Section 1.9 (Through Theorem 10) The Matrix of a Linear Transformation

Identity Matrix I_n is an $n \times n$ matrix with 1's on the main left to right diagonal and 0's elsewhere. The ith column of I_n is labeled $\mathbf{e}_i$.

EXAMPLE:

$$I_3 = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that

$$I_3\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \underline{\qquad}\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} + \underline{\qquad}\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} + \underline{\qquad}\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \underline{\qquad}.$$

In general, for $\mathbf{x}$ in $\mathbf{R}^n$,

$$I_n\mathbf{x} = \underline{\quad}$$

From Section 1.8, if $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation, then $T(c\mathbf{u} + d\mathbf{v}) = c\mathbf{T}(\mathbf{u}) + d\mathbf{T}(\mathbf{v})$.

Generalized Result:

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p).$$

EXAMPLE: The columns of $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^3$ where

$$T(\mathbf{e}_1) = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} \text{ and } T(\mathbf{e}_2) = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}.$$

Compute $T(\mathbf{x})$ for any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Solution: A vector $\mathbf{x}$ in $\mathbf{R}^2$ can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{\qquad}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \underline{\qquad}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underline{\qquad}\mathbf{e}_1 + \underline{\qquad}\mathbf{e}_2$$

Then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) = \underline{\qquad}T(\mathbf{e}_1) + \underline{\qquad}T(\mathbf{e}_2)$$

$$= \underline{\qquad}\begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} + \underline{\qquad}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \qquad\qquad\qquad \\ \\ \\ \end{bmatrix}.$$

Note that

$$T(\mathbf{x}) = \begin{bmatrix} \qquad\qquad \\ \\ \\ \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

So

$$T(\mathbf{x}) = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix}\mathbf{x} = A\mathbf{x}$$

To get A, replace the identity matrix $\begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$ with $\begin{bmatrix} T(\mathbf{e}_2) & T(\mathbf{e}_2) \end{bmatrix}$.

Theorem 10

Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation. Then there exists a unique matrix A such that

$$T(\mathbf{x}) = A\mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the jth column of the identity matrix in $\mathbf{R}^n$.

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad \cdots \quad T(\mathbf{e}_n)]$$

$$\uparrow$$

standard matrix for the linear transformation T

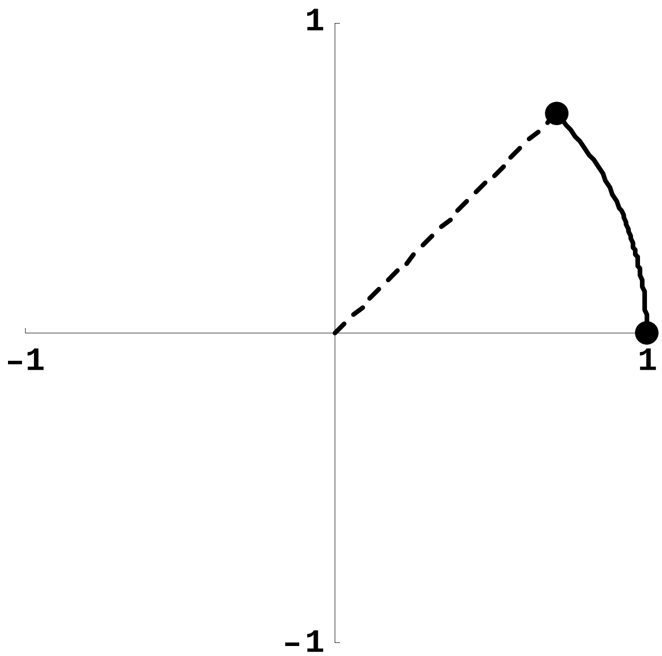
EXAMPLE: $$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ 4x_1 \\ 3x_1 + 2x_2 \end{bmatrix}$$

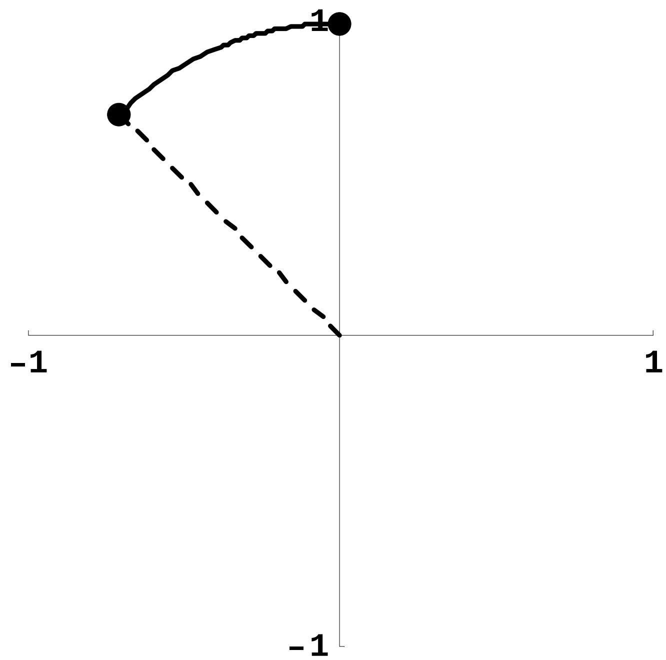
Solution:

$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} =$ standard matrix of the linear transformation T

$$\begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix} = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} =$$

EXAMPLE: Find the standard matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates a point about the origin through an angle of $\frac{\pi}{4}$ radians (counterclockwise).





$$T(\mathbf{e}_1) = \begin{bmatrix} \quad & \\ & \quad \end{bmatrix} \qquad T(\mathbf{e}_2) = \begin{bmatrix} \quad & \\ & \quad \end{bmatrix}$$

$$\Rightarrow \qquad A = \begin{bmatrix} \qquad & \qquad \\ & \end{bmatrix}$$