4.7 CHANGE OF BASIS

When a basis $\mathcal{B}$ is chosen for an n-dimensional vector space V, the associated coordinate mapping onto $\mathbb{R}^n$ provides a coordinate system for V. Each $\mathbf{x}$ in V is identified uniquely by its $\mathcal{B}$-coordinate vector $[\,\mathbf{x}\,]_{\mathcal{B}}$.[1]

In some applications, a problem is described initially using a basis $\mathcal{B}$, but the problem's solution is aided by changing $\mathcal{B}$ to a new basis $\mathcal{C}$. (Examples will be given in Chapters 5 and 7.) Each vector is assigned a new $\mathcal{C}$-coordinate vector. In this section, we study how $[\,\mathbf{x}\,]_{\mathcal{C}}$ and $[\,\mathbf{x}\,]_{\mathcal{B}}$ are related for each $\mathbf{x}$ in V.

To visualize the problem, consider the two coordinate systems in Fig. 1. In Fig. 1(a), $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, while in Fig. 1(b), the same $\mathbf{x}$ is shown as $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$. That is,

$$[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \text{and} \quad [\,\mathbf{x}\,]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Our problem is to find the connection between the two coordinate vectors. Example 1 shows how to do this, provided we know how $\mathbf{b}_1$ and $\mathbf{b}_2$ are formed from $\mathbf{c}_1$ and $\mathbf{c}_2$.

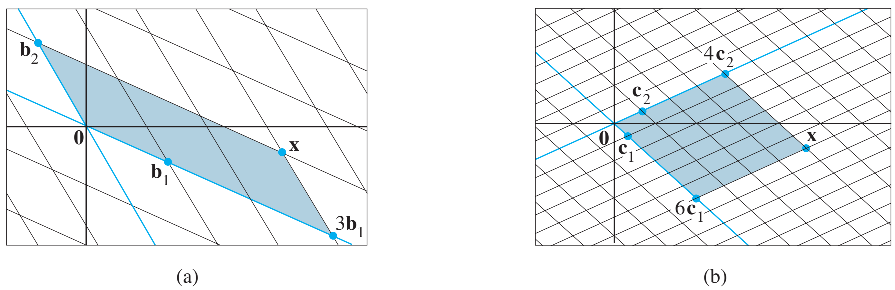

FIGURE 1 Two coordinate systems for the same vector space.

EXAMPLE 1 Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \quad \text{and} \quad \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2 \tag{1}$$

Suppose

$$\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2 \tag{2}$$

That is, suppose $[\,\mathbf{x}\,]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Find $[\,\mathbf{x}\,]_{\mathcal{C}}$.

SOLUTION Apply the coordinate mapping determined by $\mathcal{C}$ to $\mathbf{x}$ in (2). Since the coordinate mapping is a linear transformation,

$$\begin{aligned} [\,\mathbf{x}\,]_{\mathcal{C}} &= [\,3\mathbf{b}_1 + \mathbf{b}_2\,]_{\mathcal{C}} \\ &= 3[\,\mathbf{b}_1\,]_{\mathcal{C}} + [\,\mathbf{b}_2\,]_{\mathcal{C}} \end{aligned}$$

We can write this vector equation as a matrix equation, using the vectors in the linear combination as the columns of a matrix:

$$[\,\mathbf{x}\,]_{\mathcal{C}} = \begin{bmatrix} [\,\mathbf{b}_1\,]_{\mathcal{C}} & [\,\mathbf{b}_2\,]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \tag{3}$$

[1] Think of $[\,\mathbf{x}\,]_{\mathcal{B}}$ as a "name" for $\mathbf{x}$ that lists the weights used to build $\mathbf{x}$ as a linear combination of the basis vectors in $\mathcal{B}$.

This formula gives $[\mathbf{x}]_{\mathcal{C}}$, once we know the columns of the matrix. From (1),

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{and} \quad [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

Thus (3) provides the solution:

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

The $\mathcal{C}$-coordinates of $\mathbf{x}$ match those of the $\mathbf{x}$ in Fig. 1. ■

The argument used to derive formula (3) can be generalized to yield the following result. (See Exercises 15 and 16.)

THEOREM 15

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} \tag{4}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis $\mathcal{B}$. That is,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix} \tag{5}$$

The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Theorem 15 is called the **change-of-coordinates matrix from $\boldsymbol{\mathcal{B}}$ to $\boldsymbol{\mathcal{C}}$**. Multiplication by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ converts $\mathcal{B}$-coordinates into $\mathcal{C}$-coordinates.[2] Figure 2 illustrates the change-of-coordinates equation (4).

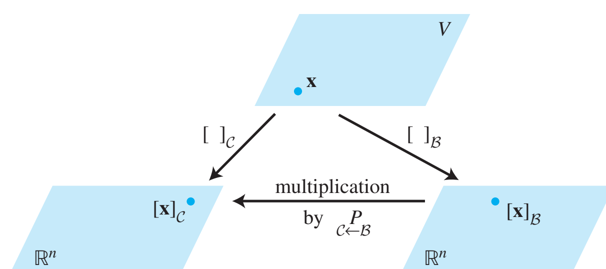

FIGURE 2 Two coordinate systems for V.

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are linearly independent because they are the coordinate vectors of the linearly independent set $\mathcal{B}$. (See Exercise 25 in Section 4.4.) Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is square, it must be invertible, by the Invertible Matrix Theorem. Left-multiplying both sides of equation (4) by $(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1}$ yields

$$(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1}[\mathbf{x}]_{\mathcal{C}} = [\mathbf{x}]_{\mathcal{B}}$$

[2] To remember how to construct the matrix, think of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$ as a linear combination of the columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. The matrix-vector product is a $\mathcal{C}$-coordinate vector, so the columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ should be $\mathcal{C}$-coordinate vectors, too.

Thus $(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1}$ is the matrix that converts $\mathcal{C}$-coordinates into $\mathcal{B}$-coordinates. That is,

$$(\underset{\mathcal{C}\leftarrow\mathcal{B}}{P})^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P} \tag{6}$$

Change of Basis in $\mathbb{R}^n$

If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{E}$ is the *standard basis* $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ in $\mathbb{R}^n$, then $[\mathbf{b}_1]_{\mathcal{E}} = \mathbf{b}_1$, and likewise for the other vectors in $\mathcal{B}$. In this case, $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ is the same as the change-of-coordinates matrix $P_{\mathcal{B}}$ introduced in Section 4.4, namely,

$$P_{\mathcal{B}} = [\,\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n\,]$$

To change coordinates between two nonstandard bases in $\mathbb{R}^n$, we need Theorem 15. The theorem shows that to solve the change-of-basis problem, we need the coordinate vectors of the old basis relative to the new basis.

EXAMPLE 2 Let $\mathbf{b}_1 = \begin{bmatrix} -9 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} 3 \\ -5 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

SOLUTION The matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ involves the $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$ and $\mathbf{b}_2$. Let $[\,\mathbf{b}_1\,]_{\mathcal{C}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $[\,\mathbf{b}_2\,]_{\mathcal{C}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then, by definition,

$$[\,\mathbf{c}_1 \quad \mathbf{c}_2\,]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{b}_1 \quad \text{and} \quad [\,\mathbf{c}_1 \quad \mathbf{c}_2\,]\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \mathbf{b}_2$$

To solve both systems simultaneously, augment the coefficient matrix with $\mathbf{b}_1$ *and* $\mathbf{b}_2$, and row reduce:

$$[\,\mathbf{c}_1 \quad \mathbf{c}_2 \;\vdots\; \mathbf{b}_1 \quad \mathbf{b}_2\,] = \left[\begin{array}{cc:cc} 1 & 3 & -9 & -5 \\ -4 & -5 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{cc:cc} 1 & 0 & 6 & 4 \\ 0 & 1 & -5 & -3 \end{array}\right] \tag{7}$$

Thus

$$[\,\mathbf{b}_1\,]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -5 \end{bmatrix} \quad \text{and} \quad [\,\mathbf{b}_2\,]_{\mathcal{C}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$$

The desired change-of-coordinates matrix is therefore

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = [\,[\,\mathbf{b}_1\,]_{\mathcal{C}} \quad [\,\mathbf{b}_2\,]_{\mathcal{C}}\,] = \begin{bmatrix} 6 & 4 \\ -5 & -3 \end{bmatrix}$$ ■

Observe that the matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ in Example 2 already appeared in (7). This is not surprising because the first column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ results from row reducing $[\,\mathbf{c}_1 \quad \mathbf{c}_2 \;\vdots\; \mathbf{b}_1\,]$ to $[\,I \;\vdots\; [\,\mathbf{b}_1\,]_{\mathcal{C}}\,]$, and similarly for the second column of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$. Thus

$$\boxed{[\,\mathbf{c}_1 \quad \mathbf{c}_2 \;\vdots\; \mathbf{b}_1 \quad \mathbf{b}_2\,] \sim [\,I \;\vdots\; \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}\,]}$$

An analogous procedure works for finding the change-of-coordinates matrix between any two bases in $\mathbb{R}^n$.

EXAMPLE 3 Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} -7 \\ 9 \end{bmatrix}$, $\mathbf{c}_2 = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$, and consider the bases for $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$.

a. Find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

b. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

SOLUTION

a. Notice that $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ is needed rather than $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, and compute

$$\left[\begin{array}{cc:cc} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{c}_1 & \mathbf{c}_2 \end{array}\right] = \left[\begin{array}{rr:rr} 1 & -2 & -7 & -5 \\ -3 & 4 & 9 & 7 \end{array}\right] \sim \left[\begin{array}{rr:rr} 1 & 0 & 5 & 3 \\ 0 & 1 & 6 & 4 \end{array}\right]$$

So

$$\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 5 & 3 \\ 6 & 4 \end{bmatrix}$$

b. By part (a) and property (6) above (with $\mathcal{B}$ and $\mathcal{C}$ interchanged),

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = (\underset{\mathcal{B}\leftarrow\mathcal{C}}{P})^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -3 \\ -6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3/2 \\ -3 & 5/2 \end{bmatrix} \quad \blacksquare$$

Another description of the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ uses the change-of-coordinate matrices $P_\mathcal{B}$ and $P_\mathcal{C}$ that convert $\mathcal{B}$-coordinates and $\mathcal{C}$-coordinates, respectively, into standard coordinates. Recall that for each $\mathbf{x}$ in $\mathbb{R}^n$,

$$P_\mathcal{B}[\mathbf{x}]_\mathcal{B} = \mathbf{x}, \quad P_\mathcal{C}[\mathbf{x}]_\mathcal{C} = \mathbf{x}, \quad \text{and} \quad [\mathbf{x}]_\mathcal{C} = P_\mathcal{C}^{-1}\mathbf{x}$$

Thus

$$[\mathbf{x}]_\mathcal{C} = P_\mathcal{C}^{-1}\mathbf{x} = P_\mathcal{C}^{-1}P_\mathcal{B}[\mathbf{x}]_\mathcal{B}$$

In $\mathbb{R}^n$, the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ may be computed as $P_\mathcal{C}^{-1}P_\mathcal{B}$. Actually, for matrices larger than 2×2, an algorithm analogous to the one in Example 3 is faster than computing $P_\mathcal{C}^{-1}$ and then $P_\mathcal{C}^{-1}P_\mathcal{B}$. See Exercise 12 in Section 2.2.

PRACTICE PROBLEMS

1. Let $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2\}$ and $\mathcal{G} = \{\mathbf{g}_1, \mathbf{g}_2\}$ be bases for a vector space V, and let P be a matrix whose columns are $[\,\mathbf{f}_1\,]_\mathcal{G}$ and $[\,\mathbf{f}_2\,]_\mathcal{G}$. Which of the following equations is satisfied by P for all $\mathbf{v}$ in V?

(i) $[\,\mathbf{v}\,]_\mathcal{F} = P[\,\mathbf{v}\,]_\mathcal{G}$ (ii) $[\,\mathbf{v}\,]_\mathcal{G} = P[\,\mathbf{v}\,]_\mathcal{F}$

2. Let $\mathcal{B}$ and $\mathcal{C}$ be as in Example 1. Use the results of that example to find the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$.

4.7 EXERCISES

1. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$.

a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

b. Find $[\,\mathbf{x}\,]_\mathcal{C}$ for $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$. Use part (a).

2. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ be bases for a vector space V, and suppose $\mathbf{b}_1 = -2\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 3\mathbf{c}_1 - 6\mathbf{c}_2$.

a. Find the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$.

b. Find $[\,\mathbf{x}\,]_\mathcal{C}$ for $\mathbf{x} = 2\mathbf{b}_1 + 3\mathbf{b}_2$.