ISOMETRIES OF $\mathbf{R}^n$

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1. Introduction

An *isometry* of $\mathbf{R}^n$ is a function $h\colon \mathbf{R}^n \to \mathbf{R}^n$ which preserves the distance between vectors:

$$||h(v) - h(v')|| = ||v - v'||$$

for all v and v' in $\mathbf{R}^n$, where $||(x_1, \dots, x_n)|| = \sqrt{x_1^2 + \cdots + x_n^2}$.

Example 1.1. The identity transformation: $\mathrm{id}(v) = v$ for all $v \in \mathbf{R}^n$.

Example 1.2. Negation on $\mathbf{R}^n$: $-\mathrm{id}(v) = -v$ for all $v \in \mathbf{R}^n$.

Example 1.3. A translation on $\mathbf{R}^n$: fix $w \in \mathbf{R}^n$ and let $t_w(v) = v + w$. Easily $||t_w(v) - t_w(v')|| = ||v - v'||$.

Since $t_{\mathbf{0}}$ is the identity, $t_w \circ t_{w'} = t_{w+w'}$ and $t_w^{-1} = t_{-w}$, the translations on $\mathbf{R}^n$ form a group under composition. In fact, this group is isomorphic to the additive group $\mathbf{R}^n$ by $t_w \leftrightarrow w = t_w(\mathbf{0})$.

Example 1.4. Rotations around points and reflections across lines in the plane are isometries of $\mathbf{R}^2$. Formulas for these isometries will be given in Example 3.3 and Section 5.

Two basic properties of isometries are:

- the composition of isometries of $\mathbf{R}^n$ is an isometry,
- if an isometry is invertible, its inverse is also an isometry.

This nearly shows, along with Example 1.1, that the set $\mathrm{Iso}(\mathbf{R}^n)$ of all isometries of $\mathbf{R}^n$ is a group under composition. We still need to check that any isometry is invertible to have a group.

In Section 2, we will see how to study isometries using dot products instead of distances. The dot product is a more convenient device to use than distance because of its algebraic properties. Section 3 introduces the matrix transformations on $\mathbf{R}^n$ (orthogonal matrices) which are isometries. In Section 4 we will see that the translations and the matrix transformations from Section 3 combine to give us all isometries of $\mathbf{R}^n$. In particular, we will see that every isometry of $\mathbf{R}^n$ is invertible and $\mathrm{Iso}(\mathbf{R}^n)$ is a group. Section 5 discusses the isometries of $\mathbf{R}$ and $\mathbf{R}^2$. In Appendix A, we will look more closely at reflections.

2. Isometries and dot products

Using translations, we can reduce the study of isometries of $\mathbf{R}^n$ to the case of isometries fixing $\mathbf{0}$.

Theorem 2.1. *Every isometry of $\mathbf{R}^n$ is the composite of a translation and an isometry fixing the origin.*

Proof. Let $h\colon \mathbf{R}^n \to \mathbf{R}^n$ be an isometry. Let $t\colon \mathbf{R}^n \to \mathbf{R}^n$ be the translation $t(v) = v + h(\mathbf{0})$. Then

$$h = t \circ t^{-1}h,$$

where $(t^{-1}h)(\mathbf{0}) = t^{-1}(h(\mathbf{0})) = h(\mathbf{0}) - h(\mathbf{0}) = \mathbf{0}$. This shows h is the composite of a translation (t) and an isometry fixing the origin ($t^{-1}h$). □

Theorem 2.2. *For a function $h\colon \mathbf{R}^n \to \mathbf{R}^n$, the following are equivalent:*

(1) *h is an isometry and $h(\mathbf{0}) = \mathbf{0}$,*
(2) *$h(v) \cdot h(v') = v \cdot v'$ for all $v, v' \in \mathbf{R}^n$.*

Proof. The second condition implies $h(\mathbf{0}) = \mathbf{0}$: set $v = v' = \mathbf{0}$ to get $h(\mathbf{0}) \cdot h(\mathbf{0}) = 0$.

The link between length and dot product is the formula

$$||v||^2 = v \cdot v.$$

For any vectors v and v' in $\mathbf{R}^n$,

$$||h(v) - h(v')|| = ||v - v'|| \iff ||h(v) - h(v')||^2 = ||v - v'||^2.$$

Writing the right side in terms of dot products makes it

$$(h(v) - h(v')) \cdot (h(v) - h(v')) = (v - v') \cdot (v - v').$$

Carrying out the multiplication, $||h(v) - h(v')|| = ||v - v'||$ if and only if

$$h(v) \cdot h(v) - 2h(v) \cdot h(v') + h(v') \cdot h(v') = v \cdot v - 2v \cdot v' + v' \cdot v'. \tag{2.1}$$

Both conditions in the lemma imply

$$h(v) \cdot h(v) = v \cdot v \tag{2.2}$$

for all $v \in \mathbf{R}^n$. Indeed, the first condition implies (2.2) by using $v' = \mathbf{0}$ in (2.1). The second condition implies (2.2) by using $v' = v$. So, whether we assume either condition of the lemma, we have $h(v) \cdot h(v) = v \cdot v$ and $h(v') \cdot h(v') = v' \cdot v'$ in (2.1). Cancelling these in (2.1), we get that

$$||h(v) - h(v')|| = ||v - v'|| \text{ and } h(\mathbf{0}) = \mathbf{0} \iff h(v) \cdot h(v') = v \cdot v'.$$

□

Corollary 2.3. *The only isometry of $\mathbf{R}^n$ fixing $\mathbf{0}$ and the standard basis is the identity.*

Proof. Let $h \in \mathrm{Iso}(\mathbf{R}^n)$ satisfy

$$h(\mathbf{0}) = \mathbf{0},\ h(e_1) = e_1,\ \ldots,\ h(e_n) = e_n.$$

Theorem 2.2 says

$$h(v) \cdot h(v') = v \cdot v'$$

for all v and v' in $\mathbf{R}^n$. Fix $v \in \mathbf{R}^n$ and let v' be successively the standard basis vectors $e_1, e_2, \ldots, e_n$, so we see

$$h(v) \cdot h(e_i) = v \cdot e_i.$$

Since h fixes each e_i,

$$h(v) \cdot e_i = v \cdot e_i.$$

Writing $v = c_1e_1 + \cdots + c_ne_n$, we get

$$h(v) \cdot e_i = c_i$$

for all i, so $h(v) = c_1e_1 + \cdots + c_ne_n = v$. As v was arbitrary, h is the identity on $\mathbf{R}^n$. □

It is essential in Corollary 2.3 that the isometry fixes $\mathbf{0}$. An isometry of $\mathbf{R}^n$ which fixes the standard basis only need not be the identity! For example, reflection across the line $x+y=1$ in $\mathbf{R}^2$ is an isometry fixing $(1,0)$ and $(0,1)$ but not $\mathbf{0}=(0,0)$.

If we knew all isometries of $\mathbf{R}^n$ are invertible, then it follows from Corollary 2.3 that two isometries f and g taking the same values at $\mathbf{0}$ and the standard basis are equal: just apply Corollary 2.3 to the isometry $f^{-1}\circ g$ to see this composite is the identity, so $f=g$. But we do not yet know that all isometries are invertible: that is one of our main tasks.

3. Orthogonal matrices

A large supply of isometries of $\mathbf{R}^n$ which fix $\mathbf{0}$ come from orthogonal matrices.

Definition 3.1. An $n\times n$ matrix A is called *orthogonal* if $AA^\top = I_n$, or equivalently if $A^\top A = I_n$.

A matrix is orthogonal when its transpose is its inverse. Since $\det(A^\top)=\det A$, any orthogonal matrix A satisfies $(\det A)^2=1$, so $\det A=\pm 1$.

Example 3.2. The orthogonal 1×1 matrices are ± 1. These are the functions $h(x)=x$ and $h(x)=-x$ on $\mathbf{R}$, which are the identity and reflection through the origin on $\mathbf{R}$.

Example 3.3. For $n=2$, algebra shows $AA^\top = I_2$ if and only if $A=\left(\begin{smallmatrix} a & -\varepsilon b\\ b & \varepsilon a\end{smallmatrix}\right)$, where $a^2+b^2=1$ and $\varepsilon=\pm1$. Writing $a=\cos\theta$ and $b=\sin\theta$, we get the matrices $\left(\begin{smallmatrix} \cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{smallmatrix}\right)$. Algebraically, these types of matrices are distinguished by their determinants: the first has determinant 1 and the second has determinant -1.

The matrices also have different meanings geometrically. The matrix $\left(\begin{smallmatrix} \cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)$ is a counterclockwise rotation by angle θ around the origin. The matrix $\left(\begin{smallmatrix} \cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{smallmatrix}\right)$ is a reflection across the line through the origin at angle $\theta/2$ with respect to the positive x-axis. (That $\left(\begin{smallmatrix} \cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{smallmatrix}\right)^2=\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right)$ is consistent with this interpretation: reflections have order 2.)

Let's explain why $\left(\begin{smallmatrix} \cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{smallmatrix}\right)$ is a reflection at angle $\theta/2$. Pick a line L through the origin, say at an angle φ with respect to the positive x-axis. To find a formula for reflection across L, use a basis with one vector on L and the other vector perpendicular to L. The unit vector $u_1=\left(\begin{smallmatrix} \cos\varphi\\ \sin\varphi\end{smallmatrix}\right)$ lies on L and the unit vector $u_2=\left(\begin{smallmatrix} -\sin\varphi\\ \cos\varphi\end{smallmatrix}\right)$ is perpendicular to L. For any $v\in\mathbf{R}^2$, write $v=c_1u_1+c_2u_2$ with $c_1,c_2\in\mathbf{R}$. The reflection of v across L is $s(v)=c_1u_1-c_2u_2$. Writing $a=\cos\varphi$ and $b=\sin\varphi$ (so $a^2+b^2=1$), in standard coordinates

$$v=c_1u_1+c_2u_2=\begin{pmatrix} c_1a-c_2b\\ c_1b+c_2a\end{pmatrix}=\begin{pmatrix} a & -b\\ b & a\end{pmatrix}\begin{pmatrix} c_1\\ c_2\end{pmatrix} \tag{3.1}$$

and

$$\begin{aligned} s(v) &= c_1u_1-c_2u_2\\ &= \begin{pmatrix} c_1a+c_2b\\ c_1b-c_2a\end{pmatrix}\\ &= \begin{pmatrix} a & b\\ b & -a\end{pmatrix}\begin{pmatrix} c_1\\ c_2\end{pmatrix}\\ &= \begin{pmatrix} a & b\\ b & -a\end{pmatrix}\begin{pmatrix} a & -b\\ b & a\end{pmatrix}^{-1} v \quad \text{by (3.1)}\\ &= \begin{pmatrix} a^2-b^2 & 2ab\\ 2ab & -(a^2-b^2)\end{pmatrix} v. \end{aligned}$$

By the sine and cosine duplication formulas, s has matrix $\left(\begin{smallmatrix}\cos(2\varphi) & \sin(2\varphi)\\ \sin(2\varphi) & -\cos(2\varphi)\end{smallmatrix}\right)$. Therefore $\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{smallmatrix}\right)$ is a reflection across the line through the origin at angle $\theta/2$.

We return to the general case of any $n \geq 1$. The geometric meaning of the condition $A^\top A = I_n$ is that the columns of A are mutually perpendicular unit vectors (check!). Therefore we can "see" how to create orthogonal matrices. There are as many as there are orthonormal bases of $\mathbf{R}^n$.

Let $\mathrm{O}_n(\mathbf{R})$ denote the set of $n \times n$ orthogonal matrices:

$$\mathrm{O}_n(\mathbf{R}) = \{A \in \mathrm{GL}_n(\mathbf{R}) : AA^\top = I_n\}. \tag{3.2}$$

Theorem 3.4. *The set* $\mathrm{O}_n(\mathbf{R})$ *is a group under matrix multiplication.*

Proof. Clearly $I_n \in \mathrm{O}_n(\mathbf{R})$. For $A \in \mathrm{O}_n(\mathbf{R})$, the inverse of A^{-1} is $(A^{-1})^\top$ since

$$(A^{-1})^\top = (A^\top)^\top = A.$$

Therefore $A^{-1} \in \mathrm{O}_n(\mathbf{R})$. If A_1 and A_2 are in $\mathrm{O}_n(\mathbf{R})$, then

$$(A_1A_2)(A_1A_2)^\top = A_1A_2A_2^\top A_1^\top = A_1A_1^\top = I_n,$$

so $A_1A_2 \in \mathrm{O}_n(\mathbf{R})$. $\square$

Theorem 3.5. *If* $A \in \mathrm{O}_n(\mathbf{R})$*, then the transformation* $h_A \colon \mathbf{R}^n \to \mathbf{R}^n$ *given by* $h_A(v) = Av$ *is an isometry of* $\mathbf{R}^n$ *which fixes* $\mathbf{0}$*.*

Proof. Trivially the function h_A fixes $\mathbf{0}$. By Theorem 2.2, it suffices now to show

$$Av \cdot Av' = v \cdot v' \tag{3.3}$$

for all $v, v' \in \mathbf{R}^n$.

The fundamental link between the dot product and transposes is

$$v \cdot Av' = A^\top v \cdot v'. \tag{3.4}$$

Thus, for an $n \times n$ real matrix A and $v, v' \in \mathbf{R}^n$,

$$Av \cdot Av' = A^\top(Av) \cdot v' = (A^\top A)v \cdot v'.$$

This is equal to $v \cdot v'$ for all v and v' precisely when $A^\top A = I_n$. $\square$

Example 3.6. Negation on $\mathbf{R}^n$ comes from the matrix $-I_n$, which is orthogonal: $-\mathrm{id} = h_{-I_n}$.

The proof of Theorem 3.5 gives us a description of $\mathrm{O}_n(\mathbf{R})$ which is more geometric than (3.2):

$$\mathrm{O}_n(\mathbf{R}) = \{A \in \mathrm{GL}_n(\mathbf{R}) : Av \cdot Av' = v \cdot v' \text{ for all } v, v' \in \mathbf{R}^n\}. \tag{3.5}$$

Remark 3.7. Equation (3.5) tells us how to define orthogonal transformations on a subsace $W \subset \mathbf{R}^n$: they are the linear transformations $W \to W$ which preserve dot products between vectors in W.

The label "orthogonal matrix" suggests a matrix which preserves orthogonality of vectors:

$$v \cdot v' = 0 \Longrightarrow Av \cdot Av' = 0 \tag{3.6}$$

for all v and v' in $\mathbf{R}^n$. While orthogonal matrices do satisfy (3.6), since (3.6) is just a special case of (3.3), (3.6) is actually a weaker property than (3.3). That is, a matrix (even an invertible matrix) can satisfy (3.6) without satisfying (3.3).

A simple example is a scaling transformation $A(v) = cv$ with $c \neq \pm 1$: $(cv)\cdot(cv') = c^2(v\cdot v')$ and $c^2 \neq 1$. (The case $c = 0$ is silly, but the other cases are invertible transformations.) More generally, a scalar multiple of an orthogonal matrix satisfies (3.6). And we now prove this exhausts the possibilities.

Theorem 3.8. *If A is an $n \times n$ real matrix which satisfies* (3.6)*, then A is a scalar multiple of an orthogonal matrix.*

Proof. By (3.6), the basis vectors $Ae_1, \dots, Ae_n$ are mutually perpendicular, so the columns of A are perpendicular to each other. We want to show that they have the same size too.

Note that $e_i + e_j \perp e_i - e_j$ when $i \neq j$, so by (3.6) and linearity $Ae_i + Ae_j \perp Ae_i - Ae_j$. Writing this in the form $(Ae_i + Ae_j) \cdot (Ae_i - Ae_j) = 0$ and expanding, we are left with $Ae_i \cdot Ae_i = Ae_j \cdot Ae_j$, so $||Ae_i|| = ||Ae_j||$. Therefore the columns of A are mutually perpendicular vectors with the same length. Calling this common length c, the matrix $(1/c)A$ has an orthonormal basis for its columns, so it is an orthogonal matrix. Therefore A is a scalar multiple of an orthogonal matrix. □

4. Iso($\mathbf{R}^n$) is a group

We now establish the converse to Theorem 3.5.

Theorem 4.1. *Any isometry $h\colon \mathbf{R}^n \to \mathbf{R}^n$ fixing $\mathbf{0}$ is an orthogonal transformation: there is an $A \in \mathrm{O}_n(\mathbf{R})$ such that $h(v) = Av$ for any $v \in \mathbf{R}^n$.*

Proof. By Theorem 2.2,

$$h(v) \cdot h(v') = v \cdot v'$$

for all $v, v' \in \mathbf{R}^n$. What does this say about the effect of h on the standard basis? Taking $v = v' = e_i$,

$$h(e_i) \cdot h(e_i) = e_i \cdot e_i,$$

so $||h(e_i)||^2 = 1$. Therefore $h(e_i)$ is a unit vector. Taking $v = e_i$ and $v' = e_j$ with $i \neq j$, we get

$$h(e_i) \cdot h(e_j) = e_i \cdot e_j = 0.$$

Therefore the vectors $h(e_1), \dots, h(e_n)$ are mutually perpendicular unit vectors (an orthonormal basis of $\mathbf{R}^n$).

Let A be the $n \times n$ matrix with i-th column equal to $h(e_i)$. Since the columns are mutually perpendicular unit vectors, $A^\top A$ equals I_n, so A is an orthogonal matrix and thus acts as an isometry of $\mathbf{R}^n$ by Theorem 3.5. By the definition of A, $A(e_i) = h(e_i)$ for all i. Therefore A and h are isometries with the same values at the standard basis. Moreover, we know A is invertible since it is an orthogonal matrix.

Consider now the isometry $A^{-1} \circ h$. It fixes $\mathbf{0}$ as well as the standard basis. By Corollary 2.3, $A^{-1} \circ h$ is the identity, so $h(v) = Av$ for all $v \in \mathbf{R}^n$: h is given by an orthogonal matrix. □

Theorem 4.2. *Every isometry $h\colon \mathbf{R}^n \to \mathbf{R}^n$ has the form*

$$h_{A,w}(v) = Av + w = (t_w A)(v)$$

for a unique $A \in \mathrm{O}_n(\mathbf{R})$ and $w \in \mathbf{R}^n$. Moreover, this formula always defines an isometry.

Proof. The indicated formula always gives an isometry, since it is the composite of a translation and orthogonal transformation, which are both isometries.

If $h_{A,w} = h_{A',w'}$, then evaluating both isometries at $\mathbf{0}$ gives $w = w'$. Therefore $Av + w = A'v + w$ for any v, so $Av = A'v$ for any v, which means $A = A'$.

It remains to show any isometry of $\mathbf{R}^n$ has the form $h_{A,w}$ for some A and w. Let $h\colon \mathbf{R}^n \to \mathbf{R}^n$ be an isometry. By Theorem 2.1, $h(v) = g(v) + w$, where g is an isometry fixing $\mathbf{0}$ and $w = h(\mathbf{0}) \in \mathbf{R}^n$. Theorem 4.1 tells us there is an $A \in \mathrm{O}_n(\mathbf{R})$ such that $g(v) = Av$ for all $v \in \mathbf{R}^n$, so

$$h(v) = Av + w.$$

$\square$

Theorem 4.3. *The set* $\mathrm{Iso}(\mathbf{R}^n)$ *of isometries of* $\mathbf{R}^n$ *is a group under composition.*

Proof. The only point that has to be checked is invertibility. By Theorem 4.2, we can write any isometry h as $h(v) = Av + w$. Its inverse is $k(v) = A^{-1}v - A^{-1}w$. $\square$

Corollary 4.4. *Two isometries of* $\mathbf{R}^n$ *which are equal at* $\mathbf{0}$ *and at a basis of* $\mathbf{R}^n$ *are the same.*

Proof. Since isometries are invertible, it suffices to show an isometry which fixes $\mathbf{0}$ and a basis of $\mathbf{R}^n$ is the identity. This is Corollary 2.3. $\square$

Let's take a look at the composition law on $\mathrm{Iso}(\mathbf{R}^n)$ when we write its elements in the form $h_{A,w}$. We have

$$\begin{aligned} h_{A,w}(h_{A',w'}(v)) &= A(A'v + w') + w \\ &= AA'v + Aw' + w \\ &= h_{AA',Aw'+w}. \end{aligned}$$

This is similar to the multiplication law in the $ax + b$ group:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa' & ab' + b \\ 0 & 1 \end{pmatrix}.$$

In fact, if we write an isometry $h_{A,w} \in \mathrm{Iso}(\mathbf{R}^n)$ as an $(n+1) \times (n+1)$ matrix $\left(\begin{smallmatrix} A & w \\ 0 & 1 \end{smallmatrix}\right)$, where the 0 in the bottom is a row vector of n zeros, then the composition law in $\mathrm{Iso}(\mathbf{R}^n)$ is multiplication of the corresponding matrices, so $\mathrm{Iso}(\mathbf{R}^n)$ can be viewed as a subgroup of $\mathrm{GL}_{n+1}(\mathbf{R})$.

5. Isometries of the line and plane

We conclude the main part of this handout on isometries of $\mathbf{R}^n$ with a look at the special cases $n = 1$ and $n = 2$.

Consider $\mathrm{Iso}(\mathbf{R})$. Since $\mathrm{O}_1(\mathbf{R}) = \{\pm 1\}$, the isometries of $\mathbf{R}$ are the functions $h(x) = x + c$ and $h(x) = -x + c$ for $c \in \mathbf{R}$. (Of course, this case can be worked out easily from scratch without all the earlier preliminary material. Try it.)

Now consider $\mathrm{Iso}(\mathbf{R}^2)$. We know from Example 3.3 that $\mathrm{O}_2(\mathbf{R})$ consists of rotation and reflection matrices, depending on the determinant. Write $h \in \mathrm{Iso}(\mathbf{R}^2)$ in the form $h(v) = Av + w$.

There turn out to be four possibilities for h: a translation, a rotation, a reflection, and a glide reflection. A *glide reflection* is the composite of a reflection and a non-zero translation in the direction parallel to the line of reflection. (The action of a glide reflection is illustrated by a point on the edge of a screw which, during a forward screw motion, repeatedly passes through a fixed plane containing the axis of the screw.)

Before we derive these possibilities for h, we collect the results in Table 1. They say how the geometry of h is determined by $\det A$ and the shape of its fixed points (solutions to $h(v) = v$). The table also shows that a description of the fixed points can be obtained algebraically from A and w.

Isometry	Condition	Fixed pts
Identity	$A = I_2, w = 0$	$\mathbf{R}^2$
Translation	$A = I_2, w \neq 0$	$\emptyset$
Rotation	$\det A = 1, A \neq I_2$	$(I_2 - A)^{-1}w$
Reflection	$\det A = -1,\ Aw = -w$	$w/2 + \ker(A - I_2)$
Glide reflection	$\det A = -1,\ Aw \neq -w$	$\emptyset$

TABLE 1. Isometries of $\mathbf{R}^2$: $h(v) = Av + w$

Now we justify the information in the table, considering first the case $\det A = 1$ and then $\det A = -1$.

Suppose $\det A = 1$, so $A = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$. If $A = I_2$, then $h(v) = v + w$, which means h is translation by w. If $A \neq I_2$, we show h is a rotation. First of all, h has a unique fixed point: $v = Av + w$ precisely when $w = (I_2 - A)v$. Since $\det(I_2 - A) = 2(1 - \cos\theta) \neq 0$, $I_2 - A$ is invertible and $p = (I_2 - A)^{-1}w$ is the fixed point of h. Then $w = (I_2 - A)p = p - Ap$, so

$$h(v) = Av + (p - Ap) = A(v - p) + p. \tag{5.1}$$

Since A is a rotation around the origin, (5.1) shows h is the same rotation around the point p.

Now suppose $\det A = -1$, so $A = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$ and $A^2 = I_2$. We again look at fixed points of h. As before, $h(v) = v$ if and only if $w = (I_2 - A)v$. But unlike the previous case, now $\det(I_2 - A) = 0$ (check!), so $I_2 - A$ is not invertible and therefore not every w is in the image of $I_2 - A$. When w is in the image, h will be a reflection. Otherwise it will be a glide reflection.

Suppose h has a fixed point. Then $w/2$ must be a fixed point. Indeed, let p be a fixed point, so $p = Ap + w$. Then, since $A^2 = I_2$,

$$Aw = A(p - Ap) = Ap - p = -w,$$

so

$$h\left(\frac{w}{2}\right) = \frac{1}{2}Aw + w = \frac{w}{2}.$$

Thus h has a fixed point if and only if $Aw = -w$, in which case

$$h(v) = Av + w = A\left(v - \frac{w}{2}\right) + \frac{w}{2}. \tag{5.2}$$

Since A is a reflection across a line through the origin, (5.2) says h is a reflection across the parallel line passing through $w/2$. Let's describe the line of reflection for h algebraically. The line of reflection for A is the solution set to $Av = v$, namely the kernel of $A - I_2$, so the line of reflection for h is $w/2 + \ker(A - I_2)$. (As a consistency check, since $A - I_2$ is not invertible and not identically 0, its kernel really is 1-dimensional.)

Now assume h has no fixed point, so $Aw \neq -w$. We will show h is a glide reflection. (The formula $h = Av + w$ shows h is the composite of a reflection and a translation, but

w need not be parallel to the line of reflection of A, which is $\ker(A - I_2)$, so this formula does not show h is a glide reflection.) Our arguments will take stronger advantage of the fact that $A^2 = I_2$.

Since $O = A^2 - I_2 = (A - I_2)(A + I_2)$ and $A \neq \pm I_2$ (after all, $\det A = -1$), $A + I_2$ and $A - I_2$ are not invertible. Therefore the subspaces

$$W_1 = \ker(A - I_2), \quad W_2 = \ker(A + I_2)$$

are both non-zero, and neither is the whole plane, so W_1 and W_2 are both one-dimensional. We already noted that W_1 is the line of reflection of A (fixed points of A form the kernel of $A - I_2$). It turns out that W_2 is the line perpendicular to W_1. To see why, pick $w_1 \in W_1$ and $w_2 \in W_2$, so

$$Aw_1 = w_1, \quad Aw_2 = -w_2.$$

Then, since $Aw_1 \cdot Aw_2 = w_1 \cdot w_2$ by orthogonality of A, we have

$$w_1 \cdot (-w_2) = w_1 \cdot w_2.$$

Thus $w_1 \cdot w_2 = 0$, so $w_1 \perp w_2$.

Now we are ready to show h is a glide reflection. Pick non-zero vectors $w_i \in W_i$ for $i = 1, 2$, and use $\{w_1, w_2\}$ as a basis of $\mathbf{R}^2$. Write w in terms of this basis: $w = c_1 w_1 + c_2 w_2$. To say there are no fixed points for h is the same as $Aw \neq -w$, so $w \notin W_2$. That is, $c_1 \neq 0$. Then

$$h(v) = (Av + c_2 w_2) + c_1 w_1. \tag{5.3}$$

Since $A(c_2 w_2) = -c_2 w_2$, our previous discussion shows $v \mapsto Av + c_2 w_2$ is a reflection across the line $c_2 w_2/2 + W_1$. Since $c_1 w_1 \in W_1$, (5.3) exhibits h as the composite of a reflection and a translation in a direction parallel to the line of reflection, so h is a glide reflection. This concludes the justification of Table 1.

Lemma 5.1. *In* $\mathrm{Iso}(\mathbf{R}^2)$*, a product of two reflections is a translation or a rotation.*

Proof. The product of two matrices with determinant -1 has determinant 1, so the product of two reflections has the form $v \mapsto Av + w$ where $\det A = 1$. Such isometries are translations or rotations by Table 1 (consider the identity to be a trivial translation or rotation). □

Theorem 5.2. *Each element of* $\mathrm{Iso}(\mathbf{R}^2)$ *is a product of at most* 2 *reflections except for glide reflections, which are a product of* 3 (*and no fewer*) *reflections.*

Proof. We check the theorem for each type of isometry in Table 1, except of course reflections.

The identity is the square of any reflection.

For a translation $t(v) = v + w$, let A be the matrix representing the reflection across the line $w^\perp$. Then $Aw = -w$. Set $s_1(v) = Av + w$ and $s_2(v) = Av$. Both s_1 and s_2 are reflections, and $(s_1 \circ s_2)(v) = A(Av) + w = v + w$ since $A^2 = I_2$.

Now consider a rotation, say $h(v) = A(v - p) + p$ for some $A \in \mathrm{O}_2(\mathbf{R})$ with $\det A = 1$ and $p \in \mathbf{R}^2$. We have $h = t \circ r \circ t^{-1}$, where t is translation by p and $r(v) = Av$ is a rotation around the origin. Let A' be any reflection matrix (*e.g.*, $A' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$). Set $s_1(v) = AA'v$ and $s_2(v) = A'v$. Both s_1 and s_2 are reflections and $r = s_1 \circ s_2$ (check). Therefore

$$h = t \circ r \circ t^{-1} = (t \circ s_1 \circ t^{-1}) \circ (t \circ s_2 \circ t^{-1}). \tag{5.4}$$

The conjugate of a reflection by a translation (or by any isometry, for that matter) is another reflection, as an explicit calculation using Table 1 shows. Thus, (5.4) expresses the rotation h as a product of 2 reflections.

Finally we consider glide reflections. Since this is the product of a translation and a reflection, it is a product of 3 reflections. We can't use fewer reflections to get a glide reflection, since a reflection is not a glide reflection and a product of two reflections is either a translation or a rotation by Lemma 5.1. □

That each element of $\mathrm{Iso}(\mathbf{R}^2)$ is a product of at most 3 reflections can be proved geometrically too, without recourse to a prior classification of all isometries of the plane. We will give a rough sketch of the argument. We will take for granted (!) that an isometry which fixes at least two points is a reflection across the line through those points or is the identity. (This is related to Corollary 2.3 when $n = 2$.) Now pick any $h \in \mathrm{Iso}(\mathbf{R}^2)$. We may suppose h is not a reflection or the identity (the identity is the square of any reflection), so h has at most one fixed point. If h has one fixed point, say P, choose $Q \neq P$. Then $h(Q) \neq Q$ and the points Q and $h(Q)$ lie on a common circle centered at P (because $h(P) = P$). Let s be the reflection across the line through P which is perpendicular to the line connecting Q and $h(Q)$. Then sh fixes P and Q, so sh is the identity or is a reflection. Solving for h, we see h is a reflection or a product of two reflections. If h has no fixed points, pick any point P. Let s be the reflection across the perpendicular bisector of the line connecting P and $h(P)$, so sh fixes P. Thus sh has a fixed point, so our previous argument shows sh is either the identity, a reflection, or the product of two reflections. Thus h is the product of at most 3 reflections.

A byproduct of this argument, which did not use the classification of isometries, is another proof that all isometries are invertible: any isometry is a product of reflections and reflections are invertible.

Appendix A. Reflections

A reflection is a transformation of $\mathbf{R}^n$ which fixes a chosen hyperplane and interchanges the position of points along each line perpendicular to the hyperplane at equal distance from the hyperplane. These isometries play a role in $\mathrm{Iso}(\mathbf{R}^n)$ which is analogous to that of transpositions in the symmetric group. Reflections, like transpositions, have order 2.

We begin with reflections across hyperplanes that contain the origin. Let H be a hyperplane containing the origin through which we wish to reflect. Let $L = H^\perp$, so L is a linear subspace. Every $v \in \mathbf{R}^n$ can be written uniquely in the form $v = w + u$, where $w \in H$ and $u \in L$. The reflection across H, by definition, is the function

$$s(v) = w - u. \tag{A.1}$$

That is, s fixes $H = v^\perp$ and acts like -1 on $L = \mathbf{R}v$. From the formula defining s, it is linear in v. Since $w \perp u$, $||s(v)|| = ||w|| + ||u|| = ||v||$, so by linearity s is an isometry: $||s(v) - s(v')|| = ||s(v - v')|| = ||v - v'||$.

Since s is linear, it can be represented by a matrix. To write s as a matrix in the simplest manner, pick an orthogonal basis $\{v_1, \ldots, v_{n-1}\}$ of H and let v_n be a vector orthogonal to H: $H = v_n^\perp$. Then reflection across H negates the last coordinate in this basis:

$$s(c_1v_1 + \cdots + c_nv_n) = c_1v_1 + \cdots + c_{n-1}v_{n-1} - c_nv_n.$$

This is a diagonal matrix transformation with 1's along the diagonal except for -1 in the last position:

$$\begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ -c_n \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \\ c_n \end{pmatrix}. \tag{A.2}$$

The matrix in (A.2) represents s relative to a convenient choice of basis. In particular, from the matrix representation we see $\det s = -1$: every reflection in $\mathrm{O}_n(\mathbf{R})$ has determinant -1. Notice the analogy with transpositions in the symmetric group, which have sign -1.

We now derive another formula for s, which will look more complicated than what we have seen so far but should be considered more fundamental. Fix a non-zero vector u_0 on L. For any $v \in \mathbf{R}^n$, write $v = w + u$ as before: $w \in H$ and $u \perp H$. Then $u = cu_0$ for some c. Since $w \perp L$, $v \cdot u_0 = u \cdot u_0 = c(u_0 \cdot u_0)$, so $c = (v \cdot u_0)/(u_0 \cdot u_0)$. Then

$$s(v) = w + u - 2u = v - 2u = v - 2\frac{v \cdot u_0}{u_0 \cdot u_0}u_0. \tag{A.3}$$

It is standard to label the reflection across a hyperplane through the origin by a vector in the orthogonal complement, in this case u_0. So we write $s = s_{u_0}$. It means: reflect in the hyperplane $u_0^\perp$. So $s_{u_0}(u_0) = -u_0$. By (A.3), $s_{cu_0} = s_{u_0}$ for any $c \in \mathbf{R} - \{0\}$. This makes geometric sense, since $(cu_0)^\perp = u_0^\perp$.

With the key formula (A.3), we can get a formula for reflection across a hyperplane which does not contain the origin. Any such hyperplane has the form $H + x_0$ where H is a hyperplane containing the origin and x_0 is a vector perpendicular to H. To reflect points across the hyperplane $H + x_0$, geometric intuition suggests we can subtract x_0, reflect across H, and then add x_0 back. Therefore reflection across $H + x_0$, *by definition*, is

$$s'(v) = s(v - x_0) + x_0,$$

where s is reflection across H. Since s is linear,

$$s'(v) = s(v) - s(x_0) + x_0 = s(v) + 2x_0 = v - 2\frac{v \cdot x_0}{x_0 \cdot x_0}x_0 + 2x_0. \tag{A.4}$$

Example A.1. We use (A.4) to show any translation $t_x(v) = v + x$ is the composite of two reflections. Set $H = x^\perp$ and write s_x for the reflection across H and s'_x for the reflection across $H + x$. By (A.4) and the linearity of s_x,

$$(s'_x \circ s_x)(v) = s'_x(s_x(v)) = s_x(s_x(v)) + 2x = v + 2x,$$

so $t_x = s'_{x/2} \circ s_{x/2} = s'_{x/2} \circ s_x$. Alternatively, $(s_x \circ s'_x)(v) = v - 2x$, so $t_x = s_{x/2} \circ s'_{-x/2} = s_x \circ s'_{-x/2}$.

These formulas show any translation is a composite of two reflections across hyperplanes perpendicular to the direction of the translation, and either the first or second hyperplane can be chosen to pass through the origin.

As an explicit example, the reader should check the horizontal translation $(a, b) \mapsto (a + 1, b)$ is the composite of reflections across two vertical lines, with one of the lines being the y-axis.

We have already noted that reflections in $\mathrm{O}_n(\mathbf{R})$ are analogous to transpositions in the symmetric group S_n. The next theorem, due to E. Cartan, is analogous to the generation of S_n by transpositions.

Theorem A.2 (Cartan). *The group* $\mathrm{O}_n(\mathbf{R})$ *is generated by reflections. More precisely, for* $n \geq 2$ *any element of* $\mathrm{O}_n(\mathbf{R})$ *is a composite of at most* n *reflections in* $\mathrm{O}_n(\mathbf{R})$.

Proof. We argue by induction on n. The theorem is trivial when $n = 1$, since $\mathrm{O}_1(\mathbf{R}) = \{\pm 1\}$ is generated by the reflection -1 on $\mathbf{R}$. Let $n \geq 2$. (While the case $n = 2$ was treated in Theorem 5.2, we will reprove it here.)

Pick $h \in \mathrm{O}_n(\mathbf{R})$ and any non-zero $v \in \mathbf{R}^n$, so $v \cdot v \neq 0$. We will find a reflection $s \in \mathrm{O}_n(\mathbf{R})$ such that $(sh)(v) = v$. We will then be able to reduce the problem to a study of isometries in $v^\perp$, which has dimension $n - 1$, and then we can wrap up the argument using the inductive hypothesis. (Thus, to make the induction argument work, we really should be proving the theorem not just for orthogonal transformations of the spaces $\mathbf{R}^n$, but for orthogonal transformations of their subspaces as well. See Remark 3.7 for the definition of orthogonal transformation on subspaces, and use (A.3) – rather than a matrix formula – to define a reflection across a hyperplane in a subspace.

Given any two distinct vectors of the same length in $\mathbf{R}^n$, let L be the line in the plane of the vectors which bisects the angle between them. Let L' be the line orthogonal to L in this plane. The reflection in $\mathbf{R}^n$ across the hyperplane orthogonal to L' transforms one vector into the other. The reader can fill in the algebraic details of this argument.

Since v and $h(v)$ have the same length, it follows that some reflection s has $s(h(v)) = v$. Since $sh \in \mathrm{O}_n(\mathbf{R})$,

$$w \perp v \Longrightarrow (sh)(w) \cdot (sh)v = w \cdot v = 0,$$

so sh sends the hyperplane $H = v^\perp$ back to itself. Its dimension is $n - 1$.

If $n = 2$, then H is a line, so $sh|_H = \pm\,\mathrm{id}_H$. If $sh|_H = \mathrm{id}_H$, then sh is the identity on $\mathbf{R}^2$, so $h = s$ is a reflection. If $sh|_H = -\,\mathrm{id}_H$, then sh is reflection across the line $\mathbf{R}v$, so h is a product of 2 reflections.

Now we may take $n > 2$ and assume by induction (*i.e.*, applying the inductive hypothesis to the vector space H) that there are reflections $\overline{s}_1, \ldots, \overline{s}_m$ within H such that

$$sh|_H = \overline{s}_1 \overline{s}_2 \cdots \overline{s}_m,$$

and $m \leq n - 1$. Any reflection $: H \to H$ which fixes $\mathbf{0}$ extends naturally to a reflection of $\mathbf{R}^n$, by declaring it to be the identity on the line $H^\perp$ and extending by linearity. (If $\overline{s}$ is a reflection in the orthogonal complement of the line $\mathbf{R}x$ in H, then this lifting is the reflection in the orthogonal complement of $\mathbf{R}x$ in $\mathbf{R}^n$.) Consider now the two isometries

$$sh, \quad s_1 s_2 \cdots s_m.$$

These agree on $H = v^\perp$ and each one fixes v. Thus, by linearity, we have equality as functions on $\mathbf{R}^n$:

$$sh = s_1 s_2 \cdots s_m.$$

Multiplying both sides on the left by s expresses h as a product of $m + 1$ reflections in $\mathrm{O}_n(\mathbf{R})$, and

$$m + 1 \leq n - 1 + 1 = n.$$

□

Corollary A.3. *Every element of* $\mathrm{Iso}(\mathbf{R}^n)$ *is a product of at most* $n + 1$ *reflections in* $\mathrm{Iso}(\mathbf{R}^n)$.

Proof. The case $n = 1$ is trivial, so let $n \geq 2$. Let $h \in \mathrm{Iso}(\mathbf{R}^n)$. If $h(\mathbf{0}) = \mathbf{0}$, then $h \in \mathrm{O}_n(\mathbf{R})$ and we are done by Cartan's theorem. If $h(\mathbf{0}) \neq \mathbf{0}$, then we can write $h = t \circ k$, where t is a translation and k is an isometry fixing $\mathbf{0}$. By Cartan's theorem, k is a product of at most n

reflections in $\mathrm{O}_n(\mathbf{R})$, and Example A.1 tells us t is a product of two reflections. Therefore h is a product of at most $n+2$ reflections. But the bound we are supposed to get is $n+1$!

We would have a bound of $n+1$ if k is a product of at most $n-1$ reflections: $2(n-1)+1 = 2n-1$. Therefore, the only case we need to analyze closely is when k is a product of n reflections in $\mathrm{O}_n(\mathbf{R})$:

$$k = s_1 s_2 \cdots s_n.$$

From Example A.1, $t = s\widetilde{s}$ where $\widetilde{s} \in \mathrm{O}_n(\mathbf{R})$. If we were so lucky to have $s_1 = \widetilde{s}$, then

$$h = tk = s\widetilde{s}s_1 s_2 \cdots s_n = s s_2 \cdots s_n$$

is a product of n reflections.

Well, we are lucky: a product of n reflections in $\mathrm{O}_n(\mathbf{R})$ can always be rewritten as a product of reflections with any reflection from $\mathrm{O}_n(\mathbf{R})$ as its left-most factor. Let's see why in terms of our k which is a product of n reflections in $\mathrm{O}_n(\mathbf{R})$. Since each reflection in $\mathrm{O}_n(\mathbf{R})$ has determinant -1, $\det k = (-1)^n$. Pick any $\sigma \in \mathrm{O}_n(\mathbf{R})$. We want k to be a product of n reflections in $\mathrm{O}_n(\mathbf{R})$ with σ as the left-most factor. Well, write

$$\sigma k = s_1' s_2' \cdots s_m'$$

with $m \leq n$ by Cartan's theorem. Taking determinants of both sides, $-(-1)^n = (-1)^m$, so m and n have opposite parity. Therefore $m \leq n-1$ and $n-1-m$ is even. Since σ has order 2, we can write

$$k = \sigma s_1' s_2' \cdots s_m' = \sigma \sigma^{n-(m+1)} s_1' s_2' \cdots s_m',$$

and the last product has n reflections with σ as the left-most factor. Applying this result with $\sigma = s_1$ above, h is a product of n reflections. $\square$

The proof of Corollary A.3 shows an isometry of $\mathbf{R}^n$ is a product of at most n reflections except possibly when it is the product of a translation and a product of $n-1$ reflections. Then $n+1$ reflections may be required.