

**SHOW ALL YOUR WORK!** Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! No calculators are to be used, but you may bring one  $8.5'' \times 11''$  notecard to class with any notes you like.

1. Define  $T : \mathbb{P}_2 \rightarrow \mathbb{R}^3$  by  $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$ .
  - (a) Find the image under  $T$  of  $\mathbf{p}(t) = 5 + 3t$ .
  - (b) Show that  $T$  is a linear transformation.
  - (c) Find the matrix for  $T$  relative to the basis  $\{1, t, t^2\}$  for  $\mathbb{P}_2$  and the standard basis for  $\mathbb{R}^3$ .
  - (d) Is  $T$  one to one? Is  $T$  onto? Explain!
2. Find the characteristic polynomial and the eigenvalues of the matrix  $\begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$ .
3. Show that if  $T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ , then  $\det T = (b - a)(c - a)(c - b)$ .
4. Prove or Disprove and Salvage if possible:
  - (a) If  $A = Q R$ , where  $Q$  has orthonormal columns, then  $R = Q^T A$ .
  - (b) If  $S = \{u_1, \dots, u_p\}$  is an orthogonal set of vectors in  $\mathbb{R}^n$ , then  $S$  is linearly independent.
  - (c) Each eigenvector of a square matrix  $A$  is also an eigenvector of  $A^2$ .
  - (d) There exists a  $2 \times 2$  matrix that has no eigenvectors in  $\mathbb{R}^2$ .
  - (e) If  $A$  is row equivalent to the identity matrix  $I$ , then  $A$  is diagonalizable.
5. Decide whether each statement below is True or False. Justify your answer.
  - (a) If  $\mathbf{y}$  is in a subspace  $W$ , then the orthogonal projection of  $\mathbf{y}$  onto  $W$  is  $\mathbf{y}$  itself.
  - (b) For an  $m \times n$  matrix  $A$ , vectors in  $\text{Nul } A$  are orthogonal to vectors in  $\text{Row } A$ .
  - (c) The matrices  $A$  and  $A^T$  have the same eigenvalues, counting multiplicities.
  - (d) A nonzero vector can correspond to two different eigenvalues of  $A$ .
  - (e) The sum of two eigenvectors of a square matrix  $A$  is also an eigenvector of  $A$ .
6. If a  $n \times n$  matrix  $A$  satisfies  $A^2 = A$ , what can you say about the determinant of  $A$ ?

7. Lay #4.6.4 (p. 269), which gives  $A \sim B$  and asks for bases and dimensions of  $\text{Col } A$ ,  $\text{Nul } A$ , and  $\text{Row } A$ .
8. Find the maximum value of  $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$  subject to the constraint  $x_1^2 + x_2^2 = 1$ . (You do not need to compute a vector at which this maximum is attained.)
9. What would you have to know about the solution set of a homogenous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss!
10. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points.
11. Here are some specific tasks (modified from a list of Prof. Leibowitz) I expect you to be able to perform **with demonstrated understanding**:
  - (a) Given a matrix  $A$ , find the dimensions of and bases for  $\text{Col } A$ ,  $\text{Nul } A$ , and  $\text{Row } A$ . Use the relations among rank, dimension of nullspace, and size of a matrix to understand properties of the associated linear transformation (one-to-one, onto, kernel, range).
  - (b) Use row reduction to solve linear equations, show that a given column vector is in the span of a given set of vectors, or compute the inverse of a matrix.
  - (c) Use row operations to reduce a matrix  $A$  to triangular form in order to calculate  $\det A$ . Use properties of determinants to compute the determinant of related matrices.
  - (d) Diagonalize a given matrix and use the  $A = PDP^{-1}$  factorization to calculate a power of  $A$ .
  - (e) Orthogonally diagonalize a real symmetric matrix, possibly representing a quadratic form, and compute constrained extrema of the form.
  - (f) Project a vector onto the subspace spanned by a given set of vectors, after verifying that they form an orthogonal set.
  - (g) Understand how to use the LU factorization of an  $m \times n$  matrix.
  - (h) Understand the theory of the course well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.
  - (i) Show that a certain set of vectors is a subspace of a given space, or are eigenvectors of a certain matrix.
  - (j) Use various forms of the Invertible Matrix Theorem in context.