Math 3250/Roby **Practice Final Solutions** December 2008

4. Compute the answer to each enumeration problem below. Express your answer both in a way that has mathematical meaning (e.g., a difference of binomial coefficients) and as a nonnegative integer. If a problem is a direct consequence of an entry in the twelvefold way, explain which one (e.g., "this is equivalent to putting indistinct balls in distinct boxes surjectively").

   (a) How many subsets of the set $[10] = \{1, 2, \ldots, 10\}$ contain at least one odd integer? $2^{10} - 2^5 = 992$.

   (b) In how many ways can seven people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (not necessarily on the same side)? $\frac{1}{2}(7-1)! = 360$.

   (c) How many permutations $p : [6] \to [6]$ satisfy $p(1) \neq p(2)$? $5 \cdot 5!$ (or $6! - 5!$)$= 600$.

   (d) How many permutations of [6] have exactly two cycles? $\binom{6}{1}4! + \binom{6}{2}3! + \frac{1}{2}\binom{6}{3}2!^2 = 274$.

   (e) How many partitions of [6] have exactly three blocks? $\binom{6}{4} + \binom{6}{1}\binom{5}{2} + \frac{1}{3!}\binom{6}{2}\binom{4}{2} = 90$.

   (f) There are four men and six women. Each man marries one of the women. In how many ways can this be done? $(6)_4 = 360$.

   (g) Ten people split up into five groups of two each. In how many ways can this be done? $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 = 945$.

   (h) How many compositions of 19 use only the parts 2 and 3? $\binom{7}{2} + \binom{8}{3} + \binom{9}{1} = 86$.

   (i) How many partitions of 7 are there into odd parts? **Five:** $7, 511, 331, 31111, 1111111$.

   (j) In how many different ways can the letters of the word MISSISSIPPI be arranged if the four S's cannot appear consecutively? $\binom{11}{1,2,4,4} - \binom{8}{1,1,2,4} = 33810$

   (k) How many sequences $(a_1, a_2, \ldots, a_{12})$ are there consisting of four 0's and eight 1's if no two consecutive terms are both 0's? $\binom{8+1}{4} = 126$.

   (l) A box is filled with three azure socks, three brown socks, and four chartreuse socks. Eight socks are pulled out, one at at time. In how many ways can this be done? (Socks of the same color are indistinguishable.) $2\binom{8}{1,3,4} + 3\binom{8}{2,3,3} + \binom{8}{2,2,4} = 2660$.

   (m) How many trees are there on the vertex set [6]? How many rooted forests? $6^{6-2} = 1296$; $(6+1)^{6-1} = 16,807$.

   (n) How many closed walks of length 7 are there on the complete (labeled) graph with vertex set [5] which do not start at the first vertex? $\frac{4}{5}\left((5-1)^7 + (5-1)(-1)^7\right) = 13,104$.

5. For each statement below, either explain why it's true (full proofs unnecessary) or give a counterexample. Try to salvage any incorrect statements so that they are true.

(a) Any graph with n vertices and $n-1$ edges is a tree. *FALSE unless assume graph is connected.*

(b) The number of partitions of n into distinct odd parts is equal to the number of self-conjugate partitions of n. *TRUE: see Bona Thm 5.18.*

(c) A forest F on n vertices with k connected components has $n-k+1$ edges. *FALSE: Prop 10.6 says just $n-k$.*

(d) A simple graph posessing a closed Eulerian walk must also have a Hamiltonian cycle. *FALSE: attach two 4-cycles at a single point.*

6. How many positive integers are there less than or equal to a million that are neither perfect squares, perfect cubes, nor perfect fourth powers? $10^6 - 10^3 - 10^2 + 10^1 = 998,910$.

7. A permutation p is called an *involution* if $p^2 = 1$, the identity permutation. Prove that for $n > 1$, the number of involutions in S_n is even.

   Pair each permutation p with p^{-1}. Then p is paired with a distinct permutation iff p is NOT an involution. (WHY?) So the total number of involutions is $n! - 2*$(number of pairs), which is even for $n > 2$.

8. Call i a *descent* of a permutation $p = p_1p_2\cdots p_n$ if $p_i > p_{i+1}$, e.g., $p = 426713985$ has descent set $\{1, 4, 7, 8\}$.

   (a) How many 8-permutations have a descent set that is a **subset** of $\{1, 4, 6\}$? *Such a permutation must have $p_2 < p_3 < p_4$, $p_5 < p_6$, and $p_7 < p_8$, and there are no other restrictions. So we get such a permutation by splitting $[8]$ into four subsets of sizes 1,3,2,2, arrange each subset in increasing order, and concatenate the four strings. This can be done in $\binom{8}{1}\binom{7}{2}\binom{4}{2}\binom{2}{2} = 1680$ ways.*

   (b) How many 8-permutations have a descent set **precisely** $\{1, 4, 6\}$? *See Bona Solution 7.10 (p. 142).*

9. Let G be a simple graph with $n > 1$ vertices, none of which is isolated, and let A be the adjacency matrix of G. Suppose that the entries $(A^5)_{i,j}$ and $(A^6)_{i,j}$ are both positive for some fixed indices $i < j$. Prove that G contains a cycle of odd length. *Let W and W' be two walks from i to j that are of lengths 5 and 6 (resp.) Then the symmetric difference of W and W' (i.e., the edges that are contained in exactly one of W and W') is a set of cycles that that have altogether an odd number of edges, specifically $11 - 2e$ where $e = |W \cap W'|$. Hence, one of these cycles must contain an odd number of edges.*

10. Find the ordinary generating fuction of the sequence $p_k(0), p_k(1), p_k(2)\cdots$, where $p_k(n)$ is the number of partitions of n into exactly k parts. *By Bona Ex. 5.6–7, $p_k(n) = p_{\le k}(n-k)$. Hence,*

$$\sum_{n\ge0} p_k(n)x^n = x^k \sum_{n\ge0} p_{\le k}(n)x^n = \frac{x^k}{(1-x)(1-x^2)\cdots(1-x^k)}.$$