University of California, Los Angeles
Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

Homework 1

yamon yucel
605704529

Exercise 1

a. Assume that the variance of the returns of security A is 0.16 and the variance of security B is 0.25. The variance of a portfolio consisting of 50%A and 50%B is 0.0525. What is the covariance between securities A and B?

$$Var(x_A R_A + x_B R_B) = 0.0525$$

$$= x_A^2 Var(R_A) + x_B^2 Var(R_B) + 2 x_A x_B \, cov(R_A, R_B)$$

$$x_A = 0.5 = x_B$$

$$= (0.5)^2 (0.16) + (0.5)^2 (0.25) + 2(0.5)^2 \, cov(R_A, R_B) = 0.0525$$

$$\boxed{cov(R_A, R_B) = -0.1}$$

b. Suppose you are constructing two portfolios using the same n stocks. What is the expression of the covariance between these two portfolios in summations form and in matrix/vector form.

$$P_1 : \sum_{i=1}^{n} x_i R_i = \underline{x}' \underline{R}$$

$$P_2 : \sum_{j=1}^{n} y_j R_j = \underline{y}' \underline{R}$$

$$cov(P_1, P_2) = cov\left(\sum_{i=1}^{n} x_i R_i, \sum_{j=1}^{n} y_j R_j\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i y_j \, cov(R_i, R_j)$$

$$cov(P_1, P_2) = cov(\underline{x}'\underline{R}, \underline{y}'\underline{R}) = E[(\underline{x}'\underline{R} - \underline{x}'E[\underline{R}])(\underline{y}'\underline{R} - \underline{y}'E[\underline{R}])]$$

$$= \underline{x}' E[(\underline{R} - E[\underline{R}])(\underline{R} - E[\underline{R}])'] \, \underline{y}$$

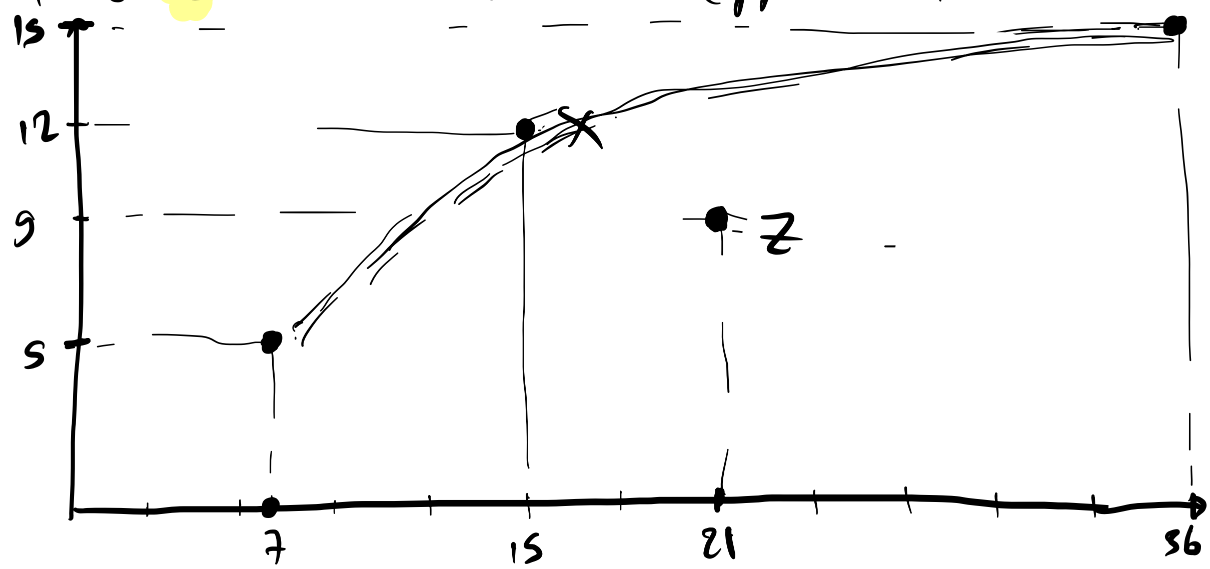
$$= \underline{x}' \underline{\Sigma} \, \underline{y}$$

Exercise 2

1. Which one of the following portfolios cannot lie on the efficient frontier as described by Markowitz? Please explain your answer.

Portfolio	Expected return (%)	Standard deviation (%)
W	15	36
X	12	15
Y	5	7
Z	9	21

Portfolio Z cannot lie on the efficient frontier.



X is a better portfolio than Z, therefore Z can not lie on effient frontier.

Return of X is higher than Z and risk of X is lower than Z, therefore Z is suboptimal

2. Suppose all stocks have $E(R) = 15\%$, $\sigma = 60\%$, and common correlation coefficient $\rho = 0.5$. What are the expected return and standard deviation of an equally weighted portfolio of $n = 25$ stocks?

$$\bar{R}_p = E[\sum x_i R_i] = \sum x_i \underbrace{E[R_i]}_{0.15} = 0.15 \underbrace{\sum x_i}_{1} = \boxed{0.15}$$

$$\sigma_p^2 = \sum_{i=1}^{n} x_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j \neq i}^{n} x_i x_j \sigma_{ij}$$

$$\sigma_{ij} = \rho \sigma_i \sigma_j = 0.5 * 0.60 * 0.60$$

$$= (0.6)^2 \frac{1}{(25)^2} 25 + 25 * 24 \frac{1}{25} \frac{1}{25} (0.5)(0.6)(0.6)$$

$$= 0.1872$$

$$\sigma_p = 0.4326661$$

Note, when equally weighted

$$\sigma_p^2 = \frac{\sigma^2}{n} + \frac{n-1}{n} \rho \sigma^2$$

3. Refer to question (2). What is the smallest number of stocks necessary to generate a portfolio with standard deviation of at most 43%?

$$\frac{\sigma^2}{n} + \frac{n-1}{n} \rho \sigma^2 < (0.43)^2$$

$$\frac{(0.6)^2}{n} + \left(\frac{n-1}{n}\right)(0.5)(0.6)^2 < (0.43)^2$$

$$(0.6)^2 + (n-1)(0.5)(0.6)^2 < (0.43)^2 n$$

$$((0.5)(0.6)^2 - (0.43)^2) n < (0.5)(0.6)^2 - (0.6)^2$$

$$-0.0049 n < -0.18 \Rightarrow n > 36.73 \Rightarrow n = 37$$

4. Refer to question (3). As n gets larger, is it true that $\sigma_p = \sigma\sqrt{\rho}$?
Please explain your answer.

$$\lim_{n\to\infty} \underbrace{\cancel{\frac{\sigma^2}{n}}}_{0} + \underbrace{\cancel{\frac{n-1}{n}}}_{1} \rho\sigma^2 = \rho\sigma^2 = \sigma_p^2$$

$\boxed{\text{Hence, } \sigma_p = \sigma\sqrt{\rho}}$ since $\sigma_p \geq 0$

Exercise 3

The mean returns and variance covariance matrix of the returns of three stocks (`C`, `XOM`, `AAPL` and the market `SP500`) are given below:

```
Mean returns:
        C          XOM         AAPL        ^GSPC
  0.005174    0.010617    0.016947    0.010846

Variance-covariance matrix:
              C       XOM      AAPL     ^GSPC
C      0.010025 0.000000 0.000000 0.000000
XOM    0.000000 0.002123 0.000000 0.000000
AAPL   0.000000 0.000000 0.005775 0.000000
^GSPC  0.000000 0.000000 0.000000 0.001217
```

Assume short sales are allowed. Compute the composition of the minimum risk portfolio using only the three stocks (do not use the `SP500`).

$$x_k = \frac{\sum_{i=1}^{3} v_{ki}}{\sum_{i=1}^{3}\sum_{j=1}^{3} v_{ij}} \quad \sum\sum v_{ij} = 743.94$$

or

$$\mathbf{x} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^T\Sigma^{-1}\mathbf{1}} = \boxed{\begin{bmatrix} 0.1341 \\ 0.6332 \\ 0.2328 \end{bmatrix}}$$

$R_p = \mathbf{x}'$ mean returns

$\boxed{= 0.0114}$

(0.011360S)

$x_1 = \frac{99.75}{743.94} = 0.13408$

$x_2 = \frac{471.031}{743.94} = 0.63315$

$x_3 = \frac{173.1601}{743.9473} = 0.232710$

Exercise 4
Assume that the average variance of the return for an individual security is 50 and that the average covariance is 10. What is the variance of an equally weighted portfolio of 5, 10, 20, 50, and 100 securities?

$$\sigma_p^2 = \frac{1}{n}\bar{\sigma}^2 + \frac{n-1}{n}\overline{cov}$$

$\overline{cov} = 10$

$\sigma^2 = 50$

n	σ_p^2
5	18
10	14
20	12
50	10.8
100	10.4

$$\frac{50}{5} + \frac{4}{5}\,10 = 18$$

$$\frac{50}{10} + \frac{9}{10}\,10 = 14$$

$$\frac{50}{20} + \frac{19}{20}\cdot 10 = 2.5 + 9.5 = 12$$

Exercise 5
What is the composition of the minimum risk portfolio using n risky stocks? Show the entire derivation using matrix/vector notation. Using this composition give the expression for the expected return and variance of the minimum risk portfolio?

$$\min \ \underline{x}'\underline{\Sigma}\underline{x} - 2\lambda(\underline{1}'\underline{x} - 1)$$

$$\frac{\partial \mathcal{L}}{\partial \underline{x}} = 2\underline{\Sigma}\underline{x} - 2\lambda\underline{1} = 0 \Rightarrow \underline{x} = \underline{\Sigma}^{-1}\lambda\underline{1}$$

$$1 = \underline{1}'\underline{x} \Rightarrow \lambda\,\underline{1}'\Sigma^{-1}\underline{1} \Rightarrow \lambda = \frac{1}{\underline{1}'\Sigma^{-1}\underline{1}}$$

$$\boxed{\underline{x} = \frac{\Sigma^{-1}\underline{1}}{\underline{1}'\Sigma^{-1}\underline{1}}}$$

$$\bar{R}_p = \left(\frac{\Sigma^{-1}\underline{1}}{\underline{1}'\Sigma^{-1}\underline{1}}\right)' E[R] = \boxed{x' E[R]} \Rightarrow \boxed{\frac{\underline{1}'\Sigma^{-1}E[R]}{\underline{1}'\Sigma^{-1}\underline{1}}}$$

symmetric so $\Sigma^{-1} = \Sigma^{-1'}$

$$\sigma_p^2 = \underline{x}'\Sigma\underline{x} \Rightarrow \frac{\underline{1}'\,\Sigma^{-1'}\,\cancel{\Sigma\Sigma^{-1}}\,1}{(\underline{1}'\Sigma^{-1}\underline{1})^2} = \boxed{\frac{1}{\underline{1}'\Sigma^{-1}\underline{1}}}$$