Chapter 10

Dynamics of Rotational Motion

Topic	Conceptual Problems (To be Discussed in Class)	In Class Problems	Assignment Problems
10.1 Torque 10.2 Torque and angular acceleration for a rigid body 10.3 Rigid body rotation about a moving axis 10.4 Work and Power in Rotational Motion 10.5 Angular momentum 10.6 Conservation of angular momentum	TYU- 10.1 TYU- 10.4 TYU- 10.5	Example-10.1 10.5 10.10	Exercise 10.9, 10.18, 10.39,

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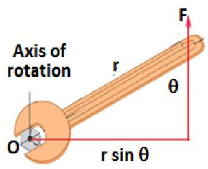
Torque

A force (**F**) **applied at** a distance (r) and at an angle (θ) will generate a torque (τ).
It causes a rotation about O and depends on:

- the magnitude **F**
- the perpendicular distance **r** between point O and the line of force. We call the distance **r** the lever arm of force **F**



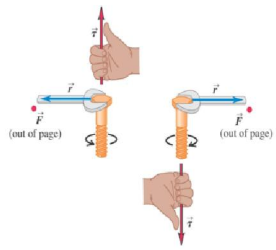
(The lever arm (or moment arm) for a force is the perpendicular distance from O to the line of action of the force)

Torque of the force **F** with respect to **O** is written as.

$\vec{\tau} = \vec{r} \times \vec{F}$

$|\vec{\tau}| = r\,F\,\sin\theta$

Where, F = force
And r = the distance between point O and the point action of force



The direction of torque is perpendicular to both *r* and *F,* which is directed along the axis of rotation, and is determined by right-hand rule.

The SI unit of torque is the **Newton-meter**.

10.2 Torque and Angular Acceleration for a Rigid Body

Let a body is made up of a large number of particles, with masses m_1, m_2, m_3, .. at distances r_1, r_2, r_3, from the axis of rotation. v_1, v_2, v_3, are the respective linear velocities.

Since the body is a rigid, the angular acceleration (α) is constant,

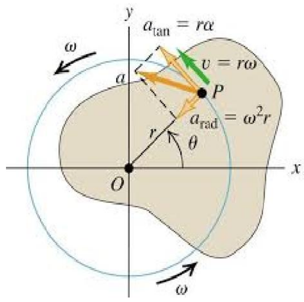
<u>For particle 1</u>
The tangential component of the force is given by:

$$F_{1,tan} = m_1\, a_{1,tan}$$

$$\Rightarrow \quad F_{1,tan} = m_1\,(r_1\,\alpha_z) \qquad (\because \quad a_{1,tan} = r_1\,\alpha_z)$$

$$\Rightarrow \quad F_{1,tan}\, r_1 = m_1\, r_1^2\, \alpha_z$$

$$\Rightarrow \quad \tau_{1z} = m_1\, r_1^2\, \alpha_z$$



For every particle in the body, we add all the torques:

$$\sum \tau_{iz} = m_1 r_1^2 \alpha_z + m_2 r_2^2 \alpha_z + ...$$

$$\sum \tau_{iz} = \left(\sum m_i r_i^2\right)\alpha_z$$

$$\sum \tau_z = I\,\alpha_z \quad \left[\text{where, } I = \sum m_i r_i^2\right]$$

Combined Translation and Rotation: Energy Relationships

Let us consider a wheel rotating about the center of mass (CM).
As the wheel rolls, CM has translational motion and the wheel has rotational motion.
The **kinetic energy** of a rigid body that has both translational and rotational motions is:

Kinetic energy = K.E. of the motion of the center of mass (**½Mv_{cm}^2**) +
K.E of the rotational motion about an axis through the center of mass (½ **$I_{cm}\omega^2$**)

$$\text{Thus,} \quad K = \frac{1}{2} M\, v_{cm}^2 + \frac{1}{2}\, I_{cm}\, \omega^2$$

Rolling without slipping

Condition for rolling without slipping is: $\mathbf{v_{cm} = R\omega}$

Kinetic energy of the rotating system is:

$$K = \frac{1}{2} m\, v_{CM}^2 + \frac{1}{2} I_{CM}\, \omega^2$$

$$\Rightarrow \quad K = \frac{1}{2} m\, v_{CM}^2 + \frac{1}{2} I_{CM} \left(\frac{v_{CM}^2}{R^2} \right)$$

Combined translation and rotation: Dynamics

When a rigid body with total mass M moves, its motion can be described by combining translational motion and rotational motion

In translational motion: $\sum \vec{F}_{ext} = M\, \vec{a}_{cm}$

The rotational motion about the center of mass: $\sum \tau_z = I_{cm}\ \alpha_z$

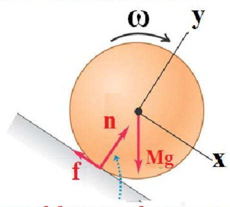
This equation is valid **even when the axis of rotation moves**, provided the following two conditions are met:

a) The axis through the center of mass must be an axis of symmetry.
b) The axis must not change direction.

Rolling Friction

- The figure shows a perfectly rigid sphere and is rolling down a perfectly rigid inclined plane.
- The line of action of the normal force passes through the center of the sphere, so its torque is zero; there is no sliding at the point of contact, so the friction force does no work.
- So, we can ignore rolling friction if both the rolling body and the surface over which it rolls are perfectly rigid.

Perfectly rigid sphere rolling on a perfectly rigid surface

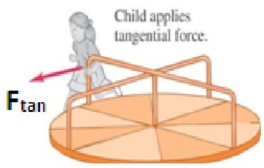


Normal force produces no torque about the center of the sphere

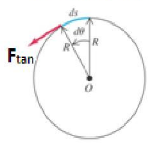
10.4 Work and Power in Rotational Motion

Let dW be the work done by the force F_{tan} while a point on the rim moves a distance ds. Then it is given by:

$dW = F_{tan} \cdot ds$.
If $d\theta$ is measured in radians, then $ds = R \cdot d\theta$



Vertical view of merry-go-round



$$dW = F_{tan}\ R\ d\theta \quad \Rightarrow \quad dW = \tau_z\ d\theta \quad \Rightarrow \quad W = \int_{\theta_1}^{\theta_2} \tau_z\ d\theta = \tau_z\ [\theta]_{\theta_1}^{\theta_2}$$

$$W = \tau_z\ (\theta_2 - \theta_1) = \tau_z\ \Delta\theta$$

Work done by a constant torque

$$W = \tau_z\ (\theta_2 - \theta_1) = \tau_z\ \Delta\theta$$

The work done by a constant torque is the product of torque and the angular displacement.

If torque is expressed in Newton-meters and angular displacement in radian, the work is in joules.
Only the tangent component of force does work, other components do no work.

When a torque does work on a rotating rigid body, the kinetic energy changes by an amount equal to the work done.

$$W_{tot} = \frac{1}{2} I \left(\omega_2^2 - \omega_1^2\right)$$

Power is the rate of doing work

$$P = \frac{dW}{dt} = \frac{\tau_z \, d\theta}{dt} = \tau_z \frac{d\theta}{dt} = \tau_z \, \omega$$

When a torque acts on a body, it rotates with angular velocity ω_z. Its power is the product of τ_z and ω_z. This is the analog of the relationship: $P = F \cdot v$

10.5 Angular Momentum

Angular momentum of a particle is a vector quantity denoted as L.

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

The value of L depends on the choice of origin O, since it involves the particle's position vector relative to O.

The unit of angular momentum are $kg \cdot m^2/s$.

L is perpendicular to the plane which contains r and v. It direction is determined by the right-hand rule.

The magnitude of L is

$$L = r\, p \sin\phi \quad \text{or,} \quad L = r\, mv \sin\phi$$

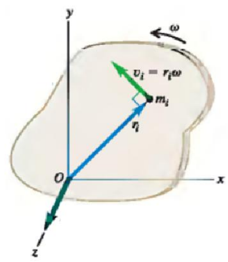
When the torque of the net force acts on a particle, its angular velocity as well as angular momentum changes:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} \quad \Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}_{net} \quad \Rightarrow \quad \frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

Angular Momentum of a Rigid Body

- When a rigid body rotates around an axis, each particle moves in a circular path centered at the origin,
- At each instant its velocity v_i is perpendicular to its position vector r_i.
- Hence $\phi = 90^o$ for every particle.
- A particle with mass m_i at a distance r_i from O has a speed $v_i = r_i\omega$.



The magnitude of its angular momentum is:

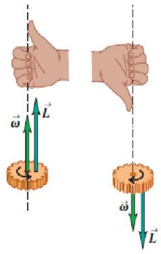
$$L_i = r_i\,(m_i\, v_i \sin\Phi) = r_i\,(m_i\, r_i\, \omega) = m_i\, r_i^2\, \omega$$

The total angular momentum is the sum $\sum L_i$ of the angular momenta L_i of the particles:

$$L = \sum L_i = \left(\sum m_i\, r_i^2\right) \omega = I\, \omega$$

Direction of Angular Momentum

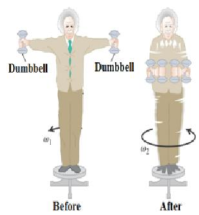
For rotation about an axis of symmetry, ω and L are parallel and along the axis. The directions of both vectors are given by the right-hand rule.



The right hand thumb points in the direction of ω. If the rotation axis is an axis of symmetry, this is also the direction of L.

10.6 Conservation of Angular Momentum

If net torque applied on a body is zero, then total angular momentum of the body remains conserved.



If, $\vec{\tau}_{net} = 0$

$$\Rightarrow \frac{d\vec{L}}{dt} = 0$$

=> L = constant.

=> I ω = constant.

A circus acrobat, a diver, and an ice skater all take advantage of this principle.

$I_1\omega_{1z} = I_2\omega_{2z}$

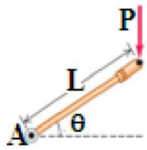
Thus, when moment of inertia (I) decreases, the angular velocity of the body increases and vice versa.

Comparison between linear and rotational dynamics

Linear	Rotational dynamics of a rigid object
$\vec{F} = m\,\vec{a}\ \ (\text{if } m = \text{constant})$	$\vec{\tau} = I\,\vec{\alpha}\ \ (\text{if } I = \text{constant})$
$\vec{p} = m\,\vec{v}$	$\vec{L} = I\,\vec{\omega}\ \left(\begin{array}{l}\text{valid for rotation}\\ \text{about a symmetry axis}\end{array}\right)$
$\vec{F} = \frac{d\vec{p}}{dt}$	$\vec{\tau} = \frac{d\vec{L}}{dt}$
For a system of object: If net $\vec{F}_{external} = 0$ Then, $\vec{p}_{total}$ is conserved	If net $\vec{\tau}_{external} = 0$ Then, $\vec{L}_{total}$ is conserved

Test Your Understanding: 10.1, 10.4, 10.5

Test Your Understanding 10.1



The figure shows a force *P* being applied to one end of a lever of length *L*. What is the magnitude of the torque of this force about point *A*?
(i) PLsin θ ; (ii) PLcos θ; (iii) PLtan θ.

Answer: (ii)

The force *P* acts along a vertical line, so the lever arm is the horizontal distance from *A* to the line of action. This is the horizontal component of the distance *L*, which is Lcosθ.
Hence the magnitude of the torque is the product of the force magnitude *P* and the lever arm Lcosθ i.e $\tau = P\,L\cos\theta$

Test Your Understanding 10.4

You apply equal torques to two different cylinders, one of which has a moment of inertial twice as large as the other cylinder. Each cylinder is initially at rest after one complete rotation, which cylinder has the greater kinetic energy?

i) The cylinder with the larger moment of inertia;
ii) The cylinder with the smaller moment of inertia;
iii) Both cylinders have the same kinetic energy.

Answer: (iii)

You apply the same torque over the same angular displacement to both cylinders. Hence, by $W = \tau\,\Delta\theta$, the same amount of work is done to both cylinders and impart the same kinetic energy to both. (The one with the smaller moment of inertia ends up with a greater angular speed, but that isn't what we are asked. Compare Conceptual Example 6.5 in Section 6.2.)

Test Your Understanding 10.5

A ball is attached to one end of a piece of string. You hold the other end of the string and whirl the ball in a circle around your hand.

a) If the ball moves at a constant speed, is its linear momentum $\vec{p}$ constant? Why or why not?
b) Is its angular momentum $\vec{L}$ constant? why or why not?

Answers: (a) no, (b) yes

1) As the ball goes around the circle:

- the magnitude of $\vec{p} = m\,\vec{v}$ remains the same (the speed is constant)
- but its direction changes,

So the linear momentum vector isn't constant.

- Again the linear momentum changes because there is a net force **F** on the ball (toward the center of the circle).

2) $\vec{L} = \vec{r} \times \vec{p}$ is constant: Because:

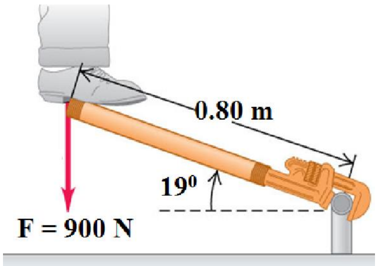
- It has a constant magnitude (the speed and the perpendicular distance from your hand to the ball are both constant) and
- Also it has a constant direction (along the rotation axis, perpendicular to the plane of the ball's motion).
- The angular momentum remains constant because $\boldsymbol{\tau}_{net} = \mathbf{0}$

The vector **r** points from your hand to the ball and the force **F** on the ball is directed toward your hand. So $\vec{\tau} = \vec{r} \times \vec{F} = 0$

Example Problems: 10.1, 10.5, 10.10

Example: 10.1: Applying a torque

To choosen a pipe fitting, a weekend plumber slips a piece of scrap pipe (a "cheater") over his wrench handle. He stands on the end of the cheater, applying his full 900 N weight at a point 0.80 m from the center of the fitting . The wrench handle and cheater makes an angle of 19^0 with the horizontal. Find the magnitude and direction of the torque he applies about the center of the fitting.

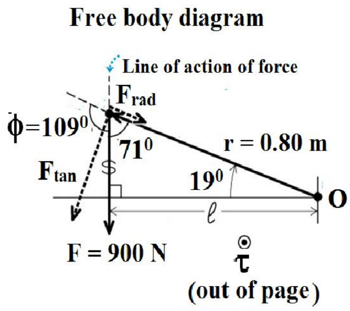


Soln:

ℓ = lever arm ($\perp^r$ distance from axis of rotation to the line of action of force)

Lever arm ℓ is given by

$\ell = r\cos\phi = (0.80\text{ m})\cos 19° = 0.76\text{ m}$



Magnitude of the torque is

$\tau = F\,\ell = (900\text{ N})(0.76\text{ m}) = 680.77\text{ N . m}$

The angle between F_{tan} and F is 19^0.

So, $F_{tan} = F\,(\cos 19°)$

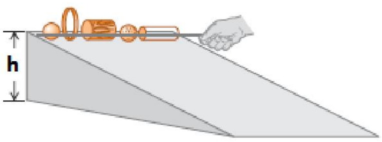
$F_{tan} = (900\text{ N})\,(\cos 19°) = 851\text{ N}.$

$\tau = F_{tan}\,r = (851\text{ N})\,(0.80\text{ m}) = 680\text{ N .m}$

τ points out of the plane (using right hand rule).

Example 10.5: Race of the rolling bodies

In a physics demonstration, an instructor "races" various bodies that roll without slipping from rest down an inclined plane (Figure). What shape should a body have to reach the bottom of the incline first?



Soln:

Kinetic friction does no work if the bodies roll without slipping.

We can also ignore the effects of rolling friction, if the bodies and the surface of the incline are rigid.

We can therefore use conservation of energy.

Each body starts from rest at the top of an incline with height h,

So, $K_1 = 0$, $U_1 = M\,g\,h$, and $U_2 = 0$

$$K_1 + U_1 = K_2 + U_2$$

$$0 + M\,g\,h = \frac{1}{2}M\,v^2 + \frac{1}{2}\left(cM\,R^2\right)\left(\frac{v}{R}\right)^2 + 0$$

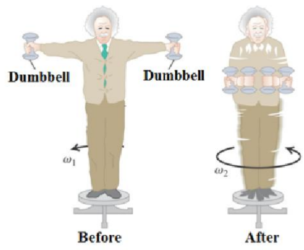
$$M\,g\,h = \frac{1}{2}(1+c)M\,v^2 \quad\Rightarrow\quad v = \sqrt{\frac{2g\,h}{1+c}}$$

For a given value of *c*, the speed after descending a distance *h* is *independent* of the body's mass *M* and radius *R*.

Hence all uniform solid cylinders have the same speed at the bottom, regardless of their mass and radii.

Example - 10.10: Anyone can be a ballerina

A person stands at the center of a frictionless turntable with arms outstretched and a 5.0-kg dumbbell in each hand (Fig.). He is set rotating about the vertical axis, making one revolution in 2.0 s. Find his final angular velocity if he pulls the dumbbells in to his stomach. His moment of inertia (without the dumbbells) is 3 $kg.m^2$ with arms outstretched and 2.2 $kg.m^2$ with his hands at his stomach. The dumbbells are 1.0 m from the axis initially and 0.20 m at the end.



Soln:

$M_{dumbbell}$ = 5.0-kg

Initial angular velocity = ω_1 = 1 rev./2.0 s = 0.5 rev/s

r = perpendicular distance from the axis to the dumbbell

$r_1 = 1.0$ m and $r_2 = 0.20$ m

I_1 = Initial moment of inertia = $I_{person} + I_{dumbbell}$

$I_1 = (3\ kg.m^2) + 2\ (M_{dumbbell}\ r^2) = (3\ kg.m^2) + 2\ (5.0\text{-}kg)(1.0\ m)^2$

$I_1 = 13\ kg.m^2$

$I_2 = 2.2\ kg.m^2 + 2\ (5.0\ kg)(0.2\ m)^2 = 4.2\ kg.m^2$

Applying conservation of angular momentum we get

$I_1\ \omega_1 = I_2\ \omega_2$

$$\omega_2 = \frac{I_1}{I_2}\omega_1 = \left(\frac{13\ kg.m^2}{2.6\ kg.m^2}\right)\omega_1 = 5\ \omega_1 = 5\ (0.5\ rev/s) = 2.5\ rev/s$$

Exercise Problems: 10.9, 10.18, 10.39

Exercise Problem 10.9

The flywheel of an engine has moment of inertia 2.50 kg.m^2 about its rotation axis. What constant torque is required to bring it up to an angular speed of 400 rev/min in 8.00 s, starting from rest?

Soln:

Moment of inertia = I = 2.50 kg.m^2

$$f = 400 \text{ rev/min} = \frac{400}{60} \text{ rev/s} = 6.667 \text{ rev/s}$$

$$\omega_z = 2\pi f = 2\pi \times 6.667 \text{ rad/s} = 41.9 \text{ rad/s}$$

$$\Sigma\tau_z = I\alpha_z = I\left(\frac{\omega_z - \omega_{0z}}{t}\right) = \left(2.5 \text{ kg.m}^2\right)\left(\frac{41.9 \text{ rad/s} - 0}{8 \text{ s}}\right) = 13.1 \text{ N.m}$$

Exercise Problem10.18

We can roughly model a gymnastic tumbler as a uniform solid cylinder of mass 75 kg and diameter 1.0 m. If this tumbler rolls forward at0.5 rev/s, (a) how much total kinetic energy does he have, and (b) what percent of his total kinetic energy is rotational?

Soln:

(a) $K = \frac{1}{2} m v_{CM}^2 + \frac{1}{2} I_{CM}\, \omega^2$

v_{cm}= Rω = (0.50 m)(3.14 rad/s)=1.57 m/s.

K_{cm} = ½ (75 kg)(1.57 m/s)^2 = 92.4 J.

$$K_{rot} = \frac{1}{2} I_{CM}\, \omega^2 = \frac{1}{2}\left(\frac{1}{2} M R^2\right)\omega^2 = \frac{1}{4} M R^2 \omega^2 = \frac{1}{4} v_{CM}\omega^2$$

$$K_{rot} = \frac{1}{4} v_{CM}\omega^2 = 46.2 \text{ J}$$

K_{tot}=92.4 J+46.2 J=140 J

(b) $\frac{K_{rot}}{K_{tot}} = \frac{46.2 \text{ J}}{140 \text{ J}} = 33\%$

Exercise Problem10.39

Find the magnitude of the angular momentum of the second hand on a clock about an axis through the center of the clock face. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Take the second hand to be a slender rod rotating with constant angular velocity about one end.

Soln:

$\omega_{\text{second hand}}$ = 1 rev/min = 1.00 (2π rad/60 s) = 0.1047 rad/s.

Moment of inertia of a slender rod, with the axis about one end is,

I = (1/3) M L^2 = (1/3) (6.0 x 10^{-3} kg)(0.15 m)^2

I = 4.5 x10^{-5} kg·m^2 .

Then,

L = I ω = (4.5 x 10^{-5} kg·m^2)(0.1047 rad/s) = 4.71 x 10^{-6} kg·m^2/s.

Direction of **L** is clockwise.