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Homework 3

Solved by Y. Y

Exercise 1
Consider two stocks A and B with expected returns $\bar{R}_1, \bar{R}_2$, variances σ_1^2, σ_2^2, and covariance σ_{12}. Suppose short sales are allowed and risk free asset R_f exists. Show that the composition of the optimal portfolio is

$$x_1 = \frac{\bar{R}_A \times \sigma_2^2 - \bar{R}_B \times \sigma_{12}}{\bar{R}_A \times \sigma_2^2 + \bar{R}_B \times \sigma_1^2 - (\bar{R}_A + \bar{R}_B) \times \sigma_{12}}$$
$$x_2 = 1 - x_1$$

Note: $\bar{R}_A = \bar{R}_1 - R_f$ and $\bar{R}_B = \bar{R}_2 - R_f$.

x's can be solved after obtaining z values which can be computed with $z = \Sigma^{-1}\bar{R}$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} = \sigma_{12} & \sigma_2^2 \end{bmatrix}, \quad \bar{R} = \begin{bmatrix} \bar{R}_1 - R_f \\ \bar{R}_2 - R_f \end{bmatrix} = \begin{bmatrix} \bar{R}_A \\ \bar{R}_B \end{bmatrix}$$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}, \text{ then}$$

$$z = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix} \begin{bmatrix} \bar{R}_A \\ \bar{R}_B \end{bmatrix}$$

$$z_1 = \frac{\bar{R}_A \sigma_2^2 - \sigma_{12}\bar{R}_B}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}, \quad z_2 = \frac{\bar{R}_B \sigma_1^2 - \sigma_{12}\bar{R}_A}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}$$

$$x_1 = \frac{z_1}{\Sigma z} = \frac{z_1}{z_1 + z_2} = \frac{\bar{R}_A \sigma_2^2 - \sigma_{12}\bar{R}_B}{\bar{R}_A \sigma_2^2 - \sigma_{12}\bar{R}_B + \bar{R}_B \sigma_1^2 - \sigma_{12}\bar{R}_A}$$

$$x_2 = \frac{z_2}{z_1 + z_2} = 1 - \frac{z_1}{z_1 + z_2} = 1 - x_1$$

Exercise 2

Given the following:

Stock	$\bar{R}$	σ
Stock A	0.12	0.20
Stock B	???	0.08

It is also given that $\rho_{AB} = 0.1$.

a. What expected return on stock B would result in an optimum portfolio of $\frac{1}{2}A$ and $\frac{1}{2}B$? Assume short sales are allowed and that $R_f = 0.04$.

Using ex 1:

$$0.5 = \frac{(0.12-0.04)(0.08)^2 - (0.1)(0.2)(0.08)(\bar{R}_B - 0.04)}{\text{same} + (\bar{R}_B - 0.04)(0.2)^2 - (0.1)(0.2)(0.08)(0.12 - 0.04)}$$

$$0.5 = \frac{(0.08)^3 - 0.096(\bar{R}_B - 0.04)}{(0.08)^3 - 0.096(\bar{R}_B - 0.04) - (0.096)(0.08) + (\bar{R}_B - 0.04)(0.2)^2}$$

$\Rightarrow \bar{R}_B = 0.0553846$ I used Wolfram to solve above

b. What expected return on stock B would mean that stock B would not be held? Assume short sales are allowed and that $R_f = 0.04$.

Set $x_A = 1$,

$$1 = \frac{(0.08)^3 - 0.096(\bar{R}_B - 0.04)}{(0.08)^3 - 0.096(\bar{R}_B - 0.04) - (0.096)(0.08) + (\bar{R}_B - 0.04)(0.2)^2}$$

$\Rightarrow \bar{R}_B = 0.0432$, I used Wolfram to solve above

Exercise 3

Use a numerical example of three stocks with a value of R_f of your choice to find the point of tangency G and then (1) combine G with R_f to find portfolio A on CAL and (2) verify that A can be obtained by using the formula for the weights $\mathbf{X}$ when the investor requires $\sum_{i=1}^{n}(\bar{R}_i - R_f)x_i + R_f = E$, where E is the expected value of portfolio A.

$$\bar{R}_1 = 0.25, \quad \bar{R}_2 = 0.3, \quad \bar{R}_3 = 0.35, \quad R_f = 0.1$$

$$\Sigma = \begin{bmatrix} 0.04 & 0.02 & 0.01 \\ 0.02 & 0.09 & 0.05 \\ 0.01 & 0.05 & 0.16 \end{bmatrix} \Rightarrow \Sigma^{-1} = \begin{bmatrix} 28.13 & -6.38 & 0.2264 \\ -6.38 & 14.89 & -4.755 \\ 0.2264 & -4.755 & 7.16 \end{bmatrix}$$

$$Z = \Sigma^{-1} R = \begin{bmatrix} 0.0125 \\ 0.0335 \\ 0.0515 \end{bmatrix}$$

$$X_G = \begin{bmatrix} 0.128 \\ 0.344 \\ 0.528 \end{bmatrix} \Rightarrow R_G = 0.32, \quad var(G) = 0.0772$$

$$\boxed{\text{Composition of } x_A \text{ is } \alpha x_G, \ \alpha < 1}$$

$$R_A = 0.5 R_G + 0.5(0.1) = 0.21$$

$$R_A = 0.5(x_G' \bar{R}) + 0.5\, 0.1)$$
$$= \underbrace{0.0641}_{x_{A1}} \bar{R}_1 + \underbrace{0.1718}_{x_{A2}} \bar{R}_2 + \underbrace{0.2641}_{x_{A3}} \bar{R}_3 + 0.5 R_f$$

Verify formula: $\sum_{i=1}^{n}(\bar{R}_i - R_f)x_i + R_f = E, = 0.21$ ✓ $\quad x_A' R + x_A' 1 R_f + R_f$

$$((\underbrace{0.25}_{\bar{R}_1} - \underbrace{0.1}_{R_f})) \underbrace{0.064}_{x_{A1}} + (\underbrace{0.30}_{\bar{R}_2} - \underbrace{0.1}_{R_f}) \underbrace{0.1718}_{x_{A2}} + (\underbrace{0.35}_{\bar{R}_3} - \underbrace{0.1}_{R_f}) \underbrace{0.264}_{x_{A3}} + \underbrace{0.1}_{R_f}$$

$$= 0.09645 + 0.03436 + 0.06625 + 0.1$$
$$= 0.1 + 0.11 = 0.21 \checkmark = R_A$$

$$x_A = \alpha\, x_G \qquad \alpha < 1 \qquad R_A = \alpha\, x_G' \bar{R} + (1-\alpha) R_f$$

$$R_A = x_A' \bar{R} + (1-\alpha) R_f = x_A' \bar{R} + R_f - \underline{\alpha} R_f$$

$$= x_A' \bar{R} + R_f - \underbrace{x_A' 1}_{\alpha} R_f$$

$$\Rightarrow \sum_{i=1}^{n} x_i R_i - R_f \sum_{i=1}^{n} x_i + R_f$$

$$= \sum_{i=1}^{n} x_i (R_i - R_f) + R_f = R_A \checkmark$$

Verify using

$$\underline{x} = \frac{(E - R_f)\, \underline{\Sigma}^{-1} (\bar{R} - R_F \underline{1})}{(\bar{R} - R_F \underline{1}) \underline{\Sigma}^{-1} (\underline{\bar{R}} - R_F \underline{1})} = \begin{bmatrix} 0.0641 \\ 0.1718 \\ 0.2641 \end{bmatrix}$$

same composition found in previous part.

Exercise 4
Answer the following questions:

a. An investor has $900000 invested in a diversified portfolio. Subsequently the investor inherits ABC company stock worth $100000. His financial adviser provided him with the following forecast information:

	$\bar{R}$ (monthly)		σ (monthly)	
Portfolio	0.67%	0.0067	2.37%	0.0237
ABC Compnay	1.25%	0.0125	2.95%	0.0295

The correlation coefficient between ABC company stock returns and the portfolio is 0.40.
Assume that the investor keeps the ABC company stock. Answer the following questions:

1. Calculate the expected return of the new portfolio which includes the ABC company stock.

Total of investor is 1m, therefore $X_p = 0.9$, $X_{ABC} = 0.1$ normalized

$$\bar{R}_{new} = \frac{0.67}{100} \cdot 0.9 + 0.1 \cdot \frac{1.25}{100} = 0.00728 \text{ or } 7280\$$$

2. Calculate the covariance between ABC company stock and the portfolio.

$$\sigma_{12} = \rho \sigma_1 \sigma_2 = 0.4 \cdot \frac{2.37}{100} \times \frac{2.95}{100} = 2.7966 \times 10^{-4}$$

3. Calculate the standard deviation of his new portfolio which includes the ABC company stock.

$$\sigma_{new}^2 = var(P)\, 0.9^2 + 2\, 0.9\, 0.1\, \sigma_{12} + var(ABC)\, (0.1)^2$$
$$= 5.14102 \times 10^{-4}$$
$$\sigma_{new} = 0.02267393$$

b. Refer to question (a). If the investor sells the ABC company stock, he will invest the proceeds in risk-free government securities yielding 0.42% per month. Calculate the:

1. Expected return of the new portfolio which includes the government securities.

$$\bar{R}_{new} = 0.9 \cdot 0.0067 + 0.1 \cdot 0.0042 = 0.00645$$

2. The standard deviation of his new portfolio which includes the government securities.

$$\sigma_{new}^2 = var(P)\, 0.9^2 + \cancel{2\, 0.9\, 0.1 (0)} + \cancel{0\, (0.1)^2}$$
$$= var(P)\, 0.9^2$$
$$= 4.549689 \times 10^{-4}$$

$$\boxed{\sigma_{new} = 0.02133}$$

Exercise 5

Answer the following questions:

a. Consider a portfolio consisting of n risky assets. When short sales allowed, the efficient frontier of all feasible portfolios which can be constructed from these n assets is defined as the locus of feasible portfolios that have the smallest variance for a prescribed expected return E is determined by solving the problem

$$\begin{array}{ll} \min & \frac{1}{2}\mathbf{x}'\boldsymbol{\Sigma}\mathbf{x} \\ \text{subject to} & \bar{\mathbf{R}}'\mathbf{x} = E \\ \text{and} & \mathbf{1}'\mathbf{x} = 1 \end{array}$$

Show that the weights of the optimal portfolio $\mathbf{x}$ is given by $\mathbf{x} = \mathbf{g} + \mathbf{h}E$, where $\mathbf{g}$ and $\mathbf{h}$ are $n \times 1$ vectors, given by

$$\begin{aligned} \mathbf{g} &= \frac{1}{D}\left[B\boldsymbol{\Sigma}^{-1}\mathbf{1} - A\boldsymbol{\Sigma}^{-1}\bar{\mathbf{R}}\right] \\ \mathbf{h} &= \frac{1}{D}\left[C\boldsymbol{\Sigma}^{-1}\bar{\mathbf{R}} - A\boldsymbol{\Sigma}^{-1}\mathbf{1}\right]. \end{aligned}$$

The scalars A, B, C, D are defined as in the paper "An Analytic Derivation of the Efficient Portfolio Frontier," by Robert Merton.

$$\underline{x} = \lambda_1 \underline{\Sigma}^{-1}\bar{R} + \lambda_2 \underline{\Sigma}^{-1}\underline{1}, \quad \lambda_1 = \frac{CE - A}{D}, \quad \lambda_2 = \frac{B - AE}{D}$$

$$\underline{x} = \frac{CE-A}{D}\Sigma^{-1}\bar{R} + \frac{B-AE}{D}\Sigma^{-1}\underline{1}$$

$$\underline{x} = \underbrace{\frac{B\Sigma^{-1}\underline{1} - A\Sigma^{-1}\bar{R}}{D}}_{g} + \underbrace{\frac{C\Sigma^{-1}\bar{R} - A\Sigma^{-1}\underline{1}}{D}}_{h}E = g + hE$$

b. Refer to question (a). Consider two portfolios a, b on the efficient frontier (other than the minimum risk portfolio). Show that the covariance between the two portfolios is given by

$$\operatorname{cov}(R_a, R_b) = \frac{C}{D}\left(E_a - \frac{A}{C}\right)\left(E_b - \frac{A}{C}\right) + \frac{1}{C}.$$

$$R_A = x_A'\bar{R} \quad \Rightarrow \quad \operatorname{cov}(R_A, R_B) = x_A'\Sigma x_B$$
$$R_B = x_B'\bar{R}$$

$$\operatorname{cov}(A,B) = (g' + h'E_A)\Sigma(g + hE_B)$$
$$= g'\Sigma g + g'\Sigma h E_B + h'\Sigma g E_A + h'\Sigma h E_A E_B$$

$$g'\Sigma g = \frac{(B\,\underline{1}'\cancel{\Sigma^{-1}} - A\,\bar{R}'\cancel{\Sigma^{-1}})\cancel{\Sigma}(B\Sigma^{-1}\underline{1} - A\Sigma^{-1}\bar{R})}{D^2}$$

$$= \frac{1}{D^2}(B\underline{1}' - AR')(B\Sigma^{-1}\underline{1} - A\Sigma^{-1}\bar{R})$$

$$= \frac{1}{D^2}(B^2 \overbrace{\underline{1}'\Sigma^{-1}\underline{1}}^{C} - AB\underbrace{\underline{1}'\Sigma^{-1}\bar{R}}_{A} - AB\underbrace{R'\Sigma^{-1}\underline{1}}_{A} + A^2\underbrace{R'\Sigma^{-1}\bar{R}}_{B}$$

$$= \frac{1}{D^2}(B^2C - 2A^2B + A^2B) = \frac{1}{D^2}(B^2C - A^2B)$$

$$g'\Sigma h = \frac{1}{D^2}\left[B1' - A\bar{R}'\right]\left[C\Sigma^{-1}\bar{R} - A\Sigma^{-1}1\right]$$

$$\begin{aligned}
\mathbf{g} &= \frac{1}{D}\left[B\boldsymbol{\Sigma}^{-1}\mathbf{1} - A\boldsymbol{\Sigma}^{-1}\bar{\mathbf{R}}\right] \\
\mathbf{h} &= \frac{1}{D}\left[C\boldsymbol{\Sigma}^{-1}\bar{\mathbf{R}} - A\boldsymbol{\Sigma}^{-1}\mathbf{1}\right].
\end{aligned}$$

$$= \frac{1}{D^2}\left[\cancel{BCA} - \cancel{ABC} - ABC + A^3\right) = \frac{1}{D^2}\left[A^3 - ABC\right]$$

$$h'\Sigma g = \frac{1}{D^2}\left[C\bar{R}' - A1'\right]\left[B\Sigma^{-1}1 - A\Sigma^{-1}\bar{R}\right]$$

$$= \frac{1}{D^2}\left(\cancel{CBA} - \cancel{CAB} - ABC + A^3\right) = \frac{1}{D^2}\left[A^3 - ABC\right]$$

$$h'\Sigma h = \frac{1}{D^2}\left[C\bar{R}' - A1'\right]\left[C\Sigma^{-1}\bar{R} - A\Sigma^{-1}1\right]$$

$$= \frac{1}{D^2}\left[C^2B - A^2C - A^2C + A^2C\right) = \frac{1}{D^2}\left[C^2B - A^2C\right]$$

$$\mathrm{cov}(R_A, R_B) = \frac{1}{D^2}\left(B^2C - A^2B + (A^3 - ABC)E_A + (A^3 - ABC)E_B + (C^2B - A^2C)E_AE_B\right.$$

$$\mathrm{cov}(R_a, R_b) = \frac{C}{D}\left(E_a - \frac{A}{C}\right)\left(E_b - \frac{A}{C}\right) + \frac{1}{C}.$$

$$\frac{1}{D^2}\left(B(BC - A^2) - A(BC - A^2)E_A - A(BC - A^2)E_B + C(BC - A^2)E_AE_B\right.$$

$$= \frac{1}{D^{\cancel{2}}}\left(B - AE_A - AE_B + CE_AE_B\right)\overset{D}{\cancel{(BC - A^2)}}$$

$$= \frac{(B - AE_A - AE_B + CE_AE_B)}{D} = \frac{C}{D}\left(\frac{B}{C} \underbrace{- \frac{AE_A}{C} - \frac{AE_B}{C} + E_AE_B}_{\Downarrow \; \left(E_a - \frac{A}{C}\right)\left(E_b - \frac{A}{C}\right) - \frac{A^2}{C^2}}\right)$$

$$= \frac{C}{D}\left(\frac{B}{C} + \left(f_a - \frac{A}{C}\right)\left(f_b - \frac{A}{C}\right) - \frac{A^2}{C^2}\right)$$

$$= \frac{\cancel{C}}{\cancel{D}}\left(\frac{\cancel{BC - A^2}}{C^2} + \left(f_A - \frac{A}{C}\right)\left(f_b - \frac{A}{C}\right)\right)$$

$$= \frac{1}{C} + \frac{C}{D}\left(f_A - \frac{A}{C}\right)\left(f_b - \frac{A}{C}\right) \checkmark$$