Chapter 9
Rotation of Rigid Bodies

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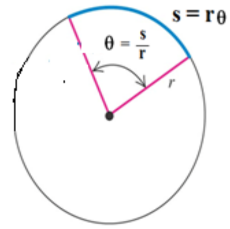
9.1 Angular displacement, Angular velocity & Angular acceleration



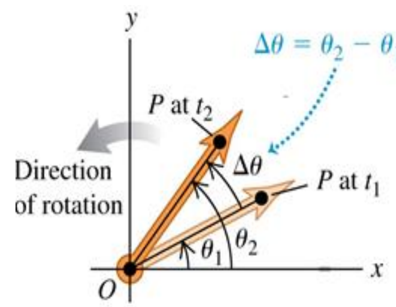
- **Angular displacement** ($\Delta\theta$) of a body is
  $\Delta\theta = \theta_2 - \theta_1$ (in radians)
- Unit of angle (or angular displacement) is radian. Other unit of angle is degree.

- One radian is the angle at which the arc **s** has the same length as the radius **r**.

  $$\theta = \frac{s}{r}$$

  2π radians = 360°

  $$1\text{ rad} = \frac{360^0}{2\pi} = 57.3^0$$



- **Angular velocity:**
  Average angular velocity (ω_{av}) of a body is

  $$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

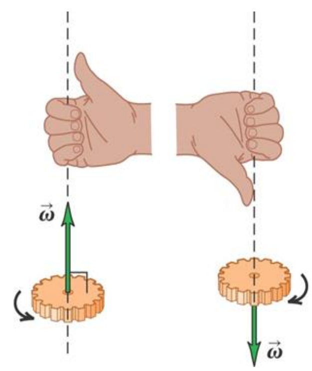
  Instantaneous angular velocity **is**

  $$\omega_z = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

  For counterclockwise rotation, $\Delta\theta > 0$, so $\frac{\Delta\theta}{\Delta t} > 0$, So, $\boldsymbol{\omega_{av}} > 0$

  For clockwise rotation, $\Delta\theta < 0$, so $\frac{\Delta\theta}{\Delta t} < 0$, So, $\boldsymbol{\omega_{av}} < 0$

  Angular velocity is defined as a vector whose direction is given by the right-hand rule.
  If the fingers of the right hand is curled in the direction of rotation then the right thumb points in the direction of $\boldsymbol{\omega_z}$.



- **Angular acceleration(α_z)**

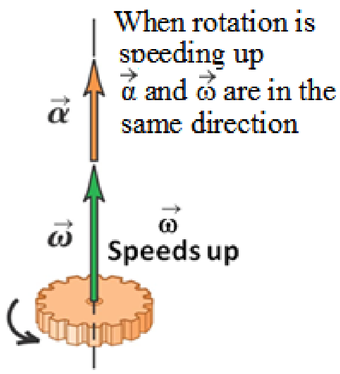
  Average angular acceleration is the rate of change in angular velocity.

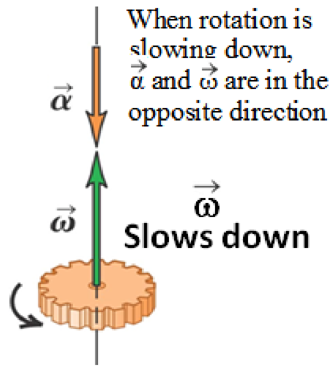
  $$\alpha_{avg\text{-}z} = \frac{\omega_{2z} - \omega_{1z}}{t_2 - t_1} = \frac{\Delta\omega_z}{\Delta t}$$

  Instantaneous angular acceleration is

  $$\alpha_z = \lim_{\Delta t \to 0} \frac{\Delta\omega_z}{\Delta t} = \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2}$$

Direction of angular acceleration is shown below:





Analogy between Linear and Angular Motion

Linear and Angular Motion with constant acceleration equations are very similar.

Sl. No	Linear Motion	Angular Motion
1	$a_x = \text{constant}$	$\alpha_z = \text{constant}$
2	$v_x = v_{0x} + a_x t$	$\omega_z = \omega_{0z} + \alpha_z t$
3	$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$	$\theta = \theta_0 + \omega_{0z}t + \frac{1}{2}\alpha_z t^2$
4	$v_x^2 = v_{0x}^2 + 2a_x(x - x_0)$	$\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$
5	$x - x_0 = \frac{1}{2}(v_{0x} + v_x)t$	$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t$

Relating linear and angular kinematics

Linear displacement and angular displacement	$x = r\,\theta$
Linear velocity and angular velocity	$v_x = r\,\omega_z$
Linear acceleration and angular acceleration	$a_x = r\,\alpha_z$

Rotational kinetic energy

Let a body is made up of a large number of particles, with masses m_1, m_2, m_3, ….. at distances r_1, r_2, r_3, from the axis of rotation. v_1, v_2, v_3, are the respective linear velocities.

Different particles have different values of r, but ω is the same for all (otherwise, the body wouldn't be rigid).

Kinetic energy of 1st particle is:

$$K_1 = \frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 r_1^2\,\omega^2 \quad (\text{using the formula } v_1 = r_1\,\omega)$$

Similarly, the kinetic energy of the 2nd, 3rd, particles are

$$K_2 = \frac{1}{2} m_2 r_2^2\, \omega^2,\; K_3 = \frac{1}{2} m_3 r_3^2\, \omega^2 \;\; ......$$

So, total kinetic energy is

$$K = \frac{1}{2} m_1 r_1^2\, \omega^2 + \frac{1}{2} m_2 r_2^2\, \omega^2 \;\; .... = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

$$K = \frac{1}{2} I\, \omega^2$$

Where, $\mathbf{I = m_1r_1^2 + m_2r_2^2 + \ldots = \Sigma m_i r_i^2}$

I = moment of inertia of a set of discrete particles

In most general sense moment of inertia can be written as

$$I = \int r^2\, dm$$

Moments of inertia

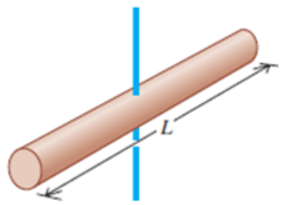
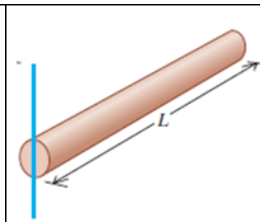
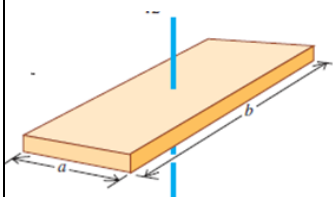
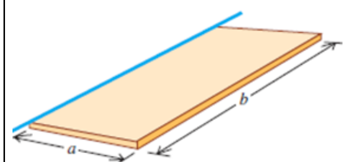
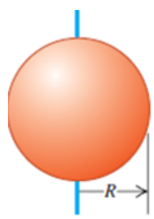
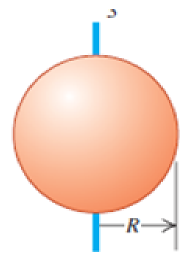
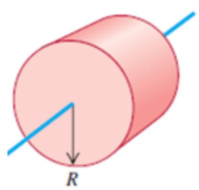
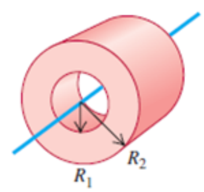
Moment of inertia of a rigid body containing 'i' number of discrete particles is the sum of the product of mass and the square of perpendicular distance from the axis of rotation. It is given by:

$I = m_1r_1^2 + m_2r_2^2 + \ldots = \Sigma m_i r_i^2$

'I' is also called as rotational inertia.

Moment of inertia also depends on the position of the axis of rotation. One body may rotate about many axes. So one body has many moment of inertia about different axes.

	Here more mass is available away from the axis of rotation. So, - 'I' will be more - more force is required to rotate, - more rotational kinetic energy.		Here more mass is available near the axis of rotation. So, - 'I' will be less - less force is required to rotate - less rotational kinetic energy

Moment of Inertia of some bodies	
 Slender rod, Axis: through center $I = \frac{1}{12} M L^2$	 Slender rod, Axis: through one end $I = \frac{1}{3} M L^2$
 Rectangular plate, Axis: through center $I = \frac{1}{12} M \left(a^2 + b^2\right)$	 Rectangular plate, Axis: along edge $I = \frac{1}{3} M a^2$
 Solid sphere $I = \frac{2}{5} M R^2$	 Thin-walled hollow Sphere $I = \frac{2}{3} M R^2$
 Solid cylinder $I = \frac{1}{2} M R^2$	 Hollow cylinder $I = \frac{1}{2} M \left(R_1^2 + R_2^2\right)$

The parallel-axis theorem

A body has infinitely many moments of inertia, because there are infinitely many axes about which it might rotate.

If,

I_{cm} = Moment of inertia about an axis passing through center of mass (CM)

I_P = Moment of inertia about any axis parallel to axis passing through center of mass (CM)

Then, $I_P = I_{cm} + M d^2$

Where, M = mass of the body

d = distance between two parallel axes

This relationship is called the **parallel-axis theorem.**

Test Your Understanding: TYU-9.2, TYU-9.4, TYU-9.6

Test Your Understanding 9.2:

Suppose the disc in Example 9.3 was initially spinning at twice the rate (55.0 rad/s^2 rather than 27.5 rad/s) and slowed down at twice the rate (-20.0 rad/s^2 rather than -10.0 rad/s^2)

(a) Compared to the situation in Example 9.3, how long would it take the disc to come to a stop?
  (i) the same amount of time; (ii) twice as much time; (iii) 4 times as much time; (iv) ½ as much time; (v) ¼ as much time.

(b) Compared to the situation in Example 9.3, through how many revolutions would the disc rotate before coming to a stop?
  (i) the same number of revolutions; (ii) twice as many revolutions; (iii) 4 times as many revolutions; (iv) ½ as many revolutions; (v) ¼ as many revolutions.

Answers: (a) (i), (b) (ii)

(a) When the disc comes to rest, $\omega_z = 0$

So $t = \dfrac{\omega_z - \omega_{0z}}{\alpha_z} = \dfrac{\omega_{0z}}{\alpha_z}$ (this is a positive time because α_z is negative).

If we double the initial angular velocity ω_{0z} and also double the angular acceleration α_z their ratio is unchanged and the rotation stops in the same amount of time.

(b) The angle through which the disc rotates is given by

$$\theta - \theta_0 = \frac{1}{2}(\omega_{0z} + \omega_z)t = \frac{1}{2}\omega_{0z}t$$

The initial angular velocity ω_{0z} has been doubled but the time t is the same, so the angular displacement $\theta - \theta_0$ (and hence the number of revolutions) has doubled.

We can also come to the same conclusion using expression $\omega_z^2 = \omega_{0z}^2 + 2\alpha_z(\theta - \theta_0)$

Test Your Understanding 9.4:

Suppose the cylinder and block in Example 9.8 have the same mass, so m = M. Just before the block strikes the floor, which statement is correct about the relationship between the kinetic energy of the falling block and the rotational kinetic energy of the cylinder?

(i) The block has more kinetic energy than the cylinder.
(ii) The block has less kinetic energy than the cylinder.
(iii) The block and the cylinder have equal amounts of kinetic energy.

9.4 Answer: (i)

The kinetic energy in the falling block is $\frac{1}{2}m v^2$ and

The kinetic energy in the rotating cylinder is $\frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v}{R}\right)^2 = \frac{1}{4}m v^2$

Hence the total kinetic energy of the system is $\frac{3}{4}m v^2$, of which two-thirds is in the block and one-third is in the cylinder.

Test Your Understanding 9.6:

Two hollow cylinders have the same inner and outer radii and the same mass, but they have different lengths. One is made of low-density wood and the other of high-density lead. Which cylinder has the greater moment of inertia around its axis of symmetry?

(i) the wood cylinder; (ii) the lead cylinder; (iii) the two moments of inertia are equal.

Answer: (iii)

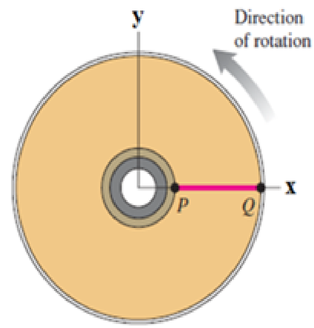
Our result from Example 9.10 does not depend on the cylinder length L. The moment of inertia depends only on the radial distribution of mass, not on its distribution along the axis.

Example Problems 9.3, 9.8, 9.10

Example 9.3:

You have finished watching a movie on Blu-ray and the disc is slowing to a stop. The disc's angular velocity at t = 0 is 27.5 rad/s and its angular acceleration is a constant -10 rad/s^2. A line PQ on the disc's surface lies along + x – axis at t = 0.

(a) What is the disc's angular velocity at t = 0.3 s?
(b) What angle does the line PQ make with the + x-axis at this time?



Soln:

(a) At t = 0.3 s we have

$\omega_z = \omega_{0z} + \alpha_z t$

$\omega_z = 27.5 \text{ rad/s} + (-10.0 \text{ rad/s}^2)(0.300 \text{ s}) = 24.5 \text{ rad/s}$

(b) Using eq^n $\theta = \theta_0 + \omega_{0z} t + \frac{1}{2}\alpha t^2$

$$\theta = 0 + (27.5 \text{ rad/s})(0.300 \text{ s}) + \frac{1}{2}(-10.0 \text{rad/s}^2)(0.3 \text{ s})^2$$

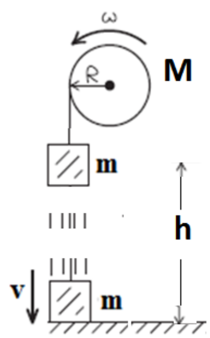
$$\theta = 7.80 \text{ rad} = 1.24 \text{ rev} = 1 \text{rev} + 0.24 \text{ rev}$$

$$\theta = 360^0 + 87^0 \quad (\because 0.24 \text{ rev} = 87^0)$$

The disc has turned through one complete revolution plus an additional 87°. Hence the line PQ makes an angle of 87^0 with the + x-axis.

Example 9.8:

We wrap a light, non-stretching cable around a solid cylinder with mass M and radius R. The cylinder rotates with negligible friction about a stationary horizontal axis. We tie the free end of the cable to a block of mass m and release the block from rest at a distance h above the floor. As the block falls, the cable unwinds without stretching or slipping. Find expressions for the speed of the falling block and the angular speed of the cylinder as the block strikes the floor.



Soln: Total energy remains constant

$$K_1 + U_1 = K_2 + U_2$$

$$0 + m g h = \frac{1}{2} m v^2 + \frac{1}{2}\left(\frac{1}{2} M R^2\right)\left(\frac{v}{R}\right)^2 + 0$$

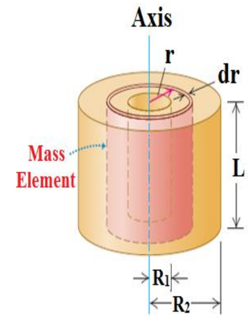
$$\Rightarrow m g h = \frac{1}{2}\left(m + \frac{1}{2} M\right) v^2 \quad \Rightarrow v = \sqrt{\frac{2 g h}{1 + M/2m}}$$

The final angular speed of the cylinder is: ω = v/R.

- When 'M' is much larger than 'm', 'v' is very small;
- when M is much smaller than m, 'v' is nearly equal to $\sqrt{2 g h}$ the speed of a body that falls freely from height h.

Example 9.10:

Find the **moment of inertia** of a **hollow cylinder** of uniform mass density with length L, inner radius $\mathbf{R_1}$ and outer radius $\mathbf{R_2}$ about its **axis of symmetry**.



Soln:

Let us consider a volume element a thin cylindrical shell of radius r, thickness dr, and length L.

The volume of the shell is very nearly that of a flat sheet with thickness dr, length L, and width $2\pi r$ (the circumference of the shell).

Volume of the elementary shell $= dV = L \times dr \times 2\pi r = 2\pi L r\, dr$

Mass of the elementary shell $= dm = \rho\, dV = \rho\, (2\pi L r\, dr)$

The moment of inertia of the hollow cylinder with inner radius $\mathbf{R_1}$ and outer radius $\mathbf{R_2}$ about its axis of symmetry is

$$I = \int_{R_1}^{R_2} r^2\, dm = \int_{R_1}^{R_2} r^2 \left(2\pi \rho L r\, dr \right) = 2\pi\rho L \int_{R_1}^{R_2} r^3\, dr = 2\pi\rho L \left[\frac{r^4}{4} \right]_{R_1}^{R_2}$$

$$I = 2\pi\rho L \left(\frac{R_2^4 - R_1^4}{4} \right) = \left(\frac{\pi \rho L}{2} \right) \left(R_2^2 - R_1^2 \right)\left(R_2^2 + R_1^2 \right)$$

Now, Volume of the cylinder $= V = \pi \left(R_2^2 - R_1^2 \right) L$

The mass of the shell $= M = \rho V = \pi \rho L \left(R_2^2 - R_1^2 \right)$

So, $I = \dfrac{1}{2} M \left(R_2^2 + R_1^2 \right)$

Special cases:

Cases- I : Solid Cylinder

For a solid cylinder we have,

Outer radius $R_2 = R$ and inner radius $R_1 = 0$,

So, moment of inertia is $I = \dfrac{1}{2} M \left(R^2 + 0^2 \right) \Rightarrow I = \dfrac{1}{2} M R^2$

Cases-II : Very thin hallow Cylinder

Here, Outer radius $R_2 = R$ and inner radius $R_1 = R$,

So, moment of inertia is

$$I = \frac{1}{2} M \left(R^2 + R^2 \right) \Rightarrow I = M R^2$$