**University of California, Los Angeles**
**Department of Statistics**

**Statistics C183/C283** **Instructor: Nicolas Christou**

**Homework 2**

Solved by Y.Y

**Exercise 1**

Refer to the lecture material on Friday, 04/02. In order to find the Lagrange multipliers $\lambda_1$ and $\lambda_2$ we must invert the matrix $\begin{pmatrix} B & A \\ A & C \end{pmatrix}$, where $A = \mathbf{1}'\mathbf{\Sigma}^{-1}\bar{\mathbf{R}}$, $B = \bar{\mathbf{R}}'\mathbf{\Sigma}^{-1}\bar{\mathbf{R}}$, and $C = \mathbf{1}'\mathbf{\Sigma}^{-1}\mathbf{1}$. Show that $BC - A^2 > 0$. Note: Begin with $(A\bar{\mathbf{R}} - B\mathbf{1})'\mathbf{\Sigma}^{-1}(A\bar{\mathbf{R}} - B\mathbf{1}) > 0$ because $\mathbf{\Sigma}$ is positive definite matrix.

$$(A\bar{R} - B1)'\Sigma^{-1}(A\bar{R} - B1) > 0$$

$$A^2 \underbrace{\bar{R}'\Sigma^{-1}\bar{R}}_{B} - AB\underbrace{\bar{R}'\Sigma^{-1}1}_{A} - BA\underbrace{1'\Sigma^{-1}\bar{R}}_{\bar{A}} + B^2\underbrace{1'\Sigma^{-1}1}_{C} > 0$$

$$A^2B - 2A^2B + B^2C > 0$$

$$B^2C - A^2B > 0$$

$$B(BC - A^2) > 0, \quad B = \bar{R}'\Sigma^{-1}\bar{R} > 0 \Rightarrow (BC - A^2) > 0$$

**Exercise 2**

In the paper "An Analytic Derivation of the Efficient Portfolio Frontier," *The Journal of Financial and Quantitative Analysis*, Vol. 7, No. 4, Robert Merton gives on page 1854 the proportion of the $k_{th}$ risky asset held in the frontier portfolio with expected return $E$ by

$$x_k = \frac{E\sum_{j=1}^m v_{kj}(CE_j - A) + \sum_{j=1}^m v_{kj}(B - AE_j)}{D}, \quad k = 1, \ldots, m. \tag{1}$$

Prove equation (1).

On the same page, it is shown that the expected return of the minimum risk portfolio is $\bar{E} = \frac{A}{C}$. Using equation (1) above show that the proportion of the $k_{th}$ risky asset of the minimum risk portfolio is $x_k = \frac{\sum_{j=1}^m v_{kj}}{C}$, $k = 1, \ldots, m$.

$$X_k = \lambda_1 \sum_j v_{kj} E_j + \lambda_2 \sum_j v_{kj}$$

substitute $\lambda_1, \lambda_2$ as $\frac{CE-A}{D}$ , $\frac{B-AE}{D}$

$$\Rightarrow X_k = \frac{CE-A}{D}\sum_j v_{kj} E_j + \frac{B-AE}{D}\sum_j v_{kj}$$

$$= \frac{E\sum_j C v_{kj} E_j - A\sum v_{kj} E_j - E\sum_j A v_{kj} + \sum B v_{kj}}{D}$$

$$\boxed{= \frac{E\sum_j v_{kj}[CE_j - A] + \sum_j v_{kj}(B - AE_j]}{D}}$$

use $\bar{E} = \frac{A}{C}$

$$= \frac{\frac{A}{C}\sum_j v_{kj}(CE_j - A) + \sum_j v_{kj}(B - AE_j]}{D}$$

$$= \frac{A\cancel{\sum_j v_{kj}E_j} - \frac{A^2}{C}\sum_j v_{kj} + \sum_j v_{kj}B - A\cancel{\sum_j v_{kj}E_j}}{D}$$

$$= \frac{\left(B - \frac{A^2}{C}\right)\sum_j v_{kj}}{D} \Rightarrow \frac{\cancel{BC-A^2}}{C\cancel{D}}\sum_j v_{kj} \Rightarrow \boxed{\frac{\sum_j v_{kj}}{C}} = x_k$$

**Exercise 3**
Find an expression of the correlation coefficient of two portfolios on the efficient frontier. See homework 1 for the covariance between two portfolios.

$$\rho_{AB} = \frac{cov(R_A, R_B)}{\sigma_A \sigma_B} = \frac{\underline{x}_A' \Sigma \underline{x}_B}{\sqrt{\underline{x}_A' \Sigma \underline{x}_A \; \underline{x}_B' \Sigma \underline{x}_B}}$$

**Exercise 4**
The covariance matrix **Q** of the returns of two stocks has the following inverse:

```
> solve(Q)
           [,1]      [,2]
[1,] 166.21139 -22.40241
[2,] -22.40241 220.41076
```

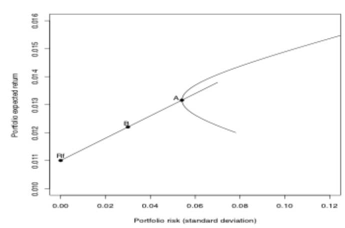
Answer the following questions:

a. Find the composition of the minimum risk portfolio.

$$\underline{x} = \frac{\Sigma^{-1}\underline{1}}{\underline{1}'\Sigma^{-1}\underline{1}} = \frac{\begin{bmatrix} 166-22 \\ -22+220.41 \end{bmatrix}}{166.21-22.402-22.402+220.410}$$

$$= \begin{bmatrix} 0.42 \\ 0.58 \end{bmatrix}$$

b. It is given that the minimum risk portfolio (point A on the graph below) has standard deviation equal to 0.05408825 and expected return equal to 0.01315856. Portfolio B (see graph below) has expected return equal to 0.01219724. What is the composition of portfolio B in terms of portfolio A and the risk free asset? Assume $R_f = 0.011$.




$\sigma_A = 0.05408825$, $\bar{R}_A = 0.013158$, $\bar{R}_B = 0.01219$, $R_F = 0.011$

$$\bar{R}_B = R_f + \left(\frac{\bar{R}_A - R_f}{\sigma_A}\right)\sigma_B \qquad \boxed{\sigma_B = 0.0300076}$$

$$\Rightarrow \bar{R}_B = x\bar{R}_A + (1-x)R_f \Rightarrow R_B - R_F = x(R_A - R_F) \Rightarrow x = \frac{R_B - R_F}{R_A - R_F}$$

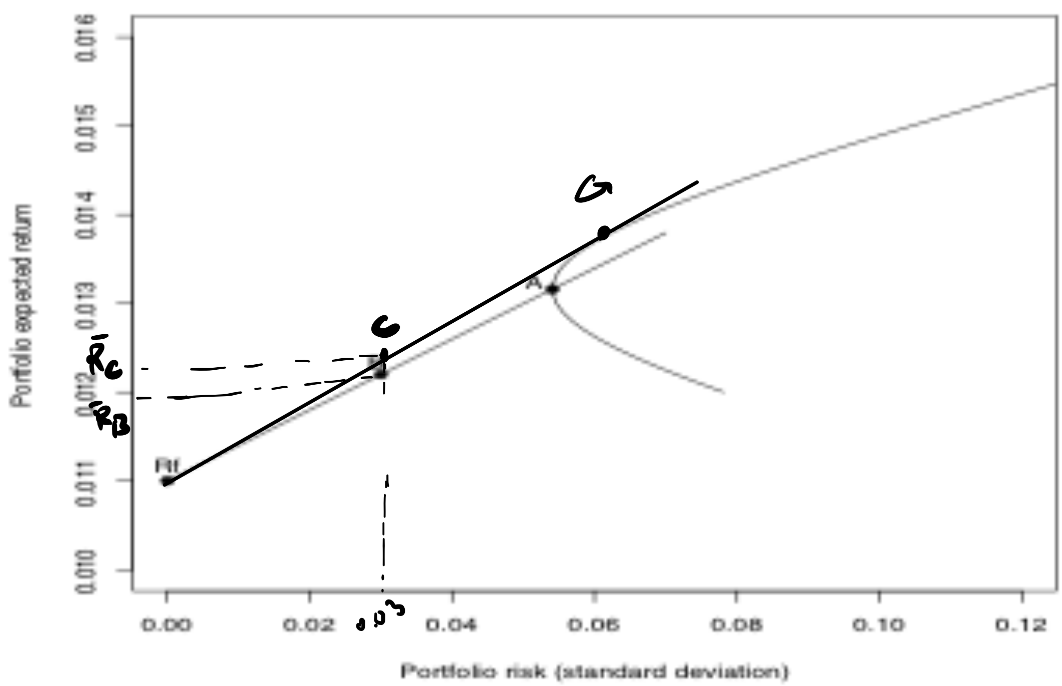
$x = 0.5514315153$

$$\boxed{R_B = 0.5514 R_A + 0.4485 R_f} = 0.5514(0.42 R_1 + 0.58 R_2) + 0.4485 R_f$$

$$= 0.2316 R_1 + 0.3198 R_2 + 0.4485 R_f$$

c. The standard deviation of portfolio B is equal to 0.03. Given this level of risk, can you do better than the expected return of portfolio B? Please explain.

YES




Note, $var(R_C) = var(R_B)$, and $E[R_C] > E[R_B]$, maximum return at that level of risk can be only achieved by finding the line that is tangent to efficient frontier.

To find $R_C$, one needs to find $G$ using the derivation at the lecture. Then, $R_C$ is combination of $R_f$ and $R_G$ $\Rightarrow R_C = x(R_G) + (1-x)R_f$

$x$ can be found by $x = \dfrac{\bar{R}_C - R_F}{R_G - R_F}$

**Exercise 5**
Show that two portfolios on the capital allocation line are perfectly correlated.

$$\rho_{AB} = \frac{\mathrm{cov}(R_A, R_B)}{\sigma_A \sigma_B}$$

$$R_A = \underline{x}_A' \underline{R} + (1 - \underline{1}' x_A) R_F$$

$$R_B = \underline{x}_B' \underline{R} + (1 - \underline{1}' x_B) R_F$$

$$\rho_{AB} = \frac{\mathrm{cov}(x_A' \underline{R} + (1 - 1' x_A) R_F,\ x_B' \underline{R} + (1 - 1' x_B) R_F)}{\sqrt{\mathrm{var}(x_A' \underline{R} + (1 - 1' x_A) R_F)\ \mathrm{var}(x_B' \underline{R} + (1 - 1' x_B) R_F)}}$$

$$= \frac{\mathrm{cov}(x_A' \underline{R},\ x_B' \underline{R})}{\sqrt{\mathrm{var}(x_A' \underline{R})\ \mathrm{var}(x_B' \underline{R})}} = \frac{x_A' \Sigma x_B}{\sqrt{x_A' \Sigma x_A \ x_B' \Sigma x_B}}$$

$$x_A = \frac{(E_A - R_F)\,\Sigma^{-1}(\underline{\bar{R}} - R_f \underline{1})}{(\underline{\bar{R}} - R_f \underline{1})' \Sigma^{-1} (\underline{\bar{R}} - R_f \underline{1})}$$

$$x_B = \frac{(E_B - R_F)\,\Sigma^{-1}(\underline{\bar{R}} - R_f \underline{1})}{(\underline{\bar{R}} - R_f \underline{1})' \Sigma^{-1} (\underline{\bar{R}} - R_f \underline{1})}$$

$$\Rightarrow \frac{(E_A - R_F)(E_B - R_F)\,\cancel{(\underline{\bar{R}} - R_f \underline{1})' \Sigma^{-1} \Sigma \Sigma^{-1} (\underline{\bar{R}} - R_f \underline{1})}}{\cancel{(\underline{\bar{R}} - R_f \underline{1})' \Sigma^{-1} (\underline{\bar{R}} - R_f \underline{1})}\ (\underline{\bar{R}} - R_f \underline{1})' \Sigma^{-1} (\underline{\bar{R}} - R_f \underline{1})}$$

upper side

$$\Rightarrow \quad (E_A - R_F)(E_B - R_F)\sqrt{\left((\bar{\underline{R}} - R_f \underline{1})' \Sigma^{-1} (\bar{\underline{R}} - R_f \underline{1})\right)^2}$$

lower

$$\Rightarrow \quad \frac{\text{upper}}{\text{lower}} = 1 = \rho_{AB},$$ A and B are perfectly correlated as expected.