Advection Equation via Finite Difference Method and Finite Volume Method

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ABSTRACT: In this report, we describe some of the methods to numerically solve the one-dimensional advection equation with spatially uniform velocity field. In particular, we first describe the finite difference method using two conditions: the FCTS condition and the upwinding condition. We also discuss the limitations of this method which is that the solutions do not conserve the field i.e. there is a numerical diffusion. To overcome this limtation, we describe the field conserving finite volume methods without using any limiter and contrast it by describing the finite volume solutions having a limiter i.e. the minmod limiter. We illustrate these four methods by solving them for two initial conditions: Top-Hat and Gaussian function. We have used Fortran to solve these equations and GNUPLOT to plot the resulting functions.

Contents

1 What is Advection?

Advection is the transportation of a conserved field due to the motion of a carrier fluid which carries the field. By conserved field, we mean that the field's volume integral over the region of the fluid remains constant as time passes. Consider a moving fluid as an example and assume that the fluid has some energy density that varies across the space where the fluid exists. Also assume that there is no loss of energy during the flow. The energy density of the fluid can be thought of as a scalar field distributed over space where fluid exists and as the fluid flows, the energy of a fluid particle is moved by the particle or the carrier as the fluid flows. Because of the flow, at each point the energy density changes as time passes and the rate of change of energy density can be related to the velocity of the fluid and the distribution of the internal energy density across space at that instant of time.

Advection Equation is a partial differential equation which mathematically describes the transportation of a conserved scalar field caused by a carrier velocity field. It is a continuity equation for the conserved scalar field. Let the scalar field be $a(x, y, z, t)$ and the velocity field be $\vec{u} \equiv \vec{u}(x, y, z, t)$

$$\partial_t a + \nabla . (\vec{u} a) = 0 \tag{1.1}$$

Assumption: Let us assume that $\vec{u}$ is uniform in space at all times i.e. $\partial_i \vec{u} = 0$. Therefore, we get:

$$\partial_t a + \vec{u} . \nabla a = 0 \tag{1.2}$$

In one dimension $\vec{u} = u \, \hat{x}$, this becomes:

$$\partial_t a + u . \partial_x a = 0 \tag{1.3}$$

This is a linear partial differential equation whose solutions are of the following form:

$$a(x,t) = f(x - ut) \tag{1.4}$$

For a positive u, it just means that the field at time $t = 0$, $a(x,0)$ is translated forward in space by a distance ut in x-direction after time t. This means that the shape $a(x,t)$ vs x will not change with time, rather it will just translate towards x-axis. Neverthless, we will see that some of the numerical solutions will get distorted due to the numerical errors that get accumulated over time.

2 Numerical Methods and Stability

Let us look at the various kinds of numerical techniques for solving this equation (1.3). We will look at 4 kinds of attempts to solve this which are as follows:

- Finite Difference Method with FCTS (Forward-time, center space discretization)
- Finite Difference Method using Upwinding Scheme
- Finite Volume Method without using any limiter
- Finite Volume Method with the Minmod limiter

Before discussing these methods individually, let us discuss the common features in all of them. We will first discretize both x and time axis and create a 2D mesh, although the way we discritize these will be different for finite difference and finite volume methods. The scalar field a will therefore be a 2D array $a \equiv a_{i,j}$ over that mesh where the first index denotes time and the second index denotes space. Let the grid spacing along x and time axis be Δx and Δt. While solving the equations numerically, if we choose Δx and Δt arbitrarily, there would be some instances where the solution will start diverging because of the accumulation of numerical errors. To prevent such divergences and to get a stable solution, we will impose a condition on Δx and Δt namely that the ratio $\frac{\Delta x}{\Delta t}$ should be less than the transport velocity u. Let us define a quantity which should be less than 1 as follows

$$C \equiv \frac{\Delta x}{u\Delta t} < 1 \qquad \text{For stable solutions}$$

This C is called the CFL number, named after Courant-Friedrichs-Lewy. So, instead of choosing the value of Δt, we will choose Δx and some $C < 1$ in order to find $\Delta t = C\frac{\Delta x}{u}$ to get stable solutions.

2.0.1 Initial and Boundary Conditions

We will solve the advection equation for two initial conditions for the 4 methods: One is the 'Top-Hat' function and another is 'Gaussian' function. Specifically, we will use the 'Top-Hat' function of a length N_x as

$$a_{\text{Top-Hat}}(x) = \begin{cases} 0 & \text{if} \quad x \in (0, \frac{N_x}{3}) \\ 1 & \text{if} \quad x \in (\frac{N_x}{3}, \frac{2N_x}{3}) \\ 0 & \text{if} \quad x \in (\frac{2N_x}{3}, N_x) \end{cases}$$

We will use the following Gaussian function as the second initial condition:

$$a_{\text{Gaussian}}(x) = e^{-\frac{(x-\frac{N_x}{2})^2}{N_x}}$$

We will be using peridic boundary conditions in all of the following methods. Periodic boundary conditions on x-direction imply that x axis can be thought of as a circular ring instead of infinite a line which implies that the if we want to go beyond the last grid point, we enter from the first grid point. In other words, if the x-grid has N_x number of points then $(N_x + 1)^{\text{th}}$ point is the first point itself and vice versa.

In addition, we have used the CFL number C equal to 0.7 in all the solutions below.

2.1 Finite Difference Method with FCTS

In this method, we divide our x axis and the t axis into N_x and N_t points respectively with equal gaps between any two consecutive points on each axis. Labelling the points by indices (i, j) with i and j denoting time and space respectively, we can discretize our field $a(t, x)$. As the name Forward-time center space discretization (FCTS) suggests, we will use the time derivative of a and the space derivative of a as following:

$$a_t = \frac{a_{i+1,j} - a_{i,j}}{\Delta t} \tag{2.1}$$

$$a_x = \frac{a_{i,j+1} - a_{i,j-1}}{2\Delta x} \tag{2.2}$$

Therefore, our equation (1.3) becomes

$$\frac{a_{i+1,j} - a_{i,j}}{\Delta t} = -u\frac{a_{i,j+1} - a_{i,j-1}}{2\Delta x} \tag{2.3}$$

$$\implies a_{i+1,j} = a_{i,j} - C\left(\frac{a_{i,j+1} - a_{i,j-1}}{2}\right) \tag{2.4}$$

We can solve this equation numerically and unfortunately, the solutions don't look like what we expect at all. The solutions are still unstable. The plot of $a(x, t)$ vs x looks like the following at different times:

Plot of a(x,t) vs x at different times for Top-Hat

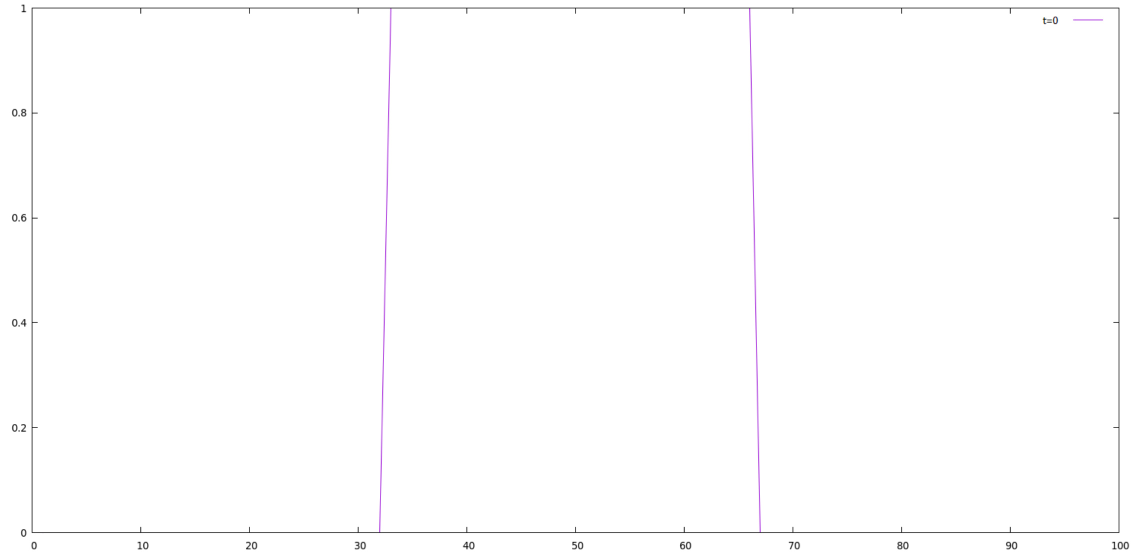

Fig 2.1.1. At t = 0

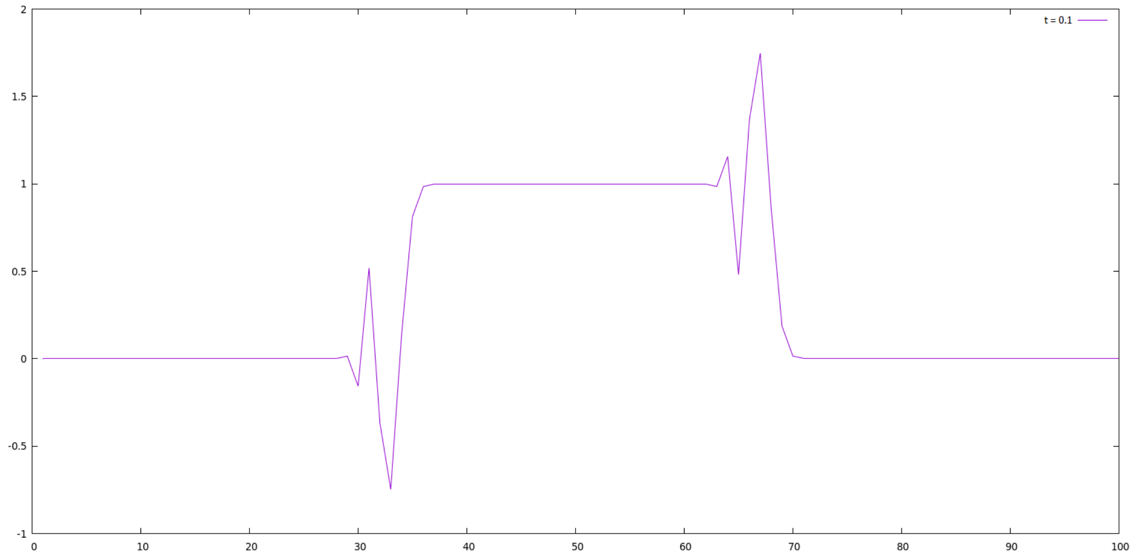

Fig 2.1.2. At t = 0.1

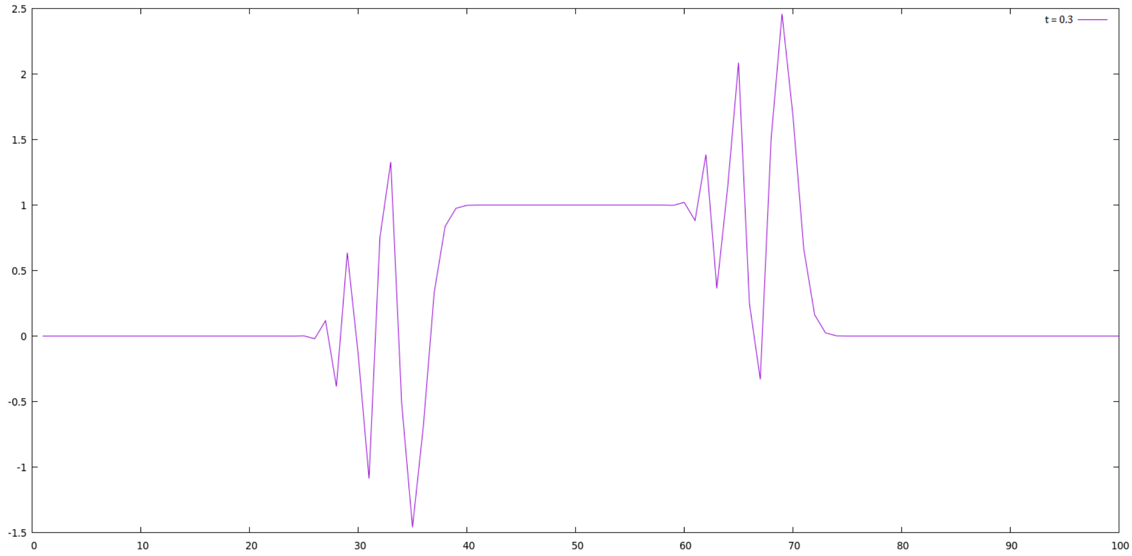

Fig 2.1.3. At t = 0.3

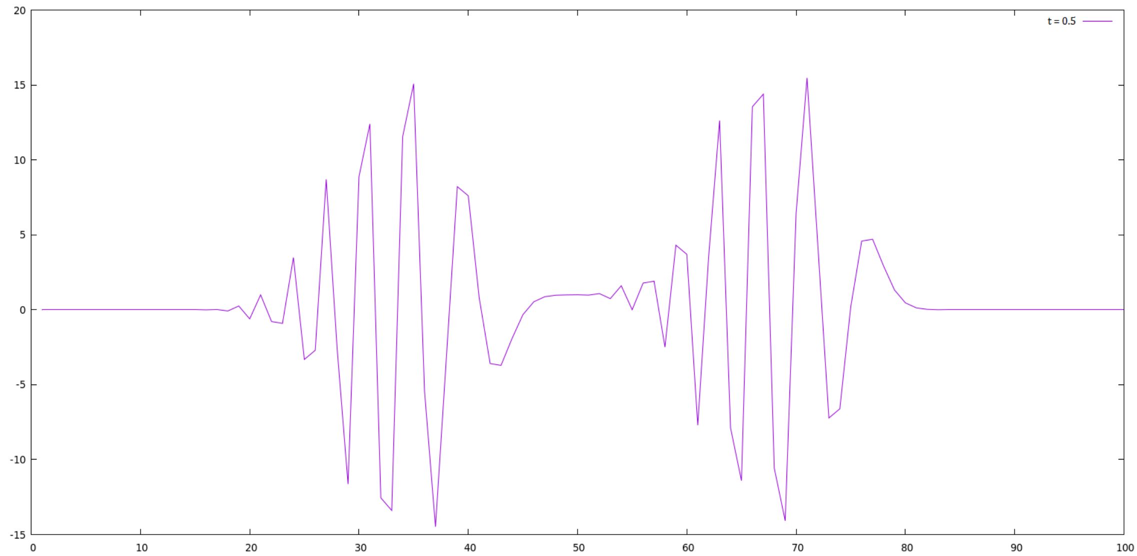

Fig 2.1.4. At t = 0.5

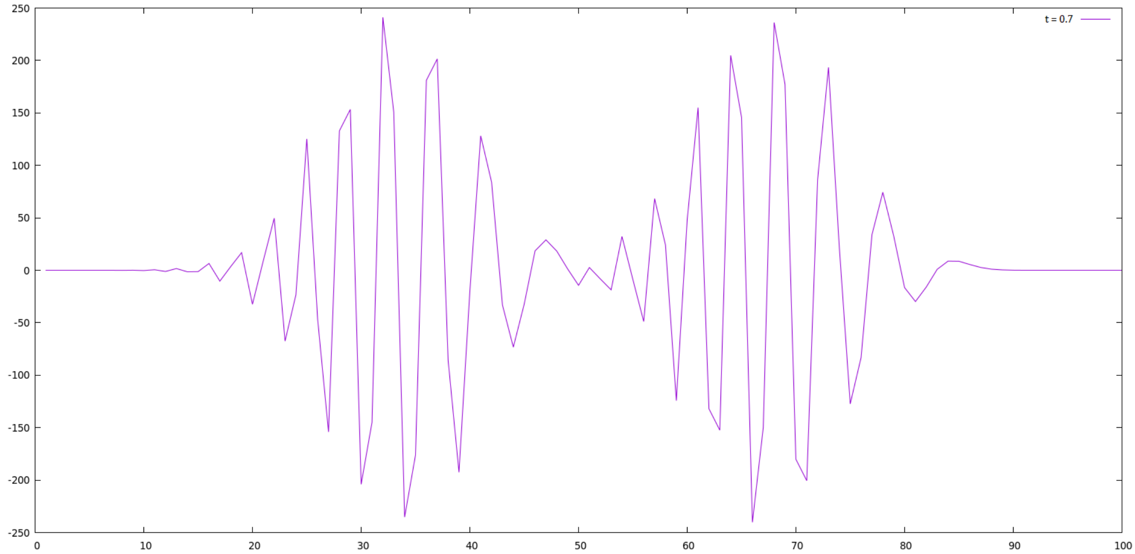

Fig 2.1.5. At t = 0.7

It is evident from the figures that the solutions become highly unstable as time goes on.

Plot of a(x,t) vs x at different times for Gaussian

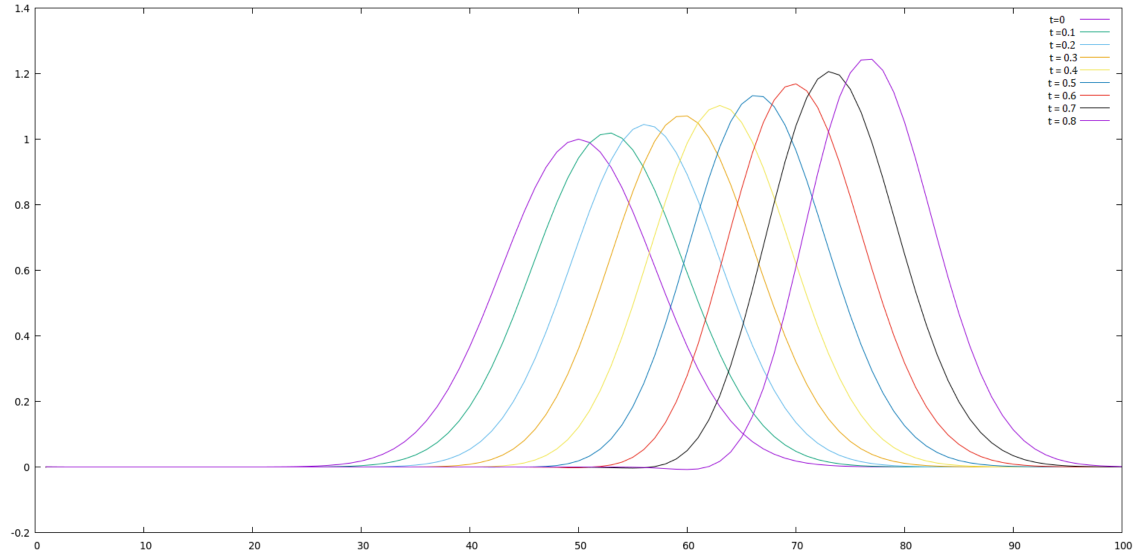

Fig 2.1.6. Gaussian propogated from at t = 0.0 to t = 0.8

It is evident from the figures that the Gaussian solution starts to diverge as time goes on. The area under curve is not conserved with time.

2.2 Finite Difference Method with Upwinding Scheme

As we saw in the previous method, there were oscillations and instabilities in the solution. In this method, we make a small change from the previous method. We use the following definition of the space derivative:

$$a_x = \frac{a_{i,j} - a_{i,j-1}}{\Delta x}, \quad u > 0 \qquad a_x = \frac{a_{i,j+1} - a_{i,j}}{\Delta x}, \quad u < 0 \tag{2.5}$$

This is called the Upwinding Scheme. This method removes the oscillation and the instability that we encountered in the previous case. But there is one drawback here. Although the field transports without oscillations, the area under the curve $a(x,t)$ vs x decreases as time goes on. We call it numerical diffusion and it can be seen from the graphs below that as time marches on, the field diffuses with time. This means that our field is not conserved in this method due to numerical errors although we are solving a continuity equation.

Plot of a(x,t) vs x at different times for Top-Hat

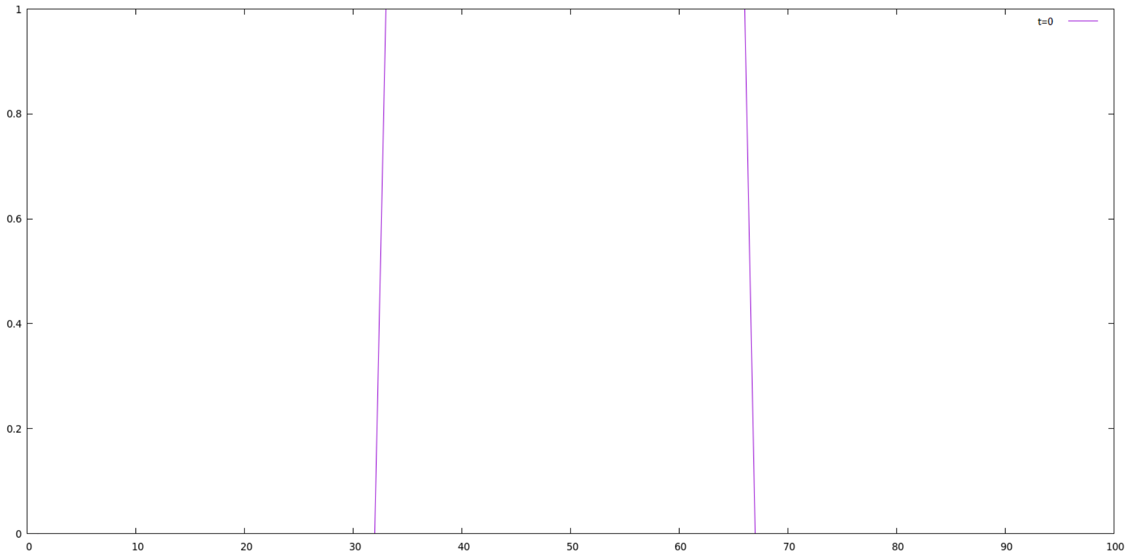

Fig 2.2.1. At t = 0

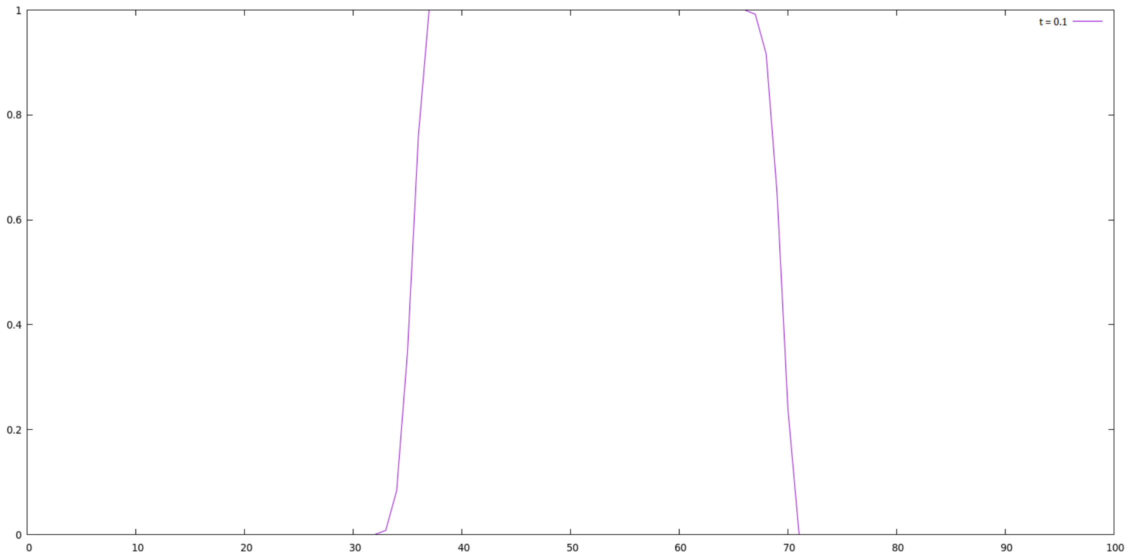

Fig 2.2.2. At t = 0.1

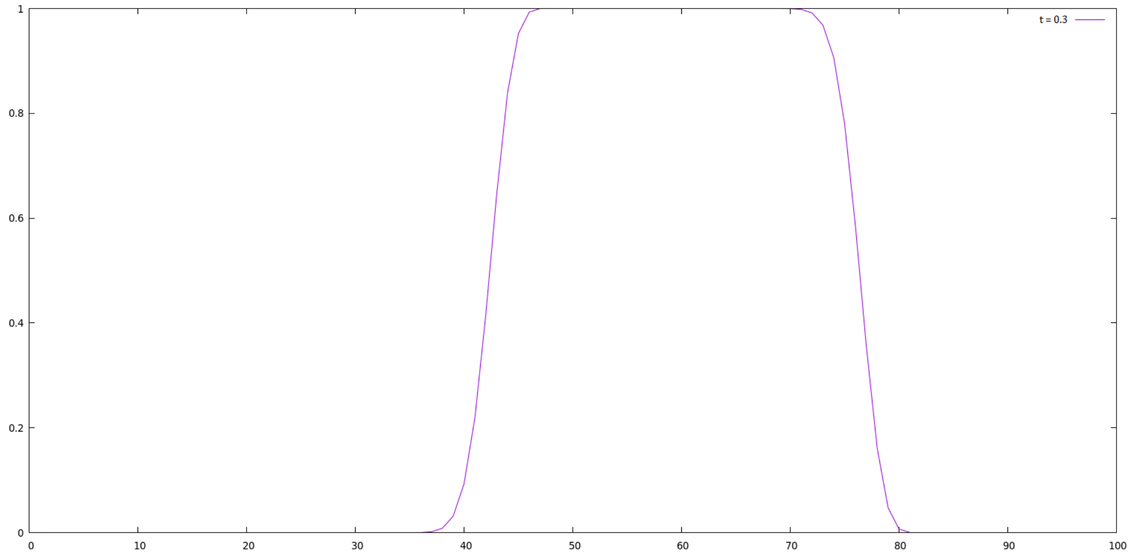

Fig 2.2.3. At t = 0.3

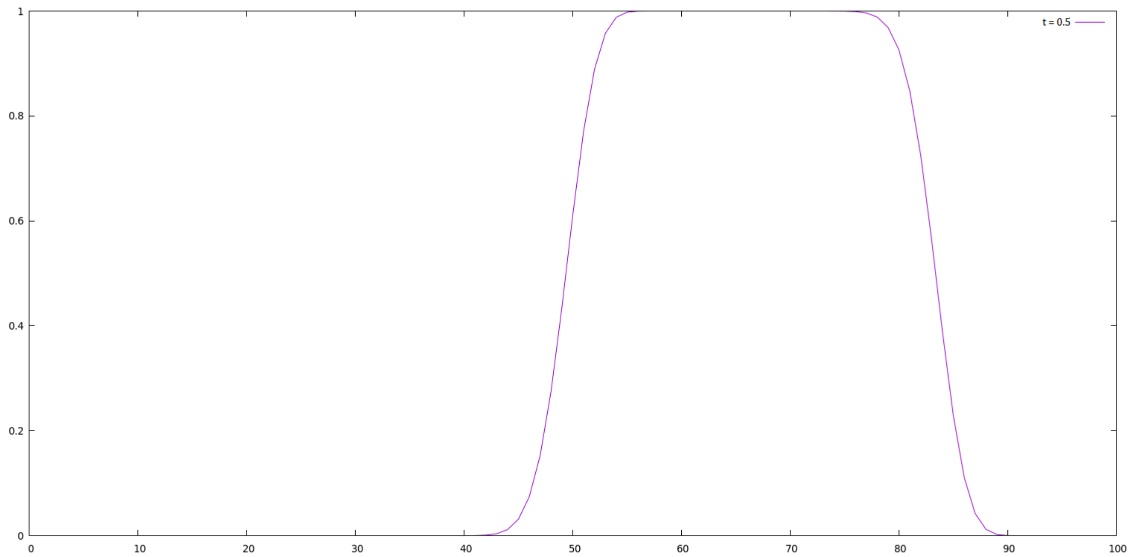

Fig 2.2.4. At t = 0.5

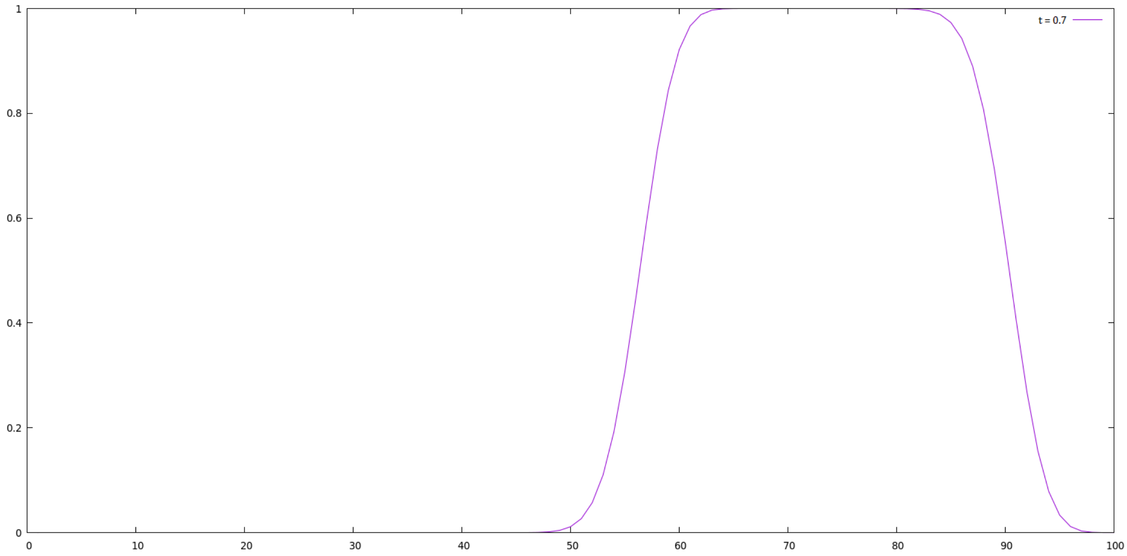

Fig 2.2.5. At t = 0.7

It is evident from the figures that the solutions are stable as time goes on however there is a numerical diffusion.

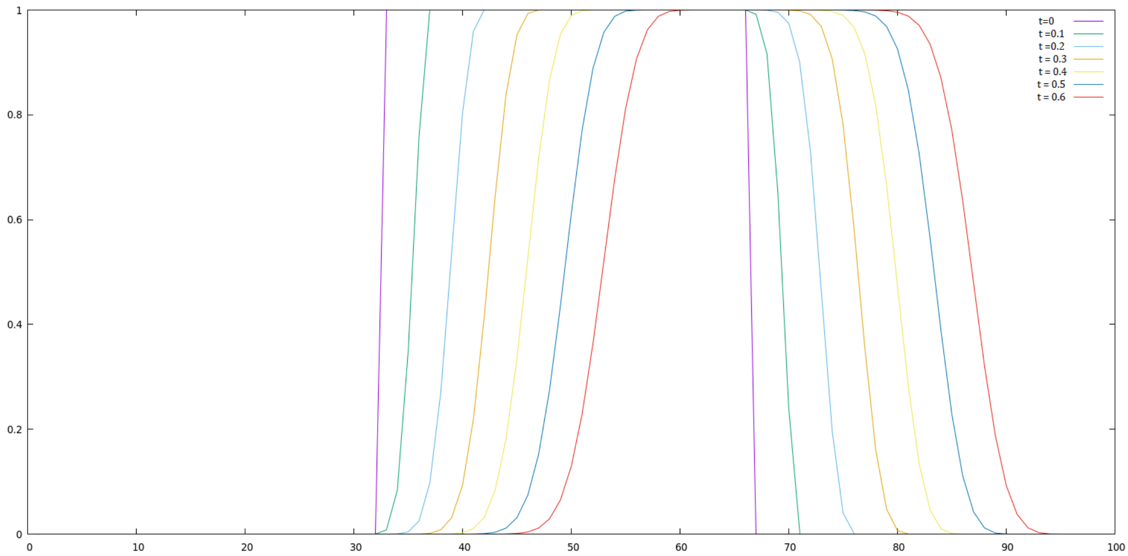

Fig. 2.2.6. We have shown all plots from 2.2.1 to 2.2.5 together to show the propogation from t=0.0 to t=0.6.

Plot of a(x,t) vs x at different times for Gaussian

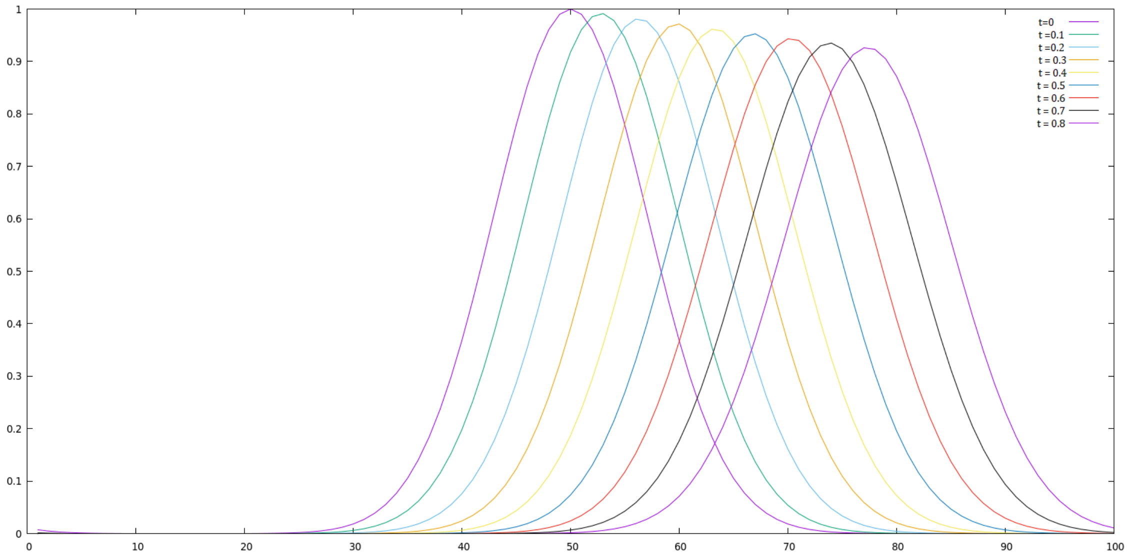

Fig 2.2.7. Gaussian propogated from at t = 0.0 to t = 0.8

It is evident from the figure above that the Gaussian solution starts to decay as time goes on. The area under curve is not conserved with time.

2.3 Finite Volume Method without any limiter

As we saw in the previous case, the numerical solution diffuses and the scalar field doesn't remain conserved with time. To tackle this problem, we implement a new technique called Finite Volume Method. In this technique, we divide the t axis as we did previously. But instead of points, we divide the x axis into Nx zones and we assign the average value of the field over a zone to its mid point. In particular, let's denote the mid points of the N_x zones with index j which starts from 1 and ends at N_x. The boundaries of the zones with midpoint j will be $j-1/2$ and $j+1/2$. We also assume that the we slice the zones so finely that the function in every zone almost looks like a straight line. Therefore, the average value of the function in each zone can be approximated by the value of the function at the mid-point of the zone itself. In other words, if we call the x coordinate of the j^{th} midpoint as x_j then:

$$\frac{1}{\Delta x}\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} a(t,x)dx = a(t,x_j) \equiv a(t)_j \tag{2.6}$$

If we take zone average of equation (1.3) as follows

$$\frac{1}{\Delta x}\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \frac{\partial a(t,x)}{\partial t}dx = -u\frac{1}{\Delta x}\int_{j-\frac{1}{2}}^{j+\frac{1}{2}} \frac{\partial a(t,x)}{\partial x}dx \tag{2.7}$$

then using (2.6) on the left side and using the fundamental theorem of calculus om the right side, we get

$$\frac{da(t)_j}{dt} = -u\frac{1}{\Delta x}\left(a(t)_{j+\frac{1}{2}} - a(t)_{j-\frac{1}{2}}\right) \tag{2.8}$$

Now, this equation looks like a ordinary differential equation in time and therefore, we can implement Runge-Kutta method for t. So following the usual lines of Runge-Kutta method, we divide the time axis into N_t number of points (not zones) and we evaluate the right hand side of the equation at mid-points of two consecutive times to get:

$$\frac{a_{i+1,j} - a_{i,j}}{\Delta t} = -u \frac{1}{\Delta x} \left(a_{i+\frac{1}{2},j+\frac{1}{2}} - a_{i+\frac{1}{2},j-\frac{1}{2}} \right) \tag{2.9}$$

Please note: The spatial index $j + \frac{1}{2}$ is the boundary of zones on x axis whereas the time index $i + \frac{1}{2}$ is midpoint of two consecutive points on the time axis.

To evaluate $a_{i+\frac{1}{2},j+\frac{1}{2}}$ we can use Taylor approximation. But because $j+\frac{1}{2}$ is a boundary between two zones, a natural question arises i.e. which zone will we choose in the Taylor expansion? This is called the Reimann problem and the solution to this is that we have to choose the zone on the left of the boundary if the wave is propogating to the right and vice versa. It is again the same upwinding method that we had used in finite difference method i.e. section 2.2. In our case, we have assumed a forward travelling wave without the loss of generality. Therefore, we can write:

$$a_{i+\frac{1}{2},j+\frac{1}{2}} = a_{i,j} + \frac{\Delta x}{2} \frac{\partial a}{\partial x} + \frac{\Delta t}{2} \frac{\partial a}{\partial t} + \text{neglecting higher order terms...} \tag{2.10}$$

Using (1.3), we can write $\frac{\partial a}{\partial t}$ as $-u \frac{\partial a}{\partial x}$ above and get:

$$a_{i+\frac{1}{2},j+\frac{1}{2}} = a_{i,j} + \frac{\Delta x}{2} \left(1 - u \frac{\Delta t}{\Delta x} \frac{\partial a}{\partial x} \right) = a_{i,j} + \frac{\Delta x}{2} (1 - C) \frac{\partial a}{\partial x} \tag{2.11}$$

Now, we need $\frac{\partial a}{\partial x}$ to find the above. We can use FCTS here i.e. we use

$$\frac{\partial a}{\partial x} = \frac{a_{i,j+1} - a_{i,j-1}}{2\Delta x} \equiv (Da)_{i,j} \tag{2.12}$$

where $(Da)_{i,j}$ is defined above as the array of the space derivative of a. Now putting everything together in (2.9), we get

$$\frac{a_{i+1,j} - a_{i,j}}{\Delta t} = -u \frac{1}{\Delta x} \left(a_{i,j} - a_{i,j-1} + \frac{\Delta x}{2} (1 - C) (Da_{i,j} - Da_{i,j-1}) \right) \tag{2.13}$$

$$\implies a_{i+1,j} = a_{i,j} - C \left(a_{i,j} - a_{i,j-1} + \frac{\Delta x}{2} (1 - C) (Da_{i,j} - Da_{i,j-1}) \right) \tag{2.14}$$

The advantage in ths method is that it will preserve the area under curve $a(t, x)$ vs x as time marches on, so the field will be conserved with time. The propogation of the gaussian pattern is a prime example of this as shown below. But for the shapes like 'Top-Hat', there is a disadvantage in this method. The solutions for the initial condition like 'Top-Hat' start developing some wavy patterns at their ends as shown below:

Plot of a(x,t) vs x at different times for Top-Hat

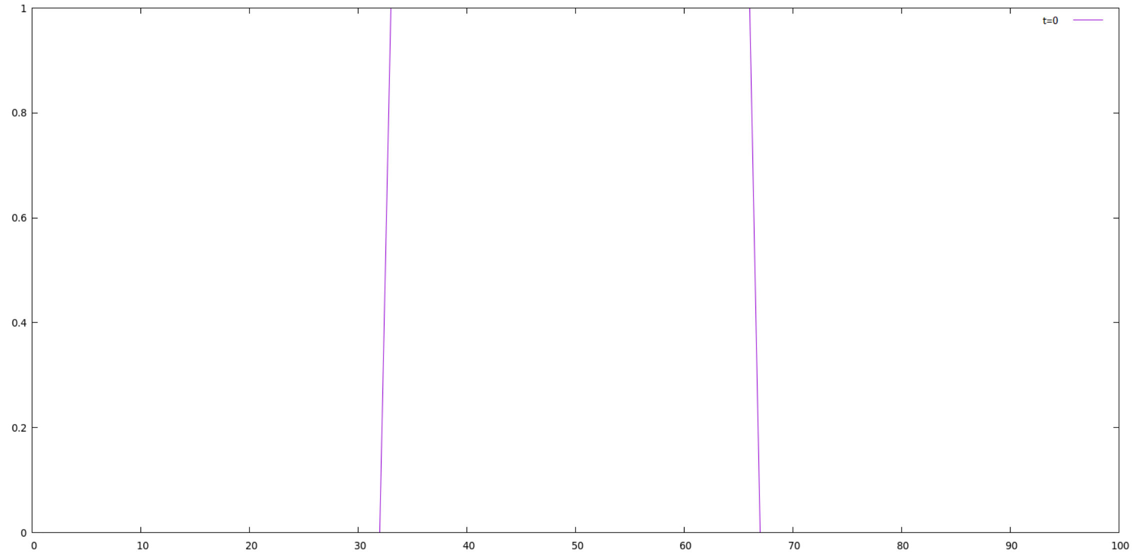

Fig 2.3.1. At t = 0

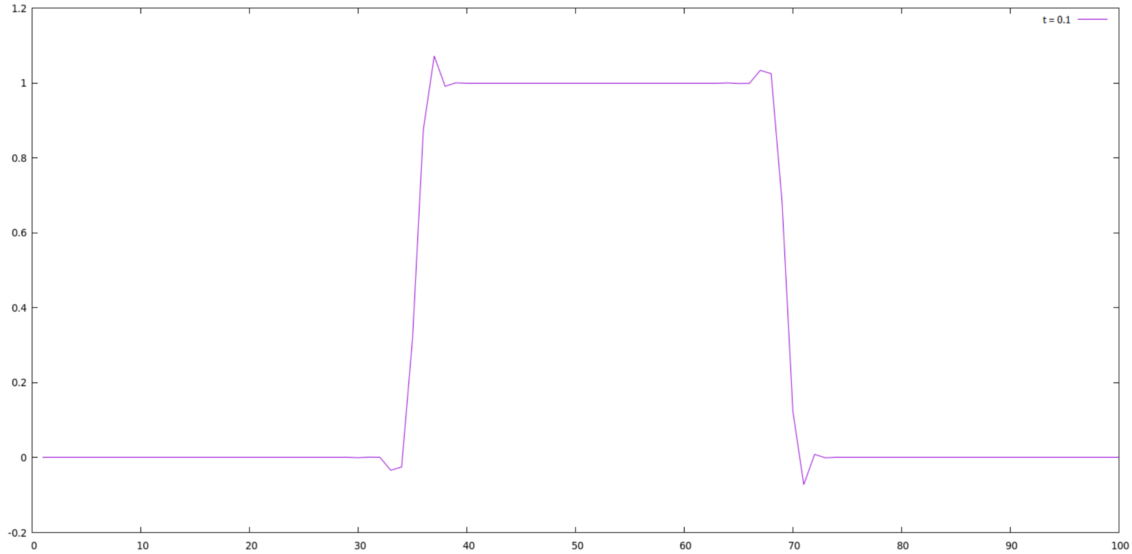

Fig 2.3.2. At t = 0.1

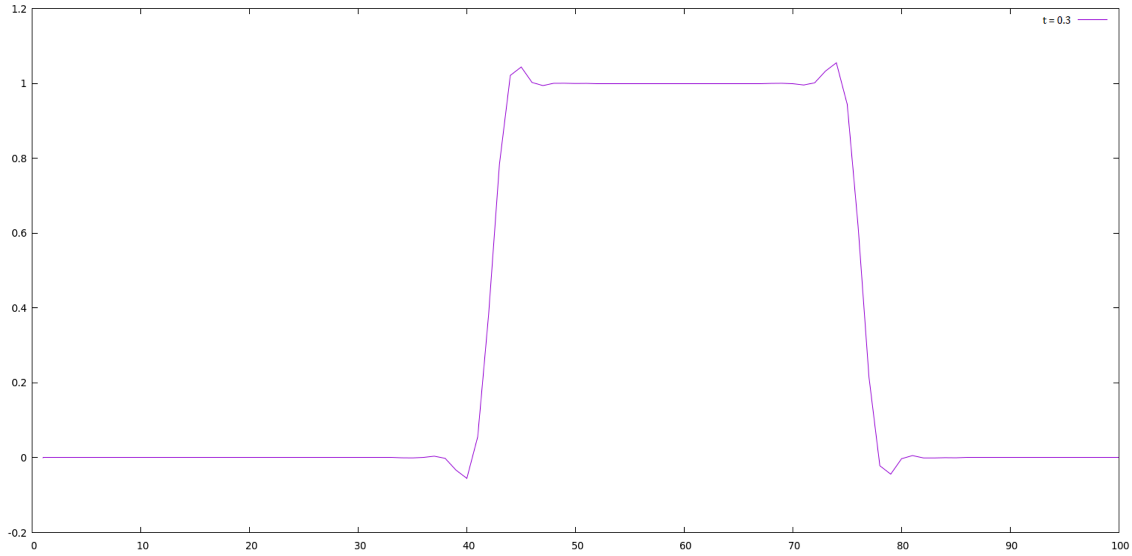

Fig 2.3.3. At t = 0.3

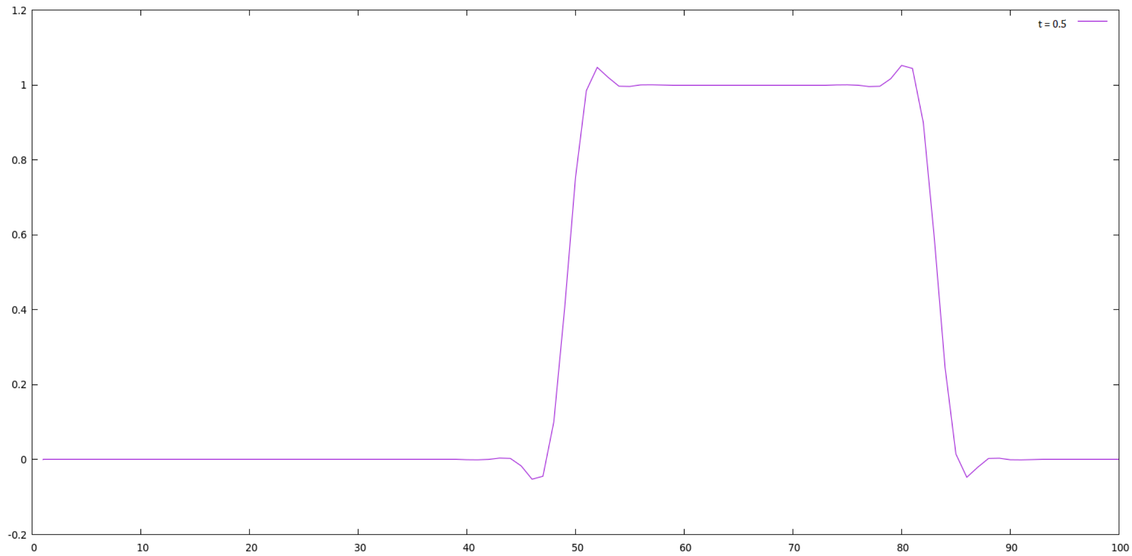

Fig 2.3.4. At t = 0.5

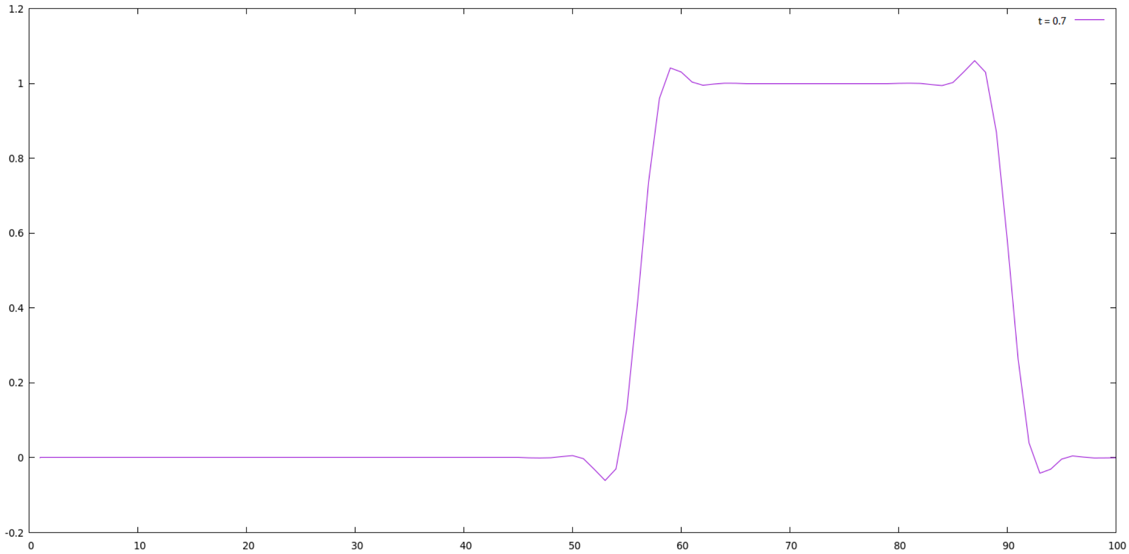

Fig 2.3.5. At t = 0.7

It is evident from the plots that the there are some wavy patterns at the ends of Top-Hat

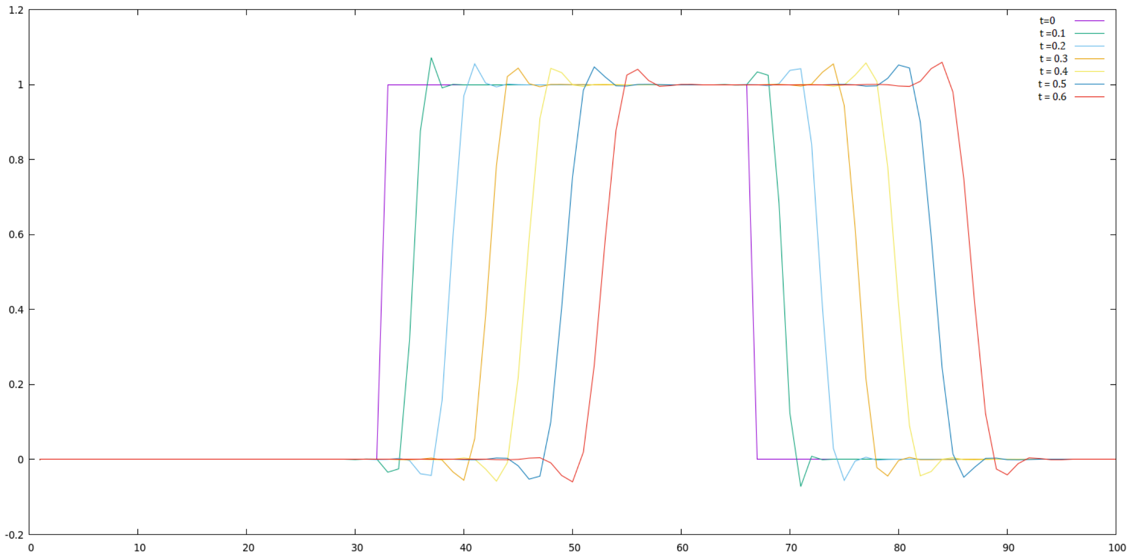

Fig. 2.2.6. We have shown all plots from 2.3.1 to 2.3.5 together to show the propogation from t=0.0 to t=0.6.

Plot of a(x,t) vs x at different times for Gaussian

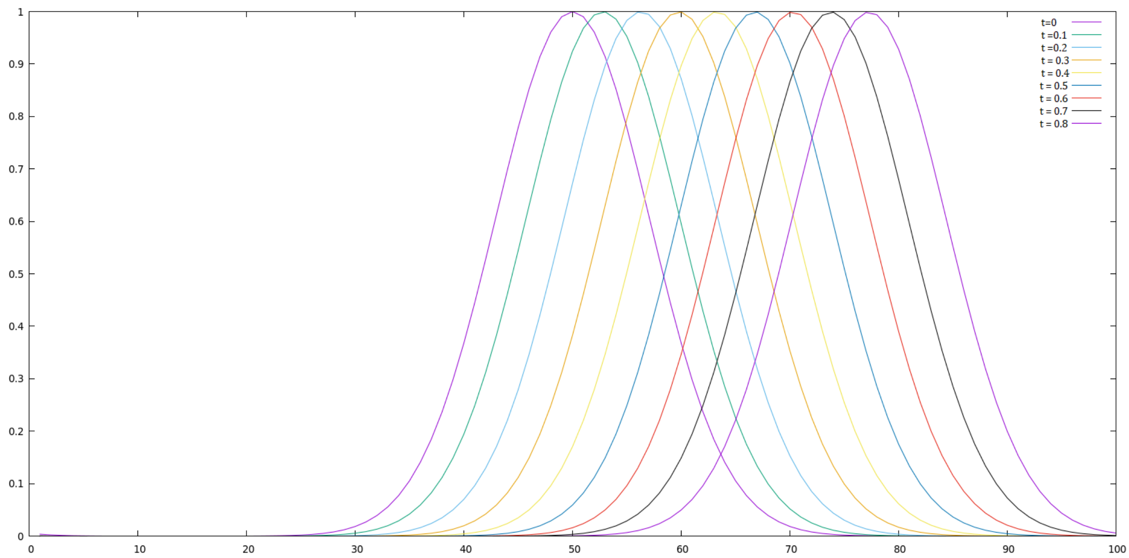

Fig 2.3.7. Gaussian propogated from at t = 0.0 to t = 0.8

It is evident from the figure above that the Gaussian solution doesn't decay at all. The area under curve is conserved with time.

2.4 Finite Volume Method using Minmod Limiter

We saw in the last section that there were some unwanted wavy patterns at the ends of the 'Top-Hat' function. To remove this, we can make a slight change in the definition of $Da_{i,j}$. For this, let us first define a function of two variables a and b called $\text{Minmod}(a,b)$ as

$$\text{Minmod}(a,b) = \begin{cases} a & \text{if } \ a.b > 0 \ \ \& \ \ |a| < |b| \\ b & \text{if } \ a.b > 0 \ \ \& \ \ |b| < |a| \\ 0 & \text{otherwise} \end{cases}$$

The way we have to modify $Da_{i,j}$ is the following

$$Da_{i,j} = \text{Minmod}\left(\frac{a_{i,j+1} - a_{i,j}}{\Delta x}, \frac{a_{i,j} - a_{i,j-1}}{\Delta x}\right) \tag{2.15}$$

We keep everything same as we did in the last section i.e. when we didn't use any limiter in the finite volume method. This Minmod limiting smoothens the sharp unwanted changes that would otherwise start to appear in our function a as time marches. Therefore, we don't get any wavy pattern as we got when we didn't use any limiter. The payoff is that we get a slight numerical diffusion but it is less than those of the finite difference methods.

Plot of a(x,t) vs x at different times for Top-Hat

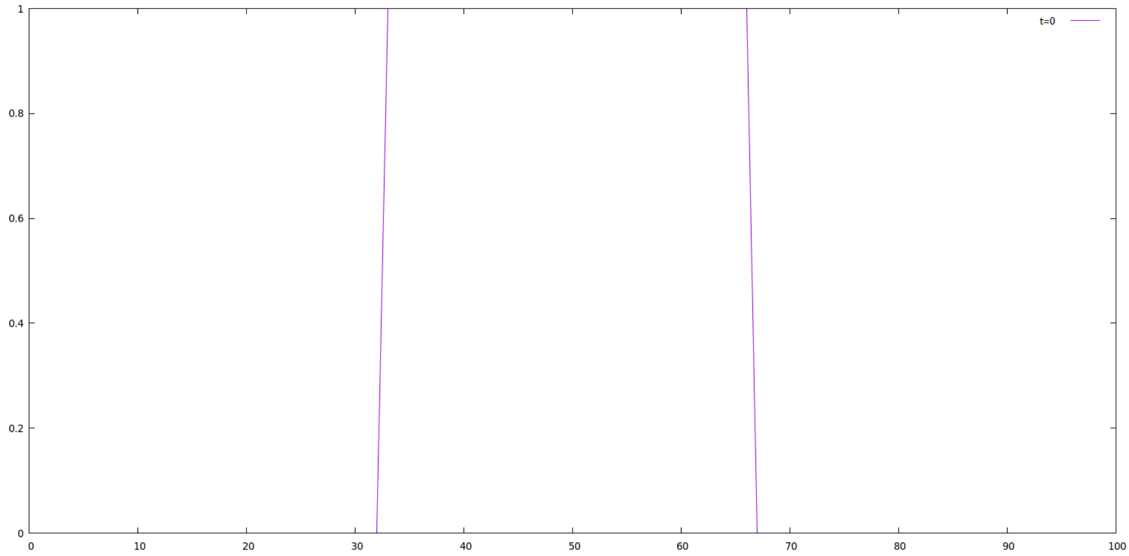

Fig 2.4.1. At t = 0

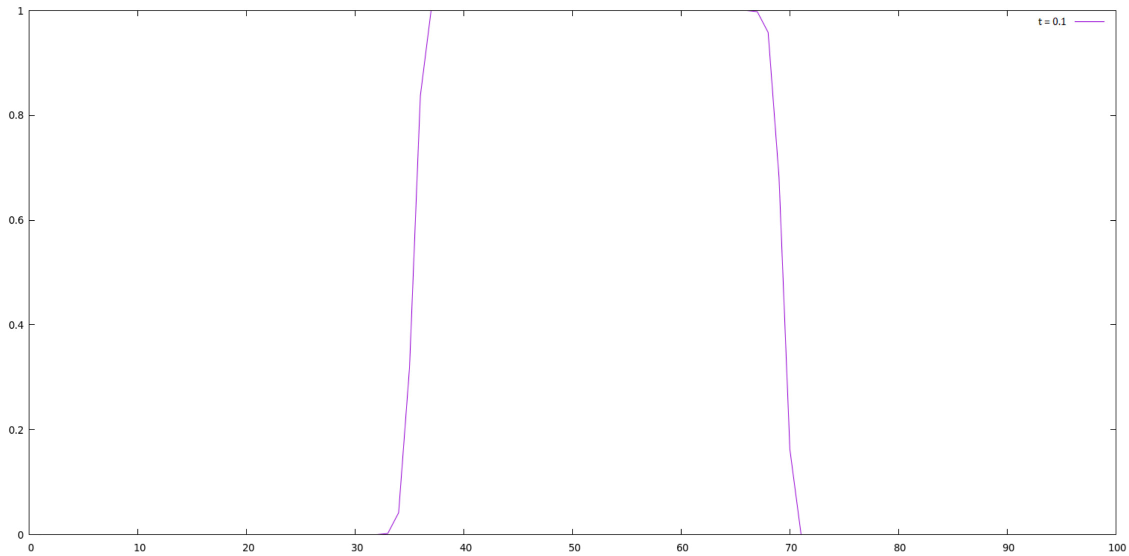

Fig 2.4.2. At t = 0.1

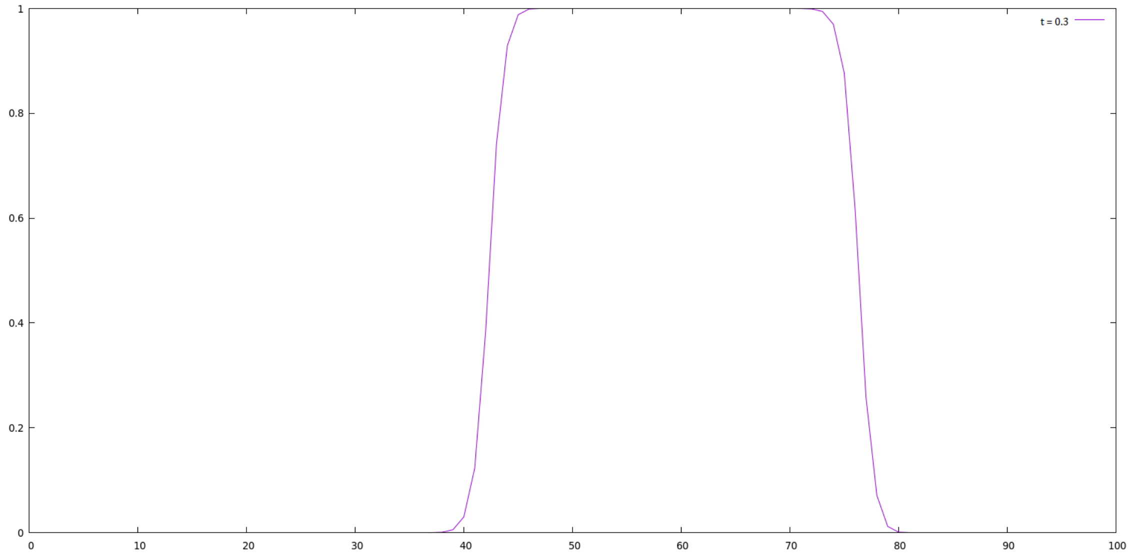

Fig 2.4.3. At t = 0.3

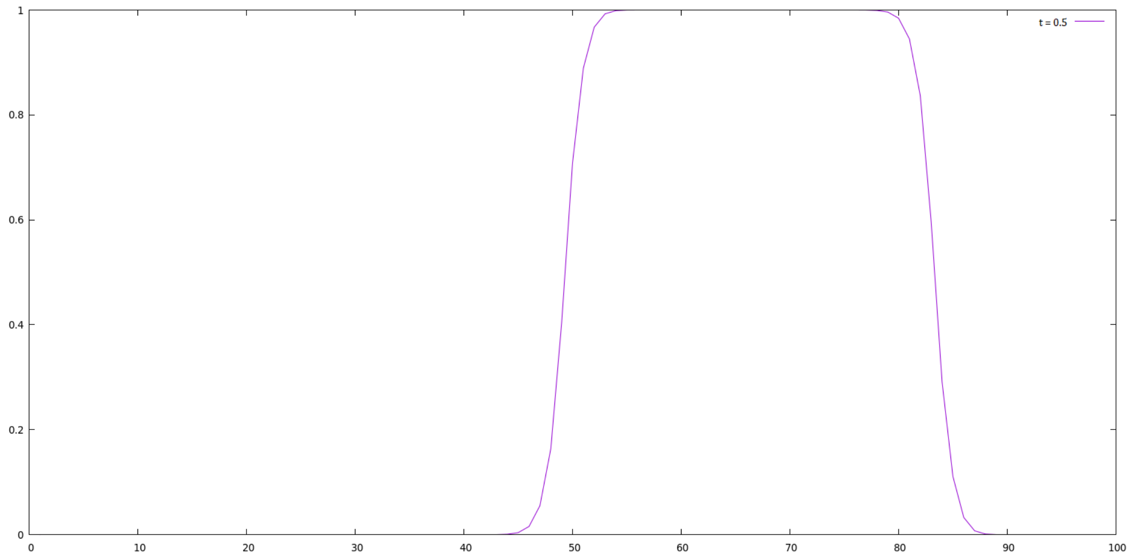

Fig 2.4.4. At t = 0.5

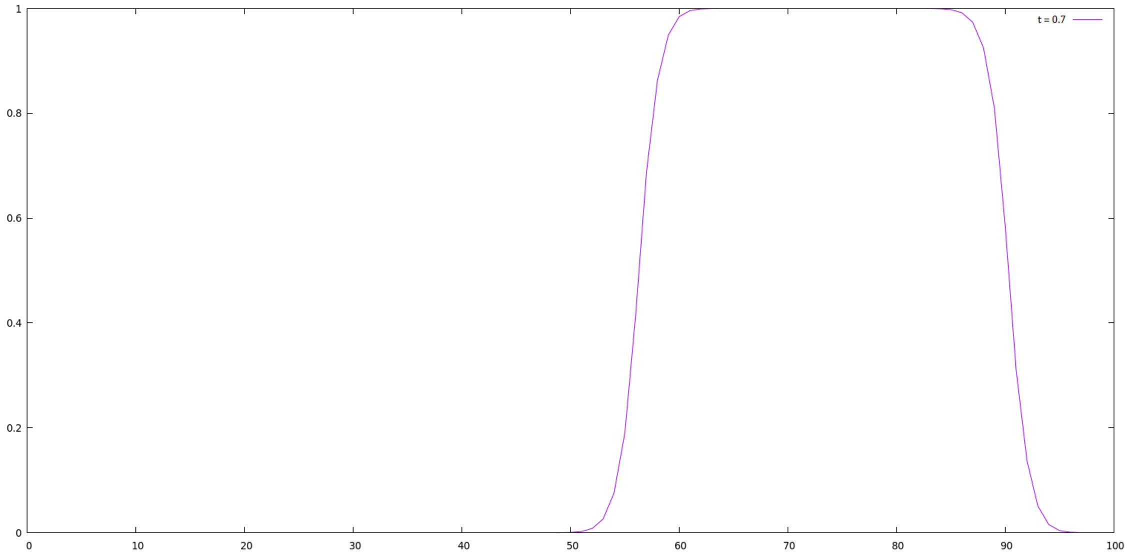

Fig 2.4.5. At t = 0.7

It is evident from the plots that the there are no wavy patterns at the ends of Top-Hat. There is sligh numerical diffusion but it is smaller than that of finite difference with upwinding method.

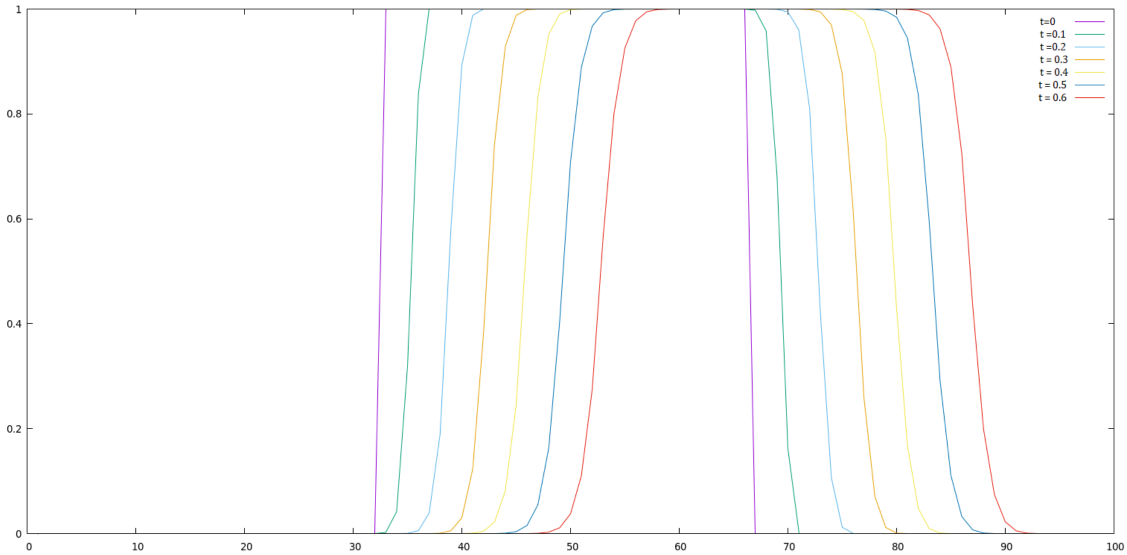

Fig. 2.4.6. We have shown all plots from 2.4.1 to 2.4.5 together to show the propogation from t=0.0 to t=0.6.

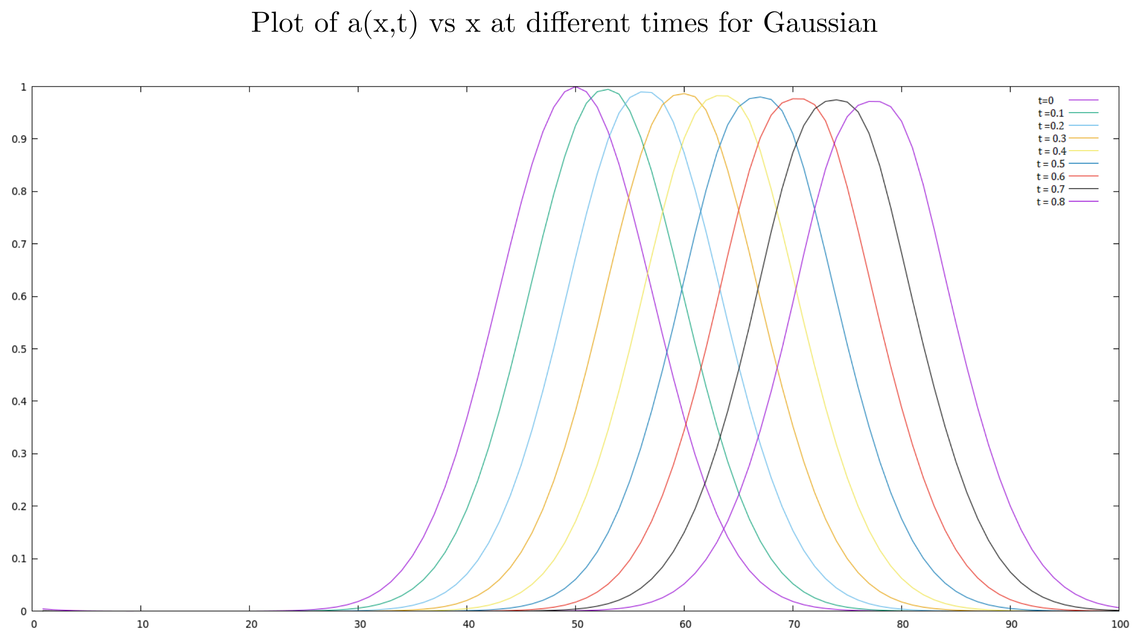

Fig 2.4.7. Gaussian propogated from at t = 0.0 to t = 0.8

It is evident from the figure above that the Gaussian solution slightly decays.

3 Conclusions

We showed 4 different attempts to solve an advection equation in one space dimension with spatial uniform velocity. We saw that the first method i.e. Finite difference failed miserably with 'Top-Hat' function. Even for the Gaussian function, we got a slowly diverging function as time moved on. We removed that instability with the Upwinding scheme but we faced another problem, namely Numerical Diffusion. To remove the numerical diffusion, we explained the area preserving techniques also known as Finite Volume Methods. The finite volume method worked perfectly as expected with the Gaussian function. However, with the Top-Hat function, we got some wavy patterns at the ends of the 'Hat'. Finally, to remove those wavy patterns, we used a limiter namely the Minmod limiter which worked as expected for 'Top-Hat' function. But this limiter created a slight decay in the peak of the Gaussian function. In conclusion, which method is the best for advection is actually dependent upon the function that is chosen as the initial conditions. For the Gaussian function, the finite volume method without any limiter turned out to be the best method and for the Top-Hat function, the Finite Volume method with Minmod limiter turned out to be the best method.