# COMPACT SUBGROUPS OF $\mathrm{GL}_n(\overline{\mathbf{Q}}_p)$

KEITH CONRAD

**Theorem 1.** *For any compact subgroup $K$ of $\mathrm{GL}_n(\overline{\mathbf{Q}}_p)$, there is a finite extension $F/\mathbf{Q}_p$ such that $K \subset \mathrm{GL}_n(F)$.*

*Proof.* The argument we will give is due to W. Sinnott. Let $G = \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ and $\overline{\mathbf{Z}}_p$ be the integers of $\overline{\mathbf{Q}}_p$. For $r \geq 1$, the subgroup

$$G_r = I_n + p^r \mathrm{M}_n(\overline{\mathbf{Z}}_p)$$

is open in $G$, so the intersection $K_r = K \cap G_r$ is an open subgroup of $K$. Any open subgroup of a compact group is closed with finite index, so $K_r$ is compact and $[K : K_r]$ is finite. If some $K_r$ is contained in $\mathrm{GL}_n(F)$ for some finite extension $F$ of $\mathbf{Q}_p$, then $K$ itself lies in $\mathrm{GL}_n(F')$ where $F'$ is the field generated over $F$ by the matrix entries from the finitely many (say, left) coset representatives for $K/K_r$ in $K$. The entries of any matrix in $K$ are all algebraic over $\mathbf{Q}_p$, so $F'$ is a finite extension field of $F$. This means $[F' : \mathbf{Q}_p]$ is finite and $K \subset \mathrm{GL}_n(F')$, so we'd be done.

Assume, to the contrary, that no $K_r$ is contained in any $\mathrm{GL}_n(F)$ where $F/\mathbf{Q}_p$ is finite. We will recursively find positive integers $d_1 < d_2 < \cdots$ and matrices $g_i \in K_{d_i}$ for each $i \geq 1$ such that

(1) for any $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, if $\sigma(g_i) \neq g_i$ then $\sigma(g_i) \not\equiv g_i \bmod p^{d_{i+1}}$, where the modulus here is really $p^{d_{i+1}}\mathrm{M}_n(\overline{\mathbf{Z}}_p)$,
(2) the field generated over $\mathbf{Q}_p$ by the entries in $g_i$ has degree at least $i$ over $\mathbf{Q}_p$.

To start, choose $d_1 \geq 1$ and $g_1 \in K_{d_1}$ arbitrarily. The second condition is obvious for $i = 1$. Since $g_1$ has only finitely many Galois conjugates, we can choose $d_2 > d_1$ to make the first condition true for $i = 1$. Next, suppose $g_1, \ldots, g_j$ and $d_1, \ldots, d_{j+1}$ have been chosen to satisfy the above two conditions for $i = 1, \ldots, j$. Then we can choose $g_{j+1} \in K_{d_{j+1}}$ to satisfy the second condition, and since $g_{j+1}$ has only finitely many Galois conjugates we can choose $d_{j+2} > d_{j+1}$ to satisfy the first condition for $i = j + 1$.

We want to work with the infinite product $h := g_1 g_2 \cdots$. To check it converges and to approximate it using partial products, we switch our focus to the subgroups $G_{d_i}$, which shrink to the identity in a controlled way through the powers of $p$ defining them. Since $g_i \in G_{d_i} \subset K$, $d_i \to \infty$, and $K$ is closed, the product $h := g_1 g_2 \cdots$ converges in $K$. We are going to look at automorphisms $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ which fix $h$. For any such $\sigma$,

$$\sigma(g_1)\sigma(g_2) \cdots = g_1 g_2 \cdots .$$

Suppose $\sigma(g_i) \neq g_i$ for some $i$. Let $\ell$ be the least such integer (it depends on $\sigma$). Then $\sigma(g_i) = g_i$ for all $i < \ell$, which means

$$\sigma(g_\ell)\sigma(g_{\ell+1})\cdots = g_\ell g_{\ell+1}\cdots .$$

For all $i > \ell$, $g_i \in G_{d_i} \subset G_{d_{\ell+1}}$ and $\sigma(g_i) \in G_{d_i} \subset G_{d_{\ell+1}}$, so reducing this equation modulo $p^{d_{\ell+1}}\mathrm{M}_n(\overline{\mathbf{Z}}_p)$ implies $\sigma(g_\ell) \equiv g_\ell \bmod p^{d_{\ell+1}}$. Then the first condition above implies $\sigma(g_\ell) = g_\ell$, which is a contradiction. Therefore $\sigma(g_i) = g_i$ for all $i$. In other words, the subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ fixing $h$ fixes every entry of every $g_i$, and the second condition above implies the subgroup fixing $h$ has a fixed field which is an infinite extension of $\mathbf{Q}_p$. However, all the entries of $h$ lie in a finite extension of $\mathbf{Q}_p$, so the subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ fixing $h$ has a fixed field which is a finite extension of $\mathbf{Q}_p$. We have reached a contradiction. $\square$

**Remark 2.** Replacing $\overline{\mathbf{Q}}_p$ by its completion $\mathbf{C}_p$, it is *false* that a general compact subgroup of $\mathrm{GL}_n(\mathbf{C}_p)$ is in $\mathrm{GL}_n(F)$ for some finite extension $F/\mathbf{Q}_p$. For example, inside $\mathrm{GL}_1(\mathbf{C}_p) = \mathbf{C}_p^\times$ we can pick $x \notin \overline{\mathbf{Q}}_p$ where $|x-1|_p < 1$ and take $K = x^{\mathbf{Z}_p}$.

The proof of Theorem 1 is similar in spirit to one of the proofs [1, pp. 182–183], [2, p. 71] that $\overline{\mathbf{Q}}_p$ is not complete: consider an infinite series $\sum_{i\geq 0} c_i p^i$ where the $c_i$'s are in $\overline{\mathbf{Q}}_p$, $|c_i|_p = 1$, and $[\mathbf{Q}_p(c_i) : \mathbf{Q}_p] \to \infty$. By a suitable choice of $c_i$'s, if that infinite series converges in $\overline{\mathbf{Q}}_p$ then a contradiction can be reached by comparing the series with a $p$-adic expansion of the limit. Turning things around, we can use the ideas in the proof of Theorem 1 to prove something about compact subgroups of the additive group $\overline{\mathbf{Q}}_p$.

**Corollary 3.** *Any compact subgroup of* $\overline{\mathbf{Q}}_p$ *is inside a finite extension of* $\mathbf{Q}_p$.

*Proof.* Repeat the proof of Theorem 1 for additive groups, *e.g.*, when $K$ is a compact subgroup of $\overline{\mathbf{Q}}_p$ the intersections $K_r = K \cap p^r\overline{\mathbf{Z}}_p$ are compact subgroups of $\overline{\mathbf{Q}}_p$ with finite index in $K$ and it suffices to show some $K_r$ is in a finite extension of $\mathbf{Q}_p$. Or, more quickly, embed $\overline{\mathbf{Q}}_p$ into $\mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ as the matrices $\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ so we can just appeal to Theorem 1 when $n = 2$. $\square$

## References

[1] F. Q. Gouvea, "$p$-adic Numbers: An Introduction," 2nd ed., Springer–Verlag, New York, 1997.

[2] N. M. Koblitz, "$p$-adic Numbers, $p$-adic Analysis, and Zeta-functions," 2nd ed., Springer-Verlag, New York, 1984.