Math 3240 (Roby) **HW #1** **Due in MSB 230 envelope**
7 Sept 2012, 2 PM

All college professors are amazed at how creative their students can be when they are interpreting the words of an assignment. —Donald Knuth

- *Required Reading*: handouts "Division Theorem in **Z** and **R**[T]," "Divisibility and Greatest Common Divisors" and "Modular Arithmetic."
- At least two students in each homework group should work out numerical results *separately* and then compare, as a check on each other's work.

0. (Warm-up) Take a look at the following problems from Chapter 5 (pp. 16–17) of W. Edwin Clark's *Elementary Number Theory*, which google can find for you. You don't have to write them up, but talk about them in your group and make sure you know how to do them (e.g., if one showed up on a quiz. . .)

   (a) Use the division algorithm to prove that every integer is either even or odd, but never both (i.e., *parity* is well-defined).

   (b) Prove that n and n^2 always have the same parity, i.e., n is even iff n^2 is even.

   (c) Find the quotient q and remainder r for each of the following values of a and b: (i) $b = 3$ and $a = 0, 1, -1, 10, -10$; and (ii) $b = 345$ and $a = 0, -1, 1, 344, 7863, -7863$.

   (d) Show that $3 \mid n^3 - n$ for every integer n.

   (e) Show that the product of any three consecutive integers is always divisible by six.

1. Use mathematical induction in the proofs below.

   (a) Determine the set of positive integers that can be expressed as the sum of some nonnegative number of 3's and some nonnegative number of 10's. E.g., $3 = 3 \cdot 1 + 10 \cdot 0$ or $35 = 3 \cdot 5 + 10 \cdot 2$.

   (b) Let $\{a_n\}_{n\geq 1}$ be a sequence satisfying $a_n = 2a_{n-1} + 3a_{n-2}$ for $n \geq 3$. If a_1 and a_2 are odd, prove that a_n is odd for all $n \in \mathbf{Z}^+$.

   (c) If $a_1 = a_2 = 1$ in the above recursion, prove that $a_n = \frac{1}{2}\left(3^{n-1} - (-1)^n\right)$ for all $n \in \mathbf{Z}^+$.

   (d) Prove that for all $n \in \mathbf{Z}^+$,$(2n)!/(2^n n!)$ is always an odd integer.

2. Perfect squares in **Z**:

   (a) What are the possible remainders when a square number is divided by 3? By 5?

   (b) Prove that for all $a \in \mathbf{Z}$, $3a^2 - 1$ is never a perfect square.

   (c) Prove that no integer in the following sequence is a perfect square:

$$11, 111, 1111, 11111, \ldots$$

3. Prove or Disprove and Salvage if Possible. (For each statement below give either a careful proof or a specific numerical counterexample; in the latter case, try to create a similar statement, perhaps by adding a hypothesis, that would be true.)

   (a) $(a+b, a-b) = (2a, a-b) = (a+b, 2b)$ for all nonzero $a, b \in \mathbf{Z}$.

   (b) For all $a, b, c \in \mathbf{Z}$, if $a \mid bc$, then $a \mid b$ or $a \mid c$.

   (c) If $m, n \in \mathbf{Z}$ with $(m, n) = 1$ and mn is a perfect square, then m and n must also be perfect squares.

   (d) If $\ell \mid km$, $\ell \mid kn$, and $(m, n) = 1$, then $\ell \mid k$. True for all $\ell, k, m, n \in \mathbf{Z}$

4. Numerical Problems

   (a) Use Wolfram Alpha to compute the following:

   $$(54438, 107821), \quad 97531 \bmod 2011, \quad 3^{2010} \bmod 2011, \quad (x, y) \text{ s.t. } 119x+272y = (119, 272)\,.$$

   (b) Use Euclid's algorithm to compute the greatest common divisor of 1769 and 2378, writing out *every* equation of the algorithm as in class and then *explaining* how the calculations justify that the last nonzero remainder really is a greatest common divisor: why every common divisor of 1769 and 2378 is a factor of the last nonzero remainder and, conversely, why the last nonzero remainder is a common factor of 1769 and 2378.

   (c) Using the previous part and back-substitution, express $(1769, 2378)$ in the form $1769x + 2378y$ for some integers x and y, (Wolfram Alpha can tell you the answer, but it won't carry out the back-substitution steps.)

   (d) Carry out Euclid's algorithm for $2t^5-3t^4+5t^3-4t^2+3t-1$ and $2t^4-3t^3+4t^2-3t+2$. and factor the greatest common divisor from each polynomial. (Remember the convention that the gcd of two polynomials is always scaled to be monic, so it may not match the last nonzero remainder in Euclid's algorithm.)

   (e) A farmer purchased 100 head of livestock for a total cost of \$4000. Prices were \$120 per calf, \$50 per lamb, and \$25 per piglet. Assuming at least one of each species was bought, how many of each did the farmer buy?

5. *(Pythagorean Triples with Consecutive Legs)* The Pythagorean triples $(3, 4, 5)$ and $(20, 21, 29)$ have consecutive legs. How can we generate more of these?

   (a) Show that the equation $m^2 + (m+1)^2 = n^2$ has an integer solution (m, n) if and only if the equation $x^2 - 2y^2 = -1$ has an integer solution (x, y) with x odd.

   (b) Show that if $x, y \in \mathbf{Z}$ satisfy $x^2 - 2y^2 = -1$, then x must be odd. (So the extra condition in the previous part is automatic.)

   (c) Find the Pythagorean triples with consecutive legs corresponding to the following solutions to $x^2 - 2y^2 = -1$: $(7, 5)$, $(41, 29)$, $(239, 169)$, $(1393, 985)$.

   (d) Why did I skip the obvious solution $(1, 1)$ in part (c)?