5.4 Eigenvectors and Linear Transformations

Change-of-coordinates matrices (bases B, C)

$$\underset{\mathcal{E}\leftarrow B}{P} = P_B = [\vec{b}_1 \dots \vec{b}_n]$$

$$\underset{B\leftarrow \mathcal{E}}{P} = P_B^{-1} = [\vec{b}_1 \dots \vec{b}_n]^{-1}$$

$$\underset{C\leftarrow B}{P} = P_C^{-1} P_B = [\vec{c}_1 \dots \vec{c}_n]^{-1} [\vec{b}_1 \dots \vec{b}_n]$$ or row reduce $\begin{array}{c}[P_C \mid P_B] \\ \downarrow \\ [I \mid P_C^{-1}P_B]\end{array}$

maps (linear transformation $T: V \to W$, matrix A, bases B, C)

$$\underset{\mathcal{E}\leftarrow\mathcal{E}}{A} = A = [T(\vec{e}_1) \dots T(\vec{e}_n)]$$

$$\underset{C\leftarrow B}{M} = \left[[T(\vec{b}_1)]_C \; [T(\vec{b}_2)]_C \dots [T(\vec{b}_n)]_C \right] = \underset{C\leftarrow\mathcal{E}}{P} \cdot \underset{\mathcal{E}\leftarrow\mathcal{E}}{A} \cdot \underset{\mathcal{E}\leftarrow B}{P}$$

EX 1 Let $B = \{\vec{b}_1, \vec{b}_2\}$ be a basis for $\mathbb{R}^2$ and $D = \{\vec{d}_1, \vec{d}_2, \vec{d}_3\}$ be a basis for $\mathbb{R}^3$. Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation with the property that

$$T(\vec{b}_1) = \vec{d}_1 + \vec{d}_2 - \vec{d}_3 \quad \text{and} \quad T(b_2) = 5\vec{d}_2 + 3\vec{d}_3$$

Find the matrix for T relative to B and D.

EX2. Let $T: \mathbb{P}_2 \to \mathbb{P}_3$ be the transformation that maps a polynomial $p(t)$ to the polynomial $tp(t) + t^2 p(1)$.

a) Find the image of $p(t) = 1 + 2t + 3t^2$

b) Find the image of $p(t) = a + bt + ct^2$

c) Find the matrix of T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$.

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maps - special case (same as above but $T: V \to V$, $B = C$)

$$\underset{B \leftarrow B}{C} = \left[[T(\vec{b}_1)]_B \; [T(\vec{b}_2)]_B \cdots [T(\vec{b}_n)]_B \right] = \underset{B \leftarrow \mathcal{E}}{P} \cdot \underset{\mathcal{E} \leftarrow \mathcal{E}}{A} \cdot \underset{\mathcal{E} \leftarrow B}{P}$$

$$= \boxed{P_B^{-1} A P_B}$$

which means C is similar to A!

EX 3 $A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Find $\underset{B \leftarrow B}{C}$.

EX 4 $A = \begin{bmatrix} 1 & -3 \\ -4 & 5 \end{bmatrix}$, $\vec{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

If a basis B exists such that $P_B^{-1} A P_B$ is a diagonal matrix D, then A is diagonalizable. The basis B will be the eigenvectors of A, and the diagonal elements of D will be the eigenvalues of A.

EX 5 Find a basis for $\mathbb{R}^2$ to diagonalize $A = \begin{bmatrix} 4 & 1 \\ 3 & 6 \end{bmatrix}$