THE FUNDAMENTAL THEOREM OF ALGEBRA VIA MULTIVARIABLE CALCULUS

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This is a proof of the fundamental theorem of algebra which is due to Gauss, in 1816. It is based on [1, pp. 680–682]. The proof is accessible, in principle, to anyone who has had multivariable calculus and knows about complex numbers. The main idea will be to compute a certain double integral and then compute the integral in the other order.

We take for granted the following result from calculus, which is a special case of Fubini's theorem.

Lemma 1. *Let $[a,b] \times [c,d] \subset \mathbf{R}^2$ be a rectangle, and f be a continuous function on this rectangle, with real values. Then*

$$\int_c^d \left(\int_a^b f(x,y)\,\mathrm{d}x \right) \mathrm{d}y = \int_a^b \left(\int_c^d f(x,y)\,\mathrm{d}y \right) \mathrm{d}x.$$

Theorem 1. *Every nonconstant polynomial in $\mathbf{C}[z]$ has a complex root.*

Proof. We are going to prove the contrapositive: if

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0$$

has no complex roots, then $f(z)$ is a (nonzero) constant. Here n is the degree of f.

Write $z = re^{i\theta}$, so $z^j = r^j \cos(j\theta) + ir^j \sin(j\theta)$. Therefore the decomposition of $f(z)$ into real and imaginary parts is

$$f(z) = P(r,\theta) + iQ(r,\theta),$$

where

$$P(r,\theta) = r^n \cos(n\theta) + \cdots + \mathrm{Re}(c_0), \quad Q(r,\theta) = r^n \sin(n\theta) + \cdots + \mathrm{Im}(c_0).$$

Both P and Q are polynomials in r of degree n,with constant terms independent of θ. (In particular, a trigonometric function of θ appears in P and Q only when multiplied by positive powers of r, so the ambiguity in the definition of θ at the origin does not matter: $P(0,\theta) = \mathrm{Re}(c_0)$ and $Q(0,\theta) = \mathrm{Im}(c_0)$ for all θ.) From this observation about the constant terms,

$$\left.\frac{\partial P}{\partial \theta}\right|_{r=0} = 0, \quad \left.\frac{\partial Q}{\partial \theta}\right|_{r=0} = 0.$$

Clearly P and Q are 2π-periodic, as are $\partial P/\partial r$ and $\partial Q/\partial r$.

To say f has no complex roots is the same as saying P and Q are not simultaneously 0 anywhere. Writing $f(z) = P + iQ$ in polar coordinates, we contemplate its angular component, $\arctan(Q/P)$.

Set

$$U = \arctan\left(\frac{Q}{P}\right).$$

From the derivative formula for the arctangent,

$$\frac{\partial U}{\partial r} = \frac{1}{(1+Q/P)^2} \cdot \frac{PQ_r - QP_r}{P^2} = \frac{PQ_r - QP_r}{P^2+Q^2} \tag{1}$$

and similarly

$$\frac{\partial U}{\partial \theta} = \frac{PQ_\theta - QP_\theta}{P^2+Q^2}, \tag{2}$$

where we adopt the subscript notation for partial derivatives.

The formulas on the right side of (1) and (2) make sense everywhere, since $P^2+Q^2 \neq 0$ for all (r,θ). However, there is something mysterious about the definition of the function U as a "value" of arctangent. Usually one defines the function $\arctan x$ to take values in $(-\pi/2, \pi/2)$, with values $\pm\pi/2$ at $\pm\infty$ from the asymptotics visible on the graph of $y = \arctan x$. But this kind of definition is bad to use in the definition of U, because we can imagine wandering through a point in the plane where $P = 0$ (and thus where Q/P is "infinite") such that the *continuous* variation in arctan may demand that the function U increase above the value $\pi/2$.

This is the same kind of problem one meets when trying to define logarithms of complex numbers, but we can circumvent the trouble with U by taking the right sides of (1) and (2) as our basic functions (*i.e.*, the partial derivative notation on the left sides is purely suggestive, at least for readers who only know up to multivariable calculus). For example, the formula

$$\frac{\partial}{\partial \theta}\left(\frac{\partial U}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial U}{\partial \theta}\right) \tag{3}$$

can be checked by a direct calculation of the θ-partial of the right side of (1) and the r-partial of the right side of (2). We do not appeal to the theorem on equality of mixed partials. The common "iterated" derivative in (3) has the form $H(r,\theta)/(P^2+Q^2)^2$ for an explicit continuous function H.

Applying Lemma 1 to the rectangle $[0,R]\times[0,2\pi]$ (with $R>0$) in the (r,θ) plane, and integrating the function in (3), we have

$$\int_0^R \left(\int_0^{2\pi} \frac{\partial}{\partial r}\frac{\partial U}{\partial \theta}\,\mathrm{d}\theta\right)\mathrm{d}r = \int_0^{2\pi}\left(\int_0^R \frac{\partial}{\partial r}\frac{\partial U}{\partial \theta}\,\mathrm{d}r\right)\mathrm{d}\theta. \tag{4}$$

On the left side, we evaluate the inner integral by appealing to (3):

$$\begin{aligned}\int_0^{2\pi} \frac{\partial}{\partial r}\frac{\partial U}{\partial \theta}\,\mathrm{d}\theta &= \int_0^{2\pi} \frac{\partial}{\partial \theta}\frac{\partial U}{\partial r}\,\mathrm{d}\theta \\ &= \left.\frac{\partial U}{\partial r}\right|_{\theta=0}^{\theta=2\pi} \\ &= 0,\end{aligned}$$

since $\partial U/\partial r$ is 2π-periodic. Therefore the left side of (4) is 0 for all $R>0$.

Now we compute the right side of (4). The inside integral is

$$\int_0^R \frac{\partial}{\partial r}\frac{\partial U}{\partial \theta}\,\mathrm{d}r = \left.\frac{\partial U}{\partial \theta}\right|_{r=0}^{r=R} = \left.\frac{\partial U}{\partial \theta}\right|_{r=R}$$

since the θ-partials of P and Q vanish at $r=0$. Having separately computed the two sides of (4), we conclude that for $R>0$,

$$\left.\frac{\partial U}{\partial \theta}\right|_{r=R} = 0.$$

Now we are going to compute the value of this partial derivative by the explicit formula (2). First we look at the numerator. Because

$$P_\theta = -nr^n\sin(n\theta)+\cdots, \quad Q_\theta = nr^n\cos(n\theta)+\cdots,$$

where $\cdots$ represents terms of lower degree in r,

$$PQ_\theta - QP_\theta = nr^{2n}\cos^2(n\theta) + \cdots + nr^{2n}\sin(n\theta) + \cdots = nr^{2n} + \cdots .$$

Similarly, the denominator in (2) is $r^{2n} + \cdots$, so

$$\frac{\partial U}{\partial \theta} = \frac{nr^{2n} + \cdots}{r^{2n} + \cdots}.$$

The lower degree terms have θ appearing only inside trigonometric (and thus bounded) functions, hence

$$\lim_{R\to\infty} \left.\frac{\partial U}{\partial \theta}\right|_{r=R} = n$$

uniformly in θ. That lets us evaluate the right side of (4), and obtain

$$\int_0^{2\pi} \left(\int_0^R \frac{\partial}{\partial r}\frac{\partial U}{\partial \theta}\,\mathrm{d}r \right) \mathrm{d}\theta = \int_0^{2\pi} \left.\frac{\partial U}{\partial \theta}\right|_{r=R} \mathrm{d}\theta \to 2\pi n$$

as $R \to \infty$. On the other hand, we already computed from (4) that the integral is 0. Therefore $n = 0$. $\square$

To summarize the argument, we showed that if $f(z)$ has degree n and $f(z) \neq 0$ for all complex z, then $U = \arctan(\operatorname{Im}(f)/\operatorname{Re}(f))$ satisfies

$$0 = \int_0^R \left(\int_0^{2\pi} \frac{\partial}{\partial r}\frac{\partial U}{\partial \theta}\,\mathrm{d}\theta \right) \mathrm{d}r = \int_0^{2\pi} \left(\int_0^R \frac{\partial}{\partial r}\frac{\partial U}{\partial \theta}\,\mathrm{d}r \right) \mathrm{d}\theta \to 2\pi n$$

as $R \to \infty$, so $n = 0$, *i.e.*, f is a constant.

Since $\arctan(Q/P)$ is essentially the argument of $P + iQ = f$, this proof of Gauss is a precursor of the proof of the Fundamental Theorem of Algebra based on winding numbers, which involves the computation of $(1/2\pi i)\int_C (f'(z)/f(z))\,\mathrm{d}z$.

References

[1] G. M. Fikhtengoltz,"Course of Differential and Integral Calculus, Vol. 2," (Russian) 7th ed., Nauka, Moscow (1969).