IFT6113, Assignment 2: Theory, 40 pts

October 4, 2024

Deadline: 23:59 October 22nd

Submission: send your writeup in PDF format by a private post in Piazza. Make sure it is readable.

Problem 1 (20 pts)

Recall the Taubin matrix defined in class for approximating mesh curvature:

$$M_p = \int_{-\pi}^{\pi} k_\theta T_\theta T_\theta^\top d\theta$$

1. Prove that the surface normal at p is an eigenvector of M_p. What is the corresponding eigenvalue?

2. Show that the other two eigenvectors are the principal curvature directions. What are the corresponding eigenvalues?

Problem 2 (20 pts)

Prove that the shortest curve connecting two points on a plane is a straight line segment, using Gateaux derivative.

Suppose you are given a regular plane curve $\gamma : [0, 1] \to \mathbb{R}^2$ connecting two fixed points on the plane. Assume $\gamma(0) = A$, $\gamma(1) = B$.

The functional we want to minimize is the arc length of γ:

$$\mathcal{S}[\gamma(t)] = \int_0^1 \|\gamma'(t)\|_2 dt$$

Take an arbitrary vector field $\mathbf{v} : [0, 1] \to \mathbb{R}^2$ along γ (think of $\mathbf{v}$ as of a displacement vector, or *variation*). Let's look at a family of different curves $\gamma_h(t) = \gamma(t) + h\mathbf{v}(t)$. Here h is a scalar parameter that can be thought of how much we displace the initial curve in the direction $\mathbf{v}$. For each h, this formula will define a different curve, but their endpoints should stay on the same place. So we assume $\mathbf{v}(0) = \mathbf{0}$, $\mathbf{v}(1) = \mathbf{0}$.

1. Take the Gateaux derivative $\frac{d}{dh}\mathcal{S}[\gamma + h\mathbf{v}]\big|_{h=0}$.

2. Now recall the notion of *arc length* parameterization $s(t) = \int_0^t \|\gamma'(x)\|_2 dx$. How the functional changes if we substitute arc length parameterization? How the Gateaux derivative from (1) changes?

3. For the curve γ to be shortest, it must satisfy must satisfy $\frac{d}{dh}\mathcal{S}[\gamma + h\mathbf{v}]\big|_{h=0} = 0$ for any $\mathbf{v}$. Now use integration by parts and show that only straight lines satisfy this requirement.