TENSOR PRODUCTS

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1. Introduction

Let R be a commutative ring and M and N be R-modules. Formation of their direct sum $M \oplus N$ is an addition operation on modules. We introduce now a product operation, called the tensor product $M \otimes_R N$. To start off, we will describe roughly what a tensor product of modules looks like. The rigorous definitions will come in Section 3.

The tensor product $M \otimes_R N$ is an R-module spanned by symbols $m \otimes n$ satisfying distributive laws:

$$(m + m') \otimes n = m \otimes n + m' \otimes n, \quad m \otimes (n + n') = m \otimes n + m \otimes n'. \tag{1.1}$$

Also multiplication by any $r \in R$ is associative with $\otimes$ on both sides:

$$r(m \otimes n) = (rm) \otimes n = m \otimes (rn). \tag{1.2}$$

(So the notation $rm \otimes n$ is unambiguous: it is both $(rm) \otimes n$ and $r(m \otimes n)$.) The formulas (1.1) and (1.2) should be contrasted with those for the direct sum $M \oplus N$, where

$$(m + m', n) = (m, n) + (m', 0), \quad r(m, n) = (rm, rn).$$

In $M \oplus N$, every (m, n) decomposes as $(m, 0) + (0, n)$, but $m \otimes n$ in $M \otimes_R N$ does not break apart in any general way. While every element of $M \oplus N$ is a pair (m, n), there are *more* elements of $M \otimes_R N$ than the products $m \otimes n$. All elements of $M \otimes_R N$ are R-linear combinations[1]

$$r_1(m_1 \otimes n_1) + r_2(m_2 \otimes n_2) + \cdots + r_k(m_k \otimes n_k),$$

where $k \geq 1$, $r_i \in R$, $m_i \in M$, and $n_i \in N$. Since $r_i(m_i \otimes n_i) = (r_i m_i) \otimes n_i$, we can rename $r_i m_i$ as m_i and write the linear combination as a sum

$$m_1 \otimes n_1 + m_2 \otimes n_2 + \cdots + m_k \otimes n_k. \tag{1.3}$$

In $M \oplus N$, equality is easy to define: $(m, n) = (m', n')$ if and only if $m = m'$ and $n = n'$. But when are two sums of the form (1.3) equal in $M \otimes_R N$? This is not easy to say in terms of the description of the tensor product above, except in one special case: M and N are finite free R-modules with bases $\{e_i\}$ and $\{f_j\}$. In this case, $M \otimes_R N$ has basis $\{e_i \otimes f_j\}$: every element of $M \otimes_R N$ is a unique sum $\sum_{i,j} c_{ij} e_i \otimes f_j$ and two sums of this special form are equal only when the coefficients of like terms are equal.

To describe equality in $M \otimes_R N$ when there aren't bases, we need to use a universal mapping property of the tensor product. In fact, the tensor product is the first fundamental concept in algebra which can be used (in the most general case) *only* through its universal mapping property, which is: $M \otimes_R N$ is the universal object that turns bilinear maps out of $M \times N$ into linear maps.

[1] Compare with the polynomial ring $R[X, Y]$, whose terms are not only the products $f(X)g(Y)$, but sums of such products like $\sum_{i,j} a_{ij} X^i Y^j$. It turns out that $R[X, Y] \cong R[X] \otimes_R R[Y]$, so the comparison is not merely an analogy but a special case (Example 4.10).

After a discussion of bilinear (and multilinear) maps in Section 2, the definition and construction of tensor products is presented in Section 3. Examples of tensor products are in Section 4. In Section 5 we will show how the tensor product interacts with some other constructions on modules. Section 6 describes the important operation of base extension, which is a process of turning an R-module into an S-module where S is another ring.

2. Bilinear Maps

We described $M \otimes_R N$ as sums (1.3) subject to the rules (1.1) and (1.2). The intention is that $M \otimes_R N$ be the "freest" object built out of M and N subject to (1.1) and (1.2). The essence of (1.1) and (1.2) is bilinearity, which we'll discuss before getting back to the tensor product.

A function $B \colon M \times N \to P$, where M, N, and P are R-modules, is called *bilinear* when it is linear in each coordinate with the other one fixed:

$$B(m_1 + m_2, n) = B(m_1, n) + B(m_2, n), \quad B(rm, n) = rB(m, n),$$

$$B(m, n_1 + n_2) = B(m, n_1) + B(m, n_2), \quad B(m, rn) = rB(m, n).$$

So $B(-, n)$ is a linear map $M \to P$ for each n and $B(m, -)$ is a linear map $N \to P$ for each m. Here are some examples:

(1) The dot product $\mathbf{v} \cdot \mathbf{w}$ on $\mathbf{R}^n$ is a bilinear function $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$.
(2) For any $A \in \mathrm{M}_n(\mathbf{R})$, the function $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot A\mathbf{w}$ is a bilinear map $\mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$. It reduces to the dot product when $A = I_n$.
(3) The cross product $\mathbf{v} \times \mathbf{w}$ is a bilinear function $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$.
(4) Multiplication $R \times R \to R$ is bilinear.
(5) For an R-module M, scalar multiplication $R \times M \to M$ is bilinear.
(6) The determinant $\det \colon \mathrm{M}_2(R) \to R$ is a bilinear function of the columns.
(7) The dual pairing $M^\vee \times M \to R$ given by $(\varphi, m) \mapsto \varphi(m)$ is bilinear.
(8) The function $M^\vee \times N \to \mathrm{Hom}_R(M, N)$ given by $(\varphi, n) \mapsto [x \mapsto \varphi(x)n]$ is bilinear.
(9) If $M \times N \xrightarrow{B} P$ is bilinear and $P \xrightarrow{L} Q$ is linear, the composite $M \times N \xrightarrow{L \circ B} Q$ is bilinear. (This is a very important example. Check it!)
(10) If $f \colon M \to \mathrm{Hom}_R(N, P)$ is linear then $B \colon M \times N \to P$ given by $B(m, n) = f(m)(n)$ is bilinear.
(11) In our naive description of $M \otimes_R N$, the expression $m \otimes n$ is bilinear in m and n. That is, the function $M \times N \to M \otimes_R N$ given by $(m, n) \mapsto m \otimes n$ is bilinear.

Even though elements of $M \times N$ and $M \oplus N$ are written in the same way, as pairs (m, n), bilinear functions $M \times N \to P$ should not be confused with linear functions $M \oplus N \to P$. For example, addition as a function $R \oplus R \to R$ is linear, but as a function $R \times R \to R$ it is not bilinear. Multiplication as a function $R \times R \to R$ is bilinear, but as a function $R \oplus R \to R$ it is not linear. Linear functions are generalized additions and bilinear functions are generalized multiplications.

An extension of bilinearity is multilinearity. For R-modules $M_1, \dots, M_k$, a function $f \colon M_1 \times \cdots \times M_k \to M$ is called *multilinear* when $f(m_1, \dots, m_k)$ is linear in each m_i with the other coordinates fixed. We use the label k-multilinear if the number of factors has to be mentioned, so 2-multilinear means bilinear.

Here are a few examples of multilinear functions:

(1) The scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is trilinear $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$.
(2) The function $f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ is trilinear $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$.

(3) The function $M^\vee \times M \times N \to N$ given by $(\varphi, m, n) = \varphi(m)n$ is trilinear.
(4) If $B\colon M \times N \to P$ and $B'\colon P \times Q \to T$ are bilinear then $M \times N \times Q \to T$ by $(m, n, q) \mapsto B'(B(m, n), q)$ is trilinear.
(5) Multiplication $R \times \cdots \times R \to R$ with k factors is multilinear.
(6) The determinant $\det\colon \mathrm{M}_n(R) \to R$, as a function of the columns, is n-multilinear.
(7) If $M_1 \times \cdots \times M_k \xrightarrow{f} M$ is multilinear and $M \xrightarrow{L} N$ is linear then the composite $M_1 \times \cdots \times M_k \xrightarrow{L \circ f} N$ is multilinear.

The R-linear maps $M \to N$ are an R-module $\mathrm{Hom}_R(M, N)$ under addition of functions and R-scaling. The R-bilinear maps $M \times N \to P$ form an R-module $\mathrm{Bil}_R(M, N; P)$ in the same way. However, unlike linear maps, bilinear maps are *missing* some features:

(1) There is no "kernel" of a bilinear map $M \times N \to P$ since $M \times N$ is not a module.
(2) The image of a bilinear map $M \times N \to P$ need not form a submodule.

Example 2.1. Bilinear maps $M \times N \to R$ can be created using $M^\vee$, $N^\vee$, and multiplication in R: for $\varphi \in M^\vee$ and $\psi \in N^\vee$, set $B_{\varphi,\psi}\colon M \times N \to R$ by $B_{\varphi,\psi}(m, n) = \varphi(m)\psi(n)$. Each $B_{\varphi,\psi}$ is bilinear. Also $\mathbf{B}\colon M^\vee \times N^\vee \to \mathrm{Bil}_R(M, N; R)$ given by $\mathbf{B}(\varphi, \psi) = B_{\varphi,\psi}$ is bilinear. (So $\mathbf{B}$ is a bilinear map whose values are bilinear maps.) The sum $\varphi_1(m)\psi_1(n)+\varphi_2(m)\psi_2(n)$ is bilinear in m and n but usually not expressible as $\varphi(m)\psi(n)$, so $\mathbf{B}$ is not surjective. And $\mathbf{B}$ is generally not injective: for any $u \in R^\times$, $\mathbf{B}(\varphi, \psi) = \mathbf{B}(u\varphi, (1/u)\psi)$.

Let's look at the special case that M and N are nonzero finite free R-modules, with respective bases $\{e_1, \dots, e_k\}$ and $\{f_1, \dots, f_\ell\}$. Any bilinear map $B\colon M \times N \to R$ is determined by its values on the basis pairs (e_i, f_j) since $B(\sum_i a_i e_i, \sum_j b_j f_j) = \sum_{i,j} a_i b_j B(e_i, f_j)$. Since $B_{e_i^\vee, f_j^\vee}$ on $M \times N$ equals 1 at (e_i, f_j) and 0 at other basis pairs, $B = \sum_{i,j} B(e_i, f_j) B_{e_i^\vee, f_j^\vee}$ for every B in $\mathrm{Bil}_R(M, N; R)$: both sides agree at basis pairs, so they agree everywhere since both sides are bilinear. This shows the functions $B_{e_i^\vee, f_j^\vee} = \mathbf{B}(e_i^\vee, f_j^\vee)$ span $\mathrm{Bil}_R(M, N; R)$. They are also linearly independent: if $\sum_{i,j} c_{ij} B_{e_i^\vee, f_j^\vee} = O$ then evaluating at the basis pair $(e_{i'}, f_{j'})$ shows $c_{i'j'} = 0$, so all coefficients are 0. Thus $\mathrm{Bil}_R(M, N; R)$ has basis $\{B_{e_i^\vee, f_j^\vee}\}$.

Because $\mathbf{B}$ sends basis pairs $\{(e_i^\vee, f_j^\vee)\}$ in $M^\vee \times N^\vee$ to a basis of $\mathrm{Bil}_R(M, N; R)$, we would like to say $\mathbf{B}$ is in some sense an isomorphism, but this makes no sense: the domain of $\mathbf{B}$ is not an R-module, and $\mathbf{B}$ is neither injective nor surjective. These defects will be rectified by the tensor product in Example 4.11: $\mathbf{B}$ can be converted into a linear map $\mathbf{L}\colon M^\vee \otimes_R N^\vee \to \mathrm{Bil}_R(M, N; R)$ which is an isomorphism for finite free M and N. The non-injective feature $\mathbf{B}(\varphi, \psi) = \mathbf{B}(u\varphi, (1/u)\psi))$, for instance, will wipe out because $\varphi \otimes \psi = u\varphi \otimes (1/u)\psi$ in $M^\vee \otimes_R N^\vee$ by (1.2) even though $(\varphi, \psi) \neq (u\varphi, (1/u)\psi)$ in $M^\vee \times N^\vee$.

3. Construction of the Tensor Product

Any bilinear map $M \times N \to P$ to an R-module P can be composed with a linear map $P \to Q$ to get a map $M \times N \to Q$ which is bilinear.

$$\begin{tikzcd} & P \arrow[dd, "\text{linear}"] \\ M \times N \arrow[ur, "\text{bilinear}"] \arrow[dr, "\text{composite is bilinear!}"'] & \\ & Q \end{tikzcd}$$

We will construct the tensor product as a solution to the following universal mapping problem: find an R-module T and bilinear map $b\colon M \times N \to T$ such that all bilinear maps out of $M \times N$ are composites of the single bilinear map b with all linear maps out of T.

$$\begin{tikzcd} & T \arrow[dd, dotted, "\exists \text{ linear?}"] \\ M \times N \arrow[ur, "b"] \arrow[dr, "\text{bilinear}"'] & \\ & P \end{tikzcd}$$

This turns the task of constructing bilinear maps out of $M \times N$ to that of constructing linear maps out of T. It is analogous to the universal mapping property of the abelianization $G/[G,G]$ of a group G: homomorphisms $G \longrightarrow A$ with abelian A are "the same" as homomorphisms $G/[G,G] \longrightarrow A$ because every homomorphism $G \xrightarrow{f} A$ is the composite of the canonical homomorphism $G \xrightarrow{\pi} G/[G,G]$ with a unique homomorphism $G/[G,G] \xrightarrow{\widetilde{f}} A$.

$$\begin{tikzcd} & G/[G,G] \arrow[dd, dotted, "\widetilde{f}"] \\ G \arrow[ur, "\pi"] \arrow[dr, "f"'] & \\ & A \end{tikzcd}$$

Definition 3.1. The *tensor product* $M \otimes_R N$ is an R-module equipped with a bilinear map $M \times N \xrightarrow{\otimes} M \otimes_R N$ such that for any bilinear map $M \times N \xrightarrow{B} P$ there is a unique linear map $M \otimes_R N \xrightarrow{L} P$ making the following diagram commute.

$$\begin{tikzcd} & M \otimes_R N \arrow[dd, dotted, "L"] \\ M \times N \arrow[ur, "\otimes"] \arrow[dr, "B"'] & \\ & P \end{tikzcd}$$

While the functions in the universal mapping property for $G/[G,G]$ are all homomorphisms (out of G and out of $G/[G,G]$), functions in the universal mapping property for $M \otimes_R N$ are not all of the same type: those out of $M \times N$ are *bilinear* and those out of $M \otimes_R N$ are *linear*.

The definition of the tensor product involves not just a new module $M \otimes_R N$, but also a distinguished bilinear map to it: $M \times N \xrightarrow{\otimes} M \otimes_R N$. This is similar to the universal mapping property for the abelianization $G/[G,G]$, which requires not just $G/[G,G]$ but also the homomorphism $G \xrightarrow{\pi} G/[G,G]$ through which all homomorphisms from G to abelian groups factor. The universal mapping property makes no sense without fixing this extra piece of information.

Before constructing the tensor product, let's show all constructions of it are essentially the same. Suppose there are R-modules T and T' and bilinear maps $M \times N \xrightarrow{b} T$ and $M \times N \xrightarrow{b'} T'$ with the universal mapping property of the tensor product. From universality of $M \times N \xrightarrow{b} T$, the bilinear map $M \times N \xrightarrow{b'} T'$ factors uniquely through T: there is a unique linear map $f \colon T \to T'$ making

$$
\begin{tikzcd}
 & T \arrow[dd, dotted, "f"] \\
M \times N \arrow[ur, "b"] \arrow[dr, "b'"'] & \\
 & T'
\end{tikzcd}
\tag{3.1}
$$

commute. From universality of $M \times N \xrightarrow{b'} T'$, the bilinear map $M \times N \xrightarrow{b} T$ factors uniquely through T': there is a unique linear map $f' \colon T' \to T$ making

$$
\begin{tikzcd}
 & T' \arrow[dd, dotted, "f'"] \\
M \times N \arrow[ur, "b'"] \arrow[dr, "b"'] & \\
 & T
\end{tikzcd}
\tag{3.2}
$$

commute. We combine (3.1) and (3.2) into the commutative diagram

$$
\begin{tikzcd}
 & T \arrow[d, "f"] \\
M \times N \arrow[ur, "b"] \arrow[r, "b'"] \arrow[dr, "b"'] & T' \arrow[d, "f'"] \\
 & T
\end{tikzcd}
$$

Removing the middle, we have the commutative diagram

$$
\begin{tikzcd}
 & T \arrow[dd, "f' \circ f"] \\
M \times N \arrow[ur, "b"] \arrow[dr, "b"'] & \\
 & T
\end{tikzcd}
\tag{3.3}
$$

From universality of (T, b), a unique linear map $T \to T$ fits in (3.3). The identity map works, so $f' \circ f = \mathrm{id}_T$. Similarly, $f \circ f' = \mathrm{id}_{T'}$ by stacking (3.1) and (3.2) together in the other order. Thus T and T' are isomorphic R-modules by f and also $f \circ b = b'$, which means f identifies b with b'. So any two tensor products of M and N can be identified with each other in a unique way *compatible*[2] with the distinguished bilinear maps to them from $M \times N$.

Theorem 3.2. *A tensor product of M and N exists.*

Proof. Consider $M \times N$ simply as a set. We form the free R-module on this set:

$$F_R(M \times N) = \bigoplus_{(m,n) \in M \times N} R\delta_{(m,n)}.$$

Let D be the submodule of $F_R(M \times N)$ spanned by all the elements

$$\delta_{(m+m',n)} - \delta_{(m,n)} - \delta_{(m',n)}, \quad \delta_{(m,n+n')} - \delta_{(m,n)} - \delta_{(m,n')}, \quad \delta_{(rm,n)} - \delta_{(m,rn)},$$
$$r\delta_{(m,n)} - \delta_{(rm,n)}, \quad r\delta_{(m,n)} - \delta_{(m,rn)}.$$

The quotient module by D will serve as the tensor product: set

$$M \otimes_R N := F_R(M \times N)/D.$$

We write the coset $\delta_{(m,n)} + D$ in $M \otimes_R N$ as $m \otimes n$.

(Notice how large $F_R(M \times N)$ can be. If $R = M = N = \mathbf{R}$ then $F_{\mathbf{R}}(\mathbf{R} \times \mathbf{R})$ is a direct sum of $\mathbf{R}^2$-many copies of $\mathbf{R}$, even though we'll see in Theorem 4.8 that the quotient $\mathbf{R}$-module $\mathbf{R} \otimes_{\mathbf{R}} \mathbf{R}$ is 1-dimensional!)

We need to write down a bilinear map $M \times N \to M \otimes_R N$ and show all bilinear maps out of $M \times N$ factor uniquely through this one. From the definition of D, we get relations in $F_R(M \times N)/D$ like

$$\delta_{(m+m',n)} \equiv \delta_{(m,n)} + \delta_{(m',n)} \bmod D,$$

which is the same as

$$(m + m') \otimes n = m \otimes n + m' \otimes n$$

in $M \otimes_R N$. Similarly, $m \otimes (n + n') = m \otimes n + m \otimes n'$ and $r(m \otimes n) = rm \otimes n = m \otimes rn$ in $M \otimes_R N$. This shows the function $M \times N \xrightarrow{\otimes} M \otimes_R N$ given by $(m, n) \mapsto m \otimes n$ is bilinear. (No other function $M \times N \to M \otimes_R N$ will be considered except this one.)

Now suppose P is any R-module and $M \times N \xrightarrow{B} P$ is a bilinear map. Treating $M \times N$ simply as a *set*, so B is just a function on this set (ignore its bilinearity), the universal mapping property of free modules extends B from a function $M \times N \to P$ to a linear function $\ell\colon F_R(M \times N) \to P$ with $\ell(\delta_{(m,n)}) = B(m, n)$, so the diagram

$$\begin{tikzcd}
& F_R(M \times N) \arrow[dd, "\ell"] \\
M \times N \arrow[ur, "(m,n)\mapsto\delta_{(m,n)}"] \arrow[dr, "B"'] & \\
& P
\end{tikzcd}$$

commutes. We want to show ℓ makes sense as a function on $M \otimes_R N$, which means showing $\ker \ell$ contains D. From the bilinearity of B,

$$B(m + m', n) = B(m, n) + B(m, n'), \quad B(m, n + n') = B(m, n) + B(m, n'),$$
$$rB(m, n) = B(rm, n) = B(m, rn),$$

[2]The universal mapping property is not about modules T *per se*, but about pairs (T, b).

so

$$\ell(\delta_{(m+m',n)}) = \ell(\delta_{(m,n)}) + \ell(\delta_{(m',n)}), \quad \ell(\delta_{(m,n+n')}) = \ell(\delta_{(m,n)}) + \ell(\delta_{(m,n')}),$$

$$r\ell(\delta_{(m,n)}) = \ell(\delta_{(rm,n)}) = \ell(\delta_{(m,rn)}).$$

Since ℓ is linear, these conditions are the same as

$$\ell(\delta_{(m+m',n)}) = \ell(\delta_{(m,n)} + \delta_{(m',n)}), \quad \ell(\delta_{(m,n+n')}) = \ell(\delta_{(m,n)} + \delta_{(m,n')}),$$

$$\ell(r\delta_{(m,n)}) = \ell(\delta_{(rm,n)}) = \ell(\delta_{(m,rn)}).$$

Therefore the kernel of ℓ contains all the generators of the submodule D, so ℓ induces a linear map $L\colon F_R(M \times N)/D \to P$ where $L(\delta_{(m,n)} + D) = \ell(\delta_{(m,n)}) = B(m,n)$, so the diagram

$$\begin{tikzcd}
& F_R(M \times N)/D \arrow[dd, "L"] \\
M \times N \arrow[ur, "(m,n)\mapsto\delta_{m,n}+D"] \arrow[dr, "B"'] & \\
& P
\end{tikzcd}$$

commutes. Since $F_R(M \times N)/D = M \otimes_R N$ and $\delta_{(m,n)} + D = m \otimes n$, the above diagram is

$$\begin{tikzcd}
& M \otimes_R N \,. \arrow[dd, "L"] \\
M \times N \arrow[ur, "\otimes"] \arrow[dr, "B"'] & \\
& P
\end{tikzcd} \tag{3.4}$$

So every bilinear map B out of $M \times N$ comes from a linear map L out of $M \otimes_R N$ such that $L(m \otimes n) = B(m,n)$ for all $m \in M$ and $n \in N$.

It remains to show the linear map $M \otimes_R N \xrightarrow{L} P$ in (3.4) is the only one which makes (3.4) commute. We go back to the definition of $M \otimes_R N$ as a quotient of the free module $F_R(M \times N)$. From the construction of free modules, every element of $F_R(M \times N)$ is a finite sum

$$r_1\delta_{(m_1,n_1)} + \cdots + r_k\delta_{(m_k,n_k)}.$$

The reduction map $F_R(M \times N) \twoheadrightarrow F_R(M \times N)/D = M \otimes_R N$ is linear, so every element of $M \otimes_R N$ is a finite sum

$$r_1(m_1 \otimes n_1) + \cdots + r_k(m_k \otimes n_k), \tag{3.5}$$

This means the particular elements $m \otimes n$ in $M \otimes_R N$ span it as an R-module. Therefore linear maps out of $M \otimes_R N$ are completely determined by their values on all $m \otimes n$. So there is at most one linear map $M \otimes_R N \to P$ with the effect $m \otimes n \mapsto B(m,n)$. Since we have created a linear map out of $M \otimes_R N$ with this effect in (3.4), it is the only one. $\square$

Having shown a tensor product of M and N exists,[3] its essential uniqueness lets us call $M \otimes_R N$ "the" tensor product rather than "a" tensor product. Don't forget that the construction involves not simply the module $M \otimes_R N$ but also the distinguished bilinear map $M \times N \xrightarrow{\otimes} M \otimes_R N$ given by $(m, n) \mapsto m \otimes n$, through which all bilinear maps out of $M \times N$ factor. We call this distinguished map the *canonical* bilinear map to the tensor product. We call general elements of $M \otimes_R N$ *tensors*, and will denote them by the letter t. Tensors of the form $m \otimes n$ are called *elementary tensors*. (Other names for elementary tensors are simple tensors, decomposable tensors, and pure tensors.) Just as most elements of the free module $F_R(A)$ on a set A are *not* of the form δ_a but are linear combinations of these, most tensors in $M \otimes_R N$ are *not* elementary tensors but are linear combinations of elementary tensors. Actually, all tensors are *sums* of elementary tensors since $r(m \otimes n) = (rm) \otimes n$. This shows all elements of $M \otimes_R N$ have the form (1.3).

The role of elementary tensors among all tensors is like that of separable solutions $f(x)g(y)$ to a PDE among all solutions. Even if not all solutions to a PDE are separable, one first seeks separable solutions and then tries to form the general solution as a sum (perhaps an infinite series; it is analysis) of separable solutions.

From now on *forget* the explicit construction of $M \otimes_R N$ as the quotient of an enormous free module $F_R(M \times N)$. It will confuse you more than it's worth to try to think about $M \otimes_R N$ in terms of its construction. What is more important to remember is the universal mapping property of the tensor product.

Lemma 3.3. *Let M and N be R-modules with respective spanning sets $\{x_i\}_{i \in I}$ and $\{y_j\}_{j \in J}$. The tensor product $M \otimes_R N$ is spanned linearly by the elementary tensors $x_i \otimes y_j$.*

Proof. An elementary tensor in $M \otimes_R N$ has the form $m \otimes n$. Write $m = \sum_i a_i x_i$ and $n = \sum_j b_j y_j$, where the a_i's and b_j's are 0 for all but finitely many i and j. From the bilinearity of $\otimes$,

$$m \otimes n = (\sum_i a_i x_i) \otimes (\sum_j b_j y_j) = \sum_{i,j} a_{ij} x_i \otimes y_j$$

is a linear combination of the tensors $x_i \otimes y_j$. So every elementary tensor is a linear combination of the elementary tensors $x_i \otimes y_j$, which means every tensor is such a linear combination too. □

Example 3.4. Let $e_1, \ldots, e_k$ be the standard basis of R^k. The R-module $R^k \otimes_R R^k$ is linearly spanned by the k^2 elementary tensors $e_i \otimes e_j$. We will see later (Theorem 4.8) that these elementary tensors are a basis.

Theorem 3.5. *In $M \otimes_R N$, $m \otimes 0 = 0$ and $0 \otimes n = 0$.*

Proof. Since $m \otimes n$ is linear in n with m fixed, $m \otimes 0 = m \otimes (0 + 0) = m \otimes 0 + m \otimes 0$. Subtracting $m \otimes 0$ from both sides, $m \otimes 0 = 0$. That $0 \otimes n = 0$ follows by a similar argument. □

[3] What happens if R is a noncommutative ring? If M and N are left R-modules and B is bilinear on $M \times N$ then for any $m \in M$, $n \in N$, and r and s in R, $rsB(m, n) = rB(m, sn) = B(rm, sn) = sB(rm, n) = srB(m, n)$. Usually $rs \neq sr$, so asking that $rsB(m, n) = srB(m, n)$ for all m and n puts us in a delicate situation! The correct tensor product $M \otimes_R N$ for noncommutative R uses a *right* R-module M, a *left* R-module N, and a "middle-linear" map B where $B(mr, n) = B(m, rn)$. In fact $M \otimes_R N$ is not an R-module but just an abelian group! We will not discuss this further.

Example 3.6. If A is a finite abelian group, $\mathbf{Q} \otimes_{\mathbf{Z}} A = 0$ since every elementary tensor is 0: for $a \in A$, let $na = 0$ for some positive integer n. Then in $\mathbf{Q} \otimes_{\mathbf{Z}} A$, $r \otimes a = n(r/n) \otimes a = r/n \otimes na = r/n \otimes 0 = 0$. Every tensor is a sum of elementary tensors, so all tensors are 0. In fact we don't need finiteness of A in this argument, but rather than each element of A has finite order. The group $\mathbf{Q}/\mathbf{Z}$ has this property, so $\mathbf{Q} \otimes_{\mathbf{Z}} (\mathbf{Q}/\mathbf{Z}) = 0$. (For instance, $(2/5) \otimes (1/3) = 0$, so we can have $m \otimes n = 0$ without m or n being 0.)

By a similar argument, $\mathbf{Q}/\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} = 0$.

Since $M \otimes_R N$ is spanned additively by elementary tensors, any linear (or just additive) function out of $M \otimes_R N$ is determined at all tensors from its values on elementary tensors. This is why linear maps on tensor products will in practice only be described by their values on elementary tensors. It is similar to describing a linear map between finite free modules using a matrix, which tells you the values of the map only on basis vectors, but this information is enough to determine the linear map everywhere.

There is a key difference between elementary tensors and basis vectors: elementary tensors have lots of linear relations. Using elementary tensors to describe all tensors is like describing the vectors in $\mathbf{R}^2$ as linear combinations of $(1,0)$, $(2,3)$, $(8,4)$, and $(-1,5)$. A linear map out of $\mathbf{R}^2$ is determined by its values on these four vectors, but those values are not independent: they have to satisfy every linear relation the four vectors satisfy because a linear map preserves linear relations. Using this linearly dependent spanning set requires keeping track of linear relations to be sure constructions made in terms of this spanning set are well-defined. Nobody wants to do that, which is why using bases is so convenient. But you usually can't get away from the zillions of linear relations among elementary tensors, and that is why a random function on elementary tensors generally does not extend to a linear map on the tensor product.

For that matter, functions on elementary tensors themselves have a well-definedness issue: the "function" $f(m \otimes n) = m + n$, for instance, makes no sense since $m \otimes n = (-m) \otimes (-n)$ but $m + n$ is usually not $-m - n$. The *only* way to create linear maps on $M \otimes_R N$ is with the universal mapping property of the tensor product (it creates linear maps out of bilinear maps), because all the linear relations among elementary tensors are built into the universal mapping property of $M \otimes_R N$. There will be a lot of practice with this in Section 4. Understanding how the universal mapping property of the tensor product can be used to compute (nonzero) examples and to prove properties of the tensor product is about the best way to get used to the tensor product; after all, if you're incapable of writing down functions out of $M \otimes_R N$, you don't understand $M \otimes_R N$.

Let's address a few beginner questions about the tensor product:

<u>Questions</u>

(1) What is $m \otimes n$?
(2) What does it mean to say $m \otimes n = 0$?
(3) What does it mean to say $M \otimes_R N = 0$?
(4) What does it mean to say $m_1 \otimes n_1 + \cdots + m_k \otimes n_k = m'_1 \otimes n'_1 + \cdots + m'_\ell \otimes n'_\ell$?
(5) Is there a way to picture the tensor product?

<u>Answers</u>

(1) A logically correct answer is: $m \otimes n$ is the image of $(m, n) \in M \times N$ under the canonical bilinear map $M \times N \xrightarrow{\otimes} M \otimes_R N$ that is part of the definition of the tensor product. That may seem like too formal an answer. Here's another answer, which is not a definition but is more closely aligned with how $m \otimes n$ actually occurs

in practice: $m \otimes n$ is that element of $M \otimes_R N$ at which the linear map $M \otimes_R N \to P$ corresponding to a bilinear map $M \times N \xrightarrow{B} P$ takes the value $B(m, n)$. Review the proof of Theorem 3.2 and check this property of $m \otimes n$ really holds.

(2) We have $m \otimes n = 0$ if and only if every bilinear map out of $M \times N$ vanishes at (m, n). Indeed, if $m \otimes n = 0$ then for any bilinear map $B\colon M \times N \to P$ we have a commutative diagram

$$\begin{tikzcd} & M \otimes_R N \arrow[dd, "L"] \\ M \times N \arrow[ur, "\otimes"] \arrow[dr, "B"'] & \\ & P \end{tikzcd}$$

for some linear map L, so $B(m, n) = L(m \otimes n) = L(0) = 0$. Conversely, if every bilinear map out of $M \times N$ sends (m, n) to 0 then the canonical bilinear map $M \times N \to M \otimes_R N$, which is a particular example, sends (m, n) to 0. Since this bilinear map actually sends (m, n) to $m \otimes n$, we obtain $m \otimes n = 0$.

To show a particular elementary tensor $m \otimes n$ is not 0, find a bilinear map B out of $M \times N$ such that $B(m, n) \neq 0$. We will use this idea in Theorem 4.8.

(3) The tensor product $M \otimes_R N$ is 0 if and only if every bilinear map out of $M \times N$ is identically 0. First suppose $M \otimes_R N = 0$. Then every elementary tensor $m \otimes n$ is 0, so $B(m, n) = 0$ for any bilinear map out of $M \times N$ by the answer to the second question. Thus B is identically 0. Next suppose every bilinear map out of $M \times N$ is identically 0. Then the canonical bilinear map $M \times N \xrightarrow{\otimes} M \otimes_R N$, which is a particular example, is identically 0. Since this function sends (m, n) to $m \otimes n$, we have $m \otimes n = 0$ for all m and n. Since $M \otimes_R N$ is additively spanned by all $m \otimes n$, the vanishing of all elementary tensors implies $M \otimes_R N = 0$.

To show $M \otimes_R N \neq 0$, find a bilinear map on $M \times N$ which is not identically 0.

(4) We have $\sum_{i=1}^{k} m_i \otimes n_i = \sum_{j=1}^{\ell} m'_j \otimes n'_j$ if and only if for all bilinear maps B out of $M \times N$, $\sum_{i=1}^{k} B(m_i, n_i) = \sum_{j=1}^{\ell} B(m'_j, n'_j)$. The justification is along the lines of the previous two answers and is left to the reader.

(5) There are objects in physics that are described using tensors (stress, elasticity, electromagnetic fields), but this doesn't really lead to a picture of a tensor. Tensors arise in physics as multi-indexed quantities which are subject to multilinear effects under a change in coordinates.

The tensor product can be extended to allow more than two factors. Given k modules $M_1, \ldots, M_k$, there is a module $M_1 \otimes_R \cdots \otimes_R M_k$ which is universal for k-multilinear maps: it admits a k-multilinear map $M_1 \times \cdots \times M_k \xrightarrow{\otimes} M_1 \otimes_R \cdots \otimes_R M_k$ and every k-multilinear map out of $M_1 \times \cdots \times M_k$ factors through this by composition with a unique linear map out of $M_1 \otimes_R \cdots \otimes_R M_k$:

$$\begin{tikzcd} & & M_1 \otimes_R \cdots \otimes_R M_k \arrow[dd, dotted, "\exists \text{ unique linear}"] \\ M_1 \times \cdots \times M_k \arrow[urr, "\otimes"] \arrow[drr, "\text{multilinear}"'] & & \\ & & P \end{tikzcd}$$

The image of $(m_1, \ldots, m_k)$ in $M_1 \otimes_R \cdots \otimes_R M_k$ is written $m_1 \otimes \cdots \otimes m_k$. This k-fold tensor product can be constructed as a quotient of the free module $F_R(M_1 \times \cdots \times M_k)$. It can also be constructed using tensor products of modules two at a time:

$$(\cdots((M_1 \otimes_R M_2) \otimes_R M_3) \otimes_R \cdots) \otimes_R M_k.$$

The canonical k-multilinear map to this R-module from $M_1 \times \cdots \times M_k$ is $(m_1, \ldots, m_k) \mapsto (\cdots((m_1 \otimes m_2) \otimes m_3) \cdots) \otimes m_k$. This is not the same construction of the k-fold tensor product using $F_R(M_1 \times \cdots \times M_k)$, but it satisfies the same universal mapping property and thus can serve the same purpose (all tensors products of $M_1, \ldots, M_k$ are isomorphic to each other in a unique way compatible with the distinguished bilinear maps to them from $M_1 \times \cdots \times M_k$).

As an exercise, check from the universal mapping property that $m_1 \otimes \cdots \otimes m_k = 0$ in $M_1 \otimes_R \cdots \otimes_R M_k$ if and only if all k-multilinear maps out of $M_1 \times \cdots \times M_k$ vanish at $(m_1, \ldots, m_k)$. The module $M_1 \otimes_R \cdots \otimes_R M_k$ is spanned additively by all $m_1 \otimes \cdots \otimes m_k$. Important examples of the k-fold tensor product are *tensor powers* $M^{\otimes k}$:

$$M^{\otimes 0} = R, \quad M^{\otimes 1} = M, \quad M^{\otimes 2} = M \otimes_R M, \quad M^{\otimes 3} = M \otimes_R M \otimes_R M,$$

and so on. (The formula $M^{\otimes 0} = R$ is a convention, like $a^0 = 1$.)

4. Examples of Tensor Products

Theorem 4.1. *For positive integers a and b with $d = (a, b)$, $\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z} \cong \mathbf{Z}/d\mathbf{Z}$ as abelian groups. In particular, $\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z} = 0$ if and only if $(a, b) = 1$.*

Proof. Since 1 spans $\mathbf{Z}/a\mathbf{Z}$ and $\mathbf{Z}/b\mathbf{Z}$, $1 \otimes 1$ spans $\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z}$ by Lemma 3.3. From

$$a(1 \otimes 1) = a \otimes 1 = 0 \otimes 1 = 0 \text{ and } \; b(1 \otimes 1) = 1 \otimes b = 1 \otimes 0 = 0,$$

the additive order of $1 \otimes 1$ divides a and b, and therefore also d, so $\#(\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z}) \leq d$.

To show $\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z}$ has size at least d, we *create* a $\mathbf{Z}$-linear map from $\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z}$ onto $\mathbf{Z}/d\mathbf{Z}$. Since $d|a$ and $d|b$, we can reduce $\mathbf{Z}/a\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ and $\mathbf{Z}/b\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ in the natural way. Consider the map $\mathbf{Z}/a\mathbf{Z} \times \mathbf{Z}/b\mathbf{Z} \xrightarrow{B} \mathbf{Z}/d\mathbf{Z}$ which is reduction mod d in each factor followed by multiplication: $B(x \bmod a, y \bmod b) = xy \bmod d$. This is $\mathbf{Z}$-bilinear, so the universal mapping property of the tensor product says there is a (unique) $\mathbf{Z}$-linear map

$f\colon \mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ making the diagram

$$\begin{tikzcd} & \mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z} \arrow[dd, "f"] \\ \mathbf{Z}/a\mathbf{Z} \times \mathbf{Z}/b\mathbf{Z} \arrow[ur, "\otimes"] \arrow[dr, "B"'] & \\ & \mathbf{Z}/d\mathbf{Z} \end{tikzcd}$$

commute, so $f(x \otimes y) = xy$. In particular, $f(x \otimes 1) = x$, so f is onto. Therefore $\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z}$ has size at least d, so the size is d and we're done. $\square$

Example 4.2. The abelian group $\mathbf{Z}/3\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/5\mathbf{Z}$ is 0. Equivalently, every $\mathbf{Z}$-bilinear map $\mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}/5\mathbf{Z} \to A$ to an abelian group is identically 0.

In $\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z}$ all tensors are elementary tensors: $x \otimes y = xy(1 \otimes 1)$ and a sum of multiples of $1 \otimes 1$ is again a multiple, so $\mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z} = \mathbf{Z}(1 \otimes 1) = \{x \otimes 1 : x \in \mathbf{Z}\}$.

Notice in the proof of Theorem 4.1 how the map $f\colon \mathbf{Z}/a\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/b\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ was created from the bilinear map $B\colon \mathbf{Z}/a\mathbf{Z} \times \mathbf{Z}/b\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ and the universal mapping property of tensor products. Quite generally, to define a linear map out of $M \otimes_R N$ that sends all elementary tensors $m \otimes n$ to particular places, *always* back up and start by defining a bilinear map out of $M \times N$ sending (m, n) to the place you want $m \otimes n$ to go. Make sure you show the map is bilinear! Then the universal mapping property of the tensor product gives you a linear map out of $M \otimes_R N$ sending $m \otimes n$ to the place where (m, n) goes, which gives you what you wanted: a (unique) linear map on the tensor product with specified values on the elementary tensors.

Theorem 4.3. *For ideals I and J in R, there is a unique R-module isomorphism*

$$R/I \otimes_R R/J \cong R/(I + J)$$

where $\overline{x} \otimes \overline{y} \mapsto \overline{xy}$. In particular, taking $I = J = 0$, $R \otimes_R R \cong R$ by $x \otimes y \mapsto xy$.

For $R = \mathbf{Z}$ and nonzero I and J, this is Theorem 4.1.

Proof. Start with the function $R/I \times R/J \to R/(I + J)$ given by $(x \bmod I, y \bmod J) \mapsto xy \bmod I + J$. This is well-defined and bilinear, so from the universal mapping property of the tensor product we get a linear map $f\colon R/I \otimes_R R/J \to R/(I + J)$ making the diagram

$$\begin{tikzcd} & R/I \otimes_R R/J \arrow[dd, "f"] \\ R/I \times R/J \arrow[ur, "\otimes"] \arrow[dr, "(x \bmod I, y \bmod J) \mapsto xy \bmod I+J"'] & \\ & R/(I + J) \end{tikzcd}$$

commute, so $f(x \bmod I \otimes y \bmod J) = xy \bmod I + J$. To write down the inverse map, let $R \to R/I \otimes_R R/J$ by $r \mapsto r(\overline{1} \otimes \overline{1})$. This is linear, and when $r \in I$ the value is $\overline{r} \otimes \overline{1} = \overline{0} \otimes \overline{1} = 0$. Similarly, when $r \in J$ the value is 0. Therefore $I + J$ is in the kernel, so we get a linear map $g\colon R/(I + J) \to R/I \otimes_R R/J$ by $g(r \bmod I + J) = r(\overline{1} \otimes \overline{1}) = \overline{r} \otimes \overline{1} = \overline{1} \otimes \overline{r}$.

To check f and g are inverses, a computation in one direction shows

$$f(g(r \bmod I+J)) = f(\overline{r}\otimes\overline{1}) = r \bmod I+J.$$

To show $g(f(t)) = t$ for all $t \in R/I \otimes_R R/J$, we show all tensors are scalar multiples of $\overline{1}\otimes\overline{1}$. Any elementary tensor has the form $\overline{x}\otimes\overline{y} = x\overline{1}\otimes y\overline{1} = xy(\overline{1}\otimes\overline{1})$, which is a multiple of $\overline{1}\otimes\overline{1}$, so sums of elementary tensors are multiples of $\overline{1}\otimes\overline{1}$ and thus all tensors are multiples of $\overline{1}\otimes\overline{1}$. We have

$$g(f(r(\overline{1}\otimes\overline{1})) = rg(1 \bmod I+J) = r(\overline{1}\otimes\overline{1}) = r(\overline{1}\otimes\overline{1}).$$

$\square$

Remark 4.4. For two ideals I and J, we know a few operations that produce new ideals: $I+J$, $I\cap J$, and IJ. The intersection $I\cap J$ is the kernel of the linear map $R \to R/I\oplus R/J$ where $r \mapsto (\overline{r},\overline{r})$. Theorem 4.3 tells us $I+J$ is the kernel of the linear map $R\to R/I\otimes_R R/J$ where $r\mapsto r(\overline{1}\otimes\overline{1})$.

Theorem 4.5. *For an ideal I in R and R-module M, there is a unique R-module isomorphism*

$$(R/I)\otimes_R M \cong M/IM$$

where $\overline{r}\otimes m \mapsto \overline{rm}$. In particular, taking $I=(0)$, $R\otimes_R M\cong M$ by $r\otimes m\mapsto rm$, so $R\otimes_R R\cong R$ as R-modules by $r\otimes r' \mapsto rr'$.

Proof. We start with the bilinear map $(R/I)\times M \to M/IM$ given by $(\overline{r},m)\mapsto\overline{rm}$. From the universal mapping property of the tensor product, we get a linear map $f\colon (R/I)\otimes_R M \to M/IM$ where $f(\overline{r}\otimes m) = \overline{rm}$.

$$\begin{tikzcd}
 & (R/I)\otimes_R M \arrow[dd, "f"] \\
(R/I)\times M \arrow[ur, "\otimes"] \arrow[dr, "(\overline{r},m)\mapsto\overline{rm}"'] & \\
 & M/IM
\end{tikzcd}$$

To create an inverse map, start with the function $M\to(R/I)\otimes_R M$ given by $m\mapsto\overline{1}\otimes m$. This is linear in m (check!) and kills IM (generators for IM are products rm for $r\in I$ and $m\in M$, and $\overline{1}\otimes rm = \overline{r}\otimes m = \overline{0}\otimes m = 0$), so it induces a linear map $g\colon M/IM \to (R/I)\otimes_R M$ given by $g(\overline{m}) = \overline{1}\otimes m$.

To check $f(g(\overline{m})) = \overline{m}$ and $g(f(t)) = t$ for all $\overline{m}\in M/IM$ and $t\in(R/I)\otimes_R M$, we do the first one by a direct computation:

$$f(g(\overline{m})) = f(\overline{1}\otimes m) = \overline{1\cdot m} = \overline{m}.$$

To show $g(f(t)) = t$ for all $t\in M\otimes_R N$, we show all tensors in $R/I\otimes_R M$ are elementary. An elementary tensor looks like $\overline{r}\otimes m = \overline{1}\otimes rm$, and a sum of tensors $\overline{1}\otimes m_i$ is $\overline{1}\otimes\sum_i m_i$. Thus all tensors look like $\overline{1}\otimes m$. We have $g(f(\overline{1}\otimes m)) = g(\overline{m}) = \overline{1}\otimes m$. $\square$

Example 4.6. For any abelian group A, $(\mathbf{Z}/n\mathbf{Z})\otimes_{\mathbf{Z}} A \cong A/nA$ as abelian groups by $\overline{m}\otimes a\mapsto\overline{ma}$.

Remark 4.7. Saying $R \otimes_R M \cong M$ by $r \otimes m \mapsto rm$ is another way of saying the R-bilinear maps B out of $R \times M$ can be identified with the linear maps out of M, and that's clear because $B(r, m) = B(1, rm)$ when B is bilinear, and $B(1, -)$ is linear in the second component.

The next theorem justifies the discussion in the introduction about bases for tensor products of free modules.

Theorem 4.8. *If F and F' are free R-modules then $F \otimes_R F'$ is a free R-module. If $\{e_i\}_{i \in I}$ and $\{e'_j\}_{j \in J}$ are bases of F and F' then $\{e_i \otimes e'_j\}_{(i,j) \in I \times J}$ is a basis of $F \otimes_R F'$.*

Proof. The result is clear if F or F' is 0, so let them both be nonzero free modules (hence $R \neq 0$ and F and F' have bases). By Lemma 3.3, $\{e_i \otimes e'_j\}$ spans $F \otimes_R F'$ as an R-module.

To show this spanning set is linearly independent, suppose $\sum_{i,j} c_{ij} e_i \otimes e'_j = 0$, where all but finitely many c_{ij} are 0. We want to show every c_{ij} is 0. Pick two basis vectors e_{i_0} and e'_{j_0} in F and F'. To create a "coordinate function" for $e_{i_0} \otimes e'_{j_0}$ in $F \otimes_R F'$, define the function $F \times F' \to R$ by $(v, w) \mapsto a_{i_0} b_{j_0}$, where $v = \sum_i a_i e_i$ and $w = \sum_j b_j e'_j$. This function is bilinear, so by the universal mapping property of tensor products there is a linear map $f_0 \colon F \otimes_R F' \to R$ such that $f_0(v \otimes w) = a_{i_0} b_{j_0}$ on all elementary tensors $v \otimes w$.

$$\begin{tikzcd}
 & F \otimes_R F' \arrow[dd, "f_0"] \\
F \times F' \arrow[ur, "\otimes"] \arrow[dr, "(v,w) \mapsto a_{i_0} b_{j_0}"'] & \\
 & R
\end{tikzcd}$$

In particular, $f_0(e_{i_0} \otimes e'_{j_0}) = 1$ and $f_0(e_i \otimes e'_j) = 0$ for $(i, j) \neq (i_0, j_0)$. Applying f_0 to the equation $\sum_{i,j} c_{ij} e_i \otimes e'_j = 0$ in $F \otimes_R F'$ tells us $c_{i_0 j_0} = 0$ in R. Since i_0 and j_0 are arbitrary, all the coefficients are 0. $\square$

Theorem 4.8 has a concrete meaning in terms of bilinear maps out of $F \times F'$. It says that any bilinear map out of $F \times F'$ is determined by its values on the pairs (e_i, e'_j) and that any assignment of values to these pairs extends in a unique way to a bilinear map. (The uniqueness of the extension is connected to the linear independence of the elementary tensors $e_i \otimes e'_j$.) This is the bilinear analogue of the existence and uniqueness of a linear extension of a function from a basis of a free module to the whole module.

Example 4.9. Let $R \neq 0$ and $F = Re_1 \oplus Re_2$ be free of rank 2. Then $F \otimes_R F$ is free of rank 4 with basis

$$e_1 \otimes e_1, \quad e_1 \otimes e_2, \quad e_2 \otimes e_1, \quad e_2 \otimes e_2.$$

(The tensors $e_1 \otimes e_2$ and $e_2 \otimes e_1$ are *not* equal, since they're different terms in a basis, but there's also an explanation of this using bilinear maps: if $e_1 \otimes e_2 = e_2 \otimes e_1$ then all bilinear maps out of $F \times F$ takes the same value at (e_1, e_2) and (e_2, e_1), and that is not true. Consider the bilinear map $F \times F \to R$ given by $(ae_1 + be_2, ce_1 + de_2) \mapsto ad$. So $e_1 \otimes e_2 \neq e_2 \otimes e_1$.)

The sum $e_1 \otimes e_2 + e_2 \otimes e_1$ in $F \otimes_R F$ is an example of a tensor that is provably *not* an elementary tensor. Any elementary tensor in $F \otimes_R F$ has the form

$$(ae_1 + be_2) \otimes (ce_1 + de_2) = ace_1 \otimes e_1 + ade_1 \otimes e_2 + bce_2 \otimes e_1 + bde_2 \otimes e_2. \tag{4.1}$$

If this equals $e_1 \otimes e_2 + e_2 \otimes e_1$ then

$$ac = 0, \quad ad = 1, \quad bc = 1, \quad bd = 0.$$

Since $ad = 1$ and $bc = 1$, a and c are invertible, but that contradicts $ac = 0$. So $e_1 \otimes e_2 + e_2 \otimes e_1$ is not an elementary tensor.[4]

Example 4.10. As an R-module, $R[X] \otimes_R R[Y]$ is free with basis $\{X^i \otimes Y^j\}_{i,j \geq 0}$, so this tensor product is isomorphic to $R[X, Y]$ as R-modules by $\sum c_{ij}(X^i \otimes Y^j) \mapsto \sum c_{ij} X^i Y^j$. More generally, $R[X_1, \ldots, X_k] \cong R[X]^{\otimes k}$ as R-modules with X_i corresponding to the tensor $1 \otimes \cdots \otimes X \otimes \cdots \otimes 1$ where X is in the ith position. The difference between ordinary products and tensor products is like the difference between multiplying polynomials as $f(T)g(T)$ and as $f(X)g(Y)$.

Example 4.11. We return to Example 2.1. For φ in $M^\vee$ and ψ in $N^\vee$, $B_{\varphi,\psi}(m, n) = \varphi(m)\psi(n)$ is a bilinear map $M \times N \to R$. Set $\mathbf{B}\colon M^\vee \times N^\vee \to \mathrm{Bil}_R(M, N; R)$ by $\mathbf{B}(\varphi, \psi) = B_{\varphi,\psi}$. This is bilinear, so we get a linear map $\mathbf{L}\colon M^\vee \otimes_R N^\vee \to \mathrm{Bil}_R(M, N; R)$ where $\mathbf{L}(\varphi \otimes \psi) = B_{\varphi,\psi}$.

When M and N are finite free R-modules, let's show $\mathbf{L}$ is an isomorphism. We may suppose M and N are nonzero, with respective bases $\{e_1, \ldots, e_k\}$ and $\{f_1, \ldots, f_\ell\}$. By Theorem 4.8, $M^\vee \otimes_R N^\vee$ is free of rank $k\ell$ with basis $\{e_i^\vee \otimes f_j^\vee\}$. By Example 2.1, $\mathrm{Bil}_R(M, N; R)$ is free of rank $k\ell$ with basis $\{B_{e_i^\vee, f_j^\vee}\}$. Since $B_{e_i^\vee, f_j^\vee} = \mathbf{L}(e_i^\vee \otimes f_j^\vee)$, $\mathbf{L}$ sends a basis to a basis, so it's an isomorphism when M and N are finite free.

It is *not* true that $\mathbf{L}$ is an isomorphism for all M and N. When p is prime, $R = \mathbf{Z}/p^2\mathbf{Z}$, and $M = \mathbf{Z}/p\mathbf{Z}$, show as an exercise that $\mathbf{L}\colon M^\vee \otimes_R M^\vee \to \mathrm{Bil}_R(M, M; R)$ is identically 0 but $M^\vee \otimes_R M^\vee$ and $\mathrm{Bil}_R(M, M; R)$ both have size p.

Theorem 4.12. *Let F be a free R-module with basis $\{e_i\}_{i \in I}$. For any $k \geq 1$, the kth tensor power $F^{\otimes k}$ is free with basis $\{e_{i_1} \otimes \cdots \otimes e_{i_k}\}_{(i_1, \ldots, i_k) \in I^k}$.*

Proof. This is similar to the proof of Theorem 4.8. $\square$

Theorem 4.13. *If M is an R-module and F is a free R-module with basis $\{e_i\}_{i \in I}$, then every element of $M \otimes_R F$ has a unique representation in the form $\sum_{i \in I} m_i \otimes e_i$, where all but finitely many m_i equal 0.*

Proof. Using M as a spanning set of M and $\{e_i\}_{i \in I}$ as a spanning set for F as R-modules, by Lemma 3.3 every element of $M \otimes_R F$ is a linear combination of elementary tensors $m_i \otimes e_i$, where $m_i \in M$. Since $r(m_i \otimes e_i) = (rm_i) \otimes e_i$, we can write every tensor in $M \otimes_R F$ as a sum of elementary tensors of the form $m_i \otimes e_i$. So we have a surjective linear map $f\colon \bigoplus_{i \in I} M \mapsto M \otimes_R F$ given by $f((m_i)_{i \in I}) = \sum_{i \in I} m_i \otimes e_i$. (All but finitely many m_i are 0, so the sum makes sense.)

To create an inverse to f, consider the function $M \times F \to \bigoplus_{i \in I} M$ where $(m, \sum_i r_i e_i) \mapsto (r_i m)_{i \in I}$. This function is bilinear (check!), so there is a linear map $g\colon M \otimes_R F \to \bigoplus_{i \in I} M$ where $g(m \otimes \sum_i r_i e_i) = (r_i m)_{i \in I}$.

To check $f(g(t)) = t$ for all t in $M \otimes_R F$, we can't expect that all tensors in $M \otimes_R F$ are elementary (an idea used in the proofs of Theorems 4.3 and 4.5), but we only need to check $f(g(t)) = t$ when t is an elementary tensor since f and g are additive and the elementary

[4]From (4.1), a necessary condition for $\sum_{i,j=1}^2 c_{ij} e_i \otimes e_j$ to be elementary is that $c_{11}c_{22} = c_{12}c_{21}$. When $R = K$ is a field this condition is also sufficient, so the elementary tensors in $K^2 \otimes_K K^2$ are characterized among all tensors by a polynomial equation of degree 2. For a generalization, see [2].

tensors additively span $M \otimes_R F$. (We will use this kind of argument a lot to reduce the proof of an identity involving functions of all tensors to the case of elementary tensors even though most tensors are not themselves elementary. The point is all tensors are sums of elementary tensors and the formula we want to prove involves additive functions.) Any elementary tensor looks like $m \otimes \sum_i r_i e_i$, and

$$\begin{aligned} f\left(g\left(m \otimes \sum_{i \in I} r_i e_i\right)\right) &= f((r_i m)_{i \in I}) \\ &= \sum_{i \in I} r_i m \otimes e_i \\ &= \sum_{i \in I} m \otimes r_i e_i \\ &= m \otimes \sum_{i \in I} r_i e_i. \end{aligned}$$

These sums have finitely many terms ($r_i = 0$ for all but finitely many i), from the definition of direct sums. Thus $f(g(t)) = t$ for all $t \in M \otimes_R F$.

For the composition in the other order,

$$g(f((m_i)_{i \in I})) = g\left(\sum_{i \in I} m_i \otimes e_i\right) = \sum_{i \in I} g(m_i \otimes e_i) = \sum_{i \in I} (\dots, 0, m_i, 0, \dots) = (m_i)_{i \in I}.$$

Now that we know $M \otimes_R F \cong \bigoplus_{i \in I} M$, with $\sum_{i \in I} m_i \otimes e_i$ corresponding to $(m_i)_{i \in I}$, the uniqueness of coordinates in the direct sum implies the sum representation $\sum_{i \in I} m_i \otimes e_i$ is unique. $\square$

Example 4.14. For any ring $S \supset R$, elements of $S \otimes_R R[X]$ have unique expressions of the form $\sum_{n \geq 0} s_n \otimes X^n$, so $S \otimes_R R[X] \cong S[X]$ as R-modules by $\sum_{n \geq 0} s_n \otimes X^n \mapsto \sum_{n \geq 0} s_n X^n$.

Remark 4.15. Although you can check an additive identity $f(g(t)) = t$ by only checking it on elementary tensors, it would be really dumb to think you have proved injectivity of some linear map $f \colon M \otimes_R N \to P$ by only looking at elementary tensors.[5] That is, if $f(m \otimes n) = 0 \Rightarrow m \otimes n = 0$, there is no reason to believe $f(t) = 0 \Rightarrow t = 0$ for all $t \in M \otimes_R N$, since injectivity of a linear map is *not* an additive identity.[6] This is the number one reason that proving that a linear map out of a tensor product is injective can require technique. As a special case, if you want to prove a linear map out of a tensor product is an isomorphism, it might be easier to construct an inverse map and check the composite in both orders is the identity than to show the original map is injective and surjective.

Theorem 4.16. *If M is nonzero and finitely generated then $M^{\otimes k} \neq 0$ for all k.*

Proof. Write $M = Rx_1 + \cdots + Rx_d$, where $d \geq 1$ is minimal. Set $N = Rx_1 + \cdots + Rx_{d-1}$ ($N = 0$ if $d = 1$), so $M = N + Rx_d$ and $x_d \notin N$. Set $I = \{r \in R : rx_d \in N\}$, so I is an ideal in R and $1 \notin I$, so I is a proper ideal. From the definition of I, the function $M^k \to R/I$ given by

$$(n_1 + r_1 x_d, \dots, n_k + r_k x_d) \mapsto r_1 \cdots r_d \bmod I$$

[5]Unless every tensor in $M \otimes_R N$ is elementary, which is usually not the case.

[6]Here's an example. Let $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C} \to \mathbf{C}$ be the $\mathbf{R}$-linear map with the effect $z \otimes w \mapsto zw$ on elementary tensors. If $z \otimes w \mapsto 0$ then $z = 0$ or $w = 0$, so $z \otimes w = 0$, but the map is not injective: $1 \otimes i - i \otimes 1 \mapsto 0$ but $1 \otimes i - i \otimes 1 \neq 0$ since $1 \otimes i$ and $i \otimes 1$ belong to a basis of $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}$ by Theorem 4.8.

is well-defined. It's multilinear (check!), so there is a linear map $M^{\otimes k} \to R/I$ such that $\underbrace{x_d \otimes \cdots \otimes x_d}_{k \text{ terms}} \mapsto 1$. That shows $M^{\otimes k} \neq 0$. $\square$

Tensor powers of non-finitely generated modules could vanish: $\mathbf{Q}/\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z} = 0$ (Example 3.6).

The rest of this section is concerned with properties of tensor products over domains.

Theorem 4.17. *Let R be a domain with fraction field K. There is a unique R-module isomorphism $K \otimes_R K \cong K$ where $x \otimes y \mapsto xy$. In particular, $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$.*

Proof. Multiplication is a function $m \colon K \times K \to K$. It is R-bilinear, so the universal mapping property of tensor products says there is an R-linear function $f \colon K \otimes_R K \to K$ where $f(x \otimes y) = xy$ on elementary tensors. That says the diagram

$$\begin{tikzcd} & K \otimes_R K \arrow[dd, "f"] \\ K \times K \arrow[ur, "\otimes"] \arrow[dr, "m"'] & \\ & K \end{tikzcd}$$

commutes. Since $f(x \otimes 1) = x$, f is onto.

To show f is one-to-one, first we show every tensor in $K \otimes_R K$ looks like $1 \otimes y$. For an elementary tensor $x \otimes y$, write $x = a/b$ with a and b in R where $b \neq 0$. Then $x \otimes y = x \otimes by/b = bx \otimes y/b = a \otimes y/b = 1 \otimes ay/b = 1 \otimes xy$. (We can't say $x \otimes y = 1 \otimes xy$ from bilinearity of $\otimes$, as $\otimes$ is R-bilinear and $x \in K$.) Thus all elementary tensors have the form $1 \otimes y$, so every tensor is $1 \otimes y$ for some $y \in K$ since every tensor is a sum of elementary tensors. Now we can show f has trivial kernel: if $f(t) = 0$ then, writing $t = 1 \otimes y$, we get $y = 0$, so $t = 1 \otimes 0 = 0$. $\square$

Remark 4.18. Multiplication $K \times K \to K$ is also K-bilinear, not just R-bilinear, and there is a K-vector space isomorphism $K \otimes_K K \cong K$ (tensor product over K, not R).

For any R-module M, every tensor in $K \otimes_R M$ is elementary from manipulations with tensors: any finite set of tensors of the form $a_i/b_i \otimes m_i$ can be provided with a common denominator for the b_i's, say $a_i/b_i = a_i'/b$, so $\sum_i a_i/b_i \otimes m_i = \sum_i 1/b \otimes a_i' m_i = 1/b \otimes (\sum_i a_i' m_i)$, which is an elementary tensor. What we can't easily do by manipulations with tensors alone is determine when an elementary tensor $x \otimes m$ is 0 in $K \otimes_R M$.

Theorem 4.19. *Let R be a domain with fraction field K and M be any nonzero R-module in K. There is a unique R-module isomorphism $K \otimes_R M \cong K$ where $x \otimes y \mapsto xy$. In particular, $K \otimes_R I \cong K$ for every nonzero ideal I in R.*

Proof. The proof is largely like the previous one, but it needs a few extra steps because the second module is an R-module in K rather than K itself.[7] Multiplication gives a function $K \times M \to K$ which is R-bilinear, so we get an R-linear map $f \colon K \otimes_R M \to K$ where $f(x \otimes y) = xy$. To show f is onto, we can't look at $f(x \otimes 1)$ since 1 is usually not in M. Instead we can argue with a choice of nonzero $m \in M$: for any $x \in K$, $f(x/m \otimes m) = x$.

[7] Theorem 4.17 is just a special case of Theorem 4.19, but we worked it out separately since some of the technicalities are simpler.

To show f is injective, suppose $f(t) = 0$. All tensors in $K \otimes_R M$ are elementary, so we can write $t = x \otimes y$. Then $xy = 0$ in K, so $x = 0$ or $y = 0$, so $t = x \otimes y = 0$. □

Theorem 4.20. *Let R be a domain and F and F' be free R-modules. If x and x' are nonzero in F and F', then $x \otimes x' \neq 0$ in $F \otimes_R F'$.*

Proof. Write $x = \sum_i a_i e_i$ and $x' = \sum_j a'_j e'_j$ using bases for F and F'. Then $x \otimes x' = \sum_{i,j} a_i a'_j e_i \otimes e'_j$ in $F \otimes_R F'$. Since x and x' are nonzero, they each have a nonzero coefficient, say a_i and a'_j. Then $a_i a'_j \neq 0$ since R is a domain, so $x \otimes x'$ has a nonzero coordinate in the basis $\{e_i \otimes e'_j\}$ of $F \otimes_R F'$ (Theorem 4.8). Thus $x \otimes x' \neq 0$. □

Theorem 4.21. *Let R be a domain with fraction field K.*

(1) *For any R-module M, $K \otimes_R M \cong K \otimes_R (M/M_{\rm tor})$ as R-modules, where $M_{\rm tor}$ is the torsion submodule of M.*
(2) *If M is a torsion R-module then $K \otimes_R M = 0$ and if M is not a torsion module then $K \otimes_R M \neq 0$.*
(3) *If $N \subset M$ is a submodule such that M/N is a torsion module then $K \otimes_R N \cong K \otimes_R M$ as R-modules by $x \otimes n \mapsto x \otimes n$.*

Proof. (1) The map $K \times M \to K \otimes_R (M/M_{\rm tor})$ given by $(x, m) \mapsto x \otimes \overline{m}$ is R-bilinear, so there is a linear map $f\colon K \otimes_R M \to K \otimes_R (M/M_{\rm tor})$ where $f(x \otimes m) = x \otimes \overline{m}$.

To go the other way, the canonical bilinear map $K \times M \xrightarrow{\otimes} K \otimes_R M$ vanishes at (x, m) if $m \in M_{\rm tor}$: when $rm = 0$ for $r \neq 0$, $x \otimes m = r(x/r) \otimes m = x/r \otimes rm = x/r \otimes 0 = 0$ (Theorem 3.5). Therefore we get an induced bilinear map $K \times (M/M_{\rm tor}) \to K \otimes_R M$ given by $(x, \overline{m}) \mapsto x \otimes m$. (The point is that an elementary tensor $x \otimes m$ in $K \otimes_R M$ only depends on m through its coset mod $M_{\rm tor}$.) The universal mapping property of the tensor product now gives us a linear map $g\colon K \otimes_R (M/M_{\rm tor}) \to K \otimes_R M$ where $g(x \otimes \overline{m}) = x \otimes m$.

The composites $g \circ f$ and $f \circ g$ are both linear and fix elementary tensors, so they fix all tensors and thus f and g are inverse isomorphisms.

(2) It is immediate from (1) that $K \otimes_R M = 0$ if M is a torsion module, since $K \otimes_R M \cong K \otimes_R (M/M_{\rm tor}) = K \otimes_R 0 = 0$. We could also prove this in a direct way, by showing all elementary tensors in $K \otimes_R M$ are 0: for $x \in K$ and $m \in M$, let $rm = 0$ with $r \neq 0$, so $x \otimes m = r(x/r) \otimes m = x/r \otimes rm = x/r \otimes 0 = 0$.

To show $K \otimes_R M \neq 0$ when M is not a torsion module, from the isomorphism $K \otimes_R M \cong K \otimes_R (M/M_{\rm tor})$, we may replace M with $M/M_{\rm tor}$ and are reduced to the case when M is torsion-free. For torsion-free M we will create a nonzero R-module and a bilinear map onto it from $K \times M$. This will require a fair bit of work (as it usually does to prove a tensor product doesn't vanish).

We want to consider formal products xm with $x \in K$ and $m \in M$. To make this precise, we will use equivalence classes of ordered pairs in the same way that a domain is enlarged to its fraction field. On the product set $K \times M$, define an equivalence relation by

$$(a/b, m) \sim (c/d, n) \iff adm = bcn \text{ in } M.$$

Here a, b, c, and d are in R and b and d are not 0. The proof that this relation is well-defined (independent of the choice of numerators and denominators) and transitive requires M be torsion-free (check!). As an example, $(0, m) \sim (0, 0)$ for all $m \in M$.

Define $KM = (K \times M)/\sim$ and write the equivalence class of (x, m) as $x \cdot m$. Give KM the addition and K-scaling formulas

$$\frac{a}{b} \cdot m + \frac{c}{d} \cdot n = \frac{1}{bd} \cdot (adm + bcn), \quad x(y \cdot m) = (xy) \cdot m.$$

It is left to the reader to check these operations on KM are well-defined and make KM into a K-vector space (so in particular an R-module). The zero element of KM is $0 \cdot 0 = 0 \cdot m$. The function $M \to KM$ given by $m \mapsto 1 \cdot m$ is *injective*, since if $1 \cdot m = 1 \cdot m'$ then $(1, m) \sim (1, m')$, so $m = m'$ in M. So $KM \neq 0$ since $M \neq 0$.

The function $K \times M \to KM$ given by $(x, m) \mapsto x \cdot m$ is R-bilinear and onto, so there is a linear map $K \otimes_R M \xrightarrow{f} KM$ such that $f(x \otimes m) = x \cdot m$, which is onto. Since $KM \neq 0$ we have $K \otimes_R M \neq 0$, and in fact $K \otimes_R M \cong KM$ by f (exercise).

(3) Since $N \subset M$, there is an obvious bilinear map $K \times N \to K \otimes_R M$, namely $(x, n) \mapsto x \otimes n$. So we get automatically a linear map $f \colon K \otimes_R N \to K \otimes_R M$ where $f(x \otimes n) = x \otimes n$. (This is not the identity function: on the left $x \otimes n$ is in $K \otimes_R N$ and on the right $x \otimes n$ is in $K \otimes_R M$.) To get an inverse map to f, let $K \times M \to K \otimes_R N$ by $(x, m) \mapsto (1/r)x \otimes rm$, where $r \in R - \{0\}$ is chosen so $rm \in N$. There is such r since M/N is a torsion module. (We can't simplify $(1/r)x \otimes rm$ by moving r through $\otimes$ since rm is in N but usually m is not.) To check this function is well-defined, if also $r'm \in N$ with $r' \in R - \{0\}$, then

$$\frac{1}{r'}x \otimes r'm = \frac{r}{rr'}x \otimes r'm = \frac{1}{rr'}x \otimes rr'm = \frac{1}{rr'}x \otimes r'(rm) = \frac{r'}{rr'}x \otimes rm = \frac{1}{r}x \otimes rm.$$

So our function $K \times M \to K \otimes_R N$ is well-defined, and the reader can check it is bilinear. It leads to a linear map $g \colon K \otimes_R M \to K \otimes_R N$ where $g(x \otimes m) = (1/r)x \otimes rm$ when $rm \in N$, $r \neq 0$. As an exercise, check $f(g(x \otimes m)) = x \otimes m$ and $g(f(x \otimes n)) = x \otimes n$, so $f \circ g$ and $g \circ f$ are both the identity by additivity. □

Corollary 4.22. *Let R be a domain with fraction field K. In $K \otimes_R M$, $x \otimes m = 0$ if and only if $x = 0$ or $m \in M_{\mathrm{tor}}$. In particular, $M_{\mathrm{tor}} = \ker(M \to K \otimes_R M)$ where $m \mapsto 1 \otimes m$.*

Proof. If $x = 0$ then $x \otimes m = 0 \otimes m = 0$. If $m \in M_{\mathrm{tor}}$, with $rm = 0$ for some nonzero $r \in R$, then $x \otimes m = (x/r)r \otimes m = (x/r) \otimes rm = (x/r) \otimes 0 = 0$.

Conversely, suppose $x \otimes m = 0$. We want to show $x = 0$ or $m \in M_{\mathrm{tor}}$. Write $x = a/b$, so $(1/b) \otimes am = 0$. If $a = 0$ then $x = 0$, so we suppose $a \neq 0$ and will show $m \in M_{\mathrm{tor}}$. Multiply by b to get $1 \otimes am = 0$. From the isomorphism $K \otimes_R M \cong K \otimes_R (M/M_{\mathrm{tor}})$, $1 \otimes \overline{am} = 0$ in $K \otimes_R (M/M_{\mathrm{tor}})$. Since M/M_{tor} is torsion-free, applying the R-linear map $K \otimes_R (M/M_{\mathrm{tor}}) \to K(M/M_{\mathrm{tor}})$ from the proof of Theorem 4.21 tells us that $1 \cdot \overline{am} = 0$ in $K(M/M_{\mathrm{tor}})$. The function $\overline{m} \mapsto 1 \cdot \overline{m}$ from M/M_{tor} to $K(M/M_{\mathrm{tor}})$ is injective, so $\overline{am} = 0$, so $am \in M_{\mathrm{tor}}$. Therefore there is nonzero $r \in R$ such that $0 = r(am) = (ra)m$. Since $ra \neq 0$, $m \in M_{\mathrm{tor}}$. □

Example 4.23. The tensor product $\mathbf{Q} \otimes_{\mathbf{Z}} A$ is 0 when A is a torsion abelian group, so we recover Example 3.6.

5. General Properties of Tensor Products

There are canonical isomorphisms $M \oplus N \cong N \oplus M$ and $(M \oplus N) \oplus P \cong M \oplus (N \oplus P)$. We want to show similar isomorphisms for tensor products: $M \otimes_R N \cong N \otimes_R M$ and $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$. Furthermore, there is a distributive property over direct sums: $M \otimes_R (N \oplus P) \cong (M \otimes_R N) \oplus (M \otimes_R P)$. How these modules are isomorphic is much more important than just that they are isomorphic.

Theorem 5.1. *There is a unique R-module isomorphism $M \otimes_R N \cong N \otimes_R M$ where $m \otimes n \mapsto n \otimes m$.*

Proof. We want to create a linear map $M \otimes_R N \to N \otimes_R M$ sending $m \otimes n$ to $n \otimes m$. To do this, we back up and start off with a map out of $M \times N$ to the desired target module $N \otimes_R M$. Define $M \times N \to N \otimes_R M$ by $(m, n) \mapsto n \otimes m$. This is a bilinear map since $n \otimes m$ is bilinear in m and n. Therefore by the universal mapping property of the tensor product, there is a unique linear map $f \colon M \otimes_R N \to N \otimes_R M$ such that $f(m \otimes n) = n \otimes m$ on elementary tensors.

$$\begin{tikzcd} & M \otimes_R N \arrow[dd, "f"] \\ M \times N \arrow[ur, "\otimes"] \arrow[dr, "(m,n)\mapsto n\otimes m"'] & \\ & N \otimes_R M \end{tikzcd}$$

Running through the above arguments with the roles of M and N interchanged, there is a unique linear map $g \colon N \otimes_R M \to M \otimes_R N$ where $g(n \otimes m) = m \otimes n$ on elementary tensors. We will show f and g are inverses of each other.

To show $f(g(t)) = t$ for all $t \in N \otimes_R M$, it suffices to check this when t is an elementary tensor, since both sides are R-linear (or even just additive) in t and $N \otimes_R M$ is spanned by its elementary tensors: $f(g(n \otimes m)) = f(m \otimes n) = n \otimes m$. Therefore $f(g(t)) = t$ for all $t \in N \otimes_R M$. The proof that $g(f(t)) = t$ for all $t \in M \otimes_R N$ is similar. We have shown f and g are inverses of each other, so f is an R-module isomorphism. □

Theorem 5.2. *There is a unique R-module isomorphism $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$ where $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$.*

Proof. By Lemma 3.3, $(M \otimes_R N) \otimes_R P$ is linearly spanned by all $(m \otimes n) \otimes p$ and $M \otimes_R (N \otimes_R P)$ is linearly spanned by all $m \otimes (n \otimes p)$. Therefore linear maps out of these two modules are determined by their values on these[8] elementary tensors. So there is at most one linear map $(M \otimes_R N) \otimes_R P \to M \otimes_R (N \otimes_R P)$ with the effect $(m \otimes n) \otimes p \mapsto m \otimes (n \otimes p)$, and likewise in the other direction.

To create such a linear map $(M \otimes_R N) \otimes_R P \to M \otimes_R (N \otimes_R P)$, consider the function $M \times N \times P \to M \otimes_R (N \otimes_R P)$ given by $(m, n, p) \mapsto m \otimes (n \otimes p)$. Since $m \otimes (n \otimes p)$ is trilinear in m, n, and p, for each p we get a linear map $f_p \colon M \otimes_R N \to M \otimes_R (N \otimes_R P)$ where $f_p(m \otimes n) = m \otimes (n \otimes p)$ on elementary tensors in $M \otimes_R N$.

Now we consider the function $(M \otimes_R N) \times P \to M \otimes_R (N \otimes_R P)$ given by $(t, p) \mapsto f_p(t)$. This is bilinear! First, it is linear in t with p fixed, since each f_p is a linear function. Next we show it is linear in p with t fixed:

$$f_{p+p'}(t) = f_p(t) + f_{p'}(t) \text{ and } f_{rp}(t) = r f_p(t)$$

[8]A general elementary tensor in $(M \otimes_R N) \otimes_R P$ is *not* $(m \otimes n) \otimes p$, but $t \otimes p$ where $t \in M \otimes_R N$ and t might not be elementary itself. Similarly, elementary tensors in $M \otimes_R (N \otimes_R P)$ are more general than $m \otimes (n \otimes p)$.

for any p, p', and r. Both sides of these identities are additive in t, so to check them it suffices to check the case when $t = m \otimes n$:

$$\begin{aligned} f_{p+p'}(m \otimes n) &= (m \otimes n) \otimes (p + p') \\ &= (m \otimes n) \otimes p + (m \otimes n) \otimes p' \\ &= f_p(m \otimes n) + f_{p'}(m \otimes n) \\ &= (f_p + f_{p'})(m \otimes n). \end{aligned}$$

That $f_{rp}(m \otimes n) = rf_p(m \otimes n)$ is left to the reader. Since $f_p(t)$ is bilinear in p and t, the universal mapping property of the tensor product tells us there is a unique linear map $f\colon (M \otimes_R N) \otimes_R P \to M \otimes_R (N \otimes_R P)$ such that $f(t \otimes p) = f_p(t)$. Then $f((m \otimes n) \otimes p) = f_p(m \otimes n) = m \otimes (n \otimes p)$, so we have found a linear map with the desired values on the tensors $(m \otimes n) \otimes p$.

Similarly, there is a linear map $g\colon M \otimes_R (N \otimes_R P) \to (M \otimes_R N) \otimes_R P$ where $g(m \otimes (n \otimes p)) = (m \otimes n) \otimes p$. Easily $f(g(m \otimes (n \otimes p))) = m \otimes (n \otimes p)$ and $g(f((m \otimes n) \otimes p)) = (m \otimes n) \otimes p$. Since these particular tensors linearly span the two modules, these identities extend by linearity (f and g are *linear*) to show f and g are inverse functions. $\square$

Theorem 5.3. *There is a unique R-module isomorphism*

$$M \otimes_R (N \oplus P) \cong (M \otimes_R N) \oplus (M \otimes_R P)$$

where $m \otimes (n, p) \mapsto (m \otimes n, m \otimes p)$.

Proof. Instead of directly writing down an isomorphism, we will put to work the essential uniqueness of solutions to a universal mapping problem by showing $(M \otimes_R N) \oplus (M \otimes_R P)$ has the universal mapping property of the tensor product $M \otimes_R (N \oplus P)$. Therefore by abstract nonsense these modules must be isomorphic. That there is an isomorphism whose effect on elementary tensors in $M \otimes_R (N \oplus P)$ is as indicated in the statement of the theorem will fall out of our work.

For $(M \otimes_R N) \oplus (M \otimes_R P)$ to be a tensor product of M and $N \oplus P$, it needs a bilinear map to it from $M \times (N \oplus P)$. Let $b\colon M \times (N \oplus P) \to (M \otimes_R N) \oplus (M \otimes_R P)$ by $b(m, (n, p)) = (m \otimes n, m \otimes p)$. This function is bilinear. We verify one part:

$$\begin{aligned} b(m, (n, p) + (n', p')) &= b(m, (n + n', p + p')) \\ &= (m \otimes (n + n'), m \otimes (p + p')) \\ &= (m \otimes n + m \otimes n', m \otimes p + m \otimes p') \\ &= (m \otimes n, m \otimes p) + (m \otimes n', m \otimes p') \\ &= b(m, (n, p)) + b(m, (n', p')). \end{aligned}$$

To show $(M \otimes_R N) \oplus (M \otimes_R P)$ equipped with the map b has the universal mapping property of $M \otimes_R (N \oplus P)$ and its canonical bilinear map $\otimes$, let $B\colon M \times (N \oplus P) \to Q$ be any bilinear map. We seek an R-linear map L making the diagram

(5.1)
$$\begin{tikzcd} & (M \otimes_R N) \oplus (M \otimes_R P) \arrow[dd, "L"] \\ M \times (N \oplus P) \arrow[ur, "b"] \arrow[dr, "B"'] & \\ & Q \end{tikzcd}$$

commute. Being linear, L would be determined by its values on the direct summands, and these values would be determined by its values on all pairs $(m \otimes n, 0)$ and $(0, m \otimes p)$ by additivity. These values are forced by commutativity of (5.1) to be

$$L(m \otimes n, 0) = L(b(m,(n,0))) = B(m,(n,0)) \text{ and } L(0, m \otimes p) = L(b(m,(0,p))) = B(m,(0,p)).$$

To construct L, we are inspired by these formulas to contemplate the maps $M \times N \to Q$ and $M \times P \to Q$ given by $(m,n) \mapsto B(m,(n,0))$ and $(m,p) \mapsto B(m,(0,p))$. Both are bilinear, so there are R-linear maps $M \otimes_R N \xrightarrow{L_1} Q$ and $M \otimes_R P \xrightarrow{L_2} Q$ where

$$L_1(m \otimes n) = B(m,(n,0)) \text{ and } L_2(m \otimes p) = B(m,(0,p)).$$

Define L on $(M \otimes_R N) \oplus (M \otimes_R P)$ by $L(t_1, t_2) = L_1(t_1) + L_2(t_2)$. (Notice we are defining L not just on ordered pairs of elementary tensors, but on *all* pairs of tensors. We need L_1 and L_2 to be defined on the whole tensor product modules $M \otimes_R N$ and $M \otimes_R P$.) The map L is linear since L_1 and L_2 are linear, and (5.1) commutes:

$$\begin{aligned}
L(b(m,(n,p))) &= L(b(m,(n,0)+(0,p))) \\
&= L(b(m,(n,0)) + b(m,(0,p))) \\
&= L((m \otimes n, 0) + (0, m \otimes p)) \text{ by the definition of } b \\
&= L(m \otimes n, m \otimes p) \\
&= L_1(m \otimes n) + L_2(m \otimes p) \text{ by the definition of } L \\
&= B(m,(n,0)) + B(m,(0,p)) \\
&= B(m,(n,0)+(0,p)) \\
&= B(m,(n,p)).
\end{aligned}$$

Now that we've shown $(M \otimes_R N) \oplus (M \otimes_R P)$ and the bilinear map b have the universal mapping property of $M \otimes_R (N \oplus P)$ and the canonical bilinear map $\otimes$, there is a unique linear map f making the diagram

$$\begin{tikzcd}
& (M \otimes_R N) \oplus (M \otimes_R P) \arrow[dd, "f"] \\
M \times (N \oplus P) \arrow[ur, "b"] \arrow[dr, "\otimes"'] & \\
& M \otimes_R (N \oplus P)
\end{tikzcd}$$

commute, and f is an isomorphism of R-modules because it transforms one solution of a universal mapping problem into another. Taking $(m,(n,p))$ around the diagram both ways,

$$f(b(m,(n,p))) = f(m \otimes n, m \otimes p) = m \otimes (n,p).$$

Therefore the inverse of f is an isomorphism $M \otimes_R (N \oplus P) \to (M \otimes_R N) \oplus (M \otimes_R P)$ with the effect $m \otimes (n,p) \mapsto (m \otimes n, m \otimes p)$. □

Theorem 5.4. *There is a unique R-module isomorphism*

$$M \otimes_R \bigoplus_{i \in I} N_i \cong \bigoplus_{i \in I} (M \otimes_R N_i)$$

where $m \otimes (n_i)_{i \in I} \mapsto (m \otimes n_i)_{i \in I}$.

Proof. We extrapolate from the case $\#I = 2$ in Theorem 5.3. The map $b\colon M \times (\bigoplus_{i\in I} N_i) \to \bigoplus_{i\in I}(M \otimes_R N_i)$ by $b((m,(n_i)_{i\in I})) = (m \otimes n_i)_{i\in I}$ is bilinear. We will show $\bigoplus_{i\in I}(M \otimes_R N_i)$ and b have the universal mapping property of $M \otimes_R \bigoplus_{i\in I} N_i$ and $\otimes$.

Let $B\colon M \times (\bigoplus_{i\in I} N_i) \to Q$ be bilinear. For each $i \in I$ the function $M \times N_i \to Q$ where $(m, n_i) \mapsto B(m, (\dots, 0, n_i, 0, \dots))$ is bilinear, so there is a linear map $L_i\colon M \otimes_R N_i \to Q$ where $L_i(m \otimes n_i) = B(m, (\dots, 0, n_i, 0, \dots))$. Define $L\colon \bigoplus_{i\in I}(M \otimes_R N_i) \to Q$ by $L((t_i)_{i\in I}) = \sum_{i\in I} L_i(t_i)$. All but finitely many t_i equal 0, so the sum here makes sense, and L is linear. It is left to the reader to check the diagram

$$\begin{tikzcd}
& \bigoplus_{i\in I}(M \otimes_R N_i) \arrow[dd, "L"] \\
M \times \bigoplus_{i\in I} N_i \arrow[ur, "b"] \arrow[dr, "B"'] & \\
& Q
\end{tikzcd}$$

commutes. Any linear map L making this diagram commute has its value on $(\dots, 0, m \otimes n_i, 0, \dots) = b(m, (\dots, 0, n_i, 0, \dots))$ determined by B, so it is unique. Thus $\bigoplus_{i\in I}(M \otimes_R N_i)$ and the bilinear map b to it have the universal mapping property of $M \otimes_R \bigoplus_{i\in I} N_i$ and the canonical map $\otimes$, so there is an R-module isomorphism f making the diagram

$$\begin{tikzcd}
& \bigoplus_{i\in I}(M \otimes_R N_i) \arrow[dd, "f"] \\
M \times \bigoplus_{i\in I} N_i \arrow[ur, "b"] \arrow[dr, "\otimes"'] & \\
& M \otimes_R \bigoplus_{i\in I} N_i
\end{tikzcd}$$

commute. Sending $(m, (n_i)_{i\in I})$ around the diagram both ways shows $f((m \otimes n_i)_{i\in I}) = m \otimes (n_i)_{i\in I}$, so the inverse of f is an isomorphism with the effect $m \otimes (n_i)_{i\in I} \mapsto (m \otimes n_i)_{i\in I}$. $\square$

Remark 5.5. The analogue of Theorem 5.4 for direct products is false. While there is a natural R-linear map

$$M \otimes_R \prod_{i\in I} N_i \to \prod_{i\in I}(M \otimes_R N_i), \tag{5.2}$$

namely $m \otimes (n_i)_{i\in I} \mapsto (m \otimes n_i)_{i\in I}$ on elementary tensors, it is not an isomorphism in general. Taking $R = \mathbf{Z}$, $M = \mathbf{Q}$, and $N_i = \mathbf{Z}/p^i\mathbf{Z}$ $(i \geq 1)$, the right side of (5.2) is 0 since $\mathbf{Q} \otimes_{\mathbf{Z}} (\mathbf{Z}/p^i\mathbf{Z}) = 0$ for all $i \geq 1$ (Example 3.6). The left side of (5.2) is $\mathbf{Q} \otimes_{\mathbf{Z}} \prod_{i\geq 1} \mathbf{Z}/p^i\mathbf{Z}$, which is not 0 by Theorem 4.21 since $\prod_{i\geq 1} \mathbf{Z}/p^i\mathbf{Z}$ is not a torsion abelian group.

In our proof of associativity of the tensor product, we started with a function on a direct product $M \times N \times P$ and collapsed this domain to an iterated tensor product $(M \otimes_R N) \otimes_R P$ using bilinearity twice. It is useful to record a rather general result in that direction, as a technical lemma for future convenience.

Theorem 5.6. *Let $M_1, \dots, M_k, N$ be R-modules, with $k > 2$, and suppose*

$$M_1 \times \cdots \times M_{k-2} \times M_{k-1} \times M_k \xrightarrow{\varphi} N$$

is a function which is bilinear in M_{k-1} and M_k when other coordinates are fixed. There is a unique function

$$M_1 \times \cdots \times M_{k-2} \times (M_{k-1} \otimes_R M_k) \xrightarrow{\Phi} N$$

that is linear in $M_{k-1} \otimes_R M_k$ when the other coordinates are fixed and satisfies

$$\Phi(m_1, \ldots, m_{k-2}, m_{k-1} \otimes m_k) = \varphi(m_1, \ldots, m_{k-2}, m_{k-1}, m_k). \tag{5.3}$$

If φ is multilinear in $M_1, \ldots, M_k$, then Φ is multilinear in $M_1, \ldots, M_{k-2}$, $M_{k-1} \otimes_R M_k$.

Proof. Assuming a function Φ exists satisfying (5.3) and is linear in the last coordinate, its value everywhere is determined by additivity in the last coordinate: write any tensor $t \in M_{k-1} \otimes_R M_k$ in the form $t = \sum_{i=1}^p x_i \otimes y_i$, and then

$$\begin{aligned} \Phi(m_1, \ldots, m_{k-2}, t) &= \Phi\left(m_1, \ldots, m_{k-2}, \sum_{i=1}^p x_i \otimes y_i\right) \\ &= \sum_{i=1}^p \Phi(m_1, \ldots, m_{k-2}, x_i \otimes y_i) \\ &= \sum_{i=1}^p \varphi(m_1, \ldots, m_{k-2}, x_i, y_i). \end{aligned}$$

It remains to show Φ exists with the desired properties.

Fix $m_i \in M_i$ for $i = 1, \ldots, k-2$. Define $\varphi_{m_1,\ldots,m_{k-2}} \colon M_{k-1} \times M_k \to N$ by

$$\varphi_{m_1,\ldots,m_{k-2}}(x, y) = \varphi(m_1, \ldots, m_{k-2}, x, y).$$

By hypothesis $\varphi_{m_1,\ldots,m_{k-2}}$ is bilinear in x and y, so from the universal mapping property of the tensor product there is a linear map $\Phi_{m_1,\ldots,m_{k-2}} \colon M_{k-1} \otimes_R M_k \to N$ such that

$$\Phi_{m_1,\ldots,m_{k-2}}(x \otimes y) = \varphi_{m_1,\ldots,m_{k-2}}(x, y) = \varphi(m_1, \ldots, m_{k-2}, x, y).$$

Define $\Phi \colon M_1 \times \cdots \times M_{k-2} \times (M_{k-1} \otimes_R M_k) \to N$ by

$$\Phi(m_1, \ldots, m_{k-2}, t) = \Phi_{m_1,\ldots,m_{k-2}}(t).$$

Since $\Phi_{m_1,\ldots,m_{k-2}}$ is a linear function on $M_{k-1} \otimes_R M_k$, $\Phi(m_1, \ldots, m_{k-2}, t)$ is linear in t when $m_1, \ldots, m_{k-2}$ are fixed.

If φ is multilinear in $M_1, \ldots, M_k$ we want to show Φ is multilinear in $M_1, \ldots, M_{k-2}$, $M_{k-1} \otimes_R M_k$. We already know Φ is linear in $M_{k-1} \otimes_R M_k$ when the other coordinates are fixed. To show Φ is linear in each of the other coordinates (fixing the rest), we carry out the computation for M_1 (the argument is similar for other M_i's): is

$$\begin{aligned} \Phi(x + x', m_2, \ldots, m_{k-2}, t) &\stackrel{?}{=} \Phi(x, m_2, \ldots, m_{k-2}, t) + \Phi(x', m_2, \ldots, m_{k-2}, t) \\ \Phi(rx, m_2, \ldots, m_{k-2}, t) &\stackrel{?}{=} r\Phi(x, m_2, \ldots, m_{k-2}, t) \end{aligned}$$

when $m_2, \ldots, m_{k-2}, t$ are fixed in $M_2, \ldots, M_{k-2}, M_{k-1} \otimes_R M_k$? In these two equations, both sides are additive in t so it suffices to verify these two equations when t is an elementary tensor, say $t = m_{k-1} \otimes m_k$. Then from (5.3), these two equations are true since we're assuming φ is linear in M_1 (fixing the other coordinates). $\square$

Theorem 5.6 is not specific to functions which are bilinear in the last two coordinates: any two coordinates can be used when the function is bilinear in those two coordinates. For instance, let's revisit the proof of associativity of the tensor product in Theorem 5.3. Define

$$\varphi\colon M \times N \times P \to M \otimes_R (N \otimes_R P)$$

by $\varphi(m,n,p) = m \otimes (n \otimes p)$. This function is trilinear, so Theorem 5.6 says we can replace $M \times N$ with its tensor product: there is a bilinear function

$$\Phi\colon (M \otimes_R N) \times P \to M \otimes_R (N \otimes_R P)$$

such that $\Phi(m \otimes n, p) = m \otimes (n \otimes p)$. Since Φ is bilinear, there is a linear function

$$f\colon (M \otimes_R N) \otimes_R P \to M \otimes_R (N \otimes_R P)$$

such that $f(t \otimes p) = \Phi(t,p)$, so $f((m \otimes n) \otimes p) = \Phi(m \otimes n, p) = m \otimes (n \otimes p)$. The construction of the functions f_p in the proof of Theorem 5.3 are now seen to be special cases of Theorem 5.6.

The remaining module properties we treat with the tensor product in this section involve its interaction with the Hom-module construction, so in particular the dual module construction ($M^\vee = \mathrm{Hom}_R(M,R)$).

Theorem 5.7. *There are R-module isomorphisms*

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P)) \cong \mathrm{Bil}_R(M,N;P) \cong \mathrm{Hom}_R(M \otimes_R N, P).$$

Proof. A member of $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P))$ is a linear map $f\colon M \to \mathrm{Hom}_R(N,P)$, and from f we can create a bilinear map $B_f\colon M \times N \to P$ by $B_f(m,n) = f(m)(n)$. In the other direction, from any bilinear map $B\colon M \times N \to P$ we get a linear map $f_B\colon M \to \mathrm{Hom}_R(N,P)$ by $f_B(m)\colon n \mapsto B(m,n)$. The correspondences $f \rightsquigarrow B_f$ and $B \rightsquigarrow f_B$ are obviously inverses, and the reader can check they are linear as well, so $\mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P)) \cong \mathrm{Bil}_R(M,N;P)$. The R-module isomorphism between $\mathrm{Bil}_R(M,N;P)$ and $\mathrm{Hom}_R(M \otimes_R N, P)$ comes from the universal mapping property of the tensor product. $\square$

Here's a high-level way of interpreting the isomorphism between the two Hom-modules in Theorem 5.7. Write $\mathbf{F}_N(M) = \mathrm{Hom}_R(N,M)$ and $\mathbf{G}_N(M) = M \otimes_R N$, so $\mathbf{F}_N$ and $\mathbf{G}_N$ turn R-modules into new R-modules. Theorem 5.7 says

$$\mathrm{Hom}_R(M, \mathbf{F}_N(P)) \cong \mathrm{Hom}_R(\mathbf{G}_N(M), P).$$

This is analogous to the relation between a matrix and its transpose inside the dot product:

$$v \cdot Aw = A^\top v \cdot w.$$

So $\mathbf{F}_N$ and $\mathbf{G}_N$ are "transposes" of each other. Actually, $\mathbf{F}_N$ and $\mathbf{G}_N$ are really called adjoints of each other ($\mathbf{F}_N$ is left adjoint to $\mathbf{G}_N$, and $\mathbf{G}_N$ is right adjoint to $\mathbf{F}_N$) because pairs of operators L and L' in linear algebra that satisfy the relation $L(v) \cdot w = v \cdot L'(w)$ are called adjoints and the relation between $\mathbf{F}_N$ and $\mathbf{G}_N$ is formally similar.

Corollary 5.8. *For R-modules M and N, there are R-module isomorphisms*

$$\mathrm{Hom}_R(M, N^\vee) \cong \mathrm{Hom}_R(N, M^\vee) \cong \mathrm{Bil}_R(M,N;R) \cong (M \otimes_R N)^\vee.$$

Proof. Let $P = R$ in Theorem 5.7: $\mathrm{Hom}_R(M, N^\vee) \cong \mathrm{Bil}_R(M,N;R) \cong (M \otimes_R N)^\vee$. By Theorem 5.1, $M \otimes_R N \cong N \otimes_R M$, so $(M \otimes_R N)^\vee \cong (N \otimes_R M)^\vee$, and the latter dual module is isomorphic to $\mathrm{Hom}_R(N, M^\vee)$ by Theorem 5.7 with the roles of M and N there reversed and $P = R$. Thus we have obtained isomorphisms between the desired modules.

Explicitly, the isomorphism between $\mathrm{Hom}_R(M, N^\vee)$ and $\mathrm{Hom}_R(N, M^\vee)$ amounts to realizing linear maps in both Hom-modules are the same as bilinear maps $B\colon M \times N \to R$. $\square$

The construction of $M \otimes_R N$ is "symmetric" in M and N in the sense that $M \otimes_R N \cong N \otimes_R M$ in a natural way, but Corollary 5.8 is *not* saying $\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(N, M)$ since those are not the Hom-modules in the corollary. For instance, if $R = M = \mathbf{Z}$ and $N = \mathbf{Z}/2\mathbf{Z}$ then $\mathrm{Hom}_R(M, N) \cong \mathbf{Z}/2\mathbf{Z}$ and $\mathrm{Hom}_R(N, M) = 0$.

Theorem 5.9. *For R-modules M and N, there is a unique linear map*

$$M^\vee \otimes_R N \to \mathrm{Hom}_R(M, N)$$

sending the elementary tensor $\varphi \otimes n$ to $[m \mapsto \varphi(m)n]$. This is an isomorphism if M and N are finite free. In particular, if F is finite free then $F^\vee \otimes_R F \cong \mathrm{End}_R(F)$ as R-modules.

Concretely, the last part is saying $(R^n)^\vee \otimes_R R^n \cong \mathrm{M}_n(R)$ as R-modules, in a natural way.

Proof. We need to make an element of $M^\vee$ and element of N act together as a linear map $M \to N$. The function $M^\vee \times N \times M \to N$ given by $(\varphi, n, m) \mapsto \varphi(m)n$ is trilinear. Here the parameter φ acts on m to give a scalar, which is then multiplied by n. By Theorem 5.6, this trilinear map induces a bilinear map $B\colon (M^\vee \otimes_R N) \times M \to N$ where $B(\varphi \otimes n, m) = \varphi(m)n$. For $t \in M^\vee \otimes_R N$, $B(t, -)$ is in $\mathrm{Hom}_R(M, N)$, so we have a linear map $f\colon M^\vee \otimes_R N \to \mathrm{Hom}_R(M, N)$ by $f(t) = B(t, -)$. (Explicitly, the elementary tensor $\varphi \otimes n$ acts on M by the rule $(\varphi \otimes n)(m) = \varphi(m)n$.)

Now let M and N be *finite free*. To show f is an isomorphism, we may suppose M and N are nonzero. Pick bases $\{e_i\}$ of M and $\{e'_j\}$ of N. Then f lets $e_i^\vee \otimes e'_j$ act on M by sending e_k to $e_i^\vee(e_k)e'_j = \delta_{ik}e'_j$. So $f(e_i^\vee \otimes e'_j) \in \mathrm{Hom}_R(M, N)$ sends e_i to e'_j and sends every other member of the basis of M to 0. Such linear maps are a basis of $\mathrm{Hom}_R(M, N)$, so f is onto.

To show f is one-to-one, suppose $f(\sum_{i,j} c_{ij} e_i^\vee \otimes e'_j) = O$ in $\mathrm{Hom}_R(M, N)$. Applying both sides to any e_k, we get $\sum_{i,j} c_{ij}\delta_{ik}e'_j = 0$, which says $\sum_j c_{kj}e'_j = 0$, so $c_{kj} = 0$ for all j and all k. Thus every c_{ij} is 0. This concludes the proof that f is an isomorphism.

Let's work out the inverse map explicitly. For $L \in \mathrm{Hom}_R(M, N)$, write $L(e_i) = \sum_j a_{ji}e'_j$, so L has matrix representation (a_{ji}). (The matrix indices here look reversed from usual practice because we use i as the index for basis vectors in M and j as the index for basis vectors in N; review how linear maps become matrices when bases are chosen. If we had indexed bases of M and N with i and j in each other's places, then $L(e_j) = \sum a_{ij}e_i^\vee$.) From the isomorphism f, let L correspond to $\sum_{i,j} c_{ij}e_i^\vee \otimes e'_j$ in $M^\vee \otimes_R N$, with the coefficients c_{ij} to be determined. Then

$$L(e_k) = \sum_{i,j} c_{ij}(e_i^\vee \otimes e'_j)(e_k) = \sum_j c_{kj}e'_j,$$

so $a_{jk} = c_{kj}$. Therefore $c_{ij} = a_{ji}$, so $L \in \mathrm{Hom}_R(M, N)$ with matrix representation (a_{ji}) corresponds to $\sum_{i,j} a_{ji}e_i^\vee \otimes e'_j$. That's nice. It just says $e_i^\vee \otimes e_j$ corresponds to the "matrix unit" E_{ji}.[9] $\square$

Example 5.10. For finite-dimensional vector spaces V and W over the field K, $V^\vee \otimes_K W \cong \mathrm{Hom}_K(V, W)$ by $(\varphi \otimes w)(v) = \varphi(v)w$. This is one of the most basic ways tensor products occur in linear algebra. *Understand this isomorphism*! When $V = W = K^2$, with standard

[9]Notice the index switch: $e_i^\vee \otimes e_j$ goes to E_{ji} and not E_{ij}.

basis e_1 and e_2, the matrix $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \mathrm{M}_2(K) = \mathrm{Hom}_K(V, W)$ corresponds to the tensor $e_1^\vee \otimes \binom{a}{c} + e_2^\vee \otimes \binom{b}{d}$ since this tensor sends e_1 to $e_1^\vee(e_1)\binom{a}{c} + e_2^\vee(e_1)\binom{b}{d} = \binom{a}{c}$, and similarly this tensor sends e_2 to $\binom{b}{d}$.

Remark 5.11. The linear map $M^\vee \otimes_R N \to \mathrm{Hom}_R(M, N)$ in Theorem 5.9 is not an isomorphism, or even injective or surjective, in general. For example, let p be prime, $R = \mathbf{Z}/p^2\mathbf{Z}$, and $M = N = \mathbf{Z}/p\mathbf{Z}$ as an R-module. Check as an exercise that $M^\vee \cong M$, $M \otimes_R M \cong M$, $\mathrm{Hom}_R(M, M) \cong M$, and the natural map $M^\vee \otimes_R M \to \mathrm{Hom}_R(M, M)$ is identically 0 (it suffices to show every elementary tensor in $M^\vee \otimes_R M$ acts on M as 0).

Remark 5.12. We could have bypassed the injectivity part of the proof, for the following reason. When M and N are finite free, $M^\vee \otimes_R N$ and $\mathrm{Hom}_R(M, N)$ are both finite free R-modules of the same rank, so they are abstractly isomorphic. Therefore a linear surjection between them has to be an isomorphism (consequence of Cayley-Hamilton theorem), so f is an isomorphism.

When M and N are *finite free* R-modules, the isomorphisms in Corollary 5.8 and Theorem 5.9 lead to a description of $M \otimes_R N$ that makes no mention of universal mapping properties. Identify M with $M^{\vee\vee}$ by double duality, so Theorem 5.9 assumes the form

$$M \otimes_R N \cong \mathrm{Hom}_R(M^\vee, N),$$

where $m \otimes n$ goes over to the linear map $\varphi \mapsto \varphi(m)n$. Since $N \cong N^{\vee\vee}$ by double duality, $\mathrm{Hom}_R(M^\vee, N) \cong \mathrm{Hom}_R(M^\vee, (N^\vee)^\vee) \cong \mathrm{Bil}_R(M^\vee, N^\vee; R)$ by Corollary 5.8. Therefore

$$M \otimes_R N \cong \mathrm{Bil}_R(M^\vee, N^\vee; R), \tag{5.4}$$

where $m \otimes n$ acts as the bilinear map $M^\vee \times N^\vee \to R$ sending (φ, ψ) to $\varphi(m)\psi(n)$. The definition of the tensor product of finite-dimensional vector spaces in [1, p. 65] and [4, p. 35] is essentially (5.4).[10] It is a good exercise to check these interpretations of $m \otimes n$ as a member of $\mathrm{Hom}_R(M^\vee, N)$ and $\mathrm{Bil}_R(M^\vee, N^\vee; R)$ are identified with each other by Corollary 5.8 and double duality.

But watch out! The "formula" (5.4) for the tensor product does not work for general modules M and N (where double duality doesn't hold). While there is always a linear map $M \otimes_R N \to \mathrm{Bil}_R(M^\vee, N^\vee; R)$ given on elementary tensors by $m \otimes n \mapsto [(\varphi, \psi) \mapsto \varphi(m)\psi(n)]$, it is generally not an isomorphism.

Example 5.13. Let p be prime, $R = \mathbf{Z}/p^2\mathbf{Z}$, and $M = \mathbf{Z}/p\mathbf{Z}$ as an R-module. The natural map $M \otimes_R M \to \mathrm{Bil}_R(M^\vee, M^\vee; R)$ is identically 0 while both sides have size p.

6. Base Extension

In algebra, there are many times a module over one ring is replaced by a related module over another ring. For instance, in linear algebra it is useful to enlarge $\mathbf{R}^n$ to $\mathbf{C}^n$, creating in this way a complex vector space by letting the real coordinates be extended to complex coordinates. In ring theory, irreducibility tests in $\mathbf{Z}[X]$ involve viewing a polynomial in $\mathbf{Z}[X]$ as a polynomial in $\mathbf{Q}[X]$ or reducing the coefficients mod p to view it in $(\mathbf{Z}/p\mathbf{Z})[X]$. We will see that all these passages to modules with new coefficients ($\mathbf{R}^n \rightsquigarrow \mathbf{C}^n, \mathbf{Z}[X] \rightsquigarrow \mathbf{Q}[X], \mathbf{Z}[X] \rightsquigarrow (\mathbf{Z}/p\mathbf{Z})[X]$) can be described in a uniform way using tensor products.

[10] Using the last isomorphism in Corollary 5.8 and double duality, $M \otimes_R N \cong \mathrm{Bil}_R(M, N; R)^\vee$ for finite free M and N, where $m \otimes n$ goes to the (evaluation) function $B \mapsto B(m, n)$. This is how tensor products of finite-dimensional vector spaces are defined in [3, p. 40], namely $V \otimes_K W$ is the dual space to $\mathrm{Bil}_K(V, W; K)$.

Let $f\colon R \to S$ be a homomorphism of commutative rings. We use f to consider any S-module N as an R-module by $rn := f(r)n$. In particular, S itself is an R-module by $rs := f(r)s$. Passing from N as an S-module to N as an R-module in this way is called *restriction of scalars*.

Example 6.1. If $R \subset S$, f can be the inclusion map (*e.g.*, $\mathbf{R} \hookrightarrow \mathbf{C}$ or $\mathbf{Q} \hookrightarrow \mathbf{C}$). This is how a $\mathbf{C}$-vector space is thought of as an $\mathbf{R}$-vector space or $\mathbf{Q}$-vector space.

Example 6.2. If $S = R/I$, f can be reduction modulo I (*e.g.*, $R = K[T]$ and $S = K[T]/(m(T))$). This is how a K-vector space V on which a linear operator A acts is turned into a $K[T]$-module by letting T act as A. First V becomes a module over the ring of operators $K[A]$ acting on V. Since $K[A] \cong K[T]/(m(T))$, where $m(T)$ is the minimal polynomial of A, V is a $K[T]/(m(T))$-module. Then V becomes a $K[T]$-module by pulling back the scalars from $K[T]/(m(T))$ to $K[T]$ using reduction mod $m(T)$.

Here is a simple illustration of restriction of scalars.

Theorem 6.3. *Let N and N' be S-modules. Any S-linear map $N \to N'$ is also an R-linear map when we treat N and N' as R-modules.*

Proof. Let $\varphi\colon N \to N'$ be S-linear, so $\varphi(sn) = s\varphi(n)$ for any $s \in S$ and $n \in N$. For $r \in R$,

$$\varphi(rn) = \varphi(f(r)n) = f(r)\varphi(n) = r\varphi(n),$$

so φ is R-linear. $\square$

As a notational convention, since we will be going back and forth between R-modules and S-modules a lot, we will write M (or M', and so on) for R-modules and N (or N', and so on) for S-modules. Since N is also an R-module by restriction of scalars, we can form the usual R-module $M \otimes_R N$, where

$$r(m \otimes n) = (rm) \otimes n = m \otimes rn,$$

with the third expression really being $m \otimes f(r)n$ since $rn := f(r)n$.

We want to reverse the process of restriction of scalars. For any R-module M we want to create an S-module of products sm which matches the old meaning of rm if $s = f(r)$. This new module is called an *extension of scalars* or *base extension*. It will be the R-module $S \otimes_R M$ equipped with a new structure of S-module.

Since S is a ring and not just an R-module, we can make $S \otimes_R M$ into an S-module by

$$s'(s \otimes m) = s's \otimes m. \tag{6.1}$$

But does this S-scaling extend unambiguously to all tensors?

Theorem 6.4. *The additive group $S \otimes_R M$ has a unique S-module structure satisfying* (6.1)*, and this is compatible with the R-module structure in the sense that $rt = f(r)t$ for all $r \in R$ and $t \in S \otimes_R M$.*

Proof. Suppose the additive group $S \otimes_R M$ has an S-module structure satisfying (6.1). We will show the S-scaling on all tensors in $S \otimes_R M$ is determined by this. Any $t \in S \otimes_R M$ is a finite sum of elementary tensors, say

$$t = s_1 \otimes m_1 + \cdots + s_k \otimes m_k.$$

For $s \in S$,

$$\begin{aligned} st &= s(s_1 \otimes m_1 + \cdots + s_k \otimes m_k) \\ &= s(s_1 \otimes m_1) + \cdots + s(s_k \otimes m_k) \\ &= ss_1 \otimes m_1 + \cdots + ss_k \otimes m_k \qquad \text{by (6.1)}, \end{aligned}$$

so st is determined, although this formula for it is not obviously well-defined. (Does a different expression for t as a sum of elementary tensors change st?)

Now we show there really is an S-module structure on $S \otimes_R M$ satisfying (6.1). Describing the S-scaling on $S \otimes_R M$ means creating a function $S \times (S \otimes_R M) \to S \otimes_R M$ satisfying the relevant scaling axioms:

$$(6.2) \qquad 1 \cdot t = t, \quad s(t_1 + t_2) = st_1 + st_2, \quad (s_1 + s_2)t = s_1 t + s_2 t, \quad s_1(s_2 t) = (s_1 s_2)t.$$

For each $s' \in S$ we consider the function $S \times M \to S \otimes_R M$ given by $(s, m) \mapsto (s's) \otimes m$. This is R-bilinear, so by the universal mapping property of tensor products there is an R-linear map $\mu_{s'} \colon S \otimes_R M \to S \otimes_R M$ where $\mu_{s'}(s \otimes m) = (s's) \otimes m$ on elementary tensors. Define a multiplication $S \times (S \otimes_R M) \to S \otimes_R M$ by $s't := \mu_{s'}(t)$. This will be the scaling of S on $S \otimes_R M$. We check the conditions in (6.2):

(1) To show $1t = t$ means showing $\mu_1(t) = t$. On elementary tensors, $\mu_1(s \otimes m) = (1 \cdot s) \otimes m = s \otimes m$, so μ_1 fixes elementary tensors. Therefore μ_1 fixes all tensors by additivity.
(2) $s(t_1 + t_2) = st_1 + st_2$ since μ_s is additive.
(3) To show $(s_1 + s_2)t = s_1 t + s_2 t$ means showing $\mu_{s_1+s_2} = \mu_{s_1} + \mu_{s_2}$ as functions on $S \otimes_R M$. Both sides are additive functions on $S \otimes_R M$, so it suffices to check they agree on tensors $s \otimes m$, where both sides have common value $(s_1 + s_2)s \otimes m$.
(4) To show $s_1(s_2 t) = (s_1 s_2)t$ means $\mu_{s_1} \circ \mu_{s_2} = \mu_{s_1 s_2}$ as functions on $S \otimes_R M$. Both sides are additive functions of t, so it suffices to check they agree on tensors $s \otimes m$, where both sides have common value $(s_1 s_2 s) \otimes m$.

Let's check the S-module structure on $S \otimes_R M$ is compatible with its original R-module structure. For $r \in R$, if we treat r as $f(r) \in S$ then scaling by $f(r)$ on an elementary tensor $s \otimes m$ has the effect $f(r)(s \otimes m) = f(r)s \otimes m$. Since $f(r)s$ is the definition of rs (this is how we make S into an R-module), $f(r)s \otimes m = rs \otimes m = r(s \otimes m)$. Thus $f(r)(s \otimes m) = r(s \otimes m)$, so $f(r)t = rt$ for all t in $S \otimes_R M$. □

By exactly the same kind of argument, $M \otimes_R S$ has a unique S-module structure where $s'(m \otimes s) = m \otimes s's$. So whenever we meet $M \otimes_R S$ or $S \otimes_R M$, we know they are S-modules in a specific way. Moreover, these two S-modules are naturally isomorphic: by Theorem 5.1, there is an isomorphism $\varphi \colon S \otimes_R M \to M \otimes_R S$ of R-modules where $\varphi(s \otimes m) = m \otimes s$. To show φ is in fact an isomorphism of S-modules, all we need to do is check S-linearity since φ is known to be additive and a bijection. To show $\varphi(s't) = s'\varphi(t)$ for all s' and t, additivity of both sides in t means we may focus on the case $t = s \otimes m$:

$$\varphi(s'(s \otimes m)) = \varphi((s's) \otimes m) = m \otimes s's = s'(m \otimes s) = s'\varphi(s \otimes m).$$

This idea of creating an S-module isomorphism by using a known R-module isomorphism that is also S-linear will be used many more times, so watch for it.

Now we must be careful to refer to R-linear and S-linear maps, rather than just linear maps, so it is clear what our scalar ring is each time.

Example 6.5. In Example 4.5 we saw $(R/I)\otimes_R M \cong M/IM$ as R-modules by $\overline{r}\otimes m \mapsto \overline{rm}$. Since M/IM is naturally an R/I-module, and now we know $(R/I)\otimes_R M$ is an R/I-module, the R-module isomorphism turns out to be an R/I-module isomorphism too since it is R/I-linear (check!).

Example 6.6. In Theorem 4.21, we looked at $K \otimes_R M$ when R is a domain with fraction field K. It was treated there as an R-module, but now we see it is also a K-vector space with the K-scaling rule $x(y \otimes m) = xy \otimes m$ on elementary tensors.

Theorem 6.7. *If F is a free R-module with basis $\{e_i\}_{i\in I}$ then $S \otimes_R F$ is a free S-module with basis $\{1 \otimes e_i\}_{i\in I}$.*

Proof. Since S is an R-module, we know from Theorem 4.13 that every element of $S \otimes_R F$ has a *unique* representation in the form $\sum_{i\in I} s_i \otimes e_i$, where all but finitely many s_i equal 0. Since $s_i \otimes e_i = s_i(1 \otimes e_i)$ in the S-module structure on $S \otimes_R F$, every element of $S \otimes_R F$ is a unique S-linear combination $\sum s_i(1 \otimes e_i)$, which says $\{1 \otimes e_i\}$ is an S-basis of $S \otimes_R F$. □

Example 6.8. As an S-module, $S \otimes_R R^n$ has S-basis $\{1 \otimes e_1, \dots, 1 \otimes e_n\}$ where $\{e_1, \dots, e_n\}$ is the standard basis of R^n, so $S^n \cong S \otimes_R R^n$ as S-modules by

$$(s_1, \dots, s_n) \mapsto \sum_{i=1}^{n} s_i(1 \otimes e_i) = \sum_{i=1}^{n} s_i \otimes e_i$$

because this map is S-linear (check!) and sends an S-basis to an S-basis. In particular, $S \otimes_R R \cong S$ as S-modules by $s \otimes r \mapsto sr$.

For instance,

$$\mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}^n \cong \mathbf{C}^n, \quad \mathbf{C} \otimes_{\mathbf{R}} \mathrm{M}_n(\mathbf{R}) \cong \mathrm{M}_n(\mathbf{C}).$$

These are isomorphisms as $\mathbf{C}$-vector spaces. For any ideal I in R, $(R/I) \otimes_R R^n \cong (R/I)^n$ as R/I-modules.

Example 6.9. As an S-module, $S \otimes_R R[X]$ has S-basis $\{1 \otimes X^i\}_{i\geq 0}$, so $S \otimes_R R[X] \cong S[X]$ as S-modules[11] by $\sum_{i\geq 0} s_i \otimes X^i \mapsto \sum_{i\geq 0} s_i X^i$.

As particular examples, $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}[X] \cong \mathbf{C}[X]$ as $\mathbf{C}$-vector spaces, $\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}[X] \cong \mathbf{Q}[X]$ as $\mathbf{Q}$-vector spaces and $(\mathbf{Z}/p\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}[X] \cong (\mathbf{Z}/p\mathbf{Z})[X]$ as $\mathbf{Z}/p\mathbf{Z}$-vector spaces.

Example 6.10. If we treat $\mathbf{C}^n$ as a real vector space, then its base extension to $\mathbf{C}$ is the complex vector space $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}^n$ where $c(z \otimes v) = cz \otimes v$ for c in $\mathbf{C}$. Since $\mathbf{C}^n \cong \mathbf{R}^{2n}$ as real vector spaces, we have a $\mathbf{C}$-vector space isomorphism

$$\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}^n \cong \mathbf{C} \otimes_{\mathbf{R}} \mathbf{R}^{2n} \cong \mathbf{C}^{2n}.$$

That's interesting: restricting scalars on $\mathbf{C}^n$ to make it a real vector space and then extending scalars back up to $\mathbf{C}$ does *not* give us $\mathbf{C}^n$ back, but instead two copies of $\mathbf{C}^n$. The point is that when we restrict scalars, the real vector space $\mathbf{C}^n$ forgets it is a complex vector space. So the base extension to $\mathbf{C}$ doesn't remember that it used to be a complex vector space.

Quite generally, if V is a finite-dimensional complex vector space and we view it as a real vector space, its base extension $\mathbf{C} \otimes_{\mathbf{R}} V$ as a complex vector space is not V but a direct sum of two copies of V. Let's do a dimension check: as real vector spaces, $\dim_{\mathbf{R}}(\mathbf{C} \otimes_{\mathbf{R}} V) = \dim_{\mathbf{R}}(\mathbf{C}) \dim_{\mathbf{R}}(V) = 2(2\dim_{\mathbf{C}} V) = 4\dim_{\mathbf{C}} V$ and $\dim_{\mathbf{R}}(V \oplus V) = 2\dim_{\mathbf{C}}(V \oplus V) =$

[11] We saw $S \otimes_R R[X]$ and $S[X]$ are isomorphic as R-modules in Example 4.14 when $S \supset R$, and it holds now for any $R \xrightarrow{f} S$.

$4 \dim_{\mathbf{C}} V$, so the two dimensions match. This match is of course not a proof that there is a natural isomorphism $\mathbf{C} \otimes_{\mathbf{R}} V \to V \oplus V$ of complex vector spaces. Work out such an isomorphism as an exercise.

To get our bearing on this example, let's compare an S-module N with the S-module $S \otimes_R N$ (where $s'(s \otimes n) = s's \otimes n$). Since N is already an S-module, should $S \otimes_R N \cong N$? If you think so, reread Example 6.10 ($R = \mathbf{R}$, $S = \mathbf{C}$, $N = \mathbf{C}^n$). Scalar multiplication $S \times N \to N$ is R-bilinear, so there is an R-linear map $\varphi \colon S \otimes_R N \to N$ where $\varphi(s \otimes n) = sn$. This map is also S-linear: $\varphi(st) = s\varphi(t)$. To check this, since both sides are additive in t it suffices to check the case of elementary tensors, and

$$\varphi(s(s' \otimes n)) = \varphi((ss') \otimes n) = ss'n = s(s'n) = s\varphi(s' \otimes n).$$

In the other direction, the function $\psi \colon N \to S \otimes_R N$ where $\psi(n) = 1 \otimes n$ is R-linear but is generally not S-linear since $\psi(sn) = 1 \otimes sn$ has no reason to be $s\psi(n) = s \otimes n$ because we're using $\otimes_R$, not $\otimes_S$. We have created natural maps $\varphi \colon S \otimes_R N \to N$ and $\psi \colon N \to S \otimes_R N$; are they inverses? It's unlikely, since φ is S-linear and ψ is generally not. But let's work out the composites and see what happens. In one direction,

$$\varphi(\psi(n)) = \varphi(1 \otimes n) = 1 \cdot n = n.$$

In the other direction,

$$\psi(\varphi(s \otimes n)) = \psi(sn) = 1 \otimes sn \neq s \otimes n$$

in general. So $\varphi \circ \psi$ is the identity but $\psi \circ \varphi$ is usually not the identity. Since $\varphi \circ \psi = \mathrm{id}_N$, ψ is a section to φ, so N is a direct summand $S \otimes_R N$. Explicitly, $S \otimes_R N \cong \ker \varphi \oplus N$ by $s \otimes n \mapsto (s \otimes n - 1 \otimes sn, sn)$ and its inverse map is $(k, n) \mapsto k + 1 \otimes n$. The phenomenon that $S \otimes_R N$ is typically larger than N when N is an S-module can be remembered by the example $\mathbf{C} \otimes_{\mathbf{R}} \mathbf{C}^n \cong \mathbf{C}^{2n}$.

Theorem 6.11. *For R-modules $\{M_i\}_{i \in I}$, there is an S-module isomorphism*

$$S \otimes_R \bigoplus_{i \in I} M_i \cong \bigoplus_{i \in I} (S \otimes_R M_i).$$

Proof. Since S is an R-module, by Theorem 5.4 there is an R-module isomorphism

$$\varphi \colon S \otimes_R \bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} (S \otimes_R M_i)$$

where $\varphi(s \otimes (m_i)_{i \in I}) = (s \otimes m_i)_{i \in I}$. To show φ is an S-module isomorphism, we just have to check φ is S-linear, since we already know φ is additive and a bijection. It is obvious that $\varphi(st) = s\varphi(t)$ when t is an elementary tensor, and since both $\varphi(st)$ and $s\varphi(t)$ are additive in t the case of general tensors follows. $\square$

The analogue of Theorem 6.11 for direct products is false. There is a natural S-linear map $S \otimes_R \prod_{i \in I} M_i \to \prod_{i \in I} (S \otimes_R M_i)$, but it need not be an isomorphism: $\mathbf{Q} \otimes_{\mathbf{Z}} \prod_{i \geq 1} \mathbf{Z}/p^i\mathbf{Z}$ is nonzero (Remark 5.5) but $\prod_{i \geq 1} (\mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Z}/p^i\mathbf{Z})$ is 0.

We now put base extensions to work. Let M be a finitely generated R-module, say with n generators. That is the same as saying there is a linear surjection $R^n \twoheadrightarrow M$. To say M contains a subset of d linearly independent elements is the same as saying there is a linear injection $R^d \hookrightarrow M$. If both $R^n \twoheadrightarrow M$ and $R^d \hookrightarrow M$, it is natural to suspect $d \leq n$, *i.e.*, the size of a spanning set should always be an upper bound on the size of a linearly independent subset. Is it really true? If R is a field, so modules are vector spaces, we can use dimension inequalities on R^d, M, and R^n to see $d \leq n$. But if R is not a field, then what? We will settle the issue in the affirmative when R is a domain, by tensoring M with the fraction field of R to reduce to the case of vector spaces. We first tensored R-modules with the fraction field of R in Theorem 4.21, but not much use was made of the vector space structure of the tensor product with a field. Now we exploit it.

Theorem 6.12. *Let R be a domain with fraction field K. For a finitely generated R-module M, $K \otimes_R M$ is finite-dimensional as a K-vector space and $\dim_K(K \otimes_R M)$ is the maximal number of R-linearly independent elements in M and is a lower bound on the size of a spanning set for M. In particular, the size of any linearly independent subset of M is less than or equal to the size of any spanning set of M.*

Proof. If $x_1, \ldots, x_n$ is any spanning set for M as an R-module then $1 \otimes x_1, \ldots, 1 \otimes x_n$ span $K \otimes_R M$ as a K-vector space, so $\dim_K(K \otimes_R M) \leq n$.

Let $y_1, \ldots, y_d$ be R-linearly independent in M. We will show $\{1 \otimes y_i\}$ is K-linearly independent in $K \otimes_R M$, so $d \leq \dim_K(K \otimes_R M)$. Suppose $\sum_{i=1}^d c_i(1 \otimes y_i) = 0$ with $c_i \in K$. Write $c_i = a_i/b$ using a common denominator b in R. Then $0 = 1/b \otimes \sum_{i=1}^d a_i y_i$ in $K \otimes_R M$. By Corollary 4.22, this implies $\sum_{i=1}^d a_i y_i \in M_{\mathrm{tor}}$, so $\sum_{i=1}^d r a_i y_i = 0$ in M for some nonzero $r \in R$. By linear independence of the y_i's over R, every ra_i is 0, so every a_i is 0 (R is a domain). Thus every $c_i = a_i/b$ is 0.

It remains to prove M has a linearly independent subset of size $\dim_K(K \otimes_R M)$. Let $\{e_1, \ldots, e_d\}$ be a linearly independent subset of M, where d is maximal. (Since $d \leq \dim_K(K \otimes_R M)$, there is a maximal d.) For every $m \in M$, $\{e_1, \ldots, e_d, m\}$ has to be linearly dependent, so there is a nontrivial R-linear relation

$$a_1 e_1 + \cdots + a_d e_d + am = 0.$$

Necessarily $a \neq 0$, as otherwise all the a_i's are 0 by linear independence of the e_i's. In $K \otimes_R M$,

$$\sum_{i=1}^d a_i(1 \otimes e_i) + a(1 \otimes m) = 0$$

and from the K-vector space structure on $K \otimes_R M$ we can solve for $1 \otimes m$ as a K-linear combination of the $1 \otimes e_i$'s. Therefore $\{1 \otimes e_i\}$ spans $K \otimes_R M$ as a K-vector space. This set is also linearly independent over K by the previous paragraph, so it is a basis and therefore $d = \dim_K(K \otimes_R M)$. $\square$

While M has at most $\dim_K(K \otimes_R M)$ linearly independent elements and this upper bound is achieved, any spanning set has at least $\dim_K(K \otimes_R M)$ elements but this lower bound is not necessarily reached. For example, if R is not a field and M is a torsion module (*e.g.*, R/I for I a nonzero proper ideal) then $K \otimes_R M = 0$ and M certainly doesn't have a spanning set of size 0 if $M \neq 0$. It is also not true that finiteness of $\dim_K(K \otimes_R M)$ implies M is finitely generated as an R-module. Take $R = \mathbf{Z}$ and $M = \mathbf{Q}$, so $\mathbf{Q} \otimes_{\mathbf{Z}} M = \mathbf{Q} \otimes_{\mathbf{Z}} \mathbf{Q} \cong \mathbf{Q}$ (Theorem 4.17), which is finite-dimensional over $\mathbf{Q}$ but M is not finitely generated over $\mathbf{Z}$.

The maximal number of linearly independent elements in an R-module M, for R a domain, is called the *rank* of M.[12] This use of the word "rank" is consistent with its usage for finite free modules: if M is free with an R-basis of size n then $K \otimes_R M$ has a K-basis of size n by Theorem 6.7.

[12] When R is not a domain, this concept of rank for R-modules is not quite the right one.

Example 6.13. A nonzero ideal I in a domain R has rank 1. We can see this in two ways. First, any two nonzero elements in I are linearly dependent over R, so the maximal number of R-linearly independent elements in I is 1. Second, $K \otimes_R I \cong K$ as K-vector spaces (in Theorem 4.19 we showed they are isomorphic as R-modules, but that isomorphism is also K-linear; check!), so $\dim_K(K \otimes_R I) = 1$.

Example 6.14. A finitely generated R-module M has rank 0 if and only if it is a torsion module, since $K \otimes_R M = 0$ if and only if M is a torsion module.

Since $K \otimes_R M \cong K \otimes_R (M/M_{\rm tor})$ as K-vector spaces (the isomorphism between them as R-modules in Theorem 4.21 is easily checked to be K-linear – check!), M and $M/M_{\rm tor}$ have the same rank.

We return to general R, no longer a domain, and see how to make the tensor product of an R-module and S-module into an S-module.

Theorem 6.15. *Let M be an R-module and N be an S-module.*

(1) *The additive group $M \otimes_R N$ has a unique structure of S-module such that $s(m \otimes n) = m \otimes sn$ for $s \in S$. This is compatible with the R-module structure in the sense that $rt = f(r)t$ for $r \in R$ and $t \in M \otimes_R N$.*
(2) *The S-module $M \otimes_R N$ is isomorphic to $(S \otimes_R M) \otimes_S N$ by sending $m \otimes n$ to $(1 \otimes m) \otimes n$.*

The point of part 2 is that it shows how the S-module structure on $M \otimes_R N$ can be described as an ordinary S-module tensor product once we know the idea of base extending M to the S-module $S \otimes_R M$.

Part 2 has both R-module and S-module tensor products. This is the first time that we must decorate the tensor product sign explicitly. Up to now it was actually unnecessary, as all the tensor products were over R.

Proof. (1) This is similar to the proof of Theorem 6.4 (which is the special case $N = S$). We just sketch the idea.

Since every tensor is a sum of elementary tensors, declaring how $s \in S$ scales elementary tensors in $M \otimes_R N$ determines its scaling on all tensors. To show the rule $s(m \otimes n) = m \otimes sn$ really corresponds to an S-module structure, for each $s \in S$ we consider the function $M \times N \to M \otimes_R N$ given by $(m, n) \mapsto m \otimes sn$. This is R-bilinear in m and n, so there is an R-linear map $\mu_s \colon M \otimes_R N \to M \otimes_R N$ such that $\mu_s(m \otimes n) = m \otimes sn$ on elementary tensors. Define a multiplication $S \times (M \otimes_R N) \to M \otimes_R N$ by $st := \mu_s(t)$. Checking the four scaling axioms to make $M \otimes_R N$ an S-module is left to the reader as an exercise.

To check $rt = f(r)t$ for $r \in R$ and $t \in M \otimes_R N$, both sides are additive in t so it suffices to check equality when $t = m \otimes n$ is an elementary tensor. In that case $r(m \otimes n) = m \otimes rn = m \otimes f(r)n = f(r)(m \otimes n)$.

(2) Let $M \times N \to (S \otimes_R M) \otimes_S N$ by $(m, n) \mapsto (1 \otimes m) \otimes n$. Let's check this is R-bilinear: Biadditivity is clear. For R-scaling, we want to check

$$(1 \otimes rm) \otimes n \stackrel{?}{=} r((1 \otimes m) \otimes n), \quad (1 \otimes m) \otimes rn \stackrel{?}{=} r((1 \otimes m) \otimes n).$$

Decorating elementary tensors with $\otimes_R$ or $\otimes_S$ to emphasize in what kind of tensor product they live, we have

$$(1 \otimes_R rm) \otimes_S n = (r(1 \otimes_R m)) \otimes_S n = (f(r)(1 \otimes_R m)) \otimes_S n = f(r)((1 \otimes_R m) \otimes_S n)$$

and

$$(1 \otimes_R m) \otimes_S rn = (1 \otimes_R m) \otimes_S f(r)n = f(r)((1 \otimes_R m) \otimes_S n).$$

Now the universal mapping property of tensor products gives an R-linear map $\varphi\colon M \otimes_R N \to (S \otimes_R M) \otimes_S N$ where $\varphi(m \otimes_R n) = (1 \otimes_R m) \otimes_S n$. This is exactly the map we were looking for, but we only know it is R-linear so far. It is also S-linear: $\varphi(st) = s\varphi(t)$. To check this, it suffices by additivity of φ to focus on the case of an elementary tensor:

$$\varphi(s(m \otimes_R n)) = \varphi(m \otimes_R sn) = (1 \otimes_R m) \otimes_S sn = s((1 \otimes_R m) \otimes_S n) = s\varphi(m \otimes_R n).$$

To show φ is an isomorphism, we create an inverse map $(S \otimes_R M) \otimes_S N \to M \otimes_R N$. The function $S \times M \times N \to M \otimes_R N$ given by $(s, m, n) \mapsto m \otimes sn$ is R-trilinear, so by Theorem 5.6 there is an R-bilinear map $B\colon S \otimes_R M \times N \to M \otimes_R N$ where $B(s \otimes m, n) = m \otimes sn$. This function is in fact S-bilinear:

$$B(st, n) = sB(t, n), \quad B(t, sn) = sB(t, n).$$

To check these equations, the additivity of both sides of the equations in t reduces us to case when t is an elementary tensor. Writing $t = s' \otimes m$,

$$B(s(s' \otimes m), n) = B(ss' \otimes m, n) = m \otimes ss'n = m \otimes s(s'n) = s(m \otimes s'n) = sB(s' \otimes m, n)$$

and

$$B(s' \otimes m, sn) = m \otimes s'(sn)m \otimes s(s'n) = s(m \otimes s'n) = sB(s' \otimes m, n).$$

Now the universal mapping property of the tensor product for S-modules tells us there is an S-linear map $\psi\colon (S \otimes_R M) \otimes_S N \to M \otimes_R N$ such that $\psi(t \otimes n) = \psi_n(t)$.

It is left to the reader to check $\varphi \circ \psi$ and $\psi \circ \varphi$ are identity functions, so φ is an S-module isomorphism. □

In addition to $M \otimes_R N$ being an S-module because N is, the tensor product $N \otimes_R M$ in the other order has a unique S-module structure where $s(n \otimes m) = sn \otimes m$, and this is proved in a similar way.

Example 6.16. For an S-module N, let's show $R^k \otimes_R N \cong N^k$ as S-modules. By Theorem 5.4, $R^k \otimes_R N \cong (R \otimes_R N)^k \cong N^k$ as R-modules, an explicit isomorphism $\varphi\colon R^k \otimes_R N \to N^k$ being $\varphi((r_1, \ldots, r_k) \otimes n) = (r_1 n, \ldots, r_k n)$. Let's check φ is S-linear: $\varphi(st) = s\varphi(t)$. Both sides are additive in t, so we only need to check when t is an elementary tensor:

$$\varphi(s((r_1, \ldots, r_k) \otimes n)) = \varphi((r_1, \ldots, r_k) \otimes sn) = (r_1 sn, \ldots, r_k sn) = s\varphi((r_1, \ldots, r_k) \otimes n).$$

To reinforce the S-module isomorphism

$$M \otimes_R N \cong (S \otimes_R M) \otimes_S N \tag{6.3}$$

from Theorem 6.15(2), we will write out the isomorphism in both directions on appropriate tensors:

$$m \otimes n \mapsto (1 \otimes m) \otimes n, \quad (s \otimes m) \otimes n \mapsto m \otimes sn.$$

Corollary 6.17. *If M and M' are isomorphic R-modules then $M \otimes_R N$ and $M' \otimes_R N$ are isomorphic S-modules, as are $N \otimes_R M$ and $N \otimes_R M'$.*

Proof. We will show $M \otimes_R N \cong M' \otimes_R N$ as S-modules. The other one is similar.

Let $\varphi\colon M \to M'$ be an R-module isomorphism. To write down an S-module isomorphism $M \otimes_R N \to M' \otimes_R N$, we will write down an R-module isomorphism which is also S-linear. Let $M \times N \to M' \otimes_R N$ by $(m, n) \mapsto \varphi(m) \otimes n$. This is R-bilinear (check!), so we get an R-linear map $\Phi\colon M \otimes_R N \to M' \otimes_R N$ such that $\Phi(m \otimes n) = \varphi(m) \otimes n$. This is also S-linear: $\Phi(st) = s\Phi(t)$. Since Φ is additive, it suffices to check this when $t = m \otimes n$:

$$\Phi(s(m \otimes n)) = \Phi(m \otimes sn) = \varphi(m) \otimes sn = s(\varphi(m) \otimes n) = s\Phi(m \otimes n).$$

Using the inverse map to φ we get an R-linear map $\Psi\colon M' \otimes_R N \to M \otimes_R N$ which is also S-linear, and a computation on elementary tensors shows Φ and Ψ are inverses of each other. $\square$

Example 6.18. We can use tensor products to prove the well-definedness of ranks of finite free R-modules when R is not the zero ring. Suppose $R^m \cong R^n$ as R-modules. Pick a maximal ideal $\mathfrak{m}$ in R (Zorn's lemma) and $R/\mathfrak{m} \otimes_R R^m \cong R/\mathfrak{m} \otimes_R R^n$ as $R/\mathfrak{m}$-vector spaces by Corollary 6.17. Therefore $(R/\mathfrak{m})^m \cong (R/\mathfrak{m})^n$ as $R/\mathfrak{m}$-vector spaces (Example 6.8), so taking dimensions of both sides over $R/\mathfrak{m}$ tells us $m = n$.

Here's a conundrum. If N and N' are both S-modules, then we can make $N \otimes_R N'$ into an S-module in two ways: $s(n \otimes n') = sn \otimes n'$ and $s(n \otimes n') = n \otimes sn'$. In the first S-module structure, N' only matters as an R-module. In the second S-module structure, N only matters as an R-module. These two S-module structures on $N \otimes_R N'$ are *not* generally the same because the tensor product is $\otimes_R$, not $\otimes_S$, so $sn \otimes n'$ need not equal $n \otimes sn'$. But are the two S-module structures on $N \otimes_R N'$ at least isomorphic to each other? In general, no.

Example 6.19. Let $R = \mathbf{Z}$, $S = \mathbf{Z}[\sqrt{-14}]$, and $\mathfrak{p} = (3, 1 + \sqrt{-14})$ be an ideal in S. We will show the two S-module structures on $S \otimes_{\mathbf{Z}} \mathfrak{p}$, coming from scaling by S on the left and the right in elementary tensors, are not isomorphic to each other.

First we show $\mathfrak{p}$ is nonprincipal. Let $\overline{\mathfrak{p}} = \{\overline{x} : x \in \mathfrak{p}\} = (3, 1 - \sqrt{-14})$. Check $\mathfrak{p}\overline{\mathfrak{p}} = 3S$. If $\mathfrak{p} = (a + b\sqrt{-14})$, then $\overline{\mathfrak{p}} = (a - b\sqrt{-14})$ and $\mathfrak{p}\overline{\mathfrak{p}} = (a + b\sqrt{-14})(a - b\sqrt{-14}) = (a^2 + 14b^2)$. For this to be (3) forces $a^2 + 14b^2 = \pm 3$, but this equation has no integral solution.

As $\mathbf{Z}$-modules, S and $\mathfrak{p}$ are both free of rank 2, with respective $\mathbf{Z}$-bases $\{1, \sqrt{-14}\}$ and $\{3, 1 + \sqrt{-14}\}$. When $S \otimes_{\mathbf{Z}} \mathfrak{p}$ is an S-module by scaling on the left, $\mathfrak{p}$ only matters as a $\mathbf{Z}$-module, so $S \otimes_{\mathbf{Z}} \mathfrak{p} \cong S \otimes_{\mathbf{Z}} \mathbf{Z}^2$ as S-modules by Corollary 6.17. By Example 6.8, $S \otimes_{\mathbf{Z}} \mathbf{Z}^2 \cong S^2$ as S-modules. So $S \otimes_{\mathbf{Z}} \mathfrak{p}$ is a free S-module of rank 2 (with S-basis $1 \otimes 3$ and $1 \otimes (1 + \sqrt{-14})$). Making $S \otimes_{\mathbf{Z}} \mathfrak{p}$ into an S-module by scaling on the right, $S \otimes_{\mathbf{Z}} \mathfrak{p} \cong \mathbf{Z}^2 \otimes_{\mathbf{Z}} \mathfrak{p} \cong \mathfrak{p}^2$ as S-modules.

It remains to show $\mathfrak{p}^2 = \mathfrak{p} \oplus \mathfrak{p}$ is not isomorphic to S^2, *i.e.*, $\mathfrak{p}^2$ is not free of rank 2 as an S-module. The only proof I know for showing $\mathfrak{p} \oplus \mathfrak{p}$ is not free of rank 2 as an S-module uses exterior powers, which lie beyond the scope of this note, so we can't discuss the proof here. However, it's important to address a natural question: since $\mathfrak{p}$ is not principal, isn't it obvious that $\mathfrak{p} \oplus \mathfrak{p}$ can't be free because $\mathfrak{p}$ is not free? No. A direct sum of two nonfree modules *can* be free. For instance, in $\mathbf{Z}[\sqrt{-5}]$ the ideals $I = (3, 1 + \sqrt{-5})$ and $J = (3, 1 - \sqrt{-5})$ are both nonprincipal but it can be shown that $I \oplus J \cong \mathbf{Z}[\sqrt{-5}] \oplus \mathbf{Z}[\sqrt{-5}]$ as $\mathbf{Z}[\sqrt{-5}]$-modules. The reason a direct sum of non-free modules could be free is that there is more room to move around in the direct sum than just within the direct summands, and this extra room might contain a basis. Therefore showing $\mathfrak{p}^2$ is not a free S-module of rank 2 requires some real work.

The upshot is that if you want to make $N \otimes_R N'$ into an S-module where N and N' are both naturally S-modules, you generally have to *specify* whether S scales on the left or the right. It would be a theorem to prove that in some particular example the two S-modules structures on $N \otimes_R N'$ are in fact identical.

The next theorem collects a number of earlier tensor product isomorphisms for R-modules and shows the same maps are S-module isomorphisms when one of the R-modules in the tensor product is an S-module.

Theorem 6.20. *Let M and M' be R-modules and N and N' be S-modules.*

(1) *There is a unique S-module isomorphism*
$$M \otimes_R N \to N \otimes_R M$$
where $m \otimes n \mapsto n \otimes m$. In particular, $S \otimes_R M$ and $M \otimes_R S$ are isomorphic S-modules.

(2) *There are unique S-module isomorphisms*
$$(M \otimes_R N) \otimes_R M' \to N \otimes_R (M \otimes_R M')$$
where $(m \otimes n) \otimes m' \mapsto n \otimes (m \otimes m')$ and
$$(M \otimes_R N) \otimes_R M' \cong M \otimes_R (N \otimes_R M')$$
where $(m \otimes n) \otimes m' \mapsto m \otimes (n \otimes m')$.

(3) *There is a unique S-module isomorphism*
$$(M \otimes_R N) \otimes_S N' \to M \otimes_R (N \otimes_S N')$$
where $(m \otimes n) \otimes n' \mapsto m \otimes (n \otimes n')$.

(4) *There is a unique S-module isomorphism*
$$N \otimes_R (M \oplus M') \to (N \otimes_R M) \oplus (N \otimes_R M')$$
where $n \otimes (m, m') \mapsto (n, m) \otimes (n, m')$.

In the first, second, and fourth parts, we are using R-module tensor products only and then endowing them with S-module structure from one of the factors being an S-module (Theorem 6.15). In the third part we have both $\otimes_R$ and $\otimes_S$.

Proof. There is a canonical R-module isomorphism $M \otimes_R N \to N \otimes_R M$ where $m \otimes n \mapsto n \otimes m$. This map is S-linear using the S-module structure on both sides (check!), so it is an S-module isomorphism. This settles part 1.

Part 2, like part 1, only involves R-module tensor products, so from earlier work we know there is an R-module isomorphism $\varphi\colon (M \otimes_R N) \otimes_R M' \to N \otimes_R (M \otimes_R M')$ where $\varphi((m \otimes n) \otimes m') = n \otimes (m \otimes m')$. Using the S-module structure on $M \otimes_R N$, $(M \otimes_R N) \otimes M'$, and $N \otimes_R (M \otimes_R M')$, φ is S-linear (check!), so it is an S-module isomorphism. To derive $(M \otimes_R N) \otimes_R M' \cong M \otimes_R (N \otimes_R M')$ from $(M \otimes_R N) \otimes_R M' \cong N \otimes_R (M \otimes_R M')$, use a few commutativity isomorphisms.

Part 3 resembles associativity of tensor products. We will in fact derive part 3 from such associativity for $\otimes_S$:
$$\begin{aligned}(M \otimes_R N) \otimes_S N' &\cong ((S \otimes_R M) \otimes_S N) \otimes_S N' \text{ by (6.3)}\\ &\cong (S \otimes_R M) \otimes_S (N \otimes_S N') \text{ by associativity of } \otimes_S\\ &\cong M \otimes_R (N \otimes_S N') \text{ by (6.3).}\end{aligned}$$
These successive S-module isomorphisms have the effect
$$\begin{aligned}(m \otimes n) \otimes n' &\mapsto ((1 \otimes m) \otimes n) \otimes n'\\ &\mapsto (1 \otimes m) \otimes (n \otimes n')\\ &\mapsto m \otimes (n \otimes n'),\end{aligned}$$
which is what we wanted.

For part 4, there is an R-module isomorphism $N \otimes_R (M \oplus M') \to (N \otimes_R M) \oplus (N \otimes_R M')$ by Theorem 5.4. Now it's just a matter of checking this map is S-linear using the S-module structure on both sides coming from N being an S-module, and this is left to the reader. As an alternate proof, we have a chain of S-module isomorphisms

$$\begin{aligned} N \otimes_R (M \oplus M') &\cong N \otimes_S \left(S \otimes_R (M \oplus M')\right) \text{ by part 1 and (6.3)} \\ &\cong N \otimes_S ((S \otimes_R M) \oplus (S \otimes_R M')) \text{ by Theorem 6.11} \\ &\cong (N \otimes_S (S \otimes_R M)) \oplus (N \otimes_S (S \otimes_R M')) \text{ by Theorem 5.4} \\ &\cong (N \otimes_R M) \oplus (N \otimes_R M') \text{ by part 1 and (6.3).} \end{aligned}$$

Of course one needs to trace through these isomorphisms to check the overall result has the effect intended on elementary tensors, and it does (exercise). □

The last part of Theorem 6.20 extends to arbitrary direct sums: the natural R-module isomorphism $N \otimes_R \bigoplus_{i\in I} M_i \cong \bigoplus_{i\in I}(N \otimes_R M_i)$ is also an S-module isomorphism.

As an application of Theorem 6.20, we can show the base extension of an R-module tensor product "is" the S-module tensor product of the base extensions:

Corollary 6.21. *For R-modules M and M', there is a unique S-module isomorphism*

$$S \otimes_R (M \otimes_R M') \to (S \otimes_R M) \otimes_S (S \otimes_R M')$$

where $s \otimes_R (m \otimes_R m') \mapsto s((1 \otimes_R m) \otimes_S (1 \otimes_R m'))$.

Proof. Since $M \otimes_R M'$ is additively spanned by all $m \otimes m'$, $S \otimes_R (M \otimes_R M')$ is additively spanned by all $s \otimes_R (m \otimes_R m')$. Therefore an S-linear (or even additive) map out of $S \otimes_R (M \otimes_R M')$ is determined by its values on the tensors $s \otimes_R (m \otimes_R m')$.

We have the S-module isomorphisms

$$\begin{aligned} S \otimes_R (M \otimes_R M') &\cong M \otimes_R (S \otimes_R M') \text{ by Theorem 6.20(2)} \\ &\cong (S \otimes_R M) \otimes_S (S \otimes_R M') \text{ by (6.3).} \end{aligned}$$

The effect of these isomorphisms on $s \otimes_R (m \otimes_R m')$ is

$$\begin{aligned} s \otimes_R (m \otimes_R m') &\mapsto m \otimes_R (s \otimes_R m') \\ &\mapsto (1 \otimes_R m) \otimes_S (s \otimes_R m') \\ &= (1 \otimes_R m) \otimes_S s(1 \otimes_R m') \\ &= s((1 \otimes_R m) \otimes_S (1 \otimes_R m')), \end{aligned}$$

as desired. The effect of the inverse isomorphism on $(s_1 \otimes_R m) \otimes_S (s_2 \otimes_R m')$ is

$$\begin{aligned} (s_1 \otimes_R m) \otimes_S (s_2 \otimes_R m') &\mapsto m \otimes_R s_1(s_2 \otimes_R m') \\ &= m \otimes_R ((s_1 s_2) \otimes_R m') \\ &\mapsto s_1 s_2 \otimes_R (m \otimes_R m'). \end{aligned}$$

□

Theorem 6.21 could also be proved by showing the S-module $S \otimes_R (M \otimes_R M')$ has the universal mapping property of $(S \otimes_R M) \otimes_S (S \otimes_R M')$ as a tensor product of S-modules. That is left as an exercise.

Corollary 6.22. *For any R-modules $M_1, \ldots, M_k$,*

$$S \otimes_R (M_1 \otimes_R \cdots \otimes_R M_k) \cong (S \otimes_R M_1) \otimes_S \cdots \otimes_S (S \otimes M_k)$$

as S-modules, where $s \otimes_S (m_1 \otimes_R \cdots \otimes_R m_k) \mapsto s((1 \otimes_R m_1) \otimes_S \cdots \otimes_S (1 \otimes_R m_k))$. In particular, $S \otimes_R (M^{\otimes_R k}) \cong (S \otimes_R M)^{\otimes_S k}$ as S-modules.

Proof. Induct on k. $\square$

Example 6.23. For any real vector space V, $\mathbf{C} \otimes_{\mathbf{R}} (V \otimes_{\mathbf{R}} V) \cong (\mathbf{C} \otimes_{\mathbf{R}} V) \otimes_{\mathbf{C}} (\mathbf{C} \otimes_{\mathbf{R}} V)$. The middle tensor product sign on the right is over $\mathbf{C}$, not $\mathbf{R}$. Note that $\mathbf{C} \otimes_{\mathbf{R}} (V \otimes_{\mathbf{R}} V) \not\cong (\mathbf{C} \otimes_{\mathbf{R}} V) \otimes_{\mathbf{R}} (\mathbf{C} \otimes_{\mathbf{R}} V)$ when $V \neq 0$, as the two sides have different dimensions over $\mathbf{R}$ (what are they?).

The base extension $M \rightsquigarrow S \otimes_R M$ turns R-modules into S-modules in a systematic way. So does $M \rightsquigarrow M \otimes_R S$, and this is essentially the same construction. This suggests there should be a universal mapping problem about R-modules and S-modules which is solved by base extension, and there is: it is the universal device for turning R-linear maps from M to S-modules into S-linear maps.

Theorem 6.24. *Let M be an R-module. For every S-module N and R-linear map $\varphi\colon M \to N$, there is a unique S-linear map $\varphi^S\colon S \otimes_R M \to N$ such that the diagram*

$$\begin{tikzcd} M \arrow[rr, "m \mapsto 1 \otimes m"] \arrow[dr, "\varphi"'] & & S \otimes_R M \arrow[dl, dotted, "\varphi^S"] \\ & N & \end{tikzcd}$$

commutes.

This says the single R-linear map $M \to S \otimes_R M$ from M to an S-module explains all other R-linear maps from M to S-modules using composition of it with S-linear maps from $S \otimes_R M$ to S-modules.

Proof. Assume there is such an S-linear map φ^S. We will derive a formula for it on elementary tensors:

$$\varphi^S(s \otimes m) = \varphi^S(s(1 \otimes m)) = s\varphi^S(1 \otimes m) = s\varphi(m).$$

This shows φ^S is unique if it exists.

To prove existence, consider the function $S \times M \to N$ by $(s, m) \mapsto s\varphi(m)$. This is R-bilinear (check!), so there is an R-linear map $\varphi^S\colon S \otimes_R M \to N$ such that $\varphi^S(s \otimes m) = s\varphi(m)$. Using the S-module structure on $S \otimes_R M$, φ^S is S-linear. $\square$

For φ in $\mathrm{Hom}_R(M, N)$, φ^S is in $\mathrm{Hom}_S(S \otimes_R M, N)$. Because $\varphi^S(1 \otimes m) = \varphi(m)$, we can recover φ from φ^S. But even more is true.

Theorem 6.25. *Let M be an R-module and N be an S-module. The function $\varphi \mapsto \varphi^S$ is an S-module isomorphism $\mathrm{Hom}_R(M, N) \to \mathrm{Hom}_S(S \otimes_R M, N)$.*

How is $\mathrm{Hom}_R(M, N)$ an S-module? Values of these functions are in N, an S-module, so S scales any function $M \to N$ to a new function $M \to N$ by just scaling the values.

Proof. For φ and φ' in $\mathrm{Hom}_R(M, N)$, $(\varphi + \varphi')^S = \varphi^S + \varphi'^S$ and $(s\varphi)^S = s\varphi^S$ by checking both sides are equal on all elementary tensors in $S \otimes_R M$. Therefore $\varphi \mapsto \varphi^S$ is S-linear. Its injectivity is discussed above (φ^S determines φ).

For surjectivity, let $h\colon S \otimes_R M \to N$ be S-linear. Set $\varphi\colon M \to N$ by $\varphi(m) = h(1 \otimes m)$. Then φ is R-linear and $\varphi^S(s \otimes m) = s\varphi(m) = sh(1 \otimes m) = h(s(1 \otimes m)) = h(s \otimes m)$, so $h = \varphi^S$ since both are additive and are equal at all elementary tensors. $\square$

The S-module isomorphism

$$\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_S(S \otimes_R M, N) \tag{6.4}$$

should be thought of as analogous to the R-module isomorphism

$$\mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) \cong \mathrm{Hom}_R(M \otimes_R N, P) \tag{6.5}$$

from Theorem 5.7, where $- \otimes_R N$ is left adjoint to $\mathrm{Hom}_R(N, -)$. (In (6.5), N and P are R-modules, not S-modules! We're using the same notation as in Theorem 5.7.) If we look at (6.4), we see $S \otimes_R -$ is applied to M on the right but nothing special is applied to N on the left. Yet there *is* something different about N on the two sides of (6.4). It is an S-module on the right side of (6.4), but on the left side it is being treated as an R-module (restriction of scalars). That changes N, but we have introduced no notation to reflect this. We still just write it as N. Let's now write $\mathrm{Res}_{S/R}(N)$ to denote N as an R-module. It is the same underlying additive group as N, but the scalars are now taken from R with the rule $rn = f(r)n$. The appearance of (6.4) now looks like this:

$$\mathrm{Hom}_R(M, \mathrm{Res}_{S/R}(N)) \cong \mathrm{Hom}_S(S \otimes_R M, N). \tag{6.6}$$

So extension of scalars (from R-modules to S-modules) is left adjoint to restriction of scalars (from S-modules to R-modules) in a similar way that $-\otimes_R M$ is left adjoint to $\mathrm{Hom}_R(M, -)$.

Using this new notation for restriction of scalars, the important S-module isomorphism (6.3) can be written more explicitly as

$$M \otimes_R \mathrm{Res}_{S/R}(N) \cong (S \otimes_R M) \otimes_S N,$$

Theorem 6.26. *Let M be an R-module and N and P be S-modules. There is an S-module isomorphism*

$$\mathrm{Hom}_S(M \otimes_S N, P) \cong \mathrm{Hom}_R(M, \mathrm{Res}_{S/R}(\mathrm{Hom}_S(N, P))).$$

Example 6.27. Taking $N = S$,

$$\mathrm{Hom}_S(S \otimes_R M, P) \cong \mathrm{Hom}_R(M, \mathrm{Res}_{S/R}(P))$$

since $\mathrm{Hom}_S(S, P) \cong P$. We have recovered $S \otimes_R -$ being left adjoint to $\mathrm{Res}_{S/R}$.

Example 6.28. Taking $S = R$, so N and P are now R-modules,

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

We have recovered $- \otimes_R N$ being left adjoint to $\mathrm{Hom}_R(N, -)$ for R-modules N.

These two consequences of Theorem 6.26 are results we have already seen, and in fact we are going to use them in the proof, so they are together equivalent to Theorem 6.26.

Proof. Since $M \otimes_R N \cong (S \otimes_R M) \otimes_S N$ as S-modules,

$$\mathrm{Hom}_S(M \otimes_R N, P) \cong \mathrm{Hom}_S((S \otimes_R M) \otimes_S N, P).$$

Since $- \otimes_S N$ is left adjoint to $\mathrm{Hom}_S(N, -)$,

$$\mathrm{Hom}_S((S \otimes_R M) \otimes_S N, P) \cong \mathrm{Hom}_S(S \otimes_R M, \mathrm{Hom}_S(N, P)).$$

Since $S \otimes_R -$ is left adjoint to $\mathrm{Res}_{S/R}$,

$$\mathrm{Hom}_S(S \otimes_R M, \mathrm{Hom}_S(N, P)) \cong \mathrm{Hom}_R(M, \mathrm{Res}_{S/R}(\mathrm{Hom}_S(N, P))).$$

Combining these three isomorphisms,

$$\mathrm{Hom}_S(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Res}_{S/R}(\mathrm{Hom}_S(N, P))).$$

Here is an explicit (overall) isomorphism. If $\varphi\colon M \otimes_R N \to P$ is S-linear there is an R-linear map $L_\varphi\colon M \to \mathrm{Hom}_S(N, P)$ by $L_\varphi(m) = \varphi(m \otimes (-))$. If $\psi\colon M \to \mathrm{Hom}_S(N, P)$ is R-linear then $M \times N \to P$ by $(m, n) \mapsto \psi(m)(n)$ is R-bilinear and $\psi(m)(sn) = s\psi(m)(n)$, so the corresponding R-linear map $\widetilde{L}_\psi\colon M \otimes_R N \to P$ where $\widetilde{L}_\psi(m \otimes n) = \psi(m)(n)$ is S-linear. The functions $\varphi \mapsto L_\varphi$ and $\psi \mapsto \widetilde{L}_\psi$ are S-linear and are inverses. $\square$

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