University of California, Los Angeles
Department of Statistics

Statistics C183/C283 **Instructor: Nicolas Christou**

Homework 5

Solved by Y.Y

Exercise 1
You are given the folowing data on 7 stocks:

Stock i	$\bar{R}_i$	σ_i	
1	0.15	0.10	1
2	0.20	0.15	2
3	0.18	0.20	
4	0.12	0.10	
5	0.10	0.05	3
6	0.14	0.10	4
7	0.16	0.20	

Using these data we ranked the stocks based on the excess return to standard deviation ratio ($\frac{\bar{R}_i - R_f}{\sigma_i}$) to compute the entries in the next table. Assume $R_f = 5\%$ and that the average correlation coefficient is $\rho = 0.50$.

Stock i	$\frac{\bar{R}_i - R_f}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	$\sum_{j=1}^{i} \frac{\bar{R}_j - R_f}{\sigma_j}$	C_i
1	1.00	0.5000	1.00	0.5000
2	1.00	0.3333	2.00	0.6667
5	1.00	0.2500	3.00	0.7500
6	0.90	0.2000	3.90	???
4	0.70	0.1667	4.60	0.7668
3	0.65	0.1429	5.25	0.7502
7	0.55	0.1250	5.80	0.7250

a. Find the missing value C_4.

$$C_4 = \frac{\rho}{1-\rho+i\rho} \sum_{j=1}^{4} \frac{R_j - R_f}{\sigma_j} = 0.2 * 3.9 = 0.78$$

b. What is the composition of the optimum portfolio assuming no short sales?

$$z_i = \frac{1}{(1-\rho)\sigma_i}\left(\frac{\bar{R}_i - R_f}{\sigma_i} - C^*\right)$$

$$z_1 = \frac{1}{(1-0.5)(0.1)}\left(\frac{0.15-0.05}{0.1} - 0.78\right) = 4.4$$

z_2, z_3, z_4 ⇒ used R to compute z_i, x_i

$x_1 = 0.2374$, $x_2 = 0.1583$, $x_3 = 0.4748$, $x_4 = 0.1295$

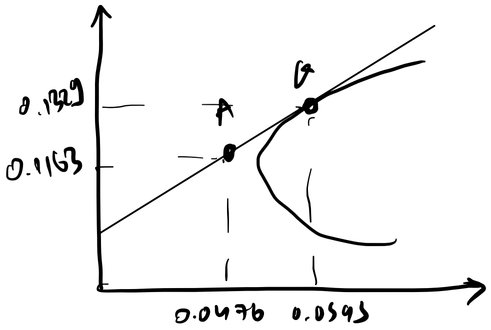
c. What is the expected return and standard deviation of the combination of the optimum portfolio with the risk free asset (80% and 20%)? Show this combination on the graph of expected return against standard deviation.

Find $\bar{R}_G$, $x'R$, σ_G $x'\Sigma x$

$\sigma_A = 0.8\sigma_G$ $\bar{R}_A = 0.8\bar{R}_G + 0.2R_f$ $\bar{R}_A = 0.1163$

$R_G = 0.1329$ used x above, $\sigma_G = 0.0595$ ⇒ $\sigma_A = 0.0476$

$$\Sigma = \begin{bmatrix} 0.005 & 0.0075 & 0.0075 & 0.0050 \\ 0.0075 & 0.0113 & 0.0038 & 0.0025 \\ 0.0025 & 0.0038 & 0.0013 & 0.0025 \\ 0.0050 & 0.0075 & 0.0025 & 0.0050 \end{bmatrix}$$



Exercise 2

Assume that $\sigma_m^2 = 10, R_f = 0.05$. You are also given $\beta_1 = 1, \beta_2 = 1.5, \beta_3 = 1, \beta_4 = 2, \beta_5 = 1, \beta_6 = 1.5, \beta_7 = 2, \beta_8 = 0.8, \beta_9 = 1, \beta_{10} = 0.6$. The table below shoes the procedure for finding the cut-off point C^*.

Stock i	$\frac{\bar{R}_i - R_f}{\beta_i}$	$\frac{(\bar{R}_i - R_f)\beta_i}{\sigma_{\epsilon i}^2}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\beta_j}{\sigma_{\epsilon j}^2}$	$\frac{\beta_i^2}{\sigma_{\epsilon i}^2}$	$\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{\epsilon j}^2}$	C_i
1	10.0	0.20	(a)	0.02000	0.02000	1.67
2	8.0	0.45	0.65	0.05625	0.07625	3.69
3	7.0	0.35	1.00	0.05000	0.12625	4.42
4	6.0	2.40	3.40	0.40000	0.52625	5.43
5	6.0	0.15	3.55	0.02500	0.55125	(c)
6	4.0	0.30	3.85	0.07500	0.62625	5.30
7	3.0	0.30	4.15	0.10000	(b)	5.02
8	2.5	0.10	4.25	0.04000	0.76625	4.91
9	2.0	0.10	4.35	0.05000	0.81625	4.75
10	1.0	0.06	4.41	0.06000	0.87625	4.52

a. Find the three missing values, (a), (b), (c) in the table above.

a) $\Sigma_{j=1}^{1}\left(\frac{(R_j - R_f)\beta_j}{\sigma_{\epsilon_j}^2}\right) = 0.2$

b) $0.62625 + 0.1 = 0.72625$

c) $C_5 = \frac{\sigma_m^2 \text{ Col } 2}{1+\sigma_m^2 \text{ Col } 4} = \frac{10(3.55)}{1+10(0.55125)} = 5.45$

b. If short sales are not allowed find the cut-off point C^* and the value of z_1.

c. If short sales are allowed find the cut-off point C^* and the value of z_1.

$C^* = \max = \boxed{5.45}$ $\quad z_1 = \frac{\beta_1}{\sigma_{\epsilon_i}^2}\left(\overbrace{\frac{R_1 - R_F}{\beta_1}}^{10} - C^*\right)$

$\frac{R_1 - R_F}{\beta_1} = 10 \Rightarrow \frac{R_1 - 0.05}{1} = 10 \quad R_1 = 10.05$ $\quad \frac{10 \cdot 1}{\sigma_{\epsilon_i}^2} = 0.2$

$z_1 = \frac{1}{50}(10 - 5.45) = \boxed{0.0910}$ $\quad \underline{\sigma_{\epsilon_i}^2 = 50}$

c. If short sales are allowed find the cut-off point C^* and the value of z_1.

$C^* = \boxed{4.52}, \; z_1 = \frac{1}{50}(10 - 4.52) = \boxed{0.1096}$

d. Find the correlation coefficient between stock 1 and the market.

$$\rho_{1m} = \frac{\sigma_{1m}}{\sigma_1\sigma_m} = \frac{\beta_1 \sigma_m^2}{\sqrt{\beta_1^2\sigma_m^2 + \sigma_{\epsilon_1}^2}\sqrt{\sigma_m^2}} = \frac{1 \; 10}{\sqrt{1 \; 10^2 + 50}\;\sqrt{10^2}} = \boxed{0.4495}$$

Exercise 3

Using the constant correlation model we completed the table below on 12 stocks. Assume $R_f = 0.05$ and average correlation $\rho = 0.45$.

Stock i	$\bar{R}_i$	σ_i	$\frac{\bar{R}_i - R_f}{\sigma_i}$	$\frac{\rho}{1-\rho+i\rho}$	$\sum_{j=1}^{i} \frac{\bar{R}_j - R_f}{\sigma_j}$	C_i
1	0.27	0.031	7.097	0.450	7.097	3.194
2	0.31	0.042	6.190	0.310	13.287	4.124
3	0.16	0.023	4.783	0.237	18.070	4.280
4	0.15	0.021	4.762	0.191	22.832	4.372
5	0.33	0.059	4.746	$a=?$	$b=?$	$c=?$
6	0.27	0.061	3.607	0.138	31.184	4.318
7	0.19	0.039	3.590	0.122	34.774	4.229
8	0.13	0.029	2.759	0.108	37.532	4.070
9	0.16	0.051	2.157	0.098	39.689	3.883
10	0.12	0.038	1.842	0.089	41.531	3.701
11	0.08	0.022	1.364	70.082	42.895	3.510
12	0.06	0.028	0.357	0.076	43.252	3.271

a. Find the three missing numbers a, b, c in the table above.

b. Find the cut-off point C^* if short sales are not allowed.

c. Find the cut-off point C^* if short sales are allowed.

d. Write down the expression in matrix form that computes the variance of the portfolio when short sales are allowed. No calculations.

e. You are given a new stock with $\bar{R} = 0.055, \sigma = 0.025$. What will change when short sales are allowed and when short sales are not allowed in terms of the portfolio allocation. Briefly explain your answer without doing all the calculations.

a) $\frac{\rho}{1-\rho+5\rho} = \overset{a)}{\boxed{0.1607}}$, $22.832 + 4.746 b = \overset{b)}{\boxed{27.578}}$, $\overset{c)}{a * b = \boxed{4.41}}$

b) $\boxed{4.41} = C^*_{max}$

c) Bottom $\boxed{3.271} = C^*$

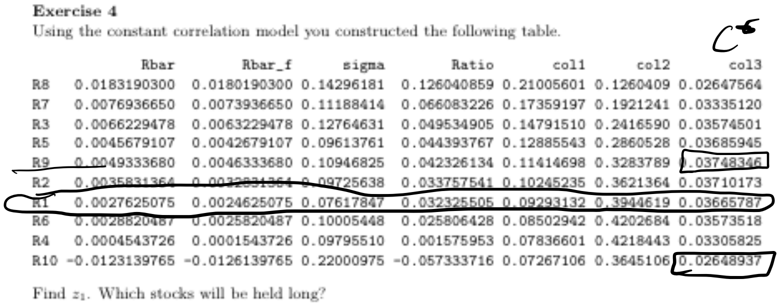
d) $\sigma_a^2 = \underline{x}'\underline{\Sigma}\,\underline{x} \Rightarrow \Sigma = \begin{bmatrix} \sigma_1^2 & \overset{\text{average cor}}{\rho}\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \end{bmatrix}$, use table or formula for x: $\boxed{Z = \Sigma^{-1}R \quad x = \frac{Z}{1'Z}}$ $R = \begin{bmatrix} \bar{R}_i - r_f \end{bmatrix}$

e) $\frac{\bar{R}_i - R_F}{\sigma_i} = \frac{0.055 - 0.05}{0.025} = 0.2$

S.S not allowed: since $0.2 < 4.41 \rightarrow$ no allocation on new

S.S allowed: new solution is required, also a new table is needed, solve as $Z = \Sigma^{-1} R$ $\boxed{x = \frac{Z}{1'Z}}$

update both



Exercise 4

Using the constant correlation model you constructed the following table.

C^*

```
              Rbar        Rbar_f      sigma         Ratio       col1      col2       col3
R8   0.0183190300  0.0180190300 0.14296181  0.126040859 0.21005601 0.1260409 0.02647564
R7   0.0076936650  0.0073936650 0.11188414  0.066083226 0.17359197 0.1921241 0.03335120
R3   0.0066229478  0.0063229478 0.12764631  0.049534905 0.14791510 0.2416590 0.03574501
R5   0.0045679107  0.0042679107 0.09613761  0.044393767 0.12885543 0.2860528 0.03685945
R9   0.0049333680  0.0046333680 0.10946825  0.042326134 0.11414698 0.3283789 0.03748346
R2   0.0035831364  0.0032831364 0.09725638  0.033757541 0.10245235 0.3621364 0.03710173
R1   0.0027625075  0.0024625075 0.07617847  0.032325505 0.09293132 0.3944619 0.03665787
R6   0.0028820487  0.0025820487 0.10005448  0.025806428 0.08502942 0.4202684 0.03573518
R4   0.0004543726  0.0001543726 0.09795510  0.001575953 0.07836601 0.4218443 0.03305825
R10 -0.0123139765 -0.0126139765 0.22000975 -0.057333716 0.07267106 0.3645106 0.02648937
```

Find z_1. Which stocks will be held long?

$\text{col1} = \frac{\rho}{1-\rho+i\rho} \Rightarrow i=1 \Rightarrow \boxed{\rho = 0.21005601}$

$C^* = 0.03748346$ if S.S not allowed

$C^* = 0.02648937$ if S.S allowed

R_8, R_7, R_3, R_5, R_9 held long

S.S allowed

$$z_1 = \frac{1}{(1-\rho)\sigma_1}\left[\frac{\bar{R}_1 - R_F}{\sigma_1} - C^*\right] = -0.0857$$

S.S not allowed

$$z_1 = \frac{1}{(1-\rho)\sigma_1}\left[\frac{\bar{R}_1 - R_F}{\sigma_1} - C^*\right] = 0$$

, not used

Exercise 5

Answer the following questions:

a. Consider the multi group model with 5 stocks in each of 5 industries. The solution is based on the calculation of the Φ_i values which are computed using $\boldsymbol{\Phi} = \boldsymbol{A}^{-1}\boldsymbol{C}$. Write the elements of the third column of matrix **A**. Write the expression for z_{13} (the z value for stock 13 that belongs in the third industry).

b. Consider the multi-index model with 5 stocks in each of 5 industries. The solution is based on the calculation of the Φ_i values which are computed using $\boldsymbol{\Phi} = \boldsymbol{M}^{-1}\boldsymbol{R}$. Write the elements of the second row of matrix **M**. Write the expression for z_4 (the z value for stock 4 that belongs in the second industry).

a)

$$A=\begin{bmatrix} & & 3 \\ \cdots & \cdots & \left(\frac{5\rho_{13}}{1-\rho_{11}}\right) \\ \vdots & \vdots & \left(\frac{5\rho_{23}}{1-\rho_{22}}\right) \\ & & \left(1+\frac{5\rho_{33}}{1-\rho_{33}}\right) \\ & & \left(\frac{5\rho_{43}}{1-\rho_{44}}\right) \\ & & \left(\frac{5\rho_{53}}{1-\rho_{51}}\right) \end{bmatrix}$$

$$z_{13} = \frac{1}{\sigma_{13}(1-\rho_{33})}\left(\frac{\bar{R}_{13}-R_F}{\sigma_{13}} - \sum_{g=1}^{5}\rho_{3g}\Phi_g\right)$$

b)

$$2)\begin{bmatrix} - \quad - \quad - \\ \mathrm{cov}(I_2,I_1)\sum_{i\in B}\frac{\beta_i^2}{\sigma_{\epsilon_i}^2}, \quad 1+\mathrm{var}(I_2)\sum_{i\in B}\frac{\beta_i^2}{\sigma_{\epsilon_i}^2}, \quad \mathrm{cov}(I_2,I_3)\sum_{i\in B}\frac{\beta_i^2}{\sigma_{\epsilon_i}^2} \end{bmatrix}$$

continued

$$\mathrm{cov}(I_2,I_4)\sum_{i\in B}\frac{\beta_i^2}{\sigma_{\epsilon_i}^2}, \quad \mathrm{cov}(I_2,I_5)\sum_{i\in B}\frac{\beta_i^2}{\sigma_{\epsilon_i}^2} \Bigg]$$

$$z_4 = \frac{\beta_4}{\sigma_{\epsilon_1}^2}\left[\frac{\bar{r}_4 - r_f}{\beta_4} - \left(\sigma_m^2\sum_{i=1}^{5} b_{1i}\Phi_i + \sigma_{c_1}^2\Phi_1\right)\right]$$

Exercise 6

Answer the following questions:

a. Suppose the single index model holds, short sales are allowed, and that the investor has access to the risk free asset. Show that the cut-off point C^* can be written as $C^* = (\bar{R}_p - R_f)\beta_p \frac{\sigma_m^2}{\sigma_p^2}$. Use your project data and your answer to project 5 (b,c) to support this result.

b. You are given the following data:

Stock i	$\bar{R}_i$	σ_i
1	0.29	0.03
2	0.19	0.02
3	0.08	0.15

1. Assume short sales are allowed, $R_f = 0.05$, and $\bar{\rho} = 0.5$. Rank the stocks based on the excess return to standard deviation ratio, find the cut-off rate C^*, and the optimal portfolio. Would the answer change if short sales are not allowed?

2. The solution when short sales allowed could have been found using the techniques that discussed earlier in class through the following:

$$\boldsymbol{Z} = \boldsymbol{\Sigma}^{-1}\boldsymbol{R} = \begin{pmatrix} 0.00090 & 0.00030 & 0.00225 \\ 0.00030 & 0.00040 & 0.00150 \\ 0.00225 & 0.00150 & 0.02250 \end{pmatrix}^{-1} \begin{pmatrix} 0.29 - 0.05 \\ 0.19 - 0.05 \\ 0.08 - 0.05 \end{pmatrix} = \begin{pmatrix} 280.00 \\ 320.00 \\ -48.00 \end{pmatrix}.$$

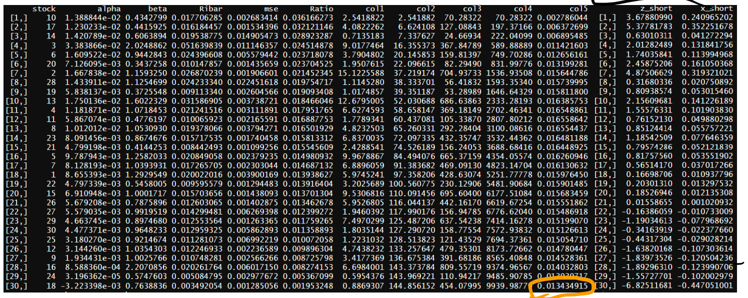
Explain what you see here and verify that the solution is the same as with part (1) when short sales are allowed.

a)

$$C^* = \sigma_m^2 \sum z_j b_j$$

$$z_j = \lambda x_j \Rightarrow \lambda = \frac{\bar{R}_p - R_F}{\sigma_p^2} \Rightarrow C^* = \sigma_m^2 \sum \frac{\bar{R}_p - R_F}{\sigma_p^2} \underbrace{x_j \beta_j}_{\beta_p}$$

$$C^* = \sigma_m^2 \frac{\bar{R}_p - R_F}{\sigma_p^2} \beta_p = \boxed{(\bar{R}_p - R_F)\beta_p \frac{\sigma_m^2}{\sigma_p^2}}$$

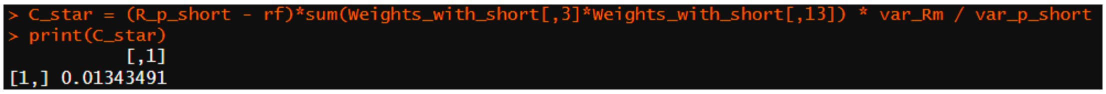


```
      stock         alpha      beta       Ribar         mse       Ratio       col1       col2      col3       col4        col5          z_short      x_short
 [1,]    10  1.388844e-02 0.4342799 0.017706285 0.002683414 0.036166273  2.5418822   2.541882  70.28322   70.28322 0.002786044  [1,]  3.67880990  0.240965202
 [2,]    17  1.230233e-02 0.4415925 0.016184457 0.001534396 0.032121146  4.0822262   6.624108 127.08843  197.37166 0.006372699  [2,]  5.37781783  0.352251678
 [3,]    14  1.420789e-02 0.6063894 0.019538775 0.014905473 0.028923287  0.7135183   7.337627  24.66934  222.04099 0.006895485  [3,]  0.63010311  0.041272294
 [4,]     3  3.383866e-02 2.0248862 0.051639839 0.011146357 0.024514878  9.0177464  16.355373 367.84789  589.88889 0.011421603  [4,]  2.01282489  0.131841756
 [5,]     6  1.609522e-02 0.9442843 0.024396608 0.005579442 0.023718078  3.7904802  20.145853 159.81397  749.70286 0.012656161  [5,]  1.74035841  0.113994968
 [6,]    20  7.126095e-03 0.3437258 0.010147857 0.001435659 0.023704525  1.9507615  22.096615  82.29490  831.99776 0.013199281  [6,]  2.45875206  0.161050368
 [7,]     2  1.667838e-02 1.1593250 0.026870239 0.001906601 0.021452345 15.1225588  37.219174 704.93733 1536.93508 0.015644786  [7,]  4.87506629  0.319321021
 [8,]    28  1.433911e-02 1.1254699 0.024233340 0.022451618 0.019754717  1.1145280  38.333701  56.41832 1593.35340 0.015739995  [8,]  0.31680336  0.020750892
 [9,]    19  5.838137e-03 0.3725548 0.009113340 0.002604566 0.019093408  1.0174857  39.351187  53.28989 1646.64329 0.015811800  [9,]  0.80938574  0.053015460
[10,]    13  1.750136e-02 1.6022329 0.031586905 0.003738721 0.018466046 12.6795005  52.030688 686.63863 2333.28193 0.016385753 [10,]  2.15609681  0.141226189
[11,]     4  1.181871e-02 1.0718453 0.021241516 0.003111891 0.017951765  6.6274593  58.658147 369.18149 2702.46341 0.016548861 [11,]  1.55576331  0.101903830
[12,]    11  5.867074e-03 0.4776197 0.010065923 0.002165591 0.016887753  1.7789341  60.437081 105.33870 2807.80212 0.016558642 [12,]  0.76152130  0.049880298
[13,]     8  1.012012e-02 1.0530930 0.019378066 0.003794271 0.016501929  4.8232503  65.260331 292.28404 3100.08616 0.016554437 [13,]  0.85124414  0.055757221
[14,]    23  8.091456e-03 0.8674676 0.015717535 0.001740458 0.015813312  6.8370035  72.097335 432.35747 3532.44362 0.016481188 [14,]  1.18542509  0.077646359
[15,]    21  4.799198e-03 0.4144253 0.008442493 0.001099256 0.015545609  2.4288541  74.526189 156.24053 3688.68416 0.016448925 [15,]  0.79574286  0.052121839
[16,]     5  9.787943e-03 1.2582033 0.020849058 0.002379235 0.014980932  9.9678867  84.494076 665.37159 4354.05574 0.016260946 [16,]  0.81757560  0.053551902
[17,]     7  8.128193e-03 1.0393931 0.017265705 0.002303044 0.014687132  6.8896059  91.383682 469.09130 4823.14704 0.016130632 [17,]  0.56514170  0.037017266
[18,]     1  8.655393e-03 1.2929549 0.020022016 0.003900169 0.013938627  5.9745241  97.358206 428.63074 5251.77778 0.015976450 [18,]  0.16698706  0.010937796
[19,]    22  4.797339e-03 0.5458005 0.009595579 0.001294483 0.013916404  3.2025689 100.560775 230.12906 5481.90684 0.015901485 [19,]  0.20301310  0.013297532
[20,]    15  6.910948e-03 1.0001717 0.015703656 0.001438093 0.013701304  9.5306816 110.091456 695.60400 6177.51084 0.015683459 [20,]  0.18526946  0.012135308
[21,]    26  5.679208e-03 0.7875896 0.012603065 0.001402875 0.013462678  5.9526805 116.044137 442.16170 6619.67254 0.015551862 [21,]  0.01558655  0.001020932
[22,]    27  5.579035e-03 0.9919519 0.014299481 0.006269398 0.012399272  1.9460392 117.990176 156.94785 6776.62040 0.015486918 [22,] -0.16386059 -0.010733009
[23,]    29  4.663745e-03 0.8974680 0.012553564 0.001263365 0.011759265  7.4970299 125.487206 637.54238 7414.16278 0.015199070 [23,] -1.19034613 -0.077968692
[24,]    30  4.477371e-03 0.9648233 0.012959325 0.005862893 0.011358893  1.8035144 127.290720 158.77554 7572.93832 0.015126613 [24,] -0.34163919 -0.022377660
[25,]    25  3.180270e-03 0.9214674 0.011281073 0.006992219 0.010072058  1.2231032 128.513823 121.43529 7694.37361 0.015054710 [25,] -0.44317304 -0.029028214
[26,]    12  3.144260e-03 1.0354303 0.012246933 0.002236589 0.009896304  4.7438232 133.257647 479.35301 8173.72662 0.014780447 [26,] -1.63820168 -0.107303614
[27,]     9  1.934431e-03 1.0025766 0.010748281 0.002566266 0.008725798  3.4177369 136.675384 391.68186 8565.40848 0.014528361 [27,] -1.83973526 -0.120504236
[28,]    16  8.588360e-04 2.2070856 0.020261764 0.006017150 0.008274153  6.6984001 143.373784 809.55719 9374.96567 0.014032803 [28,] -1.89296310 -0.123990706
[29,]    24  3.196362e-05 0.5747603 0.005084795 0.002977672 0.005367099  0.5954376 143.969221 110.94217 9485.90785 0.013930717 [29,] -1.55727701 -0.102002979
[30,]    18 -3.223398e-03 0.7638836 0.003492054 0.001285056 0.001953248  0.8869307 144.856152 454.07995 9939.9877  0.013434915 [30,] -6.82511681 -0.447051001
```

C^*

S.S allowed

$C^* = 0.01343491$

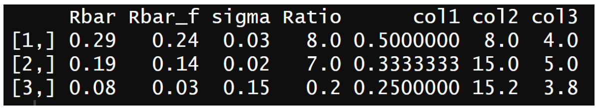


```
> C_star = (R_p_short - rf)*sum(Weights_with_short[,3]*Weights_with_short[,13]) * var_Rm / var_p_short
> print(C_star)
           [,1]
[1,] 0.01343491
```

All variables to calculate C^* are available at $\boxed{\text{Project 6}}$ Part a

C^*

b)



```
     Rbar Rbar_f sigma Ratio      col1 col2 col3
[1,] 0.29    0.24  0.03   8.0 0.5000000  8.0  4.0
[2,] 0.19    0.14  0.02   7.0 0.3333333 15.0  5.0
[3,] 0.08    0.03  0.15   0.2 0.2500000 15.2  3.8
```

1)

Ranking Ratio : $1 > 2 > 3$

$C^* = 5$ if S.S not allowed, $C^* = 3.8$ if SS allowed

$x_1 = x_2 = 0.5$ if S.S not allowed, $\left.\begin{array}{l} x_1 = 0.5072 \\ x_2 = 0.5797 \\ x_3 = -0.0470 \end{array}\right\}$ if S.S allowed

Composition change when S.S allowed, C^* also change

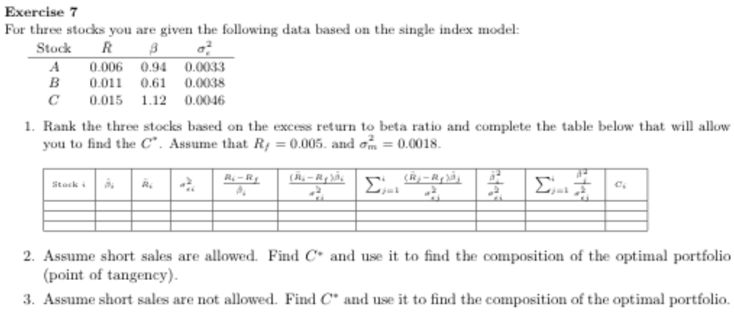
2)

$$z_i = \frac{1}{(1-\rho)\sigma_i}\left(\frac{\bar{R}_i - R_f}{\sigma_i} - C^*\right)$$ is same when C^* is chosen

to be value at the last row when stocks are ranked using $\boxed{\frac{\bar{R}_i - R_f}{\sigma_i}}$.

$z_1 = 280$, $z_2 = 370$, $z_3 = -48$, we can also find those with $z = \Sigma^{-1} R_{barf}$

$x_1 = 0.5072$, $x_2 = 0.5797$, $x_3 = -0.0870$



Exercise 7

For three stocks you are given the following data based on the single index model:

Stock	$\bar{R}$	β	σ_ϵ^2
A	0.006	0.94	0.0033
B	0.011	0.61	0.0038
C	0.015	1.12	0.0046

1. Rank the three stocks based on the excess return to beta ratio and complete the table below that will allow you to find the C^*. Assume that $R_f = 0.005$. and $\sigma_m^2 = 0.0018$.

Stock i	$\hat{\beta}_i$	$\bar{R}_i$	$\sigma_{\epsilon i}^2$	$\frac{R_i - R_f}{\beta_i}$	$\frac{(\bar{R}_i - R_f)\hat{\beta}_i}{\sigma_{\epsilon i}^2}$	$\sum_{j=1}^{i} \frac{(\bar{R}_j - R_f)\hat{\beta}_j}{\sigma_{\epsilon j}^2}$	$\frac{\hat{\beta}_i^2}{\sigma_{\epsilon i}^2}$	$\sum_{j=1}^{i} \frac{\beta_j^2}{\sigma_{\epsilon j}^2}$	C_i

2. Assume short sales are allowed. Find C^* and use it to find the composition of the optimal portfolio (point of tangency).
3. Assume short sales are not allowed. Find C^* and use it to find the composition of the optimal portfolio.

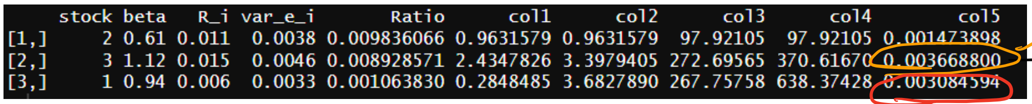
$1 \Leftrightarrow A, \quad 2 \Leftrightarrow B, \quad 3 \Leftrightarrow C$

$\boxed{\text{col 5 is } C^*}$

1)

C^*

S.S not allowed



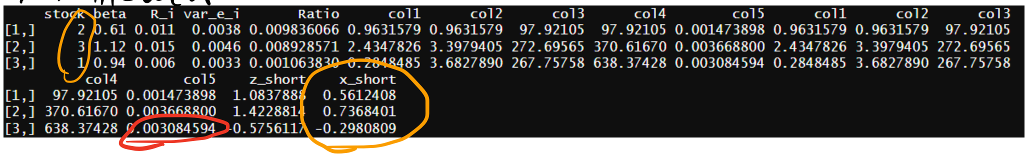
```
     stock beta   R_i var_e_i       Ratio      col1      col2      col3      col4        col5
[1,]     2 0.61 0.011  0.0038 0.009836066 0.9631579 0.9631579  97.92105  97.92105 0.001473898
[2,]     3 1.12 0.015  0.0046 0.008928571 2.4347826 3.3979405 272.69565 370.61670 0.003668800
[3,]     1 0.94 0.006  0.0033 0.001063830 0.2848485 3.6827890 267.75758 638.37428 0.003084594
```

$\rightarrow C^*$

S.S allowed

S.S allowed

2)



```
     stock beta   R_i var_e_i       Ratio      col1      col2      col3      col4        col5      col1      col2      col3
[1,]     2 0.61 0.011  0.0038 0.009836066 0.9631579 0.9631579  97.92105  97.92105 0.001473898 0.9631579 0.9631579  97.92105
[2,]     3 1.12 0.015  0.0046 0.008928571 2.4347826 3.3979405 272.69565 370.61670 0.003668800 2.4347826 3.3979405 272.69565
[3,]     1 0.94 0.006  0.0033 0.001063830 0.2848485 3.6827890 267.75758 638.37428 0.003084594 0.2848485 3.6827890 267.75758
          col4        col5    z_short    x_short
[1,]  97.92105 0.001473898  1.0837888  0.5612408
[2,] 370.61670 0.003668800  1.4228814  0.7368401
[3,] 638.37428 0.003084594 -0.5756117 -0.2980809
```

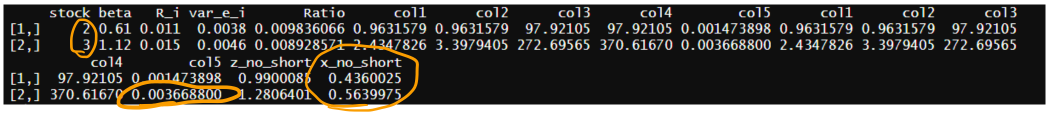
$x_2 = 0.5612408$

$x_3 = 0.7368401$

$x_1 = -0.290809$

$\rightarrow$ $\boxed{\text{col's added two times ignore}}$

3)



```
     stock beta   R_i var_e_i       Ratio      col1      col2      col3      col4        col5      col1      col2      col3
[1,]     2 0.61 0.011  0.0038 0.009836066 0.9631579 0.9631579  97.92105  97.92105 0.001473898 0.9631579 0.9631579  97.92105
[2,]     3 1.12 0.015  0.0046 0.008928571 2.4347826 3.3979405 272.69565 370.61670 0.003668800 2.4347826 3.3979405 272.69565
          col4        col5 z_no_short x_no_short
[1,]  97.92105 0.001473898  0.9900085  0.4360025
[2,] 370.61670 0.003668800  1.2806401  0.5639975
```

S.S not allowed

$x_2 = 0.4360025$

$x_3 = 0.5639975$

$x_1 = 0$

$C^* = 0.003668800$

$\boxed{\text{col's added two times ignore}}$

Exercise 8

Consider the multigroup model. In class we used an example of two stocks and two industries and then we extended the result to the general case. Use the same steps to develop the system of equations when there are are three stocks and three industries to find the elements of the equation $\mathbf{A\Phi} = \mathbf{C}$. You can begin with
$\bar{R}_1 - R_f = z_1\sigma_1^2 + z_2\sigma_{12} + z_3\sigma_{13} + z_4\sigma_{14} + z_5\sigma_{15} + z_6\sigma_{16} + z_7\sigma_{17} + z_8\sigma_{18} + z_9\sigma_{19}$.

$$\begin{bmatrix} 1+\frac{3\rho_{11}}{1-\rho_{11}} & \frac{3\rho_{12}}{1-\rho_{11}} & \frac{3\rho_{13}}{1-\rho_{11}} \\ \frac{3\rho_{21}}{1-\rho_{22}} & 1+\frac{3\rho_{22}}{1-\rho_{22}} & \frac{3\rho_{23}}{1-\rho_{22}} \\ \frac{3\rho_{31}}{1-\rho_{33}} & \frac{3\rho_{32}}{1-\rho_{33}} & 1+\frac{3\rho_{33}}{1-\rho_{33}} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{3} \frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{11})} \\ \sum_{i=4}^{6} \frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{22})} \\ \sum_{i=7}^{9} \frac{\bar{R}_i - R_f}{\sigma_i(1-\rho_{33})} \end{bmatrix}$$

Derivation:

$$\bar{R}_1 - R_f = z_1\sigma_1^2 - z_1\rho_{11}\sigma_1^2 + z_1\rho_{11}\sigma_1^2 + z_2\rho_{12}\sigma_1\sigma_2 + \dots + z_9\rho_{13}\sigma_1\sigma_9$$

$$\bar{R}_1 - R_f = z_1\sigma_1^2(1-\rho_{11}) + \sigma_1\sum_{g=1}^{3}\rho_{1g}\Phi_g$$

$$z_1 = \frac{1}{\sigma_1(1-\rho_{11})}\left[\frac{\bar{R}_1 - R_f}{\sigma_1} - \sum_{g=1}^{3}\rho_{1g}\Phi_g\right]$$

$$z_2 = \frac{1}{\sigma_2(1-\rho_{11})}\left[\frac{\bar{R}_2 - R_f}{\sigma_2} - \sum_{g=1}^{3}\rho_{1g}\Phi_g\right]$$

$$z_3 = \frac{1}{\sigma_3(1-\rho_{11})}\left[\frac{\bar{R}_3 - R_f}{\sigma_3} - \sum_{g=1}^{3}\rho_{1g}\Phi_g\right]$$

Multiply z_i with σ_i and sum

$$\underbrace{\sigma_1 z_1 + \sigma_2 z_2 + \sigma_3 z_3}_{\Phi_1} = \frac{1}{1-\rho_{11}}\left[\sum_{i=1}^{3}\frac{\bar{R}_i - R_f}{\sigma_i} - 3\sum_{g=1}^{3}\rho_{1g}\Phi_g\right]$$

$$(1-\rho_{11})\,\Phi_1 + 3\sum_{g=1}^{3}\rho_{1g}\Phi_g = \sum_{i=1}^{3}\frac{\bar{R}_i - R_f}{\sigma_i}$$

Also for other groups:

$$(1-\rho_{22})\,\Phi_2 + 3\sum_{g=1}^{3}\rho_{2g}\Phi_g = \sum_{i=4}^{6}\frac{\bar{R}_i - R_f}{\sigma_i}$$

$$(1-\rho_{33})\,\Phi_3 + 3\sum_{g=1}^{3}\rho_{3g}\Phi_g = \sum_{i=7}^{9}\frac{\bar{R}_i - R_f}{\sigma_i}$$

$$\Downarrow$$

$$\Phi_1 + \frac{3}{(1-\rho_{11})}\sum_{g=1}^{3}\rho_{1g}\Phi_g = \sum_{i=1}^{3}\frac{\bar{R}_i - R_f}{\sigma_i\,(1-\rho_{11})}$$

$$\Phi_2 + \frac{3}{(1-\rho_{22})}\sum_{g=1}^{3}\rho_{1g}\Phi_g = \sum_{i=4}^{6}\frac{\bar{R}_i - R_f}{\sigma_i\,(1-\rho_{22})}$$

$$\Phi_3 + \frac{3}{(1-\rho_{33})}\sum_{g=1}^{3}\rho_{1g}\Phi_g = \sum_{i=7}^{9}\frac{\bar{R}_i - R_f}{\sigma_i\,(1-\rho_{33})}$$

$$\Downarrow$$

$$\begin{bmatrix} 1+\frac{3\rho_{11}}{1-\rho_{11}} & \frac{3\rho_{12}}{1-\rho_{11}} & \frac{3\rho_{13}}{1-\rho_{11}} \\ \frac{3\rho_{21}}{1-\rho_{22}} & 1+\frac{3\rho_{22}}{1-\rho_{22}} & \frac{3\rho_{23}}{1-\rho_{22}} \\ \frac{3\rho_{31}}{1-\rho_{33}} & \frac{3\rho_{32}}{1-\rho_{33}} & 1+\frac{3\rho_{33}}{1-\rho_{33}} \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{3}\frac{\bar{R}_i - R_f}{\sigma_i\,(1-\rho_{11})} \\ \sum_{i=4}^{6}\frac{\bar{R}_i - R_f}{\sigma_i\,(1-\rho_{22})} \\ \sum_{i=7}^{9}\frac{\bar{R}_i - R_f}{\sigma_i\,(1-\rho_{33})} \end{bmatrix}$$