Math 297Q/Roby   **Practice Midterm**   21 March 2006

This is a closed book, closed note exam. You may use a calculator. Please do not discuss this exam with anyone other than the proctor during the exam. **SHOW ALL YOUR WORK!** Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask!

1. Go over all the old homework problems, with particularly attention to anything that gave you trouble.

2. Go over the "exercises" (with solutions) in Bona, sections 1.8, 2.8, and 3.8.

3. Go over all the entries of *The Twelvefold Way* and makes sure you understand how each entry was derived.

4. An encyclopedia has 24 volumes. How many selections of 5 volumes are there with no two consecutive volumes chosen? (The order of selection is immaterial.)

5. How many ways are there to write the digits from 0 to 9 such that each number except for the leftmost is within one of some number to the left of it?

6. How many positive integers are there less than or equal to a million that are neither perfect squares, perfect cubes, nor perfect fourth powers?

7. Given nine lattice points (points with integer coordinates) in $\mathbb{R}^3$, show that the midpoint of one of the line segments that connect these points is also a lattice point.

8. Prove that
$$\prod_{i \geq 1}(1 - qx^i)^{-1} = \sum_{k \geq 0} \frac{x^k q^k}{(1 - x)(1 - x^2) \cdots (1 - x^k)} \ .$$

9. a) Prove that the ordinary generating function for the sequence $a_n = \binom{2n}{n}$ is $(1-4x)^{-\frac{1}{2}}$.

   b) Prove that
   $$\sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i} = 4^n \ .$$

   c) (Extra credit) Can you give a combinatorial proof?