Section 4.6 Rank

The set of all linear combinations of the row vectors of a matrix A is called the **row space** of A and is denoted by Row A.

EXAMPLE: Let

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \mathbf{r}_1 &= (-1,\ 2,\ 3,\ 6) \\ \mathbf{r}_2 &= (2, -5, -6, -12) \\ \mathbf{r}_3 &= (1, -3, -3, -6) \end{aligned}.$$

$$\text{Row } A = \text{Span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\} \text{ (a subspace of } \mathbf{R}^4\text{)}$$

While it is natural to express row vectors horizontally, they can also be written as column vectors if it is more convenient. Therefore

$$\boxed{\text{Col } A^T = \text{Row } A}.$$

When we use row operations to reduce matrix A to matrix B, we are taking linear combinations of the rows of A to come up with B. We could reverse this process and use row operations on B to get back to A. Because of this, the row space of A equals the row space of B.

THEOREM 13

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as B.

EXAMPLE: The matrices

$$A = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 2 & -5 & -6 & -12 \\ 1 & -3 & -3 & -6 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 2 & 3 & 6 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are row equivalent. Find a basis for row space, column space and null space of A. Also state the dimension of each.

Basis for Row A :

$$\{ \qquad\qquad\qquad\qquad \}$$

dim Row A :_____

Basis for Col A : $\left\{ \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \begin{bmatrix} \\ \\ \\ \end{bmatrix} \right\}$

dim Col A :_____

To find Nul A, solve $A\mathbf{x} = \mathbf{0}$ first:

$$\begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 2 & -5 & -6 & -12 & 0 \\ 1 & -3 & -3 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 2 & 3 & 6 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -3 & -6 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_3 + 6x_4 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for Nul A : $\left\{ \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

and dim Nul $A =$ ______

Note the following:

$\dim \text{Col } A =$ # of pivots of $A =$ # of nonzero rows in B

$= \dim \text{Row } A$.

$\dim \text{Nul } A =$ # of free variables $=$ # of nonpivot columns of A.

DEFINITION

The **rank** of A is the dimension of the column space of A.

$$\boxed{\text{rank } A = \dim \text{Col } A = \text{\# of pivot columns of } A = \dim \text{Row } A}.$$

$$\underbrace{\text{rank } A}_{\updownarrow} + \underbrace{\dim \text{Nul } A}_{\updownarrow} = \underbrace{n}_{\updownarrow}$$

$$\left\{\begin{matrix}\text{\# of pivot}\\ \text{columns}\\ \text{of } A\end{matrix}\right\} \qquad \left\{\begin{matrix}\text{\# of nonpivot}\\ \text{columns}\\ \text{of } A\end{matrix}\right\} \qquad \left\{\begin{matrix}\text{\# of}\\ \text{columns}\\ \text{of } A\end{matrix}\right\}$$

THEOREM 14 THE RANK THEOREM

The dimensions of the column space and the row space of an $m \times n$ matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation

$$\text{rank } A + \dim \text{Nul } A = n.$$

Since Row A = Col A^T,

$$\boxed{\text{rank } A = \text{rank } A^T}.$$

EXAMPLE: Suppose that a 5×8 matrix A has rank 5. Find dim Nul A, dim Row A and rank A^T. Is Col $A = \mathbf{R}^5$?

Solution:

$$\underbrace{\text{rank } A}_{\displaystyle\updownarrow \atop \displaystyle 5} + \underbrace{\dim \text{ Nul } A}_{\displaystyle\downarrow \atop \displaystyle ?} = \underbrace{n}_{\displaystyle\updownarrow \atop \displaystyle 8}$$

$$5 + \dim \text{ Nul } A = 8 \qquad \Rightarrow \qquad \dim \text{ Nul } A = \_\_\_\_\_$$

$$\dim \text{ Row } A = \text{rank } A = \_\_\_\_\_$$

$$\Rightarrow \qquad \text{rank } A^T = \text{rank } \_\_\_\_\_ = \_\_\_\_\_$$

Since rank A = # of pivots in $A = 5$, there is a pivot in every row. So the columns of A span $\mathbf{R}^5$ (by Theorem 4, page 43). Hence Col $A = \mathbf{R}^5$.

EXAMPLE: For a 9×12 matrix A, find the smallest possible value of $\dim$ Nul A.

Solution:

$$\text{rank } A + \dim \text{ Nul } A = 12$$

$$\dim \text{ Nul } A = 12 - \underbrace{\text{rank } A}_{\text{largest possible value} = \_\_\_\_\_}$$

$$\text{smallest possible value of } \dim \text{ Nul } A = \_\_\_\_\_$$

Visualizing Row A and Nul A

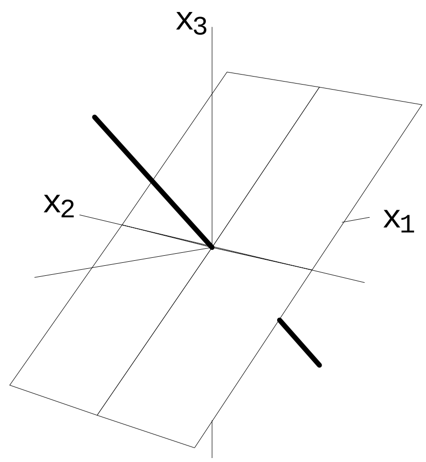
EXAMPLE: Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 2 \end{bmatrix}$. One can easily verify the following:

Basis for Nul $A = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ and therefore Nul A is a plane in $\mathbf{R}^3$.

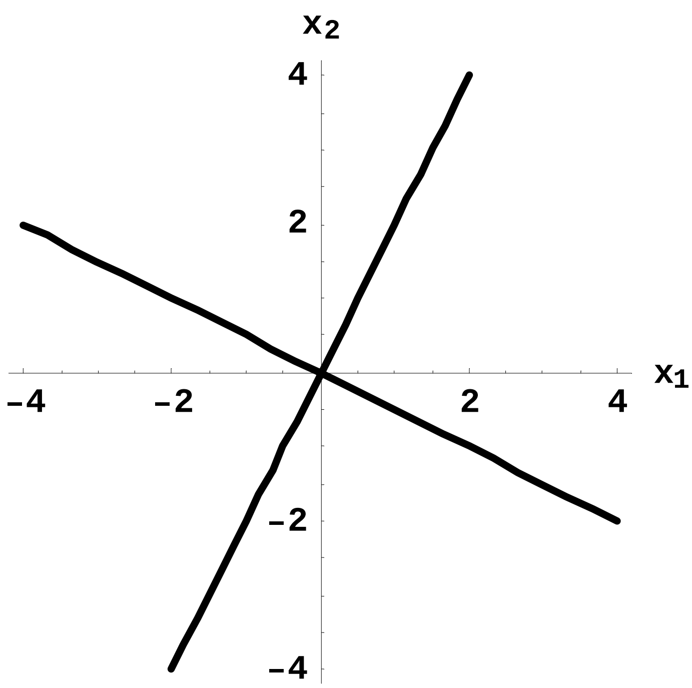
Basis for Row $A = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$ and therefore Row A is a line in $\mathbf{R}^3$.

Basis for Col $A = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ and therefore Col A is a line in $\mathbf{R}^2$.

Basis for Nul $A^T = \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$ and therefore Nul A^T is a line in $\mathbf{R}^2$.



Subspaces Nul A and Row A



Subspaces Nul A^T and Col A

The Rank Theorem provides us with a powerful tool for determining information about a system of equations.

EXAMPLE: A scientist solves a homogeneous system of 50 equations in 54 variables and finds that exactly 4 of the unknowns are free variables. Can the scientist be *certain* that any associated nonhomogeneous system (with the same coefficients) has a solution?

Solution: Recall that

rank $A = \dim$ Col A = # of pivot columns of A

$\dim$ Nul A = # of free variables

In this case $A\mathbf{x} = \mathbf{0}$ of where A is 50×54.

By the rank theorem,

rank A + _______ = _______

or

rank A = _______.

So any nonhomogeneous system $A\mathbf{x} = \mathbf{b}$ has a solution because there is a pivot in every row.

THE INVERTIBLE MATRIX THEOREM (continued)

Let A be a square $n \times n$ matrix. The the following statements are equivalent:

m. The columns of A form a basis for $\mathbf{R}^n$

n. Col $A = \mathbf{R}^n$

o. dim Col $A = n$

p. rank $A = n$

q. Nul $A = \{\mathbf{0}\}$

r. dim Nul $A = 0$