Math 3250/Roby   **Practice Midterm**   October 2008

This is a closed book, closed note exam. You may use a calculator. Please do not discuss this exam with anyone other than the proctor during the exam. **SHOW ALL YOUR WORK!** Make sure you give reasons to support your answers. If you have any questions, do not hesitate to ask!

1. Go over all the old homework problems, with particularly attention to anything that gave you trouble.

2. Go over all the problems (with solutions) in Bona listed for quiz preparation.

3. Go over all the entries of *The Twelvefold Way* and makes sure you understand how each entry was derived.

4. An encyclopedia has 24 volumes. How many selections of 5 volumes are there with no two consecutive volumes chosen? (The order of selection is immaterial.)

5. How many ways are there to write the digits from 0 to 9 such that each number except for the leftmost is within one of some number to the left of it?

6. Consider the sequence defined recursively by $r_0 = 3$, $r_1 = 4$, and $r_n = r_{n-1} + 6r_{n-2}$, for $n \geq 2$. Find a closed form expression for the ordinary generating function $R(x)$ and use this to find a closed form expression for r_n itself.

7. Let r be any irrational number. Prove that there exists a positive integer n such that the distance of nr from the closest integer is less than 10^{-10}.

8. Let $p(n) =$ number of integer partitions of n; prove that $\displaystyle\sum_{i=1}^{n} p(i) < p(2n)$ for all $n \in \mathbb{Z}^+$.

9. Instead of studying for their combinatorics midterm, four students play Pac-Man each hour and compare their scores. How many hours will it take before they can be sure to get the same relative ordering of scores? (All kinds of ties are possible, e.g., W scores more than X and Y, who tie and whose score is greater than Z.)

10. a) Prove that the ordinary generating function for the sequence $c_n = \binom{2n}{n}$ is $(1-4x)^{-\frac{1}{2}}$.

    b) Prove that
    $$\sum_{i=0}^{n} \binom{2i}{i}\binom{2(n-i)}{n-i} = 4^n.$$

    c) (Extra credit) Can you give a combinatorial proof?