Various Properties of Integer Partitions

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The concept of integer partitions is one that, while relatively simple, has fascinated mathematicians for centuries. Even the great Leonhard Euler, one of the most influential mathematicians in history, set his mind to work deciphering the mysteries and relations surrounding this broad concept. Certainly, many fascinating results have been discovered, and the beauty of this topic is clear straight from the simplest of theorems. But, before we can hope to appreciate the aesthetics of the topic, we must begin from square one. What exactly *is* an integer partition?

Definition 1

A *partition* of an integer n is a non-increasing sequence of positive integers whose sum is n.$\square$

This is a very nice definition, but what exactly does it mean? Well, to illustrate, here are a few examples of integer partitions:

$$1 = 1$$
$$2 = 2 = 1 + 1$$
$$3 = 3 = 2 + 1 = 1 + 1 + 1$$
$$4 = 4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$$

This, we hope, clearly shows what an integer partition is. The first number all the way to the left is the integer to be partitioned, and each subsequent entry is a partition of that integer. For example, if we let $n = 4$ (i.e. let 4 be the integer to be partitioned), then 4 can be partitioned into one part as 4, or into 2 parts as $3 + 1$ or $2 + 2$, etc.

A natural question to ask is this: how many partitions are there of a given integer? We denote the total number of partitions of an integer n by $p(n)$. For instance, $p(1) = 1$ and $p(4) = 5$, as we saw in the above example. While an explicit formula for $p(n)$ is a bit beyond the scope of this paper, we can easily understand $p(n)$ for small integers simply by writing all of the partitions out.

We also introduce the following notation, for convenience. If we wish to introduce any constraints on $p(n)$, then we will do so by adding a vertical bar after n and specifying any constraints before we close the parenthesis. For instance, if we wish to look at the number of partitions of n into odd parts, we will denote it as $p(n \mid \text{odd parts})$.

Now that we have a definition of integer partitions, it becomes natural to ask what kinds of relations might we find among different sets of partitions.

The following simple theorem was discovered by Euler in 1748. While the nature of the theorem is not particularly complex, it illustrates a method of discovering and writing proofs that is integral to the study of integer partitions. We now present the theorem.

Theorem 1 [Euler's Identity]
Given any integer $n \geq 1$, $p(n \mid \text{odd parts}) = p(n \mid \text{distinct parts})$.

Proof:
Essentially, we want to show that there is a bijection between the set of partitions of n into odd parts and the set of partitions of n into distinct parts. We can define the bijection in the following manner:

If we are given a partition in odd parts, there are two possibilities. Either all the odd numbers in the partition are different, in which case we have a distinct partition, or some of the numbers are the same. In the first case, there is nothing to do. In the second case we can create a partition in distinct parts simply by combining the like terms in pairs until we reach a partition in distinct parts. For example:

$$\begin{aligned} 5+3+3+3+1+1+1+1 &= 5+(3+3)+3+(1+1)+(1+1) \\ &= 6+5+3+2+2 \\ &= 6+5+3+(2+2) \\ &= 6+5+4+3 \end{aligned}$$

Clearly, this method gives us a partition in distinct parts. Working this idea in reverse, we see that this method gives us an onto function from distinct parts to odd parts. Given a partition in distinct parts, we can always break down any even parts into odd parts simply by dividing the even parts by two until we obtain odd parts. Therefore, we have an onto mapping from odd parts to distinct parts. We also know that this function is one-to-one. Clearly, since we always combine parts in the same manner, if two distinct partitions are equal, then they must have come from the same odd partition.$\square$

This is a particularly interesting type of function. This idea can be used to create bijections between all kinds of sets of partitions, simply by combining or splitting parts. One of the most accessible applications of this method gives us a way of expressing the binary numbers.

It is clear that given an integer, $p(n \mid \text{every part is one}) = 1$. That is, if we are given any integer, say n, it can be expressed as a sum of ones in one way, namely $1 + 1 + 1 + ... + 1$ for n ones. Now, if we use the method described above in the proof of Theorem 1, we can combine the ones in pairs until we get a distinct partition of n into powers of 2. For example:

$$\begin{aligned} 13 &= (1+1)+(1+1)+(1+1)+(1+1)+(1+1)+(1+1)+1 \\ &= (2+2)+(2+2)+(2+2)+1 \\ &= (4+4)+4+1 \\ &= 8+4+1 \\ &= 2^3+2^2+2^0 \end{aligned}$$

By this example, and the proof of Theorem 1, it is clear that there is a bijection between

$$p(n\ |\text{every part is one}) \text{ and } p(n\ |\text{parts are distinct powers of 2})$$

i.e., there is a unique way to express every integer as a sum of distinct powers of 2. And, of course, this is where the expressions for binary come from. If we substitute the coefficients of each power of 2 into our expressions, we obtain the corresponding binary number for that integer, i.e.:

$$13 = (1)(2^3) + (1)(2^2) + (0)(2^1) + (1)(2^0) = (1101)$$

Thus far, you may have noticed that our method of representing integer partitions is very cumbersome. You might also have noticed that it can be very difficult to visualize exactly what's going on when we create our partitions. You may have asked yourself, 'is there a way to geometrically represent these partitions?' And, of course, the answer is yes.

Definition 2

A *Young diagram* of an integer partition $p(n) = k_1 + k_2 + ... + k_m$ is a set of n boxes arranged into m left-justified rows, where the i^{th} row has k_i boxes.$\square$

As this definition is loaded with jargon, we will provide an example to illustrate the definition here. Given the partition $11 = 4 + 3 + 3 + 1$, its Young diagram looks like this:

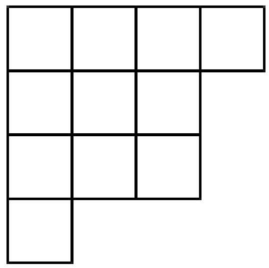

Figure 1: $p(11) = 4 + 3 + 3 + 1$

Now that we have a definition of what a Young diagram is, we can easily visualize integer partitions geometrically. In fact, this idea of boxes being used to visualize integer partitions gives us a number of surprising and fascinating results about relations among different sets of partitions. For instance, our example above is a partition known as a *self-conjugate* partition. This means that if we flip the partition along its NW-SE axis, we obtain the same partition that we had before. There are many interesting questions about self-conjugates, but we will not explore them here. Instead, we will move on to another area, one that illustrates more fundamental ideas about partitions, and gives us a good feel for how Young diagrams are used.

Definition 3

$L(m, n)$ is the set of all partitions with at most m parts and largest part at most n.$\square$

This is a pretty simple definition. As an example, let's look at $L(4, 2)$:

$$L(4, 2) = \{\emptyset, 1, 2, 22, 21, 11, 222, 221, 211, 111, 2222, 2221, 2211, 2111, 1111\}$$

There is also a diagram that is affiliated with $L(m, n)$, and this diagram is known as a *lattice*. This is simply a diagram of what partitions' Young diagrams 'fit inside' others'. For instance, the Young diagram of the partition $1 = 1$ fits inside the partition of $4 = 2 + 2$. Let's look at an example of the lattice for $L(2, 3)$.

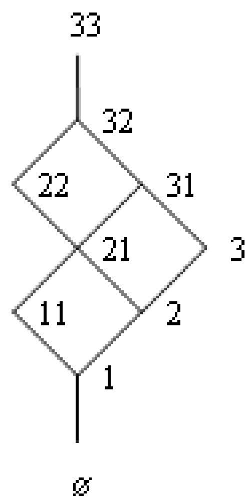

Figure 2: Lattice of $L(2, 3)$

The figure here shows the structure of $L(2, 3)$ very clearly. Starting at the bottom, the empty set fits inside of every other partition. Then, of course, $1 = 1$ fits inside of everything above it, and $2 = 2$ fits inside everything except $2 = 1 + 1$, etc. Each level of the lattice is the set of partitions of an integer satisfying the restrictions of $L(m, n)$, and the lines connect the lower partitions to the ones that they fit inside of. For instance, the fourth level is the set of partitions in $L(2, 3)$ whose sum is 3. Then, $2, 1$ fits inside of $2, 2$ and $3, 1$. They, in turn, have sums of 4, and fit inside $3, 2$, and so on.

We can now plainly see that $L(m, n)$ is a subset of Y, the set of all partitions. However, we can say much more about $L(m, n)$. We present one idea here.

Remark 1

$L(m, n)$ is the set of Young diagrams fitting inside of an $m \times n$ rectangle. Also, $L(m, n)$ and $L(n, m)$ are *isomorphic*, where the isomorphism is an interchange of the rows and columns.□

This is pretty plain to see. Since each part can be at most n, and it can have at most m parts, it's obvious that $L(m,n)$ fits inside the given rectangle. The second part states that $L(m,n)$ and $L(n,m)$ are *exactly* the same if we flip one or the other along the NW-SE diagonal. Now we have a better idea of where $L(m,n)$ fits into our picture of integer partitions.

As one last interesting point about $L(m,n)$, we can count the number of partitions that satisfy the $L(m,n)$ restriction in any given case, and the proof of this is in a style frequently used in combinatorics: counting lattice paths. We assume that the reader has seen this style of proving before. The given application, however, puts an interesting spin on the idea. We now present the theorem.

Theorem 2
$|L(m,n)| = \binom{m+n}{n}$

Proof:
There is a bijection between the Young diagrams fitting inside an $m \times n$ rectangle and sequences of N's and E's which correspond to all northeastern lattice paths which trace out the border of the diagrams. Begin by drawing an $m \times n$ rectangle. If we begin one square below the SW corner of the diagram, we may proceed along any northeastern lattice path to one square to the right of the NE corner and produce a partition that fits inside the $m \times n$ rectangle. We may draw these diagrams by choosing where we take our eastward steps. Since we must make a total of $m+n$ steps (m to the east and n to the north) , we know that we can choose our eastward steps in $\binom{m+n}{n}$ ways. Since each of these choices gives us a valid partition, we are done.□

Now that we have been exposed to a variety of ideas about integer partitions, we would like to close with a theorem that we will not prove here. Instead, we intend to explore the beauty of the theorem, while demonstrating the great difficulties encountered when trying to prove such novel ideas. (Add Andrews, Eriksson reference HERE)

We begin by recalling Theorem 1. Hopefully, the reader will remember that in it we showed that $p(n \mid \text{odd parts}) = p(n \mid \text{distinct parts})$. A natural question to ask is this: what other partition identities of this type are there?

Let us consider the set $p(n \mid 2\text{-distinct parts})$. This is the set of all integer partitions such that the parts are 2-distinct, that is, they are distinct and differ by 2 or more. For instance, $p(7 \mid 2\text{-distinct parts}) = 3$. They are 7, $6+1$, and $5+2$. In Euler's Identity, we were able to show that $p(n \mid \text{distinct parts})$ was equal to $p(n \mid \text{odd parts})$, that is, we were able to look at every partition of an integer n into distinct parts, and using only the set of odd integers, create a set of partitions of n with an equivalent cardinality. Can we do something like this for $p(n \mid 2\text{-distinct parts})$? Let's begin by looking at a table of a few example partitions.

n	Partitions of n into 2-distinct parts
1	1
2	2
3	3
4	4, 3+1
5	5, 4+1
6	6, 5+1, 4+2
7	7, 6+1, 5+2
8	8, 7+1, 6+2, 5+3
9	9, 8+1, 7+2, 6+3, 5+3+1
10	10, 9+1, 8+2, 7+3, 6+4, 6+3+1

Now let's consider a way that we could develop a set of numbers, call it N, so that the number of partitions we get for n in that set is equal to the number of partitions of n we get under our 2-distinct restriction. A logical way would be to start from 1!

There must be a single partition of 1. We get that from 1, so let's put 1 in N.

There must also be a single partition of 2. We get that from $1 + 1$, so let's not include 2 in N.

There must be a single partition of 3. We get that from $1+1+1$, so let's not include 3 in N.

There must be 2 partitions of 4. We get one from $1+1+1+1$, but that's all. Thus, we need 4 in N.

There must be 2 partitions of 5. We get one from $1 + 1 + 1 + 1 + 1$, and another from $4 + 1$, so we don't need 5 in N.

There must be 3 partitions of 6. We get one from $1+1+1+1+1+1$, and one from $4+1+1$, but that's it. Thus, we need to include 6 in our set N.

There must be 3 partitions of 7. We get one from $1+1+1+1+1+1+1$, and one from $4+1+1+1$, and one from $6+1$, so we don't need 7 in N.

There must be 4 partitions of 8. We get one from $1+1+1+1+1+1+1+1$, one from $4+1+1+1+1$, one from $4+4$, and one from $6+1+1$, so we don't need 8 in N.

There must be 5 partitions of 9. We get one from $1+1+1+1+1+1+1+1+1$, one from $4+4+1$, one from $4+1+1+1+1+1$, and one from $6+1+1+1$. Therefore, we need 9 in N.

There must be 6 partitions of 10. We get one from $1+1+1+1+1+1+1+1+1+1$, one from $4+4+1+1$, one from $4+1+1+1+1+1+1$, one from $6+4$, one from $6+1+1+1+1$, and one from $9+1$. Therefore, we don't need 10 in our set N.

If we were to continue in such a manner, it seems that we get the set of numbers $\{1,4,6,9,11,14,16,...\}$. This can be verified, with some work, for a long list of numbers. The pattern should be clear. If not, consider the union of the disjoint sets $\{1,6,11,16,21,...\}$ and $\{4,9,14,19,24,...\}$. These are simply the sets of numbers whose remainder upon division by 5 is 1 or 4. Thus, we present the following conjecture, without proof.

Conjecture 1 [Rogers-Ramanujan Identity]
$p(n\ |\text{2-distinct parts}) = p(n\ |\text{parts} \equiv 1 \text{ or } 4 \pmod 5)$.$\square$

The reader should be aware that this is, indeed, a proven theorem, but because the length and depth of the proof is well beyond that of this paper, we merely present it as an idea here. In fact, the first published proof of this conjecture was 50 pages long! Shorter proofs have since been written, but the idea is still interesting. We have many interesting ideas to work with when we begin to investigate integer partitions, but the verification of those ideas is often a long and arduous process.

The method outlined above is also interesting because this is the method by which many important discoveries have been made (try using it to prove Euler's Identity!). One of the best ways to discover things about numbers is to play with them, to experiment with different ideas until something falls out, and then we can attempt to prove it. Constructing sets like N that work to satisfy some condition on the parts in $p(n)$ has been done dozens of time over, and has yielded a number of surprising and fascinating results.