Chapter 7

Potential Energy and Energy Conservation

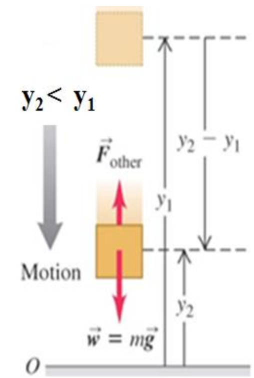
Topic	Conceptual Problems (To be Discussed in Class)	In Class Problems	Assignment Problems
7.1 Gravitational Potential Energy, 7.2 Elastic Potential Energy (Elastic Forces only) Conservation of Mechanical Energy (gravitational forces only),	TYU-7.2	Example-7.7	Exercise 7.14, 7.27, 7.44
7.3 Conservative and Non conservative forces, 7.4 Force and Potential Energy, 7.5 Energy Diagrams	TYU-7.3	Example-7.14	

Gravitational potential energy

Energy associated with a body due to its position is called potential energy. The gravitational potential energy possessed by a body is given by

$U_{grav} = m\,g\,y$ Where, m = mass of the body, y = height

Body falling downward



A body moves downward

When a body falls without air resistance, its gravitational potential energy decreases and the kinetic energy increases.

From work-energy theorem, the kinetic energy of the falling body increases because work is done on the body by the gravitational force of the earth. Here gravity does positive work and gravitational potential energy decrease.

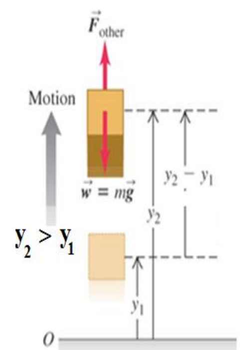
$W_{grav} = m\,g\,(y_1 - y_2)$

$W_{grav} = m\,g\,y_1 - m\,g\,y_2 = \Delta U_{grav}$

As $y_2 - y_1 < 0$

$\Delta U_{grav} < 0$. Thus the gravitational potential energy decreases.

Body moves upward



A body moves upward

When a body moves upward, gravity does negative work and gravitational potential energy increases and kinetic energy decreases.

$W_{grav} = m\,g\,(y_1 - y_2)$

$W_{grav} = m\,g\,y_1 - m\,g\,y_2 = \Delta U_{grav}$

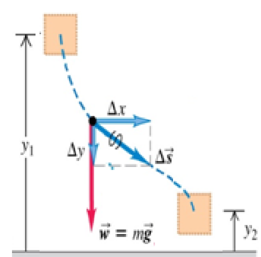
As $y_2 - y_1 > 0$

$\Delta u_{grav} > 0$. Thus the gravitational potential energy increases.

Work and energy along a curved path

We can use the same expression for gravitational potential energy whether the body's path is curved or straight.

The work done by the gravitational force over small segment is:



Work done = (force).(Displacement)

$$W = m\vec{g} \cdot \Delta\vec{s}$$

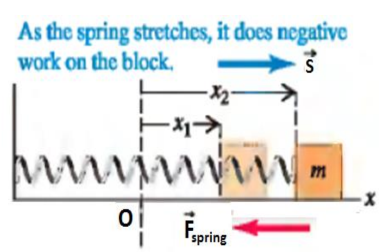
$$W = m\left(g\,\hat{j}\right)\cdot\left(\Delta x\,\hat{i} + \Delta y\,\hat{j}\right) = -m\,g\,\Delta y$$

The work done by the gravitational force depends only on the vertical component of displacement (Δy).

$W_{grav} = -\ mg\,(\Delta y) = mg\,y_1\ -\ mgy_2 = U_{grav,1} - U_{grav,2}$

So even if the path a body follows between two points is curved, the total work done by the gravitational force depends only on the difference in height between the two points of the path.

Elastic potential energy



A body is elastic if it returns to its original shape after being deformed. Elastic potential energy is the energy stored in an elastic body, such as a spring.

The elastic potential energy stored in an ideal spring is

$\mathbf{U_{el} = \frac{1}{2}\,kx^2}$.

Where ‘x’ is the extension or compression of the spring.

Elastic potential energy U_{el} is never negative.

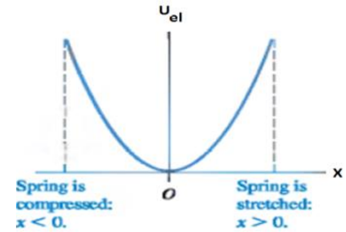


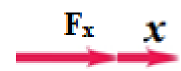
Figure shows a graph of the elastic potential energy for an ideal spring. This is parabolic in nature.

Let a spring be stretched by a distance ‘x’,

The force exerted on it is: $\mathbf{F_x = kx}$,

Where k = force constant of the spring.

If the spring is elongated from $\mathbf{x_1}$ **to** $\mathbf{x_2}$ then some work has to be done. The work done on the spring is given by

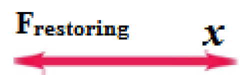


$$W = \int_{x_1}^{x_2} F\, x \cos\theta \, dx \quad \Rightarrow \quad W = \int_{x_1}^{x_2} (k\,x)(1)\, dx \quad \left[\text{Here, } \theta = 0^0\right]$$

$$\Rightarrow \quad W = k \int_{x_1}^{x_2} x \, dx \quad \Rightarrow \quad W = k\left[\frac{x^2}{2}\right]_{x_1}^{x_2}$$

$$W = \frac{1}{2} k\, x_2^2 - \frac{1}{2} k\, x_1^2 \quad \text{(work done on the spring)}$$

Thus work done on the spring is **positive**.



However, the work done by the spring on the spring is due to the restoring force (**$F_{restoring}$ = - kx**) and is **negative, because the restoring force ($F_{restoring}$) acts opposite to the displacement (x).**

Total work done and Energy relation:

In addition to both the gravitational and the elastic forces, other forces such as air resistance acts on a body then the total work W_{tot} is the sum of W_{grav}, W_{el}, and W_{other}. So,

$W_{tot} = W_{grav} + W_{el} + W_{other}$.

The work-energy theorem gives:

$W_{grav} + W_{el} + W_{other} = K_2 – K_1$

But we know that,

$W_{grav} = U_{grav,1} – U_{grav,2}$ and

$W_{el} = U_{el,1} – U_{el,2}$

So, the work-energy theorem become

$U_{grav,1} – U_{grav,2} + U_{el,1} – U_{el,2} + W_{other} = K_2 – K_1$

$K_1 + U_{grav,1} + U_{el,1} + W_{other} = K_2 + U_{grav,2} + U_{el,2}$

$K_1 + U_1 + W_{other} = K_2 + U_2$

If $W_{other} = 0$, then $K_1 + U_1 = K_2 + U_2$ => $K + U = \text{constant}$

E = constant

Thus, total mechanical energy remains constant.

Conservative and non conservative forces

- A conservative force allows conversion between kinetic and potential energy.
- Gravitational force and the spring force are conservative in nature.
- Work is always reversible by conservative forces.

- A body may move from point 1 to point 2 by various paths, but the work done by a conservative force is the same for of these paths.

The work done by a conservative force has following properties:

1. It can be expressed as the difference between the initial and final values of a potential-energy function.
2. It is reversible.
3. It is independent of the path of the body and depends only on the starting and ending points.
4. When the starting and ending points are the same, the total work is zero.
5. The total mechanical energy [E = K + U] is constant when work is done by conservative forces,.
6. A force (such as friction) that is not conservative is called a non conservative force, or a dissipative force.

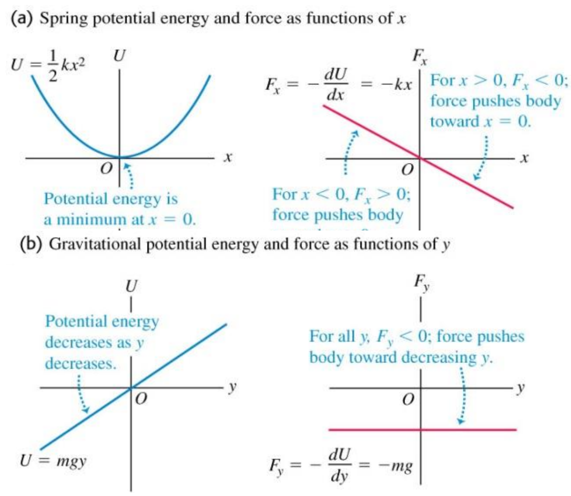
Force and potential energy

Let us consider the motion of a body along a straight line.

$F_x(x)$ = x-component of force

U(x) =. Potential energy

We know that the work done by a **conservative** force equals the negative of the change in potential energy(ΔU):



$W = -\Delta U$

$\Rightarrow \quad F_x \, \Delta x = -\Delta U$

$$\Rightarrow \quad F_x = -\frac{\partial U}{\partial x}$$

For elastic potential energy, we have

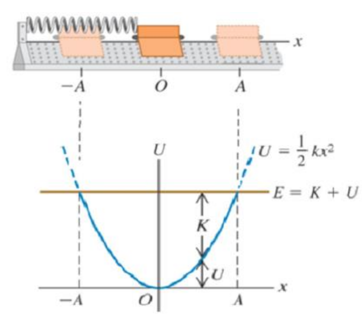
$$U(x) = \frac{1}{2} k x^2 \quad \text{So,} \quad F_x = -\frac{dU}{dx} = -\frac{d}{dx}\left(\frac{1}{2} k x^2\right)$$

$$\Rightarrow \quad F_x = -k x$$

Similarly, for gravitational potential energy we have U(y) = m g y

We get

$$F_y = -\frac{dU}{dy} = -\frac{d}{dy}(mg\, y) \quad \Rightarrow \quad F_y = -m g$$



A glider is attached to a spring. The motion of the spring has a limit between x = A and x = - A.

On the graph, the limits of motion are the points where the U curve intersects the horizontal line representing total mechanical energy E.

The vertical distance between the U and E graph at each point represents the difference (E – U), equal to the kinetic energy K at that point.

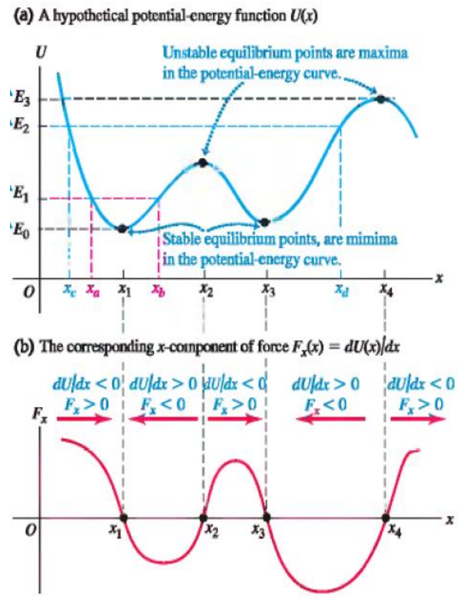
K_{max} is at x = 0, and it is zero at x =   A.

Thus the speed v is greatest at x = 0, and it is zero at x = A,

Let us consider the motion of a body such that its variation of potential with displacement is as shown in the figure.

The slope of the U(x) curve gives: $F_x = -\frac{\partial U}{\partial x}$

- When the particle is at x = 0, the slope and the force are zero, so this is an equilibrium position.
- When x is positive, the slope of the U(x) curve is positive and the force F_x is negative, directed toward the origin.
- When x is negative, the slope is negative and F_x is positive, again toward the origin. Such a force is called a restoring force;
- Any minimum in a potential energy curve is a stable equilibrium position.

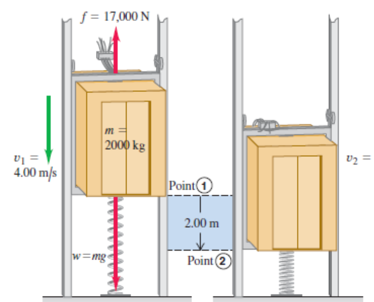


Let's consider that x_1 and x_3 are stable equilibrium points. When the particle is displaced to either side, the force pushes back toward the equilibrium points.
The slope of the U(x) curve is also zero at points x_2 and x_4, and these are also equilibrium points. But when the particle is displaced a little to the either side of both points, the particle tends to move away from the equilibrium. So, points x_2 and x_4 are called unstable equilibrium points; any maximum in a potential-energy curve is an unstable equilibrium position.

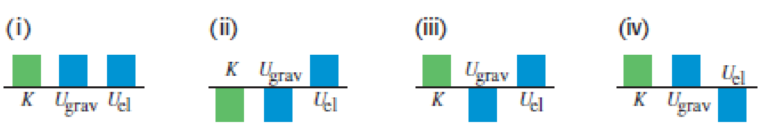
Test your Understanding (TYU)

Test Your Understanding of Section 7.2:

A 2000-kg elevator with broken cables in a test rig is falling at when it contacts a cushioning spring at the bottom of the shaft. The spring is intended to stop the elevator, compressing 2.00 m as it does so. During the motion a safety clamp applies a constant 17,000-N frictional force to the elevator.



Which of the energy bar graphs in the figure most accurately shows the kinetic energy *K*, gravitational potential energy and elastic potential energy U_{el} at the instant when the elevator is still moving downward and the spring is compressed by 1.00 m.



Answer: (iii)
The elevator is still moving downward, so the kinetic energy K is positive (remember that K can never be negative); the elevator is below point 1, so $y < 0$, and $U_{grav} < 0$, the spring is compressed, so $U_{el} > 0$.

Test Your Understanding of Section 7.3:

In a hydroelectric generating station, falling water is used to drive turbines ("water wheels"), which in turn run electric generators. Compared to the amount of gravitational potential energy released by the falling water, how much electrical energy is produced?

(i) the same; (ii) more; (iii) less.

Answer: (iii)

Because of friction in the turbines and between the water and turbines, some of the potential energy goes into raising the temperatures of the water and the mechanism

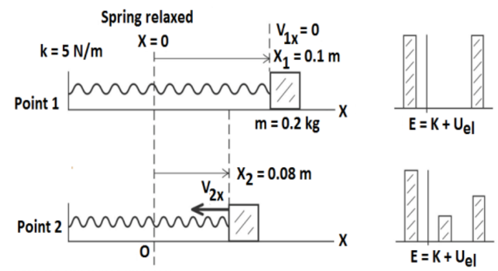
Example Problems: 7.7, 7.14

Example 7.7 : Motion with elastic potential energy

A glider with mass m = 0.2 kg sits on a frictionless horizontal air track, connected to a spring with force constant k = 5 N/m. You pull on the glider, stretching the spring 0.100 m, and release it from rest. The glider moves back toward its equilibrium position (x=0). What is its x-velocity when x = 0.080 m?

Solution:

As the glider starts to move, elastic potential energy is converted to kinetic energy. The glider remains at the same height throughout the motion, so gravitational potential energy is not a factor



So, $U = U_{el} = \frac{1}{2} k\, x^2$

Only the spring force does work on the glider, So $W_{other} = 0$

$x_1 = 0.100$ m; $x_2 = 0.080$ m; $v_{1x} = 0$; $v_{2x} = ?$

$$K_1 = \frac{1}{2} m\, v_{1x}^2 = \frac{1}{2}(12\text{ kg})(0)^2 = 0$$

$$U_1 = \frac{1}{2} k\, x_1^2 = \frac{1}{2}(5\text{ N/m})(0.1\text{m})^2 = 0.025\text{J}$$

$$U_2 = \frac{1}{2} k\, x_2^2 = \frac{1}{2}(5\text{ N/m})(0.08\text{m})^2 = 0.016\text{ J}$$

$$K_2 = K_1 + U_1 - U_2 = 0 + 0.025\text{ J} - 0.016\text{ J} = 0.009\text{ J}$$

$$V_{2x} = \pm\sqrt{\frac{2K_2}{m}} = \pm\sqrt{\frac{2\,(0.09\text{J})}{0.2\text{ kg}}} = \pm 0.3\text{ m/s}$$

We choose the negative root because the glider is moving in the -x-direction. Our answer is $v_{2x} = -0.30$ m/s.

Eventually the spring will reverse the glider's motion, pushing it back in the x-direction (see Fig. 7.13d). The solution $v_{2x} =$ 0.30 m/s. tells us that when the glider passes through x = 0.080 m on this return trip, its speed will be 0.30 m/s, just as when it passed through this point while moving to the left.

Example 7.14 Force and potential energy in two dimensions

A puck with coordinates x and y slides on a level, frictionless air hockey table. It is acted on by a conservative force described by the potential-energy function $U(x,y) = \frac{1}{2} k\,(x^2 + y^2)$. Find a vector expression for the force acting on the puck, and find an expression for the magnitude of the force.

Solution:

$$F_x = -\frac{\partial U}{\partial x} = -kx,\quad F_y = -\frac{\partial U}{\partial y} = -k\,y$$

$$\text{Now, } \vec{F} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j}\right) \quad \Rightarrow \quad \vec{F} = (-kx)\,\hat{i} + (-ky)\,\hat{j}$$

$$\Rightarrow F = \sqrt{(-kx)^2 + (-ky)^2} = k\sqrt{x^2 + y^2} = kr$$

Solution to the Assignment Problems: 7.14, 7.27, 7.44

Exercise-7.14

An ideal spring of negligible mass is 12 cm long when nothing is attached to it. When you hang a 3.15-kg weight from it, you measure its length to be 13.4 cm. If you wanted to store 10 J of potential energy in this spring, what would be its total length? Assume that it continues to obey Hooke's law.

Solution:

Mass of the body = 3.15kg

Original length of the spring = 12cm = 0.12m

Final length of the spring = 13.4cm = 0.314m

Elastic Energy stored = U_{el} = 10 J

$$F_x = kx \quad \Rightarrow \quad k = \frac{F_x}{x} \quad \Rightarrow \quad k = \frac{mg}{x} \quad \Rightarrow \quad x = \frac{(3.15\text{kg})(9.8\text{m/s}^2)}{0.134\text{m} - 0.12\text{m}} = 2205\text{N/m}$$

$$U_{el} = \frac{1}{2} k\, x^2 \quad \Rightarrow \quad x = \pm\sqrt{\frac{2U_{el}}{k}} = \Rightarrow \quad x = \pm\sqrt{\frac{2(10\text{ J})}{2205\text{ N/m}}} = \pm 0.0952\text{m}$$

Exercise-7.27

A 10 kg box is pulled by a horizontal wire in a circle on a rough horizontal surface for which the coefficient of kinetic friction is 0.25. Calculate the work done by friction during one complete circular trip if the radius is (a) 2 m and (b) 4 m. (c) On the basis of the results you just obtained, would you say that friction is a conservative or non-conservative force? Explain.

Solution:

Mass of the box = 10 kg, $\mu_k = 0.25$

For a circular trip the distance traveled is $d = 2\pi r$.

At each point in the motion the friction force and the displacement are in opposite directions.

So, $\varphi = 180°$

Frictional force acting is given by

$f_k = \mu_k\, m\, g$

(a) Now the work done by frictional force is

$$W_{fk} = f_k s \cos\varphi \quad \Rightarrow \quad W_{fk} = (\mu_k\, mg)(2\pi r) s \cos 180^0$$

$$\Rightarrow \quad W_{fk} = -\,(0.25)(10\text{ kg})(9.8\text{m/s}^2)(2\pi)(2\text{m})$$

$$\Rightarrow \quad W_{fk} = -308\text{ J}$$

(b) The distance along the path doubles so the work done doubles and becomes double and is $W_{fk} = -616$ J.

(c) The work done for a round trip displacement is not zero and friction is a non-conservative force. The direction of the friction force depends on the direction of motion of the object and that is why friction is a non-conservative force.

Exercise-7.44

On a horizontal surface; a crate with mass 50 kg is placed against a spring that stores 360 J of energy. The spring is released, and the crate slides 5.6 m before coming to rest. What is the speed of the crate when it is 2 m from its initial position?

Solution:

Mass of the crate = 50 kg

Work done by friction against the crate brings it to a halt: $U_1 = -W_{other}$
Potential energy of compressed spring = $f_k\, x$ = 360 J

$$f_k = \frac{360\ \text{J}}{5.6\ \text{m}} = 64.29\ \text{N}$$

The friction force working over a 2m distance does work equal to
$W = -f_k\, x = -(64.29\text{N})(2\ \text{m}) = -128.6\ \text{J}$.
The kinetic energy of the crate at this point is thus
K.E. = 360 J − 128.6 J = 231.4 J

$$\Rightarrow\ \frac{1}{2}mv^2 = 231.4\ \text{J}$$

$$\Rightarrow\ v = \sqrt{\frac{2(231.4\ \text{J})}{m}} \quad \Rightarrow\ v = \sqrt{\frac{2(231.4\ \text{J})}{50\ \text{kg}}} = 3.04\text{m/s}$$