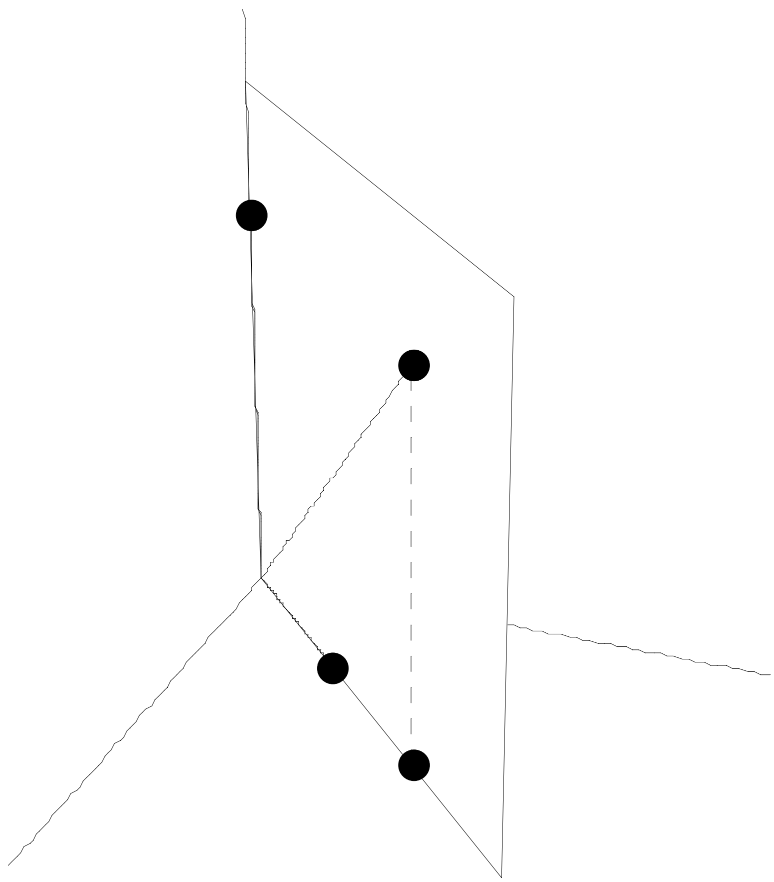
## Section 6.4 The Gram-Schmidt Process

**Goal:** Form an orthogonal basis for a subspace $W$.

**EXAMPLE:** Suppose $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$. Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for $W$.



Let

$$\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\hat{\mathbf{y}} = \text{proj}_{\mathbf{v}_1}\mathbf{x}_2 = \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

and

$$\mathbf{v}_2 = \mathbf{x}_2 - \hat{\mathbf{y}} = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \frac{4}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

(component of $\mathbf{x}_2$ orthogonal to $\mathbf{x}_1$)

**EXAMPLE:** Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for a subspace $W$ of $\mathbf{R}^4$. Describe an orthogonal basis for $W$.

*Solution:* Let

$$\mathbf{v}_1 = \mathbf{x}_1 \text{ and } \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1.$$

$\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthogonal basis for Span$\{\mathbf{x}_1, \mathbf{x}_2\}$.

Let

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

(component of $\mathbf{x}_3$ orthogonal to Span$\{\mathbf{x}_1, \mathbf{x}_2\}$)

Note that $\mathbf{v}_3$ is in $W$. Why?

$\Rightarrow$ $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for $W$.

**THEOREM 11 THE GRAM-SCHMIDT PROCESS**

Given a basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ for a subspace $W$ of $\mathbf{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p \cdot \mathbf{v}_{p-1}}{\mathbf{v}_{p-1} \cdot \mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for $W$ and

$$\text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_p\} = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$$

**EXAMPLE** Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$, where

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$, is a basis for a

subspace $W$ of $\mathbf{R}^4$. Describe an orthogonal basis for $W$.

*Solution:* $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{5}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{bmatrix}$$

Replace $\mathbf{v}_2$ with $14\mathbf{v}_2$ : $\mathbf{v}_2 = 14\begin{bmatrix} \frac{9}{14} \\ \frac{9}{7} \\ -\frac{15}{14} \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}$

(optional step - to make $\mathbf{v}_2$ easier to work with in the next step)

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \frac{9}{630}\begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{14}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} - \frac{1}{70}\begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ -\frac{2}{5} \\ 0 \\ 1 \end{bmatrix}$$

Rescale (optional): $\mathbf{v}_3 = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 5 \end{bmatrix}$

Orthogonal Basis for $W$:

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 9 \\ 18 \\ -15 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 0 \\ 5 \end{bmatrix} \right\}$$

## Orthonomal Basis

Suppose the following is an orthogonal basis for subspace $W = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\2\\3\end{bmatrix}\right\}$:

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\end{bmatrix}\right\}$$

Rescale to form unit vectors:

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\\0\end{bmatrix}$$

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \frac{1}{3}\begin{bmatrix}0\\0\\3\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Orthonormal basis for $W$ : $\{\mathbf{u}_1, \mathbf{u}_2\}$

## QR Factorization

**THEOREM 12** (The QR Factorization)

If $A$ is an $m \times n$ matrix with linearly independent columns, then $A$ can be factored as $A = QR$, where $Q$ is an $m \times n$ matrix whose columns form an orthogonal basis for Col $A$ and $R$ is an $n \times n$ upper triangular invertible matrix with positive entries on its main diagonal.

**EXAMPLE** Find the QR factorization of $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix}$.

*Solution:* Use the Gram Schmidt process to find

$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ which is an orthonomal basis for

col $A =$span$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$. So $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}$.

Since $U$ has orthonormal columns, $Q^T Q = I$. So if $A = QR$, then

$$\_\_\_\_A = \_\_\_\_QR$$

$$R = Q^T A = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 3 \end{bmatrix}.$$