Math 2210Q (Roby) **Practice Final** Spring 2016

SHOW ALL YOUR WORK! Give reasons to support your answers. No calculators allowed, but you may use one 8.5″ × 11″ sheet of notes with anything you like written on it.

1. Define a linear transformation $T : \mathbb{P}_2 \to R^3$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.
   (a) Find the image under T of $\mathbf{p}(t) = 5 + 3t$.
   (b) Show that T is a linear transformation.
   (c) Find the matrix for T relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$.
   (d) Is T one-to-one? Is T onto? Explain!

2. Find the characteristic polynomial and the eigenvalues of the matrix $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$.

3. Show that if $T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$, then $\det T = (b-a)(c-a)(c-b)$.

4. Prove or (use a counterexample to) Disprove and Salvage if possible:
   (a) If $A = QR$, where Q has orthonormal columns, then $R = Q^T A$.
   (b) If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of vectors in $\mathbb{R}^n$, then S is linearly independent.
   (c) If A and B are invertible $n \times n$ matrices, then AB is similar to BA.
   (d) Each eigenvector of a square matrix A is also an eigenvector of A^2.
   (e) There exists a 2×2 matrix with real entries that has no eigenvectors in $\mathbb{R}^2$.
   (f) If A is row equivalent to the identity matrix I, then A is diagonalizable.

5. Decide whether each statement below is True of False. Justify your answer. For False statements, a counterexample is usually best. Extra credit for good salvages and more for proofs thereof!
   (a) If $\mathbf{y}$ is in a subspace W, then the orthogonal projection of $\mathbf{y}$ onto W is $\mathbf{y}$ itself.
   (b) For an $m \times n$ matrix A, vectors in $\operatorname{Nul} A$ are orthogonal to vectors in $\operatorname{Row} A$.
   (c) The matrices A and A^T have the same eigenvalues, counting multiplicities.
   (d) A nonzero vector can correspond to two different eigenvalues of A.
   (e) The sum of two eigenvectors of a square matrix A is also an eigenvector of A.
   (f) The singular values of an $m \times n$ matrix can be $3, 1, -1, -3$.

6. If a $n \times n$ matrix A satifies $A^2 = A$, what can you say about the determinant of A?

7. Assume that matrices A and B below are row equivalent:

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Without calculations, list $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$. Then find bases for $\operatorname{Col} A$, $\operatorname{Row} A$, and $\operatorname{Nul} A$.

8. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$ subject to the constraint $x_1^2 + x_2^2 = 1$. (You do not need to compute a vector at which this maximum is attained.)

9. What would you have to know about the solution set of a homogenous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss!

10. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points.

11. Here are some specific tasks (modified from a list of Prof. Leibowitz) I expect you to be able to perform **with demonstrated understanding**:

    (a) Given a matrix A, find the dimensions of and bases for $\operatorname{Col} A$, $\operatorname{Nul} A$, and $\operatorname{Row} A$. Use the relations among rank, $\dim \operatorname{Nul} A$, and size of A to understand properties of the associated linear transformation (one-to-one, onto, kernel, range).

    (b) Use row reduction to solve linear equations, show that a given column vector is in the span of a given set of vectors, or compute the inverse of a matrix.

    (c) Use row operations to reduce a matrix A to triangular form in order to calculate $\det A$. Use properties to compute the determinant of related matrices.

    (d) Diagonalize a given matrix and use the $A = PDP^{-1}$ factorization to calculate a power of A.

    (e) Orthogonally diagonalize a real symmetric matrix, possibly representing a quadratic form, and compute constrained extrema of the form.

    (f) Project a vector onto the subspace spanned by a given set of vectors, after verifying that they form an orthogonal set.

    (g) Understand how to construct and use the the singular value decomposition of an $m \times n$ matrix.

    (h) Understand the theory of the course well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.

    (i) Show that a certain set of vectors is a subspace of a given space, or are eigenvectors of a certain matrix.

    (j) Use various forms of the Invertible Matrix Theorem in context.