j \neq i}^M N_j(\alpha_i - \alpha_j) \right)}{\gamma \sigma^2(1-\rho)(1+\rho(N-1))}, \quad i = 1, \dots, M. \quad (\text{A22})$$

We will make use of this fact in the proof.

First consider the case where $\hat{\mu} > s(M)$. Then the optimal portfolio weights are:

$$\pi_{\{1,\dots,M\}} = \frac{1}{\gamma} \tilde{\Sigma}_M^{-1} \begin{bmatrix} N_1(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_1) \\ N_2(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_2) \\ \vdots \\ N_M(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_M) \end{bmatrix}, \quad (\text{A23})$$

provided that $\pi_i > 0$ for all $i = 1, \dots, M$. We verify that this is indeed the case. By (A22)

$$\pi_i = \frac{\hat{\mu}(1-\rho) - \sigma_{\hat{\mu}} \left((1-\rho)v_i + \rho \sum_{j \neq i}^M N_j(v_i - v_j) \right)}{\gamma \sigma^2(1-\rho)(1+\rho(N-1))}, \quad i = 1, \dots, M. \quad (\text{A24})$$

Notice that $\pi_1 \geq \pi_2 \geq \dots \geq \pi_M$. Since $\hat{\mu} > s(M)$, $\pi_M > 0$

Next consider the case where $s(M-1) < \hat{\mu} \leq s(M)$. We conjecture that the optimal portfolio weights are given by

$$\pi_{\{1,\dots,M-1\}} = \frac{1}{\gamma} \tilde{\Sigma}_{M-1}^{-1} \begin{bmatrix} N_1(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_1) \\ \vdots \\ N_{M-1}(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_{M-1}) \end{bmatrix}, \quad (\text{A25})$$

$$\pi_M = 0. \quad (\text{A26})$$

Suppose to the contrary that the optimal $\pi_{1,\dots,M}$ in (A18) is such that $\pi_M \neq 0$. There are four subcases to consider. (i) Consider first the subcase where $\pi_M < 0$. Then in (A18), $v_M = -\alpha_M$. Since $\alpha_i \leq \alpha_j$ for $i \leq j$, (A22) implies that $\pi_M > 0$, a contradiction. (ii) Next consider the subcase where $\pi_M > 0$, and there is at least one $j < M$ such that $\pi_j < 0$. Then for the largest j such that $\pi_j < 0$, (A22) again implies that $\pi_j > 0$, a contradiction. (iii) In the third subcase, $\pi_M > 0$, and there is at least one $j < M$ such that $\pi_j = 0$. Fixing π_j at zero, the original maxmin problem reduces to a problem with N assets with $M-1$ classes. By assumption, the claim of the proposition is true for a problem with N assets with $M-1$ classes. But $\pi_M > 0$ is inconsistent with the claim of the proposition for $M-1$ asset classes, a contradiction. (iv) In the fourth subcase, which is the only remaining possibility, all $\pi_i > 0$, $i = 1, \dots, M$, which implies that in (A18) $v_i = \alpha_i$. But then (A22) and $\hat{\mu} \leq s(M)$ would imply $\pi_M < 0$, a contradiction. The four subcases together imply that $\pi_M \neq 0$ is impossible. Thus, for the optimal portfolio weight vector, (A26) holds. Fixing π_M at zero, the original maxmin problem reduces to a problem with N assets with $M-1$ classes. Then, by the claim of the proposition, (A25) holds.

Next consider the case where $s(m) < \hat{\mu} \leq s(m+1)$, $m = 1, \dots, M-1$. In this case, $\hat{\mu} \leq s(M)$ as well. By what is shown above, $\pi_M = 0$, and the problem can therefore be reduced to the case of N assets with $M-1$ class. Thus, the claim of the proposition holds.

Proof of Proposition 4

The relative portfolio weights are $\omega_F = \frac{\pi_F}{\pi_F + \pi_U^\top \mathbf{1}_{N-1}}, \quad \omega_U = \frac{\pi_U}{\pi_F + \pi_U^\top \mathbf{1}_{N-1}}.$

Then the variance of the portfolio of all risky assets is $(\omega_F, \omega_U)\Sigma(\omega_F, \omega_U)^\top$. As in the proof of Proposition 1, define

$$\tilde{\Sigma} = \begin{bmatrix} \sigma^2 & (N-1)\rho\sigma^2 \\ (N-1)\rho\sigma^2 & (N-1)[1+(N-2)\rho]\sigma^2 \end{bmatrix}.$$

Then the variance of the portfolio, as $N \rightarrow \infty$, is given by,

$$\rho\sigma^2 + \sigma_{\hat{\mu}}^2 \frac{(\alpha_U - \alpha_F)^2}{(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_U)^2} \frac{\rho^2}{(1-\rho)^2} (1-\rho)\sigma^2 = \sigma_S^2 + \frac{\sigma_{\hat{\mu}}^2 \sigma^2 \rho^2 (\alpha_U - \alpha_F)^2}{(1-\rho)(\hat{\mu} - \sigma_{\hat{\mu}}\alpha_U)^2}.$$

For Case II, by Corollary 1, $\omega_F = 1$ and $\omega_U = \mathbf{0}_{N-1}$. Thus the variance is 2. For Case III, the variance is zero.

Proof of Proposition 5

The result is obtained by taking derivatives of the relative weights in Corollary 1.

Proof of Proposition 6

The result is obtained by taking derivatives with respect to σ_S and σ_U of the weights in Equations (30). To show that (45) is negative we need to show that the numerator is positive. The numerator, which can be written as $\sigma \left[2\frac{\hat{\mu}\sqrt{T}}{\sigma} - \frac{\sigma^2 + \sigma_U^2}{\sigma^2} \alpha_U \right]$, is positive because $T \geq 1$ and, because we are considering Case I of Proposition 1, $\hat{\mu}/\sigma_{\hat{\mu}} > \alpha_U$.

References

- Ahn, D., S. Choi, D. Gale, and S. Kariv, 2010, "Estimating Ambiguity Aversion in a Portfolio Choice Experiment," Working Paper, University of California, Berkeley.
- Anderson, E. W., E. Ghysels, and J. L. Juergens, 2009, "The Impact of Risk and Uncertainty on Expected Returns," *Journal of Financial Economics*, 94, 233-263.
- Barber, B. M., and T. Odean, 2000, "Trading Is Hazardous to Your Wealth: The Common Stock Investment Performance of Individual Investors," *Journal of Finance*, 55, 773-806.
- Barberis, N., and M. Huang, 2008, "Stocks as Lotteries: The Implications of Probability Weighting for Security Prices," *American Economic Review*, 98, 2066-2100.
- Benartzi, S., 2001, "Excessive Extrapolation and the Allocation of 401(k) Accounts to Company Stock," *Journal of Finance*, 56, 1747-1764.
- Best, M. J., and R. R. Grauer, 1991, "On the Sensitivity of Mean-Variance-Efficient Portfolios to Changes in Asset Means: Some Analytical and Computational Results," *Review of Financial Studies*, 4, 315-342.
- Bewley, T., 1988, "Knightian Decision Theory and Econometric Inference," Working Paper, Yale University.
- Blume, M., and I. Friend, 1975, "The Asset Structure of Individual Portfolios and Some Implications for Utility Functions," *Journal of Finance*, 30, 585-603.
- Bossaerts, P., P. Ghirardato, S. Guarnaschelli, and W. R. Zame, 2010, "Ambiguity in Asset Markets: Theory and Experiment," *Review of Financial Studies*, 23, 1325-1359.
- Brennan, M. J., 1975, "The Optimal Number of Securities in a Risky Asset Portfolio When There Are Fixed Costs of Transacting: Theory and Some Empirical Results," *Journal of Financial and Quantitative Analysis*, 10, 483-496.

- Brunnermeier, M. K., and L. H. Pedersen, 2009, "Market Liquidity and Funding Liquidity," *Review of Financial Studies*, 22, 2201-2199.
- Calvet, L. E., J. Y. Campbell, and P. Sodini, 2007, "Down or Out: Assessing the Welfare Costs of Household Investment Mistakes," *Journal of Political Economy*, 115, 707-747.
- Campbell, J. Y., 2006, "Household Finance," *Journal of Finance*, 61, 1553-1604.
- Cao, H. H., T. Wang, and H. H. Zhang, 2005, "Model Uncertainty, Limited Market Participation, and Asset Prices," *Review of Financial Studies*, 18, 1219-1251.
- Chen, H., N. Ju, and J. Miao, 2009, "Dynamic Asset Allocation with Ambiguous Return Predictability," Working Paper, Boston University.
- Chen, Z., and L. Epstein, 2002, "Ambiguity, Risk, and Asset Returns in Continuous Time," *Econometrica*, 70, 1403-1443.
- Cohen, L., 2009, "Loyalty-Based Portfolio Choice," *Review of Financial Studies*, 22, 1213-1245.
- Conine, Jr., T. E., and M. J. Tamarkin, 1981, "On Diversification Given Asymmetry in Returns," *Journal of Finance*, 36, 1143-1155.
- Cooper, I., and E. Kaplanis, 1994, "Home Bias in Equity Portfolios, Inflation Hedging and International Capital Market Equilibrium," *Review of Financial Studies*, 7, 45-60.
- Coval, J., and T. Moskowitz, 1999, "Home Bias at Home: Local Equity Preference in Domestic Portfolios," *Journal of Finance*, 54, 2045-2073.
- DeMiguel, V., L. Garlappi, and R. Uppal, 2009, "Optimal Versus Naive Diversification: How Inefficient Is the 1/N Portfolio Strategy?," *Review of Financial Studies*, 22, 1915-1953.
- Dow, J., and S. R. d. C. Werlang, 1992, "Uncertainty Aversion, Risk Aversion, and the Optimal Choice of Portfolio," *Econometrica*, 60, 197-204.
- Easley, D., and M. O'Hara, 2009, "Ambiguity and Nonparticipation: The Role of Regulation," *Review of Financial Studies*, 22, 1817-1844.
- Easley, D., and M. O'Hara, 2010, "Liquidity and Valuation in an Uncertain World," *Journal of Financial Economics*, 97, 1-12.
- Epstein, L., and M. Schneider, 2007, "Learning Under Ambiguity," *Review of Economic Studies*, 74, 1275-1303.
- Epstein, L. G., and J. Miao, 2003, "A Two-Person Dynamic Equilibrium under Ambiguity," *Journal of Economic Dynamics and Control*, 27, 1253-1288.
- Epstein, L. G., and T. Wang, 1994, "Intertemporal Asset Pricing Under Knightian Uncertainty," *Econometrica*, 62, 283-322.
- Feng, L., and M. S. Seasholes, 2004, "Correlated Trading and Location," *Journal of Finance*, 59, 2117-2144.
- Fox, C., and A. Tversky, 1995, "Ambiguity Aversion and Comparative Ignorance," *Quarterly Journal of Economics*, 110, 585-603.
- French, K., and J. Poterba, 1990, "Japanese and U.S. Cross-Border Common Stock Investments," *Journal of the Japanese and International Economies*, 4, 476-493.
- Frost, P. A., and J. E. Savarino, 1986, "An Empirical Bayes Approach to Efficient Portfolio Selection," *Journal of Financial and Quantitative Analysis*, 21, 293-305.
- Frost, P. A., and J. E. Savarino, 1988, "For Better Performance Constrain Portfolio Weights," *Journal of Portfolio Management*, 15, 29-34.

- Garlappi, L., R. Uppal, and T. Wang, 2007, "Portfolio Selection with Parameter and Model Uncertainty: A Multi-Prior Approach," *Review of Financial Studies*, 20, 41-81.
- Gilboa, I., and D. Schmeidler, 1989, "Maxmin Expected Utility Theory with Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141-153.
- Goetzman, W. N., and A. Kumar, 2008, "Equity Portfolio Diversification," *Review of Finance*, 12, 433-463.
- Grinblatt, M., and M. Keloharju, 2001, "How Distance, Language, and Culture Influence Stockholdings and Trades," *Journal of Finance*, 56, 1053-1073.
- Harvey, C. R., and A. Siddique, 2000, "Conditional Skewness in Asset Pricing Tests," *Journal of Finance*, 55, 1263-1295.
- Hayashi, T., and J. Miao, 2010, "Intertemporal Substitution and Recursive Smooth Ambiguity Preferences," Working Paper, Boston University.
- Heath, C., and A. Tversky, 1991, "Preferences and Beliefs: Ambiguity and Competence in Choice Under Uncertainty," *Journal of Risk and Uncertainty*, 4, 5-28.
- Huberman, G., 2001, "Familiarity Breeds Investment," *Review of Financial Studies*, 14, 659-680.
- Huberman, G., and W. Jiang, 2006, "Oering vs. Choice by 401(k) Plan Participants: Equity Exposure and Number of Funds," *Journal of Finance*, 61, 763-801.
- Illeditsch, P. K., 2009, "Ambiguous Information, Risk Aversion, and Asset Pricing," Working Paper, Wharton, University of Pennsylvania.
- Ivkovic, Z., and S. Weisbenner, 2005, "Local Does as Local Is: Information Content of the Geography of Individual Investors' Common Stock Investments," *Journal of Finance*, 60, 267-306.
- Ju, N., and J. Miao, 2010, "Ambiguity, Learning, and Asset Returns," Working Paper, Boston University.
- Keynes, J. M., 1921, *A Treatise on Probability*, MacMillan, London, 1st edn.
- Keynes, J. M., 1983, *The Collected Writings of John Maynard Keynes*, MacMillan, Cambridge University Press, New York, Moggridge, Donald Edward, ed.
- Klibano, P., M. Marinacci, and S. Mukerji, 2005, "A Smooth Model of Decision Making Under Ambiguity," *Econometrica*, 73, 1849-1892.
- Knight, F. H., 1921, *Risk, Uncertainty and Profit*, Houghton Mifflin, Boston.
- Kraus, A., and R. H. Litzenberger, 1976, "Skewness Preference and the Valuation of Risk Assets," *Journal of Finance*, 31, 1085-1100.
- LeRoy, S. F., and L. D. Singell, 1987, "Knight on Risk and Uncertainty," *Journal of Political Economy*, 95, 395-406.
- Litterman, R., 2003, *Modern Investment Management: An Equilibrium Approach*, Wiley, New York.
- Liu, H., 2009, "Portfolio Insurance and Underdiversification," Working Paper, Olin Business School, Washington University.
- Lo, A. W., 2001, "Risk Management for Hedge Funds: Introduction and Overview," *Financial Analysts Journal*, 57, 16-33.
- Loeb, J., 1950, *The Battle for Investment Survival*, John Wiley and Sons, New York.
- Longin, F., and B. Solnik, 1995, "Is the Correlation in International Equity Returns Constant: 1960-1990?," *Journal of International Money and Finance*, 14, 3-26.

- Maenhout, P. J., 2004, "Robust Portfolio Rules and Asset Pricing," *Review of Financial Studies*, 17, 951-983.
- Markowitz, H. M., 1952, "Portfolio Selection," *Journal of Finance*, 7, 77-91.
- Markowitz, H. M., 1959, *Portfolio Selection*, Wiley, New York.
- Massa, M., and A. Simonov, 2006, "Hedging, Familiarity and Portfolio Choice," *Review of Financial Studies*, 19, 633-685.
- Mehra, R., and E. Prescott, 1985, "The Equity Premium Puzzle," *Journal of Monetary Economics*, 15, 145-161.
- Merton, R. C., 1980, "On Estimating the Expected Return on the Market: An Exploratory Investigation," *Journal of Financial Economics*, 8, 323-361.
- Merton, R. C., 1987, "A Simple Model of Capital Market Equilibrium with Incomplete Information," *Journal of Finance*, 42, 483-510.
- Meulbroeck, L., 2005, "Company Stock in Pension Plans: How Costly Is It?," *Journal of Law and Economics*, 48, 443-474.
- Michaud, R. O., 1989, "The Markowitz Optimization Enigma: Is 'Optimized' Optimal?," *Financial Analysts Journal*, 45, 31-42.
- Mitchell, O., and S. Utkus, 2002, "Company Stock and Retirement Plan Diversification," Pension Research Council Working Paper 2002-4.
- Mitton, T., and K. Vorkink, 2007, "Equilibrium Underdiversification and the Preference for Skewness," *Review of Financial Studies*, 20, 1255-1288.
- Odean, T., 1999, "Do Investors Trade Too Much?," *American Economic Review*, 89, 1279-1298.
- Polkovnichenko, V., 2005, "Household Portfolio Diversification: A Case for Rank Dependent Preferences," *Review of Financial Studies*, 18, 1467-1502.
- Poon, S., M. Rockinger, and J. A. Tawn, 2004, "Extreme Value Dependence in Financial Markets: Diagnostics, Models, and Financial Implications," *Review of Financial Studies*, 17, 581-610.
- Routledge, B., and S. Zin, 2009, "Model Uncertainty and Liquidity," *Review of Economic Dynamics*, 12, 543-566.
- Sarkissian, S., and M. J. Schill, 2004, "The Overseas Listing Decision: New Evidence of Proximity Preference," *Review of Financial Studies*, 17, 769-809.
- Scholes, M., 2000, "Crisis and Risk Management," in L. Rahl (ed.), *Risk Budgeting: A New Approach to Investing*, Risk Books.
- Sharpe, W. F., 1970, *Portfolio Theory and Capital Markets*, McGraw-Hill, New York.
- Shefrin, H., and M. Statman, 2000, "Behavioral Portfolio Theory," *Journal of Financial and Quantitative Analysis*, 35, 127-151.
- Uppal, R., and T. Wang, 2003, "Model Misspecification and Underdiversification," *Journal of Finance*, 58, 2465-2486.
- Van Nieuwerburgh, S., and L. Veldkamp, 2006, "Inside Information and the Own Company Stock Puzzle," *Journal of the European Economic Association Papers and Proceedings*, 4, 623-633.
- Van Nieuwerburgh, S., and L. Veldkamp, 2009, "Information Immobility and the Home Bias Puzzle," *Journal of Finance*, 64, 1187-1215.
- Van Nieuwerburgh, S., and L. Veldkamp, 2010, "Information Acquisition and Portfolio Under-Diversification," *Review of Economic Studies*, 77, 779-805.

- Vayanos, D., 2004, "Flight to Quality, Flight to Liquidity, and the Pricing of Risk," Working Paper, London School of Economics.
- Vissing-Jorgensen, A., 2003, "Perspectives on Behavioral Finance: Does "Irrationality" Disappear with Wealth? Evidence from Expectations and Actions," *NBER Macroeconomics Annual*, pp. 139-193.
- Welch, I., 2000, "Views of Financial Economists on the Equity Premium and Other Issues," *Journal of Business*, 73, 501-537.

EDHEC-Risk Institute is part of EDHEC Business School, one of Europe's leading business schools and a member of the select group of academic institutions worldwide to have earned the triple crown of international accreditations (AACSB, EQUIS, Association of MBAs). Established in 2001, EDHEC-Risk Institute has become the premier European centre for applied financial research.

In partnership with large financial institutions, its team of 80 permanent professors, engineers and support staff implements six research programmes and ten research chairs focusing on asset allocation and risk management in the traditional and alternative investment universes. The results of the research programmes and chairs are disseminated through the three EDHEC-Risk Institute locations in London, Nice, and Singapore.

EDHEC-Risk Institute validates the academic quality of its output through publications in leading scholarly journals, implements a multifaceted communications policy to inform investors and asset managers on state-of-the-art concepts and techniques, and forms business partnerships to launch innovative products. Its executive education arm helps professionals to upgrade their skills with advanced risk and investment management seminars and degree courses, including the EDHEC-Risk Institute PhD in Finance.

Copyright © 2012 EDHEC-Risk Institute



For more information, please contact:
Carolyn Essid on +33 493 187 824
or by e-mail to: carolyn.essid@edhec-risk.com

EDHEC-Risk Institute

393-400 promenade des Anglais
BP 3116
06202 Nice Cedex 3 - France

EDHEC Risk Institute—Europe

10 Fleet Place - Ludgate
London EC4M 7RB - United Kingdom

EDHEC Risk Institute—Asia

1 George Street - #07-02
Singapore 049145

www.edhec-risk.com