2.2 The Inverse of a Matrix

The inverse of a real number a is denoted by a^{-1}. For example, $7^{-1} = 1/7$ and

$$7 \cdot 7^{-1} = 7^{-1} \cdot 7 = 1$$

An $n \times n$ matrix A is said to be **invertible** if there is an $n \times n$ matrix C satisfying

$$CA = AC = I_n$$

where I_n is the $n \times n$ identity matrix. We call C the **inverse** of A .

FACT If A is invertible, then the inverse is unique.

Proof: Assume B and C are both inverses of A. Then

$$B = BI = B(\_\_\_\_\_) = (\_\_\_\_\_\_)\_\_\_\_\_\_ = I\_\_\_\_\_\_ = C.$$

So the inverse is unique since any two inverses coincide.■

The inverse of A is usually denoted by A^{-1}.

We have

$$\boxed{AA^{-1} = A^{-1}A = I_n}$$

Not all $n \times n$ ***matrices are invertible.*** A matrix which is *not* invertible is sometimes called a **singular** matrix. An invertible matrix is called **nonsingular** matrix.

Theorem 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

Assume A is any invertible matrix and we wish to solve $A\mathbf{x} = \mathbf{b}$. Then

$$\_\_\_\_\_A\mathbf{x} = \_\_\_\_\_\mathbf{b} \qquad \text{and so}$$

$$I\mathbf{x} = \_\_\_\_\_\_\_\_ \text{ or } \mathbf{x} = \_\_\_\_\_\_\_\_.$$

Suppose $\mathbf{w}$ is also a solution to $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{w} = \mathbf{b}$ and

$$\_\_\_\_\_A\mathbf{w} = \_\_\_\_\_\mathbf{b} \qquad \text{which means} \qquad \mathbf{w} = A^{-1}\mathbf{b}.$$

So, $\mathbf{w} = A^{-1}\mathbf{b}$, which is in fact the same solution.

We have proved the following result:

Theorem 5

If A is an invertible $n \times n$ matrix, then for each $\mathbf{b}$ in $\mathbf{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.

EXAMPLE: Use the inverse of $A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}$ to solve

$$\begin{array}{rcrcr} -7x_1 & + & 3x_2 & = & 2 \\ 5x_1 & - & 2x_2 & = & 1 \end{array}.$$

Solution: Matrix form of the linear system:

$$\begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14-15}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}.$$

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Theorem 6 Suppose A and B are invertible. Then the following results hold:

a. A^{-1} is invertible and $(A^{-1})^{-1} = A$ (i.e. A is the inverse of A^{-1}).

b. AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

c. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

Partial proof of part b:

$$(AB)(B^{-1}A^{-1}) = A(\_\_\_\_\_\_\_\_\_\_)A^{-1}$$

$$= A(\_\_\_\_\_\_\_\_\_\_\_)A^{-1} = \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_.$$

Similarly, one can show that $(B^{-1}A^{-1})(AB) = I$.

Theorem 6, part b can be generalized to three or more invertible matrices:

$$(ABC)^{-1} = \_\_\_\_\_\_\_\_\_\_\_$$

Earlier, we saw a formula for finding the inverse of a 2×2 invertible matrix. How do we find the inverse of an invertible $n \times n$ matrix? To answer this question, we first look at **elementary** matrices.

Elementary Matrices

Definition

An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

EXAMPLE: Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ and $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.

E_1, E_2, and E_3 are elementary matrices. Why?

Observe the following products and describe how these products can be obtained by elementary row operations on A.

$$E_1A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$$

$$E_2A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix}$$

$$E_3A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 3a+g & 3b+h & 3c+i \end{bmatrix}$$

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operations on I_m.

Elementary matrices are *invertible* because row operations are *reversible*. To determine the inverse of an elementary matrix E, determine the elementary row operation needed to transform E back into I and apply this operation to I to find the inverse.

For example,

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \qquad E_3^{-1} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}$. Then

$$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3(E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So

$$\boxed{E_3 E_2 E_1 A = I_3}.$$

Then multiplying on the right by A^{-1}, we get

$$E_3 E_2 E_1 A\underline{\hspace{2cm}} = I_3\underline{\hspace{2cm}}.$$

So

$$\boxed{E_3 E_2 E_1 I_3 = A^{-1}}$$

The elementary row operations that row reduce A to $\mathbf{I}_n$ are the same elementary row operations that transform $\mathbf{I}_n$ into $\mathbf{A}^{-1}$.

Theorem 7

An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduces A to I_n will also transform I_n to A^{-1}.

Algorithm for finding $\mathbf{A}^{-1}$

Place A and I side-by-side to form an augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. Then perform row operations on this matrix (which will produce identical operations on A and I). So by Theorem 7:

$$\begin{bmatrix} A & I \end{bmatrix} \text{ will row reduce to } \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

or A is not invertible.

EXAMPLE: Find the inverse of $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, if it exists.

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{bmatrix}$$

So $A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 \\ \frac{3}{2} & 1 & 0 \end{bmatrix}$

Order of multiplication is important!

EXAMPLE Suppose A, B, C, and D are invertible $n \times n$ matrices and $A = B(D - I_n)C$.

Solve for D in terms of A, B, C and D.

Solution:

$$\underline{\hspace{3em}}A\underline{\hspace{3em}} = \underline{\hspace{3em}}B(D - I_n)C\underline{\hspace{3em}}$$

$$D - I_n = B^{-1}AC^{-1}$$

$$D - I_n + \underline{\hspace{2.5em}} = B^{-1}AC^{-1} + \underline{\hspace{2.5em}}$$

$$D = \underline{\hspace{12em}}$$