# ASYMPTOTIC GROWTH

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## 1. Introduction

Let $f(x)$ and $g(x)$ be two functions which are positive when $x$ is large. They are said to be *asymptotically equal*, or just *asymptotic*, if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

We then write $f(x) \sim g(x)$. What this means is that $f(x)$ and $g(x)$ grow in the same way when $x$ gets large.

**Example 1.1.** $x^3 + 10x + 7 \sim x^3$, $3x + \sqrt{x} \sim 3x$, $1/(x^2 - x) \sim 1/x^2$, and $x + \sin x \sim x$.

In these examples, the function on the right side of the $\sim$ relation is simpler than the function on the left side. There is no intrinsic reason this has to be the case, *e.g.*, $x \sim x+\sin x$. But *in practice* the whole point of thinking about asymptotically equal functions is to make a function simpler without affecting its growth. Replacing a function $f(x)$ with another function which is asymptotic to it and in a sense simpler than $f(x)$ amounts to the idea of extracting the "main term" which describes the growth of $f(x)$.

When $f(x) \sim g(x)$ it is *not* usually true that the difference $f(x) - g(x)$ tends to 0. For instance, $x^3 + 10x + 7 \sim x^3$ but $(x^3 + 10x + 7) - x^3 = 10x + 7$ tends to $\infty$ as $x \to \infty$. Being asymptotic does not mean the "absolute error" $|f(x) - g(x)|$ goes to 0, but rather that the "relative error" $|f(x)/g(x) - 1|$ tends to 0. Notice also that we write $1/(x^2 - x) \sim 1/x^2$ even though both sides make no sense at $x = 0$ and the left side is not positive for $0 < x < 1$; all that matters is behavior as $x \to \infty$, where both sides make sense and are positive.

If you gain a decent understanding of both orders of growth (*e.g.*, exponentials dominate all power functions) and this new asymptotic relation, you will be much more confident in your study of the convergence of infinite series and improper integrals.

## 2. Properties of asymptotically equal functions

The following theorem shows that certain operations on asymptotic functions maintain the asymptotic relation.

**Theorem 2.1.** *Suppose $f(x) \sim g(x)$. Then the following additional asymptotic relations hold:*

- $g(x) \sim f(x)$,
- $f(x)^r \sim g(x)^r$ *for any exponent $r$ (possibly negative),*
- $\log f(x) \sim \log g(x)$ *if $g(x) \to \infty$ as $x \to \infty$.*

*Proof.* We are assuming $f(x)/g(x) \to 1$ as $x \to \infty$. Then $g(x)/f(x) = (f(x)/g(x))^{-1} \to 1^{-1} = 1$ and $f(x)^r/g(x)^r = (f(x)/g(x))^r \to 1^r = 1$ (as $x \to \infty$).

The relation between logarithms of asymptotically equal functions is a little trickier to explain. Since $f(x)/g(x) \to 1$, taking logarithms tells us

$$\log f(x) - \log g(x) \to \log 1 = 0. \tag{2.1}$$

Why must $\log f(g) \sim \log g(x)$? Since $g(x) \to \infty$, also $\log g(x) \to \infty$, so $1/\log g(x) \to 0$. Dividing (2.1) by $\log g(x)$ gives us

$$\frac{\log f(x)}{\log g(x)} - 1 \to 0,$$

so $\log f(x)/\log g(x) \to 1$. Thus $\log f(x) \sim \log g(x)$. $\square$

**Remark 2.2.** It is *false* that if $f(x) \sim g(x)$ then $e^{f(x)} \sim e^{g(x)}$. For instance, $x^2 + x \sim x^2$, but $e^{x^2+x} \not\sim e^{x^2}$ since the ratio $e^{x^2+x}/e^{x^2}$ is $e^x$, which does not tend to 1 as $x \to \infty$. Thus, while powers and logarithms of asymptotic functions are asymptotic, exponentials of asymptotic functions are (usually) not asymptotic.

It is also *false* that asymptotic functions have asymptotic derivatives. Here is a basic counterexample: $x^2 + x\sin x \sim x^2$ but if we take derivatives of both sides then $2x + x\cos x + \sin x \not\sim 2x$ (the ratio is $1 + (1/2)\cos x + (\sin x)/2x$, which does not tend to 1).

The following examples apply the ideas in Theorem 2.1.

**Example 2.3.** $(3x^2 - x)^2 \sim 9x^4$ since $3x^2 - x \sim 3x^2$ and we square both sides.

**Example 2.4.** $\sqrt{x^3 + 2x^2} \sim x^{3/2}$ since $x^3 + 2x^2 \sim x^3$ and we take square roots of both sides (raise to the power 1/2).

**Example 2.5.** $\log(x + \sqrt{x}) \sim \log x$ since $x + \sqrt{x} \sim x$ and we take the logarithm of both sides.

**Theorem 2.6.** *If $f(x) \sim g(x)$ and $h(x) \sim k(x)$, then $f(x)h(x) \sim g(x)k(x)$ and $f(x)/h(x) \sim g(x)/k(x)$.*

*Proof.* For the products,

$$\frac{f(x)h(x)}{g(x)k(x)} = \frac{f(x)}{g(x)} \cdot \frac{h(x)}{k(x)} \to 1 \cdot 1 = 1.$$

For the ratios,

$$\frac{f(x)/h(x)}{g(x)/k(x)} = \frac{f(x)/g(x)}{h(x)/k(x)} \to \frac{1}{1} = 1.$$

$\square$

**Example 2.7.** $(3x^6 + 2x)/(5x^2 + 9) \sim (3/5)x^4$ since $3x^6 + 2x \sim 3x^6$ and $5x^2 + 9 \sim 5x^2$. Now divide.

## 3. Integrals of asymptotic functions

We saw in Remark 2.2 that differentiation does not behave nicely on asymptotic functions. The story with integration is different, essentially because you can integrate inequalities while you can't differentiate them. The following result is the main *application* of asymptotic functions to calculus in this handout.

**Theorem 3.1.** *Suppose $f(x)$ and $g(x)$ are continuous positive functions on $[a, \infty)$ and $f(x) \sim g(x)$. Then $\int_a^\infty f(t)\mathrm{d}t$ converges if and only if $\int_a^\infty g(t)\mathrm{d}t$ converges.*

*Proof.* We will show that convergence of $\int_a^\infty g(t)\mathrm{d}t$ implies convergence of $\int_a^\infty f(t)\mathrm{d}t$; the argument in the other direction follows by interchanging the roles of $f(x)$ and $g(x)$, which is allowed since $f(x) \sim g(x)$ if and only if $g(x) \sim f(x)$.

Since $f(x)/g(x) \to 1$, for large $x$ we have $f(x)/g(x) \leq 2$. Then, since $g(x) > 0$ for large $x$, we multiply through by $g(x)$ to get

$$f(x) \leq 2g(x) \tag{3.1}$$

for all large $x$, say for $x \geq c > a$. Since $\int_a^\infty g(t)\mathrm{d}t$ converges, so does $\int_c^\infty g(t)\mathrm{d}t$ (just subtract $\int_a^c g(t)\mathrm{d}t$ from the first improper integral). Therefore $\int_c^\infty 2g(t)\mathrm{d}t$ converges, so by the comparison test $\int_c^\infty f(t)\mathrm{d}t$ converges. Finally, adding $\int_a^c f(t)\mathrm{d}t$ to this, we get convergence of $\int_a^\infty f(t)\mathrm{d}t$. $\square$

Using Theorem 3.1, it becomes a snap to decide convergence of nearly all integrals to $\infty$ of *positive* functions which show up in calculus books, as long as you know a few basic examples. For instance, since $\int_1^\infty \mathrm{d}x/x^p$ converges for $p > 1$ and not for $0 < p \leq 1$, an integral out to $\infty$ of any function which is *asymptotically equal* to a function that decays faster than $1/x^p$ for $p > 1$ converges. If a function is *asymptotically equal* to a function that decays slower than $1/x$, say, its integral out to $\infty$ diverges.

**Example 3.2.** Consider $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{\sqrt{1+x^2}}$. The integrand is $1/\sqrt{1+x^2} \sim 1/x$. Since $\int_1^\infty \mathrm{d}x/x$ diverges, so does the initial integral.

**Example 3.3.** Consider $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{1+x^2}$. The integrand is $1/(1+x^2) \sim 1/x^2$. Since $\int_1^\infty \mathrm{d}x/x^2$ converges, so does the initial integral.

**Example 3.4.** Consider $\displaystyle\int_1^\infty \frac{x\mathrm{d}x}{1+x^2}$. The integrand is $x/(1+x^2) \sim 1/x$. Since $\int_1^\infty \mathrm{d}x/x$ diverges, so does the initial integral.

This does not mean convergence of all improper integrals out to $\infty$ in a calculus book can be checked in your head. Sometimes work is needed. Here are two examples.

**Example 3.5.** Consider $\displaystyle\int_2^\infty \frac{\mathrm{d}x}{x\log x}$. The function $x\log x$ grows faster than $x$, but slower than any $x^r$ for $r > 1$. Therefore $1/x\log x$ decays faster than $1/x$ and slower than $1/x^r$ for any $r > 1$. This means we *can't* decide by comparison with power functions whether or not the integral converges. But we can do a direct calculation in this case:

$$\begin{aligned}
\int_2^\infty \frac{\mathrm{d}x}{x\log x} &= \lim_{b\to\infty}\int_2^b \frac{\mathrm{d}x}{x\log x} \\
&= \lim_{b\to\infty} \log(\log x)\Big|_2^b \\
&= \lim_{b\to\infty} (\log\log b - \log\log 2) \\
&= \infty.
\end{aligned}$$

Thus $\int_1^\infty \mathrm{d}x/x\log x$ diverges.

**Example 3.6.** Consider $\displaystyle\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^2}$. The integrand $1/x(\log x)^2$ decays faster than $1/x$ but slower than $1/x^r$ for any $r > 1$. It decays *slightly* faster than $1/x\log x$, whose integral we just saw diverges. Once again is not clear by any comparison with known growth whether or not this new improper integral converges. But we can directly compute it to find out how things go:

$$\begin{aligned}
\int_2^\infty \frac{\mathrm{d}x}{x(\log x)^2} &= \lim_{b\to\infty}\int_2^b \frac{\mathrm{d}x}{x(\log x)^2} \\
&= \lim_{b\to\infty} -\left.\frac{1}{\log x}\right|_2^b \\
&= \lim_{b\to\infty} -\frac{1}{\log b}+\frac{1}{\log 2} \\
&= \frac{1}{\log 2}.
\end{aligned}$$

Thus $\int_1^\infty \mathrm{d}x/x(\log x)^2$ converges, and in fact equals $1/\log 2$.

**Theorem 3.7.** *Integrals of asymptotic functions are asymptotic if one of the integrals diverges. That is, if $f(x)\sim g(x)$ and $\int_a^\infty g(t)\mathrm{d}t=\infty$ then $\int_a^x f(t)\mathrm{d}t \sim \int_a^x g(t)\mathrm{d}t$.*

*Proof.* The reason that integrals of asymptotic functions are asymptotic is L'Hopital's rule along with the Fundamental Theorem of Calculus. Here are the details. Set $F(x)=\int_a^x f(t)\mathrm{d}t$ and $G(x)=\int_a^x g(t)\mathrm{d}t$. Then $F'(x)=f(x)$ and $G'(x)=g(x)$ by the Fundamental Theorem of Calculus. Our hypothesis that $\int_a^\infty g(t)\mathrm{d}t=\infty$ means $G(x)\to\infty$ as $x\to\infty$. By Theorem 3.1, $\int_a^\infty f(t)\mathrm{d}t=\infty$ too, so $F(x)\to\infty$ as $x\to\infty$. Thus the ratio

$$\frac{\int_a^x f(t)\mathrm{d}t}{\int_a^x g(t)\mathrm{d}t}=\frac{F(x)}{G(x)}$$

has the indeterminate form $\infty/\infty$. The ratio of the derivatives is

$$\frac{F'(x)}{G'(x)}=\frac{f(x)}{g(x)},$$

which tends to 1 as $x\to\infty$ since $f(x)\sim g(x)$. Thus, by L'Hopital's rule we have $F(x)/G(x)\to 1$ as $x\to\infty$, so $F(x)\sim G(x)$. This is exactly the asymptotic equality of the integrals which we wanted to prove. $\square$

**Example 3.8.** Consider $x^2+x\sim x^2$. By a direct calculation, $\int_0^x (t^2+t)\mathrm{d}t = x^3/3+x^2/2$ and $\int_0^x (t^2+t)\mathrm{d}t = x^3/3$. These integrals are asymptotic to each other: $x^3/3+x^2/2\sim x^3/3$.