PLANE ISOMETRIES AND THE COMPLEX NUMBERS

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1. Introduction

In $\mathbf{R}^2$, we can think about the vector $\binom{x}{y}$ as the complex number $x + iy$. Algebraically, vector addition and complex number addition correspond to each other. Geometrically, the length of the vector agrees with the absolute value of the complex number: $||\binom{x}{y}|| = |x + iy| = \sqrt{x^2 + y^2}$. Therefore a distance-preserving function on $\mathbf{R}^2$ can be viewed as a function $h\colon \mathbf{C} \to \mathbf{C}$ preserving the absolute value of the difference: $|h(z_1)-h(z_2)| = |z_1-z_2|$ for all z_1 and z_2 in $\mathbf{C}$. These functions are called *isometries*. Examples of isometries include translations like $h(z) = z + 3 - i$ and rotations like $h(z) = iz$ (a rotation by 90 degrees counterclockwise around 0). We will use the algebra of complex numbers to describe all the isometries of the plane.

The absolute value on $\mathbf{C}$ behaves nicely with respect to multiplication in $\mathbf{C}$ and complex conjugation:

$$|z_1 z_2| = |z_1||z_2| \text{ and } |\overline{z}| = |z|. \tag{1.1}$$

Furthermore, multiplication in $\mathbf{C}$ and complex conjugation are closely related to isometries of the plane:

- *Multiplication* by $\cos\theta + i\sin\theta$, a complex number of absolute value 1, is a counterclockwise *rotation* around the origin by an angle of θ: sending $z \in \mathbf{C}$ to $(\cos\theta + i\sin\theta)z$ rotates z by the angle θ. In particular, for any non-zero $\gamma \in \mathbf{C}$ the pair $\{\gamma, i\gamma\}$ is an orthogonal basis of $\mathbf{C}$ as a real vector space.
- *Complex conjugation* is a *reflection* across the x-axis.

These are the reasons that we will be able to describe isometries of the plane using complex numbers. In Section 2, we algebraically derive formulas for all isometries of the plane. These formulas will be interepreted geometrically in Section 3.

2. Algebraic arguments

Consider the two formulas

$$h(z) = \alpha z + \beta \text{ or } h(z) = \alpha\overline{z} + \beta, \tag{2.1}$$

where $|\alpha| = 1$ and β is arbitrary. These are both isometries. Indeed, subtract two values of each formula:

$$h(z_1) - h(z_2) = \alpha(z_1 - z_2) \text{ or } \alpha\overline{(z_1 - z_2)}.$$

By (1.1), $|h(z_1) - h(z_2)| = |z_1 - z_2|$ in either case.

We will show every isometry of the plane is given by one of the formulas in (2.1).

Lemma 2.1. *An isometry of the plane which fixes* 0, 1, *and* i *is the identity.*

Proof. Let h be such an isometry. Since $h(0) = 0$, $h(1) = 1$, and $h(i) = i$, for any $z \in \mathbf{C}$ we have

$$|h(z)| = |z|, \quad |h(z) - 1| = |z - 1|, \quad |h(z) - i| = |z - i|. \tag{2.2}$$

This says $h(z)$ and z have the same distances from the three numbers 0, 1, and i. Geometrically, it is plausible that a complex number is completely determined by its distances from 0, 1, and i: circles around these three points can't intersect in more than one common point. Granting this, we conclude from (2.2) that $h(z) = z$.

To check this algebraically, we want to show for w and z in $\mathbf{C}$ that

$$|w| = |z|, \ |w - 1| = |z - 1|, \ |w - i| = |z - i| \Longrightarrow w = z. \tag{2.3}$$

Square the three equations in (2.3) and use the formula $|u|^2 = u\overline{u}$ for $u \in \mathbf{C}$:

$$w\overline{w} = z\overline{z}, \ \ (w - 1)(\overline{w} - 1) = (z - 1)(\overline{z} - 1), \ \ (w - i)(\overline{w} + i) = (z - i)(\overline{z} + i).$$

Expanding the second and third equations, and feeding in the first, we obtain after simplifying

$$w + \overline{w} = z + \overline{z}, \quad w - \overline{w} = z - \overline{z}.$$

Therefore w and z have equal real parts and equal imaginary parts, so $w = z$. □

Theorem 2.2. *Any isometry of the plane is given by one of the formulas $h(z) = \alpha z + \beta$ or $h(z) = \alpha\overline{z} + \beta$, where $|\alpha| = 1$.*

Proof. In both formulas, $\beta = h(0)$ and $\alpha = h(1) - h(0)$. This suggests that, given an unknown isometry $h \colon \mathbf{C} \to \mathbf{C}$, we *define*

$$\beta = h(0), \quad \alpha = h(1) - h(0).$$

Then $|\alpha| = |h(1) - h(0)| = |1 - 0| = 1$. Now consider the function

$$k(z) := \frac{h(z) - \beta}{\alpha} = \frac{h(z) - h(0)}{h(1) - h(0)}. \tag{2.4}$$

We *expect* this function is either z or $\overline{z}$.

Note first of all that $k(z)$ is an isometry:

$$|k(z) - k(w)| = \left|\frac{(h(z) - \beta) - (h(w) - \beta)}{\alpha}\right| = |h(z) - h(w)| = |z - w|.$$

With Lemma 2.1 in mind, we compute $k(z)$ at $z = 0, 1,$ and i. Easily $k(0) = 0$ and $k(1) = 1$. What about $k(i)$?

Since k fixes 0 and 1,

$$|k(i)| = |i| = 1, \quad |k(i) - 1| = |i - 1| = \sqrt{2},$$

so $k(i)$ lies on both the unit circle and the circle around 1 of radius $\sqrt{2}$. There are only two such points: i and $-i$. If $k(i) = i$, then k is an isometry fixing 0, 1, and i, so $k(z) = z$ for all z by Lemma 2.1. If $k(i) = -i$, then $\overline{k(z)}$ is an isometry fixing 0, 1, and i, so Lemma 2.1 tells us $\overline{k(z)} = z$ for all z. Conjugating, $k(z) = \overline{z}$ for all z.

Since $h(z) = \alpha k(z) + \beta$, the two possible formulas for $k(z)$ lead to the desired possible formulas for $h(z)$. □

Corollary 2.3. *Any isometry of the plane is invertible.*

Proof. We have explicit formulas for any isometry, so we just check those formulas are invertible. One choice is $h(z) = \alpha z + \beta$ with $|\alpha| = 1$, whose inverse is $(1/\alpha)z - \beta/\alpha$ with $|1/\alpha| = 1$. The other choice is $h(z) = \alpha\overline{z} + \beta$ with $|\alpha| = 1$, whose inverse is $(1/\overline{\alpha})\overline{z} - \overline{\beta/\alpha}$ with $|1/\overline{\alpha}| = 1$. Both inverses are isometries. □

3. Geometric arguments

Now we want to understand the geometry behind the formulas in (2.1). We will see that the different isometries we have found fall into five types: the identity, translations, rotations, reflections, and glide reflections. A *glide reflection* is a composite of a reflection and a translation in a direction parallel to the line of reflection. We will look at the functions in (2.1) and see how they have these geometric effects in terms of conditions on the parameters α and β in each case.

The key idea is to think about fixed points. A fixed point of a function $h\colon \mathbf{C} \to \mathbf{C}$ is a complex number z_0 such that $h(z_0) = z_0$. The identity has the whole plane as fixed points, a reflection has a line of fixed points, a non-trivial rotation has one fixed point (the center of the rotation), and a non-trivial translation and a glide reflection have no fixed points. Notice translations and glide reflections are not distinguished from each other by their fixed points.

Table 1 summarizes the relations we will prove between the geometric description of an isometry and the formula for the isometry. In the table, $|\alpha| = 1$ and $\gamma^2 = \alpha$.

Isometry	Formula	Fixed points
Identity	z	$\mathbf{C}$
Translation	$z + \beta,\ \beta \neq 0$	$\emptyset$
Rotation	$\alpha z + \beta, \alpha \neq 1$	$\beta/(1-\alpha)$
Reflection	$\alpha\overline{z} + \beta,\ \beta^2/\alpha \leq 0$	$\beta/2 + \mathbf{R}\gamma$
Glide reflection	$\alpha\overline{z} + \beta,\ \beta^2/\alpha \not\leq 0$	$\emptyset$

Table 1. Isometries of $\mathbf{R}^2$

Now we turn to the justification of the information in Table 1. Consider the fixed points of $h(z) = \alpha z + \beta$, where $|\alpha| = 1$. If $\alpha = 1$, then $h(z)$ is a translation and has either no fixed points if $\beta \neq 0$ or has every point as a fixed point if $\beta = 0$ (in the second case, h is the identity). If $\alpha \neq 1$, then $h(z)$ has the unique fixed point $z_0 = \beta/(1-\alpha)$. We expect $h(z)$ should be a rotation around z_0. To prove it, express $h(z)$ in terms of its fixed point z_0:

$$h(z) = \alpha z + (1-\alpha)z_0 = \alpha(z - z_0) + z_0. \tag{3.1}$$

Since multiplication by α (of absolute value 1) is a rotation around 0, (3.1) tells us h is a rotation around z_0: the process of subtracting z_0, multiplying by α, and then adding back z_0 amounts to rotating around z_0 by the angle determined by α as a rotation around 0. The first three rows of Table 1 are explained.

Now consider fixed points of $h(z) = \alpha\overline{z} + \beta$. That is, we seek solutions to the equation $z = \alpha\overline{z} + \beta$. There may be some fixed points or there may be no fixed points. For instance, the equation $z = \overline{z}$ has the real line as its fixed points, while the equation $z = \overline{z} + 1$ has no fixed points. It will turn out that $h(z)$ has either a whole line of fixed points (in which case h is the reflection across that line) or it has no fixed points (in which case it is a glide reflection).

The number z_0 is a fixed point of h when $z_0 = \alpha\overline{z}_0 + \beta$. For $z \in \mathbf{C}$, let

$$g_\alpha(z) = z - \alpha\overline{z},$$

so h has a fixed point precisely when β is in the image of g_α. The importance of g_α is that is is $\mathbf{R}$-*linear*. We will show its kernel and image are both non-zero, so each has dimension 1.

The key is to use a square root of α. Write $\alpha = \gamma^2$, so $|\gamma| = 1$. (Recall $|\alpha| = 1$.) Then $\overline{\gamma} = 1/\gamma$, so $\alpha\overline{\gamma} = \gamma$. Then $g_\alpha(\gamma) = \gamma - \alpha\overline{\gamma} = 0$ and $g_\alpha(i\gamma) = 2i\gamma \neq 0$. Thus $\ker(g_\alpha) = \mathbf{R}\gamma$ and the image of g_α is $\mathbf{R}i\gamma$.

Suppose β is in the image of g_α: $\beta = ci\gamma$ for some $c \in \mathbf{R}$. Equivalently, $\beta^2/\alpha = -c^2$ is a non-positive real number. Then $\alpha\overline{\beta} = \alpha(-ci\overline{\gamma}) = -ci\gamma = -\beta$, and $\beta/2$ is a fixed point of $h(z)$:

$$h\left(\frac{\beta}{2}\right) = \alpha \cdot \frac{\overline{\beta}}{2} + \beta = -\frac{\beta}{2} + \beta = \frac{\beta}{2}.$$

In particular, we showed that if h has any fixed point, then $\beta/2$ is a fixed point. (The reader can check $h(\beta/2) = \beta/2$ exactly when $\beta^2/\alpha \leq 0$.) Using $\beta/2$ as our fixed point, the set of all fixed points of $h(z)$ is $\{z : g_\alpha(z) = \beta\}$, which is the line

$$\frac{\beta}{2} + \ker(g_\alpha) = \frac{\beta}{2} + \mathbf{R}\gamma.$$

We will prove h is the reflection across this line.

Write

$$h(z) = \alpha\overline{z} + \beta = \alpha\overline{\left(z - \frac{\beta}{2}\right)} + \frac{\beta}{2},$$

so

$$h(z) - \frac{\beta}{2} = \alpha\overline{\left(z - \frac{\beta}{2}\right)}.$$

To show h is the reflection across the line $\beta/2 + \mathbf{R}\gamma$, it suffices to show the isometry $z \mapsto \alpha\overline{z}$ is the reflection across the line $\mathbf{R}\gamma$. To get a formula for this reflection, use the orthogonal basis $\{\gamma, i\gamma\}$ for $\mathbf{C}$ over $\mathbf{R}$. Writing $z = a\gamma + b(i\gamma)$ with real a and b, reflection across the line $\mathbf{R}\gamma$ sends z to $a\gamma - b(i\gamma)$. Then $z = (a + bi)\gamma$ and

$$a\gamma - b(i\gamma) = (a - bi)\gamma = \frac{\overline{z}}{\overline{\gamma}}\gamma = \overline{z}\gamma^2 = \alpha\overline{z}.$$

So h is the desired kind of reflection.

The remaining type of isometry is $h(z) = \alpha\overline{z} + \beta$ with no fixed points. That is, β is not in $\mathbf{R}i\gamma$ (or β^2/α is not a non-positive real number). This is not a rotation or reflection, since those have fixed points. It is also not a translation (translations are $z \mapsto z + c$). Before we show h is a glide reflection, we look at two examples.

Example 3.1. Let $h(z) = \overline{z} + 1$. This is a reflection across the x-axis composed with a translation in the *direction* of the line of reflection.

Example 3.2. Let $h(z) = \overline{z} + 1 + i = (\overline{z} + i) + 1$. Since $i = w - \overline{w}$ for $w = i/2$, $\overline{z} + i$ is a reflection across the horizontal line $i/2 + \mathbf{R}$. Thus, again h is a composite of a reflection and a translation (by 1) in the direction of the line of reflection.

To see that $h(z) = \alpha\overline{z} + \beta$ describes a glide reflection when it has no fixed points, write $\alpha = \gamma^2$. Using $\{\gamma, i\gamma\}$ as an $\mathbf{R}$-basis of $\mathbf{C}$, write $\beta = a\gamma + bi\gamma$ with real a and b. Then

$$h(z) = (\alpha\overline{z} + bi\gamma) + a\gamma. \tag{3.2}$$

Since $bi\gamma \in \mathbf{R}i\gamma$ is in the image of g_α, our earlier discussion of reflections tells us that $\alpha\overline{z} + bi\gamma$ is a reflection across a line parallel to $\mathbf{R}\gamma$. Since $\beta \notin \mathbf{R}i\gamma$, $a \neq 0$. Therefore adding $a\gamma$ in (3.2) amounts to composition of the reflection with a non-zero translation in a direction parallel to the line of reflection. Thus, h is a glide reflection.

This completes the justification of Table 1.

We now draw two conclusions about plane isometries from Table 1.

Theorem 3.3. *The conjugate of any type of plane isometry is another isometry of the same type.*

Proof. For a translation $t(z) = z + c$ and an isometry $g(z) = \alpha z + \beta$, an explicit calculation shows $(gtg^{-1})(z) = z + \alpha c$. If we have $\overline{z}$ in place of z in $g(z)$, then $(gtg^{-1})(z) = z + \alpha\overline{c}$. Thus, a conjugate of a translation is a translation.

For plane isometries g and h, h fixes z if and only if ghg^{-1} fixes $g(z)$. Any isometry takes a point or a line to a point or a line, respectively, so rotations conjugate to rotations and reflections conjugate to reflections. Since a glide reflection is a non-translation without fixed points, and we already know conjugation brings translations to translations and fixed points to fixed points, the conjugate of a glide reflection is a glide reflection. □

Theorem 3.4. *Any isometry of $\mathbf{R}^2$ is a composite of at most 2 reflections, except for glide reflections, which are composites of 3 (and no fewer) reflections.*

Proof. We check the result for translations, rotations, and glide reflections, Any non-zero translation $t_c(z) = z + c$ is a composite of 2 reflections: let $s(z) = -(c/\overline{c})\overline{z}$ and $\widetilde{s}(z) = -(c/\overline{c})\overline{z} + c$. Both are reflections (*e.g.*, for $\widetilde{s}(z)$, we have $\beta^2/\alpha = -c\overline{c} = -|c|^2 \leq 0$) and $(\widetilde{s} \circ s)(z) = z + c$. The zero translation is the identity, which is the square of any reflection.

A rotation around 0 can be written as $h(z) = \alpha z$. Take $k(z) = \overline{z}$ and $\ell(z) = \overline{\alpha} \cdot \overline{z}$. Then k and ℓ are reflections and $h = k \circ \ell$. An arbitrary rotation has the form $h(z) = \alpha(z - z_0) + z_0$, where z_0 is the center of rotation and $|\alpha| = 1$. Then $h = t \circ r \circ t^{-1}$, where $t = t_{z_0}$ is translation by z_0 and $r(z) = \alpha z$ is a rotation around 0. Write $r = k \circ \ell$ for reflections k and ℓ. Then $h = (tkt^{-1}) \circ (t\ell t^{-1})$, which is a product of two reflections (Theorem 3.3).

Lastly, a glide reflection is a composite of a reflection and a translation, so it is a composite of 3 reflections. The number of reflections in the composite can't be reduced. Indeed, a reflection is not a glide reflection since their fixed points are different. A product of two reflections has the formula

$$\alpha_1\overline{\alpha_2\overline{z} + \beta_2} + \beta_1 = \alpha_1\overline{\alpha}_2 z + \alpha_1\overline{\beta}_2 + \beta_1,$$

so it is a translation or rotation (possibly the identity), not a glide reflection. □