ISOMORPHISM OF SPLITTING FIELDS

KEITH CONRAD

Using tensor products, we will give a slick proof that any two splitting fields of a polynomial are (non-canonically) isomorphic over the base field.

Theorem 1. *Let K be a field and $f(X) \in K[X]$ be nonconstant. Any two splitting fields of $f(X)$ over K are K-isomorphic.*

Proof. Let $n = \deg f \geq 1$ and let L_1 and L_2 be splitting fields of $f(X)$ over K, so

$$L_1 = K(\alpha_1, \dots, \alpha_n), \quad L_2 = K(\beta_1, \dots, \beta_n),$$

where the α_i's and β_j's are full sets of roots of $f(X)$. (Some α_i's and some β_j's may be repeated since $f(X)$ might not be separable.) We want to show there is a field isomorphism $L_1 \to L_2$ which fixes the elements of K.

Since L_1 and L_2 are not zero, the ring $L_1 \otimes_K L_2$ is not zero because the tensor product of nonzero vector spaces is not zero. Since L_1/K and L_2/K are algebraic, $L_1 = K[\alpha_1, \dots, \alpha_n]$ and $L_2 = K[\beta_1, \dots, \beta_n]$. Thus $L_1 \otimes_K L_2$ is generated *as a K-algebra* by the elementary tensors $\alpha_i \otimes \beta_j = (\alpha_i \otimes 1)(1 \otimes \beta_j)$. Pick a maximal ideal $\mathfrak{m}$ in $L_1 \otimes_K L_2$ and consider the composite map

$$L_1 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m}.$$

where the first map is $x \mapsto x \otimes 1$ and the second map is the natural reduction. Both are K-algebra homomorphisms, so the composite is as well. Since L_1 is a field, the composite map is injective, so we can regard $(L_1 \otimes_K L_2)/\mathfrak{m}$ as a field extension of L_1. The α_i's are a full set of roots of $f(X)$ in L_1, so the only roots of $f(X)$ in $(L_1 \otimes_K L_2)/\mathfrak{m}$ are the $\alpha_i \otimes 1 \bmod \mathfrak{m}$. Each $1 \otimes \beta_j \bmod \mathfrak{m}$ is a root of $f(X)$, so $1 \otimes \beta_j \equiv \alpha_i \otimes 1 \bmod \mathfrak{m}$ for some i. Therefore $(L_1 \otimes_K L_2)/\mathfrak{m}$ is generated as a K-algebra by all $\alpha_i \otimes 1 \bmod \mathfrak{m}$, which proves the above map $L_1 \to (L_1 \otimes_K L_2)/\mathfrak{m}$ is surjective, and hence is a K-algebra isomorphism.

We get a K-algebra isomorphism $L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m}$ in a similar way. Composing $L_1 \to (L_1 \otimes_K L_2)/\mathfrak{m}$ with the inverse of $L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m}$ gives us a K-algebra isomorphism from L_1 to L_2. $\square$

Remark 2. Each $\alpha_i \otimes 1$ and $1 \otimes \beta_j$ in $L_1 \otimes_K L_2$ is a solution to $f(t) = 0$. This gives us $2n$ solutions when $f(X)$ is separable, so the ring $L_1 \otimes_K L_2$ is not a field. Thus we could anticipate a collapsing of these roots into each other when we reduce $L_1 \otimes_K L_2$ modulo a maximal ideal and get a field, where $f(X)$ always has at most n roots.

It might at first seem curious that the construction of a K-algebra isomorphism $L_1 \to L_2$ succeeded using any maximal ideal in $L_1 \otimes_K L_2$. In fact, different maximal ideals provide us with all the different isomorphisms. Let's look at an example before proving the general result.

Example 3. Two splitting fields for $X^2 - 2$ over $\mathbf{Q}$ are $L_1 = \mathbf{Q}[T]/(T^2 - 2)$ and $L_2 = \mathbf{Q}(\sqrt{2})$ (a subfield of $\mathbf{R}$). There are two $\mathbf{Q}$-isomorphisms $L_1 \to L_2$, determined by the identification of T in L_1 with $\pm\sqrt{2}$ in L_2. The tensor product of L_1 and L_2 over $\mathbf{Q}$ is

$$L_1 \otimes_{\mathbf{Q}} L_2 = \mathbf{Q}[T]/(T^2 - 2) \otimes_{\mathbf{Q}} \mathbf{Q}(\sqrt{2}) \cong \mathbf{Q}(\sqrt{2})[T]/(T^2 - 2) = \mathbf{Q}(\sqrt{2})[T]/(T - \sqrt{2})(T + \sqrt{2}).$$

Using the Chinese remainder theorem,

$$\mathbf{Q}(\sqrt{2})[T]/(T-\sqrt{2})(T+\sqrt{2}) \cong \mathbf{Q}(\sqrt{2})[T]/(T-\sqrt{2})\times\mathbf{Q}(\sqrt{2})[T]/(T+\sqrt{2}) \cong \mathbf{Q}(\sqrt{2})\times\mathbf{Q}(\sqrt{2}),$$

where T on the left corresponds to $(\sqrt{2}, -\sqrt{2})$ on the right. The ring $\mathbf{Q}(\sqrt{2}) \times \mathbf{Q}(\sqrt{2})$ has two maximal ideals, $\{0\} \times \mathbf{Q}(\sqrt{2})$ and $\mathbf{Q}(\sqrt{2}) \times \{0\}$. The quotient by each of these maximal ideals is isomorphic to $\mathbf{Q}(\sqrt{2})$, with one sending T to $\sqrt{2}$ and the other sending T to $-\sqrt{2}$.

Theorem 4. *With notation as in the proof of Theorem* 1*, the set of maximal ideals in $L_1 \otimes_K L_2$ is in bijection with the set of K-algebra isomorphisms $L_1 \to L_2$.*

Proof. We want to describe a bijection between the sets

$$\{K\text{-algebra isomorphisms } L_1 \to L_2\} \longleftrightarrow \{\text{Maximal ideals in } L_1 \otimes_K L_2\}.$$

Let $L_1 \xrightarrow{\varphi} L_2$ be a K-algebra isomorphism. To construct from φ a maximal ideal in $L_1 \otimes_K L_2$, we will construct a homomorphism from $L_1 \otimes_K L_2$ onto a field and then take its kernel. The function $L_1 \times L_2 \to L_2$ where $(x, y) \mapsto \varphi(x)y$ is K-bilinear, so there is a K-linear map

$$L_1 \otimes_K L_2 \xrightarrow{f_\varphi} L_2$$

where $f_\varphi(x \otimes y) = \varphi(x)y$. This is onto since $f_\varphi(1 \otimes y) = y$. A computation shows f_φ is multiplicative on products of elementary tensors, so f_φ is a K-algebra homomorphism. Since f_φ is surjective and L_2 is a field, the kernel of f_φ is a maximal ideal. Set

$$\mathfrak{m}_\varphi = \ker f_\varphi.$$

The correspondence $\varphi \rightsquigarrow \mathfrak{m}_\varphi$ is a mapping from K-algebra isomorphisms $L_1 \to L_2$ to maximal ideals in $L_1 \otimes_K L_2$.

To show $\varphi \rightsquigarrow \mathfrak{m}_\varphi$ is injective, we will recover φ from $\mathfrak{m}_\varphi$. The composite of the natural maps

$$L_1 \longrightarrow L_1 \otimes_K L_2 \longrightarrow (L_1 \otimes_K L_2)/\mathfrak{m}_\varphi \xrightarrow{\overline{f_\varphi}} L_2$$

has the effect $x \mapsto x \otimes 1 \mapsto x \otimes 1 \bmod \mathfrak{m}_\varphi \mapsto \varphi(x) \cdot 1 = \varphi(x)$.

Now we show $\varphi \rightsquigarrow \mathfrak{m}_\varphi$ is surjective. Let $\mathfrak{m}$ be any maximal ideal in $L_1 \otimes_K L_2$. We want to find a K-algebra isomorphism $L_1 \xrightarrow{\varphi} L_2$ such that $\mathfrak{m} = \mathfrak{m}_\varphi$. We know from the proof of Theorem 1 that for any maximal ideal $\mathfrak{m}$ of $L_1 \otimes_K L_2$, the natural composite maps

$$L_1 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m} \quad \text{and} \quad L_2 \to L_1 \otimes_K L_2 \to (L_1 \otimes_K L_2)/\mathfrak{m}$$

are K-algebra isomorphisms. Call the first one $\varphi_{1,\mathfrak{m}}$ and call the second one $\varphi_{2,\mathfrak{m}}$. Set $\varphi = \varphi_{2,\mathfrak{m}}^{-1} \circ \varphi_{1,\mathfrak{m}}$, so φ is a K-algebra isomorphism from L_1 to L_2. We will show the diagram

$$\begin{tikzcd} & L_1 \otimes_K L_2 \arrow[dl, "f_\varphi"'] \arrow[dr, "\text{redn.}"] & \\ L_2 \arrow[rr, "\varphi_{2,\mathfrak{m}}"'] & & (L_1 \otimes_K L_2)/\mathfrak{m} \end{tikzcd}$$

commutes. Then since $\varphi_{2,\mathfrak{m}}$ is an isomorphism, the kernels of the two maps out of $L_1 \otimes_K L_2$ would be equal, so we get $\mathfrak{m}_\varphi = \ker f_\varphi = \mathfrak{m}$, as desired.

To verify commutativity of the diagram, it suffices (by additivity of all the maps) to focus on elementary tensors $x \otimes y$ in $L_1 \otimes_K L_2$, where we want to check

$$\varphi_{2,\mathfrak{m}}(f_\varphi(x \otimes y)) \stackrel{?}{\equiv} x \otimes y \bmod \mathfrak{m}.$$

The left side is

$$\begin{aligned}
\varphi_{2,\mathfrak{m}}(f_\varphi(x\otimes y)) &= \varphi_{2,\mathfrak{m}}(\varphi(x)y)) \\
&= \varphi_{2,\mathfrak{m}}(\varphi(x))\varphi_{2,\mathfrak{m}}(y) \\
&= \varphi_{2,\mathfrak{m}}(\varphi_{2,\mathfrak{m}}^{-1}(\varphi_{1,\mathfrak{m}}(x)))\varphi_{2,\mathfrak{m}}(y) \\
&= \varphi_{1,\mathfrak{m}}(x)\varphi_{2,\mathfrak{m}}(y) \\
&= (x\otimes 1)(1\otimes y) \bmod \mathfrak{m} \\
&= x\otimes y \bmod \mathfrak{m}.
\end{aligned}$$

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