Diagonalization of Symmetric Matrices

Tom Roby

1 December 2016

Introduction

Recall that:

- Diagonalizing a matrix is an extremely useful way to understand how it operates on vectors, but not every matrix can be diagonalized.
- *Orthogonal* bases make it much easier to compute things, e.g., expressing a given vector in terms of the basis.
- The material in chapter 7 brings together the algebraic side of linear algebra (through chapter 5) and geometric (chapter 6, dot products, lengths, angles) to make it really useful. It also leads to the Singular Value Decomposition (SVD, § 7.4), which is one of the most applied linear algebra tools and our final goal for the class.
- The intro to Chapter 7 and § 7.5 discuss *principal component analysis*, which someone who needs it for Senior Design just asked me about after class. I encourage those of you interested in applications to read through that material, though we (just barely) won't get to it.

The main idea of this lecture is that a special class of matrices, (real) symmetric ones, are always diagonalizable, and have an amazing *spectral decomposition* in terms of eigenstuff. Such matrices appear frequently in applications, so it's great that they always have such nice ways of being understood.

Symmetric & orthogonal matrices

Definition 1. A matrix A is **symmetric** if $A = A^T$.

Example 2. Which of the following are symmetric? (If not, make them so.)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 3 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} x & x-y & x-2y \\ x-y & y & y-z \\ x-2y & z-y & z \end{bmatrix}$$

Recall that an $n \times n$ matrix P is **orthogonal** if $PP^T = I_n$, or equivalently, if the columns of P form an *orthonormal* basis for $\mathbb{R}^n$.

EG: Which of the following are orthogonal? (If not, how would you make them so?)

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \quad P = \frac{1}{5} \cdot \begin{bmatrix} 3 & 4 & 3 \\ -4 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad Q = \frac{1}{2} \cdot \begin{bmatrix} 1 & 1 & \sqrt{2} & a \\ 1 & -1 & 0 & b \\ 1 & 1 & -\sqrt{2} & c \\ 1 & -1 & 0 & d \end{bmatrix}$$

Diagonalizing symmetric matrices

Example 3. (Lay Example 2) Diagonalize the matrix $A = \begin{bmatrix} 6 & -2 & -1 \\ -2 & 6 & -1 \\ -1 & -1 & 5 \end{bmatrix}$.

Standard calculations give a characteristic polynomial which factors as $-(\lambda-8)(\lambda-6)(\lambda-3)$, giving eivenvalues 8, 6, and 3, with corresponding eigenvectors

$$\lambda = 8: \quad \mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \quad \lambda = 6: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}; \quad \lambda = 3: \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix};$$

This gives $A = PDP^{-1}$ where $P =$ ____________ and $D =$ ___________.

Note that the columns of P have the property of being __________. We might also consider an even better condition on the columns. Renormalizing the columns above, we get

$$\lambda = 8: \quad \mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}; \quad \lambda = 6: \quad \mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{6} \\ -1\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}; \quad \lambda = 3: \quad \mathbf{u}_3 = \begin{bmatrix} 1\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix};$$

Now if we let $Q = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]$, then we get $A = QDQ^{-1}$, where $Q^{-1} = Q^T$, i.e., Q is orthogonal. Among other advantages, this makes Q^{-1} trivial to compute! The next theorem shows that there's a good reason that we got orthogonal eigenvectors in this case.

Theorem 4. *If A is a (real) symmetric matrix, then any two eigenvectors from different eigenspaces are orthogonal.*

Proof. Let $\mathbf{v}$ correspond to λ and $\mathbf{w}$ to μ with $\lambda \neq \mu$.

■

Example 5. Diagonalize $A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 4 \end{bmatrix}$.

Q: What's an obvious eigenvalue/vector pair?

So

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Are the columns of P orthogonal? . What should we do?

CHECK: Q is an orthogonal matrix.

So even in the case where eigenvalues are not all distinct, we were able to diagonalize it by an orthogonal matrix. We say such a matrix is **orthogonally diagonalizable**.

Q: Can any matrix that is diagonalizable being orthogonally diagonalized?

Even more surprising, this condition is sufficient. The proof of this theorem is just a bit too hard to prove, though a proof is sketched (via some exercises) in the text.

Theorem 6. *An $n \times n$ matrix A is orthogonally diagonalizable if and only if A is symmetric.*

Amazing. Earlier we have seen that it's quite delicate to see whether a matrix is diagonalizable or not, but here we see that a large class of matrices is automatically diagonalizable.

In fact, we can say a bit more.

Theorem 7 (Spectral Thm for Symmetric Matrices)**.** *An $n \times n$ symmetric matrix A satisfies the following:*

1. *A has n **real** eigenvalues, counting multiplicities.*
2. *For each eigenvector λ, $\dim E_\lambda =$ multiplicity of λ (as root of characteristic polynomial).*
3. *For distinct eigenvalues $\lambda \neq \mu$, $E_\lambda \perp E_\mu$.*
4. *A is orthogonally diagonalizable.*
5. *We have the **spectral decomposition***

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \lambda_2 \mathbf{u}_2 \mathbf{u}_2^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T$$

The point of the last part is that it breaks the matrix A up into a sum of $n \times n$ matrices of rank 1. **(Why rank 1?)**

Each $\mathbf{u}_i \mathbf{u}_i^T$ is a **projection matrix** onto the subspace spanned by $\mathbf{u}_j$ (Exercise 35).

Example 8. Construct the spectral decomposition of the matrix A that has the orthogonal diagonalization

$$A = \begin{bmatrix} 7 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Clicker Questions

1. Every symmetric matrix is orthogonally diagonalizable.
2. An orthogonal matrix is orthogonally diagonalizable.
3. If $B = PDP^T$ where $P^T = P^{-1}$ and D is a diagonal matrix, then B is a symmetric matrix.
4. The dimension of an eigenspace of a symmetric matrix equals the multiplicity of the corresponding eigenvalue.