THE GAUSSIAN INTEGRAL

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1. Introduction

We want to give several proofs of the formula

$$\int_0^\infty e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}. \tag{1.1}$$

This formula can also be described in the following equivalent ways by a change of variables:

(1) $\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}$,
(2) $\int_{-\infty}^{\infty} e^{-\pi x^2}\,\mathrm{d}x = 1$.

The first form is how (1.1) often appears in probability theory. The second form is how (1.1) is used in certain parts of number theory.

Let $I = \int_0^\infty e^{-x^2}\,\mathrm{d}x$. A common pattern in all the proofs we give of (1.1) is that they compute I^2, not I directly. We of course then obtain I by taking a (positive) square root.

2. Polar coordinates

The most widely known proof of (1.1) expresses I^2 as a double integral and then passes to polar coordinates:

$$\begin{aligned} I^2 &= \int_0^\infty e^{-x^2}\,\mathrm{d}x \int_0^\infty e^{-y^2}\,\mathrm{d}y \\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned}$$

This is a double integral over the first quadrant. In polar coordinates, $x^2+y^2=r^2$ and $\mathrm{d}x\,\mathrm{d}y = r\,\mathrm{d}r\,\mathrm{d}\theta$. In polar coordinates the region of integration is $r \geq 0$ and $0 \leq \theta \leq \pi/2$. Therefore

$$\begin{aligned} I^2 &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\,\mathrm{d}r\,\mathrm{d}\theta \\ &= \int_0^\infty r e^{-r^2}\,\mathrm{d}r \cdot \int_0^{\pi/2} \mathrm{d}\theta \\ &= -\frac{1}{2}e^{-r^2}\Big|_0^\infty \cdot \frac{\pi}{2} \\ &= \frac{1}{2}\cdot\frac{\pi}{2} \\ &= \frac{\pi}{4}. \end{aligned}$$

Taking square roots, $I = \sqrt{\pi}/2$.

3. A second change of variables

Our next proof uses another change of variables to compute I^2, but this will only rely on single-variable calculus. The idea goes back to Laplace.

As before, we have

$$I^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,\mathrm{d}x\,\mathrm{d}y,$$

but now we make a change of variables $x = yt$ with $\mathrm{d}x = y\,\mathrm{d}t$, so

$$\begin{aligned} I^2 &= \int_0^\infty \int_0^\infty e^{-y^2(t^2+1)} y\,\mathrm{d}t\,\mathrm{d}y \\ &= \int_0^\infty \left(\int_0^\infty ye^{-y^2(t^2+1)}\,\mathrm{d}y \right) \mathrm{d}t. \end{aligned}$$

Since $\int_0^\infty ye^{-ay^2}\,\mathrm{d}y = \frac{1}{2a}$ for $a > 0$, we have

$$\begin{aligned} I^2 &= \int_0^\infty \frac{\mathrm{d}t}{2(t^2+1)} \\ &= \frac{1}{2} \cdot \frac{\pi}{2} \\ &= \frac{\pi}{4}, \end{aligned}$$

so $I = \sqrt{\pi}/2$.

4. Differentiating under the integral sign

For $t > 0$, set

$$A(t) = \left(\int_0^t e^{-x^2}\,\mathrm{d}x \right)^2, \quad B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x.$$

The integral we want to calculate is $A(\infty)$ and then take a square root. While $A(t)$ is not such an unreasonable integral to consider in the context of calculating I^2, $B(t)$ may seem to come out of nowhere *except* for the fact that in the previous proof we saw such an integrand floating around. Without that proof, it is hard to imagine how anyone would have thought of looking at $B(t)$ for a proof of (1.1).

Differentiating with respect to t,

$$A'(t) = 2\int_0^t e^{-x^2}\,\mathrm{d}x \cdot e^{-t^2}$$

and

$$\begin{aligned} B'(t) &= \int_0^1 -2te^{-t^2(1+x^2)}\,\mathrm{d}x \\ &= -2\int_0^1 te^{-t^2x^2}\,\mathrm{d}x \cdot e^{-t^2} \\ &= -2\int_0^t e^{-u^2}\,\mathrm{d}u \cdot e^{-t^2} \quad (u = tx,\ \mathrm{d}u = t\,\mathrm{d}x) \\ &= -A'(t). \end{aligned}$$

Thus $A'(t) = -B'(t)$ for all $t > 0$, so there is a constant C such that

$$A(t) = -B(t) + C \tag{4.1}$$

for all $t > 0$. To find C, we let $t \to 0^+$ in (4.1). The left side tends to $(\int_0^0 e^{-x^2}\,\mathrm{d}x)^2 = 0$ while the right side tends to $\int_0^1 \mathrm{d}x/(1+x^2) + C = \pi/4 + C$. Thus $C = -\pi/4$, so (4.1) becomes

$$\left(\int_0^t e^{-x^2}\,\mathrm{d}x\right)^2 = \frac{\pi}{4} - \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x.$$

Let $t \to \infty$ in this equation. The integral on the right goes to 0 and we are left with

$$\left(\int_0^\infty e^{-x^2}\,\mathrm{d}x\right)^2 = \frac{\pi}{4},$$

and taking (positive) square roots gives $\int_0^\infty e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}/2$.