Math 2710 (Roby) **Practice Final** Spring 2013

This is a closed book, closed note exam, except that you may have one **4 × 6** inch notecard with anything you like written on it front and back. You may use a calculator. Please do not discuss this exam with anyone other than the proctor during the exam.

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask!

1. Go over all the midterm, practice midterm, and midterm rewrite problems. Make sure you know how to do all of these, and understand any places where you lost points.

2. Go over your old homework problems. Make sure you know how to do all of these, and understand any places where you lost points or needed to rewrite.

3. Make sure you can state and give careful proofs of the following important theorems.

   (a) *Euclid's Lemma:* For any $a, b \in \mathbb{Z}$ and any prime p: $p \mid ab \implies p \mid a$ or $p \mid b$. (You may state without proof that the Euclidean algorithm gives solutions to certain linear diophantine equations.)

   (b) *Fermat's little theorem* and its generalization by Euler.

   (c) *Fundamental Theorem of Arithmetic*, i.e., unique factorization into primes in $\mathbb{Z}$.

   (d) *The Binomial Theorem*

   (e) The rational numbers are countable, but the real numbers are uncountable.

   (f) The limit of the sum of two sequences is the sum of the limits.

   (g) If $\sum a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

4. Prove or Disprove and Salvage if Possible:

   (a) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is onto.

   (b) Two functions $f, g : \mathbb{R} \to \mathbb{R}$ are bounded iff fg is bounded.

   (c) If two functions $f, g : \mathbb{R} \to \mathbb{R}$ are monotone then $g \circ f$ is monotone.

   (d) Suppose that a real number r satisfies $r \leq M + \epsilon$ for all $\epsilon > 0$. Then $r \leq M$.

   (e) If a sequence $\{x\}$ of real numbers converges, then there exists $n \in \mathbb{N}$ such that $|x_{n+1} - x_n| < 1/2^n$.

   (f) The sequence $\{x^n\}$ defined by $x_1 = 1$ and for every $n \in \mathbb{N}$

   $$x_{n+1} = \frac{1}{x_1 + \cdots + x_n}$$

   converges. (No need to find the limit.)

   (g) If $a_n \to 0$ and $b_n \to 0$, then $\sum a_n b_n$ converges.

(h) The sequence $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ converges.

5. Prove that for any sets A and B, $(A \cup B) \cap A^c = B - A$.

6. Let $\{a_n\}$ be a sequence with $a_1 = 1$ and $a_{n+1} = a_n + 3n(n+1)$ for $n \in \mathbb{N}$. Prove that $a_n = n^3 - n + 1$ for all $n \in \mathbb{N}$.

7. Give the negation of the following statements, avoiding locutions like "It is not the case that…".

   (a) Fred goes bowling and says "Yabba-Dabba-Doo!"

   (b) If I take the test, I'll fail.

   (c) Every action has an equal and opposite reaction.

   (d) The product of any two odd numbers is prime.

8. Let $f : A \to B$ and $g : B \to C$. Set $h = g \circ f$. Prove or Disprove each statement:

   (a) if h is injective, then f is injective.

   (b) if h is injective, then g is injective.

   (c) if h is surjective, then f is surjective.

   (d) if h is surjective, then g is surjective.

9. (a) Define carefully what it means for a sequence of real numbers to be a *Cauchy* sequence.

   (b) Prove that any sequence of real numbers that converges must be a Cauchy sequence.

10. A runaway train is hurtling toward a brick wall at the speed of 100 miles per hour. When it is two miles from the wall, a fly begins to fly repeatedly between the trains and the wall at the speed of 200 miles per hour. Determine how far the fly travels before it is smashed.