Math 3240/Roby **Practice Midterm** October 2010

SHOW ALL YOUR WORK! Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask!

1. How many solutions does each of the following equations have?
   (a) $17x \equiv 23 \pmod{100}$. (b) $15x \equiv 23 \pmod{100}$. (c) $15x \equiv 25 \pmod{100}$.
2. Find all the solutions to each of the equations above.
3. Compute $2^{1999} \pmod{17}$ and $\pmod{72}$. Explain why your solutions work.
4. Find all solutions to the following systems of congruences:
   a) $x \equiv 7 \pmod{15}$, $x \equiv 19 \pmod{21}$.
   b) $x \equiv 11 \pmod{15}$, $x \equiv 8 \pmod{21}$, $x \equiv 14 \pmod{35}$.
5. How many elements does $\mathbf{Z}_{49}^{\times}$ have? Does $\mathbf{Z}_{49}^{\times}$ have a generator (i.e., a primitive root)? Explain.
6. Prove or Disprove and Salvage if possible:
   (a) If $a \mid b$ and $b \mid c$, then $a \mid c$.
   (b) $ac \equiv bc \pmod{n} \Rightarrow a \equiv b \pmod{n}$.
   (c) The product of the first n positive primes plus one is also a prime for all $n \geq 1$.
   (d) The square of an odd number is always congruent to $1 \pmod{8}$.
   (e) The number 111 (all 1's) with 899 digits is prime in $\mathbf{Z}$.
   (f) The number $13 + 2i$ is prime in $\mathbf{Z}[i]$.
7. Make a table of logarithms for $\mathbf{Z}_{19}^{\times}$. Use your table to find all solutions of the following equations in $\mathbf{Z}_{19}$:
   (a) $6x^6 + 5 \equiv 9 \pmod{19}$. (b) $3x^4 - 4 \equiv 1 \pmod{19}$
8. Is it possible to have 35 successive integers all be composite numbers? Why or why not?
9. Be prepared to state the following theorems (carefully!) and give a proof (of all but the last one).
   (a) Fundamental Theorem of Arithmetic
   (b) Euclid's theorem on the infinitude of primes
   (c) Fermat's Little Theorem
   (d) Wilson's Theorem.
   (e) There exist numbers which are not rational.
   (f) Dirichlet's Theorem on primes in arithmetic progressions
10. State and prove tests for divisibility by k for $k = 1, 2, \ldots 11$.
11. Go back over your old homework and make sure you understand any problem on which you lost points.