Utrecht University

Advanced Quantum Field Theory

Yamamoto Sho

June 19, 2024

Contents

1 free field generating functional $\mathbf{Z}_0[J]$

We begin by reviewing the triumvirate of the generating functionals $Z[J]$, $W[J]$, and$\Gamma[\Phi]$.

We begin by defining the free field scalar action,

$$S_0 = \int \mathrm{d}^4x\mathcal{L}_0, \quad \mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \tag{1}$$

The corresponding generating function $Z_0[J]$ then reads,

$$Z_0[J] = \mathcal{N}\int D\phi\exp\left\{i\int \mathrm{d}^4x(-\frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}m^2\phi + \underbrace{J\phi}_{\text{fictitious field}})\right\} \tag{2}$$

Here $\mathcal{N}$ is related to the vacuum state via

$$\mathcal{N}^{-1} = Z_0[J=0] = \langle 0|0\rangle \tag{3}$$

Note that we do not set $J = 0$ when we speak of the generating function, we do so only when compute for the n-point correlation function, which will come later.

1.1 N-point correlation function

The n-point correlation function can thus be defined as,

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle = \frac{1}{i^n}(\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)})Z_0[J]|_{J=0} \tag{4}$$

2 Obtaining the Feynman propagator

It is naturally to next derive for the Feynman propagator of the theory. But before this we introduce some Fourier formalism.

2.1 Fourier formalism

Fourier transform are defined as

$$\begin{cases} \tilde{\phi}(k) = \int \mathrm{d}^4 x e^{-ikx} \phi(x) \\ \phi(k) = \int \frac{\mathrm{d}^4 x}{(2\pi)^4} e^{ikx} \tilde{\phi}(x) \end{cases} \tag{5}$$

For quadratic in ϕ theories, we can do the following procedure, with the **main** step being the functional Gaussian integral. The steps are as follow:

1. do a shift transform on $\tilde{\phi}(k) \to \tilde{\phi} - \frac{\tilde{J}(k)}{k^2+m^2}$

2. $Z_0[J]$ then reads, after replacing and I.B.P., we get

$$Z_0[J] = \underbrace{\mathcal{N} \int D\tilde{\phi} \exp\Big\{ -\frac{i}{2} \int \frac{\mathrm{d}^4 x}{(2\pi)^4} \tilde{\phi(k)k^2 + m^2 \phi(-k)} \Big\}}_{Z_0[J=0]=1} \exp\Big(\frac{i}{2} \int \frac{\mathrm{d}^4 x}{(2\pi)^4} \frac{\tilde{J}(k)\tilde{J}(-k)}{k^2+m^2} \Big) \tag{6}$$

3. We further define the propagator as

$$\Delta_F(x-y) = \int \frac{\mathrm{d}^4 x}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}, \tag{7}$$

, In momentum space, this becomes

$$\tilde{\Delta}_F(k) := \frac{1}{k^2+m^2-i\epsilon} \tag{8}$$

and rewrite (6) as

$$Z_0[J] = \exp\Big(\frac{i}{2} \int \mathrm{d}^4 x \mathrm{d}^4 y J(x) \delta_F(x-y) J(y) \Big) \tag{9}$$

Note that, to get from (6) to (9) and to get (7) to (8), we have used the identity,

$$\tag{10}$$

3 triumvirate relations

This section summarizes the triumvirate action relations.

We have the generating/partition function $W[J]$, the cumulant generating function $W[J]$, and the quantum effective action $\Gamma[\Phi]$. They are related by the two equations as below:

$$W[J] = -i\ln(Z) \quad Z = \exp(iW) \tag{11}$$

and

$$W[J] = \Gamma[\Phi] + \int \mathrm{d}^D x J(x)\Phi(x) \tag{12}$$

$$\Gamma[\Phi] = W[J] - \int \mathrm{d}^D x J(x)\Phi(x) \tag{13}$$

4 QED subtleties

We have that the mapping from a field to functional derivative for QED as follows:

$$-i\frac{\delta}{\delta\bar{\eta}} = \psi \tag{14}$$

$$i\frac{\delta}{\delta\eta} = \bar{\psi} \tag{15}$$

$$-i\frac{\delta}{\delta J_\mu} = A^\mu \tag{16}$$

And the triumvirate relations as:

For the accumulate generating function, we have

$$\frac{\delta W}{\delta J_\mu} = A^\mu \tag{17}$$

$$\frac{\delta W}{\delta\psi} = -\bar{\psi} \tag{18}$$

$$\frac{\delta W}{\delta\bar{\psi}} = \psi \tag{19}$$

$$\frac{\delta\Gamma}{\delta A_\mu} = -J_\mu \tag{20}$$

$$\frac{\delta\Gamma}{\delta\psi} = \bar{\eta} \tag{21}$$

$$\frac{\delta\Gamma}{\delta\bar{\psi}} = -\eta \tag{22}$$

5 Non-abelian gauge theory

Previously we defined

$$\mathcal{L} = \int \mathrm{d}^4 x \bar{\psi}(i\partial - m)\psi \tag{23}$$

and now with have the local symmetry: $\psi \to \psi'(x) = \mathcal{U}(x)\psi(x)$

$\mathcal{U}(x) \in$ non-abelian group (SU(N)),

such that

$$D_\mu\psi(x) \mapsto D'_\mu\psi'(x) + \mathcal{U}(x)D_\mu\psi(x) \tag{24}$$

And now we parametrize elements of SU(N) as

$$\mathcal{U}(x) = \exp(i \underbrace{\alpha^a t_a}_{\text{vector in SU(N)}}) \tag{25}$$

Same as the abelian-case, we try to make the ansatz for the covariant derivative as

$$D_\mu \psi(x) = (\partial_\mu - igA^a_\mu t_a)\psi(x) \tag{26}$$

$$\implies (\partial_\mu - igA^a_\mu t_a)\mathcal{U}(x)\psi(x) \tag{27}$$

$$= \mathcal{U}(x)(\partial_\mu - igA^a_\mu t_a)\psi(x) \tag{28}$$

$$\implies (\partial_\mu \mathcal{U})\cancel{\psi} - igA'^a_\mu t_a \mathcal{U}\cancel{\psi} = -ig\mathcal{U}A^a_\mu t_a \cancel{\psi} \tag{29}$$

$$\partial(\mathcal{U}\mathcal{U}^\dagger) = 0 \implies (\partial\mathcal{U})\mathcal{U}^\dagger = -\mathcal{U}\partial\mathcal{U}^\dagger \tag{30}$$

$$\implies A'^a_\mu(x)t_a = U(x)(A^a_\mu(x)t_a + \frac{i}{g}\partial_\mu)\mathcal{U}^\dagger(x) \tag{31}$$

$$\text{with the abelian } \mathcal{U}(x) = e^{2\Lambda(x)} \tag{32}$$

$$\text{explicitly: } U_{ij}(x)A^a_\mu(x)t^{jk}_a\psi_k(x), \quad A_\mu = A^a_\mu t_a \tag{33}$$

Comment: the $j, k \in [1, N]$ are the "matrix" or so called the "color indices". and the $a \in [1, N^2 - 1]$ are the "gauge indices", i.e. when defining a specific adjoint representation.

5.1 Infinitesimal form of gauge transformation

Now consider the infinitesimal form of the gauge transformation,

$$\psi'(x) \approx (1 + i\alpha^a(x)t_a)\psi(x) \tag{34}$$

With,

$$A'^a_\mu(x)t_a = \exp(i\alpha^b(x)t_b)(A^a_\mu t_a + \frac{i}{g}\partial_\mu)\exp(-i\alpha^c(x)t_c) \tag{35}$$

$$\approx A^a_\mu(x)t_a + i[\alpha^b(x)t_b A^\alpha_\mu(x)t_a] + \frac{1}{g}\partial_\mu\alpha^c(x), \quad \text{keep only } O(\alpha)t_c \tag{36}$$

Next we recall the generators satisfying $[T_a, T_b] = if^c_{ab}T_c$, with the structure constants $f^c_{ab} = f^a_{bc} = f^b_{ca}, f^c_{ab} = -f^c_{ba}$

then, we get

$$= A^a_\mu(x)t_a - \alpha^b(x)A^a_\mu(x)f^c_{ba}t_c + \frac{1}{g}\partial_\mu\alpha^c(x)t_c \tag{37}$$

$$\implies \text{relabel the dummy indices} \implies \tag{38}$$

$$A'^a_\mu = A^a_\mu + \underbrace{f^a_{bc}A^b_\mu\alpha^c}_{\text{new term}} + \frac{1}{g}\partial_\mu\alpha^a \tag{39}$$

To interpret the new term, it not only transform through from one bundle to another bundle, but now the bundles have vectorial information as well, thus the new term takes this into account how the vector rotates through the gauge transformation.

Next we need to generate the eom for A_μ

Naturally we compute,

$$[D_\mu, D_\nu]\psi(x) = (\partial_1 - A_1)(\partial_2 - A_2)\psi - (1 \leftrightarrow 2) \tag{40}$$

$$= -ig([\partial_\mu, A^a_\nu] - [\partial_\nu, A^a_\mu t_a])\psi(x) - g^2 A^a_\mu A^b_\nu [t_a, t_b]\psi(x) \tag{41}$$

$$= -ig(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)t_a\psi(x) - ig^2 A^b_\mu A^c_\nu f^{bca} t_a\psi(x) \tag{42}$$

$$= -igF^a_{\mu\nu}t_a\psi(x), \tag{43}$$

where $F^a_{\mu\nu} := (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + gA^b_\mu A^c_\nu f^a_{bc}$, and please note $F_{\mu\nu} \equiv F^a_{\mu\nu}t_a$

Up to this point, also with the exercise, we will see that $F_{\mu\nu}$ is now the adjoint:

$$F_{\mu\nu} \mapsto F'^a_{\mu\nu}t_a = \mathfrak{U}(x)F^a_{\mu\nu}t_a\mathfrak{U}^\dagger(x) \tag{44}$$

If we linearize R.H.S.,

$$O(\alpha) : \text{R.H.S.} = (1 + i\alpha^b t_b)F^a_{\mu\mu}t_a(1 - i\alpha^c t_c) \tag{45}$$

$$= (F^a_{\mu\nu} - f^a_{bc}\alpha^b(x)F^c_{\mu\nu})t_a \tag{46}$$

, $\forall t_a : F^a_{\mu\nu} \mapsto F^a_{\mu\nu} - f^a_{bc}\alpha^b F^c_{\mu\nu}$

So the conclusion is that $F^a_{\mu\nu}$ is not gauge invariant, and F is not an observable.

5.2 Non-abelian action

After all the previous discussion, we are in the position to define the Non-abelian action, the Yang-Mills theory.

$$S_{YM} = -\frac{1}{2}\int \mathrm{d}^4x \mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \tag{47}$$

, where $\mathrm{tr}(F'_{\mu\nu}F'^{\mu\nu}) = \mathrm{tr}((\mathcal{U}F\mathcal{U}^\dagger)(\mathcal{U}F\mathcal{U}^\dagger)) \implies \mathrm{tr}(F^a_{\mu\nu}t_aF^{\mu\nu b}t_b) = F^a_{\mu\nu}F^{\mu\nu b}\mathrm{tr}(t_at_b)$, so

$$\mathrm{tr}(t_at_b) = \frac{1}{2}\delta_{ab}, \quad \text{normalization} \tag{48}$$

. And now we further get

$$\begin{aligned} S_{YM} &= -\frac{1}{4}\int \mathrm{d}^4xF^a_{\mu\nu}F^{\mu\nu}_a & (49)\\ &= -\frac{1}{2}\int \mathrm{d}^4xF^a_{\mu\nu}\delta F^{\mu\nu}_a & (50)\\ &= -\frac{1}{2}\int \mathrm{d}^4x(F^a_{\mu\nu}\frac{\partial F^{\mu\nu}_a}{\partial A^i_\alpha} - \partial_\beta(F^a_{\mu\nu}\frac{\partial F^{\mu\nu}_a}{\partial(\partial_\beta A^i_\alpha)}))\delta A^i_\alpha & (51) \end{aligned}$$

. doing more steps and magic and finally we arrive at $\delta S_{YM} = 0 \implies \partial_\mu F^{\mu\nu a} + gA^b_\mu F^{\mu\nu c}f^a_{bc} = 0$

Suppose now Φ^a transforms in the adjoint. then

$$\begin{aligned} D_\mu\Phi^a = (\partial_\mu - igA^b_\mu T_b)\Phi^c, \quad \text{with}(T_b)_{jk} = if_{ajk} & \quad (52)\\ = \partial_\mu\Phi^a + gf^a_{bc}A^b_\mu\Phi^c & \quad (53) \end{aligned}$$

5.3 N=3, QCD

in the N=3 case, $a \in \{1, \cdots, 8\}$ gluons, and $i, j \in \{1, 2, 3\}$ colors. And in pure YM, we get terms like $gA^b_\mu A^c_\nu\partial^\mu A^{\nu a}, g^2A^bA^cA^dA^e$.

Now we pick a gauge fix path.

$$G^a_\omega(A) = \partial_\mu A^{\mu a} - \omega^a(x) \tag{54}$$

And impose the gauge fixing condition by multiplying by 1 in the form.

$$1 = \int \mathcal{D}\alpha \delta[G_\omega(A^\alpha)] \det\Big(\frac{\delta G_\omega(A^\alpha)}{\delta\alpha}\Big) \tag{55}$$

So now we evaluate

$$Z = \int \mathcal{D}A\mathcal{D}\alpha\delta[G_\omega(A^\alpha)]\det\Big(\frac{\delta G_\omega(A^\alpha)}{\delta\alpha}\Big)e^{iS_{YM}} \tag{56}$$

Formally:

$$\begin{aligned} \frac{\delta G_\omega(A^\alpha)}{\delta\alpha} &= \frac{\delta}{\delta\alpha}(\partial_\mu(A^\alpha)^\mu - \omega) && (57)\\ &= \frac{\delta}{\delta\alpha}[\partial_\mu(A^\mu + \frac{1}{g}D^\mu\alpha) - \omega] && (58)\\ &= \frac{1}{g}\partial_\mu D^\mu && (59) \end{aligned}$$

Going through pset 6, we will get

$$Z = \int DA \det(\frac{1}{g}\partial_\mu D^\mu) \exp{(iS_{YM} - i\int d^4x \frac{1}{2\xi}\partial_\mu A^\mu)} \tag{60}$$

And note that in HW2 we have shown the multidimensional gaussian integral.

So,

$$\det{(\partial_\mu D^\mu)} = \int dc\, d\bar{c} \exp\Big\{ - i\int d^4x \bar{c}(\partial_\mu D^\mu)c\Big\} \tag{61}$$

, where c^a is a Grassman field (scalar). and note that from ???? physics, $c, \bar{c}$ are non-physical Faddeev-

Popov ghost fields.

In this theory, the propagator then reads

$$\hat{G}^{ab}_{\mu\nu}(k) = -\frac{1}{k^2}(\eta_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2})\delta^{ab} \tag{62}$$

And the ghosts:

$$\mathcal{L}[c,\bar{c}] = -\int \mathrm{d}^4x\, \bar{c}^a(\delta^{ac}\partial^2 + g\partial^\mu f^a_{bc}A^b_\mu)c^c, \tag{63}$$

and $\hat{G}^{ab}_{\text{ghost}}(k) = \frac{i}{k^2}\delta^{ab}$

6 Renormalization

$$S_\Lambda[\phi] = \int \mathrm{d}^D x\, [\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \sum_I \Lambda_0^{D-D_i} \underbrace{g_{i0}}_{\text{dimensionless}} O(x)], \tag{64}$$

s.t. $O(x) = \phi^{14}\partial^7\phi$, $[\phi] = D - 2/2$, with the regularized partition function

7 RG flow

It is the trajectory from UV to IR thoery in the (∞- dim) space of theories.

7.1 RG fixed point

We start with RG fixed point. An RG fixed point is at which all β functions vanish. Interpretation: β function is like the velocity to the coupling. so it implies at the fixed point, the couplings g is independent of Λ.

To give more intuition, we consider the propagator $\Delta_F(x-y) \sim e^{-m|\mathbf{x}-\mathbf{y}|}$, with $\mathbf{x}, \mathbf{y}$ spacelike.

Now suppose we change the energy scale from $\Lambda \to \Lambda'$, then $m(\Lambda') = f[m(\Lambda); \Lambda]$, then at the fixed point, we must have $m(\Lambda') = m(\Lambda)$. This can only happen when we have either (i) m = 0, or (ii) m = ∞. But the case, $m \to \infty$ is boring (trivial). While the m = 0 case gives the critical point, where we will get a power law decays. These critical points are also called divergent correlation length, or structure on effective scales.