4.5 The Dimension of a Vector Space

THEOREM 9

If a vector space V has a basis $\beta = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Proof: Suppose $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is a set of vectors in V where $p > n$. Then the coordinate vectors $\left\{[\mathbf{u}_1]_\beta, \cdots, [\mathbf{u}_p]_\beta\right\}$ are in $\mathbf{R}^n$. Since $p > n$, $\left\{[\mathbf{u}_1]_\beta, \cdots, [\mathbf{u}_p]_\beta\right\}$ are linearly dependent and therefore $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$are linearly dependent. ■

THEOREM 10

If a vector space V has a basis of n vectors, then every basis of V must consist of n vectors.

Proof: Suppose β_1 is a basis for V consisting of exactly n vectors. Now suppose β_2 is any other basis for V. By the definition of a basis, we know that β_1 and β_2 are both linearly independent sets.

By Theorem 9, if β_1 has more vectors than β_2, then ______ is a linearly dependent set (which cannot be the case).

Again by Theorem 9, if β_2 has more vectors than β_1, then ______ is a linearly dependent set (which cannot be the case).

Therefore β_2 has exactly n vectors also. ■

DEFINITION

If V is spanned by a finite set, then V is said to be **finite-dimensional**, and the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be 0. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.

EXAMPLE: The standard basis for $\mathbf{P}_3$ is $\{ \qquad\qquad\qquad\qquad \}$. So $\dim \mathbf{P}_3 =$ ______.

$$\boxed{\text{In general, } \dim \mathbf{P}_n = n + 1.}$$

EXAMPLE: The standard basis for $\mathbf{R}^n$ is $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the columns of I_n. So, for example, $\dim \mathbf{R}^3 = 3$.

EXAMPLE: Find a basis and the dimension of the subspace

$$W = \left\{ \begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} : a,b,c,d \text{ are real} \right\}.$$

Solution: Since

$$\begin{bmatrix} a+b+2c \\ 2a+2b+4c+d \\ b+c+d \\ 3a+3c+d \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

$$W = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$$

where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

- Note that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, so by the Spanning Set Theorem, we may discard $\mathbf{v}_3$.
- $\mathbf{v}_4$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. So $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is a basis for W. Also, dim $W =$____.

EXAMPLE: ***Dimensions of subspaces of R***^{3}

0-*dimensional subspace* contains only the zero vector

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

1-*dimensional subspaces.* Span$\{$**v**$\}$ where **v** $\neq$ **0** is in **R**^{3}.

These subspaces are _______________ through the origin.

2-*dimensional subspaces.* Span$\{$**u**, **v**$\}$ where **u** and **v** are in **R**^{3} and are not multiples of each other.

These subspaces are _______________ through the origin.

3-*dimensional subspaces.* Span$\{$**u**, **v**, **w**$\}$ where **u, v**, **w** are linearly independent vectors in **R**^{3}. This subspace is **R**^{3} itself because the columns of $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} & \mathbf{w} \end{bmatrix}$ span **R**^{3} according to the IMT.

THEOREM 11

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V.$$

EXAMPLE: Let $H = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$. Then H is a subspace of $\mathbf{R}^3$ and $\dim H < \dim \mathbf{R}^3$.

We could expand the spanning set $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$ to $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ to form a basis for $\mathbf{R}^3$.

THEOREM 12 THE BASIS THEOREM

Let V be a p – dimensional vector space, $p \geq 1$. Any linearly independent set of exactly p vectors in V is automatically a basis for V. Any set of exactly p vectors that spans V is automatically a basis for V.

EXAMPLE: Show that $\{t, 1-t, 1+t-t^2\}$ is a basis for $\mathbf{P}_2$.

Solution: Let $\mathbf{v}_1 = t, \mathbf{v}_2 = 1-t, \mathbf{v}_3 = 1+t-t^2$ and $\beta = \{1, t, t^2\}$.

Corresponding coordinate vectors

$$[\mathbf{v}_1]_\beta = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [\mathbf{v}_2]_\beta = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, [\mathbf{v}_3]_\beta = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$[\mathbf{v}_2]_\beta$ is not a multiple of $[\mathbf{v}_1]_\beta$

$[\mathbf{v}_3]_\beta$ is not a linear combination of $[\mathbf{v}_1]_\beta$ and $[\mathbf{v}_2]_\beta$

$\Rightarrow$ $\left\{[\mathbf{v}_1]_\beta, [\mathbf{v}_2]_\beta, [\mathbf{v}_3]_\beta\right\}$ is linearly independent and therefore $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also linearly independent.

Since dim $\mathbf{P}_2 = 3$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbf{P}_2$ according to The Basis Theorem.

Dimensions of Col A and Nul A

Recall our techniques to find basis sets for column spaces and null spaces.

EXAMPLE: Suppose $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$. Find dim Col A and dim Nul A.

Solution

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} \quad \\ \quad \end{bmatrix}, \begin{bmatrix} \quad \\ \quad \end{bmatrix} \right\}$ is a basis for Col A and dim Col $A = 2$.

Now solve $A\mathbf{x} = \mathbf{0}$ by row-reducing the corresponding augmented matrix. Then we arrive at

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 7 & 8 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -2x_2 - 4x_4 \\ x_3 &= 0 \end{aligned}$$

and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A and

dim Nul $A = 2$.

Note

$$\boxed{\textit{dim Col } A = \textit{number of pivot columns of } A}$$

$$\boxed{\textit{dim Nul } A = \textit{number of free variables of } A}.$$