Chapter 13
Gravitation

Topic	Conceptual Problems (To be Discussed in Class)	In Class Problems	Assignment Problems
13.1 Newton's law of Gravitation, 13.2 Weight, 13.3 Gravitational Potential Energy 13.4 Motion of Satellites, 13.5 Kepler's laws and Motion of Planets	TYU-13.1 TYU-13.3 TYU-13.4 TYU-13.5	Example- 13.8,	Exercise 13.11, 13.16, 13.29

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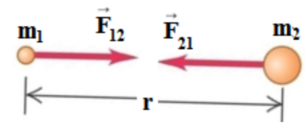
Newton's law of gravitation:

Every particle of matter attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

The gravitational force can be expressed mathematically as



$$F_g = G\frac{m_1 m_2}{r^2}$$ where G = gravitational constant

$G = 6.674 \times 10^{-11}\ N.m^2.kg^{-2}$

The presently accepted value is G was measured by Henry Cavendish in 1798 using torsion balance.

According to Newton's law of gravitation, we have,

Even if the particles have different masses, the gravitational forces they exert on each other are equal in strength:

F_{12} = gravitational force on 2nd body due to 1st body.

F_{21} = gravitational force on 1st body due to 2nd body.

$F_{12} = F_{21}$

=> F = constant, => ma = constant

=> $a \propto \frac{1}{m}$

Thus larger mass has less acceleration and vice versa. Therefore we are accelerated towards the earth, but the earth does not accelerate towards us.

Weight:

The weight of a body is the total gravitational force exerted on it by all other bodies in the universe. At the surface of the earth, we can neglect all other gravitational forces, so a body's weight is

$$w = F_g = \frac{G\ m_E\ m}{R_E^2}$$ Where, m_E = mass of the earth, m = mass of the body,

R_E = radius of the earth

If g = acceleration of the body near the surface of the earth, then weight is given by

w = m g

$$\Rightarrow \frac{G\ m_E\ m}{R_E^2} = m\,g \quad \Rightarrow g = \frac{G\ m_E}{R_E^2}$$

Thus 'g' varies with R_E on the surface of the earth.

Polar radius of earth is minimum, so 'g' is maximum at pole

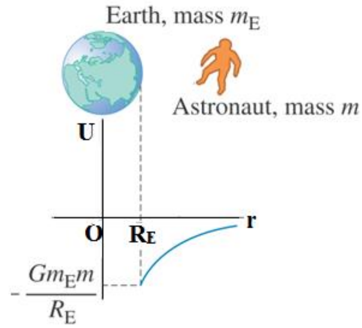
Equatorial radius of earth is maximum, so 'g' is minimum at equator.

Gravitational potential energy:

Gravitational potential energy is the amount of work done by taking the body from infinity against the gravitational force of attraction. So

$$W_{grav} = \int_{\infty}^{r} F_r dr = \int_{\infty}^{r} \frac{G\, m_E m}{r^2} dr \quad \Rightarrow \quad W_{grav} = G\, m_E m \int_{\infty}^{r} \frac{dr}{r^2} = G\, m_E m \left[\frac{-1}{r}\right]_{\infty}^{r}$$

$$\Rightarrow \quad U = -\frac{G\, m_E m}{r} - \left(-\frac{G\, m_E m}{\infty}\right) \quad \Rightarrow \quad U = -\frac{G\, m_E m}{r}$$



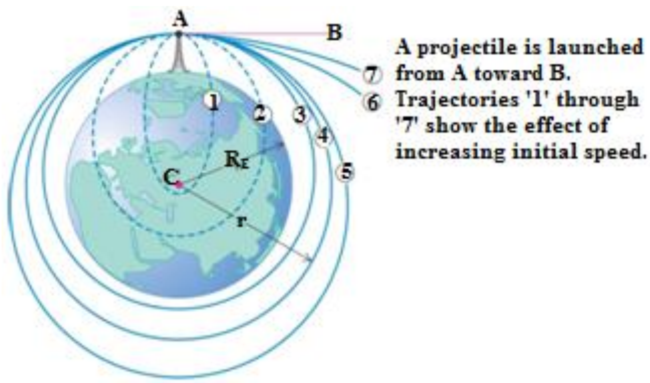
The gravitational potential energy of the earth-astronaut system is given by

$$U = -\frac{G\, m_E m}{r_E}, \quad m = \text{mass of the astronaut}$$

The gravitational potential energy increases as the astronaut moves away from the earth, because as 'r' increases gravitational force does negative work and 'U' increases (becomes less negative)

The gravitational potential energy decreases as the body 'falls' towards earth, because as 'r' decreases gravitational force does positive work and 'U'

The motion of satellites



For a circular orbit, the speed of a satellite is just right to keep its distance from the center of the earth constant. (See Figure)

The gravitational force provides the required centripetal force to the satellite

$$\frac{G\, m_E m}{r^2} = \frac{mv^2}{r}$$

$$\text{Thus,} \quad v = \sqrt{\frac{G\, m_E}{r}}$$

The period of revolution of satellite is

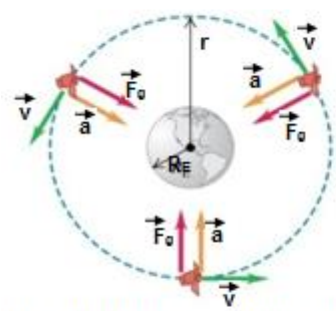
$$T = \frac{2\pi r}{V} = 2\pi r \sqrt{\frac{r}{G\, m_E}} = \frac{2\pi r^{3/2}}{\sqrt{G\, m_E}}$$

Total mechanical energy of the satellite is

$$E = K + U = \frac{1}{2} m v^2 - \frac{G\, m_E m}{r}$$

$$E = \frac{1}{2} m \left(\frac{G\, m_E}{r}\right) - \frac{G\, m_E m}{r}$$

$$E = -\frac{G\, m_E m}{2r}$$



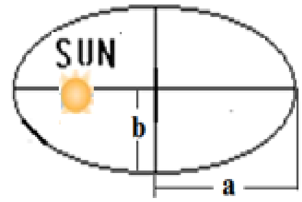
The satellite is in a circular orbit: Its acceleration *a* is always perpendicular to its velocity *v*, so its speed *v* is constant.

A satellite is constantly falling around the earth. Astronauts inside the satellite in orbit are in a state of apparent weightlessness because they are falling with the satellite.

Kepler's laws and planetary motion:

Law 1: The Law of Orbits:

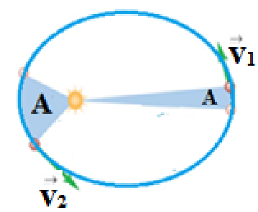
Each planet moves in an elliptical orbit with the sun at one focus.



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is the equation of an ellipse.

Law 2:The Law of Areas:

A line from the sun to a given planet sweeps out equal areas in equal times (see Figure at the right).
dA = Area swept by the line joining a planet and the sun in time dt



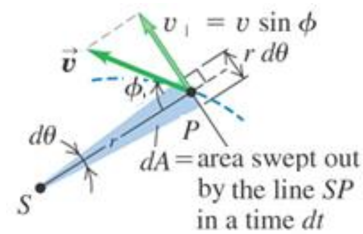
$$dA = \frac{1}{2}(\text{base x height}) = \frac{1}{2} r (r\, d\theta) = \frac{1}{2} r^2\, d\theta$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} \quad \text{where, } \frac{dA}{dt} \text{ is called the sector velocity}$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \omega \quad \text{where, } \frac{d\theta}{dt} = \omega, \text{ angular velocity}$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \frac{v}{r} = \frac{1}{2} r v \quad \text{where, } v = r\omega$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{2m} r(mv) = \frac{1}{2m} r p = \frac{L}{2m}$$



where, $L = r p$, angular momentum

$$\Rightarrow \frac{dA}{dt} = \frac{L}{2m}$$

According to Kepler's 2nd law

$$\text{Sector velocity} \left(\frac{dA}{dt}\right) = \text{constant} \qquad \Rightarrow \frac{L}{2m} = \text{constant} \quad \Rightarrow L = \text{constant}$$

$\Rightarrow$ Angular Momntum = constant

Therefore, when planets revolve around the sun their angular momentum remains constant.

Law 3:The Law of Periods:

The square of the time period of any planet about the Sun is proportional to the cube of the semi-major axis of its orbit.

$$T^2 = \left(\frac{4\pi^2}{G\, m_S}\right) a^3 \quad \Rightarrow \quad T^2 \;\alpha\; a^3$$

T = Time period of the planet about the Sun

a = semi-major axis of the orbit

Test Your Understanding: 13.1, 13.3, 13.4, 13.5

Test Your Understanding of Section 13.1

The planet Saturn has about 100 times the mass of the earth and is about 10 times farther from the sun than the earth is. Compared to the acceleration of the earth caused by the sun's gravitational pull, how great is the acceleration of Saturn due to the sun's gravitation?

(i) 100 times greater; (ii) 10 times greater;

(iii) the same; (iv) 1/10 as great; (v) 1/100 as great

Answer: (v)

The gravitational force of the sun mass m_1 on a planet mass m_2 a distance r away has magnitude

$$F_g = G\frac{m_1\, m_2}{r^2}$$

Compared to the earth, Saturn has a value r^2 of that is 10^2 = 100 times greater and a value of m_2 that is also 100 times greater.

Hence the force that the sun exerts on Saturn has the same magnitude as the force that the sun exerts on earth. The acceleration of a planet equals the net force divided by the planet's mass: Since Saturn has 100 times more mass than the earth; its acceleration is 1/100 as great as that of the earth.

Test Your Understanding of Section 13.3

Is it possible for a planet to have the same surface gravity as the earth (that is, the same value of *g* at the surface) and yet have a greater escape speed?

Answer: Yes

This is possible because surface gravity and escape speed depend in different ways on the planet's mass m_p and radius R_p.

The value of g at the surface is $G\, m_p/R_p^2$ while the escape speed is $\sqrt{(2Gm_P/R_P)}$

For the planet Saturn, for example, is about 100 times the earth's mass and R_p is about 10 times the earth's radius. The value of g is different than on earth by a factor of 1 (i.e., it is the same as on earth), while the escape speed is greater by a factor of 3.2

Test Your Understanding of Section 13.4

Your personal spacecraft is in a low-altitude circular orbit around the earth. Air resistance from the outer regions of the atmosphere does negative work on the spacecraft, causing the orbital radius to decrease slightly. Does the speed of the spacecraft (i) remain the same, (ii) increase, or (iii) decrease?

Answer: (ii)

Equation (13.10) shows that in a smaller radius orbit, the spacecraft has a faster speed. The negative work done by air resistance decreases the total mechanical energy E = K + U; the kinetic energy K increases (becomes more positive), but the gravitational potential energy U decreases (becomes more negative) by a greater amount

Test Your Understanding of Section 13.5

The orbit of Comet X has a semi-major axis that is four times longer than the semi-major axis of Comet Y. What is the ratio of the orbital period of X to the orbital period of Y?

(i) 2; (ii) 4; (iii) 8; (iv) 16; (v) 32; (vi) 64.

Answer: (iii)

We know that,

T^2 is proportional to $a^{3/2}$. Where, T = time period, a = semi-major axis.

Hence the orbital period of Comet X is longer than that of Comet Y by a factor of $4^{3/2} = 8$.

Example 13.8 Kepler's third law

The asteroid Pallas has an orbital period of 4.62 years and an orbital eccentricity of 0.233. Find the semi-major axis of its orbit.

Solution

This problem is based on Kepler's third law, which relates the period 'T' and the semi-major axis 'a' for an orbiting object (such as an asteroid).

From Appendix we have $m_s = 1.99 \times 10^{30}$ kg,

From Appendix we have $T = (4.62\ \text{yr})(3.156 \times 10^{7}\ \text{s/yr}) = 1.46 \times 10^{8}\ \text{s}$

We don't need the value of the eccentricity.

$$T^2 = \left(\frac{4\pi^2}{G m_S}\right) a^3 \quad \Rightarrow \quad a = \left(\frac{G m_S T^2}{4\pi^2}\right)^{1/3} = 4.15 \times 10^{11}\,\text{m}$$

Exercise Problem: 13.11, 13.16, 13.29

Exercise -13.11:

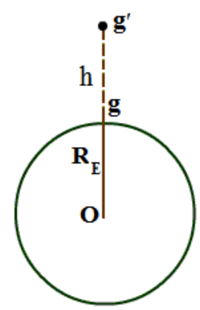
At what distance above the surface of the earth is 0.98 m/s^2 the acceleration due to the earth's gravity if the acceleration due to gravity at the surface has magnitude 9.80 m/s^2?

Solution:

$R_E = 6.38 \times 10^6$ m = radius of the earth.

And $r = h + R_E$

Where h = distance of the object above the surface of the earth



Acceleration due to gravity on the surface = $g = \dfrac{G m_E}{R_E^2}$

Acceleration due to gravity at a height 'h' = $g' = \dfrac{G m_E}{(R_E + h)^2}$

Now,

$$\frac{g}{g'} = \frac{\dfrac{G m_E}{R_E^2}}{\dfrac{G m_E}{(R_E + h)^2}} = \frac{(R_E + h)^2}{R_E^2} = \left(\frac{R_E + h}{R_E}\right)^2 = \left(1 + \frac{h}{R_E}\right)^2$$

$$\sqrt{\frac{g}{g'}} = 1+\frac{h}{R_E} \quad \Rightarrow \quad \sqrt{\frac{g}{g'}} - 1 = \frac{h}{R_E} \quad \Rightarrow \quad h = R_E\left(\sqrt{\frac{g}{g'}} - 1\right)$$

$$\Rightarrow \quad h = \left(6.38 \times 10^6 m\right)\left(\sqrt{\frac{9.8}{0.98}} - 1\right)$$

$$\Rightarrow \quad h = \left(6.38 \times 10^6 m\right)\left(\sqrt{10} - 1\right)1.38\times10^7 m$$

$$\Rightarrow \quad h = 1.38\times10^7 m$$

a_y is directed downward midway between A and B

Exercise-13.16

Estimate the gravitational potential energy of the earth with respect to the sun. (Given: M_{sun}=1.99 x 10^{30} kg, M_{earth} =5.98 x 10^{24} kg, mean distance between the sun and the earth = 1.50 x 10^6 km)

Solution:

M_{sun}=1.99 x 10^{30} kg

M_{earth} =5.98 x 10^{24} kg

r = mean distance between the sun and the earth = 1.50 x 10^6 km = 1.50 x 10^9 m

The gravitational potential energy of the earth-sun system is given by

$$U = -\frac{G\,M_{Sun}M_{Earth}}{r} = -\frac{\left(6.674 \text{ x } 10^{-11} \text{ N.m}^2\text{.kg}^{-2}\right)\left(1.99 \text{ x } 10^{30}\text{kg}\right)\left(5.98 \text{ x } 10^{24}\text{kg}\right)}{1.50 \text{ x } 10^9\text{m}} = -5.2916 \text{ x } 10^{36} \text{ Joule}$$

Exercise-13.29

The star Rho[1] Cancri is 57 light-years from the earth and has a mass 0.85 times that of our sun. A planet has been detected in a circular orbit around Rho[1] Cancri with an orbital radius equal to 0.11 times the radius of the earth's orbit around the sun. What are

(a) the orbital speed the planet of Rho[1] Cancri?

(b) the orbital period of the planet of Rho[1] Cancri?

Solution:

M_{sun}=1.99 x 10^{30} kg $\qquad$ M_{Rho}= 0.85 x (1.99 x 10^{30} kg)

r_E = orbital radius of the earth = 1.50 x 10^6 km

r_{Rho} = orbital radius of the Rho = 0.11 x (1.50 x 10^6 km)

(a) $$v = \sqrt{\frac{G\,M_R}{r}} = \sqrt{\frac{\left(6.674 \text{ x } 10^{-11} \text{ N.m}^2\text{.kg}^{-2}\right)\left(0.85 \text{ x } 1.99 \text{ x } 10^{30}\text{kg}\right)}{0.11 \text{ x}\left(1.50 \text{ x } 10^6\text{km}\right)}} = 8.27 \text{ m/s}$$

(b) The orbital period of revolution of planet is

$$T = \frac{2\,\pi\,r}{V} = \frac{2 \text{ x } 3014 \text{ x } 0.11 \text{ x}\left(1.50 \text{ x } 10^6\text{km}\right)}{8.27 \text{ m/s}} = 1.25 \text{ x } 10^6\text{s} = 14.5 \text{ days (about two weeks)}$$