- given:
  - : c classes: $\omega_1, \omega_2, \ldots, \omega_c$
  - : c+1 possible actions:
    - $\alpha_1, \alpha_2, \ldots, \alpha_c$ (decide classes $1, 2, \ldots, c$)
    - $\alpha_{(c+1)}$ (reject/refuse to classify)
  - : Loss Function $\lambda(\alpha_i, \omega_j)$:

$$\lambda(\alpha_i, \omega_j) = \begin{cases} 0 & \text{if } i=j \quad \text{(correct classification)} \\ \lambda_r & \text{if } i=c+1 \quad \text{(rejection)} \\ \lambda_s & \text{otherwise} \quad \text{(substitution error)} \end{cases}$$

where: $\lambda_r$ = loss incurred for rejecting

$\lambda_s$ = " " " misclassification

$i, j = \{1, 2, \ldots, c\}$

- find: determine the decision rule that minimizes the expected risk

- conditional risk: $R(\alpha_i|x) = \sum_{i=1}^{c} \{\lambda(\alpha_i|\omega_j) P(\omega_j|x)\}$ (expected loss for action $\alpha_i$)
- risk for deciding class $i$: $R(\alpha_i|x) = \lambda(\alpha_i|\omega_i) P(\omega_i|x) + \sum_{j \neq i} \lambda(\alpha_i|\omega_j) P(\omega_j|x)$

$$= 0 \cdot P(\omega_i|x) + \sum_{j \neq i} \lambda_s P(\omega_j|x)$$

$$= \lambda_s \sum_{j \neq i} P(\omega_j|x)$$

$\therefore$ since $\sum_{j=1}^{c} P(\omega_j|x) = 1$:

$\Longrightarrow R(\alpha_i|x) = \lambda_s [1 - P(\omega_i|x)]$

- risk for rejection: $R(\alpha_{c+1}|x) = \sum_{j=1}^{c} \lambda(\alpha_{c+1}|\omega_j) P(\omega_j|x)$

$\therefore$ since $\lambda(\alpha_{c+1}|\omega_j) = \lambda_r$ for all $j$:

$\Longrightarrow R(\alpha_{c+1}|x) = \lambda_r \sum_{j=1}^{c} P(\omega_j|x) = \lambda_r \cdot 1 = \lambda_r$

- Decision Rule: Decide $\omega_i$ if:
  1. $R(\alpha_i|x) \leq R(\alpha_j|x)$ for all $j \neq i$
  2. $R(\alpha_i|x) < R(\alpha_{c+1}|x)$

  : otherwise reject

$\therefore$ Condition 1:

: $\lambda_s[1 - P(\omega_i|x)] \leq \lambda_s[1 - P(\omega_j|x)]$ for all $j \neq i$

: $P(\omega_i|x) \geq P(\omega_j|x)$ for all $j$ (assuming $\lambda_s > 0$)

$\therefore$ Choose the class w/ the highest posterior probability

$\therefore$ Condition 2:

: $\lambda_s[1 - P(\omega_i|x)] < \lambda_r$

: $P(\omega_i|x) \geq 1 - \lambda_r/\lambda_s$ ($\geq$ for boundary case)

$$\boxed{\therefore \text{ Decide } \omega_i \text{ if: } \begin{cases} 1. \; P(\omega_i|x) \geq P(\omega_j|x) \\ 2. \; P(\omega_i|x) \geq 1 - \lambda_r/\lambda_s \\ \text{otherwise reject} \end{cases}}$$

if $\lambda_r = 0$ : rejecting costs nothing

$\therefore$ threshold becomes : $P(\omega_i|x) \geq 1 - 0/\lambda_s$

: $P(\omega_i|x) \geq 1$

$\therefore$ Analysis :

: need to be absolutely certain (100%) to not reject

$\Longrightarrow$ likely never satisfied : ALWAYS REJECT

if $\lambda_r > \lambda_s$ : loss for rejection is greater than loss for substitution error

$\therefore$ threshold becomes : $P(\omega_i|x) \geq 1 - \lambda_r/\lambda_s$

: since $\lambda_r > \lambda_s$ : $\lambda_r/\lambda_s > 1$

$\therefore$ $1 - \lambda_r/\lambda_s < 0$

$\therefore$ Analysis :

: threshold is negative

$\Longrightarrow$ since probabilities are always non-negative, this condition is ALWAYS met

: Result: As long as $\omega_i$ has the highest posterier, ALWAYS classify

(if rejection costs more than missclassification, best strategy is to guess)