Introduction

- Quadratic forms generalize taking the inner (dot, scalar) product of a vector with itself.
- They come up in engineering applications involving optimization and signal processing, utility functions in economics, confidence ellipsoids in statistics, etc.
- There is a natural progression from diagonalization of symmetric matrics through this topic to the SVD.
- Key result: By an (orthogonal) change of variable, any quadratic form is equivalent to a form without cross terms;
- Key result: The "sign" of a quadratic form (positive definite, negative definite, indefinite) is determined by the sign of its *spectrum* (i.e., eigenvalues);

Quadratic forms from symmetric matrices

Definition 1. A **quadratic form** $\mathcal{Q}$ on $\mathbb{R}^n$ is a function $\mathcal{Q}:\mathbb{R}^n \to \mathbb{R}$ of the form $\mathcal{Q}(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$, where A is a *symmetric matrix*, called the **matrix of the quadratic form**.

Example 2. What are the quadratic forms corresponding to the following matrices?

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -3 \\ -3 & 4 \end{bmatrix}, \quad \text{and } C = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix}.$$

Example 3. What is the matrix A of the quadratic form $\mathcal{Q}:\mathbb{R}^3 \to \mathbb{R}$ given by $\mathcal{Q}(\mathbf{x}) = -4x_1^2 + 7x_2^2 - 5x_3^2 - 6x_1x_2 + 3x_2x_3$?

Change of variables & Principal Axes Theorem

Recall that any $n \times n$ *invertible* matrix P (whose columns $\mathcal{B}$ are a basis for $\mathbb{R}^n$), represents a change of basis from standard $\mathcal{E}$-coordinates to $\mathcal{B}$-coordinates:

$$\mathbf{x} = P\mathbf{y} \iff \mathbf{y} = P^{-1}\mathbf{x} \iff [\mathbf{y}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{E}}$$

Question 4. *What happens if we change variable in a quadratic form? Why is this so great? How can we use that A is symmetric?*

Example 5. Orthogonally diagonalize the QF on $\mathbb{R}^2$ given by $\mathcal{Q}(\mathbf{x}) = 2x_1^2 - 4x_1x_2 + 5x_2^2$.

Theorem 6. *Let $\mathcal{Q}$ be a quadratic form on $\mathbb{R}^n$ corresponding to the (symmetric) matrix A. Then we can find an orthogonal (change of basis) matrix P such that $\mathbf{x} = P\mathbf{y}$, transforming $\mathcal{Q}(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ to $\mathcal{Q}(\mathbf{y}) = \mathbf{y}^\top D\mathbf{y}$, with no cross term. ($D$ is diagonal.)*

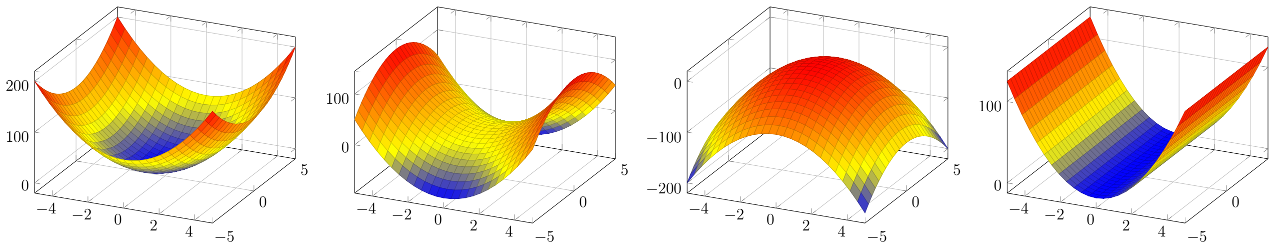
The columns of P are called the **principal axes** of the quadratic form $\mathcal{Q}$.

See the text for pictures explaining how quadratic forms correspond to conic sections. Those with nonzero cross terms are rotated relative to the standard position. Eliminating the cross terms is equivalent to the rotation-of-axes change of variables technique that used to be taught in some HS "college algebra" classes.

Classifying quadratic forms

Pictured below are graphs of the quadratic forms

(a) $5x_1^2 + 3x_2^2$, (b) $5x_1^2 - 3x_2^2$, (c) $5x_1^2 - 3x_2^2$ and (d) $5x_1^2$



Definition 7. Call a quadratic form $\mathcal{Q}$:

a. **positive definite** if $\mathcal{Q}(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$;
b. **positive semidefinite** if $\mathcal{Q}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \neq \mathbf{0}$;
c. **negative definite** if $\mathcal{Q}(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$;
d. **negative semidefinite** if $\mathcal{Q}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \neq \mathbf{0}$; and
e. **indefinite** if $\mathcal{Q}$ assume both positive and negative values.

Example 8. Classify each of the quadratic forms above.

Theorem 9. *For an $n \times n$ symmetric matrix A, the quadratic form $Q(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$ is:*

a. *positive definite if and only if all the eigenvalues of A are positive.*
b. *positive semidefinite if and only if all the eigenvalues of A are ≥ 0.*
c. *negative definite if and only if all the eigenvalues of A are negative.*
d. *negative semidefinite if and only if all the eigenvalues of A are ≤ 0.*
e. *indefinite if and only if A has both positive and negative eigenvalues.*

Proof. Use the Principal Axes Theorem.

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Example 10. Classify the QF $\mathcal{Q} = x_1^2 + 3x_2^2 + x_3^2 + 2x_1x_2 + 6x_1x_3 + 2x_2x_3$.