DIFFERENTIATING UNDER THE INTEGRAL SIGN

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> I had learned to do integrals by various methods shown in a book that my high school physics teacher Mr. Bader had given me. [It] showed how to differentiate parameters under the integral sign – it's a certain operation. It turns out that's not taught very much in the universities; they don't emphasize it. [If] guys at MIT or Princeton had trouble doing an integral, I come along and try differentiating under the integral sign, and often it worked. So I got a great reputation for doing integrals, only because my box of tools was different from everybody else's, and they had tried all their tools on it before giving the problem to me. Richard Feynman

1. Introduction

The method of differentiation under the integral sign concerns integrals depending on a parameter, such as $\int_0^1 x^2 e^{-tx}\,\mathrm{d}x$. Here t is the extra parameter. (Since x is the variable of integration, x is *not* a parameter.) In general, we might write such an integral as

$$\int_a^b f(x,t)\,\mathrm{d}x, \tag{1.1}$$

where $f(x,t)$ is a function of two variables like $f(x,t) = x^2 e^{-tx}$.

Example 1.1. Let $f(x,t) = (x+t)^2$. Then

$$\int_0^1 f(x,t)\,\mathrm{d}x = \int_0^1 (x+t)^2\,\mathrm{d}x.$$

An anti-derivative of $(x+t)^2$ with respect to x is $\frac{1}{3}(x+t)^3$, so

$$\begin{aligned}\int_0^1 (x+t)^2\,\mathrm{d}x &= \left.\frac{(x+t)^3}{3}\right|_{x=0}^{x=1} \\ &= \frac{(1+t)^3 - t^3}{3} \\ &= \frac{1}{3} + t + t^2.\end{aligned}$$

This answer is a function of t, which makes sense since the integrand depends on t. We integrate over x and are left with something that depends only on t, not x.

Since an integral like $\int_a^b f(x,t)\,\mathrm{d}x$ is a function of t, we can ask about its t-derivative, assuming that $f(x,t)$ is sufficiently nicely behaved. The rule is: the t-derivative of the integral of $f(x,t)$ is the integral of the t-derivative of $f(x,t)$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_a^b f(x,t)\,\mathrm{d}x = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} f(x,t)\,\mathrm{d}x. \tag{1.2}$$

This procedure is called differentiation under the integral sign. Since you are used to thinking mostly about functions with one variable, not two, keep in mind that (1.2) involves integrals and derivatives with respect to *separate* variables: integration is with respect to x and differentiation is with respect to t.

Example 1.2. We saw in Example 1.1 that $\int_0^1 (x+t)^2 \, \mathrm{d}x = 1/3 + t + t^2$, whose t-derivative is $1 + 2t$. According to (1.2), we can also compute the t-derivative of the integral like this:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (x+t)^2 \, \mathrm{d}x &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} (x+t)^2 \, \mathrm{d}x \\ &= \int_0^1 2(x+t) \, \mathrm{d}x \\ &= \int_0^1 (2x + 2t) \, \mathrm{d}x \\ &= x^2 + 2tx \Big|_{x=0}^{x=1} \\ &= 1 + 2t. \end{aligned}$$

The answers agree.

2. Euler's factorial integral in a new light

For integers $n \geq 0$, we have already seen Euler's integral formula

$$\int_0^\infty x^n e^{-x} \, \mathrm{d}x = n! \tag{2.1}$$

by repeated integration by parts starting from the formula

$$\int_0^\infty e^{-x} \, \mathrm{d}x = 1 \tag{2.2}$$

when $n = 0$. Now we are going to derive (2.1) by repeated differentiation from (2.2) after introducing a parameter t into (2.2).

For any $t > 0$, let $x = tu$. Then $\mathrm{d}x = t \, \mathrm{d}u$ and (2.2) becomes

$$\int_0^\infty t e^{-tu} \, \mathrm{d}u = 1.$$

Dividing by t and writing u as x (why is this not a problem?), we get

$$\int_0^\infty e^{-tx} \, \mathrm{d}x = \frac{1}{t}. \tag{2.3}$$

This is a parametric form of (2.2), where both sides are now functions of t. We need $t > 0$ in order that e^{-tx} is integrable over the region $x \geq 0$.

Now we bring in differentiation under the integral sign. Differentiate both sides of (2.3) with respect to t, using (1.2) to treat the left side. We obtain

$$\int_0^\infty -x e^{-tx} \, \mathrm{d}x = -\frac{1}{t^2},$$

so

$$\int_0^\infty x e^{-tx} \, \mathrm{d}x = \frac{1}{t^2}. \tag{2.4}$$

Differentiate both sides of (2.4) with respect to t, again using (1.2) to handle the left side. We get

$$\int_0^\infty -x^2 e^{-tx}\,\mathrm{d}x = -\frac{2}{t^3}.$$

Taking out the sign on both sides,

$$\int_0^\infty x^2 e^{-tx}\,\mathrm{d}x = \frac{2}{t^3}. \tag{2.5}$$

If we continue to differentiate each new equation with respect to t a few more times, we obtain

$$\int_0^\infty x^3 e^{-tx}\,\mathrm{d}x = \frac{6}{t^4},$$

$$\int_0^\infty x^4 e^{-tx}\,\mathrm{d}x = \frac{24}{t^5},$$

and

$$\int_0^\infty x^5 e^{-tx}\,\mathrm{d}x = \frac{120}{t^6}.$$

Do you see the pattern? It is

$$\int_0^\infty x^n e^{-tx}\,\mathrm{d}x = \frac{n!}{t^{n+1}}. \tag{2.6}$$

We have used the presence of the extra variable t to get these equations by repeatedly applying $\mathrm{d}/\mathrm{d}t$. Now specialize t to 1 in (2.6). We obtain

$$\int_0^\infty x^n e^{-x}\,\mathrm{d}x = n!,$$

which is our old friend (2.1). Voilá!

The idea that made this work is introducing a second parameter t, using calculus on t, and then setting t to a particular value so it disappears from the final formula. In other words, sometimes *to solve a problem it is useful to solve a more general problem.* Compare (2.1) to (2.6).

3. A damped sine integral

We are going to compute another definite integral, but not one you have done before in another way:

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2}. \tag{3.1}$$

This is important in signal processing and Fourier analysis. Since $(\sin x)/x$ is even, an equivalent formula over the whole real line is $\int_{-\infty}^{\infty} \frac{\sin x}{x}\,\mathrm{d}x = \pi$.

We will work not with $(\sin x)/x$, but with $f(x,t) = e^{-tx}(\sin x)/x$, where $t \geq 0$. Note $f(x,0) = (\sin x)/x$. Set

$$g(t) = \int_0^\infty e^{-tx}\frac{\sin x}{x}\,\mathrm{d}x$$

for $t \geq 0$. Our goal is to show $g(0) = \pi/2$, and we are going to get this by studying $g'(t)$ for variable t.

Using differentiation under the integral sign, for $t > 0$ we have

$$\begin{aligned} g'(t) &= \int_0^\infty -xe^{-tx}\frac{\sin x}{x}\,\mathrm{d}x \\ &= -\int_0^\infty e^{-tx}(\sin x)\,\mathrm{d}x. \end{aligned}$$

The integrand $e^{-tx}\sin x$, as a function of x, is something you have seen earlier in the course. Using integration by parts,

$$\int e^{ax}\sin x\,\mathrm{d}x = \frac{(a\sin x - \cos x)}{1+a^2}e^{ax}.$$

Applying this with $a = -t$ and turning the indefinite integral into a definite integral,

$$g'(t) = \frac{(t\sin x + \cos x)}{1+t^2}e^{-tx}\Bigg|_{x=0}^{x=\infty}.$$

As $x \to \infty$, $t\sin x + \cos x$ oscillates a lot, but in a bounded way (since $\sin x$ and $\cos x$ are bounded functions), while the term e^{-tx} decays exponentially to 0 since $t > 0$. So the value at $x = \infty$ is 0. We are left with the negative of the value at $x = 0$, giving

$$g'(t) = -\frac{1}{1+t^2}.$$

We *know* an explicit antiderivative of $1/(1+t^2)$, namely $\arctan t$. Since $g(t)$ has the same t-derivative as $-\arctan t$ for $t > 0$, they differ by a constant: $g(t) = -\arctan t + C$ for $t > 0$. Let's write this out explicitly:

$$\int_0^\infty e^{-tx}\frac{\sin x}{x}\,\mathrm{d}x = -\arctan t + C. \tag{3.2}$$

Notice we obtained (3.2) by seeing both sides have the same t-derivative, *not* by actually finding an antiderivative of $e^{-tx}(\sin x)/x$.

To pin down C in (3.2), let $t \to \infty$ in (3.2). The integrand on the left goes to 0, so the integral on the left vanishes. Since $\arctan t \to \pi/2$ as $t \to \infty$ we get

$$0 = -\frac{\pi}{2} + C,$$

so $C = \pi/2$. Feeding this back into (3.2),

$$\int_0^\infty e^{-tx}\frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2} - \arctan t. \tag{3.3}$$

Now let $t \to 0^+$ in (3.3). Since $e^{-tx} \to 1$ and $\arctan t \to 0$, we obtain

$$\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x = \frac{\pi}{2},$$

so we're done!

Notice again the convenience introduced by a second parameter t. After doing calculus with the second parameter, we let it go this time to a boundary value ($t = 0$) to remove it and solve our problem.

4. The Gaussian integral

The improper integral formula

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi} \tag{4.1}$$

is fundamental to probability theory and Fourier analysis (and number theory!). The function $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is called a Gaussian, and (4.1) says the integral of the Gaussian over the whole real line is 1.

The physicist Lord Kelvin (after whom the absolute temperature scale is named) once wrote (4.1) on the board in a class and said "A mathematician is one to whom that [pointing at the formula] is as obvious as twice two makes four is to you." Our derivation of (4.1) will not make it seem terribly obvious, alas. If you take further courses you may learn more natural derivations of (4.1) so that the result really does become obvious. For now, just try to follow the argument here step-by-step.

We are going to aim not at (4.1), but an equivalent formula over the range $x \geq 0$ (after replacing x with $\sqrt{2}x$):

$$\int_{0}^{\infty} e^{-x^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2}. \tag{4.2}$$

Make sure you see why (4.1) and (4.2) are equivalent before proceeding.

For $t > 0$, consider (!) the functions

$$A(t) = \left(\int_0^t e^{-x^2}\,\mathrm{d}x\right)^2, \quad B(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x.$$

Notice $A(t)$ has t as an upper variable of integration, while $B(t)$ has t inside the integrand. If I denotes the integral on the left side of (4.2), then $A(\infty) = I^2$.

We are going to compare $A'(t)$ and $B'(t)$. First we compute $A'(t)$. By the chain rule and the fundamental theorem of calculus,

$$A'(t) = 2\int_0^t e^{-x^2}\,\mathrm{d}x \cdot \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e^{-x^2}\,\mathrm{d}x = 2\int_0^t e^{-x^2}\,\mathrm{d}x \cdot e^{-t^2}.$$

To calculate $B'(t)$ we use differentiation under the integral sign:

$$\begin{aligned} B'(t) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{e^{-t^2(1+x^2)}}{1+x^2}\right)\mathrm{d}x \\ &= \int_0^1 -2te^{-t^2(1+x^2)}\,\mathrm{d}x \\ &= -2e^{-t^2}\int_0^1 te^{-t^2x^2}\,\mathrm{d}x \\ &= -2e^{-t^2}\int_0^t e^{-u^2}\,\mathrm{d}u \quad (u = tx,\ \mathrm{d}u = t\,\mathrm{d}x). \end{aligned}$$

This is the same as $A'(t)$ except for an overall sign. Thus $A'(t) = -B'(t)$ for all $t > 0$, so there is a constant C such that

$$A(t) = -B(t) + C \tag{4.3}$$

for all $t > 0$.

To find C, we let $t \to 0^+$ in (4.3). The left side tends to $(\int_0^0 e^{-x^2}\,\mathrm{d}x)^2 = 0$ while the right side tends to $-\int_0^1 \mathrm{d}x/(1+x^2) + C = -\pi/4 + C$. Thus $C = \pi/4$, so (4.3) becomes

$$\left(\int_0^t e^{-x^2}\,\mathrm{d}x\right)^2 = \frac{\pi}{4} - \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}\,\mathrm{d}x \tag{4.4}$$

for all $t > 0$. We already gained information by letting $t \to 0^+$. Now we look in the other direction: let $t \to \infty$. The integrand on the right side of (4.4) goes to 0, so the integral becomes 0 and (4.4) turns into

$$\left(\int_0^\infty e^{-x^2}\,\mathrm{d}x\right)^2 = \frac{\pi}{4}.$$

Taking (positive) square roots gives (4.2).

5. Higher moments of the Gaussian

For every integer $n \geq 0$ we want to compute a formula for

$$\int_{-\infty}^{\infty} x^n e^{-x^2}\,\mathrm{d}x. \tag{5.1}$$

(Integrals of the type $\int x^n f(x)\,\mathrm{d}x$ for $n = 0, 1, 2, \ldots$ are called the *moments* of $f(x)$, so (5.1) is the n-th moment of the Gaussian, more or less.) When n is odd, (5.1) vanishes since $x^n e^{-x^2}$ is an odd function. What if $n = 0, 2, 4, \ldots$ is even?

The first case, $n = 0$, is essentially the Gaussian integral (4.2) we just calculated:

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}. \tag{5.2}$$

(The right side is twice the value of (4.2) since the integration is carried out over all real x, not just $x \geq 0$.) To get formulas for (5.1) when $n \neq 0$, we follow the same strategy as our treatment of the factorial integral in Section 2: stick a t into the exponent of e^{-x^2} and then differentiate repeatedly with respect to t.

For $t > 0$, replacing x with $\sqrt{t}x$ in (5.2) gives

$$\int_{-\infty}^{\infty} e^{-tx^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{\sqrt{t}}. \tag{5.3}$$

Differentiate both sides of (5.3) with respect to t, using differentiation under the integral sign on the left:

$$\int_{-\infty}^{\infty} -x^2 e^{-tx^2}\,\mathrm{d}x = -\frac{\sqrt{\pi}}{2t^{3/2}},$$

so

$$\int_{-\infty}^{\infty} x^2 e^{-tx^2}\,\mathrm{d}x = \frac{\sqrt{\pi}}{2t^{3/2}}. \tag{5.4}$$

Differentiate both sides of (5.4) with respect to t. After removing a common minus sign on both sides, we get

$$\int_{-\infty}^{\infty} x^4 e^{-tx^2}\,\mathrm{d}x = \frac{3\sqrt{\pi}}{4t^{5/2}}. \tag{5.5}$$

Differentiating both sides of (5.5) with respect to t a few more times, we get

$$\int_{-\infty}^{\infty} x^6 e^{-tx^2}\,\mathrm{d}x = \frac{3\cdot 5\sqrt{\pi}}{8t^{7/2}},$$

$$\int_{-\infty}^{\infty} x^8 e^{-tx^2}\,\mathrm{d}x = \frac{3\cdot 5\cdot 7\sqrt{\pi}}{16t^{9/2}},$$

and

$$\int_{-\infty}^{\infty} x^{10} e^{-tx^2}\,\mathrm{d}x = \frac{3\cdot 5\cdot 7\cdot 9\sqrt{\pi}}{32t^{11/2}}.$$

Quite generally, when n is even

$$\int_{-\infty}^{\infty} x^n e^{-tx^2}\,\mathrm{d}x = \frac{1\cdot 3\cdot 5\cdots (n-1)}{(2t)^{n/2}}\sqrt{\frac{\pi}{t}},$$

where the numerator is the product of the positive odd integers from 1 to $n-1$ (understood to be the empty product 1 when $n=0$).

In particular, taking $t=1$ we have computed (5.1):

$$\int_{-\infty}^{\infty} x^n e^{-x^2}\,\mathrm{d}x = \frac{1\cdot 3\cdot 5\cdots (n-1)\sqrt{\pi}}{2^{n/2}}.$$

As an application of (5.4), we now compute $(\frac{1}{2})! := \int_0^\infty x^{1/2}e^{-x}\,\mathrm{d}x$. Set $u = x^{1/2}$, so $x = u^2$ and $\mathrm{d}x = 2u\,\mathrm{d}u$. Then

$$\begin{aligned}\int_0^{\infty} x^{1/2}e^{-x}\,\mathrm{d}x &= \int_0^{\infty} ue^{-u^2}(2u)\,\mathrm{d}u \\ &= 2\int_0^{\infty} u^2 e^{-u^2}\,\mathrm{d}u.\end{aligned}$$

From (5.4) at $t=1$, $\int_0^\infty x^{1/2}e^{-x}\,\mathrm{d}x = 2\sqrt{\pi}/2 = \sqrt{\pi}$, so $(\frac{1}{2})! = \sqrt{\pi}$.

6. A cosine transform of the Gaussian

We are going to use differentiation under the integral sign to compute

$$\int_{-\infty}^{\infty} \cos(tx)e^{-x^2/2}\,\mathrm{d}x.$$

Here we are including t in the integral from the beginning. (The corresponding integral with $\sin(tx)$ in place of $\cos(tx)$ is zero since $\sin(tx)e^{-x^2/2}$ is an odd function of x.)

Call the desired integral $I(t)$. We calculate $I(t)$ by looking at its t-derivative:

$$I'(t) = \int_{-\infty}^{\infty} -x\sin(tx)e^{-x^2/2}\,\mathrm{d}x. \tag{6.1}$$

This looks *good* from the viewpoint of integration by parts since $-xe^{-x^2/2}$ is the derivative of $e^{-x^2/2}$. So we apply integration by parts to (6.1):

$$u = \sin(tx), \quad \mathrm{d}v = -xe^{-x^2}\,\mathrm{d}x$$

and

$$\mathrm{d}u = t\cos(tx)\,\mathrm{d}x, \quad v = e^{-x^2/2}.$$

Then

$$\begin{aligned} I'(t) &= \int_{-\infty}^{\infty} u \, \mathrm{d}v \\ &= uv\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} v \, \mathrm{d}u \\ &= \frac{\sin(tx)}{e^{x^2/2}}\Big|_{x=-\infty}^{x=\infty} - t\int_{-\infty}^{\infty} \cos(tx)e^{-x^2/2} \, \mathrm{d}x \\ &= \frac{\sin(tx)}{e^{x^2/2}}\Big|_{x=-\infty}^{x=\infty} - tI(t). \end{aligned}$$

As $x \to \pm\infty$, $e^{x^2/2}$ blows up while $\sin(tx)$ stays bounded, so $\sin(tx)/e^{x^2/2}$ goes to 0 in both limits. Therefore

$$I'(t) = -tI(t).$$

We *know* the solutions to this differential equation: constant multiples of $e^{-t^2/2}$. So $I(t) = Ce^{-t^2/2}$ for some constant C:

$$\int_{-\infty}^{\infty} \cos(tx)e^{-x^2/2} \, \mathrm{d}x = Ce^{-t^2/2}.$$

To find C, set $t = 0$. The left side is $\int_{-\infty}^{\infty} e^{-x^2/2} \, \mathrm{d}x$, which we computed to be $\sqrt{2\pi}$ in Section 4. The right side is C. Thus $C = \sqrt{2\pi}$, so we are done:

$$\int_{-\infty}^{\infty} \cos(tx)e^{-x^2/2} \, \mathrm{d}x = \sqrt{2\pi}e^{-t^2/2}.$$

Amazing!

7. Nasty logs, part I

Consider the following integral over a finite interval:

$$\int_0^1 \frac{x^t - 1}{\log x} \, \mathrm{d}x.$$

Since $1/\log x \to 0$ as $x \to 0^+$, the integrand vanishes at $x = 0$. As $x \to 1^-$, $(x^t - 1)/\log x \to 0$. Therefore the integrand makes sense as a continuous function on $[0, 1]$, so it is not an improper integral.

The t-derivative of this integral is

$$\int_0^1 \frac{x^t \log x}{\log x} \, \mathrm{d}x = \int_0^1 x^t \, \mathrm{d}x = \frac{1}{1+t},$$

which we recognize as the t-derivative of $\log(1 + t)$. Therefore

$$\int_0^1 \frac{x^t - 1}{\log x} \, \mathrm{d}x = \log(1 + t) + C.$$

To find C, set $t = 0$. The integral and the log term vanish, so $C = 0$. Thus

$$\int_0^1 \frac{x^t - 1}{\log x} \, \mathrm{d}x = \log(1 + t).$$

For example,

$$\int_0^1 \frac{x - 1}{\log x} \, \mathrm{d}x = \log 2.$$

Notice we computed this definite integral *without* computing an anti-derivative of $(x-1)/\log x$.

8. Nasty logs, part II

We now consider the integral

$$I(t) = \int_2^\infty \frac{dx}{x^t \log x}$$

where $t > 1$. The integral converges by comparison with $\int_2^\infty dx/x^t$. We know that "at $t = 1$" the integral diverges to ∞:

$$\begin{aligned}
\int_2^\infty \frac{dx}{x \log x} &= \lim_{b\to\infty} \int_2^b \frac{dx}{x\log x} \\
&= \lim_{b\to\infty} \log\log x \Big|_2^b \\
&= \lim_{b\to\infty} \log\log b - \log\log 2 \\
&= \infty.
\end{aligned}$$

So we expect that as $t \to 1^+$, $I(t)$ should blow up. But *how* does it blow up? By analyzing $I'(t)$ and then integrating back, we are going to show $I(t)$ behaves essentially like $-\log(t-1)$ as $t \to 1^+$.

Using differentiation under the integral sign,

$$\begin{aligned}
I'(t) &= \int_2^\infty \frac{d}{dt}(x^{-t}) \frac{dx}{\log x} \\
&= \int_2^\infty x^{-t}(-\log x)\frac{dx}{\log x} \\
&= -\int_2^\infty \frac{dx}{x^t} \\
&= -\frac{x^{-t+1}}{-t+1}\Big|_{x=2}^{x=\infty} \\
&= \frac{2^{1-t}}{1-t}.
\end{aligned}$$

We want to bound this derivative from above and below when $t > 1$. Then we will integrate to get bounds on the size of $I(t)$.

For $t > 1$, the difference $1 - t$ is negative, so $2^{1-t} < 1$. Dividing both sides by $1 - t$, which is negative, reverses the sense of the inequality and gives

$$\frac{2^{1-t}}{1-t} > \frac{1}{1-t}.$$

This is a lower bound on $I'(t)$. To get an upper bound on $I'(t)$, we want to use a lower bound on 2^{1-t}. For this purpose we use a tangent line calculation. The function 2^x has the tangent line $y = (\log 2)x + 1$ at $x = 0$ and the graph of $y = 2^x$ is everywhere above this tangent line, so

$$2^x \geq (\log 2)x + 1$$

for all x. Taking $x = 1 - t$,

$$2^{1-t} \geq (\log 2)(1-t) + 1. \tag{8.1}$$

When $t > 1$, $1 - t$ is negative, so dividing (8.1) by $1 - t$ reverses the sense of the inequality:

$$\frac{2^{1-t}}{t-1} \leq \log 2 + \frac{1}{1-t}.$$

This is an upper bound on $I'(t)$. Combining both bounds,

$$\frac{1}{1-t} < I'(t) \leq \log 2 + \frac{1}{1-t} \tag{8.2}$$

for all $t > 1$.

We are concerned with the behavior of $I(t)$ as $t \to 1^+$. Let's integrate (8.2) from a to 2, where $1 < a < 2$:

$$\int_a^2 \frac{\mathrm{d}t}{1-t} < \int_a^2 I'(t)\,\mathrm{d}t \leq \int_a^2 \left(\log 2 + \frac{1}{1-t}\right) \mathrm{d}t.$$

Using the Fundamental Theorem of Calculus,

$$-\log(t-1)\Big|_a^2 < I(t)\Big|_a^2 \leq ((\log 2)t - \log(t-1))\Big|_a^2,$$

so

$$\log(a-1) < I(2) - I(a) \leq (\log 2)(2-a) + \log(a-1).$$

Manipulating to get inequalities on $I(a)$, we have

$$(\log 2)(a-2) - \log(a-1) + I(2) \leq I(a) < -\log(a-1) + I(2)$$

Since $a - 2 > -1$ for $1 < a < 2$, $(\log 2)(a-2)$ is greater than $-\log 2$. This gives the bounds

$$-\log(a-1) + I(2) - \log 2 \leq I(a) < -\log(a-1) + I(2)$$

Writing a as t, we get

$$-\log(t-1) + I(2) - \log 2 \leq I(t) < -\log(t-1) + I(2),$$

so $I(t)$ is a bounded distance from $-\log(t-1)$ when $1 < t < 2$. In particular, $I(t) \to \infty$ as $t \to 1^+$.

9. Smoothly dividing by t

Let $h(t)$ be an infinitely differentiable function for all real t such that $h(0) = 0$. The ratio $h(t)/t$ makes sense for $t \neq 0$, but it also can be given a reasonable meaning at $t = 0$: from the very definition of the derivative, when $t \to 0$ we have

$$\frac{h(t)}{t} = \frac{h(t) - h(0)}{t - 0} \to h'(0).$$

Therefore the function

$$r(t) = \begin{cases} h(t)/t, & \text{if } t \neq 0, \\ h'(0), & \text{if } t = 0 \end{cases}$$

is continuous for all t. We can see immediately from the definition of $r(t)$ that it is better than continuous when $t \neq 0$: it is infinitely differentiable when $t \neq 0$. The question we want to address is this: is $r(t)$ infinitely differentiable at $t = 0$ too?

If $h(t)$ has a power series representation around $t = 0$, then it is easy to show that $r(t)$ is infinitely differentiable at $t = 0$ by working with the series for $h(t)$. Indeed, write

$$h(t) = c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

for all small t. Here $c_1 = h'(0)$, $c_2 = h''(0)/2!$ and so on. For small $t \neq 0$, we divide by t and get

$$r(t) = c_1 + c_2 t + c_3 t^3 + \cdots, \tag{9.1}$$

which is a power series representation for $r(t)$ for all small $t \neq 0$. The value of the right side of (9.1) at $t = 0$ is $c_1 = h'(0)$, which is also the defined value of $r(0)$, so (9.1) is valid for all small x (including $t = 0$). Therefore $r(t)$ has a power series representation around 0 (it's just the power series for $h(t)$ at 0 divided by t). Since functions with power series representations around a point are infinitely differentiable at the point, $r(t)$ is infinitely differentiable at $t = 0$.

However, this is an *incomplete* answer to our question about the infinite differentiability of $r(t)$ at $t = 0$ because we know by the key example of e^{-1/t^2} (at $t = 0$) that a function can be infinitely differentiable at a point *without* having a power series representation at the point. How are we going to show $r(t) = h(t)/t$ is infinitely differentiable at $t = 0$ if we don't have a power series to help us out? Maybe there's actually a counterexample?

The way out is to write $h(t)$ in a very clever way using differentiation under the integral sign. Start with

$$h(t) = \int_0^t h'(u)\,\mathrm{d}u.$$

(This is correct since $h(0) = 0$.) For $t \neq 0$, introduce the change of variables $u = tx$, so $\mathrm{d}u = t\,\mathrm{d}x$. At the boundary, if $u = 0$ then $x = 0$. If $u = t$ then $x = 1$ (we can divide the equation $t = tx$ by t because $t \neq 0$). Therefore

$$h(t) = \int_0^1 h'(tx)t\,\mathrm{d}x = t\int_0^1 h'(tx)\,\mathrm{d}x.$$

Dividing by t when $t \neq 0$, we get

$$r(t) = \frac{h(t)}{t} = \int_0^1 h'(tx)\,\mathrm{d}x.$$

The left and right sides don't have any t in the denominator. Are they equal at $t = 0$ too? The left side at $t = 0$ is $r(0) = h'(0)$. The right side is $\int_0^1 h'(0)\,\mathrm{d}x = h'(0)$ too, so

$$r(t) = \int_0^1 h'(tx)\,\mathrm{d}x \tag{9.2}$$

for all t, including $t = 0$. This is a formula for $h(t)/t$ where there is no longer a t being divided!

Now we're set to use differentiation under the integral sign. The way we have set things up here, we want to differentiate with respect to t; the integration variable on the right is x. We can use differentiation under the integral sign on (9.2) when the integrand is differentiable. Since the integrand is infinitely differentiable, $r(t)$ is infinitely differentiable!

Explicitly,

$$r'(t) = \int_0^1 vh''(tx)\,\mathrm{d}x$$

and

$$r''(t) = \int_0^1 vh''(tx)\,\mathrm{d}x$$

and more generally

$$r^{(k)}(t) = \int_0^1 x^k h^{(k+1)}(tx)\,\mathrm{d}x.$$

In particular, $r^{(k)}(0) = \int_0^1 x^k h^{(k+1)}(0)\,\mathrm{d}x = \frac{h^{(k+1)}(0)}{k+1}$.

10. A counterexample

We have seen many examples where differentiation under the integral sign can be carried out with interesting results, but we have not actually stated conditions under which (1.2) is valid. The following example shows that *some* hypothesis is needed beyond just the fact that the integrals on both sides of (1.2) exist.

For any real numbers x and t, let

$$f(x,t) = \begin{cases} \dfrac{xt^3}{(x^2+t^2)^2}, & \text{if } x \neq 0 \text{ or } t \neq 0, \\ 0, & \text{if } x = 0 \text{ and } t = 0. \end{cases}$$

Let

$$F(t) = \int_0^1 f(x,t)\,\mathrm{d}x.$$

For instance, $F(0) = \int_0^1 f(x,0)\,\mathrm{d}x = \int_0^1 0\,\mathrm{d}x = 0$. When $t \neq 0$,

$$\begin{aligned} F(t) &= \int_0^1 \frac{xt^3}{(x^2+t^2)^2}\,\mathrm{d}x \\ &= \int_{t^2}^{1+t^2} \frac{t^3}{2u^2}\,\mathrm{d}u \quad (\text{where } u = x^2+t^2) \\ &= -\frac{t^3}{2u}\bigg|_{u=t^2}^{u=1+t^2} \\ &= -\frac{t^3}{2(1+t^2)} + \frac{t^3}{2t^2} \\ &= \frac{t}{2(1+t^2)}. \end{aligned}$$

This formula also works at $t = 0$, so $F(t) = t/(2(1+t^2))$ for all t. Therefore $F(t)$ is differentiable and

$$F'(t) = \frac{1-t^2}{2(1+t^2)^2}$$

for all t. In particular, $F'(0) = \frac{1}{2}$.

Now we compute $\frac{\mathrm{d}}{\mathrm{d}t} f(x,t)$ and then $\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(x,t)\,\mathrm{d}x$. Since $f(0,t) = 0$ for all t, $f(0,t)$ is differentiable in t and $\frac{\mathrm{d}}{\mathrm{d}t} f(0,t) = 0$. For $x \neq 0$, $f(x,t)$ is differentiable in t and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} f(x,t) &= \frac{(x^2+t^2)^2(3xt^2) - xt^3 \cdot 2(x^2+t^2)2t}{(x^2+t^2)^4} \\ &= \frac{xt^2(x^2+t^2)(3(x^2+t^2)-4t^2)}{(x^2+t^2)^4} \\ &= \frac{xt^2(3x^2-t^2)}{(x^2+t^2)^3}. \end{aligned}$$

Combining both cases ($x = 0$ and $x \neq 0$),

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x,t) = \begin{cases} \frac{xt^2(3x^2-t^2)}{(x^2+t^2)^3}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases} \tag{10.1}$$

In particular $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(x,t) = 0$. Therefore at $t = 0$ the left side of (1.2) is $F'(0) = 1/2$ and the right side of (1.2) is $\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} f(x,t)\,\mathrm{d}x = 0$. The two sides are unequal!

The problem in this example is that $\frac{\mathrm{d}}{\mathrm{d}t} f(x,t)$ is not a continuous function of (x,t). Indeed, the denominator in the formula in (10.1) is $(x^2+t^2)^3$, which has a problem near $(0,0)$. Specifically, while this derivative vanishes at $(0,0)$, it we let $(x,t) \to (0,0)$ along the line $x = t$, then here $\frac{\mathrm{d}}{\mathrm{d}t} f(x,t)$ has the value $1/4x$, which does not tend to 0 as $(x,t) \to (0,0)$.

Theorem 10.1. *The two sides of* (1.2) *both exist and are equal at a point $t = t_0$ provided the following two conditions hold:*

- *$f(x,t)$ and $\frac{\mathrm{d}}{\mathrm{d}t} f(x,t)\,\mathrm{d}x$ are continuous for x in the range of integration and t in an interval around t_0,*
- *there are upper bounds $|f(x,t)| \le A(x)$ and $|\frac{\mathrm{d}}{\mathrm{d}t} f(x,t)| \le B(x)$ independent of t such that $\int_a^b A(x)\,\mathrm{d}x$ and $\int_a^b B(x)\,\mathrm{d}x$ converge.*

In Table 1 we include choices for $A(x)$ and $B(x)$ for each of the functions we have treated. Since the calculation of a derivative at a point only depends on an interval around the point, we have replaced a t-range such as $t > 0$ with $t \ge c > 0$ in some cases to obtain choices for $A(x)$ and $B(x)$.

Section	$f(x,t)$	x range	t range	$A(x)$	$B(x)$
2	$x^n e^{-tx}$	$[0,\infty)$	$t \ge c > 0$	$x^n e^{-cx}$	$x^{n+1} e^{-cx}$
3	$e^{-tx} \frac{\sin x}{x}$	$[0,\infty)$	$t \ge c > 0$	e^{-cx}	e^{-cx}
4	$\frac{e^{-t^2(1+x^2)}}{1+x^2}$	$[0,1]$	$0 \le t \le c$	$\frac{1}{1+x^2}$	$2c$
5	$x^n e^{-tx^2}$	$\mathbf{R}$	$t \ge c > 0$	$x^n e^{-cx^2}$	$x^{n+2} e^{-cx^2}$
6	$\cos(tx) e^{-x^2/2}$	$\mathbf{R}$	$\mathbf{R}$	$e^{-x^2/2}$	$\|x\| e^{-x^2/2}$
7	$\frac{x^t - 1}{\log x}$	$(0,1]$	$t \ge c > 0$	??	1
8	$\frac{1}{x^t logx}$	$[2,\infty)$	$t \ge c > 1$	$\frac{1}{x^2 \log x}$	$\frac{1}{x^c}$
9	$x^k h^{(k+1)}(tx)$	$[0,1]$	$\mathbf{R}$	??	??

TABLE 1. Summary

11. Exercises

1. Starting with the indefinite integral formulas

$$\int \cos(tx)\,\mathrm{d}x = \frac{\sin(tx)}{t}, \qquad \int \sin(tx)\,\mathrm{d}x = -\frac{\cos(tx)}{t},$$

compute formulas for $\int x^n \sin x\,\mathrm{d}x$ and $\int x^n \cos x\,\mathrm{d}x$ for $n = 2, 3, 4$.