Utrecht University

Gravitational waves and observations

Yamamoto Sho

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Contents

1 sonic of Landau-Lifschitz Formalism of EFE

Define the metric density (which later referred as gothic) as $\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu}$ and $\mathfrak{g}_{\mu\nu} = \frac{1}{\sqrt{-g}}g_{\mu\nu}$, and of course the kronecker relation $\mathfrak{g}^{\nu\alpha}\mathfrak{g}_{\mu\nu} = \delta^{\alpha}_{\mu}$

Next, if we chug this whole thing in EFE and make use of these properties, one should obtain the the EFE in gothic formalism, ***Note that:*** this a very important formula, with the kronecker relation, we can partial the whole relation and get,

$$\mathfrak{g}_{\mu\nu}\partial_{\rho}\mathfrak{g}^{\mu\alpha} + \mathfrak{g}^{\mu\alpha}\partial_{\rho}\mathfrak{g}_{\mu\nu} = 0 \tag{1}$$

Then contract with $\mathfrak{g}_{\alpha\beta}$ and the kronecker relation once again to obtain,

$$\partial_{\rho}\mathfrak{g}_{\beta\nu} = -\mathfrak{g}_{\alpha\beta}\mathfrak{g}_{\mu\nu}\partial_{\rho}\mathfrak{g}^{\mu\alpha} \tag{2}$$

Note 2 that: It is also beneficial to obtain the relation of gothic versus metric, this can be achieved by,

$$0 = \partial_{\mu}(g_{\alpha\beta}g^{\alpha\beta}) = \partial_{\mu}\left(g_{\alpha\beta}\frac{\mathfrak{g}^{\alpha\beta}}{\sqrt{-g}}\right) \tag{3}$$

After some manipulation, we can then get the important conversion relation,

$$g^{\alpha\beta}\partial_{\mu}g_{\alpha\beta} = \mathfrak{g}^{\alpha\beta}\partial_{\mu}\mathfrak{g}_{\alpha\beta} \tag{4}$$

Note 3 that: At last, we consider a mixed indices contraction,

$$0 = g_{\beta\gamma}\partial_{\mu}(\mathfrak{g}^{\alpha\beta}g_{\alpha\rho}) \implies 0 = -\frac{1}{2}g^{\alpha\beta}g_{\gamma\rho}\partial_{\mu}g_{\alpha\beta} + \partial_{\mu}g_{\gamma\rho} + g_{\alpha\rho}\mathfrak{g}_{\beta\gamma}\partial_{\mu}\mathfrak{g}^{\alpha\beta} \tag{5}$$

where in the second expression we have used conversions between metric and gothics.

Packing everything up, we get the "ultimate" relation,

$$\partial_\mu g_{\gamma\rho} = -\sqrt{-g}\mathfrak{g}_{\alpha\rho}\mathfrak{g}_{\beta\gamma}\partial_\mu \mathfrak{g}^{\alpha\beta} + \frac{1}{2}\sqrt{-g}\mathfrak{g}_{\alpha\beta}\mathfrak{g}_{\gamma\rho}\partial_\mu \mathfrak{g}^{\alpha\beta} \tag{6}$$

1.1 EFE in gothic

With all the relation above, we can churn through it all, which is done in tutorial 2, to get the EFE (also the Christoffel symbols before)

$$\mathfrak{g}^{\mu\nu}\partial_\mu\partial_\nu \mathfrak{g}^{\alpha\beta} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} + \underbrace{\Sigma^{\alpha\beta}[\mathfrak{g}, \partial\mathfrak{g}]}_{\text{nonlinear terms}} . \tag{7}$$

And forget about $\Sigma^{\alpha\beta}$, its just nonlinear 9 terms.

We then decompose the gothic as $\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$, with $\eta^{\mu\nu}$ the flat gothic metric.

Next, we impose the harmonic coordinate condition,

$$\partial_\mu h^{\mu\alpha} = 0 \tag{8}$$

Finally get the EFEs in a different form

$$\Box_\eta h^{\alpha\beta} = \frac{16\pi G}{c^4}\tau^{\alpha\beta}, \tag{9}$$

where $\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + \frac{c^4}{16\pi G}\mathcal{N}^{\alpha\beta}(h, \partial h, \partial^2 h)$

2 energy carried by GWs

We define GW in the following context (more like a pictorial sense of).

Note: So the caveat here is that GW is only defined only when there exists a separation of scales.

Figure 1: Regardless of the background metric, as long as there is a distinct periodicity at some characteristic length, it is a GW signal.

Now let

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu}_{(1)} + \epsilon h^{\mu\nu}_{(1)} + \epsilon^2 h^{\mu\nu(2)}. \tag{10}$$

, note that everything here are gothics but not metric tensors.

Next we impose the gauge fix $\Box_\eta \eta^{\alpha\beta} = 0 \implies \eta = \text{const}$.

And we can tackle the EFE by recursive perturbation, which reads for the first two orders as

$$\begin{cases} \Box_\eta h^{\alpha\beta}_{(1)} &= 0 \\ \Box_\eta h^{\alpha\beta}_{(2)} &= \mathcal{N}^{\alpha\beta}[h_{(1)}]. \end{cases} \tag{11}$$

The first order corresponds to simply the waves, with purely oscillation. The second order however corresponds to the second order term, which has both energy and oscillation perturbation, this is because second order terms goes like $\Box^2 h$, which indeed appears in EFE (9). "Explicitly", we can denote these solutions as $h^{\alpha\beta}_{(1)} = \hat{h}^{\alpha\beta}_{(1)}$, and $\hat{h}^{\alpha\beta}_{(2)} = \hat{h}^{\alpha\beta}_{(2)} + \langle h^{\alpha\beta}_{(2)}\rangle$ and in particular the averaged term indeed does show up in the EM tensor.

2.1 Lightning review of $\langle\cdot\rangle$ operator

· It commutes with $\Box_\eta$

i.e.$[\Box_\eta, \langle\rangle] = 0$ which implies $\Box(\langle h^{\alpha\beta}_{(1)}\rangle + \hat{h}^{\alpha\beta}_{(2)}) = 0$

So we naturally obtain

$$\Box(\cancelto{0}{\eta^{\alpha\beta}} + \epsilon\cancelto{0}{h^{\alpha\beta}_{(1)}} + \epsilon^2\langle h^{\alpha\beta}_{(2)}\rangle) = \epsilon^2\langle\mathcal{N}^{\alpha\beta}_{(2)} \equiv \frac{32\pi G}{c^4}\tau^{\text{eff,GW}}_{\alpha\beta} \tag{12}$$

3 Slightly different from linearized gravity

In our previous formalism, GWs source are not corrected to ”background” at 2nd order. And thus we missed out on the coarse-grain effects of GWs.

3.1 Packaging up the formalism

In order to obtain the energy-momentum tensor contributed by the gravitational wave, we can follow the procedure as follows

1. Assume TT-gauge and implement average over $\mathcal{L} \gg \lambda$ and use I.B.P. multiple throughout the calculation below,

$$\langle h^{\alpha\beta}\partial_\alpha\partial_\beta h^{\mu\nu}\rangle = \langle\partial_\alpha(h^{\alpha\beta\partial_\eta h^{\mu\nu}}) - \langle(\cancelto{0}{\partial_\alpha h^{\alpha\beta}})(\partial_\beta h^{\mu\nu})\rangle. \tag{13}$$

, the 2nd term is zero due to harmonic gauge.

2. From here, we further massage (13) via the three tricks below:

$$\begin{cases} \partial_\mu h^{\alpha\beta}\partial_\gamma h^{\mu\delta} & = \partial_\mu(h^{\alpha\beta}\partial_\gamma h^{\mu\delta}) - h^{\alpha\beta}\cancelto{0}{(\partial_\gamma\partial_\mu h^{\mu\delta})} \\ \partial_\mu h^{\mu\beta} & = 0 \\ \partial_\alpha\partial^\alpha h^{\mu\nu}_{(1)} & = 0 \end{cases} \tag{14}$$

1st is simply I.B.P., 2nd is harmonic gauge condition, and 3. is ?

3. Keep working then one will eventually obtain the expression

$$t^{\text{GW}}_{\mu\nu} = \langle\frac{c^4}{32\pi G}\partial^\mu h^{(1)}_{\alpha\beta}\partial^\nu h^{\alpha\beta}_{(2)}\rangle \tag{15}$$

And here t_{GW}^{00} is defined as the GW energy density, which written explicitly as

$$t_{\mathrm{GW}}^{00} = \frac{c^2}{32\pi G}\langle \dot{\hat{h}}_{ij}\dot{\hat{h}}^{ij}\rangle = \frac{c^2}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2\rangle \tag{16}$$

The c^2 is suppressed by $\partial_0 = 1/c\partial$

The GW power is then defined as,

$$\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_v \mathrm{d}^3x\, t_{00}^{\mathrm{GW}} = c\int_v \mathrm{d}^3x \partial_0 t_{\mathrm{GW}}^{00} \tag{17}$$

Next, we can impose conservation law such that $\partial_\mu t_{\mathrm{GW}}^{\mu\nu} = 0$, which implies $\partial_0 t_{00}^{\mathrm{GW}} + \partial_0 t_{\mathrm{GW}}^{0i} = 0$, and it implies finally $\partial_0 t_{\mathrm{GW}}^{00} = -\partial_0 t_{\mathrm{GW}}^{0i}$

Therefore we perform gauss law on (17),

$$\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}t} = c\int_v \mathrm{d}^3x(-\partial_i t_{\mathrm{GW}}^{0i}) \tag{18}$$

$$= c\int_S \mathrm{d}^2x(u^i t_{\mathrm{GW}}^{0i}) \tag{19}$$

where u^i is the unit normal to the surface S.

The pictorial interpretation is as follow

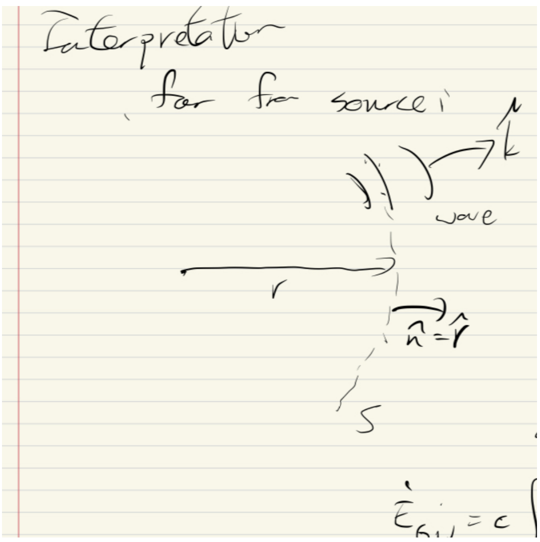


Figure 2: this is the pictorial representation of (18).

So we approximately get $h_{ij} \approx h_{ij}(t - r/c) \implies \partial_i h(t - r/c) = 1/c\partial_t h(t - r/c) = -\partial_0 h(t - r/c)$, and in this case $t^{0i}_{\mathrm{GW}} = -t^{00}_{\mathrm{GW}}$, and finally arriving at

$$\dot{E}_{\mathrm{GW}} = c \int_S \mathrm{d}^2\Omega r^2 t^{00}_{\mathrm{GW}} = \underbrace{\frac{c^3}{16\pi G}}_{\text{crazy huge term } c^3/G \sim 10^{35}\mathrm{kg/s}} r^2 \int \mathrm{d}^2\Omega \langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \tag{20}$$

And the flux can be simply defined as

$$F_{\mathrm{GW}} = \frac{c^3}{16\pi G}\langle \dot{h}_+^2 + \dot{h}_\times^2 \rangle \iff \text{GW also carry Energy Momentum} \tag{21}$$

Figure 3: Here lies a picture of thomas.

4 BBH GWs

In this section, we solely look into all the GWs possibly produced by studying a BBH scenario.

4.1 Level-Matching of asymptotic expansion

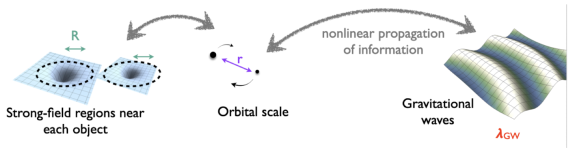


Figure 4: Schematic information flow from a binary source to GWs. Different descriptions are required in different patches of the spacetime.

The upshot of this figure is that, spacetime behaves very different on different scales, and this is reflected in during the matched asymptotic expansion.

4.1.1 Simple example of asymptotic expansion

We aim to solve the ODE:

$$\epsilon y^{(1)}(x) + y^{(1)}(x) + 3y(x) = 0, \qquad \epsilon \ll 1 \tag{22}$$

, with b.c. $y(0) = 0,\ y(1) = 1$.

Now we approximate in the far zone via

$$y = y_0(x) + \epsilon y_1(x) + \mathcal{O}(\epsilon^2) \tag{23}$$

Then we solve the ODE by neglecting $O(\epsilon^2)$ to get

$$\epsilon y_0'' + y_0' + \epsilon y_1' + 3y_0 + 3\epsilon y_1 = O(\epsilon^2) \tag{24}$$

by collecting the ϵ in terms of their orders we get

$$\begin{cases} O(\epsilon^0) : y_0' + 3y = 0 \to y_0 = c_1 e^{-3x} \\ O(\epsilon^1) : y_1' + 3y_1 = -y_0'' \to y_1 = c_2 e^{-3x} - 9c_1 x e^{-3x} \\ O(\epsilon^2) : \cdots \end{cases} \tag{25}$$

and of course we get the coefficients by imposing b.c. and we get $c_1 = e^3,\ c_2 = 9e^3$ Then we plug these back into (1) to get the near zero-zone approx of y

$$y = e^{3-3x} + 9\epsilon e^{3-3x}(1-x) + O(\epsilon^2) \tag{26}$$

Comments:

· Note that this expression fails at far zone, but fails at $x \to 0$ fails.

Now, we aim for the zero-zone expression,

$$X = \frac{X}{\epsilon} = O(1) \quad \text{for } x = O(t) \tag{27}$$

The interpretation for this variable is that, it "streches" the region of small x.

Now consider the limit $\epsilon \to 0$ at fixed X. We will obtain the ansatz,

$$y^{\text{Inner}} = y_0(x) + \epsilon y_1(x) + O(\epsilon^2) \tag{28}$$

, with the derivative swapped as

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}X}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{\epsilon}\frac{\mathrm{d}}{\mathrm{d}x} \tag{29}$$

The ODE now reads:

$$\frac{1}{\epsilon}Y_0'' + Y_1'' + \frac{1}{\epsilon}y_0' + y_1' + 3y_0 = O(\epsilon) \tag{30}$$

And if we keep track of the order of $O(\epsilon)$ we further get,

$$\begin{cases} O(\epsilon^{-1}) : y_0'' + y_0' = 0 \\ O(1) : y_1'' + y_1' = -3y_0 \end{cases} \tag{31}$$

The ansatz is thus,

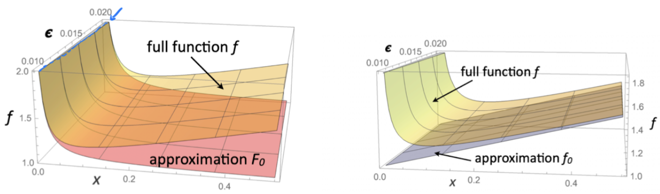
$$y_0 = c_1 e^{-x} + c_2 \tag{32}$$

And imposing b.c., we further get $y_0(0) = 0 \rightarrow c_2 = -c_1$ And finally

$$\begin{cases} y^{\text{outer}} = e^{3-3x} + \epsilon_1 9e^{3-3x}(1-x) + O(\epsilon^2) \\ y_{\text{inner}} = c_1(e^{-x} - 1) + B_1 c(x-1) + e^{2x}(\cdots) \end{cases} \tag{33}$$

Comment: Please notice the ϵ dependence in the equation, this means that this approximation does depend on one extra parameter now, kinda magical to say.

We can plot these as



(a) (b)

Figure 5: (a) The leading order outer approximation, (b) the leading order inner approximation

One can easily see that the extra ϵ degrees of freedom when doing the approximation.

4.2 intermediate regime

Now we can see that the approximation fails at both ends, and the natural question to ask is whether there exists an approximation such that we can speak of the overlapping domain. And this section talks about such approach.

So in the intermediate regime of x small, and X large, we expect both expansions are valid. We can formalize this by introducing a function $\eta(\epsilon)$ with $\epsilon \ll \eta(\epsilon) \ll 1$, and note that this naturally has to be a range, e.g. $\epsilon^{1/2} \ll \eta \ll \epsilon$.

Thus we define in the intermediate domain:

$$\begin{cases} x &= O(\eta), \quad \text{small as } \epsilon \to 0 \\ x &= O(\frac{\eta}{\epsilon}), \quad \text{large as } \epsilon \to 0 \end{cases} \tag{34}$$

We also introduce a scaled coordinate that is $O(1)$ in this regime:

$$x_\eta = \frac{x}{\eta} \tag{35}$$

And by imposing the matching condition, as $\epsilon \to 0$

$$\lim_{\epsilon \to 0} [y^{\text{inner}}(x_\eta) - y^{\text{outer}}(\eta)] = 0 \tag{36}$$

Next, we convert solutions to x_η.

The inner solution reads,

$$y^{\text{inner}}(x_\eta) = C_0(e^{-\frac{\eta(\epsilon)}{\epsilon}x_\eta} - 1) + 3C_0\eta x_\eta(1 + e^{-\frac{\eta(\epsilon)}{\epsilon}x_\eta}) + \epsilon(3C_0 + C_1)(e^{-\frac{\eta(\epsilon)}{\epsilon}x_\eta} - 1) + \cdots \tag{37}$$

$$\approx -C_0 + 3C_0\eta x_\eta - \epsilon(3C_0 + C_1) + O(e^{-\frac{\eta(\epsilon)}{\epsilon}}) + \cdots , \tag{38}$$

and the outer solution reads,

$$y^{\text{outer}}(x_\eta) = e^{3-3\eta(\epsilon)x_\eta} + 9\epsilon e^{3-3\eta(\epsilon)x_\eta} - 9\epsilon\eta x_\eta e^{3-3\eta(\epsilon)x_\eta} + \cdots \tag{39}$$

$$= e^3 - 3e^3\eta x_\eta + 9\epsilon e^3 + O(\epsilon\eta, \eta^2) + \cdots \tag{40}$$

And by imposing the criteria of (36), or intuitively, both (37) and (39) are in terms of the new variable x_η, thus the comparison is reasonable, and taking the limit $\epsilon \to 0$, simply (1) is the only for us to conduct the matching; (2) it is technically imposing $\eta \to 0$ at the same time.

Now we hunt down the coefficient by matching the $O(\epsilon)$ orders.

$$\begin{cases} O(1)\text{terms}: & C_0 = -e^3 \\ O(\epsilon)\text{terms}: & C_1 = -6e^3 \end{cases} \tag{41}$$

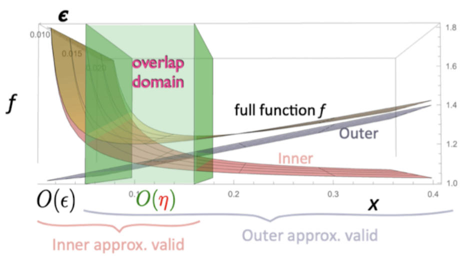


Figure 6: shows all the approximation region with the extra η parameter space.

4.3 Composite expansion and comparisons

To obtain the composite solution, we essentially add distinct terms from y^{inner} and y^{outer}, and then subtract the common terms. If we fall back to the previous example then we are trying to add (37) and (39) and then subtracting the double-counting of the common terms, which gives

$$y^{\text{composite}} = e^3(1 - e^{-\frac{x}{\epsilon}}) + e^{3(1-x)} - e^3 + \cdots = e^{3-3x} - e^{3-\frac{x}{\epsilon}} + \cdots \tag{42}$$

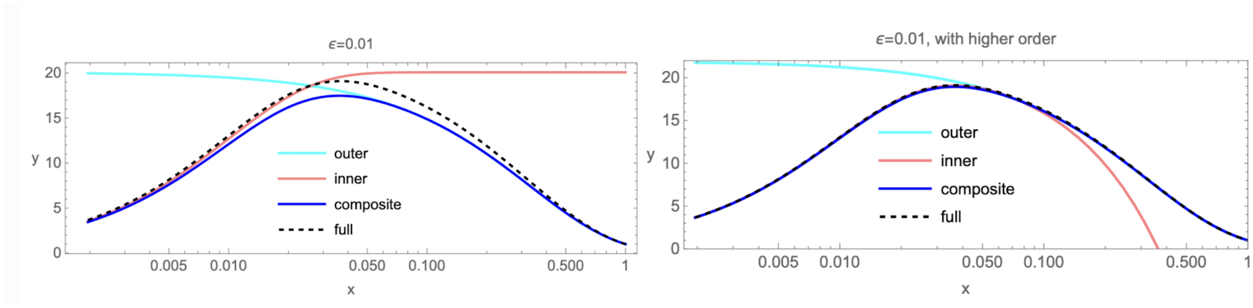


Figure 7: shows the plots with different approximation at a given ϵ

Comment: Of course there is the natural question to ask why there is no turning points within the function, in fact, it may or may not have at the end of the day, and if we figure so, then we have to do further inner-inner approximation (or any combination).

5 Post-Newtonian (PN) theory for GWs

In this section we aim to study compact-objects binaries in the regime where post-newtonian theory is needed.

To start off, it is instructive to ask what is the regime where PN approximation is needed. PN approximation is valid for semi-relativistic systems that are graivtationally bound, where the typical velocities are $v^2/c^2 \sim GM/rc^2 \ll 1$, and it is also combined with an approximation in the distant wave zone known as the *multipolar post-Minkowski expansion.*

It is somewhat useful to define the following dimensionless characteristic parametrs for the binary system, they are:

mass ratio: m/M, objects' internal gravity: GM/rc^2,

objects'size compared to orbit: R/r, gravitational interaction potential: GM/rc^2, velocity: v/c.

And the PN limit we are describing are that (i) it is semi-relativistic $v^2/c^2 \ll 1$ and weakly gravitating $GM/rc^2 \ll 1$ and gravitationally bound ($v^2/c^2 \sim GM/rc^2$).

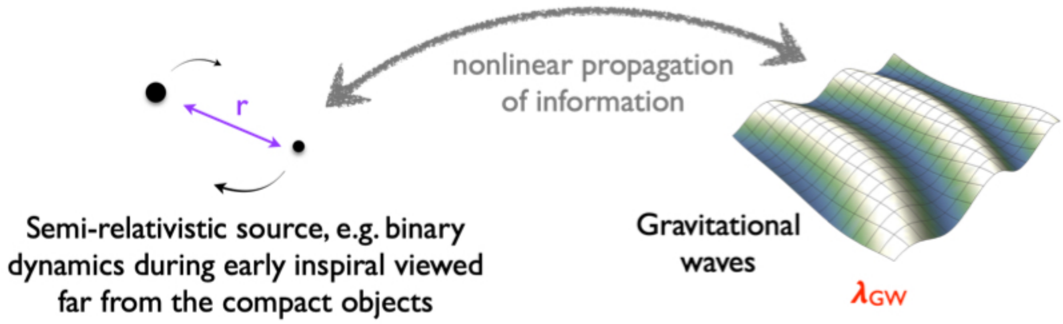


Figure 8: The post-Newtonian (PN) approximation is suitable for describing the dynamics and GWs of a point-mass binary system during the early inspiral, when the motion is semi-relativistic. THe term ’PN’ is often loosely used for the entire description, which involves matching PN expansion in the regions near the source to a multipolar post-Minkowski approximation in adpated to the wavezone at large distances from the source.

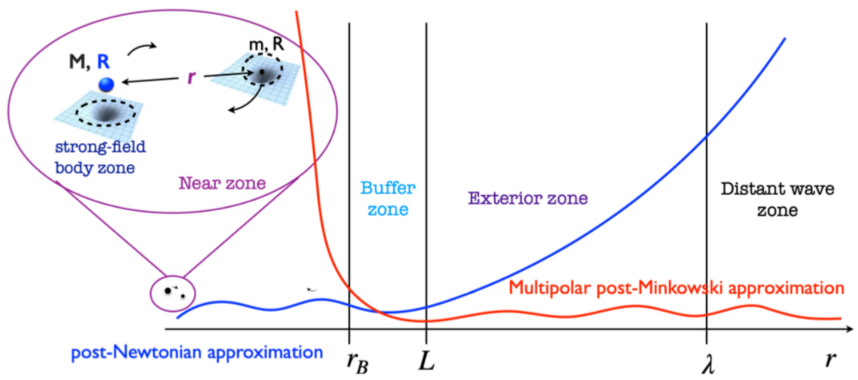


Figure 9: Example of the hierachy of scales in the two-body problem and the application of the method of matched asymptotic expansion connecting a post-Newtonian expansion in the near-zone to the post-Minkowski expansion in the wave zone.

5.1 Commence the PN analysis

Formally, we will use $1/c$ as the expansion parameter, and for each factor of $1/c^2$ we treat it as 1 PN order.

And the matter source behaves like,

$$T^{\alpha\beta} \sim \text{ Newt. rel. corr., i.e. } T^{00} \sim \rho c^2 \tag{43}$$
$$\sim c^2 T^{\alpha\beta}_{(0)} + O(c) \tag{44}$$

And if we put this into the gothic EFE formalism,

$$\Box h^{\alpha\beta} = \frac{16\pi G}{c^4}[-g(c^2 T^{\alpha\beta}_{(0)} + O(c)) + \mathcal{N}^{\alpha\beta}] \tag{45}$$
$$\sim \frac{16\pi G}{c^2}\tau^{\alpha\beta}_{(0)} + O(\frac{1}{c^3}) \tag{46}$$

The upshot with looking at PN theory in gothic EFE is that the gothich EFE becomes a perturbative system in $O(1/c)$, i.e. of the form $\Box h^{\alpha\beta}_{(0)} = 16\pi G T^{\alpha\beta}_{(0)} \cdots$, and you can go up to higher-order as one pleases.

And the solution for the 0-th order via integration (green-function is obtained) takes the form,

$$h^{\alpha\beta}_{(0)} = -4G \int \mathrm{d}t'\, \mathrm{d}^3\mathbf{x}'\ \frac{\tau^{\alpha\beta}_{(0)}(t', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\delta(t' - t + \frac{\mathbf{x} - \mathbf{x}'}{c}) \tag{47}$$
$$= -4G \int_{\mathbb{R}} \mathrm{d}^3\mathbf{x}'\ \underbrace{\tau^{\alpha\beta}_{(0)}\frac{(t - \frac{\mathbf{x}-\mathbf{x}'}{c}, \mathbf{x}')}{(\mathbf{x} - \mathbf{x}')}}_{I^{\alpha\beta}\text{, quadrupole moment.}} \tag{48}$$

And one can see the retardation time appearing in this expression.

5.2 Near-zone expansion:

Near-zone, as its name suggests describes the dynamics close the the binary where PN limit is adopted, and note that in this regime the retardation effects are small and the potentials are nearly instantaneous, so the equation of motion reduce to some U-potential. and we get something for the quadrupole moment like,

$$I_{nz} \approx \frac{\tau(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'} + \frac{\dot{\tau}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}(-\frac{|\mathbf{x}-\mathbf{x}'}{c}) + \frac{1}{2}\frac{\ddot{\tau}}{|\mathbf{x}-\mathbf{x}'|}\frac{|\mathbf{x}-\mathbf{x}'|^2}{c^2} \tag{49}$$

Now note that the potentials are instantaneous, so we can further do

$$h_{(0)}^{\alpha\beta,\mathrm{NZ}} = -4G\int \mathrm{d}^3\mathbf{x}'\,\frac{\tau_{(0)}^{\alpha\beta}(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}, \qquad \mathrm{c.f.}T_{(0)}^{00} = \rho \tag{50}$$

Remarks: Note that the quadrupole moment here diverges at spatial infinity! This reflects the nature that the near-zone expansion does not consider b.c. at infinity, which naturally calls for exterior zone expansion. Also note that in the main lecture notes, spherical harmonics is used to expand the quadrupole moment but not during the lecture, but anyhow, this section is not too important for exams anyways, though conceptional.

5.3 Exterior zone

For exterior zone, we notice that the size of source is small, then we can expand the quadrupole moment spatially, thus we get

$$I_{\mathrm{EZ}} = \frac{\tau(t-|\mathbf{x}|,\mathbf{x}')}{|\mathbf{x}|} + \mathbf{x}^i\frac{\partial}{\partial x^i}\left(\frac{\tau(t-\frac{|\mathbf{x}-\mathbf{x}'|}{c},\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\right) + \cdots \tag{51}$$

And what not surprisingly, as from what we learnt in the remarks for the near-zone section, the integral diverges at $|\mathbf{x}| \to 0$, i.e. it does not consider for the near-zone b.c..

5.4 Multipole Expansions

This section is important to study, as it plays the dominant role in PN theory.

Other than the regular Taylor expansion, we can do it in a more fancier method, i.e. introducing multipole languages. And the Taylor expansion looks like for a function $f(x)$ expanded at a reference point z,

$$f(\mathbf{x}) = \sum_{l=0}^{\infty} \frac{1}{l!}(x-z)^L \partial_L f(\mathbf{x})\Big|_{\mathbf{x}=\mathbf{z}}, \tag{52}$$

, where L denotes the string of l indices, $x^L = x^i x^j \cdots x^{a_l}$, and the same for the ∂_L you know the drill. In short L denotes the index set.

5.5 Example: exterior zone

We can expand $\mathbf{x}'$ around $\mathbf{z}$ like this,

$$h_{(0)}^{00,\mathrm{NZ}} = -4G \int \mathrm{d}^3\mathbf{x}'\, \rho(t,\mathbf{x}') \frac{1}{|\mathbf{x}-\mathbf{x}'|} \tag{53}$$

$$= -4G \int \mathrm{d}^3\mathbf{x}'\, \rho(t,\mathbf{x}') \sum_{l=0}^{\infty} \frac{1}{l!}(x-z)^L \underbrace{\partial_L \frac{1}{|\mathbf{x}-\mathbf{x}'|}\Big|_{\mathbf{x}'=\mathbf{z}}}_{=(-1)^l \frac{\partial}{\partial x^L}\frac{1}{|\mathbf{x}-\mathbf{x}'|}} \tag{54}$$

observe that we can set $\mathbf{x}'$ of the underbrace term to $\mathbf{z}$ as it is indep. of ∂_L (55)

$$= -4G \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \frac{1}{|\mathbf{x}-\mathbf{z}|} \underbrace{\int \mathrm{d}^3\mathbf{x}'\, \rho(t,\mathbf{x}')(x'-z)^L}_{\text{Newtonian mass multipole}} \tag{56}$$

We can further simplify the expression by letting $\mathbf{x}^i = x^i - z^i$, $\mathbf{r} = \sqrt{\delta_{ij}\mathbf{x}^i\mathbf{x}^j}$

And with the result from the tutorial, which we found:

$$\partial_i\partial_j \frac{1}{r} = \frac{3}{\mathbf{r}}(\mathbf{n}^i\mathbf{n}^j - \frac{1}{3}\delta^{ij}), \quad \text{with}\mathbf{n}^i = \frac{\mathbf{x}^i}{r} \tag{57}$$

5.6 Symmetric trace-free tensors (STF)

It is crucial to introduce STF if we were to eventually try to do further computation, by definition, STF are irreducible representations of the rotational group SO(3).

$$n^{\langle ij\rangle} = n^i n^j - \frac{1}{3}\delta^{ij} \tag{58}$$

<u>**Comment:**</u> please not that $n^i n^j$ by default is symmetrized so we don't need to write $n^{(i}n^{j)}$

There is also this notation but not often used,

$$n^{\langle ivj\rangle} = \frac{1}{2}(n^i v^j + v^i n^j) - \frac{1}{3}(\mathbf{v}\cdot\mathbf{n})\delta^{ij} \tag{59}$$

With this we can also compactify the expression for the derivative of 1/r as,

$$\partial_L(\frac{1}{r}) = (-1)^l(2l-1)!!\frac{n^{\langle L\rangle}}{r^{l+1}} \tag{60}$$

5.6.1 properties of STF tensors

Below marks the properties of STF tensors:

- $\mathrm{T}_L S^{\langle L\rangle} = T_{\langle L\rangle}S^{\langle L\rangle} = T_{\langle L\rangle}S^L$

5.7 Example (continued): exterior zone

Now we can go back to the previous calculation, So, we can rewrite (53) as,

$$h^{00}_{(0)} = -4G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}(\partial_L\frac{1}{r})M^{\langle L\rangle} = \sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}\frac{\mathbf{n}^{\langle L\rangle}}{\mathbf{r}^{l+1}}M^{\langle L\rangle} \tag{61}$$

, where the boldfont indicates, we are now adopting spherical coordinates.

And thus we can introduce the spherical harmonics, $Y_{lm}(\theta,\phi)N_{lm}P_l(\cos(\theta))e^{im\phi}$

Now the linear combination of the spherical harmonics can be expressed with the help of SFT tensors as, $Y_{lm} = \mathcal{Y}^{\langle L\rangle}_{lm}\cdot n^L$

and the inverse is obtained via,

$$n^{\langle L\rangle} = \frac{4\pi l!}{(2l+1)!!}\sum_{m=-l}^{l} \mathcal{Y}_{lm}^{\langle L\rangle} Y_{lm}^{*}(\theta,\phi) \tag{62}$$

and the curly $\mathcal{Y}$ is the conversion tensor with the form,

$$\mathcal{Y}_{lm}^{\langle L\rangle} = \frac{(2l+1)!!}{4\pi}\int \mathrm{d}\Omega\, Y_{lm}(\theta,\phi) n^{\langle L\rangle} \tag{63}$$

Also note that:

$$(n^{\langle L\rangle})^{*} = \frac{4\pi l!}{(2l+1)!!}\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\mathcal{Y}_{lm}^{*\langle L\rangle} \tag{64}$$

Finally, at last! we obtain the 0-th order PN expansion in spherical harmonics (or multipole expansion) as,

$$h_{(0)}^{00} \sim -4G\sum_{l,m}\frac{4\pi}{(2l+1)}Y_{lm}(\theta,\phi)\frac{Q_{lm}}{\mathbf{r}^{l+1}} \tag{65}$$

, with $Q_{lm} = \mathcal{Y}_{lm}^{*\langle L\rangle} M^{\langle L\rangle}$

5.8 Wave-zone solutions

It is quite whacky to put this section here but that's what the lecturer did at the end of lecture 4 so here it is.

Far form the source, recall that we adopt the PM expansion $h^{\alpha\beta} = \sum_{G=1}^{\infty} G^{(n)} h_{(n)}^{\alpha\beta PM}$, and that the solution of EFE takes the form,

$$\Box h_{(1)}^{\alpha\beta,PM} = -\frac{1}{c^2}\partial_t^2 + \nabla^2 = 0, \text{ in some "radiative" coordinates} \tag{66}$$

Now we need to come up with solution that matches the source, so we consider

$$\Box\left(\frac{F_L^{\alpha\beta}(T - R/c)}{R}\right) = -\frac{1}{c^2}\partial_T^2\left(\frac{F_l}{R}\right) + \frac{1}{R^2}\partial_R[R^2\partial_R\left(\frac{F_L}{R}\right)] \tag{67}$$

$$= -\frac{1}{c^2}\frac{F''^L}{R} + \frac{1}{R^2}\partial_R[-\frac{R}{c}F'_L - F_L] \tag{68}$$

$$= 0 \tag{69}$$

Now since ∂_L commutes with $\Box$, we further get,

$$h_{(1)}^{\alpha\beta,PM} = \sum_{l=0}^{\infty}\partial_L\left(\frac{F_L^{\alpha\beta}(\mathcal{U})}{R}\right), \quad \text{with } \mathcal{U} = T - R/c \tag{70}$$

, in retarded time in radial coordinates. and the the EFE takes the rough form,

$$\Box h_{(1)}^{\alpha\beta,PM} = \mathcal{N}^{\alpha\beta}[\text{lower than n order } h] \tag{71}$$

, but this is certainly not meaningful to write as the terms diverges for $R \to 0$, so we need regularization.

$$h_{(n)}^{\alpha\beta} = \underbrace{(\text{FP}, B \to 0)}_{\text{finite part}} \Box_{\text{ref.}}[\underbrace{\left(R/R_{\text{cnst.}}\right)^B}_{\text{regularization constant}} \mathcal{N}_{(n)}^{\alpha\beta}] \tag{72}$$

, the assumption we made for the approximation here is that the neutrinos have zero chemical potential, and of course the non-relativistic limit setting to get to (**??**)

5.8.1 Properties of Spherical Harmonics

Below marks some properties of spherical harmonics that also applied to the conversion tensor.

· They form the orthogonal bases

$$\int \mathrm{d}\Omega Y^*_{lm}(\theta,\phi)Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \tag{73}$$

∘ They are normalized by

$$Y^{*L}_{lm}Y^{L}_{lm'} = \frac{(2l+1)!!}{4\pi l}\delta_{m,m'} \tag{74}$$

5.9 Non linear features in PN theory

After introducing multipole expansion and adding PN correction to the GWs. We get the first order correction to the GW as

$$h^{(1)\mu\nu}(\mathbf{x},t) = \sum_{l=0}^{\infty} N_L h^{(1)\mu\nu}_L(r,t), \tag{75}$$

. And in order to further get the exact for of the wave solutions (note that the above expression is the approximation for the wave-zone), we have to expand and match the near-zone solution in PN series, and the wave-zone solution in PN series, and show that there is overlapping domain among the two. The result we get is that

$$h^{\mathrm{TT}}_{ij} = \frac{G}{c^2 d}\Lambda_{ijkl}\sum_{l=2}^{\infty}\frac{1}{c^l}N_{L-2}\underbrace{\mathfrak{I}_{klL-2}}_{\substack{radiative\\ mass\\ moment}} + \frac{1}{c^{l+1}N_{aL-1}}\Big(\epsilon_{abk}\mathfrak{J}_{lbL-2} + \epsilon_{abl}\underbrace{\mathfrak{J}_{kbL-2}}_{\substack{radiative\\ current\\ moments}}\Big) + O(d^{-2}). \tag{76}$$

Now, note that the source moments is nonlinear, and for example it looks something like

$$\begin{aligned}\mathfrak{I}_{ij} &\approx \ddot{M}_{\langle ij\rangle}(t_{\mathrm{ret}}) \\ &+ \underbrace{\frac{GM}{c^3}\int_0^\infty \mathrm{d}t'\ \ddddot{M}_{\langle ij\rangle}(t-t')[2\log(\frac{t'}{2t_0}) + \frac{11}{6}]}_{\text{'tail'}} \\ &+ \frac{G}{c^5}\Big[-\frac{2}{7}\underbrace{\int_0^\infty \mathrm{d}t'\ \ddot{M}_{ai}(t-t')\dddot{M}_{ja}(t-t')}_{\text{nonlinear memory}} + \cdots\Big]_{\mathrm{STF}}\end{aligned} \tag{77}$$

5.9.1 Tail effects

The second term of the expression (77) is the *tail effect* term, it arises from the scattering of GWs off of the spacetime curvature produced by the total mass of the binary system. In other words, the evolution history of the binary system also influences the measurement result. **Remarks:** Please note that it is not the tail is not the GWs signal that are scattered through the potential from the source to Earth, but solely the potential of the binary system.

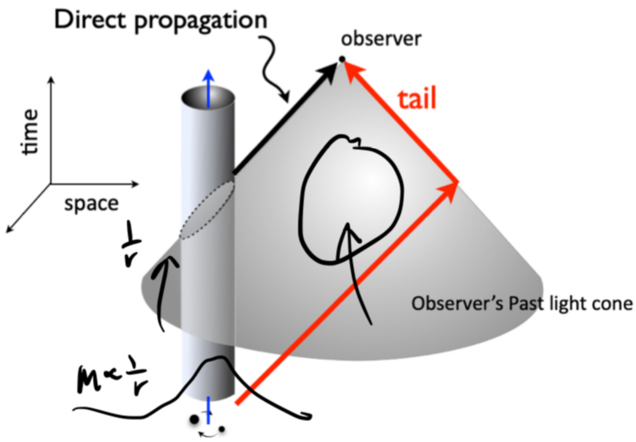


Figure 10: The leading-order GW tail arises from GWs scatterin g off the spacetime curvature due to the total mass and cause the GWs observed at time t to depend on the entire past history of the source

5.9.2 Memory effects

As for the *Nonlinear memory effects*, it arises from GWs that are sourced by previously emitted GWs. This leads to a shift in the oscillation amplitude and it is a non-oscillatory effect. To interpret this in a more **intuitive sense**, we can think of it as the previous GWs are not picked up by the detectors until some certain time (the merger GW signal is a ”short” term signal).

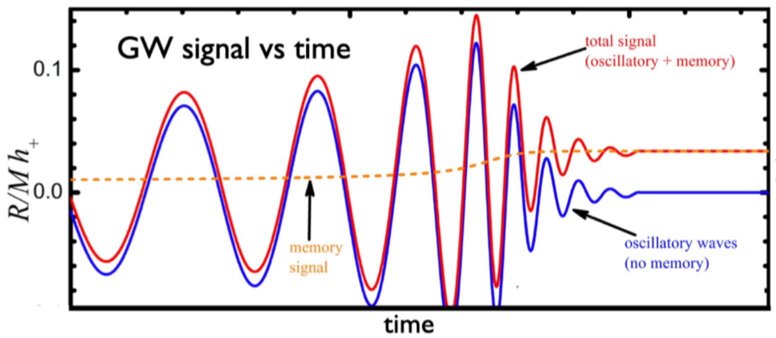


Figure 11: The nonlinear memory effect leads to a displacement amplitude before and after a burst of GWs, here a binary merger.

6 Application to a binary system

Let us now look at some source region where the following action applies, $S = S_g + S_m$, with

$$S_m = -\sum_{\alpha=1,2} m_a \int \mathrm{d}\tau_\alpha \tag{78}$$

and the corresponding energy momentum tensor is

$$T^{\alpha\beta} = \frac{1}{\sqrt{-g}} m \frac{\mathrm{d}\tau}{\mathrm{d}t}\frac{\mathrm{d}z^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}z^\beta}{\mathrm{d}\tau}\delta^{(3)}(\mathbf{x}-\mathbf{z}(t))|_a \tag{79}$$

, where z^α denotes the worldline. and $\frac{\mathrm{d}z^\alpha}{\mathrm{d}t}$ is the tangent vector to n^α.

And now under the 3+1 decomposition, we obtain infinite self-fields that do not contribute to dynamics of the action,

$$S_a = -m_a c^2 \int \mathrm{d}t \sqrt{-g_{00}^{(b)} - 2g_{0i}^{(b)}\frac{v_a^i}{c} - g_{ij}^{(b)}\frac{v_a^i v_a^j}{c^2}} \tag{80}$$

, where each terms inside the square root can be substituted perturbatively from solving EFE, also $v_a^i \equiv \frac{\mathrm{d}z^i}{\mathrm{d}t}$. After the perturbation, the action is now reduced to the so-called *Fokker effective action* for dynamics of a.

We can then move to the center of mass frame of the system and solve the eom up to 1 PN The eom will then read

$$\frac{L}{\mu} = \frac{v^2}{2} + \frac{GM}{r} + \frac{1}{c^2}\left\{\frac{(1-3\nu)}{8}v^4 + \frac{GM}{r}[\frac{\nu}{2}\dot{r}^2 + \frac{(3+\nu)}{2}v^2] - \frac{G^2M^2}{2r^2}\right\} \tag{81}$$

<u>**Radiation under this action:**</u>

Under this action, we will get the radiation as

$$h_{ij}^{\mathrm{TT}} = \frac{G}{c^4 R}\Lambda_{ijkl}^{\mathrm{TT}}[I_{kl}^{(2)} + \frac{1}{c}N_a(\epsilon_{abck}J_{lb}^{(2)} + I_{kca}^{(2)}) + \cdots] \tag{82}$$

, with $I_{ij} = \mu r_{\langle ij\rangle} + \mathrm{O}(1\overline{_{c^2)}}$

Note that the radiation in the observer below,

$$x_{\mathrm{obs}}^{j} = R_{ij}x_{\mathrm{source}}^{j}, \qquad R_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\iota) & -\sin(\iota) \\ 0 & \sin(\iota) & \cos(\iota) \end{pmatrix} \tag{83}$$

, one can reduce the orbital parameter to simply $\mathbf{x}_{\mathrm{source}} = r(\cos(\phi), \sin(\phi), 0)$

6.0.1 EOM of lagrangian in circular orbits

The eom of Lagrangian in circular orbits takes the following form. With first the three components being

$$\ddot{r} = \dot{r} = 0, \qquad \dot{\phi} = \omega, \qquad \dot{\omega} = 0, \tag{84}$$

We can readily obtain the radial eom with the first equality

$$0 = -\frac{GM}{r^2} + \omega^2 + \frac{1}{c^2}[\frac{G^2M^2}{r^4} + \frac{GM\omega^2(3+r)}{2} + \frac{1}{2}r^2\omega^4(1-3r)] \tag{85}$$

, where here r is gauge dependent and therefore we would like to express GW in terms of frequency instead, i.e. find $r(\omega)$.

To do so, we simply solve perturbatively $r = r_0(\omega)[1+\frac{1}{c^2}\delta r_1(\omega)+\cdots]$, note that this is an ansatz as always.

We then substitute this in to eom and collect only $1/r$ terms

$$O(c^0) : -\frac{GM}{r_0^2} + \omega^2 = 0 \tag{86}$$

$$O(\frac{1}{c^2}) : \frac{3GM\delta r_1}{r_0^2} + [\cdots]_{r_0}. \tag{87}$$

where $r_0 = (GM)^{1/3}\omega^{-2/3}$, $\delta r_1 = \frac{1}{3}(GM\omega)^{2/3}(\nu - 3)$

6.1 Inspiral frequency correction at 1PN order

In this section, we look at the correction made to the inspiral system at 1PN order.

The first immediate change is the expression for the polarizations

$$h_+ = -\frac{G\mathcal{M}}{dc^2}\Big(\frac{5G\mathcal{M}}{c^3\tau}\Big)^{1/4}\frac{(1+\cos(\iota^2))}{2}\cos\Big[2\left(\frac{c^3\tau}{5G\mathcal{M}}\right)^{5/8} + 2\varphi_c\Big] \tag{88}$$

$$h_\times = -\frac{G\mathcal{M}}{dc^2}\Big(\frac{5G\mathcal{M}}{c^3\tau}\Big)^{1/4}\cos(\iota)\sin\Big[2\left(\frac{c^3\tau}{5G\mathcal{M}}\right)^{5/8} + 2\varphi_c\Big]. \tag{89}$$

6.1.1 Inspiral time evaluation

It turns out that in the new PN order, we now have dissipative effect appearing in the near-zone metric at 2.5PM, which has been shown to be equivalent to using flux balance

$$\langle\frac{\mathrm{d}E_{\text{source}}}{\mathrm{d}t}\rangle = -\dot{E}_{GW} \tag{90}$$

, with

$$E_{\text{source}} \equiv v^i\frac{\partial\mathcal{L}}{\partial v^i} - \mathcal{L} \underbrace{=}_{\text{1PN}} -\frac{1}{2}\mu c^2 x[1 + (-\frac{3}{4} - \frac{\nu}{12}x)]. \tag{91}$$

Note that the term after 1 are overall negative in value, this means that this 1PN order correction lowers the energy of the source and the R.H.S. of (90) reads

$$\dot{E}_{\rm GW} = \frac{32G^{7/3}}{5c^5}(\mathcal{M}\omega)^{10/3}\Big[1 + \Big(-\frac{1247}{336} - \frac{35\nu}{12}\Big)x\Big], \tag{92}$$

And we can then get the infamous orbital frequency via

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{\dot{E}_{GW}}{\mathrm{d}E\,/\,\mathrm{d}\omega} = \frac{96}{5c^5}\pi^{8/3}\mathcal{M}^{5/3}\omega^{11/3}\Big[1 - \Big(\frac{743}{336} + \frac{11}{4}\nu\Big)x\Big]. \tag{93}$$

, where we have defined the chirp mass as $\mathcal{M} = \mu^{3/5}M^{2/5}$. Also note that the dimensionless symmetric mass ratio is defined as

$$\nu = \frac{\mu}{M} \in [0, 1/4] \tag{94}$$

, and note that it is 1/4 for equal masses and smaller for unequal masses.

7 Compact objects

In this section, we adopt the *Geometric units*, $G = c = 1$. We begin by defining a dimensionless quantity called compactness

$$\frac{GM}{Rc^2} = \frac{M}{R} \tag{95}$$

.

Black holes (BHs): The most renowned types of compact objects is none other than the BHs, which believed to obey the ”no-hair” conjecture

$$Q_l + iJ_l = M(i\chi M)^2 \tag{96}$$

, with $\chi = S/M$, $|\chi| \leq 1$. Currently, we know only up to the horizon of the BHs, which it itself also generates deep puzzles within *physics*, namely *the information paradox.* In attempts to tackle this paradox, some introduce the idea of *BH mimickers*, which are proposed solutions to Quantum Black Holes (QMBHs), such feats can be done by modifying solutions to BHs up to the horizon scale, though not as far as the photosphere as observations pose strong constraint regarding that, such modification includes introducing non-local quantum effects, and the idea of AdS/CFT from *string theory*.

Neutron stars: Next up on the list we have neutron stars, they are the densest states of matter, which calls for a lot of unexplored physics, their usual compactness is around $C \sim 0.2$.

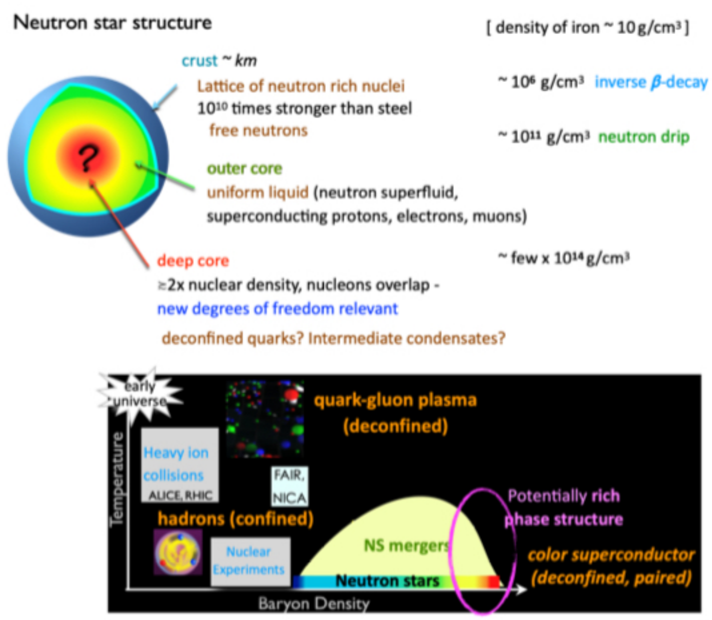


Figure 12: Neutron star interiors involve a rich variety of physics and probe unexplored regimes of the QCD phase diagram.

White dwarfs: These are usually around $C \sim 10^{-4}$, and they counteract gravity via *electron degeneracy*, whatever.

Dark "matter" objects: They are usually called gravitational condensates of beyond Standard Model (BSM). These BSM fields arise in different DM models and standard model extensions of particle physics.

7.1 Mathematical description of Compact Objects

Now we commence the mathematical description of these objects, we first describe them via EFE and EM tensor conservation

$$G^{\mu\nu} = 8\pi G T^{\mu\nu}, \quad \nabla_\mu T^{\mu\nu} = 0 \tag{97}$$

, and for non-spinning objects, they are well described via Schwarzschild coordinates, which takes the form

$$\mathrm{d}s_0^2 = A(t)\,\mathrm{d}t^2 + B(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2 \tag{98}$$

, now to solve for the whole field configuration, we separate to the case of interior and exterior of the compact object, and match the two solutions ($T^\mu,\ g_{\mu\nu}^{int}$) with $g_{\mu\nu}^{ext}$ = Schwarzschild solution for any objects at $r = R$.

7.1.1 Neutron stars

We can treat the interior of a neutron star as perfect fluid.

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{99}$$

, and we also need an equation of state to connect $p(\rho)$, though usually this exhibits great uncertainties as it depends on the interior microphysics of the neutron star.

7.1.2 Binary system tidal fields

For binary compact objects system, they are immersed in external tidal fields due to the companion/counterpart. Thus a quadrupole called tidal tensor (or "gravitoelectric" STF moments) $\epsilon_{ij} = C_{0i0j}$ is introduced in the object's rest frame, which is one of the irreducible pieces (of the Weyl tensor) with even parity, while $C_{\mu\nu\alpha\beta}$ denotes the Weyl curvature tensor. The other irreducible pieces, which

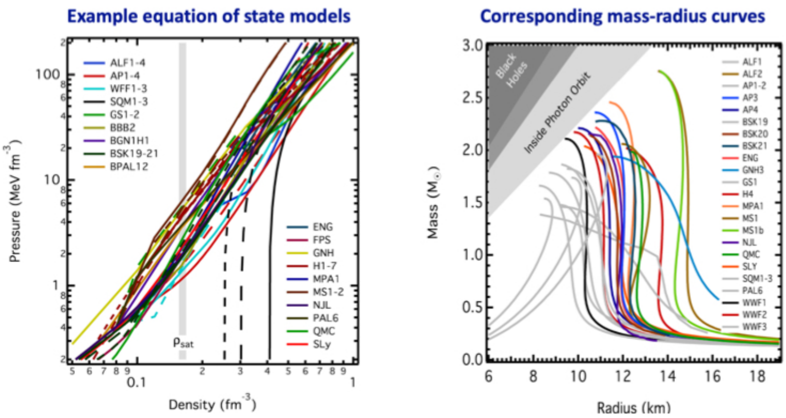


Figure 13: For neutron stars, the large uncertainty in the equation of state leads to a wide range of possible radii for a star with a given mass. There is a maximum mass beyond which the pressure can no longer counteract gravitational collapse, which depends on the equation of state. Each point on a curve represents a different neutron star, with decreasing central densities from left to right along a curve.

is of odd parity, is the "gravitomagnetic" STF tidal fields $\mathcal{B}_{ij}$ **c.f.:** In the Newtonian limit, the even parity piece reduces to l derivatives of the Newtonian gravitational potential due to the companion point mass

$$\epsilon^{L}_{\text{Newt}} = -\partial_L \frac{m_2}{r} \tag{100}$$

While there is **no** odd-parity tidal field in the Newtonian limit, and for non-Newtonian case, it is given by

$$\mathcal{B}_L = \frac{3}{2(l+1)(l-2)!} \epsilon_{\langle a_1 jk} C^{jk}_{a_2 0; a_3 \cdots a_l \rangle}. \tag{101}$$

, and they can be interpreted as the relativistic frame-dragging fields.

7.2 Compact objects - Non-spinning case

perturbing the metric:

To solve for the tidal perturbation, we are essentially perturbing the metric in the form of

$$\mathrm{d}s^2 = \mathrm{d}s_0^2 + \delta g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu, \tag{102}$$

and similarly $T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu}$. Result from Regge-Wheeler in 1957 has shown that the metric perturbation can be splitted into 2 sectors with each of different parity, i.e. changing signs under the reflectional map $(\theta, \phi) \mapsto (\pi - \theta, \pi + \phi)$. They are decomposed into 6 parts (in radial coordinate) as

$$\delta g_{\mu\nu}^{\text{even}}\mathrm{d}x^\mu \mathrm{d}x^\nu = \sum_{l,m}\Big[- e^{-2\Phi}H_0^{lm}\mathrm{d}t^2 + 2H_1^{lm}\mathrm{d}t\mathrm{d}r + e^\gamma H_2^{lm}\mathrm{d}r^2 + r^2 k^{lm}\mathrm{d}\Omega^2\Big]Y^{lm}(\theta, \phi). \tag{103}$$

, where all the H depend on (t, r) in general.

perturbing EM tensor:

Now we need to do the same for $T^{\mu\nu}$, though we simply need to decompose it in terms of the spherical harmonics via

$$\nabla^2 Y_{lm} = -\frac{l(l+1)}{r^2}Y_{lm} \tag{104}$$

, and then we will find that different (l, m) modes decouple.

In order to get something GW-like, we need to obtain the multipoles of the spacetime metric, we can get the definition of the relative multipole moments in ACMC (asymptotic cartesian mass centered) coordinates.

7.3 Linear tidal response

We begin by defining the linear tidal response,

$$Q_{ij} = -\lambda\epsilon_{ij} \tag{105}$$

, where λ is the tidal Love number, that are widely studied in the literature.

7.4 Quadrupole effects in binaries

To obtain the quadrupole effects, we can try to solve for the action for a test particle

$$S = m\int z\mathrm{d}t + \int z\mathrm{d}\tau\Big[-\frac{1}{2}Q_{ij}\epsilon^{ij} + L^{\mathrm{internal}}(Q,\dot{Q})\Big] \tag{106}$$

, which here Q_{ij} are the small deviations from equilibrium.

If we vary the action over Q_{ij}, we get the eom

$$-\frac{1}{2}\epsilon_{ij} - 2c_i Q_{ij} = 0 \tag{107}$$

, which implies $c_i = 1/4\lambda$

One can also define the effective/reduced worldline action, obtained by integrating out the quadrupole dof from the action by the results above, that does not depend on interal dof as

$$S^{\mathrm{red}} = m\int z\mathrm{d}\sigma + \frac{\lambda}{4}\int z\mathrm{d}\sigma\tilde{\epsilon}_{ij}\tilde{\epsilon}_{ij} + \cdots \tag{108}$$

, and note that all the information about the interior object here is encoded in λ.

A BMS-group

There is a group of people who are working on specific type of GR under the Bondi-Metzner Sachs (BMS) group $\mathfrak{B}$, which essentially involves extending the regular Poincare group to symmetries for null-infinity, supertranslation and superrotational symmetry groups, please see [1] for more.

References

[1] Abhay Ashtekar, Miguel Campiglia, and Alok Laddha. Null infinity, the bms group and infrared issues. *General Relativity and Gravitation*, 50:1–23, 2018.