Source: Carlyle

A Brief Review of Discrete Time Markov Chains

Given a Markov chain defined over a set of states $1, 2, \ldots, n$, and specified by a stochastic transition matrix, $P = \{p_{i,j}\}$, we wish to determine the probability, π_i, of the chain being in each state i once it "reaches" steady state. Equivalently, given a large population, each element of which is always in one of the states, we wish to determine the steady state proportion of the population in each state at any given point in the distant future. By steady state, we mean that in any time period if the vector π represents the proportions of the population in each of the states, then after allowing each member of the population to transition once, the expected proportions in the next time period should be unchanged. This can be stated succinctly as:

$$P^{\mathrm{T}}\pi = \pi,$$

or, less succinctly, as:

$$\begin{pmatrix} p_{1,1} & p_{2,1} & \cdots & p_{n,1} \\ p_{1,2} & p_{2,2} & \cdots & p_{n,2} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1,n} & p_{2,n} & \cdots & p_{n,n} \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{pmatrix} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_n \end{pmatrix}$$

Each row j in the transposed transition matrix represents all the ways members of the population can end up in state j after one transition, and the equation says that one transition should leave steady state proportions unchanged. Rewriting:

$$(P^{\mathrm{T}} - I)\pi = 0.$$

This system of equations is underdetermined; it has rank at most $n-1$, and it has an infinite number of solutions. But, if P is irreducible, positive recurrent, and aperiodic, then if we remove one equation (any equation will do; here we remove the last row) and add in the additional requirement that the proportions must sum to one, we get a nonsingular system of equations:

$$\begin{pmatrix} p_{1,1}-1 & p_{2,1} & \cdots & p_{n-1,1} & p_{n,1} \\ p_{1,2} & p_{2,2}-1 & \cdots & p_{n-1,2} & p_{n,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{1,n-1} & p_{2,n-1} & \cdots & p_{n-1,n-1}-1 & p_{n,n-1} \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_{n-1} \\ \pi_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

We write this modified system as $G\pi = g$, for appropriate definitions of G and g, and obtain the solution to this system of equations by using Gaussian elimination, or some fancy implementation thereof. The crude way to solve for the steady state probabilities is to just invert G:

$$\pi = G^{-1}g$$

and we can do this quickly in Excel with an array formula, but this is *not numerically stable* and should never be used to solve real problems:

```
MMULT(MINVERSE(G_MATRIX),g_vector)
```

This is an *array formula* in Excel, and must be entered using a special keystroke. For a PC, you press `CTRL-SHIFT-ENTER`, and on a Mac you press `COMMAND-ENTER`.