LOCAL COMPACTNESS OF THE DUAL GROUP USING ASCOLI

Let G be a locally compact abelian group and $\widehat{G}$ be its dual group with the compact-open topology. The usual proof that the dual group is locally compact proceeds through Banach algebras, Alaoglu's theorem, L^1-L^∞ duality, and a comparison between the compact-open topology and the topology of pointwise convergence on $\widehat{G}$. We will give here a proof that the dual group is locally compact which uses only the compact-open topology on the dual group (no other topologies) and no Banach algebras, *etc.* Our main tool is the standard theorem describing when sets of functions are compact: Ascoli's theorem.

Lemma 1. *Fix $f \in L^1(G)$. For $y \in G$, let $L_y f\colon G \to \mathbf{C}$ by $(L_y f)(x) = f(yx)$. The map $y \mapsto L_y f$ from G to $L^1(G)$ is continuous.*

The lemma is proved by checking it directly on the dense subset $C_c(G) \subset L^1(G)$ and then extending it to all of $L^1(G)$ by an approximation argument. Details are left to the reader.

Next we prove a result about the decay of Fourier transforms. For $f \in L^1(G)$, its Fourier transform is $\widehat{f}(\chi) = \int_G f(x)\overline{\chi}(x)\,\mathrm{d}x$, where $\mathrm{d}x$ is some choice of Haar measure on G. Using the compact-open topology on $\widehat{G}$, $\widehat{f}\colon \widehat{G} \to \mathbf{C}$ is (uniformly) continuous.

Theorem 2. *If $f \in L^1(G)$ then $\widehat{f}\colon \widehat{G} \to \mathbf{C}$ "vanishes at ∞": for any $\varepsilon > 0$ there is a compact set $C \subset \widehat{G}$ such that $|\widehat{f}(\chi)| < \varepsilon$ for $\chi \notin C$.*

Proof. Since $\widehat{f}\colon \widehat{G} \to \mathbf{C}$ is continuous, our task is the same as showing for any $\varepsilon > 0$ that the (closed) set

$$\{\chi \in \widehat{G} : |\widehat{f}(\chi)| \geq \varepsilon\}$$

is compact in $\widehat{G}$ using the compact-open topology.

Since $\widehat{G}$ is a closed subset of the space $C(G, S^1)$ of continuous functions from G to S^1, what we need to do is show the above set is compact in $C(G, S^1)$. For this, Ascoli's theorem tells us exactly what has to be checked: equicontinuity of our set of characters at each point of G. Since we're dealing with characters and the compact-open topology, it is enough to check equicontinuity of our set of characters at the identity e of G. So for each $\delta > 0$ we want to find an open neighborhood $U = U_\delta$ of e such that

$$y \in U, \quad |\widehat{f}(\chi)| \geq \varepsilon \Longrightarrow |\chi(y) - 1| < \delta.$$

It's not evident how to turn a lower bound on the Fourier transform at χ into an upper bound on $\chi(y) - 1$. The trick is to get a bound on $|\chi(y) - 1|$ where y doesn't show up in $\chi(y)$ anymore.

For any $\chi \in \widehat{G}$ such that $|\widehat{f}(\chi)| \geq \varepsilon$ and any $y \in G$, we have

$$\begin{aligned}
\varepsilon|\chi(y)-1| &\leq |(\overline{\chi}(y)-1)\widehat{f}(\chi)| \\
&= \left|(\overline{\chi}(y)-1)\int_G f(x)\overline{\chi}(x)\,\mathrm{d}x\right| \\
&= \left|\int_G f(x)\overline{\chi}(xy)\,\mathrm{d}x - \int_G f(x)\overline{\chi}(x)\,\mathrm{d}x\right| \\
&= \left|\int_G f(xy^{-1})\overline{\chi}(x)\,\mathrm{d}x - \int_G f(x)\overline{\chi}(x)\,\mathrm{d}x\right| \\
&= \left|\int_G (f(xy^{-1}) - f(x))\overline{\chi}(x)\,\mathrm{d}x\right| \\
&\leq \int_G \left|f(xy^{-1}) - f(x)\right|\,\mathrm{d}x \\
&= |L_{y^{-1}}f - f|_1,
\end{aligned}$$

so

$$|\chi(y)-1| \leq \frac{1}{\varepsilon}|L_{y^{-1}}f - f|_1.$$

From continuity of $y \mapsto L_y f$ and continuity of inversion on G, $|L_{y^{-1}}f - f|_1 \to 0$ as $y \to e$ in G. Therefore $|\chi(y)-1| < \delta$ for all y near e, and that level of nearness to e gives us the desired set U. $\square$

Remark 3. Theorem 2 is a generalization of the Riemann–Lebesgue lemma, which is the special case $G = S^1 = \mathbf{R}/2\pi\mathbf{Z}$: if $f\colon \mathbf{R} \to \mathbf{C}$ is 2π-periodic and integrable then its Fourier coefficients $\widehat{f}(n) = \int_0^{2\pi} f(x)e^{-2\pi inx}\,\mathrm{d}x$ tend to 0 as $|n| \to \infty$.

Corollary 4. *When G is locally compact, so is $\widehat{G}$ in the compact-open topology: if $0 < \varepsilon < 1$ and $K \subset G$ is compact then the neighborhood $N(K, \varepsilon) = \{\chi \in \widehat{G} : |\chi(x) - 1| < \varepsilon$ for all $x \in K\}$ of the trivial character has compact closure in $\widehat{G}$.*

Proof. We copy the argument from Hewitt & Ross vol. 1, page 362.

Let $f = \xi_K$ be the characteristic function of K, so $f \in L^1(G)$. For any $\chi \in \widehat{G}$,

$$\mu(K) = \int_G \xi_K\,\mathrm{d}x = \int_G \xi_K \cdot (1-\chi)\,\mathrm{d}x + \int_G \xi_K \cdot \chi\,\mathrm{d}x.$$

Taking absolute values, if $\chi \in N(K, \varepsilon)$ then $\mu(K) \leq \mu(K)\varepsilon + |\widehat{\xi}_K(\chi)|$, so

$$|\widehat{\xi}_K(\chi)| \geq (1-\varepsilon)\mu(K).$$

By Theorem 2, the set of χ fitting this inequality is a compact set, so $N(K, \varepsilon)$ has compact closure in $\widehat{G}$. $\square$