1.7 Linear Independence

A homogeneous system such as

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be viewed as a vector equation

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The vector equation has the trivial solution ($x_1 = 0$, $x_2 = 0$, $x_3 = 0$), but is this the *only solution*?

Definition

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exists weights $c_1, \ldots, c_p,$ not all 0, such that

$$\underbrace{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}}.$$

linear dependence relation
(when weights are not all zero)

EXAMPLE Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$.

a. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.
b. If possible, find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Solution: (a)

$$x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free variable $\Rightarrow$ there are nontrivial solutions.

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set

(b) Reduced echelon form: $\begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & 1 & -18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{ll} x_1 & = \\ x_2 & = \\ x_3 & \end{array}$

Let $x_3 =$ _____ (any nonzero number).

Then $x_1 =$ _____ and $x_2 =$ _____.

$$\_\_\_\_\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + \_\_\_\_\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + \_\_\_\_\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\mathbf{v}_2 + \_\_\_\_\mathbf{v}_3 = \mathbf{0}$$

(one possible linear dependence relation)

Linear Independence of Matrix Columns

A linear dependence relation such as

$$-33\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 18\begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + 1\begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

can be written as the matrix equation:

$$\begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix}\begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.

The columns of matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

Special Cases

Sometimes we can determine linear independence of a set with minimal effort.

1. *A Set of One Vector*

Consider the set containing one nonzero vector: $\{\mathbf{v}_1\}$

The only solution to $x_1\mathbf{v}_1 = 0$ is $x_1 =$ _____.

So $\{\mathbf{v}_1\}$ is linearly independent when $\mathbf{v}_1 \neq \mathbf{0}$.

2. *A Set of Two Vectors*

EXAMPLE Let

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

a. Determine if $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly dependent set or a linearly independent set.

b. Determine if $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set or a linearly independent set.

Solution: (a) Notice that $\mathbf{u}_2 =$ _____$\mathbf{u}_1$. Therefore

$$\_\_\_\_\_\mathbf{u}_1 + \_\_\_\_\_\mathbf{u}_2 = 0$$

This means that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a linearly ____________________ set.

(b) Suppose

$$c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}.$$

Then $\mathbf{v}_1 = $ ______$\mathbf{v}_2$ if $c \neq 0$. But this is impossible since $\mathbf{v}_1$ is

_______ a multiple of $\mathbf{v}_2$ which means $c = $ _____.

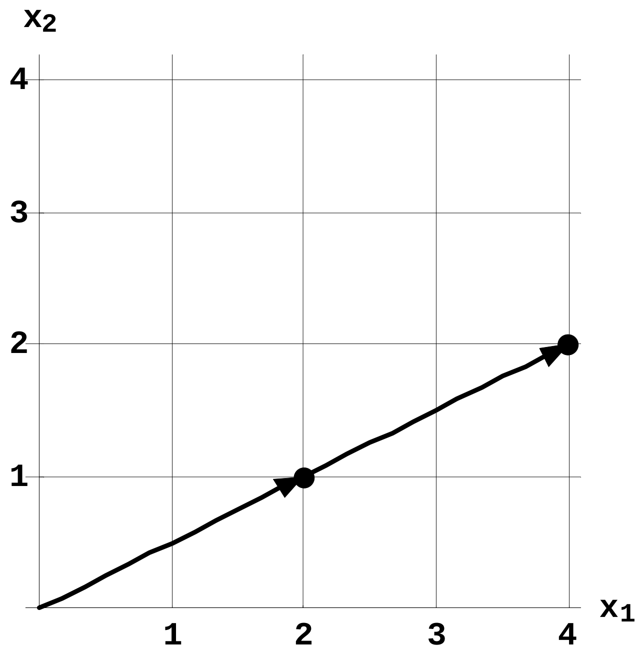
Similarly, $\mathbf{v}_2 = $ ______$\mathbf{v}_1$ if $d \neq 0$.

But this is impossible since $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$ and so $d = 0$.

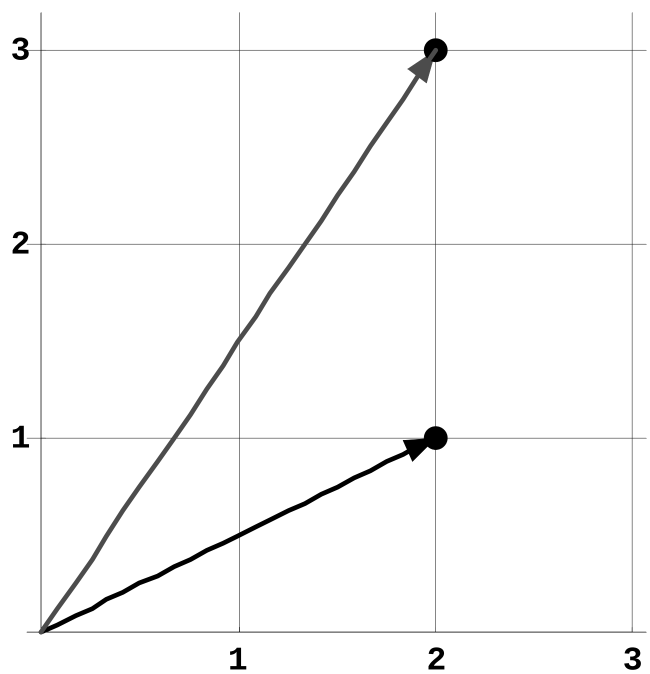
This means that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly ____________________ set.

A set of two vectors is linearly dependent if at least one vector is a multiple of the other.

A set of two vectors is linearly independent if and only if neither of the vectors is a multiple of the other.



linearly ______________________



linearly ______________________

3. ***A Set Containing the* 0 *Vector***

Theorem 9

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ containing the zero vector is linearly dependent.

Proof: Renumber the vectors so that $\mathbf{v}_1 =$ ____. Then

$$\_\_\_\_\mathbf{v}_1 + \_\_\_\_\_\mathbf{v}_2 + \cdots + \_\_\_\_\_\mathbf{v}_p = \mathbf{0}$$

which shows that S is linearly ________________.

4. ***A Set Containing Too Many Vectors***

Theorem 8

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. I.e. any set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is linearly dependent if $p > n$.

Outline of Proof:

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_p \end{bmatrix} \text{ is } n \times p$$

Suppose $p > n$.

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has more variables than equations

$\Rightarrow A\mathbf{x} = \mathbf{0}$ has nontrivial solutions

$\Rightarrow$columns of A are linearly dependent

EXAMPLE With the least amount of work possible, decide which of the following sets of vectors are linearly independent and give a reason for each answer.

a. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$

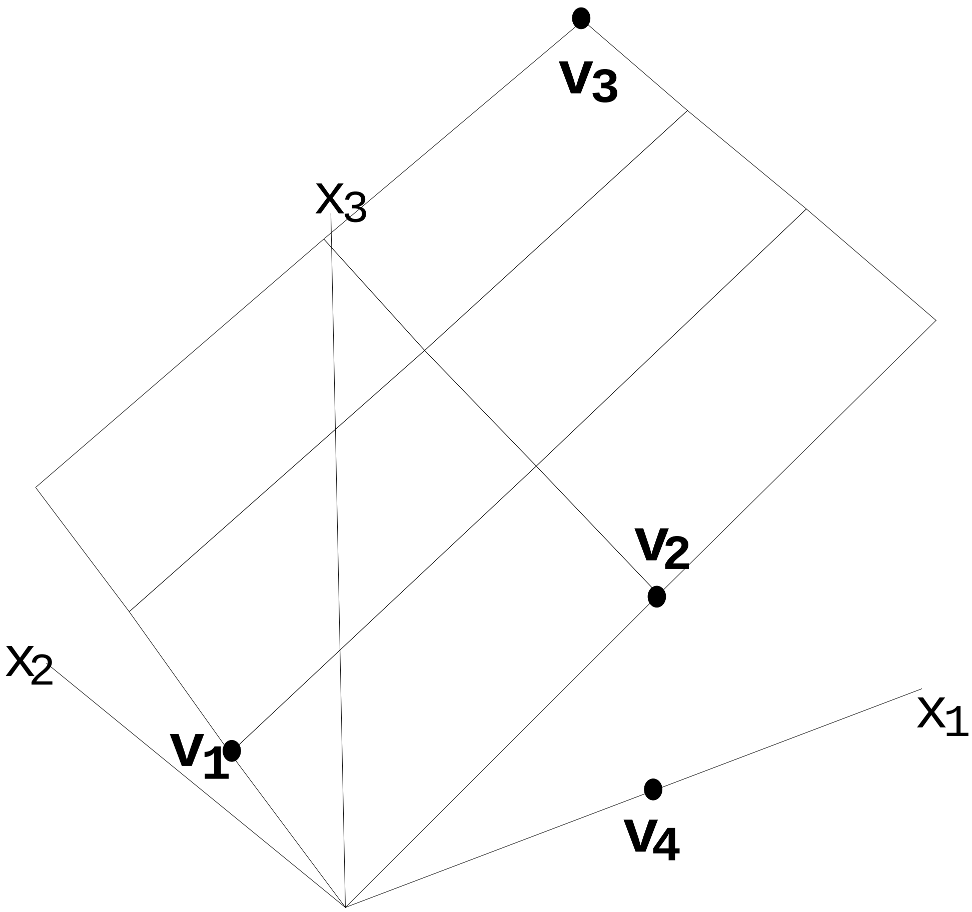
b. Columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 & 8 \end{bmatrix}$

c. $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

d. $\left\{ \begin{bmatrix} 8 \\ 2 \\ 1 \\ 4 \end{bmatrix} \right\}$

Characterization of Linearly Dependent Sets

EXAMPLE Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ in $\mathbf{R}^3$ in the following diagram. Is the set linearly dependent? Explain



Theorem 7

An indexed set $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent, and $\mathbf{v}_1 \neq \mathbf{0}$, then some vector $\mathbf{v}_j$ ($j \geq 2$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.