University of California, Los Angeles
Department of Statistics

Statistics C183/C283 — Instructor: Nicolas Christou

Homework 8

Exercise 1
Use a one-step binomial tree to explain the no-arbitrage and risk-neutral valuation of a European call option. Please provide a numerical example.

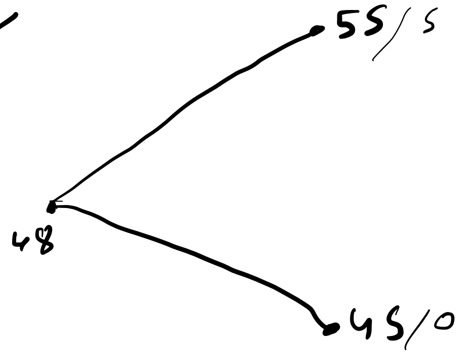
no arbitrage

$(\alpha S_0 - C) = (-C_u + \alpha u S_0)e^{-rt} = [-C_d + \alpha d S_0)e^{-rt}$

risk-neutral valuation

$C = [p C_u + (1-p) C_d)]e^{-rt}$

Ex



$r = 10\%$ $\Rightarrow$ $u = \frac{55}{48} = 1.1458$

$t = 3$ months

$d = \frac{45}{48} = 0.9375$

$E = 50$

$p = 0.4216$ $1-p = 0.5784$

$\alpha = \frac{}{55-45} = \frac{1}{2}$

For no-arbitrage: $24 - 2.056 = (-5 + \frac{55}{2})e^{-\frac{3}{12}0.1})$

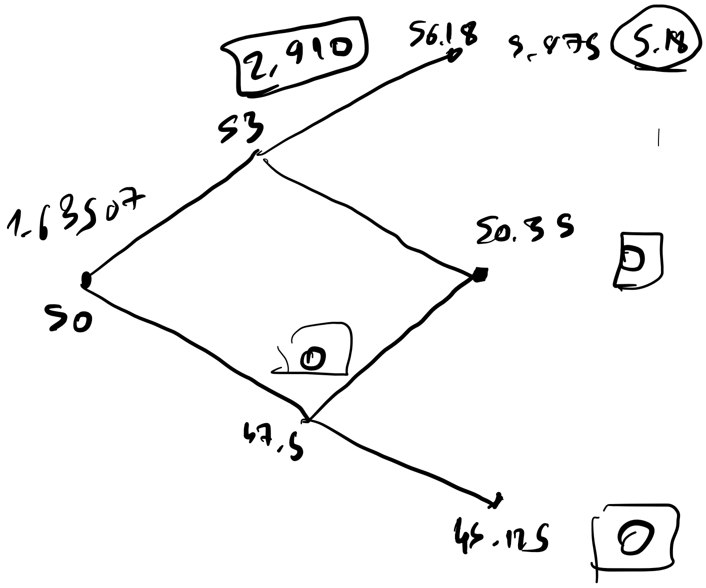
21.9447

$C = (5p + 0(1-p))e^{-\frac{3}{12} \times 0.1} = 2.056$

for risk-neutral

Exercise 2

A stock price is currently \$50. Over each of the next two 3-month periods it is expected to go up by 6% or down by 5%. The risk-free interest rate is 5% per year with continuous compounding. What is the value of a 6-month European call option with strike price of \$51? Complete the entire binomial tree diagram for the 2 periods. Place the price of the stock and the price of the call at each node on the binomial tree diagram.



$u = 1.06 \qquad d = 0.95$

$r = 5\%$ use e^{-rt}

$E = 51$

one step

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{0.05\frac{3}{12}} - 0.95}{0.11}$$

$p = 0.5688 \Rightarrow 1 - p = 0.431104$

$$C = (p^2\, 5.18)\, e^{-rt} = p^2\, 5.18\, e^{-r\frac{6}{12}}$$

$$= \boxed{1.6350711}$$

$$C_u = \left(p\, C_{u^2} + (1-p)\, C_{ud}\right) e^{-r\frac{3}{12}}$$

$C_d = 0$

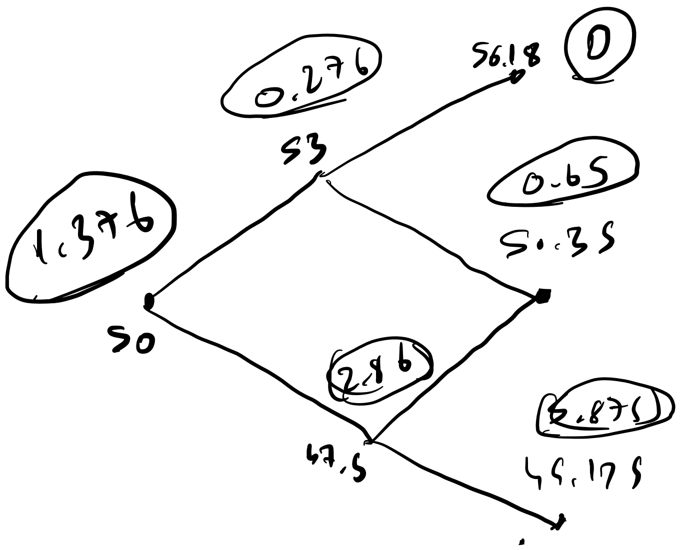
$\Downarrow$

$$C = \left(p\, C_u + (1-p)\, C_d\right) e^{-r\frac{3}{12}} = 1.6350711$$

Check code at the back

Exercise 3

Refer to exercise 2. What is the value of a 6-month European put option with strike price of \$51? Complete the entire binomial tree diagram for the 2 periods. Place the price of the stock and the price of the put at each node on the tree diagram. Verify that the European call and the European put prices satisfy the put-call parity.



$u = 1.06 \quad d = 0.95$

$r = 5\%$ use e^{-rt}

$E = 51$

one step

$$p = \frac{e^{rt} - d}{u - d} = \frac{e^{0.05\frac{3}{12}} - 0.95}{0.11}$$

$p = 0.5688 \Rightarrow 1 - p = 0.431104$

$P = 1.37587665$

Check code at the back

Put-call parity:

$$c + E e^{-rt} = p + S_0$$

$$1.635 + 51 e^{-r\frac{1}{2}} = 1.3758\cancel{7}8 + 50$$

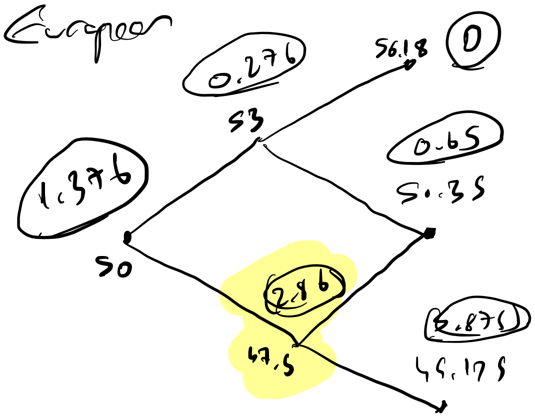
$$= \boxed{51.37587} \Rightarrow \text{same both sides}$$

$\boxed{\text{dif} = -7e^{-15} \text{ negligible}}$

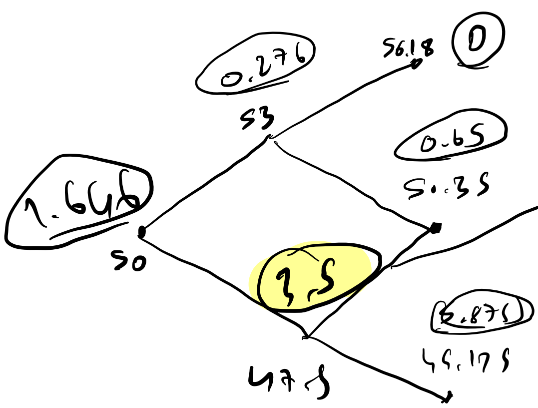
Exercise 4

Refer to exercise 3. If the put option were American, would it ever be optimal to exercise it early at any of the nodes on the binomial tree diagram? Find the value of this American put option.

European



American



optimal to exercise at node C

$$\left(p(0.276) + (1-p)3.5\right)e^{-rt} = 1.646$$

much expensive since there is early exercise

$3.5 > 2.86$

Exercise 5

You want to find the value of a European call option for the following data: $S_0 = \$50, E = \$60, u = 1.2, d = \frac{1}{u}$, $r = 0.10$ (for each period), and number of periods to expiration $n = 10$. Using the binomial option pricing model:

a. Find the value of k, the number of up movements of the stock, so that the call is "in the money" at the end of the 10_{th} period.

$$k = \text{ceil}\left(\frac{\log\left(\frac{E}{d^n S_0}\right)}{\log\left(\frac{u}{d}\right)}\right) = 6$$

$$p = \frac{1+r-d}{u-d} = 0.\overline{72}$$

$$p' = \frac{pu}{1+r} = 0.7953684$$

b. Draw the binomial tree diagram and place the price of the stock at each node of the binomial tree (only at the end of 10_{th} period).

	Price of call
$u^{10} S_0 = 309.58$	249.5868
$u^9 d S_0 = 214.9908$	154.9908
$u^8 d^2 S_0 = 149.29920$	89.2992
$u^7 d^3 S_0 = 103.680$	43.68
$u^6 d^4 S_0 = 72$	12
$u^5 d^5 S_0 = 50$	0
$u^4 d^6 S_0 = 34.72$	0
$u^3 d^7 S_0 = 24.11$	0
$u^2 d^9 S_0 = 16.74$	0
$u^1 d^9 S_0 = 11.12$	0
$d^{10} S_0 = 8.07$	0

Check code at the back

c. What is the intrinsic value of the call at each node of the 10_{th} period?

d. Find the price of the call at $t = 0$ by:

1. Using the binomial formula:

$$C = S_0 \sum_{j=k}^{n} \binom{n}{j} p'^j (1-p')^{n-j} - \frac{E}{(1+r)^n} \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

p, p' found above

$C = 27.486 = S_0 *$ pbinom(k-1, n, p', lower_tail = False) $- \left(\frac{E}{(1+r)^n}\right)$

Same with p

2. Discounting the expected value of the call at the end of the 10_{th} period.

Note: (1) and (2) must give the same answer.

$C{-}2 = 27.48629$

Exercise 6
We have seen that for the one-step binomial tree the probability of upward movement of the stock is given by $p = \frac{e^{rt}-d}{u-d}$. Now refer to page 3 of handout #48, "Binomial option pricing model", to show that p for the second period is also given by $p = \frac{e^{rt}-d}{u-d}$. This holds because u and d are assumed to be constant for each step on the binomial tree.

hedged if : $-C_{u^2} + \alpha u^2 S_0 = -C_{ud} + \alpha u d S_0$

$$\alpha = \frac{C_{u^2} - C_{ud}}{u^2 S_0 - u d S_0}$$

Also; discount interest : $(-C_u + \alpha u S_0)(1+r) = \alpha u d S_0 - C_{ud}$

solve for C_u

$(-C_u)(1+r) = -(\alpha u S_0)(1+r) + \alpha u d S_0 - C_{ud}$

$C_u(1+r) = (\alpha u S_0)(1+r) - \alpha u d S_0 + C_{ud}$ ⟹ plug in α

$$C_u = \frac{\left(\frac{C_{u^2} - C_{ud}}{u^{\cancel{2}} \cancel{S_0} - \cancel{u} d \cancel{S_0}}\right) \cancel{u} \cancel{S_0}(1+r) - \left(\frac{C_{u^2} - C_{ud}}{u^{\cancel{2}} \cancel{S_0} - \cancel{u} d \cancel{S_0}}\right) \cancel{u} d \cancel{S_0} + C_{ud}}{1+r}$$

$$= \frac{\frac{C_{u^2} - C_{ud}}{u-d}(1+r) - \frac{(C_{u^2} - C_{ud})}{u-d} d + C_{ud}}{(1+r)}$$

$$= \frac{\left[(C_{u^2} - C_{ud})\left(\frac{1+r-d}{u-d}\right)\right] + C_{ud}}{1+r}$$

$$= \frac{C_{u^2}\left(\frac{1+r-d}{u-d}\right) + C_{ud}\left(1 - \frac{1+r-d}{u-d}\right)}{1+r} \Rightarrow P = \frac{1+r-d}{u-d}$$

$$= \frac{C_{u^2}\, p + C_{ud}\,(1-p)}{1+r}$$

p does not change

Since u, d same

hw8 Solution to Ex2, Ex3, Ex5

Yaman Yucel

```
#Q2
u  =1.06
d = 0.95
s0 = 50
r = 0.05
E = 51
months = 3
t1 = months/12
t2 = 2*months/12
sU = u*s0
sU2 = u*sU
sD = d*s0
sD2 = d*sD
sUD = u*d*s0

p = (exp(r*t1) - d)/(u-d)
p_not = 1-p

cu2 = max(sU2 - E,0)
cud = max(sUD - E,0)
cd2 = max(sD2 - E,0)

C = (p^2*cu2 + 2*p*(p_not)*cud + p_not^2*cd2)*exp(-r*t2)
cu = (p*cu2 + p_not*cud)*exp(-r*t1)
cd = (p*cud + p_not*cd2)*exp(-r*t1)
c_2 = (cu*p + cd *p_not)*exp(-r*t1)

print(sU)
```

```
## [1] 53
```

```
print(sU2)
```

```
## [1] 56.18
```

```
print(sD)
```

```
## [1] 47.5
```

```
print(sD2)
```

```
## [1] 45.125
```

```
print(sUD)
```

```
## [1] 50.35
```

```
print(p)
```

```
## [1] 0.568895
```

```
print(p_not)
```

```
## [1] 0.431105
```

```
print(cu2)
```

```
## [1] 5.18
```

```
print(cud)
```

```
## [1] 0
```

```
print(cd2)
```

```
## [1] 0
```

```
print(cu)
```

```
## [1] 2.910269
```

```
print(cd)
```

```
## [1] 0
```

```
print(C)
```

```
## [1] 1.635071
```

```
print(c_2)
```

```
## [1] 1.635071
```

```
#Q3
u  =1.06
d = 0.95
s0 = 50
r = 0.05
E = 51
months = 3
t1 = months/12
t2 = 2*months/12
sU = u*s0
sU2 = u*sU
sD = d*s0
sD2 = d*sD
sUD = u*d*s0

p = (exp(r*t1) - d)/(u-d)
p_not = 1-p

pu2 = max(E - sU2,0)
pud = max(E - sUD ,0)
pd2 = max(E - sD2,0)

P = (p^2*pu2 + 2*p*(p_not)*pud + p_not^2*pd2)*exp(-r*t2)
pu = (p*pu2 + p_not*pud)*exp(-r*t1)
```

```r
pd = (p*pud + p_not*pd2)*exp(-r*t1)
P_2 = (pu*p + pd *p_not)*exp(-r*t1)

parity_right = C + E*exp(-r*t2)
parity_left = P + s0
dif = parity_right - parity_left

print(sU)
```

```
## [1] 53
```

```r
print(sU2)
```

```
## [1] 56.18
```

```r
print(sD)
```

```
## [1] 47.5
```

```r
print(sD2)
```

```
## [1] 45.125
```

```r
print(sUD)
```

```
## [1] 50.35
```

```r
print(p)
```

```
## [1] 0.568895
```

```r
print(p_not)
```

```
## [1] 0.431105
```

```r
print(pu2)
```

```
## [1] 0
```

```r
print(pud)
```

```
## [1] 0.65
```

```r
print(pd2)
```

```
## [1] 5.875
```

```r
print(pu)
```

```
## [1] 0.2767373
```

```r
print(pd)
```

```
## [1] 2.866468
```

```r
print(P)
```

```
## [1] 1.375877
```

```r
print(P_2)
```

```
## [1] 1.375877
```

```
print(dif)
```

```
## [1] -7.105427e-15
```

```
u = 1.2
d = 1/u
E = 60
s0 = 50
n = 10
r = 0.1 # for each period, not continuous compounding

k = ceiling(log(E/(d^n*s0))/log(u/d)) # 6

p = ((1+r) - d)/(u-d)
p_not = 1-p

last_price_list = rep(0,n + 1)
call_list = rep(0,n+1)

for (j in 0:n){
  last_price_list[j+1] = u^(j) * d^(n-j) * s0
  call_list[j+1] = max(last_price_list[j+1] - E,0)
}
p_mark = p*u/(1+r)
p_mark_not = 1-p_mark

c = s0*pbinom(k-1,n,p_mark, lower.tail=FALSE) - (E/(1+r)^n)*pbinom(k-1, n, p, lower.tail=FALSE)

get_comb <- function(n,k){ factorial(n)/(factorial(k)*factorial(n-k))}
c_2 = 0
for (i in k:n){
  c_2 = c_2 + get_comb(n,i) * p^i*(1-p)^(n-i)*call_list[i+1]
}
c_2 = c_2 / ((1+r)^n)

dif = c - c_2

print(k)
```

```
## [1] 6
```

```
print(p)
```

```
## [1] 0.7272727
```

```
print(p_not)
```

```
## [1] 0.2727273
```

```
print(last_price_list)
```

```
##  [1]   8.075279  11.628402  16.744899  24.112654  34.722222  50.000000
##  [7]  72.000000 103.680000 149.299200 214.990848 309.586821
```

```
print(call_list)
```

```
##  [1]   0.0000   0.0000   0.0000   0.0000   0.0000   0.0000  12.0000  43.6800
```

```
##  [9]  89.2992 154.9908 249.5868
```

```
print(p_mark)
```

```
## [1] 0.7933884
```

```
print(p_mark_not)
```

```
## [1] 0.2066116
```

```
print(c)
```

```
## [1] 27.48628
```

```
print(c_2)
```

```
## [1] 27.48628
```

```
print(dif)
```

```
## [1] -3.552714e-15
```