CIE

Further Pure Mathematics 1

分类真题

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Chapter 1

Roots of Polynomial Equations

Q1: 9231/11/S20

2 The cubic equation $6x^3 + px^2 - 3x - 5 = 0$, where p is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are α^2, β^2, γ^2. [3]

(b) It is given that $\alpha^2 + \beta^2 + \gamma^2 = 2(\alpha + \beta + \gamma)$.

(i) Find the value of p. [3]

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. [2]

Q2: 9231/13/S20

1 The cubic equation $7x^3 + 3x^2 + 5x + 1 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^{-1}, \beta^{-1}, \gamma^{-1}$. [3]

(b) Find the value of $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$. [2]

(c) Find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$. [2]

Q3: 9231/12/W20

1 The cubic equation $x^3 + bx^2 + cx + d = 0$, where b, c and d are constants, has roots α, β, γ. It is given that $\alpha\beta\gamma = -1$.

(a) State the value of d. [1]

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(b) Find a cubic equation, with coefficients in terms of b and c, whose roots are $\alpha + 1$, $\beta + 1$, $\gamma + 1$. [3]

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(c) Given also that $\gamma + 1 = -\alpha - 1$, deduce that $(c - 2b + 3)(b - 3) = b - c$. [4]

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Q4: 9231/11/S21

3 The equation $x^4 - 2x^3 - 1 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\alpha^3, \beta^3, \gamma^3, \delta^3$ and state the value of $\alpha^3 + \beta^3 + \gamma^3 + \delta^3$. [4]

(b) Find the value of $\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\gamma^3}+\frac{1}{\delta^3}$. [3]

(c) Find the value of $\alpha^4+\beta^4+\gamma^4+\delta^4$. [2]

Q5: 9231/13/S21

2 The cubic equation $2x^3 - 4x^2 + 3 = 0$ has roots α, β, γ. Let $S_n = \alpha^n + \beta^n + \gamma^n$.

(a) State the value of S_1 and find the value of S_2. [3]

(b) **(i)** Express S_{n+3} in terms of S_{n+2} and S_n. [1]

(ii) Hence, or otherwise, find the value of S_4. [2]

(c) Use the substitution $y = S_1 - x$, where S_1 is the numerical value found in part **(a)**, to find and simplify an equation whose roots are $\alpha+\beta$, $\beta+\gamma$, $\gamma+\alpha$. [3]

(d) Find the value of $\dfrac{1}{\alpha+\beta}+\dfrac{1}{\beta+\gamma}+\dfrac{1}{\gamma+\alpha}$. [2]

Q6: 9231/11/W21

1 It is given that

$$\alpha+\beta+\gamma=3, \quad \alpha^2+\beta^2+\gamma^2=5, \quad \alpha^3+\beta^3+\gamma^3=6.$$

The cubic equation $x^3+bx^2+cx+d=0$ has roots α, β, γ.

Find the values of b, c and d. [6]

Q7: 9231/11/S22

4 The cubic equation $2x^3 + 5x^2 - 6 = 0$ has roots α, β, γ.

(a) Find a cubic equation whose roots are $\frac{1}{\alpha^3}, \frac{1}{\beta^3}, \frac{1}{\gamma^3}$. [3]

(b) Find the value of $\frac{1}{\alpha^6} + \frac{1}{\beta^6} + \frac{1}{\gamma^6}$. [3]

(c) Find also the value of $\frac{1}{\alpha^9}+\frac{1}{\beta^9}+\frac{1}{\gamma^9}$. [2]

Q8: 9231/11/W22

1 The cubic equation $x^3 + bx^2 + d = 0$ has roots α, β, γ, where $\alpha = \beta$ and $d \neq 0$.

(a) Show that $4b^3 + 27d = 0$. [5]

(b) Given that $2\alpha^2 + \gamma^2 = 3b$, find the values of b and d. [3]

Q9: 9231/12/W22

2 The equation $x^4 + 3x^2 + 2x + 6 = 0$ has roots $\alpha, \beta, \gamma, \delta$.

(a) Find a quartic equation whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}, \frac{1}{\delta^2}$ and state the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2}$.

[4]

(b) Find the value of $\beta^2\gamma^2\delta^2+\alpha^2\gamma^2\delta^2+\alpha^2\beta^2\delta^2+\alpha^2\beta^2\gamma^2$. [3]

(c) Find the value of $\frac{1}{\alpha^4}+\frac{1}{\beta^4}+\frac{1}{\gamma^4}+\frac{1}{\delta^4}$. [2]

Chapter 2

Rational Functions

Q1: 9231/11/S20

1 Let a be a positive constant.

(a) Sketch the curve with equation $y = \frac{ax}{x+7}$. [2]

(b) Sketch the curve with equation $y = \left|\frac{ax}{x+7}\right|$ and find the set of values of x for which $\left|\frac{ax}{x+7}\right| > \frac{a}{2}$.

[4]

Q2: 9231/11/S20

3 The curve C has equation $y = \dfrac{x^2}{2x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [3]

(c) Sketch *C*. [3]

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Q3: 9231/13/S20

6 The curve C has equation $y = \dfrac{10 + x - 2x^2}{2x - 3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that C has no turning points. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{10+x-2x^2}{2x-3}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{10+x-2x^2}{2x-3}\right| < 4$. [6]

Q4: 9231/11/W20

6 The curve C has equation $y = \dfrac{x^2 + x - 1}{x - 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 5$. [4]

(c) Find the coordinates of the intersections of C with the axes, and sketch C. [3]

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(d) Sketch the curve with equation $y = \left| \dfrac{x^2 + x - 1}{x - 1} \right|$. [2]

Q5: 9231/12/W20

6 Let a be a positive constant.

(a) The curve C_1 has equation $y = \dfrac{x-a}{x-2a}$. [2]

Sketch C_1.

The curve C_2 has equation $y = \left(\dfrac{x-a}{x-2a}\right)^2$. The curve C_3 has equation $y = \left|\dfrac{x-a}{x-2a}\right|$.

(b) (i) Find the coordinates of any stationary points of C_2. [3]

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(ii) Find also the coordinates of any points of intersection of C_2 and C_3. [3]

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(c) Sketch C_2 and C_3 on a single diagram, clearly identifying each curve. Hence find the set of values of x for which $\left(\frac{x-a}{x-2a}\right)^2 \leqslant \left|\frac{x-a}{x-2a}\right|$. [5]

Q6: 9231/11/S21

7 The curve C has equation $y = \dfrac{x^2 + x + 9}{x + 1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x^2+x+9}{x+1}\right|$ and find the set of values of x for which $2\left|x^2+x+9\right| > 13\left|x+1\right|$. [5]

Q7: 9231/13/S21

7 The curve C has equation $y = \dfrac{x^2 - x - 3}{1 + x - x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [3]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{x^2 - x - 3}{1 + x - x^2}\right|$ and find in exact form the set of values of x for which $\left|\dfrac{x^2 - x - 3}{1 + x - x^2}\right| < 3$. [6]

Q8: 9231/11/W21

7 The curve C has equation $y = \dfrac{4x+5}{4-4x^2}$.

(a) Find the equations of the asymptotes of C. [2]

(b) Find the coordinates of any stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

(d) Sketch the curve with equation $y = \left|\dfrac{4x+5}{4-4x^2}\right|$ and find in exact form the set of values of x for which $4|4x+5| > 5|4-4x^2|$. [6]

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Q9: 9231/12/W21

6 The curve C has equation $y = \frac{x^2}{x-3}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $0 < y < 12$. [4]

(c) Sketch C. [2]

(d) **(i)** Sketch the graphs of $y = \left|\dfrac{x^2}{x-3}\right|$ and $y = |x| - 3$ on a single diagram, stating the coordinates of the intersections with the axes. [4]

(ii) Use your sketch to find the set of values of c for which $\left|\dfrac{x^2}{x-3}\right| \leqslant |x| + c$ has no solution. [1]

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Q10: 9231/11/S22

5 The curve C has equation $y = \dfrac{2x^2 - x - 1}{x^2 + x + 1}$.

(a) Show that C has no vertical asymptotes and state the equation of the horizontal asymptote of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of the intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right|$ and state the set of values of k for which $\left|\dfrac{2x^2 - x - 1}{x^2 + x + 1}\right| = k$ has 4 distinct real solutions. [2]

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Q11: 9231/13/S22

1 **(a)** Sketch the curve with equation $y = \dfrac{x+1}{x-1}$. [2]

(b) Sketch the curve with equation $y = \dfrac{|x|+1}{|x|-1}$ and find the set of values of x for which $\dfrac{|x|+1}{|x|-1} < -2$. [4]

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Q12: 9231/13/S22

3 A curve C has equation $y = \dfrac{ax^2 + x - 1}{x - 1}$, where a is a positive constant.

(a) Find the equations of the asymptotes of C. [3]

(b) Show that there is no point on C for which $1 < y < 1 + 4a$. [4]

(c) Sketch C. You do not need to find the coordinates of the intersections with the axes. [3]

Q13: 9231/11/W22

7 The curve C has equation $y = \dfrac{5x^2}{5x-2}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the coordinates of the stationary points on C. [4]

(c) Sketch C. [3]

(d) Sketch the curve with equation $y = \left|\frac{5x^2}{5x-2}\right|$ and find in exact form the set of values of x for which $\left|\frac{5x^2}{5x-2}\right| < 2$. [6]

Q14: 9231/12/W22

7 The curve C has equation $y = \dfrac{x^2 - x}{x+1}$.

(a) Find the equations of the asymptotes of C. [3]

(b) Find the exact coordinates of the stationary points on C. [4]

(c) Sketch C, stating the coordinates of any intersections with the axes. [3]

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(d) Sketch the curve with equation $y = \left|\frac{x^2 - x}{x+1}\right|$ and find in exact form the set of values of x for which $\left|\frac{x^2 - x}{x+1}\right| < 6$. [5]

Chapter 3

Summation of Series

Q1: 9231/11/S20

4 **(a)** By first expressing $\frac{1}{r^2-1}$ in partial fractions, show that

$$\sum_{r=2}^{n} \frac{1}{r^2-1} = \frac{3}{4} - \frac{an+b}{2n(n+1)},$$

where a and b are integers to be found. [5]

(b) Deduce the value of $\sum_{r=2}^{\infty} \frac{1}{r^2-1}$. [1]

(c) Find the limit, as $n \to \infty$, of $\sum_{r=n+1}^{2n} \frac{n}{r^2-1}$. [4]

Q2: 9231/13/S20

3 Let $S_n = 2^2 + 6^2 + 10^2 + \ldots + (4n-2)^2$.

(a) Use standard results from the List of Formulae (MF19) to show that $S_n = \frac{4}{3}n(4n^2 - 1)$. [4]

(b) Express $\frac{n}{S_n}$ in partial fractions and find $\sum_{n=1}^{N} \frac{n}{S_n}$ in terms of N. [4]

(c) Deduce the value of $\sum_{n=1}^{\infty} \frac{n}{S_n}$. [1]

Q3: 9231/11/W20

2 **(a)** Use standard results from the List of Formulae (MF19) to show that

$$\sum_{r=1}^{n}(7r+1)(7r+8) = an^3 + bn^2 + cn,$$

where a, b and c are constants to be determined. [3]

(b) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(7r+1)(7r+8)}$ in terms of n. [4]

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Q4: 9231/12/W20

3 **(a)** By simplifying $\left(x^n - \sqrt{x^{2n}+1}\right)\left(x^n + \sqrt{x^{2n}+1}\right)$, show that $\dfrac{1}{x^n - \sqrt{x^{2n}+1}} = -x^n - \sqrt{x^{2n}+1}$. [1]

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Let $u_n = x^{n+1} + \sqrt{x^{2n+2}+1} + \dfrac{1}{x^n - \sqrt{x^{2n}+1}}$.

(b) Use the method of differences to find $\sum_{n=1}^{N} u_n$ in terms of N and x. [3]

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(c) Deduce the set of values of x for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

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Q5: 9231/11/S21

2 **(a)** Use standard results from the List of formulae (MF19) to find $\sum_{r=1}^{n}\left(1-r-r^2\right)$ in terms of n, simplifying your answer. [3]

(b) Show that

$$\frac{1-r-r^2}{(r^2+2r+2)(r^2+1)} = \frac{r+1}{(r+1)^2+1} - \frac{r}{r^2+1}$$

and hence use the method of differences to find $\sum_{r=1}^{n} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$. [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1-r-r^2}{(r^2+2r+2)(r^2+1)}$. [1]

Q6: 9231/13/S21

1 **(a)** Show that

$$\tan(r+1) - \tan r = \frac{\sin 1}{\cos(r+1)\cos r}.$$ [2]

Let $u_r = \dfrac{1}{\cos(r+1)\cos r}$.

(b) Use the method of differences to find $\sum_{r=1}^{n} u_r$. [3]

(c) Explain why the infinite series $u_1 + u_2 + u_3 + \ldots$ does not converge. [1]

Q7: 9231/11/W21

2 **(a)** Use standard results from the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer. [3]

(b) Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions and hence use the method of differences to find

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}.$$ [5]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$. [1]

Q8: 9231/12/W21

3 Let $S_n = \sum_{r=1}^{n} \ln \frac{r(r+2)}{(r+1)^2}$.

(a) Using the method of differences, or otherwise, show that $S_n = \ln \frac{n+2}{2(n+1)}$. [4]

Let $S = \sum_{r=1}^{\infty} \ln \frac{r(r+2)}{(r+1)^2}$.

(b) Find the least value of n such that $S_n - S < 0.01$. [3]

Q9: 9231/12/W21

4 The cubic equation $x^3 + 2x^2 + 3x + 3 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [2]

(b) Show that $\alpha^3 + \beta^3 + \gamma^3 = 1$. [2]

(c) Use standard results from the list of formulae (MF19) to show that

$$\sum_{r=1}^{n}\left((\alpha+r)^3+(\beta+r)^3+(\gamma+r)^3\right)=n+\tfrac{1}{4}n(n+1)\left(an^2+bn+c\right),$$

where a, b and c are constants to be determined. [6]

Q10: 9231/11/S22

1 Let a be a positive constant.

(a) Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(ar+1)(ar+a+1)}$ in terms of n and a. [4]

(b) Find the value of a for which $\sum_{r=1}^{\infty} \frac{1}{(ar+1)(ar+a+1)} = \frac{1}{6}$. [3]

Q11: 9231/13/S22

4 Let $u_r = e^{rx}(e^{2x} - 2e^x + 1)$.

(a) Using the method of differences, or otherwise, find $\sum_{r=1}^{n} u_r$ in terms of n and x. [3]

(b) Deduce the set of non-zero values of x for which the infinite series

$$u_1 + u_2 + u_3 + \ldots$$

is convergent and give the sum to infinity when this exists. [3]

(c) Using a standard result from the list of formulae (MF19), find $\sum_{r=1}^{n} \ln u_r$ in terms of n and x. [3]

Q12: 9231/11/W22

3 **(a)** By considering $(2r+1)^3-(2r-1)^3$, use the method of differences to prove that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$ [5]

Let $S_n = 1^2 + 3\times 2^2 + 3^2 + 3\times 4^2 + 5^2 + 3\times 6^2 + \ldots + \left(2+(-1)^n\right)n^2$.

(b) Show that $S_{2n} = \frac{1}{3}n(2n+1)(an+b)$, where a and b are integers to be determined. [3]

(c) State the value of $\lim\limits_{n\to\infty} \dfrac{S_{2n}}{n^3}$. [1]

Q13: 9231/12/W22

1 **(a)** Use the list of formulae (MF19) to find $\sum_{r=1}^{n} r(r+2)$ in terms of n, simplifying your answer. [2]

(b) Express $\frac{1}{r(r+2)}$ in partial fractions and hence find $\sum_{r=1}^{n} \frac{1}{r(r+2)}$ in terms of n. [4]

(c) Deduce the value of $\sum_{r=1}^{\infty} \frac{1}{r(r+2)}$. [1]

Chapter 4

Matrices

Q1: 9231/13/S20

4 The matrix **A** is given by

$$\mathbf{A} = \begin{pmatrix} k & 0 & 2 \\ 0 & -1 & -1 \\ 1 & 1 & -k \end{pmatrix},$$

where k is a real constant.

(a) Show that **A** is non-singular. [3]

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The matrices **B** and **C** are given by

$$\mathbf{B} = \begin{pmatrix} 0 & -3 \\ -1 & 3 \\ 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} -3 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

It is given that $\mathbf{CAB} = \begin{pmatrix} -2 & -\frac{3}{2} \\ -1 & -\frac{3}{2} \end{pmatrix}$.

(b) Find the value of k. [3]

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(c) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

Q2: 9231/11/W20

1 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$, where a and b are positive constants.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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The unit square in the x-y plane is transformed by **M** onto parallelogram $OPQR$.

(b) Find, in terms of a and b, the matrix which transforms parallelogram $OPQR$ onto the unit square. [2]

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It is given that the area of $OPQR$ is $2\,\text{cm}^2$ and that the line $x+3y=0$ is invariant under the transformation represented by **M**.

(c) Find the values of a and b. [5]

Q3: 9231/11/W20

3 The cubic equation $x^3 + cx + 1 = 0$, where c is a constant, has roots α, β, γ.

(a) Find a cubic equation whose roots are $\alpha^3, \beta^3, \gamma^3$. [3]

(b) Show that $\alpha^6 + \beta^6 + \gamma^6 = 3 - 2c^3$. [3]

(c) Find the real value of c for which the matrix $\begin{pmatrix} 1 & \alpha^3 & \beta^3 \\ \alpha^3 & 1 & \gamma^3 \\ \beta^3 & \gamma^3 & 1 \end{pmatrix}$ is singular. [5]

Q4: 9231/12/W20

4 The matrices **A** and **B** are given by

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}.$$

(a) Give full details of the geometrical transformation in the x-y plane represented by **A**. [1]

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(b) Give full details of the geometrical transformation in the x-y plane represented by **B**. [2]

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The triangle DEF in the x-y plane is transformed by **AB** onto triangle PQR.

(c) Show that the triangles DEF and PQR have the same area. [3]

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(d) Find the matrix which transforms triangle PQR onto triangle DEF. [2]

(e) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **AB**. [5]

Q5: 9231/11/S21

4 The matrix **M** represents the sequence of two transformations in the x-y plane given by a rotation of 60° anticlockwise about the origin followed by a one-way stretch in the x-direction, scale factor d ($d \neq 0$).

(a) Find **M** in terms of d. [4]

(b) The unit square in the x-y plane is transformed by **M** onto a parallelogram of area $\frac{1}{2}d^2$ units^2.

Show that $d = 2$. [2]

The matrix **N** is such that $\mathbf{MN} = \begin{pmatrix} 1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.

(c) Find **N**. [3]

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(d) Find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **MN**. [5]

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Q6: 9231/13/S21

4 The matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 2 & k & k \\ 5 & -1 & 3 \\ 1 & 0 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \quad \mathbf{C} = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 2 & 0 \end{pmatrix},$$

where k is a real constant.

(a) Find **CAB**. [3]

(b) Given that **A** is singular, find the value of k. [3]

(c) Using the value of k from part **(b)**, find the equations of the invariant lines, through the origin, of the transformation in the x-y plane represented by **CAB**. [5]

Q7: 9231/11/W21

4 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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(b) Find the values of θ, for $0 \leqslant \theta \leqslant \pi$, for which the transformation represented by **M** has exactly one invariant line through the origin, giving your answers in terms of π. [9]

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Q8: 9231/12/W21

1 **(a)** Give full details of the geometrical transformation in the x-y plane represented by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [1]

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Let $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 2 & 2 \end{pmatrix}$.

(b) The triangle DEF in the x-y plane is transformed by $\mathbf{A}$ onto triangle PQR.

Given that the area of triangle DEF is $13\,\text{cm}^2$, find the area of triangle PQR. [2]

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(c) Find the matrix $\mathbf{B}$ such that $\mathbf{AB} = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$. [2]

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(d) Show that the origin is the only invariant point of the transformation in the x-y plane represented by $\mathbf{A}$. [4]

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Q9: 9231/11/S22

7 The matrix **A** is given by $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & k & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

(a) Find the set of values of k for which **A** is non-singular. [3]

(b) Given that **A** is non-singular, find, in terms of k, the entries in the top row of $\mathbf{A}^{-1}$. [4]

(c) Given that $\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, give an example of a matrix $\mathbf{C}$ such that $\mathbf{BAC} = \begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$. [4]

(d) Find the set of values of k for which the transformation in the x-y plane represented by $\begin{pmatrix} 2 & 1 \\ k & 4 \end{pmatrix}$ has two distinct invariant lines through the origin. [6]

Q10: 9231/13/S22

2 The cubic equation $x^3 + 5x^2 + 10x - 2 = 0$ has roots α, β, γ.

(a) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

(b) Show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

Q11: 9231/11/W22

5 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, where k is a constant.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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(b) The triangle ABC in the x–y plane is transformed by **M** onto triangle DEF.

Find, in terms of k, the single matrix which transforms triangle DEF onto triangle ABC. [2]

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(c) Find the set of values of k for which the transformation represented by **M** has no invariant lines through the origin. [7]

Q12: 9231/12/W22

3 The matrix **M** is given by $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$, where k is a constant and $k \neq 0$ or 1.

(a) The matrix **M** represents a sequence of two geometrical transformations.

State the type of each transformation, and make clear the order in which they are applied. [2]

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(b) Write $\mathbf{M}^{-1}$ as the product of two matrices, neither of which is **I**. [2]

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(c) Show that the invariant points of the transformation represented by **M** lie on the line $y = \frac{k^2}{1-k}x$. [4]

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(d) The triangle ABC in the x-y plane is transformed by $\mathbf{M}$ onto triangle DEF.

Find the value of k for which the area of triangle DEF is equal to the area of triangle ABC. [2]

Chapter 5

Polar Coordinates

Q1: 9231/11/S20

7 The curve C_1 has polar equation $r = \theta\cos\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) The point on C_1 furthest from the line $\theta = \frac{1}{2}\pi$ is denoted by P. Show that, at P,

$$2\theta\tan\theta - 1 = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

The curve C_2 has polar equation $r = \theta\sin\theta$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. The curves C_1 and C_2 intersect at the pole, denoted by O, and at another point Q.

(b) Find the polar coordinates of Q, giving your answers in exact form. [2]

(c) Sketch C_1 and C_2 on the same diagram. [3]

(d) Find, in terms of π, the area of the region bounded by the arc OQ of C_1 and the arc OQ of C_2. [7]

Q2: 9231/13/S20

5 The curve C has polar equation $r = a\tan\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C and state the greatest distance of a point on C from the pole. [2]

(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = \frac{1}{4}\pi$. [4]

(c) Show that C has Cartesian equation $y = \dfrac{x^2}{\sqrt{a^2 - x^2}}$. [3]

(d) Using your answer to part **(b)**, deduce the exact value of $\displaystyle\int_0^{\frac{1}{2}a\sqrt{2}} \frac{x^2}{\sqrt{a^2 - x^2}}\,\mathrm{d}x$. [2]

Q3: 9231/11/W20

7 **(a)** Show that the curve with Cartesian equation

$$(x^2+y^2)^{\frac{5}{2}} = 4xy(x^2-y^2)$$

has polar equation $r = \sin 4\theta$. [4]

The curve C has polar equation $r = \sin 4\theta$, for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the equation of the line of symmetry. [3]

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(c) Find the exact value of the area of the region enclosed by C. [4]

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(d) Using the identity $\sin 4\theta \equiv 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$, find the maximum distance of C from the line $\theta = \frac{1}{2}\pi$. Give your answer correct to 2 decimal places. [6]

Q4: 9231/12/W20

5 The curve C has polar equation $r = \ln(1+\pi-\theta)$, for $0 \leqslant \theta \leqslant \pi$.

(a) Sketch C and state the polar coordinates of the point of C furthest from the pole. [3]

(b) Using the substitution $u = 1+\pi-\theta$, or otherwise, show that the area of the region enclosed by C and the initial line is

$$\tfrac{1}{2}(1+\pi)\ln(1+\pi)\big(\ln(1+\pi)-2\big)+\pi.$$ [6]

(c) Show that, at the point of C furthest from the initial line,

$$(1+\pi-\theta)\ln(1+\pi-\theta)-\tan\theta=0$$

and verify that this equation has a root between 1.2 and 1.3. [5]

Q5: 9231/11/S21

5 The curve C has polar equation $r = a\cot\left(\frac{1}{3}\pi - \theta\right)$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{6}\pi$.

It is given that the greatest distance of a point on C from the pole is $2\sqrt{3}$.

(a) Sketch C and show that $a = 2$. [3]

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(b) Find the exact value of the area of the region bounded by C, the initial line and the half-line $\theta = \frac{1}{6}\pi$. [4]

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(c) Show that C has Cartesian equation $2\left(x+y\sqrt{3}\right)=\left(x\sqrt{3}-y\right)\sqrt{x^2+y^2}$. [3]

Q6: 9231/13/S21

5 The curve C has polar equation $r = \dfrac{1}{\pi - \theta} - \dfrac{1}{\pi}$, where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Sketch C. [3]

(b) Show that the area of the region bounded by the half-line $\theta = \frac{1}{2}\pi$ and C is $\dfrac{3 - 4\ln 2}{4\pi}$. [6]

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Q7: 9231/11/W21

6 The curve C has polar equation $r = 2\cos\theta(1+\sin\theta)$, for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(a) Find the polar coordinates of the point on C that is furthest from the pole. [5]

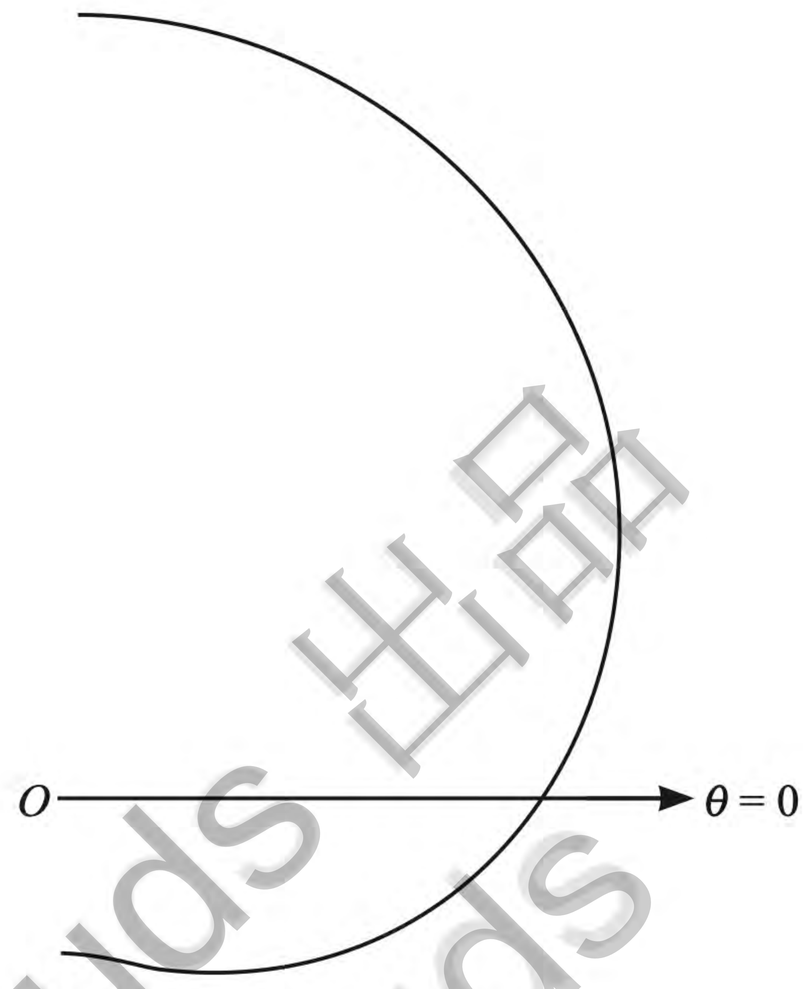
(b) Sketch C. [2]

(c) Find the area of the region bounded by C and the initial line, giving your answer in exact form. [6]

Q8: 9231/12/W21

5 The curve C has polar equation $r = 3 + 2\sin\theta$, for $-\pi < \theta \leqslant \pi$.

(a) The diagram shows part of C. Sketch the rest of C on the diagram. [1]



The straight line l has polar equation $r\sin\theta = 2$.

(b) Add l to the diagram in part **(a)** and find the polar coordinates of the points of intersection of C and l. [5]

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(c) The region R is enclosed by C and l, and contains the pole.

Find the area of R, giving your answer in exact form. [6]

Q9: 9231/11/S22

6 The curve C has polar equation $r^2 = \tan^{-1}\left(\frac{1}{2}\theta\right)$, where $0 \leqslant \theta \leqslant 2$.

(a) Sketch C and state, in exact form, the greatest distance of a point on C from the pole. [3]

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(b) Find the exact value of the area of the region bounded by C and the half-line $\theta = 2$. [5]

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Now consider the part of C where $0 \leqslant \theta \leqslant \frac{1}{2}\pi$.

(c) Show that, at the point furthest from the half-line $\theta = \frac{1}{2}\pi$,

$$(\theta^2 + 4)\tan^{-1}\left(\tfrac{1}{2}\theta\right)\sin\theta - \cos\theta = 0$$

and verify that this equation has a root between 0.6 and 0.7. [5]

Q10: 9231/13/S22

6 The curve C has Cartesian equation $x^2 + xy + y^2 = a$, where a is a positive constant.

(a) Show that the polar equation of C is $r^2 = \dfrac{2a}{2 + \sin 2\theta}$. [3]

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(b) Sketch the part of C for $0 \leqslant \theta \leqslant \frac{1}{4}\pi$. [2]

The region R is enclosed by this part of C, the initial line and the half-line $\theta = \frac{1}{4}\pi$.

(c) It is given that $\sin 2\theta$ may be expressed as $\dfrac{2\tan\theta}{1+\tan^2\theta}$. Use this result to show that the area of R is

$$\tfrac{1}{2}a\int_0^{\frac{1}{4}\pi}\frac{1+\tan^2\theta}{1+\tan\theta+\tan^2\theta}\,\mathrm{d}\theta$$

and use the substitution $t = \tan\theta$ to find the exact value of this area. [8]

Q11: 9231/11/W22

6 **(a)** Show that the curve with Cartesian equation

$$\left(x^2+y^2\right)^2 = 36\left(x^2-y^2\right)$$

has polar equation $r^2 = 36\cos 2\theta$. [3]

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The curve C has polar equation $r^2 = 36\cos 2\theta$, for $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

(b) Sketch C and state the maximum distance of a point on C from the pole. [3]

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(c) Find the area of the region enclosed by C. [2]

(d) Find the maximum distance of a point on C from the initial line, giving the answer in exact form. [6]

Q12: 9231/12/W22

5 The curve C has polar equation $r = a\sec^2\theta$, where a is a positive constant and $0 \leqslant \theta \leqslant \frac{1}{4}\pi$.

(a) Sketch C, stating the polar coordinates of the point of intersection of C with the initial line and also with the half-line $\theta = \frac{1}{4}\pi$. [3]

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(b) Find the maximum distance of a point of C from the initial line. [2]

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(c) Find the area of the region enclosed by C, the initial line and the half-line $\theta = \frac{1}{4}\pi$. [4]

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(d) Find, in the form $y = \mathrm{f}(x)$, the Cartesian equation of C. [3]

Chapter 6

Vectors

A Level Clouds 出品

Q1: 9231/11/S20

5 The lines l_1 and l_2 have equations $\mathbf{r} = 3\mathbf{i} + 3\mathbf{k} + \lambda(\mathbf{i} + 4\mathbf{j} + 4\mathbf{k})$ and $\mathbf{r} = 3\mathbf{i} - 5\mathbf{j} - 6\mathbf{k} + \mu(5\mathbf{j} + 6\mathbf{k})$ respectively.

(a) Find the shortest distance between l_1 and l_2. [5]

The plane Π contains l_1 and is parallel to the vector $\mathbf{i}+\mathbf{k}$.

(b) Find the equation of Π, giving your answer in the form $ax+by+cz=d$. [4]

(c) Find the acute angle between l_2 and Π. [3]

Q2: 9231/13/S20

7 The lines l_1 and l_2 have equations $\mathbf{r} = -5\mathbf{j} + \lambda(5\mathbf{i} + 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k} + \mu(\mathbf{j} + \mathbf{k})$ respectively. The plane Π contains l_1 and is parallel to l_2.

(a) Find the equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

(b) Find the distance between l_2 and Π. [3]

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(c) Show that P has position vector $\frac{55}{27}\mathbf{i} - 5\mathbf{j} + \frac{22}{27}\mathbf{k}$ and state a vector equation for PQ. [8]

Q3: 9231/11/W20

4 The points A, B, C have position vectors

$$-\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{i}-\mathbf{j}, \quad 2\mathbf{i}+2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) Find the acute angle between the planes OAB and ABC. [4]

Q4: 9231/12/W20

7 The points A, B, C have position vectors

$$-2\mathbf{i}+2\mathbf{j}-\mathbf{k}, \quad -2\mathbf{i}+\mathbf{j}+2\mathbf{k}, \quad -2\mathbf{j}+\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the acute angle between the planes OBC and ABC. [4]

The point D has position vector $t\mathbf{i} - \mathbf{j}$.

(c) Given that the shortest distance between the lines AB and CD is $\sqrt{10}$, find the value of t. [6]

Q5: 9231/11/S21

6 Let t be a positive constant.

The line l_1 passes through the point with position vector $t\mathbf{i}+\mathbf{j}$ and is parallel to the vector $-2\mathbf{i}-\mathbf{j}$. The line l_2 passes through the point with position vector $\mathbf{j}+t\mathbf{k}$ and is parallel to the vector $-2\mathbf{j}+\mathbf{k}$.

It is given that the shortest distance between the lines l_1 and l_2 is $\sqrt{21}$.

(a) Find the value of t. [5]

The plane Π_1 contains l_1 and is parallel to l_2.

(b) Write down an equation of Π_1, giving your answer in the form $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}+\mu\mathbf{c}$. [1]

The plane Π_2 has Cartesian equation $5x - 6y + 7z = 0$.

(c) Find the acute angle between l_2 and Π_2. [3]

(d) Find the acute angle between Π_1 and Π_2. [3]

Q6: 9231/13/S21

6 The lines l_1 and l_2 have equations $\mathbf{r} = -\mathbf{i} - 2\mathbf{j} + \mathbf{k} + s(2\mathbf{i} - 3\mathbf{j})$ and $\mathbf{r} = 3\mathbf{i} - 2\mathbf{k} + t(3\mathbf{i} - \mathbf{j} + 3\mathbf{k})$ respectively.

The plane Π_1 contains l_1 and the point P with position vector $-2\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$.

(a) Find an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. [2]

The plane Π_2 contains l_2 and is parallel to l_1.

(b) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

(c) Find the acute angle between Π_1 and Π_2. [5]

(d) The point Q is such that $\overrightarrow{OQ} = -5\overrightarrow{OP}$.

Find the position vector of the foot of the perpendicular from the point Q to Π_2. [4]

Q7: 9231/11/W21

5 The plane Π has equation $\mathbf{r} = -2\mathbf{i} + 3\mathbf{j} + 3\mathbf{k} + \lambda(\mathbf{i} + \mathbf{k}) + \mu(2\mathbf{i} + 3\mathbf{j})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l passes through the point P with position vector $2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$ and is parallel to the vector $\mathbf{k}$.

(b) Find the position vector of the point where l meets Π. [3]

(c) Find the acute angle between l and Π. [3]

(d) Find the perpendicular distance from P to Π. [3]

Q8: 9231/12/W21

7 The points A, B, C have position vectors

$$2\mathbf{i}+2\mathbf{j}, \quad -\mathbf{j}+\mathbf{k} \quad \text{and} \quad 2\mathbf{i}+\mathbf{j}-7\mathbf{k}$$

respectively, relative to the origin O.

(a) Find an equation of the plane OAB, giving your answer in the form $\mathbf{r}.\mathbf{n} = p$. [3]

The plane Π has equation $x-3y-2z=1$.

(b) Find the perpendicular distance of Π from the origin. [1]

(c) Find the acute angle between the planes OAB and Π. [3]

(d) Find an equation for the common perpendicular to the lines OC and AB. [10]

Q9: 9231/11/S22

2 The points A, B, C have position vectors

$$4\mathbf{i}-4\mathbf{j}+\mathbf{k}, \quad -4\mathbf{i}+3\mathbf{j}-4\mathbf{k}, \quad 4\mathbf{i}-\mathbf{j}-2\mathbf{k},$$

respectively, relative to the origin O.

(a) Find the equation of the plane ABC, giving your answer in the form $ax+by+cz=d$. [5]

(b) Find the perpendicular distance from O to the plane ABC. [2]

(c) The point D has position vector $2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$.

Find the coordinates of the point of intersection of the line OD with the plane ABC. [3]

Q10: 9231/13/S22

7 The position vectors of the points A, B, C, D are

$$7\mathbf{i}+4\mathbf{j}-\mathbf{k}, \qquad 11\mathbf{i}+3\mathbf{j}, \qquad 2\mathbf{i}+6\mathbf{j}+3\mathbf{k}, \qquad 2\mathbf{i}+7\mathbf{j}+\lambda\mathbf{k}$$

respectively.

(a) Given that the shortest distance between the line AB and the line CD is 3, show that $\lambda^2-5\lambda+4=0$. [7]

Let Π_1 be the plane ABD when $\lambda = 1$.

Let Π_2 be the plane ABD when $\lambda = 4$.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [2]

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(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

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(c) Find the acute angle between Π_1 and Π_2. [5]

Q11: 9231/11/W22

4 The plane Π contains the lines $\mathbf{r} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k} + \lambda(-\mathbf{i} + 2\mathbf{j} + \mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} + 4\mathbf{j} + 2\mathbf{k} + \mu(3\mathbf{i} + 2\mathbf{j} - \mathbf{k})$.

(a) Find a Cartesian equation of Π, giving your answer in the form $ax + by + cz = d$. [4]

The line l passes through the point P with position vector $2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and is parallel to the vector $\mathbf{j}+\mathbf{k}$.

(b) Find the acute angle between l and Π. [3]

(c) Find the position vector of the foot of the perpendicular from P to Π. [4]

Q12: 9231/12/W22

6 The lines l_1 and l_2 have equations $\mathbf{r} = 2\mathbf{i} + \mathbf{k} + \lambda(\mathbf{i} - \mathbf{j} + 2\mathbf{k})$ and $\mathbf{r} = 2\mathbf{j} + 6\mathbf{k} + \mu(\mathbf{i} + 2\mathbf{j} - 2\mathbf{k})$ respectively.

The point P on l_1 and the point Q on l_2 are such that PQ is perpendicular to both l_1 and l_2.

(a) Find the length PQ. [5]

The plane Π_1 contains PQ and l_1.

The plane Π_2 contains PQ and l_2.

(b) **(i)** Write down an equation of Π_1, giving your answer in the form $\mathbf{r} = \mathbf{a} + s\mathbf{b} + t\mathbf{c}$. [1]

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(ii) Find an equation of Π_2, giving your answer in the form $ax + by + cz = d$. [4]

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(c) Find the acute angle between Π_1 and Π_2. [5]

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Chapter 7

Proof by Induction

Q1: 9231/11/S20

6 Let $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$.

(a) The transformation in the x-y plane represented by $\mathbf{A}^{-1}$ transforms a triangle of area $30\,\text{cm}^2$ into a triangle of area $d\,\text{cm}^2$.

Find the value of d. [3]

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(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 2^n & 0 \\ 2^n - 1 & 1 \end{pmatrix}.$$ [5]

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(c) The line $y = 2x$ is invariant under the transformation in the x-y plane represented by $\mathbf{A}^n\mathbf{B}$, where $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 33 & 0 \end{pmatrix}$.

Find the value of n. [5]

Q2: 9231/13/S20

2 The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 1$ and $u_{n+1} = 2u_n + 1$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 2^n - 1$ for all positive integers n. [5]

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. [2]

Q3: 9231/11/W20

5 Prove by mathematical induction that, for every positive integer n,

$$\frac{\mathrm{d}^{2n-1}}{\mathrm{d}x^{2n-1}}(x\sin x) = (-1)^{n-1}\left(x\cos x + (2n-1)\sin x\right).$$ [7]

Q4: 9231/12/W20

2 Prove by mathematical induction that $7^{2n} - 1$ is divisible by 12 for every positive integer n. [5]

Q5: 9231/11/S21

1 Prove by mathematical induction that $2^{4n}+31^{n}-2$ is divisible by 15 for all positive integers n. [6]

Q6: 9231/13/S21

3 **(a)** Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\left(5r^4+r^2\right)=\tfrac{1}{2}n^2(n+1)^2(2n+1).$$ [6]

(b) Use the result given in part **(a)** together with the List of formulae (MF19) to find $\sum_{r=1}^{n} r^4$ in terms of n, fully factorising your answer. [3]

Q7: 9231/11/W21

3 The sequence of real numbers $a_1, a_2, a_3, \ldots$ is such that $a_1 = 1$ and

$$a_{n+1} = \left(a_n + \frac{1}{a_n}\right)^3.$$

(a) Prove by mathematical induction that $\ln a_n \geqslant 3^{n-1} \ln 2$ for all integers $n \geqslant 2$. [6]

[You may use the fact that $\ln\left(x + \frac{1}{x}\right) > \ln x$ for $x > 0$.]

(b) Show that $\ln a_{n+1} - \ln a_n > 3^{n-1} \ln 4$ for $n \geqslant 2$. [2]

Q8: 9231/12/W21

2 It is given that $y = xe^{ax}$, where a is a constant.

Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n y}{dx^n} = \left(a^n x + na^{n-1}\right)e^{ax}.$$ [6]

Q9: 9231/11/S22

3 The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 > 4$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{u_n^2 + u_n + 12}{2u_n}.$$

(a) By considering $u_{n+1} - 4$, or otherwise, prove by mathematical induction that $u_n > 4$ for all positive integers n. [5]

(b) Show that $u_{n+1} < u_n$ for $n \geqslant 1$. [3]

Q10: 9231/13/S22

5 Let $\mathbf{A} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$, where a is a positive constant.

(a) State the type of the geometrical transformation in the x–y plane represented by $\mathbf{A}$. [1]

(b) Prove by mathematical induction that, for all positive integers n,

$$\mathbf{A}^n = \begin{pmatrix} 1 & na \\ 0 & 1 \end{pmatrix}.$$ [5]

Let $\mathbf{B} = \begin{pmatrix} b & b \\ a^{-1} & a^{-1} \end{pmatrix}$, where b is a positive constant.

(c) Find the equations of the invariant lines, through the origin, of the transformation in the x–y plane represented by $\mathbf{A}^n\mathbf{B}$. [6]

Q11: 9231/11/W22

2 Prove by mathematical induction that, for all positive integers n, $7^{2n} + 97^n - 50$ is divisible by 48. [6]

Q12: 9231/12/W22

4 The function f is such that $f''(x) = f(x)$.

Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}\big(xf(x)\big) = xf'(x) + (2n-1)f(x).$$ [7]