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String theory

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1 Motivation

String theory has numerous prominent resultants, which are supersymmetry (i.e. extended Poincare Group), extra dimension and such. Well whatever, it's just crazy-bazy so that's why we study it, does it matter if something really is the QG at the end of the day? What is it all anyways? Why do we live? Why we have what we observe now? Is everything meaningless, or meaningless is everything?

2 Nambu-Goto Action

2.1 Relativistic Point Particle

We take for a D-dim Minkowski space: $\mathbb{R}^{D-1,1} \implies \eta_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ We define the world-like action for a string

$$S = -m \int \mathrm{d}t \sqrt{1 - \dot{\mathbf{X}}\dot{\mathbf{X}}} = -m \int \mathrm{d}t \sqrt{1 - \dot{\mathbf{X}}^{\mu}\dot{\mathbf{X}}^{\nu}\eta_{\mu\nu}} \tag{1}$$

Check:

$$\begin{cases} \mathbf{p} = \frac{m\mathbf{x}}{\sqrt{1-\mathbf{x}\cdot\mathbf{x}}} \\ E = \sqrt{m^2 + p^2} \end{cases} \tag{2}$$

However, we there is still a different treatment with space and time. Thus we reparametrize altogether that $\tau = (t, \mathbf{x})$ We also replace the derivative of the fake Nambu-Goto action (1) by $\dot{\mathbf{X}}^\mu = \frac{\partial X^\mu}{\partial \tau}$, the modified equation now reads

$$S = -m \int \mathrm{d}\tau \sqrt{1 - \dot{\mathbf{X}}\dot{\mathbf{X}}} = -m \int \mathrm{d}t \sqrt{1 - \dot{\mathbf{X}}^\mu \dot{\mathbf{X}}^\nu \eta_{\mu\nu}} \tag{3}$$

Next we look for new symmetry in this action and ***Check 2:***

$$p_\mu = \frac{\partial L}{\partial X^\mu} = \frac{m\dot{\mathbf{X}}^\mu \eta_{\mu\nu}}{\sqrt{-\dot{X}^\mu \dot{X}^\rho \eta_{\mu\rho}}} \tag{4}$$

The result we get if we compute for the e.o.m is

$$p^\mu p^\nu \eta_{\mu\nu} + m^2 = 0 \tag{5}$$

which suggests that there is the mass-shell condition the string has to obey. Note that this action is also invariant under Poincare symmetry

$$\tilde{X}^\mu = \Lambda^\mu_\nu X^\nu + c^\mu \quad \text{and} \quad \dot{\tilde{X}}^\mu = \frac{\partial \tilde{X}}{\partial \tau} = \Lambda^\mu_\nu X^\nu \tag{6}$$

Check 3:

$$\tilde{S} = -m \int \mathrm{d}\tilde{\tau} \sqrt{\Lambda^\mu_\nu \Lambda^\nu_\mu \dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}} \tag{7}$$

2.2 Einbein formalism

Turns out we can introduce the einbein formalism into the action we defined in the previous section (3) (we introduce it here solely for the historical development of string theory). Using einbein formalism to the action gives,

$$S_A = \frac{1}{2}\int \mathrm{d}\tau (e^{-1}\dot{X}^2 - em^2) \tag{8}$$

, where the einbein e is given by $e = \sqrt{-g_{\tau\tau}}$. Note that although e is related to the metric, it should be viewed as a separate field to the metric. *Advantage of the formalism*:

1. works for the case m = 0

2. easier to quantize

The above equation also reads

$$S_A = -\frac{1}{2}\int \mathrm{d}\tau \sqrt{-g_{\tau\tau}}(g^{\tau\tau}\dot{X}^2 + m^2) \tag{9}$$

3 Nambu-Goto Action

After all these build-up and motivation, we can finally begin to introduce the Nambu-Goto Action.

We take the string worldsheet (ws) picture,

So in short, a string is a map from the world into $\mathbb{R}^{D-1,1}$, i.e. $X^\mu : \Sigma \to \mathbb{R}^{D-1,1}$, $X^\mu(\tau, \sigma) = X^\mu(\sigma^\alpha)$, yes I know, very bad notation.

We then look for an action that is independent of the parametrization of , just like finding the action that is independent of coordinate change in QFT. One natural starting point is to instead of speak of the length path but of the area of the string. These calls for two changes (1) the integral and (2) an induced metric to on the worldsheet whenever we speak of an area of a curved surface

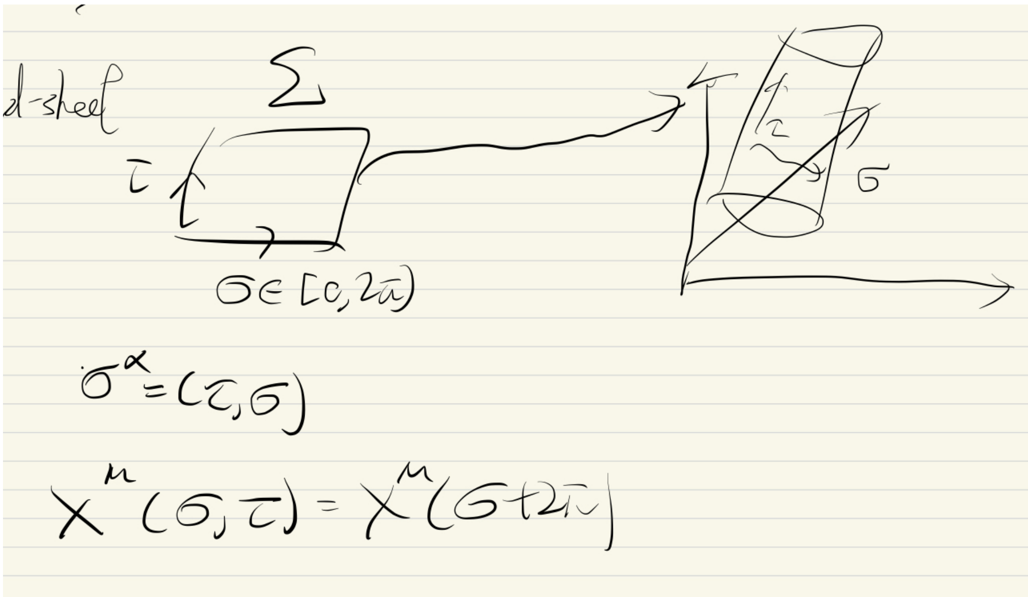

Figure 1: The string worldsheet interpretation

(worldsheet as the surface).

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu} \tag{10}$$

And by extraterrestrial black magic, we guess the Nambu-Goto action to be

$$S_{NB} = -T \int_\Sigma \mathrm{d}^2\sigma \sqrt{-\det\gamma}, \tag{11}$$

or alternatively

$$S_{NB} = -T \int \mathrm{d}^2\sigma \sqrt{-(\dot{X})^2 (X')^2 + (\dot{X} \cdot X')^2} \tag{12}$$

Sometimes, for historical reasons, we refer to T with $T = 1/2\pi\alpha'$, and α' is sometimes called the *Universal Reggae slope* from the QCD gluons.

If we then turn to the potential energy of the string, we see that it happens to be $E_{\text{pot}} = T \times$ (stringlength), which suggests that there is this spring like binding on the string that the energy grows linearly with the length of the string. And so to minimize the energy naturally, it shrinks to zero length

”classically”, and giving rise to a point-like particle. However, it is not this simple as we need to take *quantum fluctuation* into account, and this calls for quantization procedures.

3.1 Symmetries of S_{NG}

It is instructive to look at the symmetry which S_{NG} possess.

The first form of symmetry can be probed by reparametrization of $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$, while the second form of symmetry is the Poincare invariance of spacetime $\mathbb{R}^{D-1,1}$. Note that the first form implies gauge symmetry that is redundancy in the choice of σ^α and that it has no physical meaning.

3.2 Eom of Nambu-Goto action

To obtain the equation of motion for Nambu-Goto Action, we begin by introducing the momenta:

$$\Pi^\tau_\mu = \frac{\partial \mathcal{L}}{\partial x^\mu}; \quad X^\mu = \frac{\partial X^\mu}{\partial \tau}; \quad \Pi^\sigma_\mu = \frac{\partial \mathcal{L}}{\partial x^\mu}; \quad X^\mu = \frac{\partial X^\mu}{\partial \sigma}, \tag{13}$$

and apply we can obtain the eom by the Euler Lagrangian equation (over the Nambu-Goto action).

The eom then reads,

$$\frac{\partial \Pi^\tau_\mu}{\partial \tau} + \frac{\partial \Pi^\sigma_\mu}{\partial \tau} = 0. \tag{14}$$

Albeit it looks simple at first glance, it is actually very nasty, nonlinear equations. And thus it is better to obtain the eom instead by performing variation on the pull-back metric, and after doing so, we will get

$$\partial_\alpha(\sqrt{-\det\gamma}\gamma^{\alpha\beta}\partial_\beta X^\mu) = 0. \tag{15}$$

However, X^μ is still difficult to obtain, which then calls for a different action, pun intended.

Remarks: You may wonder why we bother bringing up Nambu-Goto if this is not the action for early string theory, it is in fact this action that sparked early motivation to study string theory in the first place, and it serves as a good starting point (not too difficult) to introduce string theory.

4 Polyakov Action

With the previous situation in mind, i.e. the eom for Nambu-Goto action does not yield a nice-behaved solution for X^μ, we will have to resort to defining a new action, but luckily enough, it turns out there is another form of the string action that is classically equivalent to S_{NG}, it is the *Polyakov Action*

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma \sqrt{-g}g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{16}$$

Here $g \equiv \det g$. (out of topic but Polyakov didn't discover this action)

Now we have a new field $g^{\alpha\beta}$, which is the dynamical metric on the worldsheet.

The corresponding eom of X^μ is

$$\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta X^\mu) = 0, \tag{17}$$

this is identical to (14) except $g_{\alpha\beta}$ is independent variable, and one can obtain its respective eom by varying the action over $\delta g_{\alpha\beta}$, which reads

$$g_{\alpha\beta} = 2f(\sigma)\partial_\alpha X \cdot \partial_\beta X, \tag{18}$$

and

$$f^{-1} = g^{\rho\sigma}\partial_\rho X \cdot \partial_\sigma X \tag{19}$$

and note that $f(\sigma)$ is short for $f(\sigma, \tau)$.

Remarks on $g_{\alpha\beta}$ vs $\gamma_{\alpha\beta}$: these two quantities differs by a conformal factor of f, but when we obtain the eom for Polyakov, the f cancels out in the end, which resulting in the same eom for both actions.

4.1 Symmetries of the Polyakov action

Here lies the symmetries that Polyakov action exhibits

- Poincare invariance, same as before $X^\mu \to \Lambda^\mu_\nu X^\nu + c^\mu$
- Reparametrization invariance, or so-called *diffeomorphisms*. It goes simply as leaving the action invariant under $\sigma^\alpha \to \tilde{\sigma}^\alpha(\sigma)$.

  Note that under this transformation, the field X^μ and the metric $g_{\alpha\beta}$ transforms as

$$X^\mu(\sigma) \to \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma) \tag{20}$$

$$g_{\alpha\beta}(\sigma) \to \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\gamma}{\partial\tilde{\sigma}^\beta}g_{\gamma\delta}(\sigma) \tag{21}$$

  In a particular case, i.e. under infinitesimal shift $\sigma^\alpha \to \tilde{\sigma}^\alpha = \sigma^\alpha - \eta^\alpha(\sigma)$, for some small η. We will get the following two important variations

$$\delta X^\mu(\sigma) = \eta^\alpha \partial_\alpha X^\mu \tag{22}$$

$$\delta g_{\alpha\eta}(\sigma) = \nabla_\alpha \eta_\beta + \nabla_\beta \eta_\alpha, \tag{23}$$

  where the covariant derivative is defined as $\nabla_\alpha \eta_\beta = \partial_\alpha \eta_\beta - \Gamma^\sigma_{\alpha\beta}\eta_\sigma$, and the *Levi-Civita connection* as the usual

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2}g^{\sigma\rho}(\partial_\alpha g_{\beta\rho} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}) \tag{24}$$

- Weyl Invariance. Under the Weyl symmetry, the metric goes like $g_{\alpha\beta}(\sigma) \to \Omega^2(\sigma)g_{\alpha\beta}(\sigma)$, or in infinitesimal form as writing first $\Omega^2(\sigma) = e^{2\phi(\sigma)}$ for small ϕ such that $\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma)g_{\alpha\beta}(\sigma)$. One important result we get from Weyl symmetry action is that

$$T^\alpha_\alpha = 0 \tag{25}$$

  Remarks: If we think of the reparametrization invariance as local gauge transformation, then

Weyl invariance is a local invariance such that it preserves scale (in particular angles) under such transformation. Also note that Weyl symmetry is very restrictive in itself that adding extra factors in front of the action (like potential $V(x) \implies S'_p \to S_p V(x) \neq 0$ conformal factor) already breaks the Weyl symmetry

4.2 Fixing the gauge

The eom (17) looks fairly disgusting, thus we need to choose a gauge to simplify it. It turns out that we can do two reparametrizations in our case, it goes as follows

1. Use reparametrization: we set $g_{\alpha\beta} = e^{2\phi(\sigma)}\eta_{\alpha\beta}$ under the reparametrization $\sigma^\alpha \to \tilde{\sigma}^\alpha$. We then immediately see that we reduce the 3 dof of the metric to just 1 under this *diffeomorphism*.
2. Next, to get rid of the conformal factor, we simply use the Weyl transformation $g_{\alpha\beta} \to \Omega^2 g_{\alpha\beta}$ to set $g_{\alpha\beta} = \eta_{\alpha\beta}$.
3. Sit back and relax the beauty of the symmetries this action exhibits, as it always guarantees to fix the metric to be a flat one.

4.3 eom and the stress-energy tensor

Under the aforementioned gauge, the action now simplifies to

$$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\, \partial_\alpha X \cdot \partial^\alpha X, \tag{26}$$

and the eom becomes

$$\partial_\alpha \partial^\alpha X^\mu = 0. \tag{27}$$

This is just a simple wave equation, however this is just for the eom for the D free scalar fields, and we have to also obtain the eom w.r.t. the metric. To do so, we simply perform the variation $\delta g_{\alpha\beta}$ to S_{NG}, which turns out to be the stress-energy tensor, $T_{\alpha\beta}$.

$$T_{\alpha\beta} = -\frac{2}{T}\frac{1}{\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}}. \tag{28}$$

After computing (28), and set $g_{\alpha\beta} = \eta_{\alpha\beta}$, we will get

$$T_{\alpha\beta} = 0, \tag{29}$$

or more explicitly,

$$T_{01} = \dot{X} \cdot X' = 0 \tag{30}$$

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0. \tag{31}$$

<u>**Remarks on the constraints:**</u>

- The first constraint can be thought of as we have to parametrize σ, τ such that the string always travels along τ, and there is only the transverse oscillation along the string, **no** longitudinal modes.
- The second constraint tells us the relation between the length of the string to the instantaneous velocity of the string. To do so, we fix a gauge such that $\dot{X} = 0$, and integrate over the σ loop and we get the R.H.S. as $2\pi R$, i.e. the ”hidden tail” of the string.

5 Mode Expansion

In this section, we discuss the mode expansion physics.

We shall begin by introducing the lightcone coordinates on the ws.

$$\sigma^{\pm} = \tau \pm \tau, \tag{32}$$

then the eom simply reads

$$\partial_+\partial_- X^\mu = 0. \tag{33}$$

The most general solution for this is then,

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) \tag{34}$$

, which has the periodicity condition,

$$X^\mu(\sigma,\tau) = X^\mu(\sigma+2\pi,\tau). \tag{35}$$

As well as the constraints (30) - (31).

It turns out that the left-moving and right-moving waves can be expanded in Fourier modes as

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+} \tag{36}$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^-} \tag{37}$$

Remarks: The interpretation here is that x^μ denotes the position of the center of mass (com) of the string, and p^μ the momentum of the com of the string. While the reality of X^μ requires that the coefficients (of the Fourier modes), α^μ_n and $\tilde{\alpha}^\mu_n$, obey

$$\alpha^\mu_n = (\alpha^\mu_{-n})^*, \quad \tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^*. \tag{38}$$

5.1 Virasoro Constraint

Up to this point, we have not yet discussed how to impose the two constraints (30) - (31), which turns out that in lightcone coordinates, (30) becomes

$$(\partial_+ X)^2 = (\partial_-)^2 = 0. \tag{39}$$

When we try to compute the ∂_- one, we will see that

$$\begin{aligned} \partial_- X^\mu = \partial_- X^R_\mu &= \frac{\alpha'}{2} p^\mu + \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha^\mu_n e^{-in\sigma^-} & (40) \\ &= \sqrt{\frac{\alpha'}{2}} \sum_n \alpha^\mu_n e^{-in\sigma^-}, & (41) \end{aligned}$$

where by reading off second to last equality we have defined

$$\alpha^\mu_0 \equiv \sqrt{\frac{\alpha'}{2}} p^\mu. \tag{42}$$

Going back to (39) again, we then get

$$\begin{aligned} (\partial_- X)^2 = \partial_- X^\mu \cdot \partial_- X^\nu &= \frac{\alpha'}{2} \sum_{m,p} \alpha_m \cdot \alpha_p e^{-i(m+p)\sigma^-} & (43) \\ &= \frac{\alpha'}{2} \sum_{m,n} \alpha_m \cdot \alpha_{n-m} e^{-in\sigma^-} & (44) \\ &\equiv \alpha' \sum_n L_n e^{-in\sigma^-} = 0. & (45) \end{aligned}$$

Here we have define the sum of all the oscillator modes as

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m. \tag{46}$$

The exact same can be done for the left-moving modes and we will get

$$\tilde{L}_n = \frac{1}{2}\sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m. \tag{47}$$

and the 0th mode $\tilde{\alpha}_0^\mu \equiv \sqrt{\frac{\alpha'}{2}}p^\mu$. in order for the constraints to vanish for n-th modes, this then requires

$$L_n = \tilde{L}_n = 0 \quad n \in \mathbb{Z}. \tag{48}$$

Interpretation of L_0, since L_0 and $\tilde{L}_0$ contains the square of the spacetime momentum p^μ, but this is exactly the rest mass of a particle, this means that

$$M^2 = \frac{4}{\alpha'}\sum_{n>0} \alpha_n \cdot \alpha_{-n} - \frac{4}{\alpha'}\sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}. \tag{49}$$

Note that to get (49), we use (**??**) and then move the 0th term to the R.H.S. and call it M^2, and all the rest to the R.H.S., which is what we have above.

Remarks:

- Note that we require both the right-moving oscillators α_n^μ and the left-moving oscillators $\tilde{\alpha}_n^\mu$, and that these two terms must be equivalent to each other, and this is known as the *level matching*.

6 Quantization

In this section, we aim to quantize the string with different approaches, and they are summarised as follows

- Covariant quantisation: Procedures:

  1. promote to operators
  2. impose Virasoro constraints

  Results: Lorentz invariant but not unitary unless in D = 26.

- Lightcone quantisation: Procedures:

  1. introduce lightcone coordinates ($\sigma^{\pm}$)
  2. solve the Virasoro constraints for classical theory
  3. quantize

  Results: Not Lorentz invariant unless in D = 26, but it is unitary.

- Path integral quantization: Procedures:

  1. Fadeev-Papov procedure and do not gauge fix strings

  Results: theory is only consistent in D = 26. By consistent, it means it is *anomaly free*, which means the classical symmetry is generated solely by its quantum theory counterpart.

6.1 Important stuff from covariant quantization

This section will conclude the result for computation in covariant quantization.

So from the closed string dynamics we get the constraints

$$\dot{X} \cdot X' = \dot{X}^2 + X'^2 = 0. \tag{50}$$

, Now from this we can immediately promote X and their conjugate momenta $\Pi_\mu = (1/2\pi\alpha')\dot{X}_\mu$ and then we will get the following equal-time (τ) commutation relations,

$$[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta(\sigma-\sigma')\delta^\mu_\nu \tag{51}$$

$$[X^\mu(\sigma,\tau), X_\nu(\sigma',\tau)] = [\Pi_\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = 0 \tag{52}$$

And we can also translate thee into commutation relations for the followings and get,

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \quad \text{and} \quad [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = [\alpha^\mu_n, \alpha^\nu_m] = n\,\eta^{\mu\nu}\delta_{n+m,0}. \tag{53}$$

From the commutation relation of the α and $\tilde{\alpha}$, we can observe that these are actually the harmonic oscillator creation and annihilation operators up to some different in normalisation factor, which they are

$$a_n = \frac{\alpha_n}{\sqrt{n}}, \quad \alpha^\dagger_n = \frac{\alpha_{-n}}{\sqrt{n}} \quad \forall n > 0. \tag{54}$$

With this then we can recover the familiar $[\alpha_n, \alpha^\dagger_m] = \delta_{nm}$.

Interpretation: We can think of it as each of the D scalar fields giving two infinite towers of creation and annihilation operators (a_n and $\tilde{a}_n$), and $\alpha_n and \tilde{\alpha}_n$ are simply the rescaled version these operators. The α_n are rescaled annihilation operators and the α_{-n} are the creation operators (for $n > 0$ in this context).

6.1.1 Fock space

Now we are in the position to build the Fock space of our theory. We begin by defining the vacuum as $\hat{p}_\mu|0;p\rangle = p^\mu|0,p\rangle$ and $\alpha^\mu_n|0;p\rangle$ for $n > 0$.

To build up the Fock space, we simply act on the vacuum state by both creation operators, so a generic state comes from acting with any number of them on the vacuum,

$$(\alpha^{\mu_1}_{-1})^{n_{\mu_1}}(\alpha^{\mu_2}_{-2})^{n_{\mu_2}}\cdots(\tilde{\alpha}^{\mu_1}_{-1})^{n_{\mu_1}}(\tilde{\alpha}^{\mu_2}_{-2})^{n_{\mu_2}}\cdots|0;p\rangle \tag{55}$$

6.1.2 Ghosts

Consider the probability amplitude of the state $\alpha^0_{-1}|0;p\rangle$

$$0;p|\alpha^0_1\alpha^0_{-1}|0\rangle \tag{56}$$

$$= 0;p|[\alpha^0_1,\alpha^0_{-1}]|0\rangle + \underbrace{0;p|\alpha^0_{-1}\alpha^0_1|0\rangle}_{=0} \tag{57}$$

$$= 0;p|1\times\eta^{00}|0;p\rangle \tag{58}$$

$$= \eta^{00} = -1 \tag{59}$$

$$\implies \text{ unitary violated.} \tag{60}$$

Remarks:

- Note the true ground state is just $|0\rangle$, but not $|0;p\rangle$, unlike in the field theory, since the string itself can be moving or in a potential without any vibration along the string.

6.1.3 Virasoro Constraints

Classically we have $L_m = \tilde{L}_m = 0$, but for the quantum theory, we need a ordering description, and with this then we will generate "infinite possible" constants through different orderings, thus a constraint on the ambiguity is needed.

To make the problem more apparent, for the two operators

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} i\alpha_{m-n}\cdot\alpha_n \tag{61}$$

$$\tilde{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} i\tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_n \tag{62}$$

We define the normal ordering as

$$:\alpha_n\cdot\alpha_m := \begin{cases}\alpha_n\cdot\alpha_m & m<n\\ \alpha_m\cdot\alpha_n & n<m\end{cases} \tag{63}$$

, note that this is simply a notation to say any creation and annihilation operators inside are ordered such that all annihilation are on the right, and creation on the left, nothing more.

Now the problem lies on that in the classical theory, there is no commutation relation between the $\alpha_n^\mu, \tilde{\alpha}_n^\mu$, and now that we defined the $L_0 \tilde{L}_0$ as above, we expect that there is a need to introduce a constant (which we will find out it is 1 later) to compensate for difference when commutation relation is introduced.

So in the end, we get the *Virasoro constraint* as

$$(L_0 - a)|\text{phys}\rangle = (\tilde{L}_0 - a)|\text{phys}\rangle = 0 \tag{64}$$

6.2 Normal ordering

This section is dedicated to getting more familiarity with the normal ordering notation.

We will use the commutation relation $[\alpha_n^\mu, \alpha_m^\nu] = n\delta^{\mu\nu}$ as usual.

And recall again the normal ordering notation as,

$$: \alpha_n \cdot \alpha_m := \begin{cases} \alpha_n \cdot \alpha_m & m < n \\ \alpha_m \cdot \alpha_n & n < m \end{cases}$$

Here, we want to show that,

$$\frac{1}{2} \sum_{n \in \mathbb{Z}^*} \alpha_{-n}^\mu \alpha_n^\nu = \frac{1}{2} \sum_{\mathbb{Z}^*} : \alpha_{-n}^\mu \alpha_n^\nu : + \frac{1}{2} \sum_{n>0} n\delta^{\mu\nu}, \tag{65}$$

where $\mathbb{Z}^* = \mathbb{Z} \backslash \{0\}$

The steps are then,

$$\frac{1}{2}\sum_{n\in\mathbb{Z}^*}\alpha^\mu_{-n}\alpha^\nu_n = \frac{1}{2}\sum_{n<0}\alpha^\mu_{-n}\alpha^\nu_n + \frac{1}{2}\sum_{n>0}\alpha^\mu_{-n}\alpha^\nu_n \tag{66}$$

$$= \underbrace{\frac{1}{2}\sum_{n<0}\alpha^\mu_n\alpha^\nu_{-n} + \frac{1}{2}\sum_{n<0}[\alpha^\mu_{-n}\alpha^\nu_n]}_{\text{first term}} + \frac{1}{2}\sum_{n>0}\alpha^\mu_{-n}\alpha^\nu_n \tag{67}$$

$$= \frac{1}{2}\sum_{n<0} :\alpha^\mu_n\alpha^\nu_{-n}: + \frac{1}{2}\sum_{n<0} -n\delta^{\mu\nu} + \frac{1}{2}\sum_{n>0} :\alpha^\mu_{-n}\alpha^\nu_n \tag{68}$$

$$= \frac{1}{2}\sum_{\mathbb{Z}^*} :\alpha^\mu_{-n}\alpha^\nu_n: + \underbrace{\frac{1}{2}\sum_{n>0} n\delta^{\mu\nu}}_{\text{flipped n}} \tag{69}$$

6.3 No ghost theorem

It simply states that for $a = \tilde{a} = 1$, and $D = 26$, ghosts decouple from physical states

6.4 Virasora algebra

It turns out that these operators $L_n, \tilde{L}_m$ satisfies something call *Virasoro algebra*. Some of their properties are

- $[\mathrm{L}_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$, where the second terms are the *central extension*.

  It also turns out that with this algebra, it guarantees conformal transformation (obeying).

6.5 Number operator

If we now define the number operator as

$$N = \sum_{n>0}\alpha^\mu_{-n}\alpha^\nu_n, \qquad \tilde{N} = \sum_{n>0}\tilde{\alpha}^\mu_{-n}\tilde{\alpha}^\nu_n \tag{70}$$

and consider the Virasoro constraints (64), we get

$$
\begin{aligned}
0 &= [(L_0 - a) + (\tilde{L}_0 - a)]|\text{phys}\rangle &(71)\\
&= [N + \tilde{N} + \frac{\alpha'}{2} p^\mu \cdot p^\nu - 2a]|\text{phys}\rangle &(72)\\
&= [-\frac{\alpha'}{2} M^2 + N + \tilde{N} - 2a]|\text{phys}\rangle &(73)\\
\implies M^2 &= \frac{4}{\alpha'}(\frac{1}{2}N + \frac{1}{2}\tilde{N} - a), &(74)
\end{aligned}
$$

Remarks:

- if we look at the mass of the ground state in this theory, we immediately see that the mass is negative ($-1/2a$), where $a = 1$ for the theory to work, this is a tachyon but it is not the end of the day because turns out these ground state are simply not stable, think of $M^2 \propto p^2 \propto \partial^2$, so it is at some hill of a upward potential, i.e. in an unstable equilibrium.
- Please note that mass can be negative, the norm is still 1 so unitary is conserved (by default ground state is just 1).

6.6 Important stuff from covariant quantization

This section will conclude the result for computation in covariant quantization.

So from the closed string dynamics we get

$$\dot{X} \cdot X' = \dot{X}^2 + X'^2 = 0. \tag{75}$$

, Now from this we can immediately promote X and their conjugate momenta $\Pi_\mu = (1/2\pi\alpha')\dot{X}_\mu$ and then we will get the following equal-time (τ) commutation relations,

$$[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta(\sigma - \sigma')\delta^\mu_\nu \tag{76}$$

$$[X^\mu(\sigma,\tau), X_\nu(\sigma',\tau)] = [\Pi_\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = 0 \tag{77}$$

And we can also translate thee into commutation relations for the followings and get,

$$[x^\mu, p_\nu] = i\delta^\mu_\nu \quad \text{and} \quad [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = [\alpha^\mu_n, \alpha^\nu_m] = n\,\eta^{\mu\nu}\delta_{n+m,0}. \tag{78}$$

6.7 Light cone quantization

Aim: The aim here is to find a parametrization of all classical solutions of the string, i.e. finding the classical phase space. Mathematically, we want to find physical solution of

$$\dot{X} \cdot X' = \dot{X}^2 + X'^2 = 0. \tag{79}$$

The action here is of course none other than the Polyakov action.

Here goes the "algorithm":

1. gauge fix the worldsheet metric to

$$g_{\alpha\beta} = \eta_{\alpha\beta}. \tag{80}$$

But the slight caveat here is that, there still exists conformal transformation (recall one of the exercises that you have done). However, one can always through reparametrization + Weyl to get the worldsheet metric

$$h_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{81}$$

2. we then get this variation identity on the w.s. metric for free (we did so in the exercises),

$$\delta h_{\alpha\beta} = -(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha) + 2\Lambda h_{\alpha\beta} = 0, \tag{82}$$

where here ξ_α is called the *conformal killing vector*, which carries the left over symmetry after gauge fixing.

3. Now it is naturally to ask, what are the transformation exactly to obtain such transformations, which are exactly the light cone coordinates,

$$\sigma^{\pm} = \tau \pm \sigma \tag{83}$$

$$\Longrightarrow\ ds^2 = -\mathrm{d}\sigma^{+}\mathrm{d}\sigma^{-} \tag{84}$$

$$\text{now we take} \tag{85}$$

$$\sigma^{+} \to \tilde{\sigma}^{+}(\sigma^{+}) \quad \text{and} \quad \sigma^{-} \to \tilde{\sigma}^{-}(\sigma^{-}) \tag{86}$$

$$ds^2 \to \underbrace{\frac{\partial\tilde{\sigma}^{+}}{\partial\sigma^{+}}\frac{\partial\tilde{\sigma}^{-}}{\partial\sigma^{-}}}_{\text{undone by Weyl}} ds^2 \tag{87}$$

4. We then churn through the quantization procedure as follow:

The metric now reads:

$$ds_D^2 = -2dX^{+}dX^{-} + \sum_{n=1}^{D-2} dX^i dX^i \tag{88}$$

and the index raising and lowering goes like

$$A_{\pm} = -A^{\mp} \quad A_i = A^i \quad A\cdot B = -A^{+}B^{-} - A^{-}B^{+} + A^iB^i \tag{89}$$

and the eom for X^{+} becomes

$$X^{+} = X_L^{+}(\sigma^{+}) + X_R^{+}(\sigma^{+}) \tag{90}$$

and then apply the lightcone gauge *,

$$X_L^+ = \frac{1}{2}X^+ + \frac{1}{2}\alpha' p^+ \sigma^+ \tag{91}$$

$$X_L^- = \frac{1}{2}X^- - \frac{1}{2}\alpha' p^- \sigma^- \tag{92}$$

next to solve for X^-

use the e.o.m

$$\partial_+\partial_- X^- = 0, \quad X^- = X_L^-(\sigma^-) - X_R^-(\sigma^-) \tag{93}$$

note that $(\partial_+ X)^2 = (\partial_- X)^2 = 0 \implies 2\partial_\pm X^+ \partial_\mp X^+ = \sum_{n=1}^{D-2} \partial_\pm X^i \partial_\mp X^i$

$\implies \partial_\pm X_{L/R}^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_\pm X^i \partial_\pm X^i$

6.7.1

7 D-branes

7.1 Big-branes time

This section is primarily reserved for studying branes-problem in general.

In most general D-brane problem, we are asked to find the first few excited-levels of the open strings given different D-branes sectors settings.

We will primarily use these two equations,

$$\alpha' M^2 = \frac{(x_1 - x_0)^2}{(2\pi\sqrt{\alpha'})^2} + N_{NN} + N_{DD} + N_{DN/ND} - a_{\text{open}}, \tag{94}$$

, where we have

$$x_1 - x_0: \quad \text{distance between the endpoints of the string components with} \tag{95}$$

$$\text{the (DD) boundary conditions;} \tag{96}$$

$$N_{NN}: \quad \text{level operator in the (NN) sector;} \tag{97}$$

$$N_{DD}: \quad \text{level operator in the (DD) sector;} \tag{98}$$

$$N_{DN/ND}: \text{level operator in the (DN) or (ND) sector;} \tag{99}$$

$$a_{\text{open}}: \quad \frac{n_{NN}}{24} + \frac{n_{DD}}{24} - \frac{n_{DN/ND}}{48} \tag{100}$$

, and the small n denotes the number of components of X^μ with (the subscripted) boundary conditions, but note that for (NN), it excludes $X^\pm$ In sheet 7, we are asked to work on the setting:

1. One spacetime-filling D25-brane:

2. A stack of five **coincident** D12-branes extending along the directions $\mu = +, -, \cdots, 12$

3. A stack of seven **coincident** D50branes extending along the directions $\mu = +, -, 2, 12, 13, 14$ parametrised by X^μ.

Then we can look directly at the "hardest sector" to go through, which is D5-D12, *please think about why.*

Ans: (i) The branes separation terms come into play in (94), (ii) DN/ND exists, which causes 1/2 level excitation to pop-up, and need to consider all terms of a_0.

7.2 0th level

First, it is quintessential to draw the table below,

So let's start with the 0th level: with 0 level, there is no creation operator so only the last term (the constant a_{open}) of (94) will be considered. So the mass $-\alpha M^2$ is simply $a_{\text{open}} = n_{NN}/24 + n_{DD}/24 - n_{ND/DN}/48 = 11/24 + 2/24 - (9+2)/48 = 5/16$.

dim. $\mu \rightarrow$	+	-	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\cdots$	25
1 × D25	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
5 × D12	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	×	×	×	×	×	×
7 × D5	✓	✓	✓	×	×	×	×	×	×	×	×	×	✓	✓	✓	×	×	×	×

Table 3: dimensions occupied by the various branes

So at level 0, we get a mass dependent field with $\alpha' M^2 = -5/16 + \Delta x^2_{D5-D12}/(2\pi\sqrt{\alpha'})^2$.

At level 1/2. we get 2 massive scalars with DN or ND preferential massive scalars ($\alpha M^2 = -5/16 + 1/2 + \cdots$).

At level 1, we get massive scalars and a vector as well, with the creation operators α^i_{-1} indices i belonging to (NN) directions.

Some important remarks:

1. The basic idea is that, only for states with the indices of the creation operator containing NN direction coordinates, we will get vectors, as only they will transform under Lorentz transformation.

2. If the energy level we are considering involves (DN/ND), we will get half integer excited states.

8 Conformal Field Theory (CFT)

8.1 Making the Noether currents

Final Comment: Note that it is quite disturbing that we first make the field (grav) dynamical and then at later stage we fix it such that we can formalize the EM tensor.

8.2 Noether currents in CFT

below is lecture 9 contents

currents for all conformal transformation obeys,

$$\begin{cases} z' = z + \epsilon(z) \\ \bar{z}' = \bar{z} + \bar{\epsilon}(\bar{z}), \quad \text{infin. transf.} \end{cases} \tag{101}$$

And as the same as before: $\epsilon \to \epsilon(\sigma^\alpha) \implies \epsilon(z) \to \epsilon(z, \bar{z})$, note that $\bar{z}$ is independent of z, it is its own separate variable.

Now we do the same trick:

$$\delta S = -\int d^2\sigma \frac{\partial S}{\partial g^{\alpha\beta}} \delta g^{\alpha\beta} = \frac{1}{2\pi} \int d^2\sigma T_{\alpha\beta}(\partial^\alpha \delta\sigma^\beta) \tag{102}$$

$$\underbrace{=}_{\text{CFT}, T_{z\bar{z}}=} \frac{1}{2\pi} \int d^2 z \Big[T \partial_{\bar{z}} \epsilon + \bar{T} \partial_z \bar{\epsilon} \Big] \tag{103}$$

Now if we set: $\epsilon(z, \bar{z}) = \epsilon(z) f(\bar{z})$

and treat $z, \bar{z}$ as independent variables, then $\delta z = \epsilon(z, \bar{z}), \delta\bar{z} = 0 \implies$

$$y^z = 0, y^{\bar{z}} = T(z)\epsilon(z) \tag{104}$$

and one can check $\partial_\alpha j^\alpha = 0$

also if $\delta z = 0, \delta\bar{z} = \bar{\epsilon}(z, \bar{z}) \implies$

$$\bar{y}^{\bar{z}} = 0, \bar{y}^z = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \tag{105}$$

8.3 Example: free scalar field

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\partial_\alpha X\partial^\alpha X \tag{106}$$

$$T_{\alpha\beta} = -\frac{1}{\alpha'}(\partial_\alpha X\partial_\beta X - \frac{1}{2}\delta_{\alpha\beta}(\partial X)^2) \tag{107}$$

$$T = -\frac{1}{\alpha'}\partial X\partial X \tag{108}$$

$$\text{by using e.o.m., and } \partial\bar{\partial}X = 0 \implies X = X + \bar{X} \tag{109}$$

$$\bar{T} = -\frac{1}{\alpha'}\bar{\partial}X\bar{\partial}X \tag{110}$$

and we conclude with, $T|_{\text{e.o.m}} = T(z), \bar{T}|_{\text{e.o.m}} = \bar{T}(\bar{z})$

8.4 Quantum Aspects

In CFT we study the correlation functions among fields, however please note that there is different terminology in CFT with respect to QFT. Namely the notion of what is field means something more general.

So ”fields” are all local operators, i.e. In QFT, we call ϕ a field, but in CFT)$\partial\phi, e^{i\phi}, \cdots$ are fields as well.

8.5 Operator product expansion (OPE)

In CFT, we can also do something called operator product expansion. Now here we want to ask the question of: What happens if local operators approach each other?. In most textbooks, this section is put at the last, however, this is often the key object we look for when studying CFT.

In abstract sense, we consider local operators

$$O_i(z,\bar{z})O_i(\omega,\bar{}) = \sum_k \underbrace{C^k_{ij}(z-\omega,\bar{z}-\bar{w})}_{\text{only difference is due to transl. invar.}} O_k(\omega,\bar{\omega}) \tag{111}$$

So the interpretation here is that as the two approaches, it can be expressed as a sum of operators at

$(\omega, \bar{\omega})$

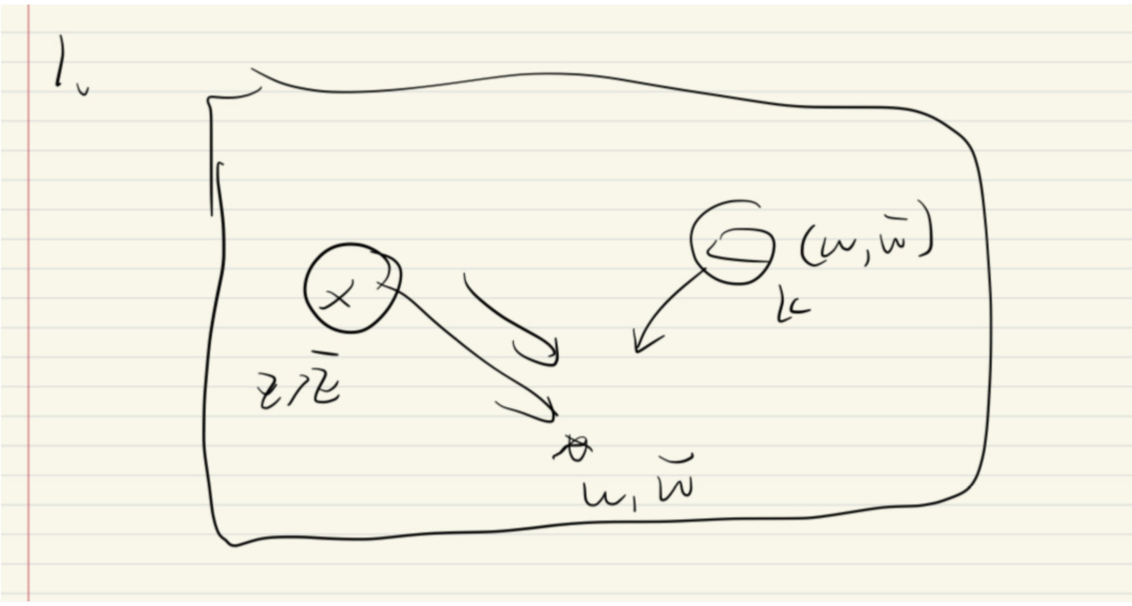

Figure 2:

Remarks: OPEs are always understood as being evaluated inside time-ordered correlation functions.

$$\langle \mathcal{O}_i(z,\bar{z})\mathcal{O}_j(\omega,\bar{\omega}) \cdot X\rangle = \sum_k C^k_{ij}(z-\omega, \bar{z}-\bar{w})\langle \mathcal{O}_k(\omega, \bar{\omega} \cdot X)\rangle \tag{112}$$

, where $\langle O\rangle = \int \mathcal{D}\phi e^{-S[\phi]}/Z$

Remarks:

· note that the correlation functions are always time-ordered (or "radially ordered", see later sections). This implies that everything commutes by default.

· operator insertions X in (112) are arbitrary but ned to be "away" from $O_1(\omega)O_2(z)$. This is intuitive as X itself is inside the contour that contains both operators, as it will influence the expansion.

· all the important information is in singular behaviour of OPE when $\omega \to z$, e.g. commutation relations, how operators transform etc. So this in a sense is more fundamental than commutation relations and transformation of operators, the "information encoder" is C_{ij}^k in particular.

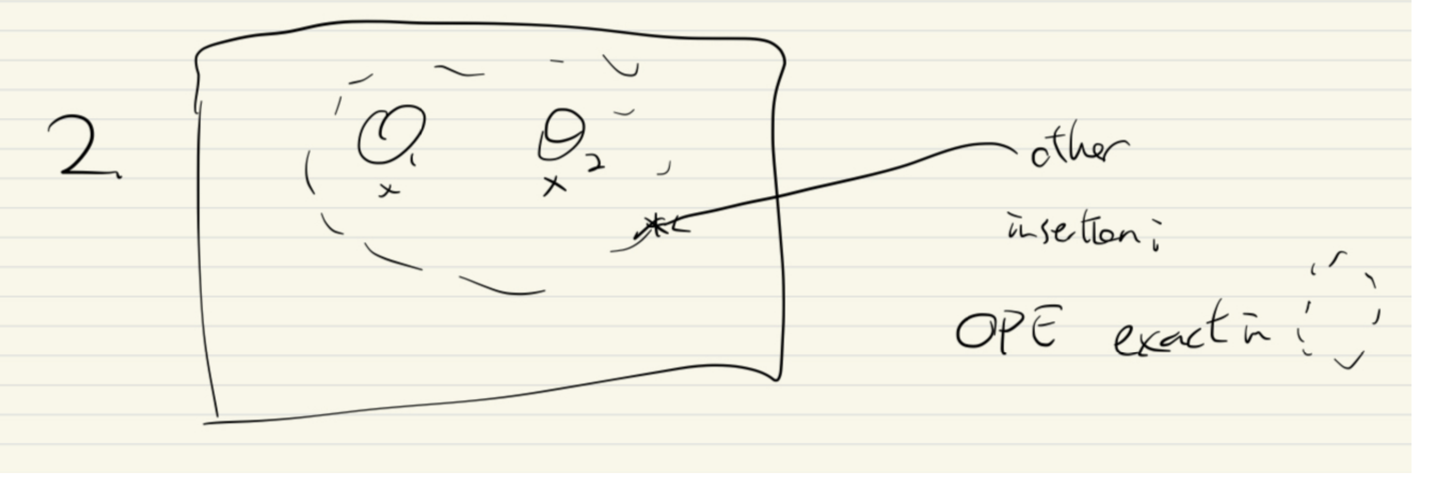

Figure 3:

8.6 Ward identities

Ward identities are conservations laws in QFT. We can find a corresponding version of this in CFT. To do so, we first find a general symmetry, then the conformal symmetry.

8.6.1 General derivation

a) **Quantum Noether**

As usual, we define

$$Z = \int \mathcal{D}\phi e^{-S[\phi]} \tag{113}$$

Imposing the symmetry: $\phi' = \phi + \epsilon\delta\phi$

Now we perform the trick: $\epsilon \to \epsilon(\sigma)$, and we see

$$Z \to \int \mathcal{D}\phi' e^{-S[\phi']} = \int \mathcal{D}\phi e^{-S[\phi]}(1 - \frac{1}{2\pi}\int y^\alpha \partial_\alpha \epsilon) \tag{114}$$

note that in general: y^α can be have contributions from the measure as well. However: In the path integral, $\phi \to \phi'$ is not changing anything in the path integral. And therefore we get the following

$$\int \mathcal{D}\phi e^{-S[\phi]}(\int y^\alpha \partial_\alpha \epsilon) = 0 \tag{115}$$

So, $\implies$ $\boxed{\langle \partial^\alpha y_\alpha \rangle = 0}$, for all $\epsilon(\sigma)$, but we can not drop the brackets (i.e.) it is only valid up to the operators (Quantum version statement).

Thus we look for a stronger statement, then we should look at time ordered correlation function with operators inserted.

$$\langle \mathcal{O}_1(\sigma_1)\cdots\mathcal{O}_n(\sigma_n)\rangle \frac{1}{Z}\int \mathcal{D}\phi e^{-S[\phi]}\mathcal{O}_1(\sigma_1)\cdots\mathcal{O}_n(\sigma_n) \tag{116}$$

, and impose symmetry: $\mathcal{O}_i \to \mathcal{O}_i + \epsilon\delta\mathcal{O}_i$, and again apply the trick: $\epsilon \to \epsilon(\sigma)$ So first, With this

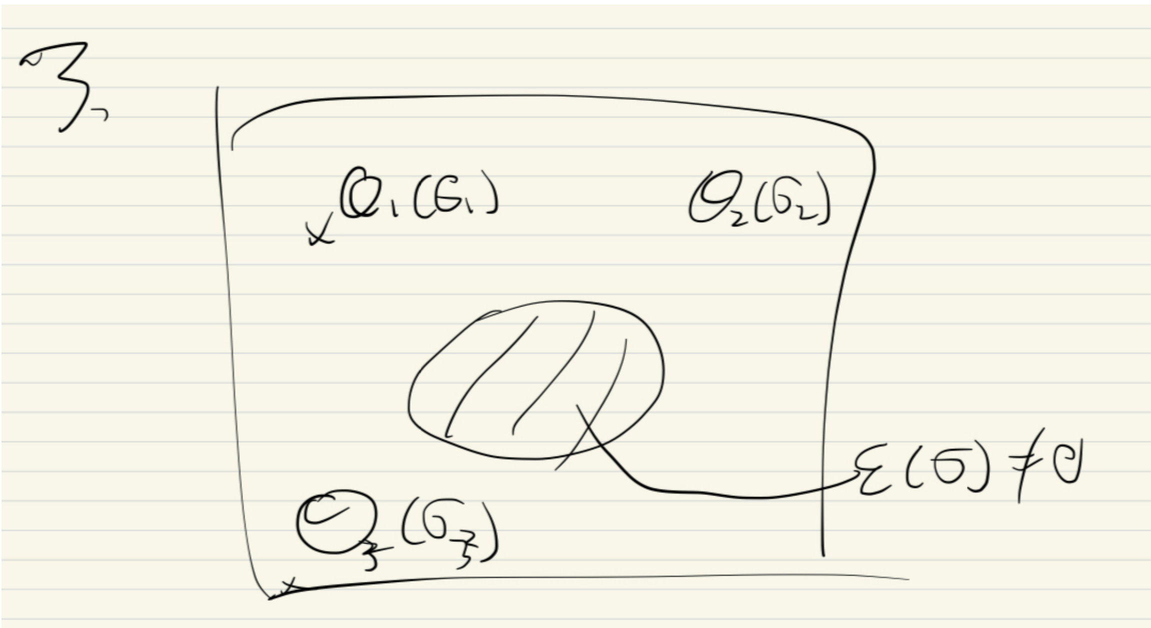

Figure 4:

interpretation, and as above, we get the implication

$$\langle \partial_\alpha y^\alpha(\sigma) \mathcal{O}_1(\sigma_1) \cdots \mathcal{O}_n(\sigma_n) \rangle = 0 \tag{117}$$

, for $\sigma \neq \sigma_i$ And next, we consider the same picture but with one operator in ϵ region which the Z now reads

$$\frac{1}{Z} \int \mathcal{D}\phi e^{-S[\phi]} (1 - \frac{1}{2\pi} \int y^\alpha \partial_\alpha \epsilon)(\mathcal{O}_1 + \epsilon \delta \mathcal{O}_1) \mathcal{O}_2 \mathcal{O}_3 \cdots \mathcal{O}_n \tag{118}$$

Now if we look at the lowest ϵ order:

$$-\frac{1}{2\pi} \int_\epsilon \partial_\alpha \langle y^\alpha(\sigma) \mathcal{O}_1(\sigma_1) \cdots \mathcal{O}_n(\sigma_n) =\rangle = \langle \delta \mathcal{O}_1(\sigma_1) \mathcal{O}_2(\sigma_2) \cdots \mathcal{O}_n(\sigma_n) \rangle \tag{119}$$

, note that in the integral bound, $\epsilon \neq 0$ region is surrounding σ_1

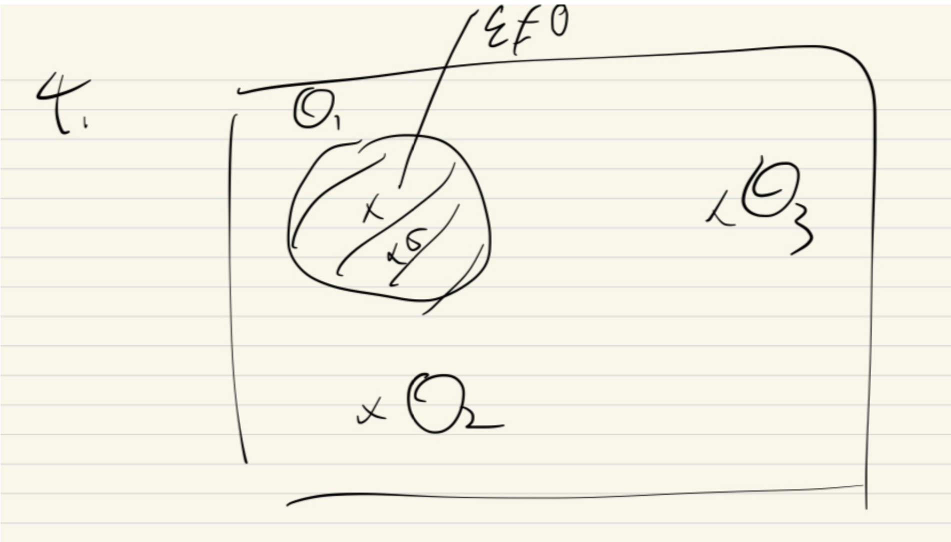

Figure 5: now we

Now we apply the Stokes Theorem:

$$\int_{\epsilon} \partial_\alpha y^\alpha =_{\partial\epsilon} y_\alpha \hat{h}^\alpha = \int_{\partial\epsilon} y_1 \mathrm{d}\sigma^2 - y_2 \mathrm{d}\sigma^1 \tag{120}$$

$$= -i_{\partial\epsilon}(y_z \mathrm{d}z - y_{\bar{z}} \mathrm{d}\bar{z}) \tag{121}$$

and recall: $y_z = \frac{1}{2}(y_1 - iy_2)$ Then we recover the Ward Identity

$$\frac{i}{2\pi}\int_{\partial\epsilon} \mathrm{d}z \langle y_z(z,\bar{z})\mathcal{O}_1(\sigma_1)\cdots\rangle - \frac{i}{2\pi}\int_{\partial\epsilon} \mathrm{d}\bar{z}(y_{\bar{z}}(z,\bar{z})\mathcal{O}_1(\sigma_1)\cdots) \tag{122}$$

$$= \langle \delta\mathcal{O}_1(\sigma_1)\rangle \tag{123}$$

8.7 Ward identities for conformal transformation

Recall we defined (106) $y_z = T(z)\epsilon(z), y_{\bar{z}} = \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$ These implies

$$\frac{i}{2\pi} \mathrm{d}z y_z \mathcal{O}_1(\sigma_1) = -\mathrm{Res}[y_z\mathcal{O}_1] \tag{124}$$

$$y_z(z)\mathcal{O}_1(\omega,\bar{\omega}) = \cdots + \frac{\mathrm{Res}[y_2\mathcal{O}_1]}{z-\omega} + \frac{(z-\omega)^0}{\cdots} + \cdots \tag{125}$$

With $z, \bar{z}$ both indep. we apply the transformation and get

$$\begin{cases} \delta z = \epsilon(z) : \delta\mathcal{O}_1(\sigma_1) = -\mathrm{Res}[y_z(z)\mathcal{O}_1(_1)] = -\mathrm{Res}[\epsilon(z)T(z)\mathcal{O}_1(\sigma_1)] \\ \text{bar everything on } \epsilon, z, y, \text{ of previous equation} \end{cases} \tag{126}$$

So, if we know OPE $T(z)\mathcal{O}_1(\sigma_1)$, then we can read off how operators transform under conformal transformation.

and if we know how to the operators transforms, then we know part of OPE with $T, \bar{T}$ known.

Remarks:

1. Note that when evaluating this operator equation, one always need to evaluate the expectation of it (as well as with other operator).

Above is the end of Lecture 9 content.

8.8 Primary operators

Next, we introduce a set of conformal operator, called primary operators which essentially are the most fundamental operators in CFT.

To do so, we can make use of the transformation properties of the conformal operators we have found previously and we need to piece these OPEs together.

We start with simple conformal transformation,

1. translations: $\delta z = \epsilon \implies \mathcal{O}(z-\epsilon) = \mathcal{O}(z) - \epsilon\partial\mathcal{O}(z)$

   $\delta\mathcal{O}(z) = -\epsilon\partial\mathcal{O}(z)$ With these, we then impose the Ward identity and get

$$T(z)\mathcal{O}(\omega,\bar{\omega}) = \cdots + \frac{\partial\mathcal{O}(\omega,\bar{\omega})}{z-\omega} + \cdots \tag{127}$$

   and the same for the $\bar{T}$ as well. with one of the laurent expansion term fixed.

2. rotations and scalings: $z \to z + \epsilon z \implies \delta z = \epsilon z; \bar{z} \to \bar{z} + \bar{\epsilon}\bar{z}$

   Remarks:

   Note that in general operators rotation or scalings does not transform well, but in QM we can pick a basis of local operators that have well defined transformation behaviour. To do so, we simply make sure these operators are in the eigenbasis representation. So we Def: Operator $\mathcal{O}$ has *weight* $h, \tilde{h}$ if it transforms under $\delta z = \epsilon z$, $\delta\bar{z} = \bar{\epsilon}\bar{z}$, such that

$$\delta\mathcal{O} = -\epsilon(h\mathcal{O}) - \bar{\epsilon}(\tilde{h}\mathcal{O} + \bar{z}\bar{\partial}\mathcal{O}) \tag{128}$$

   Also note that $h, \tilde{h}$ are real numbers in a unitary CFT, with $h, \tilde{h} > 0$

We can further define spin as $s = h - \tilde{h}$, and the scaling dimension $\Delta = h + \tilde{h}$

And we have now the Rotational operator

$$L = z\partial - \bar{z}\bar{\partial}, \tag{129}$$

and the dilation operator D, which acts as scaling, as

$$D = z\partial + \bar{z}\bar{\partial}. \tag{130}$$

Remarks:

- Δ are "often" the dimensions in dimension analysis e.g. $\Delta[\partial] = +1$

It is then instructive to compare this to the transformation law defined previously, we can do so by applying the Ward identity to the transformation law.

$$T(z)\mathcal{O}(\omega,\bar{\omega}) = \cdots + h\frac{\mathcal{O}(\omega,\bar{\omega})}{(z-\omega)^2} + \frac{\partial\mathcal{O}(\omega,\bar{\omega})}{z-\omega} + \cdots \tag{131}$$

, and similarly for the barred term.

8.9 Primary operators

To infer transformation of primary operators under general conformal transformation, we can use the transformation laws for $\delta z = \epsilon(z)$ and assume $\epsilon(z)$ has no singularity at $z = \omega$, this then implies $\epsilon(z) = \epsilon(\omega) + \epsilon'\omega)(z-\omega) + \cdots$ and we get

$$\delta\mathcal{O}(\omega,\bar{\omega}) = -h\epsilon'(\omega)\mathcal{O}(\omega,\bar{\omega}) - \epsilon(\omega)\partial\mathcal{O}(\omega,\bar{\omega}) \tag{132}$$

, infinitesimal to finite transformation, $z \to \tilde{z}(z), \bar{z} \to \bar{\tilde{z}}(\bar{z})$ So the transformation of the primary operators becomes

$$\mathcal{O}(z,\bar{z}) \to \tilde{\mathcal{O}}(\tilde{z},\bar{\tilde{z}}) = \left(\frac{\partial\tilde{z}}{\partial z}\right)^{-h}\left(\frac{\partial\bar{\tilde{z}}}{\partial z}\right)^{-\tilde{h}}\mathcal{O}(z,\bar{z}), \tag{133}$$

now these primary operator and "weight-spectrum" $(h,\tilde{h})$ is central in CFT, and we can compare to mass spectrum of QFT

8.10 Example: free scalar field

8.10.1 Propagator

Similar: We try to match

$$\langle \partial^2 X(\sigma) X(\sigma') \rangle = -2\pi a' \delta(\sigma - \sigma') \tag{134}$$

, and to do so we look at

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X(\sigma)} \Big(e^{-S} X(\sigma') \Big) \tag{135}$$

, to Solve this, then we need to use $\partial^2 \log(\sigma - \sigma')^2 = 4\pi\delta(\sigma - \sigma')$

We can then further repeat the trick as before:

$$0 = \int \mathcal{D}X \frac{\delta}{\delta X(\sigma)} [e^{-S} \mathcal{O}_1(\sigma_1) \cdots \mathcal{O}_n(\sigma_n)] \tag{136}$$

, and it implies the OPE:

$$X(\sigma) X(\sigma') = -\frac{\alpha'}{2} \log(\sigma - \sigma')^2 + \cdots \tag{137}$$

But the problem is that X is not nice, so we use ∂X to represent instead

$$\partial X(z) \partial X(\omega) = -\frac{\alpha'}{2} \frac{1}{(z - \omega)^2} + \text{non-singular} \tag{138}$$

8.10.2 Stress-energy tensor

define the stress-energy tensor as

$$T = -\frac{1}{\alpha'} \partial X \partial \tag{139}$$

, the problem with definition is that the operators are evaluated at the same point, which implies normal order problem, so we define the normal ordering as

$$\text{"normal ordering"}: \quad T(\omega) := \frac{1}{\alpha'} : \partial X \partial X :\equiv -\frac{1}{\alpha'} \lim_{z\to\omega} (\partial X(z)\partial X(\omega) - \langle \partial X(z)\partial X(\omega)\rangle) \tag{140}$$

, this implies $\langle T \rangle = 0$

important exercises:

- ∂X is primary $(h, \tilde{h}) = (1, 0)$
- $: e^{ikX} :$ is also primary $(h, \tilde{h}) = (\alpha' k^2/4, \alpha' k^2/4)$
- the two stresses product reads

$$T(z)T(\omega) = \frac{1/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \cdots \tag{141}$$

, where we have that (i) T is not primary; (ii) $(h, \tilde{h}) = (2, 0)$.

8.11 Central charge

In any CFT: T is (2, 0) iff $\Delta = 2$ since obtained by integrating over space, and $s = 2$ since T is a symmetric 2-tensor. So we have that the OPE

$$T(z)T(\omega) = \cdots + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{z-\omega} + \cdots, \tag{142}$$

similarly for $\bar{T}\bar{T}$ The remaining terms via dimensional analysis, should be of the form $\mathcal{O}_n/(z-\omega)^n$, and $\Delta[\mathcal{O}] = 4 - n$

but: In unitary CFT: $h, \tilde{h}$ implies at most singular term $(z-\omega)^4$. So we only get

$$T(z)T(\omega) = \frac{c/2}{(z-\omega)^4} + \frac{2T(\omega)}{(z-\omega)^2} + \frac{\partial T(\omega)}{(z-\omega)} + \cdots \tag{143}$$

similarly for the barred counterpart. Here the constants $c, \tilde{c}$ are the central charges ”left-moving”, ”right-moving”. **Comments:**

- very important property of CFT, e.g. for D-free scalar fields $c = \tilde{c} = D$
- the central charge can be non-integer.
- T is not primary if $c \neq 0$, and implies a unitary non-trivial CFT $c, \tilde{c} > 0$.

This marks the end of lecture ?

This is the beginning of lecture ? + 1

Since we see from (143) that they are not primary operator, then we introduce transformation of T:

$$\delta T(\omega) \underbrace{=}_{\text{Ward}} -\mathrm{Res}[\epsilon(z)T(z)T(\omega)], \tag{144}$$

$$\tag{145}$$

where for non-singular $\epsilon(z)$,

$$\epsilon(z) = \epsilon(\omega) + \epsilon'(\omega)(z-\omega) + \frac{1}{2}\epsilon'''(\omega)(z-\omega)^3 + \cdots \tag{146}$$

Then we get finally the transformation

$$\delta T(\omega) = -\epsilon(\omega)\partial T(\omega) - 2\epsilon'(\omega)T(\omega) - \frac{c}{12}\epsilon'''(\omega), \tag{147}$$

$$\tag{148}$$

which integrates to,

$$\tilde{T}(\tilde{z}) = (\frac{\partial \tilde{z}}{\partial z})^{-2}[T(z) - \frac{c}{12}S(\tilde{z}, z)], \tag{149}$$

with Schwarzian

$$S(z,\tilde{z}) = \left(\frac{\partial^3 \tilde{z}}{\partial z^3}\right)\left(\frac{\partial \tilde{z}}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 \tilde{z}}{\partial z^2}\right)^2 \left(\frac{\partial \tilde{z}}{\partial z}\right)^{-2} \tag{150}$$

8.12 c is for Casimir

S-term does not depend on T implies that it can be evaluated on all states while shifting the constant term, such a physical interpretation is shifting the energy by constant term which is *Casimir energy*.

Important example:

We can fall back to the Euclidean cylinder, $\omega = \sigma + i\tau, \quad \sigma \in [0, 2\pi)$, and with the map

$$z = e^{-i\omega}. \tag{151}$$

Then we can map each circles on the cylinder to a radius of circle on the z coordinate sheet, the point at the center of z complex plane corresponds to $\omega = \sigma + i(-\infty)$, i.e. infinite past.

To see the transformation of T, we can set $S(z,\omega) = 1/2$, and we get

$$T_{\text{cylinder}}(\omega) = -z^2 T_{\text{plane}}(z) + \frac{c}{24}. \tag{152}$$

If we have $\langle T_{\text{plane}} \rangle = 0$ then it is the ground state. We can also define the Hamiltonian as

$$H \equiv \int \mathrm{d}\sigma T_{\tau\tau} = -\int \mathrm{d}\sigma (T_{\omega\omega} + \bar{T}_{\bar{\omega}\bar{\omega}}), \tag{153}$$

And the ground state energy is

$$E = -\frac{2\pi(c+\tilde{c})}{24} = -\frac{2\pi}{12} \tag{154}$$

8.13 Weyl anomaly

The quantum expression for the trace of operator $\langle T^{\alpha}_{\alpha} \rangle = -\mathcal{R}c/12$, where $\mathcal{R}$ is the 2D Ricci scalar, where inlmcalR is the 2D Ricci scalar. One can obtain this expression through OPE caclulations, however, let us gain some insight from the result.

First, we know that on flat space $\langle T^{\alpha}_{\alpha}\rangle$ on a quantum level, but $\langle T^{\alpha}_{\alpha} \neq 0\rangle$ for $c \neq 0$ and on curved background.

Remarks:

- $\langle T^{\alpha}_{\alpha}\rangle \propto \mathcal{R}$ is natural, since

  1. same for all states, implies it can only depend on background metric
  2. it should be local and of dimension 2, which implies $\mathcal{R}$ is the only possibility.

Now, we investigate for the coefficients attached to the Ricci Scalar, we can choose the metric $g_{\alpha\beta} = e^{2\omega}\delta_{\alpha\beta}$ Then we have $\mathcal{R} = -2e^{-2\omega}\partial^2\omega$, and the physical observable $\langle T^{\alpha}_{\alpha}\rangle$ takes different values on background differing by Weyl transformation if $c \neq 0$ which implies the Weyl anomaly.

Note that this Weyl anomalies have analogues in higher dimensions, e.g.

$$\langle T^{\mu}_{\mu}\rangle_{4d} = \frac{c}{16\pi^2}C_{\rho\sigma\kappa\lambda}C^{\rho\sigma\kappa\lambda} - \frac{a}{16\pi t^2}\tilde{\mathcal{R}}_{\rho\sigma\kappa\lambda}\tilde{\mathcal{R}}^{\rho\sigma\kappa\lambda} \tag{155}$$

8.14 c is for Cardy

In CFT, the entropy of higher energy states goes like $S(E) \sim \sqrt{cE}$, which is known as Cardy's formula, where we see again c sets up the scales of this relation. (Not mentioned at all in lecture).

8.15 c has a theorem

This theorem is a result of Zamalodchikov, it states that if we look at the space of all 2d QFTs and the Renormalization group flows between them (i.e. changes in energy scale).

The procedure is as follows, first we get the UV QFT c value via $\langle T^{\alpha}_{\alpha}\rangle$, then we do the same for the IR theory (of course one has to renormalize the UV theory), and the theorem is that, $c_{\mathrm{UV}} > c_{IR}$.

Remarks: Note that for higher dimension CFT, we will get a value as well and in fact that is also the a has a theorem, which was proven not long ago.

8.15.1 More on RG group and CFT

Note that, CFTs are fixed points of RG-flow, where c becomes the central charge, so we can consider perturbing a CFT by adding extra terms to the action, i.e. $S \rightarrow S + \alpha \int d^2\sigma \mathcal{O}(\sigma)$, with $\mathcal{O}(\sigma)$ of dimension Δ. And we have the classes/cases

- $\Delta < 2 : [\alpha] = 2 - \Delta > 0$, these deformation are called *relevant* because they are important in the IR (because IR is scale dependent). So the RG flow takes us away from the original CFT, and then stop at a new CFT
- $\Delta = 2 : \alpha$ dimensioless, these are marginal deformations, which defines new CFT.
- $\Delta > 2 : \alpha$ is negative, then these deformation are scale invariant, which is irrelevant to IR (i.e. flows to the same CFT in IR), but only UV physics is effected.

9 Virasoro Algebra revisited

Let us now study the evolution of states in CFT, this quantization requires a split in ”space” and ”time”. Then the states actually live on the spatial slices, and the Hamiltonian gives the time evolution. We set up the quantization for the CFT plane as follows,

$$\omega = \sigma + i\tau, \qquad z = e^{-i\omega} \tag{156}$$

And on the cylinder, we have $H = \partial_\tau$, and after the transition to the CFT plane, the Hamiltonian becomes the dilatation operator

$$D = z\partial + \bar{z}\bar{\partial}, \tag{157}$$

which this operator lives on the constant radius circle boundary of the CFT plane, this is so-called the *radial quantization*.

These radial ordering of correlation functions (note: by using the same time ordering as we had before), are such that large radius operators is moved to the left. Because larger radius implies further in time.

The upshot here is that, the Virasoro generators,

This starts the lecture 12. Our main goal here today is to relate the virasoro algebra with the momentum tensors.

9.1 Virasoro Generators

we define the virasoro algebra as

$$L_n = \frac{1}{2\pi i}\int \mathrm{d}z z^{n+1} T(z), \quad \tilde{L}_n = \frac{1}{2\pi i}\int \mathrm{d}\bar{z}\bar{z}^{n+1}\bar{T}(\bar{z}), \tag{158}$$

and note that there are no other insertions in the contour.

Now L_n is conserved charge under conformal transformation $\delta z = z^{n+1}$

Comments:

- L$_{-1}\tilde{L}_{-1} generatetranslationsintheplane. L_0 and \tilde{L}_0 generatescalingandrotations.$

- The dilatation operator is defined, which measures the energy of states on the cylinder, by

$$D = L_0 + \tilde{L}_0 \tag{159}$$

9.2 Virasoro Algebra

The Virasoro generators obeys the algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{160}$$

Next, we can check by using TT OPE and contour deformation, that the commutator follows the right radial ordering.

More specifically, we write L_m as contour integral over dz and L_n over dω Then we get,

$$[L_m, L_n] = \Big(\frac{\mathrm{d}z}{2\pi i}\oint\frac{\mathrm{d}\omega}{2\pi i} - \oint\frac{\mathrm{d}\omega}{2\pi i}\frac{\mathrm{d}z}{2\pi i}\Big)z^{m+1}\omega^{n+1}T(z)T(\omega) \tag{161}$$

And fix ω first and we get

(c)

Show that the energy density in this homogenous mode is,

sol:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V \tag{162}$$
$$= (\frac{\phi_0}{2t})^2 + M^4 e^{-\lambda\phi/M} \tag{163}$$
$$= (\frac{M}{t\lambda})^2 + M^4 e^{-2\ln(t/\tau)} \tag{164}$$
$$= \frac{M^2}{t^2\lambda^2} + M^4(\frac{t}{\tau})^{-2} \tag{165}$$
$$= \frac{M^2}{t^2\lambda^2} + M^4(\frac{(2/\lambda^2 M^2[6/\lambda^2(M/M_P)^2 - 1])}{t^2}) \tag{166}$$
$$= \frac{12M^4}{\lambda^4 M_2^P t^2} \tag{167}$$

9.2.1 The State-Operator Map

For a general QFT, we learn that: operators are defined at points in spacetime, while states are defined on space-like slices.

And for QM, we only have 1D QFT, i.e. wave functions. And we develop it from some initial configuration to some end configuration.

Explicitly,

$$\psi_f(x_f, \tau_f) = \int \mathrm{d}x_i G(x_f, x_i)\psi_i(x_i, \tau_i) \tag{168}$$

And note that the propagator/green function here, the particles move from $x_i \to x_f$.

And moving onto QFT, we have wave functional:

$$\psi[\phi_f[\sigma], \tau_f] = \int \mathcal{D}\, \phi_i \int_{\phi_i}^{\phi_f} \mathcal{D}\phi e^{-S[\phi]} \Psi_i[\phi_i(\sigma), \tau_i] \tag{169}$$

References