Section 4.9 Applications to Markov Chains

Rent-a-Lemon has three locations from which to rent a car for one day: Airport, downtown and the valley

Daily Migration:

Rented From				
Airport	Downtown	Valley		
$.95$	$.02$	$.05$	Airport	**Returned To**
$.03$	$.90$	$.05$	Downtown	
$.02$	$.08$	$.90$	Valley	

Airport

Downtown ***Valley***

$$M = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix}$$

(migration matrix)

$$\mathbf{x}_0 = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} \quad \begin{array}{l} \text{(initial fraction of cars at } \textit{airport}\text{)} \\ \text{(initial fraction of cars } \textit{downtown}\text{)} \\ \text{(initial fraction of cars at } \textit{valley} \text{ location)} \end{array}$$

(initial distribution vector which is a *probability vector*)

Interpretation of $M\mathbf{x}_0$

$$M\mathbf{x}_0 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix} \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} =$$

$$\underbrace{.5\begin{bmatrix} .95 \\ .03 \\ .02 \end{bmatrix}}_{\uparrow} + \underbrace{.3\begin{bmatrix} .02 \\ .90 \\ .08 \end{bmatrix}}_{\uparrow} + \underbrace{.2\begin{bmatrix} .05 \\ .05 \\ .90 \end{bmatrix}}_{\uparrow}$$

Redistribution of airport cars	Redistribution of downtown cars	Redistribution of valley cars

Distribution after one day=

$$\mathbf{x}_1 = M\mathbf{x}_0 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix} \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix} = \begin{bmatrix} 0.491 \\ 0.295 \\ 0.214 \end{bmatrix}$$

$$\mathbf{x}_{k+1} = M\mathbf{x}_k \text{ for } k = 0, 1, 2, \ldots$$

(Markov Chain)

Distribution after two days=

$$\mathbf{x}_2 = M\mathbf{x}_1 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix} \begin{bmatrix} 0.491 \\ 0.295 \\ 0.214 \end{bmatrix} = \begin{bmatrix} 0.483 \\ 0.290 \\ 0.226 \end{bmatrix}$$

$$\mathbf{x}_3 = M\mathbf{x}_2 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix} \begin{bmatrix} 0.483 \\ 0.290 \\ 0.226 \end{bmatrix} = \begin{bmatrix} 0.475 \\ 0.287 \\ 0.236 \end{bmatrix}$$

$$\mathbf{x}_4 = M\mathbf{x}_3 = \begin{bmatrix} .95 & .02 & .05 \\ .03 & .90 & .05 \\ .02 & .08 & .90 \end{bmatrix} \begin{bmatrix} 0.475 \\ 0.287 \\ 0.236 \end{bmatrix} = \begin{bmatrix} 0.468 \\ 0.284 \\ 0.244 \end{bmatrix}$$

$$\vdots$$

$$\mathbf{x}_{49} = M\mathbf{x}_{48} = \begin{bmatrix} 0.417 \\ 0.278 \\ 0.305 \end{bmatrix}$$

$$\mathbf{x}_{50} = M\mathbf{x}_{49} = \begin{bmatrix} 0.417 \\ 0.278 \\ 0.305 \end{bmatrix} \text{ (long term distribution)}$$

$$\vdots$$

$$\mathbf{x} = \begin{bmatrix} 0.417 \\ 0.278 \\ 0.305 \end{bmatrix}$$ is called a **steady state vector** since $\mathbf{x} = M\mathbf{x}$

Finding the Steady State Vector

$$M\mathbf{x} = \mathbf{x}$$

$$M\mathbf{x} = I\mathbf{x}$$

$$M\mathbf{x} - I\mathbf{x} = \mathbf{0}$$

$$(M - I)\mathbf{x} = \mathbf{0}$$

Solve $(M-I)\mathbf{x} = \mathbf{0}$ to find the steady state vector. Note that the solution $\mathbf{x}$ must be a *probability vector*.

EXAMPLE: Suppose that 3% of the population of the U.S. lives in the State of Washington. Suppose the migration of the population into and out of Washington State will be constant for many years according to the following migration probabilities. What percentage of the total U.S. population will eventually live in Washington?

From :		
WA	Rest of U.S.	**To**:
$.9$	$.01$	WA
$.1$	$.99$	Rest of U.S.

Solution

$$M = \begin{bmatrix} .9 & .01 \\ .1 & .99 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} \text{\% of people in WA} \\ \text{\% in rest of U.S.} \end{bmatrix}$$

$$M - I = \begin{bmatrix} .9 & .01 \\ .1 & .99 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.1 & 0.01 \\ 0.1 & -0.01 \end{bmatrix}$$

Solve $(M-I)\mathbf{x} = \mathbf{0}$

$$\begin{bmatrix} -0.1 & 0.01 & 0 \\ 0.1 & -0.01 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -0.1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.1x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0.1 \\ 1 \end{bmatrix}$$

One solution: $\mathbf{x} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$

Solution we want has entries which add up to one:

$$\mathbf{x} = \begin{bmatrix} 1/11 \\ 10/11 \end{bmatrix} \approx \begin{bmatrix} 0.091 \\ 0.909 \end{bmatrix}$$