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Project Report

Twistor Theory

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Contents

1 Abstract

This paper gives an overview of the twistor theory, which is formulated by Roger Penrose[1]. Twistor theory (TT) is one of the many theories attempting to unite our current best candidate in explaining gravity - General Relativity (GR) and our best candidate in explaining small-scale structures - Quantum Field Theory (QFT), as there are parts of the theories which are incompatible with one another. This paper is structured as follows, in section I, several abstract mathematical concepts are introduced to provide the reader with minimum understanding to study twistor theory.

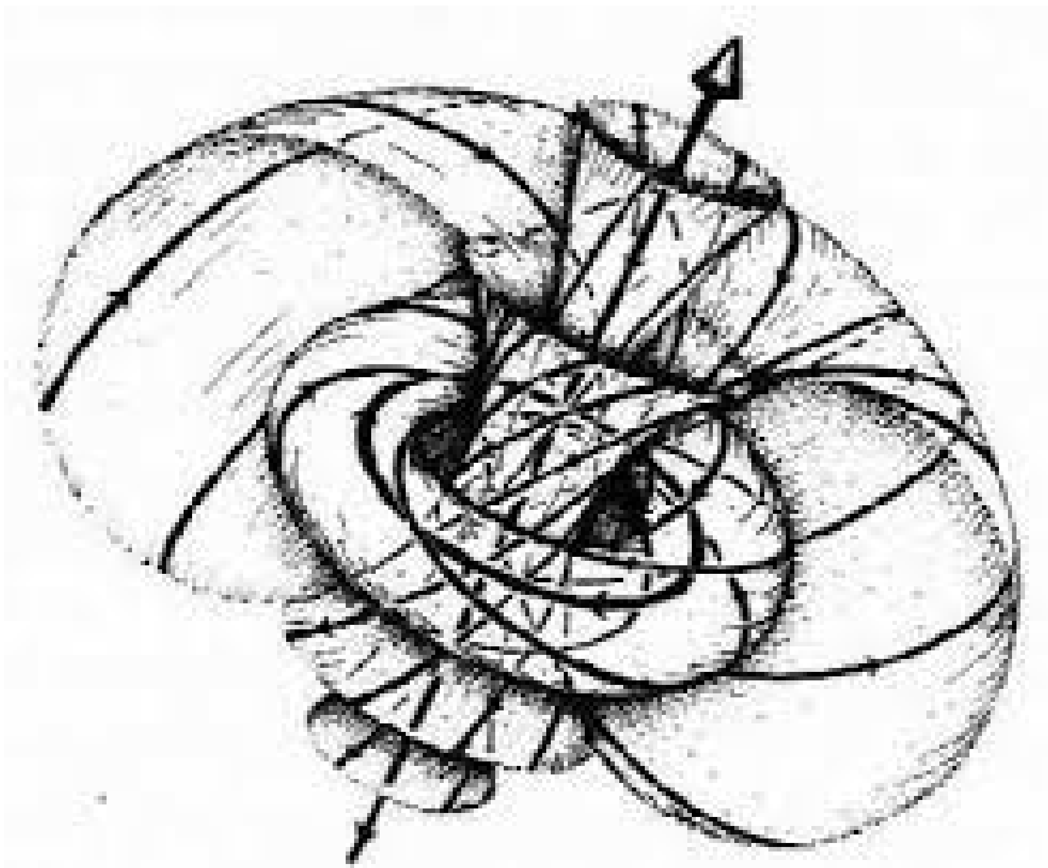

Figure 1: abstract representation of twistor by Roger Penrose.

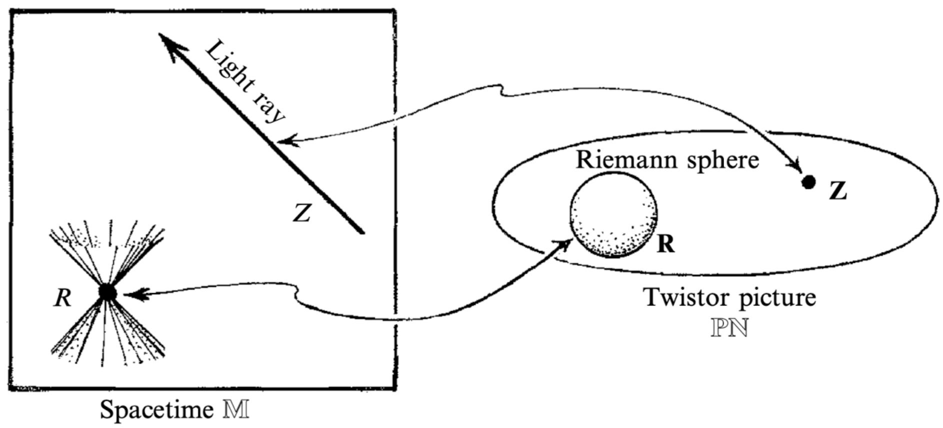

Figure 2: shows how one perceives light cone and light ray differently in twistor space. Extracted from [2].

2 Introduction

2.1 twistor theory in its essence

Before exploring the mathematical construct behind twistor theory, it is best to provide the reader here the overall idea of twistor theory to not get completely lost in mathematical abstraction.

Twistor theory aims to look at the traditional spacetime at a different perspective. In GR, we look at each event in spacetime as a point, in which one can draw a light cone to define causal relationships and define the light rays as the null vectors. However, the goal of twistor theory is to look at this differently, thus Penrose constructed the twistor space, in which each point in spacetime is now a Riemann Sphere, and the light rays become points. With this mathematical construct, one will get new results and provide a different perspective to work on physics of GR and QFT.

3 Mathematical Tools

In this section, we will briefly review some mathematical constructs that is required to understand TT.

3.1 2-spinor representation in Minkowski space

To describe a point in Minkowski space $\mathbb{M}$, we can simply use a vector $v^a = (v^0, v^1, v^2, v^3)$, but it is also possible to represent the space with spinors, in fact, each vector as a geometrical quantity can be decomposed into two vectors by following:

$$v^{\alpha\dot{\alpha}} := \frac{1}{\sqrt{2}}\sigma_a^{\alpha\dot{\alpha}}v^a = \frac{1}{\sqrt{2}}(\sigma_0^{\alpha\dot{\alpha}}v^0 + \sigma_1^{\alpha\dot{\alpha}}v^1 + \sigma_2^{\alpha\dot{\alpha}}v^2 + \sigma_3^{\alpha\dot{\alpha}}v^3) = \frac{1}{\sqrt{2}}\begin{pmatrix} v^0 + v^3 & v^1 - iv^2 \\ v^1 + iv^2 & v^0 - v^3 \end{pmatrix} \tag{1}$$

Here $\sigma_0^{\alpha\dot{\alpha}}$ is the 2 dimension identity matrix $\sigma_i^{\alpha\dot{\alpha}}$, i runs from 1 to 3, refers to the 3 Pauli matrices respectively.

This decomposition of a 4-vector into 2 pairs of spinors is called the Weyl representation[3].

To make the decomposition more mathematically rigorous. We first notice that a complex Minkowski space is SO(4, $\mathbb{C}$) which locally isomorphic to SO(2, $\mathbb{C}$)$\times$SO(2, $\mathbb{C}$). This is a stronger statement for the Lie algebra $\mathfrak{so}(4, \mathbb{C})$ is isomorphic to $\mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$. Thus we confirm this the algebraic relationship is valid, we can then identify that the decomposition is justifiable because vectors belong to SO(4, $\mathbb{C}$), and spinors belong to SL(2, $\mathbb{C}$), with 2 spinors making up one vector SO(4, $\mathbb{C}$) $\cong$ SL(2, $\mathbb{C}$) $\times$ SL(2, $\mathbb{C}$).

3.1.1 Properties in 2-spinor representation

With this 2-spinor formalism, there are few properties that provides a better description of previously known physical quantities.

First, we observe that the norm of a vector with respect to the metric becomes the determinant of the vector in spinor representation.

$$\eta_{\alpha\beta} v^{\alpha} v^{\beta} = 2 \det(v^{\alpha\dot{\alpha}}) \tag{2}$$

Thus in spinor formalism, the null vector is defined when the determinant of the vector (in 2-spinor representation vanishes), the only non-trivial cases are thus vectors of the form $v^{\alpha\dot{\alpha}} = \mu^{\alpha}\mu^{\dot{\alpha}}$ for some arbitrary non-zero spinors of opposite chirality. This is because these two spinors of opposite chirality from the same column vector and leading the determinant to be rank 1, meaning the column vector spans 1 dimension only, which means it vanishes. Note that this property will be used in later sections (namely the twistor correspondence).

3.1.2 Signatures

As the name suggests, complexified Minkowski space differs from our reality 3+1 by $3 \times (1 + i) + (1 + i)$, thus in order to recover the Minkowski space, we need to do transformations on spinors in $\mathbb{M}_{\mathbb{C}}$ such that we can recover it back to our usual physical spaces, e.g. Lorentzian, and Euclidean space. For this to happen, we need to define various operations on the spinors, here we will mention only on Lorentzian signature.

In order to make sense of the reason behind the imposed constraints, they are derived from first finding out how one recovers the signatures, for example in Lorentzian signature, starting with vectors, we can take Lorentzian space as an example. In order for $x^{\alpha\dot{\alpha}} \in \mathbb{M}_{\mathbb{C}}$ to be in Lorentzian signature, i.e. the components v^{a} defined in equation (1) needs to be all real and with the metric convention diag(-1,1,1,1), then we need to require that $x^{\alpha\dot{\alpha}}$ hermitian, i.e. $x^{\alpha\dot{\alpha}} = (x^{\alpha\dot{\alpha}})^{\dagger}$. And with this, it can be derived that the constraint for spinors in (1) is equivalent to this.

3.2 Projective spaces

To give an overview of the concept of projective spaces. It would be best first to go through the gist of it with the help of physical interpretation. In its essence, projective spaces describe the space of a n+1 dimensional space projected onto a n-dimensional space.

3.2.1 n = 1 case

For simplicity, it would be best to start with n = 1. Note that to describe n = 0 case would not give too much insight since it is trivial to thinking about, i.e. every point on $\mathbb{R}$ projects onto the origin.

The higher dimension space is called the ambient space in most mathematics literature, and in this case, it is $\mathbb{R}^2\backslash\{0\}$. The reason for excluding the origin will become apparent later. After constructing a line at some arbitrary location (see Figure 2) such that the projective space is the space of all straight lines passing through the origin. Points belonging to the same projective line will condense at the red horizontal line, that is some form of degeneracy occurs. If we consider the projective space, it is still a 2D space, however, there is one less degree of freedom in describing the coordinates in this space. Explicitly, take any coordinate on $v = (x, y) \in \mathbb{R}^2$, we can equally represent it as $v = (1, \frac{y}{x})$, with $x \neq 0$ guaranteed by our previous definition. This seemingly 2D space is exactly what projective space encodes, and if we focus to the 2-spinor formalism, spinors are $\mathbb{CP}^1$ quantities, and 2-spinors combined can form a vector of dimension 4. Thus we say, that the 2-spinors double cover a vector.

3.3 Spin Group and SO(n)

It is non-trivial at first to understand the relationship between the spin group and SO(n), especially in a more abstract way of representing this concept. Thus we will start with

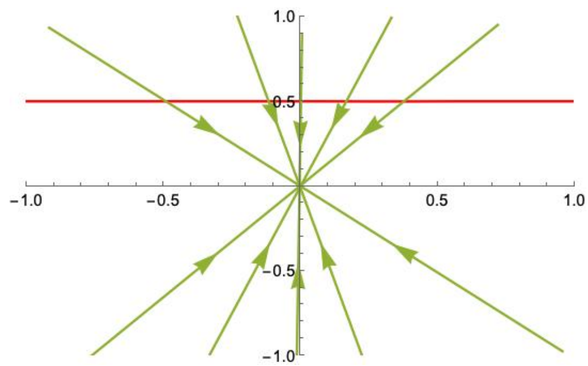

Figure 3: Graphical representation of $\mathbb{R}^2$ onto $\mathbb{R}^2$. One can easily see the all points lying on a given projective line map to the same point that is on the y = 0.5.

the representation theory approach

3.3.1 Abstract description

In a more abstract setting, there is much less structural definition given to Spin(n), and it follows as Spin(n) is a Lie group which is a double cover of the SO(n), coupled with a the short exact sequence between these lie groups as below:

$$1 \longrightarrow \mathbb{Z}^2 \longrightarrow \text{Spin}(n) \longrightarrow \text{SO(n)} \longrightarrow \mathbb{Z}^2 \longrightarrow 1$$

Figure 4: shows short exact sequence of the Spin(n) and SO(n) such that Spin(n) is well-defined for the discussion. Note that 1 is the trivial group, and $\mathbb{Z}^2$ is the 2-cyclic group, and both of them are Lie groups indeed.

3.4 groups and representation theory

The definition of a group in mathematics since it is a fundamental concept in mathematics. In its essence, a group is a set where its elements within the set obey a set of group axioms and this set encapsulates the idea of symmetry in a more abstract mathematical settings. Certain finite groups, that is groups under certain classifications and constraints, for example U(1) group, exactly corresponds to conserved quantities in physics, related by the Noether theorem (cite here).

3.4.1 SO(3) Group

SO(3) group is the essence of describing rotation while preserving the norm of vector in 3-Space. It is defined as follows:

Definition 1. *A SO(3) group is a set which contains 3x3 matrices as elements R that have the constraints:*
(1) $RR^T = 1$,
(2) $\det(R) = 1$ *while the elements in the group obey the usual group axioms, that are, (a) closure, (b) associativity, (c) identity, and (d) inverse properties. The group multiplication is defined by the usual matrix multiplication.*

To have a brief sense of the reason behind imposing the two constraint, (1) is simply the crux of rotation, that the matrix and its transpose mapping back to the identity suggests rotation along some axis, while (2) is imposed to preserve the norm while rotating objects (e.g. vectors, tensors, etc).

3.4.2 Representation Theory

Definition 2. *A representation* $\pi : G \to GL(V)$ *of a group G over a field k is a k-vector space V together with an action of G on V by linear maps. Another way to look at it is that it is a homomorphism from G to GL(V).*

Albeit the group theory lets us probe into fundamental physics - conserved quantities as well as particle pairs in QFT. (cite here) It is sometimes too abstract for physicists to work with, in particular, pure abstraction does not help in to prediction of real physical value at all. For example, to calculate the matrix element to obtain the cross section for particle decays. This is exactly where representation theory comes into play, it deals with abstract groups with matrices instead, and this can be done as long as the group we are dealing with is isomorphic to the matrix group.

3.4.3 explicit example of application of representation theory

3.5 Fibration

In this subsection, we will introduce briefly the concept of fibration, which provides a deeper understanding of studying twistor theory.

We will also provide minimal definitions that are essential to understanding the definition of fibration.

3.5.1 lift

Definition 3. *Given a mapping $f : X \to \mathbb{M}_{\mathbb{C}}$, and also a map $\pi_1 : \mathbb{PS} \to \mathbb{M}_{\mathbb{C}}$, a lift is a map $h : X \to \mathbb{PS}$ such that $\pi_1 \circ h = f$.*
The graphical way to represent this is provided below.

$$\begin{tikzcd} & \mathbb{PS} \arrow[d, "\pi_1"] \\ X \arrow[ur, dashed, "h"] \arrow[r, "f"] & \mathbb{M}_{\mathbb{C}} \end{tikzcd}$$

Figure 5: shows the commutative diagram such that the definition for lifting is established.

To understand this in a more physical sense. This is to say if a space $\mathbb{PS}$ can morph into $\mathbb{M}_{\mathbb{C}}$ under some map , and that X can also morph into $\mathbb{M}_{\mathbb{C}}$ under some map, then the lift is the map where X morph into $\mathbb{PS}$.

3.5.2 homotopy lifting properties (HLP)

Definition 4. *A mapping $\pi_1 : \mathbb{PS} \to \mathbb{M}_{\mathbb{C}}$ satisfies the HLP for a space X if (1) for every homotopy h: $X \times [0,1] \to \mathbb{M}_{\mathbb{C}}$ and (2) for every mapping (or so-called lift) $\tilde{h}_0 : X \to \mathbb{M}_{\mathbb{C}}$ with $\tilde{h}_0 = \tilde{h}|_{X\times\{0\}}$, and that $\tilde{h}$ is defined such that $h = \pi_1 \circ \tilde{h}$ (or in words, $\tilde{h}$ is defined such that $\tilde{h}$ lifts h), and there exists a homotopy $\tilde{h} : X \times [0,1]$.*

$$\begin{tikzcd}
X \times \{0\} \arrow[r, "\tilde{h}_0"] & \mathbb{PS} \arrow[d, "\pi_1"] \\
X \times [0,1] \arrow[r, "h"] \arrow[ur, dashed, "\tilde{h}"] & \mathbb{M}_{\mathbb{C}}
\end{tikzcd}$$

Figure 6: shows the commutative diagram such that the definition for HLP is well-established.

In a more simple manner, the map that takes one space to another exhibits HLP for some space X when that space X can simultaneously and continuously morph into the separate two spaces of interest. In this case, $X \times [0,1]$ can continuously morph into $\mathbb{PS}$ and $\mathbb{M}_{\mathbb{C}}$. And if one were to imagine the continuous morphing as a dynamical evolution, X will morph into the other two smoothly, and form both the spaces.

3.6 Sheaf

In this subsection, we will review the concept and we shall begin with presheaves.

3.6.1 presheaf

Definition 5. *Let X be a space equipped with a topology, so that (X, τ) defines a topological space. A presheaf $\mathcal{F}$ of abelian groups on X consist of the data:*
(a) $\forall$ open sets $U \subset X$, an abelian group $\mathcal{F}$, and
(b) $\forall$ inclusion on $V \subset X$, a morphism of abelian groups $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ subject

to the conditions:

1. $\mathcal{F}(\emptyset) = 0$
2. $\rho_{UU} = id$
3. $W \subset V \subset U \implies \rho_{UW} = \rho_{VW} \circ \rho_{UV}$

The motivation behind this definition is as follows, suppose the space X is complicated, to begin with, the map $\mathcal{F}(X)$ maps the complicated topological space X to some abelian groups, namely the $GL(V^n)$, such that it will be simpler to do calculation in this space, this corresponds to (a) in the definition, with $\mathcal{F}$ specifically being an abelian group. Next, it will be logical to map empty set to 0 (point (b) 1.) as $\mathcal{F}$ is meant to capture information on a topological subset, and empty sets contain no information. After $\mathcal{F}$ is defined, we can set up ρ, which serves as a tool to cross-check data between individual $\mathcal{F}(U), \forall U \in X$, and naturally (b) 2. is required. And finally imposing (b) 3. , we require the mapping to be sensible in the sense that it captures the order (or transitive) property of a general topological space, which is sets are ordered by the operation $\subset$, just like numbers are ordered by the operation $\leq$. In the usual context of presheaf in topology, one would define ρ simply as the restriction map, i.e. $\rho : \mathcal{F}(U) \to \mathcal{F}(V)$, with $\rho(\mathcal{F}(U)) = \rho(\mathcal{F}(U)) \mid_V$. It would also be beneficial to mention some technical terms here. In mathematical literature, the space $\mathcal{F}(U)$, for some $U \subset X$ is called a stalk (over the set U), and any one element in the stalk is called a section.

3.6.2 Sheaf

A sheaf is simply a stronger requirement of presheaf, which is its sections are determined by local data.

Definition 6.

(1) if U is an open set on X, and $\{V_i\}$ are open covers on U, and if $s \in \mathcal{F}(U)$ is an

element s.t. $s\,|_{v_i} = 0, \forall i$*, then* $s = 0$*.*

(2) if U is an open set on X, and $\{V_i\}$ *are open covers on U. If* $s_i \in \mathcal{F}(V_i)$ *for each i, s.t. for each i, j,* $s_i\,|_{v_i \cap v_j} = s_j\,|_{v_j \cap v_i} \implies \exists! s \in \mathcal{F}(U)$ *s.t.* $s\,|_{v_i} = s_i$ *for each i.*

(1) and (2) are encodes similar ideas. For (1), if the sections of open covers of some subset U are all 0, then globally the section of subset U should be 0 as well. The same thing should happen in the case where the intersections of the open covers agree in terms of sections, then there should be one global unique section on subset U such that the individual covers inside U agrees with this globally defined section.

3.7 Complex Objects

In this section, we review various important geometrical objects that is important in studying twistor theory or complex manifolds in general.

3.7.1 Differential forms on complex manifold

Similar to differential forms on real-valued manifolds. One can define differential forms on it. However, due to the nature of complex geometry, there are differences between them, which require a more careful analysis.

When complex coordinates are allowed, the usual functions that are attached to the forms carry complex coordinates as well, which is built into the setting of the Kähler manifold, see[4]. Thus one-forms become $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$, where z is some complex coordinate that takes in two real components, and it is also possible to form $\mathrm{d}\bar{z} = \mathrm{d}x - i\mathrm{d}y$. With this, any differential one form can be represented by the linear combination of these two, explicitly written as $\sum_{j=1}^{n} f_j z^j + g_j \mathrm{d}\bar{z}^j$, where f_j and g_j need to be holomorphic functions, which also explains why in literature these forms are called holomorphic differential forms.

Similar to forms on real numbers. One can define a higher dimensional forms, and they are constructed with the same wedge the product that is defined in the regular differential geometry, i.e. $\Omega^{p,q} = \Omega^{1,0} \wedge \cdots \wedge \Omega^{1,0} \wedge \Omega^{0,1} \wedge \cdots \wedge \Omega^{0,1}$, where the one $(1, 0)$ forms, which in twistor theory are called holomorphic forms, are wedged p times, and the $(0, 1)$ forms, which are called anti-holomorphic forms in twistor theory, are wedged q times. Since the wedge product is nothing but a tensor product combined with sign permutation, the total space formed by all forms of total degree n is just the direct sum of all combinations of (p, q) form that sum up to n, explicitly $E^n = \bigoplus_{p+q=n} \Omega^{p,q}$. Finally, one can define a projection map $\pi^{p,q} : E^n \to \Omega^{p,q}$ to obtain the (p, q) form from the total space, which is simply the trivial projection, that is if the total space is spanned by $E = \mathrm{d}z^3 \mathrm{d}\bar{z}^2 \oplus dz$, we can recover the $(1, 0)$ form just by ignoring the 1st direct sum space.

A holomorphic cotangent bundle can also be defined on a complex manifold if one were to collect all $(1, 0)$ and $(0, 1)$ forms for each point on the manifold, and direct sum them both to obtain such a bundle, $TP^*M = T^{1,0*}M \oplus T^{0,1*}M$.

3.7.2 Dolbeault operators

In this section, we wish to obtain some operators that behave like the usual exterior derivative in the real-valued manifold. The the result is that one can define two Dolbeault operators,

$$\partial = \pi^{p+1,q} \circ d : \Omega^{p,q} \to \Omega^{p+1,q}, \qquad \bar{\partial} = \pi^{p,q+1} \circ d : \Omega^{p,q} \to \Omega^{p,q+1} \tag{3}$$

. To see them in action, we begin by defining a form

$$\omega = \sum_{i=1}^{p} \sum_{j=1}^{q} f_{i,j} \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \in \Omega p, q \tag{4}$$

and act them of both the Dolbeault operators respectively,

$$\partial\omega = \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}\frac{\partial f_{i,j}}{\partial z^l}\mathrm{d}z^l \wedge; \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j \tag{5}$$

$$\bar{\partial}\omega = \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{l=1}\frac{\partial f_{i,j}}{\partial \bar{z}^l}\mathrm{d}\bar{z}^l \wedge \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j, \tag{6}$$

where the summation on l carries up to the number of coordinates (arguments) in $f_{i,j}(z^l)$, and from these two equations, it can be seen that the Dolbeault operators have the properties:

$$\mathrm{d} = \partial + \bar{\partial} \tag{7}$$

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0 \tag{8}$$

With this, the Dolbeault operators became the promoted version of the usual exterior derivative for Kähler manifold, in particular, the Dolbeault cohomology, which is analogous to de Rahm cohomology, can be established[5].

3.7.3 tangent vectors on complex manifold

Other than the complex differential forms on Kähler manifold, one can also define the tangent vectors, which are called holomorphic tangent vectors, for a given point p on M, one can represent the tangent vector at the point p via the tangent basis in the holomorphic tangent bundle at p, which is nothing but the dual of holomorphic cotangent bundle at p, and it becomes $v = v^i\partial_i + v^j\bar{\partial}_j$, where the two partial derivatives are

$$\partial_i = \frac{\partial}{\partial z^i}|_p \ ; \qquad \bar{\partial}_j = \frac{\partial}{\partial \bar{z}^j}|_p \ , \tag{9}$$

but not the Dobeault operators that are defined in the previous section. The holomorphic tangent bundle form as $TP^M = T^{1,0}M \oplus T^{0,1}M$.

3.7.4 Holomorphic delta function

Just as we have a delta function in real-valued space, there is a corresponding one for complex space, which is defined as,

$$\bar{\delta}(z) := \frac{1}{2\pi \mathrm{i}} \mathrm{d}\bar{z} \frac{\partial}{\partial \bar{z}} \left(\frac{1}{z} \right) = \frac{1}{2\pi \mathrm{i}} \bar{\partial} \left(\frac{1}{z} \right). \tag{10}$$

, with the property

$$\int_D \mathrm{d}z \wedge \bar{\delta}(z) f(z) = f(0), \tag{11}$$

where D is the boundary enclosing the origin.

4 Twistor Theory

With all these mathematical tools equipped, we can begin to unveil twistor theory, which it is best to start with the definition of twistor space.

In its essence, twistor theory is the study of fundamental physics, that is of QFT and GR in the twistor space, which Penrose believed it to be the right place to study. Thus we will begin with defining the twistor space.

4.1 Mathematical structure of Twistor Theory

Definition 7. *The structure of twistor theory is that one studies the double fibration of the projective spinor bundle* $\mathbb{PS} \cong \mathbb{M}_{\mathbb{C}} \times \mathbb{CP}^1$ *over the complex Minkowski space* $\mathbb{M}_{\mathbb{C}}$ *and twistor space* $\mathbb{PT}$, *with the map* $\pi_1 : \mathbb{PS} \to \mathbb{M}_{\mathbb{C}}$ *taking* $(x^{\alpha\dot{\alpha}}, \mu^{\beta}) \in \mathbb{M}_{\mathbb{C}} \times \mathbb{CP}^1 \cong \mathbb{PS} \mapsto (x^{\alpha\dot{\alpha}}) \in \mathbb{M}_{\mathbb{C}}$, *and the map* $\pi_2 : \mathbb{PS} \to \mathbb{PT}$ *taking* $(x^{\alpha\dot{\alpha}}, \mu^{\beta}) \in \mathbb{M}_{\mathbb{C}} \times$

$\mathbb{CP}^1 \cong \mathbb{PS} \mapsto (\lambda^{\dot{\alpha}}, \mu_\alpha)$ *imposing the relations on* $\lambda^{\dot{\alpha}}, \mu_\alpha$ *with* $\lambda^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\mu_\alpha$ *and* $\mu_\alpha \neq 0$, *which is so-called the incidence relation.*

$$
\begin{tikzcd}
 & A \arrow[dl, "\pi_2"'] \arrow[d, "\pi_5"] \arrow[dr, "\pi_1"] & \\
B & C \arrow[l, "\pi_3"'] \arrow[r, "\pi_4"] & D
\end{tikzcd}
$$

Figure 7: E

With this definition,

$$\mathbb{PS} = \{\mathbb{M}_{\mathbb{C}} \times \mathbb{CP}^1\}, \mathbb{M}_{\mathbb{C}} = \{\mathbb{C}^4\} \text{ and } \mathbb{PT} = \left\{(Z = \mu^{\dot{\alpha}}, \lambda_\alpha) \in \mathbb{CP}^3 \mid \lambda_\alpha \neq 0\right\} \tag{12}$$

, and Z is called a twistor.

4.2 Dual Twistor Space

The definition for dual twistor space is very much identical to the definition of dual vector spaces. We begin by the previously defined $\mathbb{PT}$, and introduce another space so-called dual twistor space $\mathbb{PT}^\wedge$ such that it is dual to $\mathbb{PT}$, such that elements from $\mathbb{PT}^\wedge$ serves as a linear mapping $W \in \mathbb{PT}^\wedge : Z \in \mathbb{PT} \mapsto \mathbb{R}$. And this space is explicitly

$$\mathbb{PT}^\wedge = \left\{(W = \tilde{\lambda}_{\dot{\alpha}}, \tilde{\mu}^\alpha) \in \mathbb{CP}^3 \mid \tilde{\mu}^\alpha \neq 0\right\}, \tag{13}$$

where W is the dual twistor One can define the inner product between dual twistor and twistor as

$$Z \cdot W = \mu^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}} + \lambda_\alpha\tilde{\mu}^\alpha = [\mu\tilde{\lambda}] + \langle\tilde{\mu}\lambda\rangle. \tag{14}$$

4.3 Twistor correspondence

At this point, this definition may seem too abstract and in fact takes quite some time to digest. Indeed, it would be best first to describe what kind of objects reside in the twistor space, namely points and Riemann Spheres which exhibit correspondence to null vectors and points in Minkowski Space. This correspondence is so-called twistor correspondence.

Twistor correspondence states that (1) points in complexified Minkowski space, $x^{\alpha\dot{\alpha}} \in \mathbb{M}_{\mathbb{C}}$, are Riemann spheres in the twistor space; (2) and points in twistor space, $X = (\lambda^{\dot{\alpha}}, \mu_\alpha) \in \mathbb{PT}$, are null vectors in complexified Minkowski space, $v^{\alpha\dot{\alpha}}_{\text{null}} \in \mathbb{M}_{\mathbb{C}}$.

To see why this is the case, the proof is as follows:

Proof. For (1), we start by fixing $x^{\alpha\dot{\alpha}} \in \mathbb{M}_{\mathbb{C}}$, and by the incidence relation, $\lambda^{\dot{\alpha}} = x^{\alpha\dot{\alpha}}\mu_\alpha$, we have $(x^{\alpha\dot{\alpha}}\mu_\alpha, \mu_\alpha) \in \mathbb{PT}$. We then observe that $\mathbb{PT} \subset \mathbb{C}^4$, as $\mathbb{PT} \subset \mathbb{CP}^3$, and since $x^{\alpha\dot{\alpha}}$ is fixed, then the only free parameter is $\mu_\alpha \subset \mathbb{C}^2$, and finally, since we start with the coordinates in $\mathbb{PT}$, therefore the projective scaling needs to be consider and it becomes $(x^{\alpha\dot{\alpha}}\mu_\alpha, \mu_\alpha) \subset \mathbb{CP}^1$. Thus a point in $\mathbb{M}_{\mathbb{C}}$ is corresponds to a Riemann Sphere ($\mathbb{CP}^1$) in $\mathbb{PT}$. For a more explicit proof, see [6].

For (2), we begin by realizing the result of (1), which tells us that we can think of a point in $\mathbb{PT}$ as the point intersection of two Riemann spheres in $\mathbb{PT}$ (these two spheres just touches each other only at one point in $\mathbb{PT}$). Now, we know that each Riemann sphere corresponds to a point in $\mathbb{M}_{\mathbb{C}}$ by the result of (1), and denote the corresponding points to be $x, y \in \mathbb{M}_{\mathbb{C}}$. Since by default these two Riemann spheres in $\mathbb{PT}$ and two points in $\mathbb{M}_{\mathbb{C}}$ are related by the incidence relation. We can subtract both equations to get

$$(x - y)^{\alpha\dot{\alpha}}\mu_\alpha = 0 \tag{15}$$

Next, we raise the μ_α with $\epsilon^{\alpha\beta}$, i.e.

$$(x-y)^{\alpha\dot{\alpha}}\mu^{\beta}\epsilon_{\beta\alpha} = 0. \tag{16}$$

since $\epsilon_{\beta\alpha}$ is an anti-symmetric tensor and thus in order to obtain a non-trivial solution for (4), x-y must be proportional to μ^α. Thus it becomes,

$$(x-y)^{\alpha\dot{\alpha}} = \mu^{\alpha}\tilde{\eta}^{\dot{\alpha}} \tag{17}$$

where $\tilde{\eta}^{\dot{\alpha}}$ is any arbitrary dotted spinor.

But recall that any vector that can be represented by 2 spinors of opposite chirality means that it is a null vector in $\mathbb{M}_{\mathbb{C}}$, see equation (2). Thus (2) is proved $\square$

5 Zero-rest-mass fields

In this section, we will look into the correspondence of z.r.m. fields in twistor theory.

5.1 Zero Rest Mass fields on $\mathbb{M}$

The immediate benefit of studying QFT in twistor space is deriving the solution of the Zero Rest Mass fields equation with a much simpler approach than what the usual QFT does. Though here we will not derive the equation from the bottom-up approach, we study for a specific example, namely the spin 1 massless field (photon).

5.1.1 geometric objects in spin 1 zero mass field

First, we know that the spin 1 zero mass field is simply a gauge potential $A_{\mu\nu}$, and with this we can also define the field strength (tensor):

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \tag{18}$$

If we represent this in the 2-spinor formalism, it becomes

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \partial_{\alpha\dot{\alpha}}A_{\beta\dot{\beta}} - \partial_{\beta\dot{\beta}}A_{\alpha\dot{\alpha}} \tag{19}$$

And notice that $F_{\mu\nu}$ is an anti-symmetric tensor under exchange of μ and ν, which in 2-spinor formalism, it will require the field strength tensor to be anti-symmetric upon exchange of $(\alpha\dot{\alpha})$ and $(\beta\dot{\beta})$. Or equivalently, we need the exchange upon α and $\dot{\alpha}$ indices symmetric, while the exchange upon β and $\dot{\beta}$ asymmetric, or the other way around. And in 2-spinor formalism, any asymmetric tensor must be proportional to the 2D levi-civita symbol. This is simply because any 2×2 anti-symmetric matrix must have its diagonal zero, and the skew-diagonals differ by a negative sign (which are simply the definition of the asymmetric tensor). Thus, after some careful consideration, one would conclude that $F_{\alpha\dot{\alpha}\beta\dot{\beta}}$ takes the general form:

$$F_{\alpha\dot{\alpha}\beta\dot{\beta}} = \epsilon_{\alpha\beta}\tilde{\Phi}_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}\Phi_{\alpha\beta}, \tag{20}$$

where $\tilde{\Phi}_{\dot{\alpha}\dot{\beta}} = \tilde{\Phi}_{(\dot{\alpha}\dot{\beta})} := \frac{1}{2}F^{\gamma}{}_{\dot{\alpha}\gamma\dot{\beta}}$; $\qquad \Phi_{\alpha\beta} = \Phi_{(\alpha\beta)} := \frac{1}{2}F_{\alpha}{}^{\dot{\gamma}}{}_{\beta\dot{\gamma}}$,

Indeed, the Φ and $\tilde{\Phi}$, which are called self-dual (SD) and anti self-dual (ASD) portions of the field strength respectively, are symmetric tensors under their own indices and ϵ are asymmetric, therefore the resulting field strength is just as what we required.

Next, we can obtain the Hodge dual of the field strength tensor, and to do this in the usual way, we take the transformation

$$\star F = \frac{1}{2}\epsilon^{abcd}F_{ab} = \epsilon^{\gamma\delta}\tilde{\Phi}^{\dot{\gamma}\dot{\delta}} - \epsilon^{\dot{\gamma}\dot{\delta}}\Phi^{\gamma\delta}, \tag{21}$$

where the first equality is the Hodge dual in the regular vector formalism, and ϵ^{abcd} is the 4D levi-civita tensor. From equation (9), we can see that $\tilde{\Phi}$ is indeed SD to the Hodge dual of F and Φ is ASD to the Hodge dual of F since they give the eigenvalue of 1 and -1 respectively under the Hodge dual transformation. Also, note that $\epsilon^{abcd} = \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\epsilon^{\dot{\alpha}\dot{\delta}}\epsilon^{\dot{\beta}\dot{\gamma}} - \epsilon^{\alpha\delta}\epsilon^{\beta\gamma}\epsilon^{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}}$ in 2-spinor formalism.

At first glance, it would seem there is little correspondence between SD and ASD to our usual physical quantities, but turns out, that one can make use of the Maxwell's equation and the Bianchi identity, which in the standard differential forms are:

$$\begin{cases} \mathrm{d}F = 0 \\ \star\mathrm{d}\star F = J, \end{cases} \tag{22}$$

to obtain the 2-spinor formalism version of the two identities, which exhibit the symmetrical form:

$$\begin{cases} \partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} + \partial^{\alpha}_{\dot{\beta}}F_{\alpha\beta} = 0 \\ \partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} - \partial^{\alpha}_{\dot{\beta}}F_{\alpha\beta} = 0 \end{cases} \tag{23}$$

One would realise that SD part corresponds to the positive helicity polarization of the gauge potential and ASD part corresponds to the negative counterpart. For a brief review of the helicity states, one can refer to Appendix A. Now, if we set either one of the SD or ASD part to be 0, one can recover the Maxwell fields equation and the Bianchi identities respectively, however, in twistor language, this corresponds to the spin-1 *zero-rest-mass* (z.r.m.) equations:

$$\partial^{\dot{\alpha}}_{\beta}\tilde{F}_{\dot{\alpha}\dot{\beta}} = 0, \qquad \partial^{\alpha}_{\dot{\beta}}F_{\alpha\beta} = 0. \tag{24}$$

This decomposition method can be further applied to z.r.m. fields with any integer or half-integer spin. For the spin-2 case, in other words, the Weyl Tensor in GR, is well-studied by Atiyah[7]. In general, a helicity h z.r.m. equations are simply:

$$h > 0 \quad \tilde{\phi}_{\dot{\alpha}_1 \cdots \dot{\alpha}_{2|h|}}, \qquad \partial^{\beta\dot{\alpha}_1}\phi_{\dot{\alpha}_1 \cdots \dot{\alpha}_{2|h|}} = 0, \tag{25}$$

$$h = 0 \quad \Phi, \qquad \Box\Phi = \partial^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\Phi = 0, \tag{26}$$

$$h < 0 \quad \phi_{\alpha_1 \cdots \alpha_{2|h|}}, \qquad \partial^{\alpha\beta}\phi_{\alpha_1 \cdots \alpha_{2|h|}} = 0, \tag{27}$$

5.1.2 conformity of z.r.m. equations

In most physics applications, it is best to have invariant equations, that is equations that are invariant under coordinate or conformal transformation. For example, Einstein's field equation or the covariant form of Maxwell's field equations. It turns out that the z.r.m. equations are also conformally invariant. To see why this is true, we can take the spin 1 case as an example.

We begin by noticing that the minkowski metric under conformal transformation becomes $\eta_{ab} \to \Omega^2(x)\eta_{ab}$. But since η_{ab} in 2-spinor representation is nothing but $\eta_{ab} \leftrightarrow \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}$, then each ϵ should get a factor of:

$$\epsilon_{\alpha\beta} \to \Omega(x)\epsilon_{\alpha\beta}, \qquad \epsilon_{\dot{\alpha}\dot{\beta}} \to \Omega(x)\epsilon_{\dot{\alpha}\dot{\beta}}. \tag{28}$$

And we know that the field strength tensor $F_{ab} = F_{\alpha\dot{\alpha}\beta\dot{\beta}}$ is conformally invariant, and by equation (19), we conclude that both the ASD and SD in (23) transforms with a factor of Ω^{-1}, and finally given $\partial_{\alpha\dot{\alpha}}\epsilon_{\beta\gamma} = 0$, which results from simply the classic anti-symmetrization rule trick, we can figure out that $\partial^{\alpha\dot{\alpha}} \to \Omega(x)\partial^{\alpha\dot{\alpha}}$, and thus combining both implies (23) is indeed conformal invariant.

A similar argument can be applied to higher helicity z.r.m. equations, and in fact, all z.r.m. fields carries a conform factor of Ω^{-1}.

5.2 positive(negative)-frequency quantities

Having introduced the idea of SD and ASD quantities, we further package the SD and ASD quantities into the more useful objects called positive and negative frequency quantities.

Sticking with the example on the field strength in electromagnetism. After we defined SD and ASD portions of the field strength, we can define the two quantities:

$$F^{+} = \frac{1}{2}(F_{\mu\nu} - iF^{*}_{\mu\nu}); \qquad F^{-} = \frac{1}{2}(F_{\mu\nu} + iF^{*}_{\mu\nu}) \tag{29}$$

They are called positive (self-dual) and negative (anti self-dual) frequency parts of the field strength tensor respectively. In a more general setting, i.e. higher rank tensorial equations, such as the Einstein's Field Equation (EFE), one can obtain for such two quantities under similar decomposition.

Once the helicity objects are defined. We have successfully divide $\mathbb{PT}$ into three parts, which is one of the key results in twistor theory. Namely, we have $\mathbb{PT}$ divided to the tree subspaces, $\mathbb{PN}$ = { set of all zero helicity objects } = Null twistor space, $\mathbb{PT}^{+}$ = { set of all positive frequency objects } = positive helicity twistor space and $\mathbb{PT}^{+}$ = { set of all positive frequency objects } = negative helicity twistor space. The above statement can be written out as:

$$\mathbb{PT}^{+} = \{Z \in \mathbb{PT} \mid Z \cdot \bar{Z} > 0\}, \tag{30}$$

$$\mathbb{PN} = \{Z \in \mathbb{PT} \mid Z \cdot \bar{Z} = 0\}, \tag{31}$$

$$\mathbb{PT}^{-} = \{Z \in \mathbb{PT} \mid Z \cdot \bar{Z} < 0\}, \tag{32}$$

$$\tag{33}$$

, where Z is the twistor mentioned in the previous section.

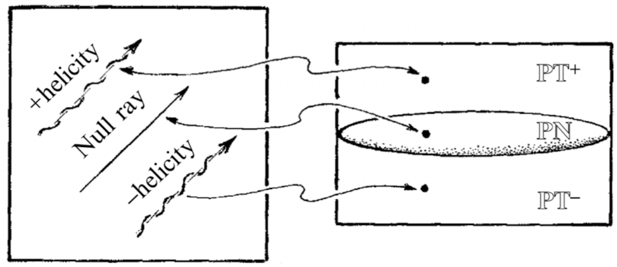

Figure 8: shows the decomposition of twistor space into the three subspaces. Extracted from [2].

6 The Penrose Transform

One major achievement in twistor theory is the Penrose Transform, which highly resembles the Radon Transform in real integral geometry[8], the result can simply be stated as any $\pm s$ helicity z.r.m. fields on twistor space are equal (corresponds) to Dolbeault cohomology group, or in the mathematical description

$$\{\phi_{\pm 2s-2}\} \cong H^{0,1}(\mathbb{PT}, \mathcal{O}(\pm 2s-2)), \text{ which obey } \bar{\partial} f = 0 \text{ and } \partial f \neq g. \tag{34}$$

Here, the notation $\mathcal{O}(k)$ is defined as the sheaf of locally holomorphic functions homogenous of weight k on $\mathbb{PT}$, that is $f(rZ) = r^k f(Z)$. In other words, the elements (0, 1)-th cohomology group of weight $\pm 2s \pm 2$ are the positive and negative helicity fields in twistor space. And in other literatures, one may prefer using sheaf cohomology for a more general settings, such as twistor space imposed with some signatures, or modified version of twistor theory[9][10]. In this section, the proof behind this correspondence will be lightly touched upon.

6.1 On finding correspondence of z.r.m field in $\mathbb{PT}$

Here, we will only prove the correspondence in one direction, i.e. z.r.m. equations imply the cohomology group and for the h < 0 case. The other way around is much more mathematically involved and it is proved by Eastwood, Michael and Penrose[11][12].

We begin by observing that fields in $\mathbb{M}_{\mathbb{C}}$ are points in $\mathbb{M}_{\mathbb{C}}$ associated with the corresponding field value, thus through twistor correspondence, we immediately see that fields in $\mathbb{PT}$ cannot be local but in fact, twistor lines (or $\mathbb{CP}^1$). It would be also intuitive to have the ansatz that the correspondence should involve integrating over the entire twistor line, or otherwise a part of the data regarding the point in $\mathbb{M}_{\mathbb{C}}$ is lost. Thus, given a $\phi_{\alpha_1\cdots\alpha_{2s}}(x)$, we need to match (1) the number of spinors indices, which are 2s undotted spinor indices, and (2) the 2s symmetrization upon exchanging the spinor indices, and (3) the weight (or homogeneity), which is 0 as we have established for a z.r.m field. Recall from.The z.r.m. of spin s correspondence in $\mathbb{PT}$ thus becomes,

$$\phi_{\alpha_1\cdots\alpha_{2s}}(x) = \int_X \langle\lambda\, \mathrm{d}\lambda\rangle \wedge \lambda_{\alpha_1}\cdots\lambda_{\alpha_{2s}} f(Z)|_{X\cong\mathbb{CP}^1}. \tag{35}$$

If we ignore the constraints on f for the moment, we see that (2) and (3) are not yet satisfied. In fact, there arises one more requirement (4) that the integrand needs to be a (1, 1)-form such that it makes sense to integrate over X to get the correspondence. If we ignore f, the number of spinor indices matches as 2s so f must not contain undotted spinor indices to satisfy (1); the λ_{α_i} naturally provides the symmetrization required in, so f contains no spinor indices (2); the weight is 2s+2, which can be obtained by counting the number of λ in the equation, so f needs to be of weight -2s-2 under projective scaling to satisfy (3); the integral of R.H.S in (29) is a (1, 0)-form provided by the dλ, so f needs to be a (0, 1)-form to satisfy (4). Furthermore, f has to be (5) a holomorphic function such that the integral is mathematically valid. With all these consideration from (1)-(5), f is a weight -2s-2 (0, 1)-form on $\mathbb{CP}^1$, therefore, it has the form:

$$f(\lambda,\bar{\lambda}) = f^{\bar{\alpha}}(\lambda,\bar{\lambda})\mathrm{d}\bar{\lambda}_{\bar{\alpha}}, \qquad f(r\lambda,\bar{r}\bar{\lambda}) = r^{-2s-2}f(\lambda,\bar{\lambda}). \tag{36}$$

However, f in equation (30) is still defined in $\mathbb{PT}$, thus one needs to impose restriction on this arbitrary holomorphic function f, namely the restriction from $\mathbb{PT}$ onto $\mathbb{CP}^1$. This can simply be achieved by treating the incidence relation as the restriction map, which pullback $\mathbb{PT}$ to $\mathbb{CP}^1$. In other words, f becomes $f(Z,\bar{Z})|_X = f(x^{\beta\dot{\alpha}}\lambda_\beta, \lambda_\alpha, \overline{x^{\beta\dot{\alpha}}\lambda_\beta}, \bar{\lambda}_{\bar{\alpha}})$. With this we get explicitly a quantity $f \in \Omega^{0,1}(\mathbb{PT}, \mathcal{O}(-2s-2))$, i.e. a (0, 1)-form on $\mathbb{PT}$ of projective weight -2s-2, thus (29) is recovered.

We can then proceed to obtain the z.r.m field equation in (19) by taking derivative on (34) using the incidence relation,

$$\partial^{\alpha_1\alpha}\phi_{\alpha_1\cdots\alpha_{2s}} = \int_X \langle\lambda\,\mathrm{d}\lambda\rangle \wedge \lambda_{\alpha_1}\cdots\lambda_{\alpha_{2s}}\Big(\lambda^{\alpha_1}\frac{\partial f}{\partial\mu_{\dot{\alpha}}}\Big|_X + \bar{\lambda}^{\alpha_1}\frac{\partial f}{\partial\bar{\mu}_{\dot{\alpha}}}\Big|_X\Big). \tag{37}$$

The first term contains the contraction $\lambda_{\alpha_1}\lambda^{\alpha_1} = \epsilon^{\alpha\beta}\lambda_{\alpha_1}\lambda_{\alpha_1} = 0$ using anti-symmetrization. And recall we have imposed that f to be holomorphic, which means independent on conjugate variables, or mathematically as $\bar{\partial}f = 0$, where $\bar{\partial}$ is the Dolbeaut operator defined previously, thus the partial derivative in the second term vanishes, and thus the z.r.m. equation is recovered. And f is indeed the cohomology class as required.

6.2 z.r.m. field structure in $\mathbb{PT}$

After performing a similar procedure in the previous subsection. We would come to conclude that the structure of helicity h z.r.m. equations are given by the integral formula:

$$h<0 \qquad \phi_{\alpha_1\alpha_{2|h|}}(x) = \int_X \langle\lambda \mathrm{d}\lambda\rangle \wedge \lambda_{\alpha_1}\cdots\lambda_{\alpha_{2|h|}} f|_X, \tag{38}$$

$$h=0 \qquad \phi(x) = \int_X \langle\lambda \mathrm{d}\lambda\rangle \wedge f|_X, \tag{39}$$

$$h>0 \qquad \tilde{\phi}_{\dot{\alpha}_1\cdots\dot{\alpha}_{2h}} = \int_X \langle\lambda \mathrm{d}\lambda\rangle \wedge \frac{\partial}{\partial\mu^{\dot{\alpha}_1}}\cdots\frac{\partial}{\partial\mu^{\dot{\alpha}_{2h}}} f\bigg|_X. \tag{40}$$

Here, we should take the remark that our discussion so far does take signatures into account, and upon adopting a signature, one would get a slightly different cohomology class and thus different constraints on f, the corresponding cohomology class for different signatures, especially split signature is discussed in. [13].

Although Penrose transform cleverly makes use of the powerful tool of cohomology in Algebraic Topology to obtain z.r.m. fields equations, it is still lacking in finding the gauge potential that is usually attached to the z.r.m field. This is where Sparling further developed it into a transform named after himself [14].

7 Ward's correspondence

The development of the Penrose transform does not end at the Sparling transform, it was later Ward started to ponder towards generalization of Penrose transform[15] to implement gauge theory into TT. To do so, it would be most natural to begin with reviewing ordinary gauge theory in $\mathbb{M}$.

7.1 Gauge theory in general setting

In gauge theory, we start by observing the existence of a gauge field, say a $A(x)$, and for a gauge field we can defined a *gauge connetion* by promoting the partial derivative

$\partial \to D = \partial + A$. Further defining

$$F_{ab} = [D_a, D_b], \tag{41}$$

that is the Lie bracket of gauge connection as field strength tensor, and take the traces of it, a local gauge-invariant quantity is formed. Thus with the connection defined, we have a tool to erase redundancy in choosing a gauge, making the theory gauge invariant after the appropriate promotion of the lagrangian of the theory of interest.

7.2 Gauge theory in TT

Suppose we do the same on $\mathbb{PT}$. One can start by constructing a gauge connection by promoting the Dolbeault operator by

$$\bar{\partial} \to \bar{D} = \bar{\partial} + a, \qquad a \in \Omega^{0,1}(\mathbb{PT}, \mathfrak{g}). \tag{42}$$

Here a is a (0, 1)-form that takes values in the adjoint of the gauge group G, and $\bar{D}$ is covariant almost complex structure in mathematics. This structure can be thought of geometrically as the deformation of complex structures in $\mathbb{PT}$, i.e. deformation of $\mathbb{PT}$ manifold itself. Similarly, we define a quantity, which is referred to as anti-holomorphic curvature of the connection

$$F^{(0,2)} = [\bar{D}, \bar{D}] \in \Omega^{0,2}(\mathbb{PT}, \mathfrak{g}), \tag{43}$$

and fix the gauge $F^{(0,2)} = 0$, which corresponds to $\bar{D}^2 = 0$, to arrive at the ward's correspondence. It states that a connection $\bar{D}$ obeying $F^{(0,2)} = 0$, it will corresponds to a SD Yang-Mills field on $\mathbb{R}^4$ with gauge group $GL(N, \mathbb{C})$, which these SD gauge fields are referred to as *instantons*[16].

8 MHV amplitudes in TT

Another major achievement in TT is providing an alternative derivation of Feynman rules for pure Yang-Mills theory, known as the MHV rules[17], through gauge theory in twistor space. In this section, we will look into the tree-level gluon scattering pure Yang-Mills theory.

8.1 Tree-level gluon scattering in pure Yang-Mills theory

Scattering amplitudes are fundamental to the study of QFT, as it is the major observable in experimental particle physics. Amongst different gauge theory in QFT, quantum-chromo-dynamics (QCD) is notoriously difficult to calculate for the cross-sections in this theory. In fact, our current observational evidence relies heavily on the leading order (LO) of the perturbative expansion in QCD, with the QCD coupling factor α_s. In fact, the calculation of gluon scattering amplitudes of higher order perturbation expansion in QCD is often handled by computers, where physicists have developed numerous method to compute the amplitudes[18][19].

Despite the active research in QCD scattering calculations, there are a few mysteries regarding the calculation, one of them being the Parke-Taylor formula for calculating $gg \to gg$ scattering. For background on this scattering calculation, see appendix B.

8.2 Parke-Taylor Formula (PTF)

The Parke-Taylor Formula essentially gives a way to computed colored order MHV amplitude for any number of gluons, that is given a fixed Feynman diagram with only the color order unfixed, we have the color ordered amplitude as

$$\tilde{\mathcal{M}}(1^+ \cdots 2^+ \cdots i^+ \cdots j^+ \cdots n^+) = \frac{\langle ij \rangle^4}{\Pi_{l=1}^{n-1} \langle l(l+1) \rangle \langle n1 \rangle} = \frac{\langle \kappa_i \kappa_j \rangle^4}{\Pi_{l=1}^{n-1} \langle \kappa_l \kappa_{l+1} \rangle \langle \kappa_n \kappa_1 \rangle}, \tag{44}$$

where the first and second equality are of the same representation (spinor helicity formalism) but the second is to emphasize that it is referring to the helicity states of the gluon particles. The denominator product is also named the cyclic product overall helicity particle states. Initially, this formula is obtained by induction on n points gluon diagrams, though later Nair[20], speculated that this can be derived alternatively from twistor theory.

8.3 Alternative derivation of Parke Taylor formula

First, we take any (n-2) positive helicity gluon representatives via Penrose transform denoted as $f^{(1)} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(0))$ and $f^{(-1)} \in H^{0,1}(\mathbb{PT}, \mathcal{O}(-4))$. And then we take 2 negative gluon helicity

Then the Parke-Taylor formula in $\mathbb{PT}$ representation becomes,

$$\text{PTF in } \mathbb{PT} = \int_{\mathbb{M}\times\mathbb{CP}^n} \mathrm{d}^4 x \Big(\prod_{k=1}^{n} \frac{D\lambda_k}{\langle \lambda_k \lambda_{k+1} \rangle} \Big) \langle \lambda_i \lambda_j \rangle^4 f_i^{(-1)} \mid_X f_j^{(-1)} \mid_X \prod_{l \neq i,j} f_j^{(1)} \mid_X, \tag{45}$$

Sometime after this discovery by Nair, Witten[21] further extended his work to what is known as Witten's conjecture.

8.4 Witten's conjecture

Witten's conjecture states essentially that the entire tree-level S-matrix of Yang-Mills theory could be written in terms of holomorphic functions from $\mathbb{CP}^1 \hookrightarrow \mathbb{PT}$. Simply stated, any tree-level The S-matrix of pure Yang-Mills theory (e.g. QCD) can be written compactly by a single formula, but not restricted to only MHV amplitude. The conjecture was proven to be true and the formula is now known as Roiban-Spradlin-Volovich (RSV) formula [22].

$$A_n = i(2\pi)^4 g_{\mathrm{YM}}^{n-2} \sum_{d=1}^{n-3} \int \mathrm{d}\mathcal{M}_{n,d} \prod_{i=1}^{n} \delta^2(\lambda_i^\alpha - \xi_i P_i^\alpha) \prod_{k=0}^{d} \delta^2\Big(\sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot{\alpha}}\Big) \delta^4\Big(\sum_{i=1}^{n} \xi_i \sigma_i^k \eta_i A\Big) \tag{46}$$

, where A_n denotes the partial amplitude of the color-stripped n-particle in the corresponding Yang-Mills theory, $(\lambda_i^\alpha, \tilde{\lambda}_i^{\dot{\alpha}}, \eta_{iA})$ denotes the i-th particle positive (λ) and negative ($\tilde{\lambda}$) chirality states, just as defined in the standard spinor helicity notation, η_A denotes the 4-component Grassman coordinate of $\mathcal{N} = 4$ superspace. degree d is given by $d = \frac{1}{2}(n - \sum h_i - 2)$. The integration measure is given by

$$\mathrm{d}\mathcal{M}_{n,d} = \frac{d^{2d+2} a d^n \sigma d^n \xi}{\mathrm{vol}(\mathrm{GL}(2))} \prod_{i=1}^{n} \frac{1}{\xi_i(\sigma_i - \sigma_{i+1})}. \tag{47}$$

9 Conclusion

To summarize, TT is one of the candidates in QG, albeit having received much less attention from the others, namely, string theory and LQG. Nonetheless, it still provides profound insight in understanding fundamental physics, to both QFT and GR. For example, Ward's correspondence to QFT and Penrose transform to both QFT and GR. It even serves to provide a useful platform for string theory, in fact, string theorists have developed incorporated twistor space to string theory, which further become the ambitwistor theory[10]. The key point of TT is that rather than developing theories in just the ordinary Minkowski space, which theorists have been doing ever since the discovery of GR, working in Twistor space often provides surprising results that we have no access to previously. Thus TT is worth remarking as not just as a candidate to QG but to theoretical physics that involves QFT or GR in general.

Appendices

A Massless spin 1 particle

In this appendix, the minimum background introduction to the theory of massless spin 1 particles is provided. The lagrangian to describe massless spin 1 particle is as follows:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{48}$$

, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Since typically we know that photons are massless spin 1 particles, and thus we can associate the theory with electromagnetic wave theory and conclude that $F_{\mu\nu}$ is the field strength tensor, and A_μ is the gauge potential.

One result of this lagrangian is that the potential is invariant under U(1) transformation, in other words, it exhibits U(1) symmetry. To show this, we can perform the transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x), \tag{49}$$

for any scalar function $\alpha(x)$. With the Euler-Lagrange equation, one can obtain the EOMs

$$A_\mu - \partial_\mu(\partial_\nu A_\nu) = 0. \tag{50}$$

Further solving the equation, one will eventually obtain Fourier space the solution of A_μ as

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}\sum_{i=1}^{2}(\epsilon_\mu^i(p)a_{p,i}e^{-ipx} + \epsilon_\mu^{i\star}(p)a_{p,i}^\dagger e^{ipx}) \tag{51}$$

where $a_{p,i}, a_{p,i}^\dagger$ are the annihilation and creation operators for momentum p and i-th component, ϵ_μ^i and $\epsilon_\mu^{\star i}$ are the basis for A_μ. Here we have chose the standard gauge (coordinate choice) with $p^\mu = (E, 0, 0, E)$, in the z-direction. It is also worth mentioning that $\epsilon_\mu^*\epsilon^\mu = -1$ and $p_\mu^\mu = 0$

For further reading, see [17].

B $gg \to gg$ scattering amplitude in Pure Yang-Mills theory

We begin by writing down the matrix element of $gg \to gg$ simply using the Feynman Diagram and Feynman Rules. In which, we obtain the following,

$$i\mathcal{M}_s(p_1p_2 \to p_3p_4) = -i\frac{g_s^2}{s}f^{abe}f^{cde}[(\epsilon_1\cdot\epsilon_2)(p_1-p_2)^\mu + \epsilon_2^\mu(p_2+q)\cdot\epsilon_1 + \epsilon_1^\mu(-q-p_1)\cdot\epsilon_2] \tag{52}$$

$$\times\ [(\epsilon_4^*\cdot\epsilon_3^*)(p_4-p_3)^\mu + \epsilon_3^{*\mu}(p_3+q)\cdot\epsilon_4^* + \epsilon_4^{*\mu}(-q-p_4)\cdot\epsilon_3^*], \tag{53}$$

where conservation of momentum tells $q = p_1 + p_2 = p_3 + p_4$, and ϵ refers to the polarizations, g is the strong interaction coupling constant, s is the s channel Mandelstam variable. However, this is just one of the many diagrams to obtain $gg \to gg$, if we consider all possible diagrams that contribute to the final averaged matrix amplitude, we need to calculate 1000 more of these diagrams, with each diagram having any of the different combinations of polarizations (abbreviated as pols.), crossed or uncrossed diagrams, and color factors (abbreviated as cols.). The resulting averaged matrix element square though takes the simple form,

$$\frac{1}{256}\sum_{\text{pols. cols.}}|\mathcal{M}|^2 = g_s^4\frac{9}{2}(3-\frac{tu}{s^2}-\frac{su}{t^2}-\frac{st}{u^2}), \tag{54}$$

where s,t and u correspond to the s, t, and u Mandelstam variables.

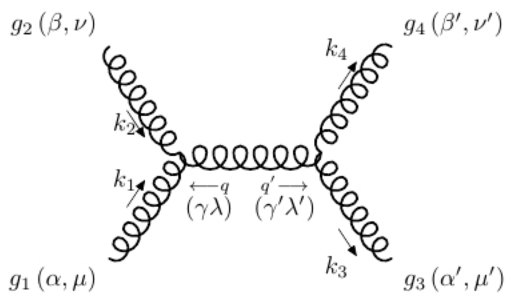

Figure 9: shows the Feynman diagram of one $gg \to gg$ scattering channel. Extracted from [23].

In this section, we will not try to recover the (48) but instead give a rough picture of how to obtain the equation. The next step is to represent the momentum quantity p in spinor formalism, which we adopt the usual spinor-helicity formalism, that is the convention used in [18] and [17] chapter 27.1. We then make use of 4-momentum conservation and decomposition of spinors, one can obtain the Schouten identity.

The remarkable result of adopting this formalism is that since $\epsilon_\mu\epsilon^\mu = 0.$, which behaves exactly as $p_\mu p^\mu = 0$. We can choose a reference momentum such that the polarizations of positive and negative helicity have the representation as

$$[\epsilon_p^-(r)]^{\alpha\dot{\alpha}} = \sqrt{2}\frac{p\rangle[r}{[pr]}, \qquad [\epsilon_p^+(r)]^{\alpha\dot{\alpha}} = \sqrt{2}\frac{r\rangle[p}{\langle pr\rangle}. \tag{55}$$

The notation of the spinor helicity formalism and the notation in 2-spinor formalism follows as (by taking $v^{\alpha\dot{\alpha}} = \lambda^{\alpha}\lambda^{\dot{\alpha}}$)

$$\lambda^{\alpha} = v\rangle, \quad \lambda_{\alpha} = \langle p, \quad \tilde{\lambda}_{\dot{\alpha}} = v], \quad = [v. \tag{56}$$

We will arrived at the result $\epsilon^{+} \cdot \epsilon^{+} = \epsilon^{-} \cdot \epsilon^{-} = \epsilon^{\pm} \cdot p = 0$ Under this representation, we will conclude that amplitudes that only have 2 positive and 2 negative helicities on the 4 legs remain, whereas the rest all vanishes. This significantly reduces the calculation, and the corresponding remnant diagrams are called maximum helicity violating (MHV) amplitudes.

Further calculation under this formalism on the color ordering of the $gg \to gg$ amplitude will yield the famous Parke-Taylor formula.

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