Math 2210Q (Roby) **Practice Final Solutions** Spring 2016

**SHOW ALL YOUR WORK!** Give reasons to support your answers. No calculators allowed, but you may use one $8.5'' \times 11''$ sheet of notes with anything you like written on it.

1. Define a linear transformation $T : \mathbb{P}_2 \to R^3$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$.

   (a) Find the image under $T$ of $\mathbf{p}(t) = 5 + 3t$. Plug in: $T(5+3t) = \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$.

   (b) Show that $T$ is a linear transformation.

   $$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} (\mathbf{p}+\mathbf{q})(-1) \\ (\mathbf{p}+\mathbf{q})(0) \\ (\mathbf{p}+\mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1)+\mathbf{q}(-1) \\ \mathbf{p}(0)+\mathbf{q}(0) \\ \mathbf{p}(1)+\mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(-1) \\ \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(-1) \\ \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix}.$$

   (c) Find the matrix for $T$ relative to the basis $\{1, t, t^2\}$ for $\mathbb{P}_2$ and the standard basis for $\mathbb{R}^3$. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$.

   (d) Is $T$ one-to-one? Is $T$ onto? Explain! By row reduction or computing the determinant, one easily sees that this matrix is nonsingular; hence, by the IMT, $T$ is both one-to-one and onto.

2. Find the characteristic polynomial and the eigenvalues of the matrix $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$.

   Best to compute $|A - \lambda I|$ by expanding along the middle row. After routine computation one gets the eigenvalues $\lambda = -4, 1,$ and $7$.

3. Show that if $T = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$, then $\det T = (b-a)(c-a)(c-b)$.

   Make sure you can explain why each equality of determinants is true!

   $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & b-a \\ 1 & c-a \end{vmatrix} \text{ etc.}$$

4. Prove or (use a counterexample to) Disprove and Salvage if possible:

   (a) If $A = QR$, where $Q$ has orthonormal columns, then $R = Q^T A$. If the columns of $Q$ are orthonormal, then by definition $Q^T Q = I$. Hence, multiplying both sides of $A = QR$ by $Q^T$ yields $Q^T A = IR = R$.

   (b) If $S = \{u_1, \ldots, u_p\}$ is an orthogonal set of vectors in $\mathbb{R}^n$, then $S$ is linearly independent. False, since $S$ may contain $\mathbf{0}$, e.g., $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{0} \right\}$ is orthogonal. However, any orthogonal set of *nonzero* vectors is linearly independent [§6.2, Thm. 4].

   (c) If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is similar to $BA$. True, since $A^{-1}(AB)A = BA$.

   (d) Each eigenvector of a square matrix $A$ is also an eigenvector of $A^2$. True. If $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero $v$, then $A^2\mathbf{v} = \lambda^2\mathbf{v}$ [Why?]. So $\mathbf{v}$ is an eigenvector for $A^2$ (corresponding to the eigenvalue $\lambda^2$).

   (e) There exists a $2 \times 2$ matrix with real entries that has no eigenvectors in $\mathbb{R}^2$. True. Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $\det(A - \lambda I) = \lambda^2 + 1$, which has no real (only complex) roots.

   (f) If $A$ is row equivalent to the identity matrix $I$, then $A$ is diagonalizable. The matrix $A$ in the example above is a counterexample.

5. Decide whether each statement below is True of False. Justify your answer. For False statements, a counterexample is usually best. Extra credit for good salvages and more for proofs thereof!

   (a) If $\mathbf{y}$ is in a subspace $W$, then the orthogonal projection of $\mathbf{y}$ onto $W$ is $\mathbf{y}$ itself. True. The projection of $\mathbf{y}$ onto $W$ is the vector in $W$ that is closest to $\mathbf{y}$. If $\mathbf{y} \in W$, then that vector will be $\mathbf{y}$ itself. One can also see this by noting that the formulae in §6.3, Thm. 8 and §6.2, Thm. 5 for expanding $\mathbf{y}$ in terms of basis for $W$ give the same coefficients.

   (b) For an $m \times n$ matrix $A$, vectors in $\operatorname{Nul} A$ are orthogonal to vectors in $\operatorname{Row} A$. True. By definition, $\mathbf{v} \in \operatorname{Nul} A$ means that $A\mathbf{v} = 0$. But this just says that the result of taking the inner product of each row of $A$ with $\mathbf{v}$ is zero. Hence, $\mathbf{v}$ is orthogonal to a basis for $\operatorname{Row} A$, hence to any vector in $\operatorname{Row} A$.

   (c) The matrices $A$ and $A^T$ have the same eigenvalues, counting multiplicities. This is true since they have the same characteristic equation: $|A - \lambda I| = |(A - \lambda I)^T| = |A^T - \lambda I|$.

   (d) A nonzero vector can correspond to two different eigenvalues of $A$. False. If $Av = \lambda v$ and $Av = \mu v$ with $\lambda \neq \mu$, then $(\lambda - \mu)v = 0 \implies \mathbf{v} = 0$, since $\lambda - \mu \neq 0$.

(e) The sum of two eigenvectors of a square matrix $A$ is also an eigenvector of $A$. False. Take $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $e_1$ is an eigenvector for $\lambda = 2$, and $e_2$ is an eigenvector for $\lambda = 3$. But $A(e_1 + e_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, which is not a multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(f) The singular values of an $m \times n$ matrix can be $3, 1, -1, -3$. False, all singular values are positive!

6. If a $n \times n$ matrix $A$ satifies $A^2 = A$, what can you say about the determinant of $A$? Since the determinant is multiplicative, we get $D = \det A = \det A^2 = (\det A)^2$. The only solutions to $D^2 = D$ are $D = 0$ or $1$, so $\det A = 0$ or $1$.

7. Assume that matrices $A$ and $B$ below are row equivalent:

$$A = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 1 & 2 & -3 & 0 & -2 & -3 \\ 1 & -1 & 0 & 0 & 1 & 6 \\ 1 & -2 & 2 & 1 & -3 & 0 \\ 1 & -2 & 1 & 0 & 2 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & -2 \\ 0 & 1 & -1 & 0 & -3 & -1 \\ 0 & 0 & 1 & 1 & -13 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Without calculations, list $\operatorname{rank} A$ and $\dim \operatorname{Nul} A$. Then find bases for $\operatorname{Col} A$, $\operatorname{Row} A$, and $\operatorname{Nul} A$. We get $\operatorname{rank} A = 5$, so $\dim \operatorname{Nul} A = 6 - 5 = 1$. A basis for $\operatorname{Col} A$ is given by columns 1, 2, 3, 5, and 6 of $A$, while a basis for $\operatorname{Row} A$ is given by all five rows of $B$ (not of $A$). To get a basis for $\operatorname{Nul} A$, we further reduce $B$ to echelon form:

$$B \sim \begin{bmatrix} 1 & 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \operatorname{Nul} A = \operatorname{Span} \left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

8. Find the maximum value of $Q(\mathbf{x}) = 7x_1^2 + 3x_2^2 - 2x_1x_2$ subject to the constraint $x_1^2 + x_2^2 = 1$. (You do not need to compute a vector at which this maximum is attained.) This is equivalent to finding the maximum value of $\vec{x}^T A \vec{x}$ subject to the constraint $\vec{x}^T \vec{x} = 1$. By Theorem 6, this is the greatest eigenvalue $\lambda_1$ of the matrix of the quadratic form, namely

$$A = \begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix} \Longrightarrow \lambda_1 = 5 + \sqrt{5} \text{ and } \lambda_2 = 5 - \sqrt{5}.$$

9. What would you have to know about the solution set of a homogenous system of 18 linear equations in 20 variables in order to know that every associated nonhomogeneous equation has a solution? Discuss! Let $A$ be the matrix that represents this homogenous system in the form $A\mathbf{x} = \mathbf{0}$. In order for the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ to be **onto** $\mathbb{R}^{18}$, the matrix $A$ must have rank 18. So by the rank-nullity theorem, $\dim \operatorname{Nul} A = 2$, which means that the solution set of the homogenous system is two-dimensional, so can be written as the span of a set of two linearly independent vectors.

10. Go back over your old homework and quizzes to review and make sure you understand any problem on which you lost points. Check!

11. Here are some specific tasks (modified from a list of Prof. Leibowitz) I expect you to be able to perform **with demonstrated understanding**:

    (a) Given a matrix $A$, find the dimensions of and bases for $\operatorname{Col} A$, $\operatorname{Nul} A$, and $\operatorname{Row} A$. Use the relations among rank, $\dim \operatorname{Nul} A$, and size of $A$ to understand properties of the associated linear transformation (one-to-one, onto, kernel, range).

    (b) Use row reduction to solve linear equations, show that a given column vector is in the span of a given set of vectors, or compute the inverse of a matrix.

    (c) Use row operations to reduce a matrix $A$ to triangular form in order to calculate $\det A$. Use properties to compute the determinant of related matrices.

    (d) Diagonalize a given matrix and use the $A = PDP^{-1}$ factorization to calculate a power of $A$.

    (e) Orthogonally diagonalize a real symmetric matrix, possibly representing a quadratic form, and compute constrained extrema of the form.

    (f) Project a vector onto the subspace spanned by a given set of vectors, after verifying that they form an orthogonal set.

    (g) Understand how to construct and use the the singular value decomposition of an $m \times n$ matrix.

    (h) Understand the theory of the course well enough to distinguish true statements from false ones, giving supporting evidence or counterexamples as appropriate.

    (i) Show that a certain set of vectors is a subspace of a given space, or are eigenvectors of a certain matrix.

    (j) Use various forms of the Invertible Matrix Theorem in context.