

2.3 Characterizations of Invertible Matrices

Theorem 8 (The Invertible Matrix Theorem)

Let A be a square $n \times n$ matrix. The the following statements are equivalent (i.e., for a given A , they are either all true or all false).

- a. A is an invertible matrix.
- b. A is row equivalent to I_n .
- c. A has n pivot positions.
- d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- e. The columns of A form a linearly independent set.
- f. The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ is one-to-one.
- g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each \mathbf{b} in \mathbf{R}^n .
- h. The columns of A span \mathbf{R}^n .
- i. The linear transformation $\mathbf{x} \rightarrow A\mathbf{x}$ maps \mathbf{R}^n onto \mathbf{R}^n .
- j. There is an $n \times n$ matrix C such that $CA = I_n$.
- k. There is an $n \times n$ matrix D such that $AD = I_n$.
- l. A^T is an invertible matrix.

EXAMPLE: Use the Invertible Matrix Theorem to determine if A is invertible, where

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix}.$$

Solution

$$A = \begin{bmatrix} 1 & -3 & 0 \\ -4 & 11 & 1 \\ 2 & 7 & 3 \end{bmatrix} \sim \dots \sim \underbrace{\begin{bmatrix} 1 & -3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 16 \end{bmatrix}}_{\text{3 pivots positions}}$$

Circle correct conclusion: Matrix A is / is not invertible.

EXAMPLE: Suppose H is a 5×5 matrix and suppose there is a vector \mathbf{v} in \mathbf{R}^5 which is not a linear combination of the columns of H . What can you say about the number of solutions to $H\mathbf{x} = \mathbf{0}$?

Solution Since \mathbf{v} in \mathbf{R}^5 is not a linear combination of the columns of H , the columns of H do not _____ \mathbf{R}^5 .

So by the Invertible Matrix Theorem, $H\mathbf{x} = \mathbf{0}$ has

_____.

Invertible Linear Transformations

For an invertible matrix A ,

$$\begin{aligned} A^{-1}A\mathbf{x} &= \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n \\ &\text{and} \\ AA^{-1}\mathbf{x} &= \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n. \end{aligned}$$

Pictures:

A linear transformation $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is said to be **invertible** if there exists a function $S : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n$$

and

$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$

Theorem 9

Let $T : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a linear transformation and let A be the standard matrix for T . Then T is invertible if and only if A is an invertible matrix. In that case, the linear transformation S given by $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n$$

and

$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbf{R}^n.$$