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Cosmology

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Contents

1 Midterm recap

In this section we review the most important concepts for the midterm.

1.1 Evolution equations

The Friedmann equations are,

$$H^2(t) = \frac{8}{3}\rho \tag{1}$$
$$\frac{\ddot{a}}{a} = -\frac{4}{3}(\rho + 3P) \tag{2}$$

, where often we take $P = \omega\rho$. We also couple these with the conservation of energy,

$$a^{-3}\frac{\mathrm{d}(\rho a^3)}{\mathrm{d}t} = -3HP \tag{3}$$

1.2 Components of the Universe

- matter: $\rho_m = \rho_{m,0}a^{-3}$

- radiation: $\rho_r = \rho_{r,0}a^{-4}$

$\rho_\Lambda = \frac{\Lambda}{8\pi G} = -P_\Lambda$ And the general expression for a perfect fluid is $P = \omega\rho$, and in the CPL parametrisation for DE: $\omega(a) = \omega_0 + \omega_a(1 - a)$

1.3 Distances in the Universe

There are numerous distance indications for cosmology, the primary ones are,

- Comoving distance:

$$\chi(a) = \int_{a(t)}^{1} \frac{\mathrm{d}a'}{a'^2 H(a')}, \quad \text{or} \quad \chi(t) = \int_{t}^{t_0} \frac{\mathrm{d}t'}{a(t')}, \tag{4}$$

- Redshift:

$$1 + z \equiv \frac{\lambda_{\text{obs}}}{\lambda_e} = \frac{1}{a(t_e)} \tag{5}$$

- Luminosity distance:

$$F \equiv \frac{L}{4\pi d_L^2(z)} \text{ then } \quad d_L(z) = \chi(z)(1 + z) \tag{6}$$

- Angular diameter distance: $\mathrm{d}_A \equiv \frac{L}{\delta\theta}$ then $\quad d_A(z) = \frac{\chi(z)}{(1+z)}$

1.4 Key concepts in thermal history

The three main key concepts in thermal history are

- Redshift of equality $\Omega_{m,0}a_{\text{eq}}^{-3} = \Omega_{\gamma,0}a_{\text{eq}}^{-4}$
- Energy density in relativistic species scales as $\rho_r \propto a^{-4}$

2 Thermal History

2.1 Planck time

We define the Planck time as $\lambda_B = \frac{h}{M_P c}, r_s = \frac{2GM_P}{c^2}$, with the Planck mass defined as $M_p = \sqrt{\frac{hc}{2G}} = 10^{19}$GeV, we can also define $t_p = \frac{2GM_P}{c^3}10^{-43}s$ (btw this is where we define GR and QM breaks down time after the big bang singularity).

2.2 Interactions

To talk about interaction, we start by defining the cross section

$$\Gamma = \frac{v}{l} = n\sigma v \tag{7}$$

and the relevant scale in the evolution of all matters in cosmology is namely comparing Γ^{-1} and H^{-1}

$$\begin{cases} \Gamma^{-1} \sim t_\Gamma \ll t_H \sim H^{-1} \implies \text{freeze out (or equilibrium)} \\ \Gamma^{-1} \sim t_\Gamma \gg t_H \sim H^{-1} \implies \text{still in interaction.} \\ \Gamma^{-1} \sim t_\Gamma \sim t_H \sim H^{-1} \implies \text{decouple from the thermal bath (particle soup).} \end{cases} \tag{8}$$

Very intuitive so I won't comment here, yet this very comment is a comment, quite contradictory indeed. For particles that are leftover, i.e. after decoupling, we coined them as relic abundance over that matter. Also, one often use temperature (of photon) as a way to indicate decoupling of a certain species in the x-axis, as in a sense it indicates time flow in the early universe.

2.3 Effective DOFs

We now introduce degrees of freedom (dof) in the early cosmology picture (not really "early" but hey ho). For coupled species of same T_γ

$$\rho = \sum_i \rho_i = \sum_B \frac{\pi^2}{30} g_i T^4 + \sum_F \frac{\pi^2}{30}\frac{7}{8} g_i T^4 = \frac{\pi^2}{30} g_\star^{th} T^4 \tag{9}$$

where,

$$g_\star^{th} = \sum_B g_i + \sum_F \frac{7}{8} g_i \tag{10}$$

And for decoupled species.

$$\rho = \sum_i \rho_i = \sum_B \frac{\pi^2}{30} g_i T_i^4 + \sum_F \frac{\pi^2}{30}\frac{7}{8} g_i T_i^4 = \frac{\pi^2}{30} g_\star^{dec} T^4 \tag{11}$$

where,

$$g_\star^{dec} = \sum_B g_i \left(\frac{T_i}{T}\right)^4 + \sum_F \frac{7}{8} g_i \left(\frac{T_i}{T}\right)^4 \tag{12}$$

Comment: We can be fairly reckless about computing (10), but not for (12).

2.4 Temperature evolution

When a system is in equilibrium, it remains at a fixed entropy, which asks for a good place to start the analysis of temperature evolution, the corresponding formula to start with is none other than the *second law of thermodynamics*.

Comment: Just so we are on the same page, we are trying to obtain the entropy density s in our following calculation, which serves numerous use in this whole temperature analysis.

$$T\mathrm{d}S = \mathrm{d}U + P\mathrm{d}V. \tag{13}$$

, where we can further define $U = \rho V$, thus

$$V\mathrm{d}\rho + \rho\mathrm{d}V = T\mathrm{d}S - P\mathrm{d}V \tag{14}$$

We then apply the energy conservation equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -3H(\rho + P). \tag{15}$$

and that $V \propto a^3$, so we can turn H into V,

Checkpoint 1:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}t}(\rho + P) \tag{16}$$

And now we finally substitute (13) into (16) and differentiating it with respect to time:

$$-\frac{\mathrm{d}V}{\mathrm{d}t}(\rho + P) + \rho\frac{\mathrm{d}V}{\mathrm{d}t} = T\frac{\mathrm{d}S}{\mathrm{d}t} - P\frac{\mathrm{d}V}{\mathrm{d}t}. \tag{17}$$

which implies,

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0. \tag{18}$$

This means the universe we assume here is a closed system which conserves entropy. And for convienience we introudce the entropy density $s \equiv S/V$, and change of variables into (17) and get.

$$V\mathrm{d}\rho + \rho\mathrm{d}V = T(s\mathrm{d}V + V\mathrm{d}s) - P\mathrm{d}V. \tag{19}$$

And rewriting further and also realizing the following important discussion.

2.4.1 Important remark on Extensive and Intensive properties of thermodynamics quatities

Rewriting and we get

$$\mathrm{d}\rho + T\mathrm{d}s = (Ts - \rho - P)\frac{\mathrm{d}V}{V}. \tag{20}$$

And note that the density and the entropy density are intensive properties on the L.H.S., this means that they do not depend on the extent of the system, and in this case depends only on the temperature, so L.H.S. $\propto \mathrm{d}T$ and R.H.S. $\propto \mathrm{d}V$. So we get something like $C_1\mathrm{d}T + C_2\mathrm{d}V = 0$, and for this to conserve then both $C_i = 0$, which means we get,

$$s = \frac{\rho + P}{T} \tag{21}$$

$$S = s(T)a^3 = \left[\frac{\rho(T) + P(T)}{T}\right]a^3 = \text{constant}. \tag{22}$$

where a is the scale factor.

Now, we know that for radiation matter, $\rho \propto T^4 \implies s \propto T^3$, then we can once again rewrite the entropy density as multiple coupled and decoupled species,

$$s(T) = \sum_i \frac{2\pi^2}{45}(g^{\rm dec}_{\star,S}(T) + g^{\rm th}_{\star,S}(T))T^3. \tag{23}$$

Now for species that are coupled, we see that $g^{\rm th}_{\star,S} = g^{\rm th}_{\star}$. And that for decoupled species, again need to consider the temperature difference from radiation, becomes

$$g^{\rm dec}_{\star,S} = \sum_{i=b} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_{i=f} g_i \left(\frac{T_i}{T}\right)^3. \tag{24}$$

2.5 Consequences of entropy conservation

- **Evolution of the temperature with the scale factor:** Since $g_{\star,S}T^3a^3 = \text{constant} \implies T \propto g_{\star,S}^{-1/3}a^{-1} \propto g_{\star,S}^{-1/3}(1+z)$. , this is only valid as long as we are far from mass thresholds where creation and annihilation cease to be efficient.

- **Time evolution of the temperature of the Universe:** In radiation-dominated Universe, the scale factor evolves as $a \propto t^{1/2} \implies T \propto g_{\star,S}^{-1/3}t^{-1/2}$.

2.6 Neutrino decoupling

Neutrino decoupling plays a big role in the overall temperature of the Universe, in this regime, we have the weak interactions that keeps neutrinios in equilibrium with other species, namely:

$$\nu_e + \bar{\nu}_e \leftrightarrow e^- + e^+ \tag{25}$$

$$e^- + \bar{\nu}_e \leftrightarrow e^+ + \nu_e \tag{26}$$

At roughly $T_{\rm dec} \approx 1{\rm MeV}$, these interactions becomes inefficient to keep in equilibrium, and the decoupled neutrinos now preserve a Fermi-Dirac distribution and their number density dilutes with the expansion of the universe as,

$$n_\nu \propto \int \mathrm{d}^3 p \, \frac{1}{\exp{(p/T_\nu)} + 1}. \tag{27}$$

Note that we also know the momentum of radiation scales as $p \propto a^{-1}$, and make a change of variable on (27) with $q \equiv pa$ and we see that

$$n_\nu \propto a^{-3} \int \mathrm{d}^3 q \, \frac{3}{\exp{(\frac{q}{aT_\nu} + 1)}}. \tag{28}$$

, but we know that $n_\nu \propto a^{-3}$ to obey particle number conservation after decoupling, so $T_\nu \propto a^{-1}$ to keep (28) true.

2.7 Positron electron annihilation

Now we move on to positron electron annihilation period, The relevant reaction is (and the temperature $T \approx m_e$),

$$e^- + e^+ \leftrightarrow \gamma + \gamma \tag{29}$$

Here we wish to calculate the ratio $T_{\gamma,0}/T_{\nu,0}$, and our starting point is getting the entropy density s, to do so we first compute the dof, which is simply $g_\star = 2 + 7/8(2 + 2 + 3 + 3)$, this is because we have the following particles present at that moment,

$$e^-, e^+, \gamma, \nu_e, \bar{\nu}_e, \nu_\tau, \bar{\nu}_\tau, \nu_\mu, \bar{\nu}_\mu \tag{30}$$

The entropy density is thus,

$$s(a_1) = \frac{43\pi^2}{90} T_1^3. \tag{31}$$

And after annihilation, we have the following particles only,

$$\gamma, \nu_e, \bar{\nu}_e, \nu_\tau, \bar{\nu}_\tau, \nu_\mu, \bar{\nu}_\mu. \tag{32}$$

Thus dof has been reduced to 2 for the photons and 6 for the neutrinos, but both at different temperatures, so that we have,

$$s(a_2) = \frac{2\pi^2}{45}\Big(2T_\gamma^3(a_2) + \frac{7}{8}6T_\nu^3(a_2)\Big). \tag{33}$$

Next, apply conservation of entropy, i.e. $s(a_1)a_1^3 = s(a_2)a_2^3$, and at the same time, we use $a_1(T_1 = T_\gamma) = a_2 T_\nu(a_2)$ So putting everything together, we shall get,

$$\frac{T_\gamma(a_2)}{T_\nu(a_2)} = \left(\frac{11}{4}\right)^{1/3} \tag{34}$$

And knowing CMB temperature, we can tell

$$T_{\nu,0} = 1.95\text{K} < T_{\gamma,0} = 2.73\text{K}. \tag{35}$$

3 Beyond equilibrium

In order to probe further information about evolution of the Universe, we should not limit ourselves to equilibrium analysis, and the main tool for us to go beyond this is the intensive study of Boltzmann equation, and the relevant processes for such are, (i) the formation of light elements during Big Bang nucleosynthesis (BBN); (ii) recombination of neutrinos and protons into neutral hydrogen; (iii) the production of dark matter.

3.1 Boltzmann equation

The boltzmann equation reads,

Figure 1: picture of the magestic Boltzmann

$$\frac{\partial f}{\partial t} = -\frac{\mathbf{p}}{m} \cdot \nabla_x f - m\mathbf{a} \cdot \nabla_p f, \tag{36}$$

Here we make the assumption that the system of interest is in the non-relativistic case and under the unperturbed FLRW metric.

And note that we can rewrite (117) in two cases,

$$\begin{cases} \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + x \cdot \nabla_x f = \mathbf{p} \cdot \nabla_x f, & \text{number conserved case} \\ \frac{\mathrm{d}f}{\mathrm{d}t} = \underbrace{C[f]}_{\text{collision terms}} & \text{number not conserved case} \end{cases} \tag{37}$$

For the 1st case, we shall get

$$\frac{\mathrm{d}n_1}{\mathrm{d}t} + 3Hn_1 = 0 \implies n_1 \propto a^{-3} \tag{38}$$

and for the 2nd we get

$$a^{-3}\frac{\mathrm{d}N_1}{\mathrm{d}t} = C_1[\{n_j\}]. \tag{39}$$

For example, we can take the collision terms as

$$a^{-3}\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\alpha n_1 n_2 + \beta n_3 n_4 \tag{40}$$

, where $\alpha = \langle\sigma v\rangle$, $\beta = \left(\frac{n_1 n_2}{n_3 n_4}\right)_{eq}\alpha$.

And now if we make a change of variable $N_i \equiv \frac{m_i}{s}(s \propto a^{-3}), \Gamma_\gamma \equiv m_2\langle\sigma v\rangle = n_2\alpha$, these are at the same time ansatz, we can express (**??**) as,

$$\frac{\mathrm{dln}(N_i)}{\mathrm{dln}(a)} = \frac{-\Gamma_1}{H}\left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq}\left(\frac{N_3 N_4}{N_1 N_2}\right)\right] \tag{41}$$

And now notice in the limit $H \ll \Gamma_1$, we have

$$-\cancelto{0}{\frac{H}{\Gamma_1}\frac{\mathrm{dln}(N_i)}{\mathrm{dln}(a)}} = \left[1 - \left(\frac{N_1 N_2}{N_3 N_4}\right)_{eq}\left(\frac{N_3 N_4}{N_1 N_2}\right)\right] \tag{42}$$

and if in the limit $\Gamma \ll H$, we have

$$\frac{\mathrm{dln}(N_i)}{\mathrm{dln}(a)} = 0 \tag{43}$$

, this implies N_i a constant, i.e. freeze-out of species N_i.

3.2 Dark matter

Now we look into the behaviour of Boltzmann equation for dark matter (assuming WIMP-like particle), which the relevant interaction should take the form

$$X + \bar{X} \to l + \bar{l} \tag{44}$$

and note that it should be much simpler as it we have the major assumptions (i) non-interacting; (ii) non-relativistic; (iii) these light particles $(l, \bar{l})$ are tightly coupled to the plasma of charged particles in the early Universe (which calls for *tightly-coupled approximation*); (iv) these light particles are in equilibrium at later stage, $n_l = n_l^{eq}$. We are also interested in predicting the relic-abundance for WIMP.

So using $N_X^{eq} \equiv n_X^{eq}/s$, the Boltzmann equation becomes

$$a^{-3}\frac{\mathrm{d}(n_x a^{-3})}{\mathrm{d}t} = -\langle\sigma v\rangle\Big(n_X^2 - (n_X^{(eq)})^2\Big) \tag{45}$$

To make our lives simpler, turns out we need to make the change of variable $Y \equiv n_X/T^3$. And also with the assumption $T \propto a^{-1}$, then

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = -\langle\sigma v\rangle T^3\Big(Y^2 - (Y_X^{(eq)})^2\Big) \tag{46}$$

And again make a change of variable $x \equiv M_x/T$ to get

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{M_x}{T}\right) = \frac{-1}{T}\frac{\mathrm{d}T}{\mathrm{d}t}x \tag{47}$$

, and remembering $T \propto a^{-1}$, we conclude with

$$\frac{\mathrm{d}x}{\mathrm{d}t} \approx Hx \tag{48}$$

The above equation is useful to remember. Now we plug all these in to the Boltzmann equation, it will reduce to the Riccati equation:

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -\frac{\lambda}{x^2}\Big(Y^2 - (Y^{(eq)})^2\Big) \tag{49}$$

, where

$$\lambda \equiv \frac{M_X^3\langle\sigma v\rangle}{H(M_X)} \tag{50}$$

To obtain a solution, we assume (otherwise numerical) the freeze-out occurs at some time $x = x_f$.

and the integration becomes

$$\int_{Y^f}^{Y^\infty} \frac{\mathrm{d}Y}{Y^2} = -\int_{x_f}^{\infty} \mathrm{d}\, x \frac{\lambda}{x^2} \tag{51}$$

$$\frac{1}{Y^\infty} - \frac{1}{Y^f} = \frac{\lambda}{x_f}. \tag{52}$$

And assuming $Y^f \gg Y^\infty$, we arrive at

$$Y^\infty \approx \frac{x_f}{\lambda}. \tag{53}$$

. The typical $x_f \sim 10$

4 Hydrogen recombination

After BBN (and photon epoch), we get the recombination period, which happens around $T > 1\text{eV}$. And the relevant processes are

$$e^- + p^+ \leftrightarrow H + \gamma \tag{54}$$

$$e^- + \gamma \leftrightarrow e^- + \gamma \tag{55}$$

And when electrons and protons combine to form neutral atoms, now the density of free electrons significantly drops, and the photons can now travel through freely without scattering via process (55). This is also where CMB photons are ”generated”.

Remarks: The term ”recombination” is a bit of misnomer as this is the first time that electrons and protons combine into neutral hydrogen.

4.1 free electron fraction

The main goal here is to predict the free electron fraction, $X_e \equiv n_e/n_b$, where n_b denotes the density of baryons. Here we are also assuming (i) non-relativistic limit, which implies

$$n_i^{eq} = g_i \left(\frac{m_i T}{2\pi}\right)^{3/2} \exp\left(\frac{\mu_i - m_i}{T}\right); \tag{56}$$

(ii) the relevant species are $i = \{e, p, H\}$

And note that the chemical potentials of the three species is $\mu_p + \mu_e = \mu_H$

With these, we can get the ratio as,

$$\frac{n_H^{eq}}{n_e^{eq} n_p^{eq}} = \frac{g_H}{g_e g_p} \left(\frac{m_H}{m_e m_p} \frac{2\pi}{T}\right)^{3/2} \exp\left(\frac{m_p + m_e - m_H}{T}\right) \tag{57}$$

, we can also further make assumptions

- $m_H \approx m_p$, but for only the prefactor,
- no net charge, $n_e = n_p$
- the binding energy of hydrogen is $B_H = m_p + m_e - m_H = 13.6eV$
- number of dof: $g_p = g_e = 2, g_H = 4$

Then everything will be simplified as

$$\left(\frac{n_H}{n_e^2}\right)_{eq} = \left(\frac{2\pi}{m_e T}\right)^{3/2} e^{B_H/T} \tag{58}$$

We can now use this equation to keep track of the free electron fraction. To do so, we first write the number density of baryon as:

$$n_b = \eta n_\gamma = \eta \frac{2\zeta(3)}{\pi^2} T^3 \tag{59}$$

, where η is the baryon-to-photon ratio that is determined observationally.

and the with the following assumption

$$n_b \approx n_p + n_H = n_e + n_H \tag{60}$$

Putting the (60) and (59) together, we get

$$\left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \frac{n_H}{n_e^2} n_b \tag{61}$$

and finally plug (61) to (58), we obtain the *Saha equation*,

$$\left(\frac{1 - X_e}{X_e^2}\right)_{eq} = \eta \frac{2\zeta(3)}{\pi^2} \left(\frac{2\pi T}{m_e}\right)^{3/2} e^{B_H/T} \tag{62}$$

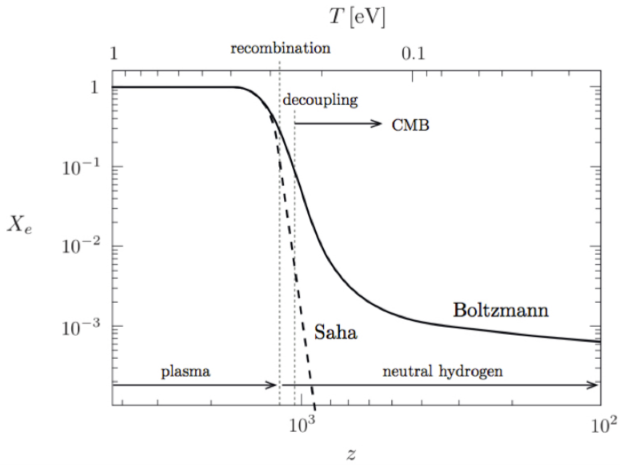

Figure 2: The electron abundance as a function of redshift in the early Universe. The Saha equation under predicts the abundance left after recombination and decoupling. To get the right abundance, one must solve the Boltzmann equation numerically.

4.1.1 Quick redshift estimation

For early universe era (up to recombination), we can abuse the relation $T \propto a^{-1}$, and say if we were to predict redshift of recombination, then we do

$$T_{rec} = T_0(1 + z_{rec}) \tag{63}$$

, and note that T_0 is the temperature of CMB (2.71K). To further obtain the age, then we do $a = (t/t_0)^{2/3}$, with $t_0 as age of the universe.$

4.2 Photon decoupling

To obtain the time for photon decoupling, which happens right after recombination, exactly when

$$\Gamma_\gamma(T_{rec}) \approx H(T_{dec}) \tag{64}$$

, and that the we know the cross-section for Thomson scattering to be $\sigma_T \approx 2 \times 10^{-3}\text{MeV}^{-2}$. Thus we can estimate with

$$\Gamma_\gamma \approx n_e \sigma_T \approx H(T_{dec}) \tag{65}$$

, which yields a temperature of $T_{dec} \approx 0.27eV$

5 Jesus christ is BBN (Not Julio)

Let us now go back in time (bruh moment), at BBN period $T_{BBN} \approx 1\text{MeV}$, and here we will focus on how helium is synthesised.

5.1 Neutron abundance in equlibrium

The relevant process here is the weak interaction

$$p^+ + e^- \leftrightarrow n + \nu_e \tag{66}$$

, you may ask why this is important. And the answer is because Helium requires neutron to form simply.

The ratio of proton over neutron in equilibrium is thus

$$\frac{n_p^{eq}}{n_n^{eq}} \approx \left(\frac{m_n}{m_p}\right)^{3/2} e^{(m_n-m_p)/T} \tag{67}$$

, the assumption we made for the approximation here is that the neutrinos have zero chemical potential, and of course the non-relativistic limit setting to get to (67), we can do the same approximation we did for the prefactor on the R.H.S., setting $m_n = m_p$ and further reducing it to the value such that we can compute, the ratio readily. Though if we consider the limits for T, we will begin to notice that around $T < T_{BBN} \approx$ 1MeV, the neutrons starts to be scarce, i.e. the ration becomes away (much larger) than 1. Thus this calls for a look into different processes that crossing our fingers it will produce more neutrons.

5.2 The deuterium bottleneck

The relevant process here is

$$n + p^+ \leftrightarrow D + \gamma \tag{68}$$

, And with Deuterium, we can further produce Helium via the following two processes,

$$D + p^+ \leftrightarrow^3 \text{He} + \gamma \tag{69}$$

$$D +^3 \text{He} \leftrightarrow^4 \text{He} + p^+ \tag{70}$$

So the corresponding process, under the assumptions on the chemical potential $\mu_\gamma = 0 \implies \mu_n + \mu_p = \mu_D$, we obtain

$$\frac{n_D^{eq}}{n_n^{eq} n_p^{eq}} = \frac{g_D}{g_n g_p}\left(\frac{m_D}{m_e m_p}\frac{2\pi}{T}\right)^{3/2} \exp\left(\frac{m_n + m_p - m_D}{T}\right) \tag{71}$$

, the dof in this expression are naturally $g_D = 3$, and $g_n = g_p = 2$, and the mass assumptions are $m_D \approx 2m_n \approx 2m_p$. Then we can reduce (**??**) to

$$\frac{n_D^{eq}}{n_p^{eq}} = \frac{3}{4} n_n^{eq}\left(\frac{4\pi}{m_p T}\right)^{3/2} \exp\left(\frac{B_D}{T}\right) \tag{72}$$

Again, we can pack the previous equation (72) to a nicer compact form by first introucing,

$$n_n \approx n_b = \eta n_\gamma = \eta \frac{2\zeta(3)}{\pi^2} T^3, \tag{73}$$

and shove this to (72) and obtain

$$\frac{n_D^{eq}}{n_p^{eq}} \approx \eta\left(\frac{T}{m_p}\right)^{3/2} \exp\left(\frac{B_D}{T}\right) \tag{74}$$

Ideally we want the ratio to be large, which is in the limit $T \ll B_D$, thus it has to wait for the Universe to cool down sufficiently, and thus called the "deuterium bottleneck", as this heavily decides the rate of formation of helium.

5.2.1 Neutrons out of equilibrium

To obtain the neutron fraction, we define

$$X_n \equiv \frac{n_n}{n_n n_p}, \tag{75}$$

and then we can use the previous (67) to obtain (some steps omitted, primarily dividing both terms by n_p^{eq}),

$$X_n^{eq}(T) = \frac{e^{-(m_n - m_p)/T}}{1 + e^{-(m_n - m_p)/T}} \tag{76}$$

, we can take $T = T_{rec} \approx 0.8$MeV to obtain $X_n^{eq}(T_{dec}) = 0.17$

5.2.2 Neutron decay

After the freeze-out of neutron, neutron start to decay via weak interactions,

$$n \rightarrow p^+ + e^- + \bar{\nu}_e \tag{77}$$

And the decay goes exponentially as

$$X_n(t) = 0.17 e^{-t/\tau_n}, \tag{78}$$

with $\tau_n = 889.7$ s.

5.3 Helium fusion

Now we are in the position to tackle the production of helium in the early Universe, and note that the reaction paths we have gone through are the dominant processes that produces helium efficiently, for they are two-body interactions. So at $T_{nuc} \approx 0.06$MeV, we get what we wish for - Helium production!

And note that to obtain the time estimate, we can make use of scaling of temperature of the Universe with time as $T \propto t^{-1/2}$, and get

$$\frac{T}{1\text{ MeV}} \approx 1.5 g_\star^{-1/4} \left(\frac{1\text{ s}}{t} \right)^{1/2} \tag{79}$$

From (79), if we plug in what we have found earlier that $T_{nuc} \approx 0.06$ MeV, and we should get

$t_{nuc} \approx 330$ s. And if we were to use (78) with this time. We will get

$$X_n(t_{nuc}) = 0.17e^{-t_{nuc}/\tau_n} \approx \frac{1}{8} \tag{80}$$

And now notice the following three items

1. $n_{\mathrm{He}} = \frac{1}{2}n_n(t_{nuc})$, that each helium requires 2 neutron to form.
2. $n_H = n_p$, the abundance of hydrogen is just abundance of proton (indirect assumption of no net charge and number conservation).

So finally we get from

$$\frac{n_{\mathrm{He}}}{n_{\mathrm{H}}} = \frac{n_{\mathrm{He}}}{n_{\mathrm{p}}} \approx \frac{\frac{1}{2}X_n(t_{nuc})}{1 - X_n(t_{nuc})} \approx \frac{1}{2}X_n(t_{nuc}) \sim \frac{1}{16} \tag{81}$$

and abundance by mass is

$$\frac{4m_{\mathrm{He}}}{m_{\mathrm{H}}} = \frac{1}{4} \tag{82}$$

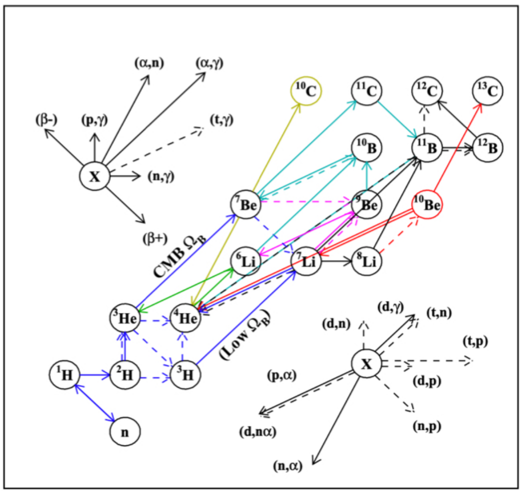

Figure 3: Chain of the *cursed* BBN reactions.

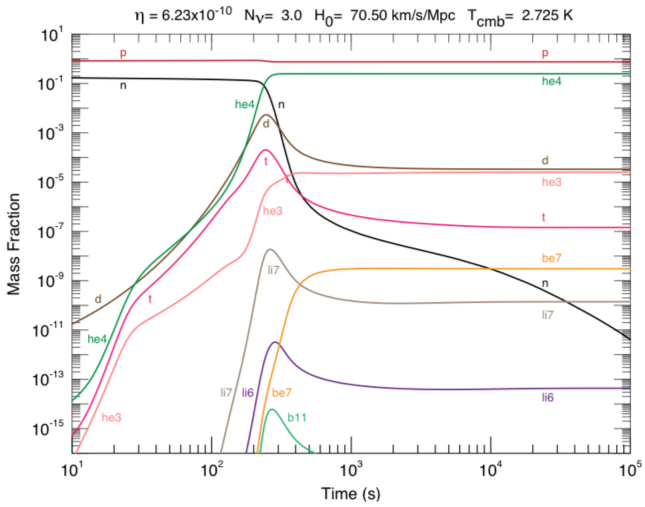

Figure 4: Predicted mass fraction by element from Big Bang nucleosynthesis chain of interactions.

5.4 Big review of Beyond Equilibrium in general

Note that for a given reaction, and if assumed non-relativistic limit.

$$A + B \leftrightarrow C + D \tag{83}$$

We can immediately obtain the equation below,

$$\frac{n_C^{eq}}{n_A^{eq} n_B^{eq}} = \frac{g_C}{g_A g_B}\left(\frac{m_C}{m_e m_p}\frac{2\pi}{T}\right)^{3/2} \exp\left(\frac{m_A + m_B - m_C}{T}\right) \tag{84}$$

, with the assumption that D is just photon, and next we compute for the dof and prefactor masses with given assumptions for the corresponding species.

6 The Boltzmann equation

In this section, we aim to recover the tiny **inhomogeneities** of the CMB at the scales of μK, which originates from the quantum fluctuations of the inflationary potential.

6.1 Boltzmann equation for perturbations

At decoupling $z \sim 1100$, we have the particles soup

$$\rho, e^{-}\nu, \gamma, \mathrm{DM} \tag{85}$$

The big picture of the system of (Boltzmann) equations we are solving is as follows,

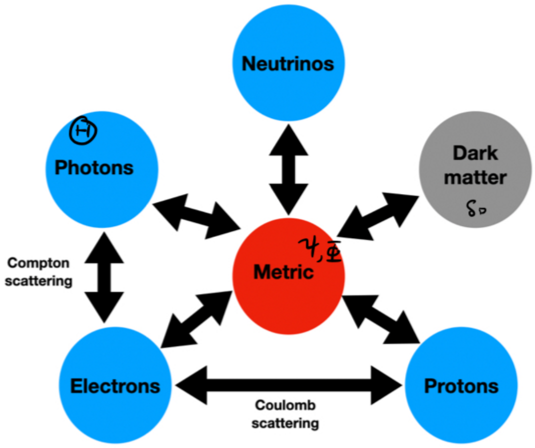

Figure 5: Species present in the Universe at the time of recombination and their interactions.

and we introduce the metric perturbations

$$g_{00} = -1 - 2\Psi(\mathbf{x}, t), \quad g_{0i} = 0, \quad g_{ij} = a^2\delta_{ij}(1 + 2\Phi(\mathbf{x}, \mathbf{t})) \tag{86}$$

.

It is also necessary to introduce the geodesic deviation to the system of equations we are trying to solve, as we have to solve for f_i in $g_{\mu\nu}$: $\frac{\mathrm{d}f_i}{\mathrm{d}t} = C[f_i, f_j]$. It is now instructive to gain more intuition about the aforementioned Boltzmann equation.

6.2 Intuition on Boltzmann equation through SHO

We can quickly gain some intuition on Boltzmann equation through SHO system, with the energy defined as the usual

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{87}$$

and note that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\rho}{m}, \qquad \frac{\mathrm{d}p}{\mathrm{d}t} = -kx \tag{88}$$

Given there is no collision term, we can then write the Boltzmann equation as,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} = 0 \tag{89}$$

$$\implies \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{p}{m}\frac{\partial f}{\partial x} - kx\frac{\partial f}{\partial p} = 0 \tag{90}$$

, where we have simply used (88) to obtain the final expression for (89).

In order to further obtain more results, we need initial condition of the particle, which we now impose they are at equilibrium.

Equilibrium: Now we suppose the system is in equilibrium, then we have $\partial f/\partial t = 0$. Then we will get

$$f(x, p) = f_{eq}(E[p, x]) \tag{91}$$

, this means the system only depends on the function of the energy of the particle, then we can replace the total derivative w.r.t E instead of t. The new expression becomes

$$\frac{p}{m}\frac{\partial f}{\partial x} - kx\frac{\partial f}{\partial p} = \frac{\mathrm{d}f}{\mathrm{d}E}\left[\frac{p}{m}\frac{\partial E}{\partial x} - kx\frac{\partial E}{\partial p}\right] = 0 \tag{92}$$

6.3 Photons, electrons, and metrics relations

In this section, we will probe the influence in interactions between electrons, photons and metrics to the Boltzmann equation. Note that the interaction goes as electrons ↔ photons ↔ themetric. Now the relevant interaction of photon and electrons are via Compton scattering.

$$e^- + \gamma \leftrightarrow e^- + \gamma \tag{93}$$

, whereby a photon encounters an electron and they exchange energy momentum.

Since we are trying to get something similar to that of (92), we need to find the momentum of the photon first, which can be done by

$$P^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \tag{94}$$

and we have the usual normalisation on the momentum in GR as

$$P^2 \equiv g_{\mu\nu} P^\mu P^\nu = -(1 + 2\Psi)(P^0)^2 + p^2 = 0. \tag{95}$$

So in the end we have the spatial and temporal part of the momentum separated, with

$$p^2 = g_{ij} P^i P^j, \quad p^i = p\hat{p}^i, \quad \delta_{ij}\hat{p}^i\hat{p}^j = 1 \tag{96}$$

After some equation merging and approximation, we will arrive at

$$P^i = p\hat{p}^i \frac{(1 - \Phi)}{a} \tag{97}$$

Packing everything into the Boltzmann equation

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \underbrace{\frac{\partial f}{\partial \hat{p}^i}\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}}_{\text{2nd order}} = C[f] \tag{98}$$

, where we can drop the 2nd order as we wish.

With this, we have obtained

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\frac{1}{\mathrm{d}t/\mathrm{d}\lambda} = \frac{P^i}{P^0} \approx \frac{\hat{p}^i}{a}(1+\Phi-\Psi) < \frac{\hat{p}^i}{a}, \tag{99}$$

note that with this expression, it means the photon climbs out of an over-dense region, and when $\Psi < 0$, and $\Phi > 0$, it slows down And the Boltzmann equation now reads

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\cdot\frac{\hat{p}^i}{a} + \frac{\partial f}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} \tag{100}$$

6.4 Implementing geodesic equation

After going through the calculations on previous sections, we are in the position to calculate the 3rd term of (100). We can begin with the time component of the geodesic equation,

$$\frac{\mathrm{d}P^0}{\mathrm{d}\lambda} = -\Gamma^0_{\alpha\beta}P^\alpha P^\beta, \tag{101}$$

And we can plug (97) to (101) and expand the total derivative of Ψ w.r.t time and obtain

$$\frac{\mathrm{d}p}{\mathrm{d}t} = p\left\{\frac{\partial\Psi}{\partial t} + \frac{\hat{p}^i}{a}\frac{\partial\Psi}{\partial x^i}\right\} - \Gamma^0_{\alpha\beta}\frac{P^\alpha P^\beta}{p}(1+2\Psi). \tag{102}$$

Further manipulation should get us

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^i}\cdot\frac{\hat{p}^i}{a} - p\frac{\partial f}{\partial p}\left[H + \frac{\partial\Phi}{\partial t} + \frac{\hat{p}^i}{a}\cdot\frac{\partial\Psi}{\partial x^i}\right] \tag{103}$$

6.5 Photon perturbation

Next, we shall focus on $\mathrm{d}f/\mathrm{d}t$, to do so, we assume the distribution function is to 0th-order a BE equilibrium function with $\mu = 0$.

Next, We define the photon temperature perturbation as

$$\Theta(\mathbf{x}, \hat{p}, t) = \frac{\delta T}{T}(\mathbf{x}, \hat{p}, t) \tag{104}$$

The distribution $f(\mathbf{x}, p, \hat{p}, t)$ now reads

$$f(\mathbf{x}, p, \hat{p}, t) = \left[\exp\left\{\frac{p}{T(t)[1 + \Theta(\mathbf{x}, \hat{p}, t)]}\right\} - 1\right]^{-1} \tag{105}$$

We then do 1st order expansion on this and get,

$$f(\mathbf{x}, p, \hat{p}, t) \approx f^{(0)} - p\frac{\partial f^{(0)}}{\partial p}\Theta. \tag{106}$$

At this point forward, we have to solve the Boltzmann equation order by order.

6.5.1 0th order

At 0th order, there are no collisions, and the equation reads

$$\frac{\mathrm{d}f^{(0)}}{\mathrm{d}t} = 0 \implies \frac{\mathrm{d}T}{T} = -\frac{\mathrm{d}a}{a} \tag{107}$$

, so we get $T \propto a^{-1}$.

6.5.2 1st order

At 1st order, we will get collision terms and the expression goes

$$\frac{\mathrm{d}f^{(1)}}{\mathrm{d}t} = -p\frac{\partial f^{(0)}}{\partial p}\left[\frac{\partial \Theta}{\partial t} + \frac{\hat{p}^i}{a} \cdot \frac{\partial \Theta}{\partial x^i} + \frac{\partial \Phi}{\partial t} + \frac{\hat{p}^i}{a} \cdot \frac{\partial \Psi}{\partial x^i}\right] = C[f(p)], \tag{108}$$

where the collision term here is the Compton scattering, which goes like

$$C[f(p)] = -p\frac{\partial f^{(0)}}{\partial p} n_e \sigma_T [\Theta_0 - \Theta(\hat{p}) + \hat{p} \cdot \mathbf{v}_b], \tag{109}$$

where

$$\Theta_0(\mathbf{x},t) \equiv \frac{1}{4\pi}\int \mathrm{d}\Omega'\,\Theta(\hat{p}',\mathbf{x},t). \tag{110}$$

Next we perform a change of variable $a\mathrm{d}\eta = \mathrm{d}t$, i.e. the conformal time, to get

$$\Theta' + \hat{p}\cdot\frac{\partial\Theta}{\partial x^i} + \Phi' + \hat{p}^i\cdot\frac{\partial\Psi}{\partial x^i} = n_e\sigma_T a[\Theta_0 - \Theta + \hat{p}\cdot\mathbf{v}_b], \tag{111}$$

where Θ' denotes derivative with respect to conformal time.

Next we perform *Fourier transform*, via

$$\Theta(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}e^{i\mathbf{k}\cdot\mathbf{x}}\tilde{\Theta}(\mathbf{k}), \tag{112}$$

which ultimately leads us to

$$\tilde{\Theta}' + ik\mu\tilde{\Theta} + ik\mu\tilde{\Psi} = -\tau'[\tilde{\Theta}_0 - \tilde{\Theta} + \mu\tilde{v}_b], \tag{113}$$

where

$$\mu \equiv \frac{\hat{p}\cdot\mathbf{k}}{k}, \tag{114}$$

defines the angle between wave vector and momentum of photon

Optical depth: Note that one can define a quantity called

$$\tau(\eta) = \int_\eta^{\eta_0}\mathrm{d}\eta'\,n_e\sigma_T a. \tag{115}$$

Important Remarks:

- Note that throughout the calculation, it seems like there is lack of the appearance of electron in the Boltzmann equation, but it is in fact encoded in the collision (Compton scattering) term.

- It is also sensible to not include a quantity for electrons since we are interested only in obtaining Θ for the CMB.

6.6 Dark matter

We now move on to describe the Boltzmann equation for dark matter, and notice that we need to start fresh with the Boltzmann equation once again because dark matter only interacts with metric, so naturally we need to call Boltzmann equation twice to get the whole system closed.

The momentum for DM can be obtained with similar method as before

$$g_{\mu\nu}P^{\mu}P^{\nu} = -m^2, \quad P^0 = E(1-\Psi), \quad P^i(p,\hat{p}^i) = p\hat{p}^i(1-\Phi)/a. \tag{116}$$

We can then write the Boltzmann equation as

$$\frac{\mathrm{d}f_{dm}}{\mathrm{d}t} = \frac{\partial f_{dm}}{\partial t} + \frac{\partial f_{dm}}{\partial x^i}\cdot\frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\partial f_{dm}}{\partial E}\frac{\mathrm{d}E}{\mathrm{d}t} + \frac{\partial f_{dm}}{\partial \hat{p}^i}\cdot\frac{\mathrm{d}\hat{p}^i}{\mathrm{d}t}. \tag{117}$$

Keeping the expression to 1st order in $\Phi, \Psi, p/E$, we get

$$\frac{\mathrm{d}f_{dm}}{\mathrm{d}t} = \frac{\partial f_{dm}}{\partial t} + \frac{\hat{p}_i}{a}\frac{p}{E}\frac{\partial f_{dm}}{\partial x^i} - \frac{\partial f_{dm}}{\partial E}\left[\frac{\mathrm{d}a/\mathrm{d}t}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial \Phi}{\partial t} + \frac{\hat{p}_i p}{a}\frac{\partial \Phi}{\partial x^i}\right] = 0 \tag{118}$$

Now, it is convenient to, similar to fluid dynamics, introduce moments (and replacing it with) to solve for the Boltzmann equation.

6.6.1 Moments in DM

The moments we introduced are the 0th order moment and 1st order moment respectively

$$n_{dm} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{dm} \tag{119}$$

$$v^i \equiv \frac{1}{n_{dm}}\int \frac{\mathrm{d}^3 p}{(2\pi)^3} f_{dm}\frac{p\hat{p}^i}{E}. \tag{120}$$

The key step here is then to integrate the Boltzmann equation by fourier integral, and substituting(119) and (120) into the equation. We also make a change of variable $\mathrm{d}E/\mathrm{d}p = p/E$ to get

$$\frac{\partial n_{dm}}{\partial t} + \frac{1}{a}\frac{\partial(n_{dm}v^i)}{\partial x^i} + 3\left[\frac{\mathrm{d}a/\mathrm{d}t}{a} + \frac{\partial\Phi}{\partial t}\right]n_{dm} = 0 \tag{121}$$

And now we perform the expansion $n_{dm} = n_{dm}^{(0)}(1 + \delta(\mathbf{x}, t)$, and keep track of the orders, we will get the following two equations: The 0th order gives

$$\frac{\partial n_{dm}^{(0)}}{\partial t} + 3\frac{\mathrm{d}a/\mathrm{d}t}{df}n_{dm}^{(0)} = 0 \tag{122}$$

And the 1st order gives

$$\frac{\partial n_{dm}^{(0)}}{\partial t} + \frac{1}{a}\frac{\partial v^i}{\partial x^i} + 3\frac{\partial\Phi}{\partial t} = 0 \tag{123}$$

Note that we also need to take the first moment of the Boltzmann equation (121), since there are 3 unknowns to solve,

$$\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{p}{E}\hat{p}^j\left\{\frac{\partial f_{dm}}{\partial t} + \frac{\hat{p}_i}{a}\frac{p}{E}\frac{\partial f_{dm}}{\partial x^i} - \frac{\partial f_{dm}}{\partial E}\left[\frac{\mathrm{d}a/\mathrm{d}t}{a}\frac{p^2}{E} + \frac{p^2}{E}\frac{\partial\Phi}{\partial t} + \frac{\hat{p}_i p}{a}\frac{\partial\Phi}{\partial x^i}\right]\right\} = 0 \tag{124}$$

And then with all these equation together we can get the first order equations in Fourier space as

$$\tilde{\delta}' + ik\tilde{v} + \tilde{\Phi}' = 0 \tag{125}$$

$$\tilde{v}' + \frac{a'}{a}\tilde{v} + ik\tilde{\Psi} = 0. \tag{126}$$

Note that to get to this point, we have used integration by parts, integrating over one of the momentum terms and assumed $n_{dm} \approx n_{dm}^{(0)}$.

6.7 Baryons and electrons and the metric

Finally we have to also implement baryons into our picture, and it is useful to only consider the density contrast of the baryons (electrons and protons).

$$\delta_b \equiv \frac{\rho_e - \rho_e^{(0)}}{\rho_e^{(0)}} = \frac{\rho_p - \rho_p^{(0)}}{\rho_p^{(0)}}. \tag{127}$$

. Now the assumption on them are that (i) they are tightly coupled via Coulomb scattering $\implies$ their velocities are the same statistically; (ii) they are non relativistic after recombination ($T \ll m_e$), their Boltzmann equation are identical to the DM one except for the collision terms, which are

$$\frac{\mathrm{d}f_e(\mathbf{x}, \mathbf{q}, \mathbf{t})}{\mathrm{d}t} = C_{ep} + C_{e\gamma} \tag{128}$$

$$\frac{\mathrm{d}f_p(\mathbf{x}, \mathbf{Q}, \mathbf{t})}{\mathrm{d}t} = C_{ep} + C_{p\gamma} \tag{129}$$

Now the Coulomb scattering between proton and electrons are neglected compared to baryons and photons due to negligible cross-section. And finally after similar calculation to DM, we arrive at

$$\tilde{\delta}_b' + ik\tilde{v}_b + 3\tilde{\Phi}' = 0 \tag{130}$$

$$\tilde{v}_b' + ik\tilde{\Psi} = \tau' \frac{4\rho_\gamma}{3\rho_b}[3i\tilde{\Theta}_1' + \tilde{v}_b] \tag{131}$$

7 Inhomogeneities

We start by defining cosmological distance (physical distance) as the distance that defines the casually connectedness among region of spaces in the Universe.

$$r_h = a(t) \int_0^t \frac{\mathrm{d}t'}{a(t')} = a(t) \int_0^t \frac{\mathrm{d}a'}{a'^2 H(a')}, \tag{132}$$

and at radiation domination era it scales as $a \propto t^{1/2}$, $d_h = 2t$, and matter domination era is $a \propto t^{2/3}$, $d_h = 3t$ We can also define the comoving horizon as

$$r_h(t) = \int_0^t \frac{\mathrm{d}a'}{a'^2 H(a')}, \tag{133}$$

and so the approximation goes $r_H = (aH)^{-1}$.

7.1 Gravitational collapse

It turns out that the gravitational potential grows over time and takes the form

$$\tilde{\Phi}(\mathbf{k}, a) = \tilde{\Phi}_p(\mathbf{k}) \times T(k) \times \frac{D_1(a)}{a}, \tag{134}$$

where T is the transfer function that tracks the evolution as modes enter the horizon and through the radiation-matter transition, and D is the growth function to describe the scale-independent growth at late time.

Note that from the Poisson equation, it is also possible to model the growth in terms of the density perturbation (of matter) as

$$\tilde{\Phi}(\mathbf{k}, a) = \frac{4\pi G \rho_m(a) a^2 \tilde{\delta}}{k^2}, \tag{135}$$

where ρ_m is the mean matter density at a given time. And thus the density perturbation takes the form

$$\tilde{\delta}(\mathbf{k}, a) = \frac{3}{5} \frac{k^2}{\Omega_m H_0^2} \tilde{\Phi}_p(\mathbf{k}) T(k) D_1(a). \tag{136}$$

7.2 Recap on Boltzmann

Now we can start to use the ingredients we have prepared in last chapter. We can take the Boltzmann equation for radiation and define the higher order multipoles of the radiation as

$$\Theta_l \equiv \frac{1}{(-i)^l}\int_{-1}^{1} \mathrm{d}\mu\, \frac{\mathcal{P}_l(\mu)}{2}\Theta(\mu). \tag{137}$$

Important Remarks:

- Note that to do all this multipole business, simply take the regular Boltzmann equation, and then multiply both terms by $\int_{-1}^{1}\mathrm{d}\mu$, and notice which terms (usually $\tilde{\Psi}$) are independent of μ, and then you can define some of the terms into, say Θ_0, Θ_1', etc.

With all these, we should get the multiple and Boltzmann equations as

$$\tilde{\Theta}'_{r,0} + k\tilde{\Theta}_{r,1} = -\tilde{\Phi}' \tag{138}$$

$$\tilde{\Theta}'_{r,1} - \frac{k}{3}\tilde{\Theta}_{r,0} = -\frac{k}{3}\tilde{\Phi} \tag{139}$$

$$\tilde{\delta}' + ik\tilde{v} + 3\tilde{\Phi}' = 0 \tag{140}$$

$$\tilde{v}' + \frac{a'}{a}\tilde{v} + ik\tilde{\Psi} = 0 \tag{141}$$

7.3 Perturbations on large scales

Before we go into the consideration of different horizon crossing modes, it is important to go through what this formalism is about.

We know that the density perturbation can be expressed in this standard form,

$$\delta_x(t,\mathbf{r}) = \frac{\delta\rho_x(t,\mathbf{r})}{\bar{\rho}_x(t)}. \tag{142}$$

However, it is better to Fourier expand this expression so that we can study in terms of modes, this is because each modes grows independently (which we get the orthogonality of Fourier modes for free mathematically). (142) then becomes,

$$\delta_x(t,\mathbf{k}) = \int \mathrm{d}^3\mathbf{r}\, e^{-\mathbf{k}\cdot\mathbf{r}} \delta_x(t,\mathbf{r}). \tag{143}$$

Also note that transformation is defined with respect to the comoving coordinates $\mathbf{r}$, thus k here is the comoving wavenumber, and the comoving wavelength is $\lambda = 2\pi/k$, and the *physical wavelength* is

$$(t) = \frac{2\pi a(t)}{k}. \tag{144}$$

In literature, we are often interested to not study individual modes, but the spectrum, which is the root-mean-square of all $\delta_x(t,\mathbf{k})$ for a given time and wavenumber, averaged over all directions. The spectrum is usually denoted as $\delta_x(t,k)$.

7.3.1 Horizon crossing $k\eta \ll 1$

We first consider the super-horizon scales, $k\eta \ll 1$. Then we can drop all terms with k in all (138). However, the equation we are left with does not close the system, so we also need to take the Poisson equation into account.

We then make careful consideration and assumption as well as the change of variable $y \equiv a/a_{eq}$

$$\frac{\mathrm{d}}{\mathrm{d}\eta} = \frac{\mathrm{d}y}{\mathrm{d}\eta}\frac{\mathrm{d}}{\mathrm{d}y} = aHy\frac{\mathrm{d}}{\mathrm{d}y}, \tag{145}$$

where $a' = a^2H$ from $(\mathrm{d}t = a\,\mathrm{d}\eta)$ is used.

After some horrible calculation, we should get

$$\text{Small y: } \Phi = \Phi(0) \tag{146}$$

$$\text{Large y: } \Phi = \frac{9}{10}\Phi(0) \tag{147}$$

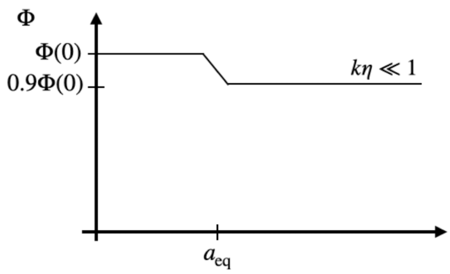

Figure 6: The potential on large scales as the Unvierse traverses matter-radiation equality.

7.3.2 Horizon crossing $k\eta \sim 1$

For this limit, if $k\eta \sim 1$, then

$$a_H \sim 0.03 \gg a_{eq}. \tag{148}$$

Now, under this assumption, i.e. $a_H \gg a_{eq}$, we can drop the photon terms as the modes of interest enter the horizon during matter domination. After resorting to the Einstein's equation and the previous systems of Fourier equations (**??**), we will eventually obtain

$$\left[\frac{2k^2}{3a^2H^2} + 3\right]\tilde{\Phi}' + \left[k^2 + \frac{9a^2H^2}{2}\right]\left(\frac{i\tilde{v}}{k} + \frac{2\tilde{\Phi}}{3aH}\right) = 0 \tag{149}$$

If we differentiate this once more and we will find that $\tilde{\Phi}$ is actually constant, which means the growth function $T(k) \approx 1 \implies \Phi(\mathbf{k}, \mathbf{a}) \approx \Phi_p(\mathbf{k})D_1(a)$.

7.4 Perturbation on small scales

Now for small scale modes, we are essentially describing them during radiation domination (that's where they first enter), and thus we only need to consider the radiation terms in the Boltzmann system, as the matter terms are suppressed by a factor ρ_{dm}/ρ_r. Again to solve this, we use (i) one of Einstein's equations, (ii) $H^2 = 8\pi G\rho_r/3$, (iii) $\eta \approx (aH)^{-1}$ during radiation era, (iv) and lots of good luck.

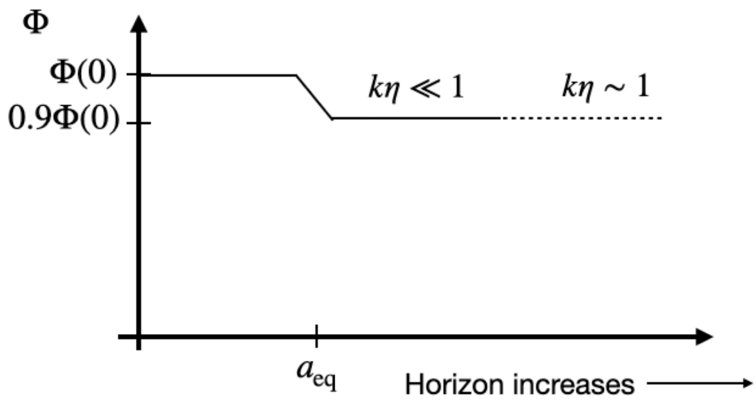

Figure 7: As modes enter the horizon during matter domination, the potential remains constant.

We then get the differential equation,

$$\tilde{\Phi}'' + \frac{4}{\eta}\tilde{\Phi}' + \frac{k^2}{3}\tilde{\Phi} = 0. \tag{150}$$

The analytic solution to this is given by

$$\tilde{\Phi} = 3\tilde{\Phi}_p\left[\frac{\sin(k\eta/\sqrt{3}) - (k\eta/\sqrt{3})\cos(k\eta/\sqrt{3})}{(k\eta/\sqrt{3})^3}\right]. \tag{151}$$

The implication here is that they enter the horizon during radiation era and then decay and oscillate.

7.5 Matter density perturbation

We can do the similar procedure (but harder) for the matter density perturbation $\tilde{\delta}$ as well, which they are of the form

$$\tilde{\delta}(k,\eta) \approx 9.6\Phi_p \ln(0.44k\eta). \tag{152}$$

The upshot here is that they enter the horizon during the radiation and grow logarithmically. Though to obtain a more accurate result, we have to also consider the fact that at later times matter grows and it will get coupled to the potential, i.e. in the limit $\rho_{dm}\tilde{\delta} > \rho_r\tilde{\Theta}_{r,0}$.

With this consideration, we need to write the Boltzmann system in terms of new variable y and consider the no anisotropic stress assumption,

$$\frac{\mathrm{d}\tilde{\delta}}{\mathrm{d}} + \frac{ikv}{aHy} = -3\frac{\mathrm{d}\tilde{\Phi}}{\mathrm{d}} \tag{153}$$

$$\frac{\mathrm{d}\tilde{v}}{\mathrm{d}y} + \frac{\tilde{v}}{y} = \frac{ik\tilde{\Phi}}{aHy}, \tag{154}$$

where $y = a/a_{eq}$.

Again we adopt the Einstein's version of the Poisson equation that links the density and potential:

$$k^2\tilde{\Phi} = \frac{3y}{2(y+1)}a^2H^2\tilde{\delta}. \tag{155}$$

Finally if we solve them all, we will obtain that $\tilde{\delta} \propto D_1(y) \approx y$. So the two solutions becomes

$$D_1(y) = y + \frac{2}{3}. \tag{156}$$

Note that we will also get second solution for $\tilde{\delta}$ as well, but it goes as $D_2(y) \propto y^{-3/2}$ and thus unphysical.

Explicitly, $D_2(y)$ goes like

$$D_2(y) = D_1(y)\ln\left[\frac{\sqrt{1+y}+1}{\sqrt{1+y}-1}\right] - 2\sqrt{1+y} \tag{157}$$

7.6 The matching era and the transfer function

It is possible to match the two solutions of $\tilde{\delta}$,

$$A\tilde{\Phi}_p\ln\left(\frac{By_m}{y_H}\right) = C_1D_1'(y_m) + C_2D_2'(y_m) \tag{158}$$

$$A\frac{\tilde{\Phi}_p}{y_m} = C_1D_1'(y_m) + C_2D_2'(y_m) \tag{159}$$

By doing so and solve for the constants, we will get a factor an extra factor for $\tilde{\delta}$, which is the transfer function

$$T(k) = \frac{12k_{eq}^2}{k^2}\ln\left[\frac{k}{8k_{eq}}\right]. \tag{160}$$

7.7 Growth function at late times

To solve for the growth function at later times, we can use the following equation

$$D_1(a) = \frac{5\Omega_{m,0}}{2}\frac{H(a)}{H_0}\int_0^a \frac{\mathrm{d}a'}{(a'H(a')/H_0)^3} \tag{161}$$

7.8 Evolution of the matter power spectrum

Now we can write down single-handedly one of the most important quantity in cosmology - the (matter) power spectrum

$$P_\delta(k,a) = 2\pi^2\delta_H\frac{k^n}{H_0^{n+3}}T^2(k)\left(\frac{D_1(a)}{D_1(a=1)}\right)^2. \tag{162}$$

Here δ_H is some normalization constant.

7.9 Numerical approaches

There are a group of physicists primarily working on simulating all these perturbations, and generally there are several assumptions made for most simulations due to technical limitations and limitations in numerical mathematics.

7.9.1 Matter-only

We can safely ignore dark energy in our simulation since it does not have any spatial fluctuations but contributes only to the background expansion. This background expansion can be taken off by simply re-scaling the physical coordinates of the particles to comoving coordinates.

7.9.2 Use of Newtonian physics

If the simulation "box" is not too big, and gravity is not too strong anywhere, then one can use Newtonian physics for the evolution. By too big, it means that the size of simulation contains non-uniform density distribution to the point where Newtonian limit (linear-limit) fails to approximate the result. And by too strong it means, quantities like black holes.

7.9.3 No formation of galaxies

The main interests here is the matter spectrum, whereas galaxies are form at late stages.

Although, they induces < 15 corrections to the power spectrum at nonlinear scales, which is still significant and it is a topic of current investigation.

8 CMB

In this section, we will delve into the physics and mathematical construction behind the CMB.

8.1 Blackbody Law

First, we can use the *blackbody law* to obtain a rough estimate on the CMB temperature. To do so, we introduce first the solid angle $B_\lambda(T)\,\mathrm{d}\lambda\,\mathrm{d}A\cos(\theta)\,\mathrm{d}\Omega$, with

$$B_\lambda(T) = \frac{2hc^2/\lambda^5}{e^{hc/\lambda kT} - 1} \tag{163}$$

$$B_\nu(T) = \frac{2h\nu^3/c^2}{e^{h\nu/kT} - 1}. \tag{164}$$

By **Wien's law**,

$$\lambda_{\max} T = 0.0029\mathrm{mK}, \tag{165}$$

we can then estimate that CMB is 2.1K.

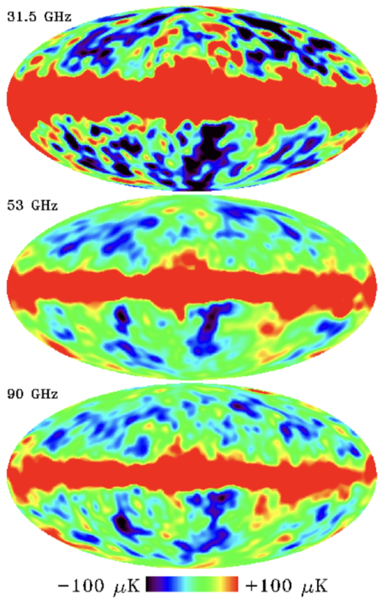

Figure 8: Temperature fluctuations in the microwave background (with respect to the mean) at different frequencies, as measured by COBE. The red area corresponds to emission from our own Galactic plane.

Comments: Whatever, just skip this...

8.2 Angular decomposition of CMB

We can begin by defining the temperature perturbation as

$$\Theta(\mathbf{x}, \hat{\mathbf{p}}, \eta) = \frac{T(\mathbf{x}, \hat{\mathbf{p}}, \eta) - \bar{T}(\eta)}{\bar{T}(\eta)}, \tag{166}$$

and decompose this with the spherical harmonics so that

$$\Theta(\mathbf{x}, \hat{p}, \eta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}(\mathbf{x}, \eta) Y_{lm}(\hat{\mathbf{p}}) \tag{167}$$

It is also necessary to Fourier transform over the perturbation as

$$\Theta(\mathbf{x}, \hat{\mathbf{p}}, \eta) = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \tilde{\Theta}(\mathbf{k}, \hat{\mathbf{p}}, \eta) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{168}$$

we do so because we can spread the CMB into different modes for study.

<u>**Remarks:**</u> Note that $\eta = \eta_0$ since this is as far as we can probe. Yes life is short, although there are goals to compare CMB at different η, which will be a century project.

8.3 connection to Boltzmann theory

To resonate with previous section's results, we have that, before recombination $\Theta = (\Theta_0, \Theta_1)$. And after recombination

$$\frac{1}{(-1)^l} \int_{-1}^{1} \mathrm{d}\mu \, P_l(H) \Theta(\mathbf{x}, \hat{\mathbf{p}}, \eta) \tag{169}$$

9 Brief History of the Universe again

9.1 Relevant scales

We define the reduced Planck mass as $M_p = \sqrt{\hbar c/8\pi G} \approx 2.4 \times 10^{18}\mathrm{GeV/c^2}$, and we adopt the natural unit. Note that the inflation ends at $H_I \sim \sqrt{\hbar\rho/3cM_p^2} \approx 10^{13}\mathrm{GeV/\bar{h}}$

9.2 slow roll parameter

The time of the universe is given by the rest frame of the CMB photon, i.e. the pull back $u|^\mu$.

We also define the principal slow roll parameter as $\epsilon = -\dot{H}/H$.

Note the from the Friedmann equation, we can find for the relation of scale factor with respect to

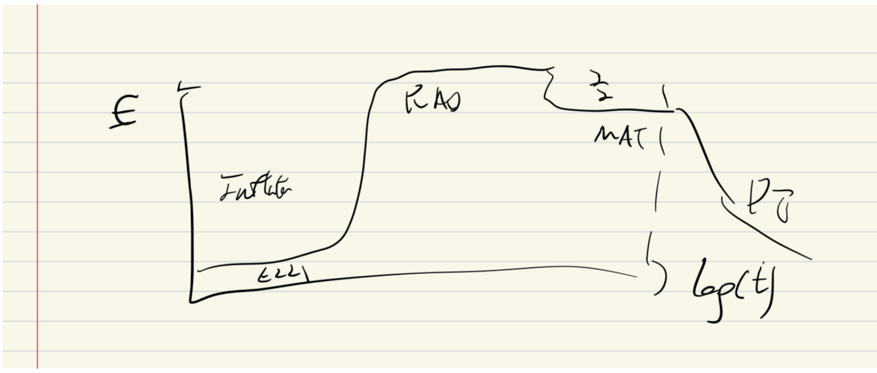

Figure 9: slow roll across Universe time, note that there is no evidence during the $\epsilon \ll 1$ regime.

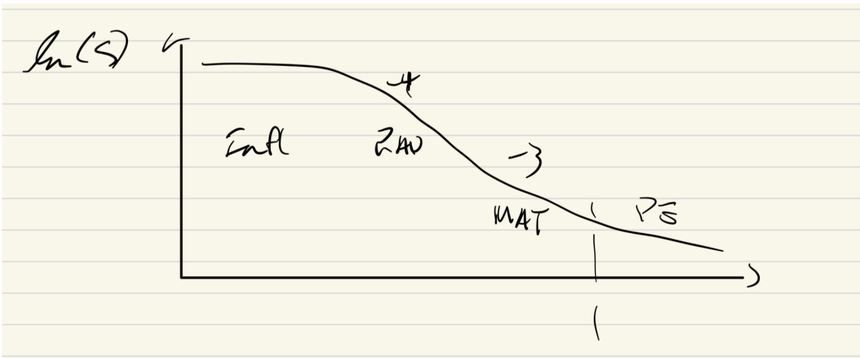

Figure 10: logarithm of slow roll across Universe time.

ρ along the state parameter ω, then we should get

$$\epsilon = \frac{3}{2}(1+\omega), \tag{170}$$

and from theory, we get that the inflation happens at $10^{-35}s$.

Typically,

$$\begin{cases} \text{Inflation :} 0 < \epsilon \ll 1; \quad \frac{\dot{\epsilon}}{\epsilon} \ll H \\ \text{Radiation :} \epsilon \approx 2, \text{the transition to the second is 100000 yrs roughly} \\ \text{Matter : } \epsilon \approx \frac{3}{2} \\ \text{Dark Energy : } \epsilon \approx 0.6 (\text{today}) \end{cases} \tag{171}$$

9.3 perfect fluid and driving mechanics

Here as usual, we take the perfect fluid assumption, then we shall get under the assumption

$$\epsilon = \frac{3}{2}(1+\omega) \approx 0 \implies -1; \quad H(t) \approx H_0 a^{-\epsilon(t)} \tag{172}$$

Now we are in the position to ask what drive inflation,

If we first assume this is fundamental; $\phi(t) = \langle \hat{\Phi}(x) \rangle$.

We can also define the fluctuation in energy-momentum tensor

$$\langle \hat{T}^{\phi}_{\mu\nu} \rangle = \langle \partial_\mu \hat{\phi} \partial_\nu \hat{\phi} \rangle + g_{\mu\nu} \langle \mathcal{L}(\hat{\phi}) \rangle (t), \tag{173}$$

note that this is a very non-trivial step but generally on a typical homogenous inflation setting, we can always guarantee to find a FFF such that the fluctutation of fields are well-defined.

We now define the energy density and pressure as

$$P = \frac{\dot{\phi}^2}{2} - v(\phi) \tag{174}$$

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{175}$$

And the slow roll parameter becomes

$$\omega = \frac{P}{\rho} = \frac{\dot{\phi}^2}{2} - V(\phi) / \frac{\dot{\phi}^2}{2} + V(\phi) \sim -1 \tag{176}$$

Now, one potential way to produce inflation is perhaps the expectation of the certain quantites like, $\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}$, i.e. higher-spin fields, can drive the inflation if we pick a potential such that it exhibits Higgs-like potential, and with the field starting at the unstable equilibrium.

* <u>How inflation ends?</u>

The short answer is that we do not know still, as there is no observation at the end of inflation.

9.4 Quick review of thermodynamics

In inflation, we assume istropic, thus $\mathrm{d}S = 0$. We also define $U = \rho V$, then we get

$$\mathrm{d}U = \mathrm{d}(\rho V) = V\,\mathrm{d}\rho + \rho\,\mathrm{d}V\,. \tag{177}$$

We can see that with this law, then $\rho\,\mathrm{d}V \approx -P\,\mathrm{d}V \implies W > 0, \mathrm{if} P < 0; \rho \approx$ const. Now, the big question is where does this energy comes from when as the Universe expands.

9.4.1 Energy comes from gravity

$$-\frac{3H^2}{8\pi G} + \rho \equiv 0, \quad \text{(00 − component of EFE)} \tag{178}$$

9.5 perturbative methods

$$\mathcal{L}_{\text{int}} = -y\phi\bar{\psi}\psi - \frac{g}{2}\phi^2\chi^2, \tag{179}$$

where all the fields here are the standard model fields, with ϕ being the inflaton.

There are two ways to go about studying the interaction, for the Yukawa term, we can do perturbative QFT, which we get the tree-level decay rates as

$$\Gamma_{\phi\to\bar{\psi}\psi} = \frac{|y|^2}{4\pi}E_\phi, \tag{180}$$

where $E_\phi = \sqrt{p_\phi^2 + m^2} \approx$, and

$$\Gamma_{\phi\to\chi\chi} = \frac{|g|^2}{4\pi}\frac{|\phi_0|^2}{E_\phi}, \tag{181}$$

Remarks: Please note that although the yukawa is 4 fields interaction, but however under the condensate assumption for the inflaton, we can write it as $\phi = \phi_0 + \delta\phi$, thus we get the resulting 3 scattering diagram, the dashed lines refers to $\delta\phi$ as always.

9.6 Non-perturbative methods

In the non-perturbative QFT analysis, one may use the technique of parametric resonance.

Say given the inflaton oscillating in the well potential, with the equation of motion as

$$(\partial_t^2 + 3H\partial_t + m_p^2)\phi_0(t) = 0 \tag{182}$$

$$(\partial_t^2 + m_p^2 - \frac{3}{2}\dot{H} - \frac{g}{2}H^2)(a^{3/2}\phi_0) = 0 \tag{183}$$

At the end, we will obtain the Mathiew equation for the equation of motion with parametric resonance. And note that the eom has a form

$$\ddot{x} + (a + 2b\cos(t))x = 0 \tag{184}$$

And in the a = 2b regime, we recover the typical simple harmonic oscillator.