Math 2210Q/Roby **Practice Final** December 2009

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! No calculators are to be used, but you may bring one $4'' \times 6''$ notecard to class with any notes you like.

1. Define $T : \mathbb{P}_2 \to R^3$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

   (a) Find the image under T of $\mathbf{p}(t) = 5 + 3t$.

   (b) Show that T is a linear transformation.

   (c) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.

   (d) Is T one to one? Is T onto? Explain!

2. Find the characteristic polynomial and the eigenvalues of the matrix $\begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$.

3. Show that if $T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$, then $\det T = (b - a)(c - a)(c - b)$.

4. Prove or Disprove and Salvage if possible:

   (a) If $A = QR$, where Q has orthonormal columns, then $R = Q^T A$.

   (b) If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of vectors in $\mathbb{R}^n$, then S is linearly independent.

   (c) Each eigenvector of a square matrix A is also an eigenvector of A^2.

   (d) There exists a 2×2 matrix that has no eigenvectors in $\mathbb{R}^2$.

   (e) If A is row equivalent to the identity matrix I, then A is diagonalizable.

5. Decide whether each statement below is True of False. Justify your answer.

   (a) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.

   (b) For an $m \times n$ matrix A, vectors in $\operatorname{Nul} A$ are orthogonal to vectors in $\operatorname{Row} A$.

   (c) The matrices A and A^T have the same eigenvalues, counting multiplicities.

   (d) A nonzero vector can correspond to two different eigenvalues of A.

   (e) The sum of two eigenvectors of a square matrix A is also an eigenvector of A.

6. If a $n \times n$ matrix A satifies $A^2 = A$, what can you say about the determinant of A?

7. Lay #4.6.4 (p. 269), which gives $A \sim B$ and asks for bases and dimensions of $\operatorname{Col} A$, $\operatorname{Nul} A$, and $\operatorname{Row} A$.

8. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$ subject to the constraint $x_1^2 + x_2^2 = 1$. (You do not need to compute a vector at which this maximum is attained.)

9. What would you have to know about the solution set of a homogenous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss!

10. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points.

11. Here are some specific tasks (modified from a list of Prof. Leibowitz) I expect you to be able to perform **with demonstrated understanding**:

    (a) Given a matrix A, find the dimensions of and bases for $\operatorname{Col} A$, $\operatorname{Nul} A$, and $\operatorname{Row} A$. Use the relations among rank, dimension of nullspace, and size of a matrix to understand properties of the associated linear transformation (one-to-one, onto, kernal, range).

    (b) Use row reduction to solve linear equations, show that a given column vector is in the span of a given set of vectors, or compute the inverse of a matrix.

    (c) Use row operations to reduce a matrix A to triangular form in order to calculate $\det A$. Use properties of determinants to compute the determinant of related matrices.

    (d) Diagonalize a given matrix and use the $A = PDP^{-1}$ factorization to calculate a power of A.

    (e) Orthogonally diagonalize a real symmetric matrix, possibly representing a quadratic form, and compute constrained extrema of the form.

    (f) Project a vector onto the subspace spanned by a given set of vectors, after verifying that they form an orthogonal set.

    (g) Understand how to use the LU factorization and singular value decompositions of $m \times n$ matrices.

    (h) Understand the theory of the course well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.

    (i) Show that a certain set of vectors is a subspace of a given space, or are eigenvectors of a certain matrix.

    (j) Use various forms of the Invertible Matrix Theorem in context.