GROUPS OF ORDER 12

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We will use semidirect products to describe all groups of order 12. There turn out to be 5 such groups: 2 are abelian and 3 are nonabelian. The nonabelian groups are an alternating group, a dihedral group, and a third less familiar group.

Theorem 1. *Any group of order* 12 *is isomorphic to* $\mathbf{Z}/(12)$, $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$, A_4, D_6, *or the nontrivial semidirect product* $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$.

In the proof, we will appeal to an isomorphism property of semidirect products: $H \rtimes_\varphi K \cong H \rtimes_{\varphi \circ f} K$ for any homomorphism $\varphi \colon K \to \mathrm{Aut}(H)$ and automorphism $f \colon K \to K$. We will refer to this by saying that precomposing an action of K on H by an automorphism of K produces an isomorphic semidirect product.

Proof. Let $\#G = 12 = 2^2 \cdot 3$. We will first show one of the Sylow subgroups is normal. From the Sylow theorems,

$$n_2 | 3, \quad n_2 \equiv 1 \bmod 2, \quad n_3 | 4, \quad n_3 \equiv 1 \bmod 3.$$

Therefore $n_2 = 1$ or 3 and $n_3 = 1$ or 4. We want $n_2 = 1$ or $n_3 = 1$.

Suppose $n_3 \neq 1$, so $n_3 = 4$. A 3-Sylow subgroup has order 3, so any two different 3-Sylow subgroups intersect trivially. Each of the four 3-Sylow subgroups of G has two elements of order 3 shared by no other 3-Sylow, so the number of elements in G of order 3 is $2 \cdot 4 = 8$. This leaves us with $12 - 8 = 4$ elements in G not of order 3. A 2-Sylow subgroup has order 4 and contains no elements of order 3, so a single 2-Sylow subgroup accounts for the remaining elements and therefore $n_2 = 1$ if $n_3 \neq 1$.

Let P be a 2-Sylow and Q be a 3-Sylow in G, so one of P or Q is normal and $G = PQ$ (since P and Q intersect trivially and have the right sizes). We have $P \cong \mathbf{Z}/(4)$ or $P \cong (\mathbf{Z}/(2))^2$, and $Q \cong \mathbf{Z}/(3)$, so G is some semidirect product

$$\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3), \quad (\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3), \quad \mathbf{Z}/(3) \rtimes \mathbf{Z}/(4), \quad \mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2.$$

Since the Sylow subgroups are abelian, the semidirect products are abelian only for the direct product (trivial action). We will determine all the semidirect products, up to isomorphism, by working out all the ways $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$ can act by automorphisms on $\mathbf{Z}/(3)$ and all the ways $\mathbf{Z}/(3)$ can act by automorphisms on $\mathbf{Z}/(4)$ and $(\mathbf{Z}/(2))^2$. First we need to know the automorphism groups of these groups: $\mathrm{Aut}(\mathbf{Z}/(4)) \cong (\mathbf{Z}/(4))^\times$, $\mathrm{Aut}((\mathbf{Z}/(2))^2) \cong \mathrm{GL}_2(\mathbf{Z}/(2))$, and $\mathrm{Aut}(\mathbf{Z}/(3)) \cong (\mathbf{Z}/(3))^\times$.

Now we take cases depending on if $n_2 = 1$ or $n_3 = 1$. First suppose $n_2 = 1$, so the 2-Sylow is normal and the 3-Sylow acts on it. We are considering

$$\mathbf{Z}/(4) \rtimes \mathbf{Z}/(3), \quad (\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3).$$

In the first semidirect product, a homomorphism $\mathbf{Z}/(3) \to (\mathbf{Z}/(4))^\times$ is trivial since the domain has order 3 and the target has order 2, so the first semidirect product has to be trivial: it's the direct product $\mathbf{Z}/(4) \times \mathbf{Z}/(3)$, which is cyclic of order 12 (generator $(1, 1)$).

In the second semidirect product, we want all homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$. The trivial homomorphism leads to the direct product $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$. What about nontrivial homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$? Inside $\mathrm{GL}_2(\mathbf{Z}/(2))$ there is one subgroup of order 3: $\{(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}), (\begin{smallmatrix}0&1\\1&1\end{smallmatrix}), (\begin{smallmatrix}1&1\\1&0\end{smallmatrix})\}$. A nontrivial homomorphism $\varphi\colon \mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ is determined by sending 1 mod 3 to one of the two matrices A with order 3 (and then letting $\varphi(k \bmod 3) = A^k$ in general). The two matrices with order 3 in $\mathrm{GL}_2(\mathbf{Z}/(2))$ are inverses of each other, and precomposing one of these homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ with negation on $\mathbf{Z}/(3)$ turns it into the other homomorphism because it turns the value at 1 mod 3 into the inverse of what it was at first. Therefore the two nontrivial homomorphisms $\mathbf{Z}/(3) \to \mathrm{GL}_2(\mathbf{Z}/(2))$ are the same up to precomposition by an automorphism of $\mathbf{Z}/(3)$, and thus they give isomorphic semidirect products. So up to isomorphism there is one nontrivial semidirect product $(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3)$, as well as the trivial semiproduct $(\mathbf{Z}/(2))^2 \times \mathbf{Z}/(3)$.

Concretely, the nontrivial semidirect product $(\mathbf{Z}/(2))^2 \rtimes \mathbf{Z}/(3)$ is isomorphic to A_4, since we showed there is only one nonabelian group of order 12 with $n_2 = 1$ and A_4 fits: its normal 2-Sylow subgroup is $\{(1), (12)(34), (13)(24), (14)(23)\}$.

Next we turn to the case $n_2 \neq 1$, so $n_2 = 3$ and $n_3 = 1$. We will find two groups up to isomorphism, both nonabelian. Our group is a semidirect product

$$\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4), \quad \mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2.$$

Since $n_2 \neq 1$, the group is nonabelian, so the semidirect product is nontrivial: we seek nontrivial homomorphisms $\mathbf{Z}/(4) \to \mathrm{Aut}(\mathbf{Z}/(3)) = (\mathbf{Z}/(3))^\times$ and $(\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$. There is only one nontrivial homomorphism $\mathbf{Z}/(4) \to (\mathbf{Z}/(3))^\times$ (1 mod 4 has to go to -1 mod 3, and everything else is determined), which in fact is $c \bmod 4 \mapsto (-1)^c$, so we get one nontrivial semidirect product $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$. Explicitly, the group law here is

$$(a, b)(c, d) = (a + (-1)^b c, b + d).$$

As for nontrivial homomorphisms $(\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$, there are three of them: $(\mathbf{Z}/(2))^2$ has a pair of generators $(1, 0)$ and $(0, 1)$ and a nontrivial homomorphism $(\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$ sends the generators to any choices of ± 1 other than sending both to 1. Using a 2×2 matrix over $\mathbf{Z}/(2)$ to move nonzero vectors around in $(\mathbf{Z}/(2))^2$, the 3 nontrivial homomorphisms $(\mathbf{Z}/(2))^2 \to (\mathbf{Z}/(3))^\times$ can all be turned into each other by precomposing one of them with automorphisms of $(\mathbf{Z}/(2))^2$. Therefore the three nontrivial semidirect products $\mathbf{Z}/(3) \rtimes (\mathbf{Z}/(2))^2$ are isomorphic, so all nonabelian groups of order 12 with $n_3 = 1$ and 2-Sylow subgroup isomorphic to $(\mathbf{Z}/(2))^2$ are isomorphic. Concretely, one such group is D_6 (with normal 3-Sylow subgroup $\{1, r^2, r^4\}$ and a 2-Sylow subgroup $\{1, r^3, s, r^3 s\}$). $\square$

If we meet a group of order 12, we can decide which of the 5 groups it is isomorphic to by checking if it is abelian or not, and in the nonabelian case seeing if there is a normal 2-Sylow subgroup (then it is isomorphic to A_4) or a normal 3-Sylow subgroup with 2-Sylow subgroups that are cyclic or not cyclic (nontrivial $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ in the first case, D_6 in the second case).

For example, four nonabelian groups of order 12 are $\mathbf{Z}/(2) \times S_3$, $\mathrm{PSL}_2(\mathbf{F}_3)$, $\mathrm{Aff}(\mathbf{Z}/(6))$, and $\mathrm{Aff}(\mathbf{F}_4)$. The group $\mathbf{Z}/(2) \times S_3$ has a normal 3-Sylow subgroup and its 2-Sylow subgroup is not cyclic, so $\mathbf{Z}/(2) \times S_3 \cong D_6$. The group $\mathrm{PSL}_2(\mathbf{F}_3)$ has a normal 2-Sylow subgroup, so $\mathrm{PSL}_2(\mathbf{F}_3) \cong A_4$. The group $\mathrm{Aff}(\mathbf{Z}/(6))$ has a normal 3-Sylow subgroup and its 2-Sylow subgroups are not cyclic, so $\mathrm{Aff}(\mathbf{Z}/(6)) \cong D_6$. Finally, $\mathrm{Aff}(\mathbf{F}_4)$ has a normal 2-Sylow subgroup, so $\mathrm{Aff}(\mathbf{F}_4) \cong A_4$.

Another way to distinguish the three nonabelian groups of order 12 is to count elements of order 2 in them: D_6 has 7 elements of order 2 (6 reflections and r^3), A_4 has 3 elements of order 2 (the permutations of type $(2, 2)$), and $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ has one element of order 2 (it is $(0, 2)$).

In abstract algebra textbooks (not group theory textbooks), $\mathbf{Z}/(3) \rtimes \mathbf{Z}/(4)$ is usually written as T but is almost never given a name to accompany the label. Should it be called the obscure group of order 12? Actually, this group belongs to a standard family of finite groups: the dicyclic groups, also called the binary dihedral groups. They are nonabelian with order $4n$ ($n \geq 2$) and each contains a unique element of order 2. The one of order 8 is Q_8, and more generally the one of order 2^m is the generalized quaternion group Q_{2^m}.