Chapter 11

Equilibrium And Elasticity

Topic	Conceptual Problems (To be Discussed in Class)	In Class Problems	Assignment Problems
11.1 Conditions for Equilibrium 11.2 Center of Gravity 11.4 Stress, Strain, and Elastic Moduli 11.5 Elasticity and Plasticity	TYU-11.4	Example-11.5	Exercise 11.6, 11.25, 11.37

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11.1: Conditions for Equilibrium

First condition: For an extended body, the center of mass of the body has zero acceleration if the net external forces acting on the body are zero.

$$\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma F_z = 0$$

Second condition: A rigid body in equilibrium can't rotate about any point, if the sum of the torques due to all external forces acting on the body is zero.

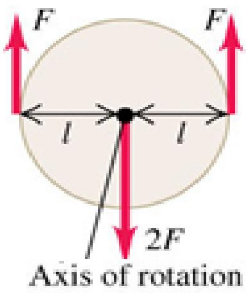
$$\Sigma \tau = 0$$

When both the 1st and 2nd conditions are satisfied, then the body is in Static Equilibrium

Case – I: The body is in static equilibrium:

First condition satisfied: If the net force = 0, then the body at rest has no tendency to start moving as a whole.

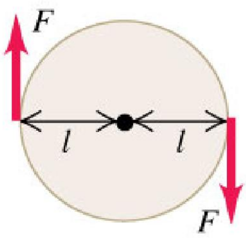
Second condition satisfied: If the net torque about the axis = 0, then the body at rest has no tendency to start rotating.



Case – II: The body has no tendency to accelerate as a whole, but it has a tendency to start rotating.

First condition satisfied: If the net force = 0, then the body at rest has no tendency to start moving as a whole.

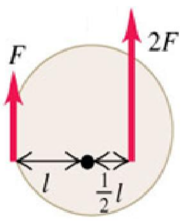
Second condition NOT satisfied: There is a net clockwise torque about the axis, so body at rest will start rotating clockwise.



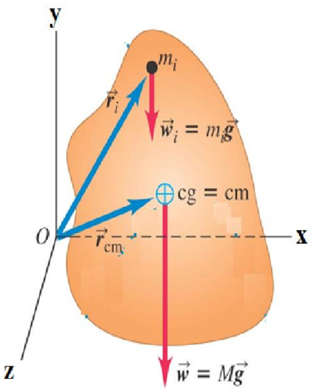
Case – III: The body has a tendency to accelerate as a whole but no tendency to start rotating.

First condition NOT satisfied: There is a net upward force, so body at rest will start moving upward.

Second condition satisfied: If the net torque about the axis = 0, then the body at rest has no tendency to start rotating.



11.2 Center of Gravity



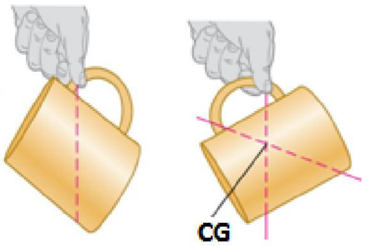
- The gravitational torque about O on a particle of mass m_i within the body is: $\tau = r_i \times w_i$.
- If **g** has the same value at all points on the body, then CG = CM.
- The net gravitational torque about O on the entire body can be found by assuming that all the weight acts at the cg:

  $\tau = r_i \times w$.
- We can assume that the total weight of the body at a single point i.e at **the center of gravity.**

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots\ldots}{m_1 + m_2 + m_3 + \ldots\ldots} = \frac{\sum_i m_i\vec{r}_i}{\sum_i m_i}$$

Finding the center of gravity of an irregularly shaped body

- Suspend the mug from any point. A vertical line extending down from the point of suspension passes through the center of gravity.
- Now suspend the mug from a different point. A vertical line extending down from this point intersects the first line at the center of gravity (which is inside the mug).

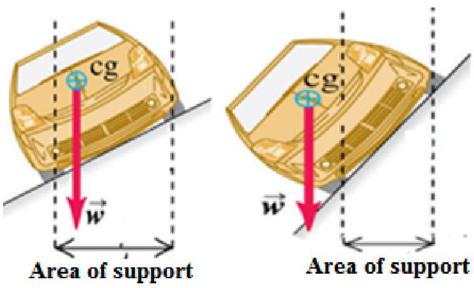


Condition for the tip over of the vehicle:

If the center of gravity is within the area bounded by the supports, then the car is in equilibrium and the car will not tip over.

The car will tip over when its centers of gravity lie outside the area of support.

The higher the center of gravity, the smaller the incline needed to tip the vehicle over.



Variation of g, CG & CM with altitude:

Consider the case of Petronas Towers in Malaysia of height 452 m.

The acceleration due to gravity at the bottom is only 0.014% greater than at the top.

The center of gravity (CG) of the towers is only about 2 cm below the center of mass (CM).

If ignore variation of gravity with altitude then Center of Gravity (CG) is same as center of mass (CM).

11.4 Stress, Strain, and Elastic Moduli

- **Stress** causes stretching, squeezing, and twisting a real body. It is defined as the force applied per unit area of the body. It is measured in N/m^2 or ‘Pascals’ (Pa)

- **Strain** is produced on the body when the **Stress** (a force) is applied on it. **Strain** is the relative change in size or shape of an object because of externally applied forces. Thus **Strain** is the fractional deformation due to the stress. Strain is dimensionless. It is just a fraction.

- **Hooke’s law**: Hooke’s law states that the stress applied on body is directly proportional to the strain produced in the body.

  Stress α Strain

  $$\frac{\text{Stress}}{\text{Strain}} = \text{Constant} = \text{Elastic Modulus}$$

- **Young’s modulus(Y)**

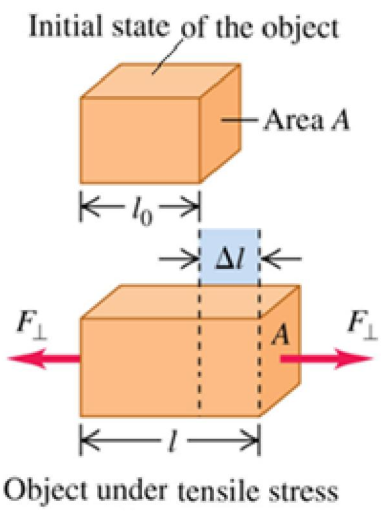
  For a sufficiently small tensile stress, stress and strain are proportional to each other and the corresponding elastic modulus is called **Young’s modulus, denoted by Y.** Thus.

  $$Y = \frac{\text{Tensile Stress}}{\text{Tensile Strain}}$$

  Let an external force ($F_\perp$) is applied so that a spring/body deforms a distance $\Delta\ell$ from its original length ℓ_0.

  We define the tensile stress at the cross section as the ratio of the force to the cross-sectional area A:

  $$\text{Tensile Stress} = \frac{F_\perp}{A}$$



The object is stretches due to tensile stress and the length changes from ℓ_0 to $\ell\ (=\ell_0+\Delta\ell)$

$$\text{Tensile Strain} = \frac{\Delta\ell}{\ell_0}$$

$$\text{Now, } Y = \frac{\text{Tensile Stress}}{\text{Tensile Strain}} \quad \Rightarrow \quad Y = \frac{F_\perp/A}{\Delta\ell/\ell_0}$$

$$\Rightarrow \quad Y = \frac{F_\perp}{A}\frac{\ell_0}{\Delta\ell}$$

A material with a large value of **Y** is relatively un-stretchable; a large stress is required for a given strain.
For example, the value of Y, for cast steel (2×10^{11} Pa) is much larger than that for rubber (5×10^{8} Pa)

Compressive Stress & Strain

When the forces on the ends of a bar are pushes rather than pulls, the bar is in **compression and the stress is a compressive stress.**

The compressive stress & compressive strain are defined similar to tensile stress and tensile strain, but $\boldsymbol{\Delta\ell}$ has the opposite direction. The Young's modulus is given by,

$$Y = \frac{F_\perp}{A}\frac{\ell_0}{\Delta\ell}$$

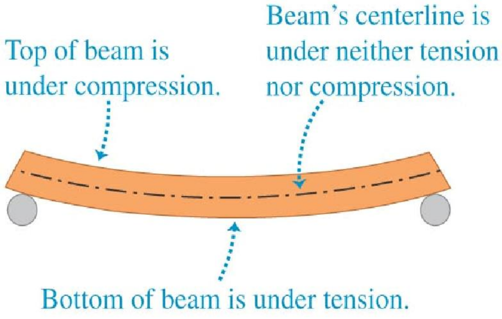
For many materials, Young's modulus has the same value for both tensile and compressive stresses.

Composite materials such as concrete and stone are an exception. They can withstand compressive stresses but fail under comparable tensile stresses.

Both Tensile & Compressive stress

Body can experience both tensile & compressive stress at the same time. Example:

Composite materials such as concrete and stone are used arches in doorways and bridges, where the weight of the overlying material compresses the stones of the arch together and does not place them under tension.



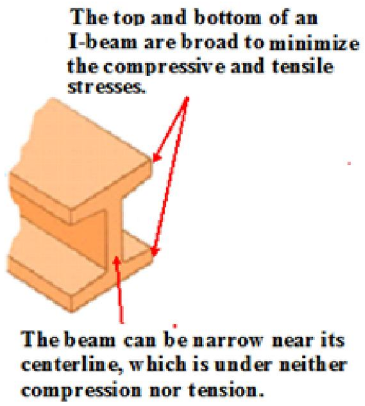
A horizontal beam supported at each end sags under its own weight. As a result,

i) the top of the beam is under compression,
ii) while the bottom of the beam is under tension.

To minimize the stress and hence the bending strain, the top and bottom of the beam are given a large cross-sectional area.

There is neither compression nor tension along the centerline of the beam, so this part can have a small cross section; this helps to keep the weight of the bar to a minimum and further helps to reduce the stress

The result is an I-beam of the familiar shape used in building construction.



Bulk (Volume) stress and strain

The force per unit area that the fluid exerts on the surface of an immersed object is called the pressure **p** in the fluid:

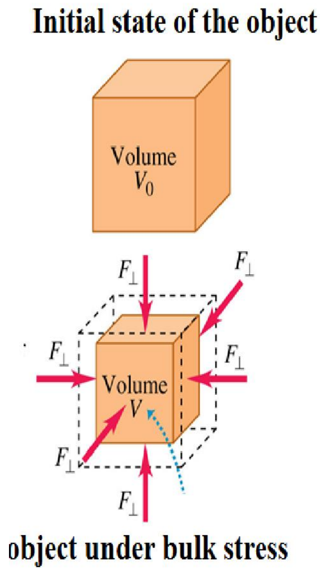
$$p = \frac{F_{\perp}}{A}$$

Pressure plays the role of stress in a volume deformation. The corresponding strain (Bulk) is the fractional change in volume that is, the ratio of the volume change ΔV to the original volume V_0

$$\text{Bulk (Volume) Strain} = \frac{\Delta V}{V_0}$$

Bulk modulus (B) is given by :

$$B = \frac{\text{Bulk Stress}}{\text{Bulk Strain}} = \frac{p_0 - (p_0 + \Delta p)}{V_0} = \frac{-\Delta p}{\Delta V / V_0}$$



Minus sign indicates that, an increase of pressure always causes a decrease in volume.

The reciprocal of the bulk modulus is called the **compressibility** and is denoted by **k**.

$$k = \frac{1}{B} = -\left(\frac{1}{V_0}\right)\frac{\Delta V}{\Delta p}$$

Sheer stress and strain

We define the shear stress as the force $F_{||}$ acting tangent to the surface divided by the area A on which it acts:

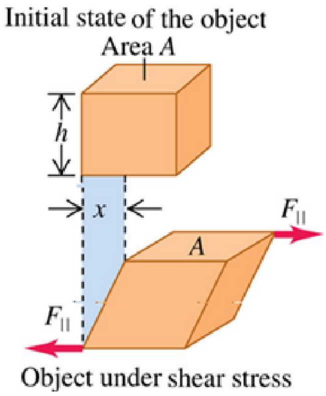
$$\text{Sheer Stress} = \frac{F_{||}}{A}$$

Shear strain is the ratio of the displacement x to the transverse dimension 'h'. Thus,

$$\text{Sheer Strain} = \frac{x}{h}$$

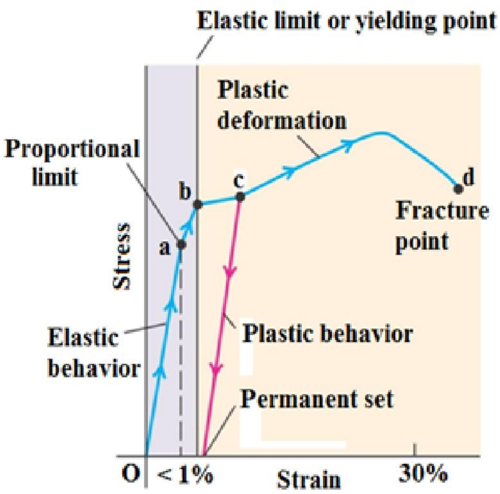
Sheer modulus (S) is sheer stress divided by sheer strain, and is given by

$$S = (F_{\parallel}/A)(h/x). \Rightarrow S = \frac{F_{\parallel}/A}{x/h} = \frac{F_{\parallel}}{A}\frac{h}{x}$$



Elasticity and plasticity:

The graph shows that the stress is a function of strain for a ductile metal under tension.



- ✓ Hooke's law applies up to point **a**.
- ✓ Hooke's law is obeyed in straight-line portion ends at point **a**. The stress at this point is called the proportional limit.

Between point O and b:

- ✓ From **a** to **b**, stress and strain are no longer proportional, and Hooke's law is not obeyed.
- ✓ if stress is removed the body will regain its original condition i.e. the path is retraced. Thus deformation is reversible. So, the forces are conservative. So, energy put into the material to cause the deformation is recovered when the stress is removed.
- ✓ In region Ob we say that the material shows elastic behavior.
- ✓ Point **b**, the end of this region, is called the **yield point**;
- ✓ the stress at the yield point is called the elastic limit.

Between point b and c :

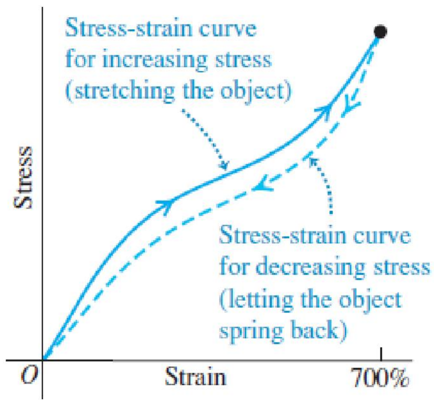
- When we increase the stress beyond point **b**, the strain continues to increase. But when removed the load, the material does not come back to its original length. It follows the red line path.
- Here the material has undergone an irreversible deformation and has acquired a permanent set.

Between point c and d :

- Further increase of load beyond **c** produces a large increase in strain for a relatively small increase in stress up to point d. Point d is called fracture point.
- The behavior of the material from **b** to **d** is called **plastic flow or plastic deformation.**
- A plastic deformation is irreversible; when the stress is removed, the material does not return to its original state
- If a large amount of plastic deformation takes place (between the elastic limit and the fracture point) then the material is said to be **ductile**.
- But if fracture occurs soon after the elastic limit is passed, the material is said to be **brittle**.
- A soft iron wire that can have considerable permanent stretch without breaking is ductile, while a steel piano string that breaks soon after its elastic limit is reached is brittle.

Elastic Hysteresis

Stress-strain diagram for vulcanized rubber



Stress-strain curve for vulcanized rubber that has been stretched by more than seven times its original length is shown in the figure.

The stress is **not** proportional to the strain, but the behavior is **elastic** because when the load is removed, the material returns to its original length.

However, the material follows different curves for increasing and decreasing stress. This is called **elastic hysteresis**.

The work done by the material when it returns to its original shape is less than the work required to deform it; there are non-conservative forces associated with internal friction. Rubber with large elastic hysteresis is very useful for absorbing vibrations, such as in engine mounts and shock-absorber bushings for cars.

Breaking Stresses

The stress required to cause actual fracture of a material is called the breaking stress, the ultimate strength, or (for tensile stress) the tensile strength. Two materials, such as two types of steel, may have very similar elastic constants but vastly different breaking stresses. Following Table gives typical values of breaking stress for several materials in tension.

Material	Breaking Stress (Pa or N/m^2)	Material	Breaking Stress (Pa or N/m^2)
Aluminum	2.0×10^8	Iron	3.0×10^8
Brass	4.7×10^8	Phosphor bronze	5.6×10^8
Glass	10.0×10^8	Steel	$5 - 20 \times 10^8$

Test Your Understanding 11.4

A copper rod of crosssectional area 0.5 cm^2 and length 1.00 m is elongated by 2×10^{-2} mm and a steel rod of the same cross-sectional area but 0.100 m in length is elongated by 2×10^{-3} mm.

(a) Which rod has greater tensile strain? (i) the copper rod; (ii) the steel rod; (iii) the strain is the same for both.

(b) Which rod is under greater tensile stress? (i) the copper rod; (ii) the steel rod; (iii) the stress is the same for both.

Answers: (a) (iii), (b) (ii)

In (a), the copper rod has 10 times the elongation $\Delta\ell$ of the steel rod, but it also has 10 times the original length ℓ_0. Hence the tensile strain $\Delta\ell/\ell_0$ is the same for both rods.

In (b), the stress is equal to Young's modulus Y multiplied by the strain. From Table 11.1, steel has a larger value of Y, so a greater stress is required to produce the same strain.

Example Problems

Example – 11.5: Tensile stress and strain

A steel rod 2.0 m long has a cross-sectional area of 0.3 cm^2 . It is hung by one end from a support, and a 550-kg milling machine is hung from its other end. Determine the stress on the rod and the resulting strain and elongation.

Solution:

$$\text{Tensile Stress} = \frac{F_\perp}{A} = \frac{(550\text{ kg})\left(9.8\text{ m/s}^2\right)}{3 \times 10^{-5}\text{ m}^2} = 1.8 \times 10^8\text{ N.m}^2$$

$$\text{Tensile Strain} = \frac{\Delta\ell}{\ell_0} = \frac{\text{Stress}}{Y} = \frac{1.8 \times 10^8\text{ N.m}^2}{20 \times 10^{10}\text{ N.m}^2} = 9 \times 10^{-4}$$

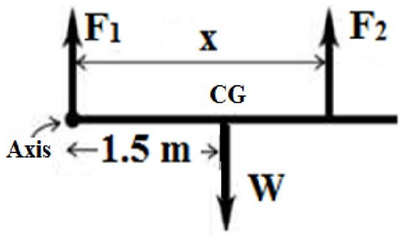
Elongation = $\Delta\ell$ = (Strain) x ℓ_0 = (9.0×10^{-4}) (2.0 m) = 1.8×10^{-3} m

This small elongation, resulting from a load of over half a ton, is a testament to the stiffness of steel.

Assignment Problems: 11.6, 11.25, 11.37

Exercise Problems: 11.6

Two people are carrying a uniform wooden board that is 3.0 m long and weighs 160 N. If one person applies an upward force equal to 60 N at one end, at what point does the other person lift? Begin with a free-body diagram of the board.



Solution:

Since the board is uniform its center of gravity is 1.50 m from each end.

$w = 160$ N, $F_1 = 60$ N

$\Sigma F_y = 0 \quad \Rightarrow F_1 + F_2 - w = 0$

$\Rightarrow \quad F_2 = w - F_1 = 160\text{ N} - 60\text{ N} = 100$ N.

$\Sigma \tau_z = 0 \quad \Rightarrow \quad F_2 x - w\ (1.50\text{ m}) = 0 \quad \Rightarrow \quad x = 2.4$ m

The other person lifts with a force of 100 N at a point 2.40 m from the end where the other person lifts.

By considering the axis at the center of gravity we can see that a larger force is applied by the person who pushes closer to the center of gravity.

Exercise Problems: 11.25

Calculate the longest length of the steel wire that can hang vertically without breaking. Breaking stress for steel is 7.982×10^8 N/m^2 and density of steel $d = 8.1 \times 10^3$ kg/m^3.

Solution:

$$\text{Breaking Stress} = \frac{F_\perp}{A} = \frac{M\,g}{A} = \frac{V\,d\,g}{A} = \frac{A\,\ell_0\,d\,g}{A} = \ell_0\,d\,g$$

$$\ell_0 = \frac{\text{Breaking Stress}}{d\,g} = \frac{\left(7.982 \times 10^8\ \text{N/m}^2\right)}{\left(8.1 \times 10^3\ \text{kg/m}^3\right)\left(9.8\ \text{m/s}^2\right)}$$

$$\ell_0 = 0.1005542958 \times 10^5\ \text{m}$$

Exercise Problems: 11.37

A square steel plate is 10.0 cm on a side and 0.500 cm thick.

(a) Find the shear strain that results if a force of magnitude 9×10^5N is applied to each of the four sides, parallel to the side.

(b) Find the displacement x in centimeters.

Solution:

$F = 9.0\times10^5$ N, $A = (0.100\text{ m})(0.500\times10^{-2}\text{ m})$, $h = 0.100$ m, $S = 7.5\times10^{10}$ Pa for steel

(a) Shear strain is

$$\text{Shearing Strain} = \frac{F}{AS} = \frac{\left(9 \times 10^5\ \text{N}\right)}{(0.1\text{ m})(0.5 \times 10^{-2}\text{m})(7.5 \times 10^{10}\text{Pa})} = 2.4 \times 10^{-2}$$

(b) x= (Shear strain).h= $(0.024)(0.100\text{ m}) = 2.4\times10^{-3}$m.