University of California, Los Angeles
Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

Homework 10

Exercise 1:
Suppose the return of the underlying stock of a European call is equal to the risk-free interest rate. Show that the probability that a European call option will be exercised at time T is equal to $\Phi(d_2)$. Assume lognormal property of stock prices. Also, time now is 0, therefore $\Delta t = T$.

$S_T > E$ since call is exercised

$\Rightarrow \log S_T > \log E$

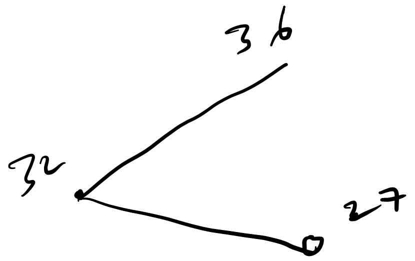
$$\log S_T \sim N\left(\log S_0 + \left(r - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right)$$

$$P(\log S_T > \log E) \Rightarrow P\left(Z > \frac{\log E - \log S_0 - \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= P\left(Z < \frac{-\log E + \log S_0 + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)$$

$$= P\left(Z < \frac{\log \frac{S_0}{E} + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right) = \Phi(d_2)$$

Exercise 2:
Consider a 6 month put option on a stock with exercise price $E = \$32$. The current stock price is \$30 and over the next 6 months it is expected to rise to \$36 or fall to \$27. The risk-free continuous interest rate s 6%. What is the risk-neutral probability of the stock rising to \$36?

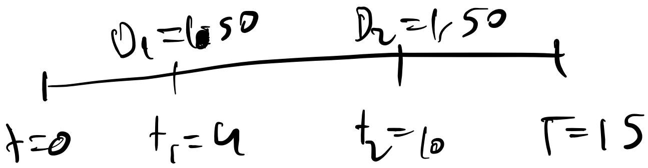


$$30 e^{0.06 \frac{6}{12}} = p36 + (1-p)27$$

$$\Downarrow$$

$$\boxed{p = 0.4348}$$

Exercise 3:
Consider an American call option on a stock. The following is given: $S_0 = \$50$, the time to expiration is 15 months, the risk-free interest rate is $r_f = 8\%$ per year, $E = \$55$, and $\sigma = 25\%$ per year. In addition, we know that the stock will pay dividends of \$1.50 in 4 months from now and another \$1.50 in 10 months from now. Show that it will never be optimal to exercise early on either of the two dividend dates. Calculate the price of the option.



$$D_2 < E\left(1-e^{-r(T-t_2)}\right) = 55\left(1-e^{-0.08\frac{5}{12}}\right) = 1,80 > D_2$$

$$D_1 < E\left(1-e^{-r(t_2-t_1)}\right) = 55\left(1-e^{-0.08\frac{6}{12}}\right) = 2,15 > D_1$$

$\rightsquigarrow$

It is not optimal to exercise early on either of the two dividend dates

$$S_0' = S_0 - \Sigma PV_i = 50 - 3 = 47$$

$$C = S_0'\Phi(d_1) - Ee^{-rt}\Phi(d_2)$$

$$\boxed{C = 4.11}$$

Exercise 4:
Answer the following questions:

a. Let $c = S^{-\frac{2r}{\sigma^2}}$. Does c satisfy the Black-Scholes-Merton differential equation?

b. Suppose the volatility for a stock goes to zero, i.e. $\sigma \to 0$. It means the stock is riskless and must earn the risk free interest rate. Therefore, at expiration time of a call option, $S_T = S_0 e^{rt}$. What is the value of the call option at time zero (now)?

c. What is the result obtained by the Black-Scholes-Merton model for the situation in (b)?

a) $c = S^{-\frac{2r}{\sigma^2}}$, B-S-M-dif eq =>

$$\frac{\partial c}{\partial t} + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}\sigma^2 S^2 + rS\frac{\partial c}{\partial S} - rc = 0$$

$$0 \qquad -\frac{2r}{\sigma^2}\left(-\frac{2r}{\sigma^2}-\frac{\sigma^2}{\sigma^2}\right)S^{-\frac{2r}{\sigma^2}-2} \qquad -\frac{2r}{\sigma^2}S^{-\frac{2r}{\sigma^2}-1}$$

$$\frac{1}{2}\cdot\frac{-2r}{\sigma^2}\left(-\frac{2r}{\sigma^2}-1\right)S^{-\frac{2r}{\sigma^2}-2}\sigma^2 S^2 + rS\left(-\frac{2r}{\sigma^2}S^{-\frac{2r}{\sigma^2}-1}\right) - rc$$

$$-r\left(-\frac{2r}{\sigma^2}-1\right)S^{-\frac{2r}{\sigma^2}} + r\left(-\frac{2r}{\sigma^2}S^{-\frac{2r}{\sigma^2}}\right) - rS^{-\frac{2r}{\sigma^2}}$$

$= 0$ ✓ satisfies

b) payoff at expiration $\max(S_0 e^{rt} - E, 0)$

$\Rightarrow$ price: $e^{-rt}\max(S_0 e^{rt} - E, 0) = \max(S_0 - Ee^{-rt}, 0)$

c) If $S_0 > Ee^{-rt}$

$d_1 \to \infty \qquad d_2 \to \infty$

$\phi(d_1) \to 1 \qquad \phi(d_2) \to 1$

$c = S_0 - Ee^{-rt}$

If $S_0 < Ee^{-rt}$

$d_1 \to -\infty \qquad d_2 \to -\infty$

$\phi(d_1) \to 0 \qquad \phi(d_2) \to 0$

$c = 0$

Exercise 5:
Determine the value of the following call using the Black-Scholes-Merton model. The stock's current price is \$95 with $\sigma = 0.6$. The call's exercise price is \$105, and it expires in 8 months from now. Assume that the continuously compounded riskless rate of interest is 0.08.

$C = S_0 \Phi(d_1) - E e^{-rt} \Phi(d_2)$

$d_1 = 0.149521$

$d_2 = -0.340377$

$\Phi(d_1) = 0.559428$

$\Phi(d_2) = 0.36678$

$\boxed{C = 16.6334}$