2.1 Matrix Operations

Matrix Notation:

Two ways to denote $m \times n$ matrix A:

In terms of the *columns* of A:

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

In terms of the *entries* of A:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & & & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Main diagonal entries:________________________

Zero matrix:

$$0 = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

THEOREM 1

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + sA$

f. $r(sA) = (rs)A$

Matrix Multiplication

Multiplying B and $\mathbf{x}$ transforms $\mathbf{x}$ into the vector $B\mathbf{x}$. In turn, if we multiply A and $B\mathbf{x}$, we transform $B\mathbf{x}$ into $A(B\mathbf{x})$. So $A(B\mathbf{x})$ is the composition of two mappings.

Define the product AB so that $A(B\mathbf{x}) = (AB)\mathbf{x}$.

Suppose A is $m \times n$ and B is $n \times p$ where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_p \end{bmatrix} \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

and

$$A(B\mathbf{x}) = A(x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p)$$

$$= A(x_1\mathbf{b}_1) + A(x_2\mathbf{b}_2) + \cdots + A(x_p\mathbf{b}_p)$$

$$= x_1A\mathbf{b}_1 + x_2A\mathbf{b}_2 + \cdots + x_pA\mathbf{b}_p = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Therefore,

$$A(B\mathbf{x}) = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}\mathbf{x}.$$

and by defining

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

we have $A(B\mathbf{x}) = (AB)\mathbf{x}$.

EXAMPLE: Compute AB where $A = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ 6 & -7 \end{bmatrix}$.

Solution:

$$A\mathbf{b}_1 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -4 \\ -24 \\ 6 \end{bmatrix}, \qquad A\mathbf{b}_2 = \begin{bmatrix} 4 & -2 \\ 3 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 26 \\ -7 \end{bmatrix}$$

$$\Rightarrow AB = \begin{bmatrix} -4 & 2 \\ -24 & 26 \\ 6 & -7 \end{bmatrix}$$

Note that $A\mathbf{b}_1$ is a linear combination of the columns of A and $A\mathbf{b}_2$ is a linear combination of the columns of A.

Each column of AB is a linear combination of the columns of A using weights from the corresponding columns of B.

EXAMPLE: If A is 4×3 and B is 3×2, then what are the sizes of AB and BA?

Solution:

$$AB = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix} \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$$

BA would be $\begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix} \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$

which is ____________________.

$\boxed{\text{If } A \text{ is } m \times n \text{ and } B \text{ is } n \times p, \text{ then } AB \text{ is } m \times p.}$

Row-Column Rule for Computing AB (alternate method)

The definition

$$\boxed{AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}}$$

is good for theoretical work.

When A and B have small sizes, the following method is more efficient when working by hand.

If AB is defined, let $(AB)_{ij}$ denote the entry in the ith row and jth column of AB. Then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

$$\begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix} \begin{bmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \\ & b_{nj} & \end{bmatrix}$$

$$= \begin{bmatrix} & & \\ & (AB)_{ij} & \\ & & \end{bmatrix}$$

EXAMPLE $A = \begin{bmatrix} 2 & 3 & 6 \\ -1 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 0 & 1 \\ 4 & -7 \end{bmatrix}$. Compute AB, if it is defined.

Solution: Since A is 2×3 and B is 3×2, then AB is defined and AB is ______$\times$______.

$$AB = \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} \mathbf{28} & \blacksquare \\ \blacksquare & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{6} \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -\mathbf{3} \\ 0 & \mathbf{1} \\ 4 & -\mathbf{7} \end{bmatrix} = \begin{bmatrix} 28 & -\mathbf{45} \\ \blacksquare & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{2} & -3 \\ \mathbf{0} & 1 \\ \mathbf{4} & -7 \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ \mathbf{2} & \blacksquare \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 6 \\ -\mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} 2 & -\mathbf{3} \\ 0 & \mathbf{1} \\ 4 & -\mathbf{7} \end{bmatrix} = \begin{bmatrix} 28 & -45 \\ 2 & -\mathbf{4} \end{bmatrix}$$

So $AB = \begin{bmatrix} 28 & -45 \\ 2 & -4 \end{bmatrix}$.

THEOREM 2

Let A be $m \times n$ and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)

b. $A(B + C) = AB + AC$ (left - distributive law)

c. $(B + C)A = BA + CA$ (right-distributive law)

d. $r(AB) = (rA)B = A(rB)$

for any scalar r

e. $I_m A = A = AI_n$ (identity for matrix multiplication)

WARNINGS

Properties above are analogous to properties of real numbers. But **NOT ALL** real number properties correspond to matrix properties.

1. It is not the case that AB always equal BA. (see Example 7, page 114)
2. Even if $AB = AC$, then B may not equal C. (see Exercise 10, page 116)
3. It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$. (see Exercise 12, page 116)

Powers of A

$$A^k = \underbrace{A \cdots A}_{k}$$

EXAMPLE:

$$\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} & \\ & \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 21 & 8 \end{bmatrix}$$

If A is $m \times n$, the **transpose** of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 8 \\ 7 & 6 & 5 & 4 & 3 \end{bmatrix} \quad \Rightarrow \quad A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 7 & 6 \\ 3 & 8 & 5 \\ 4 & 9 & 4 \\ 5 & 8 & 3 \end{bmatrix}$$

EXAMPLE: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}$. Compute AB, $(AB)^T$, A^TB^T and B^TA^T.

Solution:

$$AB = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

$$A^TB^T = \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 3 & 10 \\ 2 & 0 & -4 \\ 2 & 1 & 4 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

THEOREM 3

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$ (I.e., the transpose of A^T is A)

b. $(A+B)^T = A^T + B^T$

c. For any scalar r, $(rA)^T = rA^T$

d. $(AB)^T = B^T A^T$ (I.e. the transpose of a product of matrices equals the product of their transposes in reverse order.)

EXAMPLE: Prove that $(ABC)^T =$ _________.

Solution: By Theorem 3d,

$$(ABC)^T = ((AB)C)^T = C^T\Big(\qquad\qquad\Big)^T$$

$$= C^T\Big(\qquad\qquad\Big) = \underline{\hspace{6em}}.$$