Math 2210Q (Roby) **Practice Midterm #2** **Solutions** Fall 2016

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask! For this exam **no calculators are to be used.**

1. Let A be the matrix $A = \begin{bmatrix} 1 & 1 & 0 & -2 \\ 2 & -2 & 1 & 1 \\ 1 & 0 & -2 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

   (a) Compute $\det A$. The answer is 6. This may be computed in a couple of ways: (1) by doing row reductions to transform A to triangular form, keeping track of any moves that modify the determinant or (2) expanding by cofactors (minors) along a suitable row or column.

   (b) Compute $\det(A^{-1})$ without computing A^{-1}. Since $\det(A^{-1}) = 1/(\det A)$, the answer is $1/6$.

   (c) Use Cramer's Rule to find x_4 so that $A \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix}$.

   To apply Cramer's rule, for x_4, we replace the fourth column of A with the output vector, take the determinant, and divide that by the determinant of the original matrix (computed above to be 6). Therefore,

$$x_4 = \frac{1}{6} \begin{vmatrix} 1 & 1 & 0 & 2 \\ 2 & -2 & 1 & 2 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \end{vmatrix}$$

$$= \frac{1}{6}\left((-1) \begin{vmatrix} 1 & 0 & 2 \\ -2 & 1 & 2 \\ 0 & -2 & 0 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 1 & 2 \\ 2 & -2 & 2 \\ 1 & 0 & 0 \end{vmatrix} \right) \quad = \frac{1}{6}(-12 - 6) = -3$$

2. Find the volume of the parallelepiped determined by the vectors $\begin{bmatrix} 3 \\ 6 \\ 7 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

   Take the absolute value of the determinant of the matrix of column vectors to get:

$$\left\| \begin{matrix} 3 & 0 & 2 \\ 6 & 4 & 3 \\ 7 & 1 & 4 \end{matrix} \right\| = |-5| = 5.$$

3. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 & -1 \\ -1 & -2 & 0 & 1 & 3 \\ 2 & 4 & 4 & 2 & 2 \end{bmatrix}$. Find bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$. What should the sum of the dimensions of these two subspaces be? Does your answer check?

By row reduction we see that $A \sim \begin{bmatrix} 1 & 2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ has pivots in columns 1 and 3, so we use those columns of A as a basis for $\operatorname{Col} A$. For $\operatorname{Nul} A$, we parameterize the solutions in terms of the free variables to get the basis shown below.

$$\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} \right\}, \text{ and } \operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

4. Define a transformation $T : \mathbb{P}_3 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(2) \end{bmatrix}$.

   (a) Show that T is a linear transformation.
   Let $\mathbf{p}, \mathbf{q} \in \mathbb{P}_3$. Then

   $$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(0) \\ (\mathbf{p}+\mathbf{q})(2) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}(2)+\mathbf{q}(2) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(2) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(2) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q}).$$

   Similarly, one shows that $T(c\mathbf{p}) = cT(\mathbf{p})$ for any $c \in \mathbb{R}$.

   (b) Describe the kernel and range of this linear transformation.
   By def. $\ker T$ is the set of polys that map to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, i.e., $\{\mathbf{p} \in \mathbb{P}_3 : \mathbf{p}(0) = \mathbf{p}(2) = 0\}$, while range T is all of $\mathbb{R}^2$, since for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, we can always find a polynomial $\mathbf{p}$ with $\mathbf{p}(0) = a$ and $\mathbf{p} = b$ (e.g., by Lagrange interpolation, or less fancily by noting that matrix below has two pivot columns, so the dimension of range $T = \dim \operatorname{Col} A = 2$).

   (c) Write the matrix A of this linear transformation in terms of the standard bases for $\mathbb{P}_3$ and $\mathbb{R}_2$.

   (d) Compute a basis for $\operatorname{Nul} A$.

   (e) Compute a basis for $\operatorname{Col} A$. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 \end{bmatrix}$, which is already almost in RREF. So we get bases $\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$, and $\operatorname{Col} A = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$.

5. Find the dimensions of Nul A and Col A for the matrix $A = \begin{bmatrix} 1 & 3 & -4 & 2 & -1 & 6 \\ 0 & 0 & 1 & -3 & 7 & 0 \\ 0 & 0 & 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

   We have $\dim \operatorname{Col} A = \operatorname{rk} A = \#\text{pivot cols} = 3$, and $\dim \operatorname{Nul} A = \#\text{free vars} = 6-3 = 3$, by the rank-nullity theorem.

6. If A is a 4×3 matrix, what is the largest possible dimension of the row space of A? What is the smallest possible dimension? What if A is 3×4 matrix? Explain!

   Since Row A is spanned by 4 vectors in $\mathbb{R}^3$, it has dimension at most 3, and $A = \begin{bmatrix} I_3 \\ 0 \end{bmatrix}$, shows that dimension is achievable. The smallest possible dimension is 0, achieved by $A = 0$. Similar reasoning shows the same bounds for a 3×4 matrix.

7. Prove or Disprove and Salvage if possible:

   (a) If A is a 2×2 matrix with a zero determinant, then one column of A is multiple of the other. T

   (b) If λ is an eigenvalue of an $n \times n$ matrix M, then λ^2 is an eigenvalue of M^2. T

   (c) If A and B are $n \times n$ matrices with $\det A = 2$ and $\det B = 3$, then $\det(A+B) = 5$. F

   (d) Let A and B be $n \times n$ matrices. If B is obtained from A by adding to one row of A a linear combination of other rows of A, then $\det B = \det A$. T

   (e) $\det A^T = -\det A$. F

8. Decide whether each statement below is True of False. Justify your answer.

   (a) The number of pivot columns of a matrix equals the dimension of its column space. T

   (b) Any plane in $\mathbb{R}^3$ is a two-dimensional subspace of $\mathbb{R}^3$. F

   (c) The dimension of the vector space $\mathbb{P}^4$ is 4. F

   (d) If $\dim V = n$ and S is a linearly indpendent set in V, then S is a basis for V. F

   (e) If there exists a linearly dependent set $\{v_1, \ldots, , v_p\}$ that spans V, then $\dim V \leq p$. T

   (f) The eigenvectors of any $n \times n$ matrix are linearly independent in $\mathbb{R}^n$. F

9. Prove or Disprove and Salvage if possible:

   (a) The null space of an $m \times n$ matrix is in $\mathbb{R}^m$ F

   (b) The range of a linear transformation is a vector subpace of the codomain. T

   (c) Let A and B be $n \times n$ matrices. If B is obtained from A by adding to one row of A a linear combination of other rows of A, then $\det B = \det A$. T

(d) The row space of A^T is the same as the column space of A. T

10. Let $\mathcal{S} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$.

(a) Is $\mathcal{S}$ linearly independent in $\mathbb{P}_2$? Explain!

(b) Is $\mathcal{S}$ a basis for $\mathbb{P}_2$? Explain!

(c) Express $\mathbf{p}(t) = 3 + t - 6t^2$ as a linear combination of elements of $\mathcal{S}$.

(d) Is the expression unique? Explain!

The standard isomorphism $\mathbb{P}_2 \to \mathbb{R}^3$ given by $1 \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $t \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $t^2 \mapsto \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ takes the polynomials in S to the columns of the matrix

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

which is invertible. So the columns of C form a basis for $\mathbb{R}^3$, which means the original set $\mathcal{S}$ is a basis for $\mathbb{P}_2$ (b). In particular, this means that there is a unique way of writing any polynomial as a linear combination of the basis elements (d). The usual techniques for solving $C\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ -6 \end{bmatrix}$ give $\mathbf{x} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}$, so

$$\mathbf{p}(t) = 3 + t - 6t^2 = 7(1 - t^2) - 3(t - t^2) - 2(2 - 2t + t^2)\,.$$

as one can easily check.

11. Suppose a nonhomogeneous system of six linear equations in eight unknowns has a solution with two free variables. Is it possible to change some constants on the equations' right sides to make the new system inconsistent? Explain! No. This system is equivalent to $A\mathbf{x} = b$, where $\dim \operatorname{Nul} A = 2 \implies \dim \operatorname{Col} A = 8 - 2 = 6 \implies \mathbf{x} \mapsto A\mathbf{x}$ is *onto* $\mathbb{R}^6$; hence, every right hand side is obtainable.

12. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points. Check!