Symmetric Chain Decomposition

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Outline

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The saturated chain $x_1 < x_2 < ... < x_k$ is a **symmetric chain** in P if $r(x_1) + r(x_k) = r(P)$.

Example

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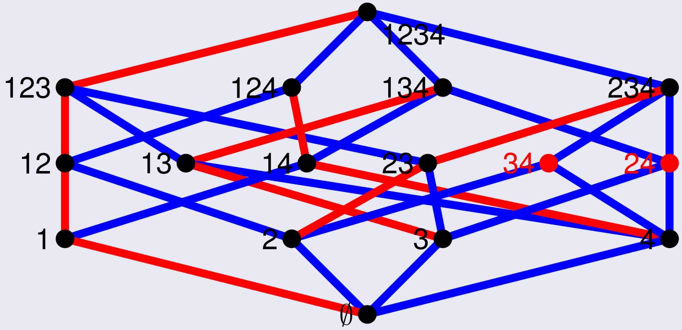
Theorem

For a x in B_n with $|U_0(x)| = k$, let $C_x = \{x, \tau(x), \tau^2(x), ..., \tau^k(x)\}$. The following is a symmetric chain decomposition of B_n:

$S = \{C_x | x \in B_n, U_1(x) = \emptyset\}$.

–Greene and Kleitman

Example



Example

Symmetric Chain:

0000-1000-1100-1110-1111

0100-0110-0111

0010-1010-1011

0001-1001-1101

00010

00001

Theorem

For all positive integers n, $B_n/(1 \ldots n)$ has a symmetric chain decomposition.

–K. K. Jordan

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Proof

Define the anti-sequence of $10a_3a_4...a_n$ is $01a_3a_4...a_n$ (and vice-versa).

If we remove all chains with anti-sequence of elements of the form $10a_3a_4..a_n$, the remain is the symmetric chain decomposition of G.

Proof

For any $s_1 = 10a_3a_4...a_n$ and $s_2 = 01a_3a_4...a_n$

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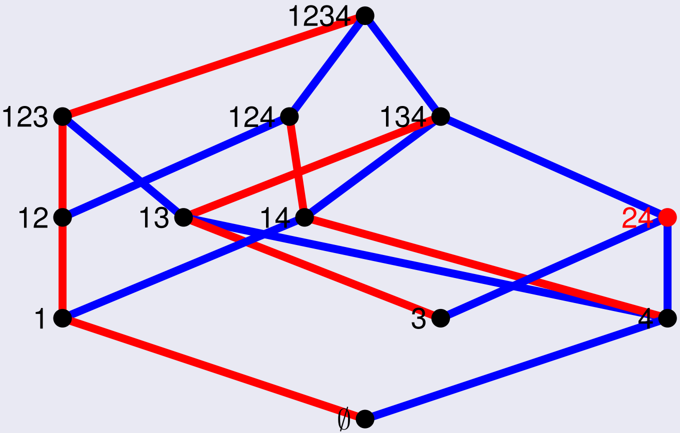
- case1 $\tau(s_1) = 11a_3a_4...a_n$
- case2 $\tau(s_1) = 10b_3b_4...b_n$

Example



Example

symmetric chain decomposition:

0000-1000-1100-1110-1111

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0101

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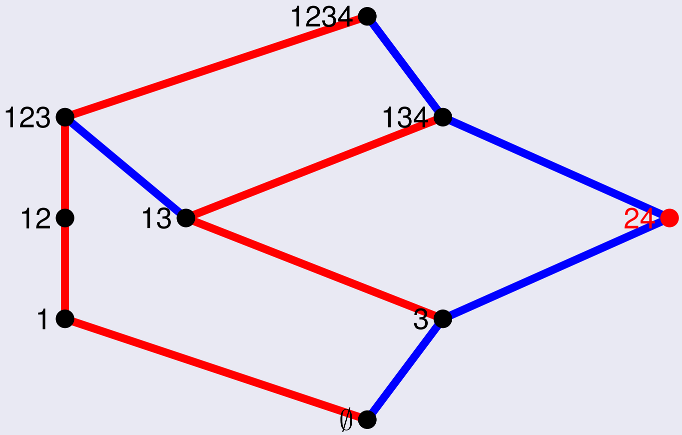
Corollary

If G is generated by $(a_1, b_1), (a_2, b_2), ..., (a_i, b_i)$, B_n/G has a symmetric chain decomposition.

Example

$B_4/(12)(34)$



Example

Symmetric chain decomposition:

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Outline

1. Symmetric chain decompositions in the Boolean lattice
2. Necklace Poset
3. Other quotient posets
   - Transposition
   - Group generated by transposition
   - $B_n/(1..n-1)$
   - i-cycle
4. Summary

Theorem

$B_n/(1..n-1)$ is two copies of $B_{n-1}/(1..n-1)$. There is an edge between two parts if and only if the vertices are $a_1 a_2 ... a_k$ and $a_1 a_2 ... a_k n$ where $a_1, ..., a_k < n$.

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2. Eliminate the top edge of every chain in the second part of $B_n/(1..n-1)$.
3. Add the edge between $a_1 a_2 ... a_k$ and $a_1 a_2 ... a_k n$ where $a_1 a_2 ... a_k$ is the top vertex of chains in the first part.

Proof

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$f : X_1 \to X_2$

$f(a_1 a_2 ... a_k) = a_1 a_2 ... a_k n$, f is bijective

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guarantee the symmetric property

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guarantee the symmetric property

$\rightarrow$ increase one edge or decrease one edge of every chain

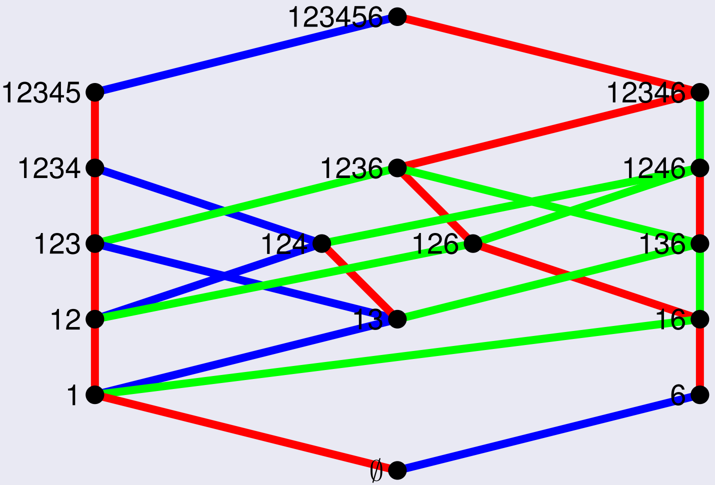
eliminate the top edge of every chain of X_2

connect every top element s of X_1 with $f(s)$

Theorem

B_n/(1..n-1) is two copies of B_{n-1}/(1..n-1). There is an edge between two parts if and only if the vertices are $a_1a_2...a_k$ and $a_1a_2...a_kn$ where $a_1, ..., a_k < n$.

Example



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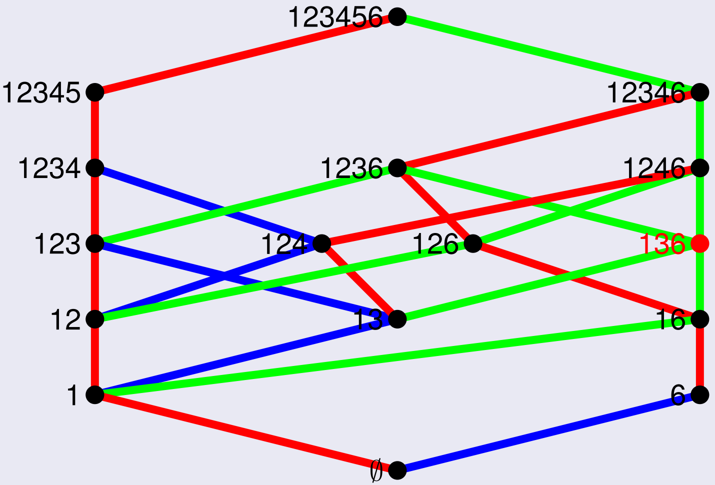

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For any $i \geq 2$, the poset $P = B_n/(123...i)$ has a symmetric chain decomposition. We can use the same way mentioned in the former theorem to generate this chains.

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Proof

Use induction to prove.

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key idea: the structure of chains cannot be changed

Thanks for listening