Chapter 14
Periodic Motion

Topic	Conceptual Problems (To be Discussed in Class)	In Class Problems	Assignment Problems
14.2 Simple harmonic Motion, 14.3 Energy in Simple Harmonic Motion	TYU-14.3,	Example-14.4,	Exercise 14.6, 14.8, 14.11

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Simple Harmonic Motion (SHM)

A body is said to execute SHM if it possess:

- To-and – fro motion (Periodic Motion)
- Acceleration is directly proportional to negative of the displacement
- Acceleration is always directed towards the mean position.

For SHM the restoring force F_x acting on the body is directly proportional to its displacement from equilibrium position 'x'.

$F_x = -kx$ Where, k is called force constant

$\Rightarrow m a_x = -kx$

$\Rightarrow m\dfrac{d^2x}{dt^2} = -kx \quad \Rightarrow \dfrac{d^2x}{dt^2} + \dfrac{k}{m}x = 0$

$\Rightarrow \dfrac{d^2x}{dt^2} + \omega_0^2 x = 0$ Where, $\omega_0 = \sqrt{\dfrac{k}{m}}$

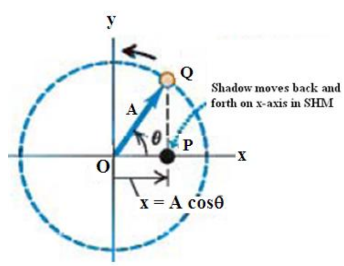
The above equation represents the differential form of SHM.

Solution of the above differential equation is :

$x(t) = A\cos(\omega_0 t + \phi)$

Where, ϕ = phase difference

Displacement



Let a body is moving in a circular path. Its projection represents the SHM. The displacement at any instant is:

$x = A\sin\theta$

Where, A = radius of the circle

Since rotation is uniform, ω is constant, So, $\theta = \omega t$

The displacement can be written as

$x = A\sin\omega t$

Maximum value of displacement (x) is $x = A$, this is called the amplitude.

Velocity

$v_x = \dfrac{dx}{dt} = \dfrac{d}{dt}(A\sin\omega t)$

$v_x = A\omega\cos\omega t$ ------------- (1)

$v_x = A\omega\sqrt{1-\sin^2\omega t} \quad \Rightarrow \quad v_x = A\omega\sqrt{1-\dfrac{x^2}{A^2}}$

$v_x = \omega\sqrt{A^2-x^2}$ ------------- (2)

Equation (1) & (2) represents the velocity of a particle executing simple harmonic motion.
v_x is maximum at x=0. It is given by $|v_{max}| = A\omega$

Acceleration

$$a_x = \frac{dv}{dt} = \frac{d}{dt}\left(A\omega \cos \omega t\right) = -A\omega^2 \sin \omega t$$

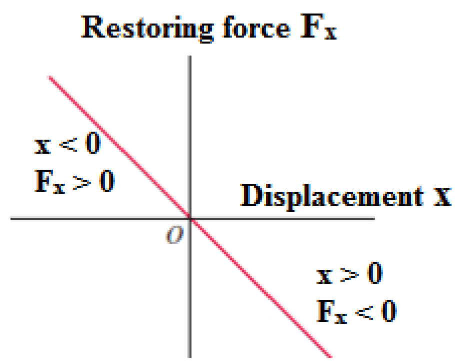
$a_x = -\omega^2 x$

$a_x \propto -x$

Thus the acceleration of a particle executing SHM is directly proportional to the displacement 'x' and the negative sign signifies that the acceleration is always directed towards the mean position.

Angular Frequency of SHM

When a particle executes SHM, the restoring force is given by



$F_x = -k\,x$ where k = force constant

$\Rightarrow m\,a_x = -k\,x$

$\Rightarrow a_x = -\dfrac{k}{m}x$ ------(1)

Again, we know that for the SHM,

$a_x = -\omega^2 x$ -----------(2)

From equation (1) and (2) we get

$$\omega^2 x = \frac{k}{m}x \Rightarrow \omega^2 = \frac{k}{m} \Rightarrow \omega = \sqrt{\frac{k}{m}}$$

ω is the angular speed (circular motion) and the angular frequency of SHM.

Now, frequency and period is given by:

$$f = \frac{\omega}{2\pi} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \text{and} \quad T = \frac{1}{f} \Rightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

Energy of in SHM

Kinetic energy of a particle is given by

$$K = \frac{1}{2}mv_x^2 = \frac{1}{2}m\,A^2\omega^2\cos^2\omega t$$

Potential energy of a particle executing SHM is

$$U = \frac{1}{2}kx^2 = \frac{1}{2}m\,\omega^2\left(A^2\sin^2\omega t\right)$$

Now the total energy is

$E = K + U$

$$\Rightarrow E = \frac{1}{2}m\,A^2\omega^2\cos^2\omega t + \frac{1}{2}m\,\omega^2\left(A^2\sin^2\omega t\right)$$

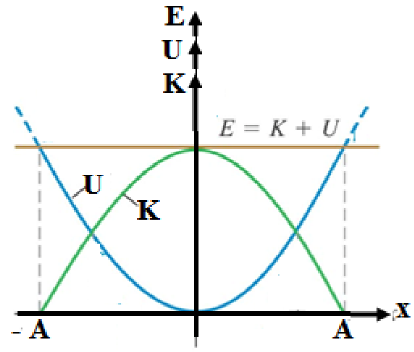
$$\Rightarrow E = \frac{1}{2}m\,\omega^2A^2 = \frac{1}{2}kA^2 = \text{constant}$$

$$\boxed{\begin{aligned} & F_x = -\frac{dU}{dx} \\ \Rightarrow\ & dU = -F_x\,dx \\ \Rightarrow\ & U = \int -F_x\,dx \\ \Rightarrow\ & U = \int -(-k\,x)\,dx \\ \Rightarrow\ & dU = \frac{1}{2}k\,x^2 \end{aligned}}$$

Thus the total energy of the body executing SHM is conserved.

Graphical representation E, K, and U in SHM

Graphical representation of the variation of kinetic energy and potential energy with displacement is shown in the figure.



At $x = \pm A$ the energy is all potential; the kinetic energy is zero.

At $x = 0$ the energy is all kinetic; the potential energy is zero.

The kinetic energy and potential energy changes but the total energy remains constant and is shown by the straight line parallel to displacement axis.

Vertical SHM

Let a body with mass 'm' hanged from a spring with force constant k. It is shown in the figure. The spring is stretched an amount Δl, and the spring's upward vertical force on the body balances its weight 'mg' and the body is in equilibrium. Thus,

$k\,\Delta l = mg$ -----------------(1)

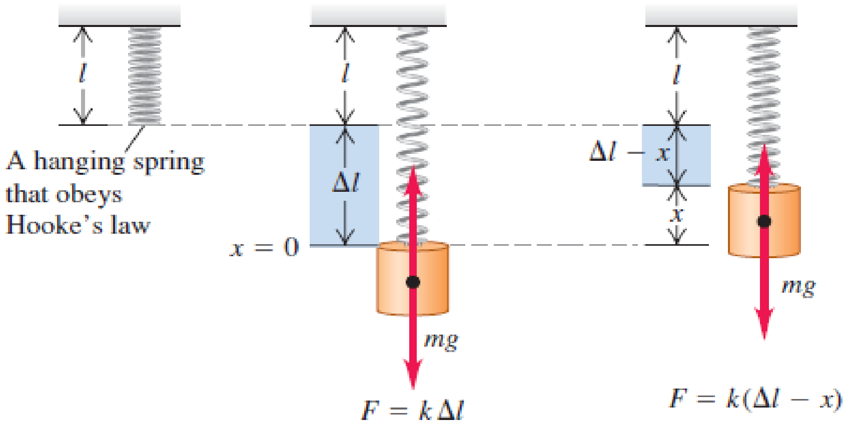
Let the body be displaced from its equilibrium position. The net force acting on the body when it is at a distance $(\Delta l - x)$ above the equilibrium position is

$F = k(\Delta l - x) + (-mg)$

=> $F = k\Delta l - k\,x - mg$

=> $F = mg - k\,x - mg$

=> $F = -k\,x$



Similarly, when the body is below the equilibrium position by $(\Delta l - x)$, then the net upward force on the body is kx.

In either case there is a restoring force with magnitude kx.
Thus, restoring force acting on the body is proportional to its displacement. So the body possess SHM in vertical direction.

Test Your Understanding of Section 14.3

a) To double the total energy for a mass-spring system oscillating in SHM, by what factor must the amplitude increase?
(i) 4; (ii) 2; (iii) 1.414 (iv) 1.189

b) By what factor will the frequency change due to this amplitude increase?
(i) 4; (ii) 2; (iii) 1.414 (iv) 1.189 (v) it does not change

Answers: (a) (iii), (b) (v)

To increase the total energy $E = \frac{1}{2} kA^2$ by a factor of 2, the amplitude 'A' must increase by a factor of $\sqrt{2}$. Because the motion is SHM, changing the amplitude has no effect on the frequency.

Example problem:

Example-14.4: Velocity, acceleration, and energy in SHM

A spring is mounted horizontally, with its left end fixed. A spring balance attached to the free end and pulled toward the right (shown in figure) indicates that the stretching force is proportional to the displacement, and a force of 6.0 N causes a displacement of 0.030 m. We replace the spring balance with a 0.50-kg glider, pull it 0.020 m to the right along a frictionless air track, and release it from rest (shown in figure).

a) Find the maximum and minimum velocities attained by the oscillating glider.

b) Find the maximum and minimum accelerations.

c) Find the velocity v_x and acceleration a_x when the glider is halfway from its initial position to the equilibrium position.

d) Find the total energy, potential energy, and kinetic energy at this position.

Soln:

a) The velocity v_x at any displacement x is

$$v_x = \omega\sqrt{A^2 - x^2}$$

Velocity of the glider will be maximum at x = 0

Thus,

$$v_{max} = \omega\sqrt{A^2 - 0^2} \quad \Rightarrow \quad v_{max} = \omega A \quad \Rightarrow \quad v_{max} = \sqrt{\frac{k}{m}}\,A$$

$$\Rightarrow \quad v_{max} = \sqrt{\frac{200\text{N/m}}{0.5\text{kg}}}\,(0.02\text{m}) = 0.4\text{m/s}$$

Its maximum and minimum (most negative) velocities are +0.40 m/s and - 0.40 m/s which occur when it is moving through x = 0 to the right and left, respectively.

b) The acceleration a_x at any displacement x is

$a_x = -\omega^2 x$

The glider's maximum (most positive) acceleration occurs at the most negative value of x, i.e x = -A

Thus,

$$a_{max} = -\omega^2 (-A) \quad \Rightarrow a_{max} = \omega^2 A \quad \Rightarrow a_{max} = \frac{k}{m} A$$

$$\Rightarrow a_{max} = \frac{200\text{N/m}}{0.5\text{kg}} (0.02\text{m}) = 8.0\text{m/s}^2$$

The minimum (most negative) acceleration occurs at x = +A = +0.020 m

$a_{min} = -8.0 \text{ m/s}^2$

c) The velocity v_x at any displacement x is $v_x = -\omega\sqrt{A^2 - x^2}$

Velocity of the glider at x = A/2 = 0.010 m is

$$v_{A/2} = -\omega\sqrt{A^2 - \left(\frac{A}{2}\right)^2} \quad \Rightarrow v_{A/2} = -\omega\sqrt{\frac{3A^2}{4}} \quad \Rightarrow v_{A/2} = -\sqrt{\frac{k}{m}}\frac{\sqrt{3}A}{2}$$

$$\Rightarrow v_{A/2} = -\sqrt{\frac{200\text{N/m}}{0.5\text{kg}}}\frac{\sqrt{3}(0.02\text{m})}{2} = -0.35 \text{ m/s}$$

The acceleration a_x at any displacement x is $a_x = -\omega^2 x$

The glider's acceleration at x = A/2 is

$$a_{A/2} = -\omega^2\left(\frac{A}{2}\right) \quad \Rightarrow a_{A/2} = -\frac{k}{m}\left(\frac{A}{2}\right)$$

$$\Rightarrow a_{A/2} = -\frac{200\text{N/m}}{0.5\text{kg}}\left(\frac{0.02\text{m}}{2}\right) = -4.0\text{m/s}^2$$

d) The total energy, potential energy, and kinetic energy at x = A/2 are

$$E = \frac{1}{2} k A^2 = \frac{1}{2}(200 \text{ N/m}) (0.02 \text{ m})^2 = 0.04 \text{ J}$$

$$U_{A/2} = \frac{1}{2} k\left(\frac{A}{2}\right)^2 = \frac{1}{2}(200 \text{ N/m}) \left(\frac{0.02 \text{ m}}{2}\right)^2 = 0.01 \text{ J}$$

$$K_{A/2} = \frac{1}{2} m v_{A/2}^2 = \frac{1}{2} (0.5 \text{ kg})(-0.35\text{m/s})^2 = 0.03 \text{ J}$$

Exercise Problems: 14.6, 14.8, 14.11

Exercise- 14.6:

A particle executing simple harmonic motion with a frequency of 1/2π Hz has a peak amplitude of 1.2 cm on either side of the equilibrium position. Determine its velocity and acceleration at a displacement of 0.6 cm.

Solution:

A = 1.2 cm , x = 0.6 cm

$$f = \frac{1}{2\pi}\text{Hz} \quad \text{So,} \quad \omega = 2\pi f = 2\pi\frac{1}{2\pi} = 1.0 \text{ rad/s}$$

$$v_x = -\omega\sqrt{A^2 - x^2} = -(1\text{rad/s})\sqrt{(1.2\text{cm})^2 - (0.6\text{cm})^2} = 1.039\text{cm/s}^2$$

$a_x = -\omega^2 x = -(1\text{rad/s})^2(0.6 \text{ cm}) = -0.6 \text{ cm/s}^2$

Exercise- 14.8

Prove that for a particle is executing simple harmonic motion the kinetic energy will more than its potential energy on just over 70% of its path.

Solution:

$$\text{K.E} = \frac{1}{2}\, m\, v_x^2 = \frac{1}{2}\, m \frac{k}{m}\left[\left(A^2 - x^2\right)\right] = \frac{1}{2}\, k\left[\left(A^2 - x^2\right)\right]$$

$$\text{P.E} = \frac{1}{2}\, k\, x^2$$

The position at which the K.E and P.E will be of equal magnitude is

$$\frac{1}{2}\, k\left[\left(A^2 - x^2\right)\right] = \frac{1}{2}\, kx^2$$

$$\Rightarrow\; A^2 - x^2 = x^2 \qquad \Rightarrow\; A^2 = 2x^2$$

$$\Rightarrow\; x = \frac{A}{\sqrt{2}} = 0.707\ A$$

After this length the K.E will be less than the P.E.

Thus, Percentage of the path length over which the K.E > P.E is 70.7%

Exercise-14.11

A small block is attached to an ideal spring and is moving in SHM on a horizontal, frictionless surface. When the block is at $x = 0.280$ m, the acceleration of the block is -5.30 m/s^2. What is the frequency of the motion?

Solution:

$$a_x = -\omega^2 x \quad \Rightarrow\; a_x = -(2\pi f)^2 x \quad \Rightarrow\; (2\pi f)^2 = -\frac{a_x}{x}$$

$$\Rightarrow\; f = \frac{1}{2\pi}\sqrt{-\frac{a_x}{x}} = \frac{1}{2\pi}\sqrt{-\frac{\left(-5.30 \text{ m/s}^2\right)}{0.280 \text{ m}}} = \frac{1}{2\pi}\sqrt{18.93 \text{ s}^{-2}} = 0.692\text{Hz}$$