The problem with wrong proofs to correct statements is that it is hard to give a counterexample.
—H. Lenstra

- *Required Reading*: Text Chapter 6 and Sections 7.4, 8.1–3, 9.1–2;
- At least two students in each homework group should work out numerical results *separately* and then compare, as a check on each other's work.

1. Give proofs by induction for the following statements.
   (a) Show that if a is an odd integer, then for every $n \in \mathbf{Z}^+$, $a^{2^n} \equiv 1 \pmod{2^{n+2}}$.
   (b) Call an integer pprime-ish if each of its prime factors occurs with power two or higher. Prove that there are infinitely many pairs of *consecutive* pprime-ish positive integers.

2. Prove or Disprove and Salvage if Possible. Try to prove your salvages.
   (a) The rational number $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is never an integer for $n > 1$.
   (b) If $m, n \in \mathbf{Z}^+$, then $\sqrt[m]{n}$ is irrational, provided that n is not a perfect mth power.
   (c) For $a \in \mathbf{Z}^+$, $a^n - 1$ is prime if and only if $a = 2$ and n is prime.
   (d) Let $\alpha, \beta \in \mathbf{Z}[i]$. Then $(N(\alpha), N(\beta)) = 1 \implies (\alpha, \beta) = 1$.
   (e) For $\alpha, \beta \in \mathbf{Z}[i]$, if $N(\alpha) \mid N(\beta)$, then $\alpha \mid \beta$.

3. Numerical Problems
   (a) Find a integer solution to $a^3 + b^3 = 743$, or prove that none can exist.
   (b) Find digits a and b such that 495 divides $273a49b5$. Find digits a, b, and c such that 792 divides $13ab45c$.
   (c) Find the values of $n \geq 1$ for which $1! + 2! + \cdots + n!$ is a perfect square in $\mathbf{Z}$.
   (d) Use Euclid's algorithm to Find the GCD of $7 + 11i$ and $3 + 5i$ in $\mathbf{Z}[i]$ and solve the diophantine equation $(7 + 11i)\xi + (3 + 5i)\rho = (7 + 11i, 3 + 5i)$ for $\xi, \rho \in \mathbf{Z}[i]$.

4. Exploration of roots and logarithms in mods.
   (a) Find all the roots of the equation $x^2 - 6x + 8 = 0$ in $\mathbf{Z}/15$. Find all the roots in $\mathbf{Z}/15$ of the equation $x^2 - 6x + 10 = 0$. Find all the roots in $\mathbf{Z}/105$ of the equation $x^2 - 6x + 8 = 0$. Any conjectures?
   (b) Compute the 17×18 table that lists $\{a^k : a \in \mathbf{Z}/17, 0 \leq k \leq 17\}$. What patterns do you notice?
   (c) Choose a base, and make a table of logarithms for mod 17. (This is a 2×16 table.
   (d) Use the logarithm table to find all the solutions of each of the following equations in $\mathbf{Z}/17$: (a) $x^2 = 2$; (b) $7x^2 = 6$; (c) $x^3 = 3$.

5. Solving congruences.

(a) Describe all x such that $x \equiv 1 \pmod 3$, $x \equiv 2 \pmod 5$, and $x \equiv 3 \pmod 7$.

(b) Describe all x such that $2x \equiv 1 \pmod 5$, $3x \equiv 9 \pmod 6$, $4x \equiv 1 \pmod 7$, and $5x \equiv 9 \pmod{11}$.

(c) When eggs in a basket were removed 2,3,4,5,6 at a time there was 1,2,3,4,5 (respectively) left over. When they were taken out 7 at a time, there were none left over. How many eggs were in the basket?

(d) Find all solutions to $3x - 7y \equiv 11 \pmod{13}$.