Donald Waters

QUANTITATIVE METHODS FOR BUSINESS

FOURTH EDITION

FT Prentice Hall
FINANCIAL TIMES

Quantitative Methods for Business

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Quantitative Methods for Business

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TO CHARLES

BRIEF CONTENTS

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PREFACE

Managers are the people who run their organisations. They need many skills for this – including problem-solving, leadership, communications, analysis, reasoning, experience, judgement, and so on. They also have to understand quantitative methods, which give essential tools for making rational decisions in complex circumstances. This does not mean that managers have to be professional mathematicians, but they do need to understand quantitative reasoning and interpret numerical results.

All students of management do a course in quantitative methods. These courses come in various guises, including quantitative analysis, decision analysis, business modelling, numerical analysis, and so on. This book describes a range of quantitative methods that are widely used in business, and which every student of management will meet somewhere in their course. It gives an introduction to quantitative methods that is appropriate for the early years of an HND, an undergraduate business course, an MBA, or many professional courses. It is aimed at anyone who wants to see how quantitative ideas are used in business.

Management students come from different backgrounds, so we cannot assume much common knowledge or interests. This book starts with the assumption that you have no previous knowledge of management or quantitative methods. It works from basic principles and develops ideas in a logical sequence, moving from core ideas through to real applications.

Management students often find quantitative ideas difficult. Typically, you are not interested in mathematical abstraction, proofs and derivations, but are more concerned with how useful a result is, and how you can apply it. This is why the book has a practical rather than a theoretical approach. We have made a deliberate decision to avoid proofs, derivations and rigorous (often tedious) mathematics. Some formal procedures are included, but these are kept to a minimum. We assume that computers – particularly spreadsheets – do the routine arithmetic, with Microsoft Excel used to illustrate many of the calculations (but you can get equivalent results from any spreadsheet). There is additional material on the Companion Website at **www.pearsoned.co.uk/waters**.

Contents

Managers can use almost any kind of quantitative methods in some circumstances, so there is an almost unlimited amount of material that we could put into the book. To keep it to a reasonable length we have concentrated on the most widely used topics, taking a balanced view without emphasising some topics at the expense of others. And the book takes a deliberately broad approach, describing many topics rather than concentrating on the details of a few.

For convenience the book is divided into five parts that develop the subject in a logical sequence.

- *Part One* gives an introduction to quantitative methods for managers. These first three chapters lay the foundations for the rest of the book, saying why managers use quantitative methods, and giving a review of essential quantitative tools.
- *Part Two* describes data collection and description. All quantitative methods need reliable data, so these chapters show how to collect this, summarise it, and present it in appropriate forms.
- *Part Three* shows how to use these quantitative ideas for solving different types of problems, including finance, performance, regression, forecasting, simultaneous equations, matrices, linear programming and calculus.
- *Part Four* describes some statistical methods, focusing on probabilities, probability distributions, sampling and statistical inference.
- *Part Five* shows how to use these statistical ideas for problems with uncertainty, including decision analysis, quality management, inventory control, project networks, queues and simulation.

Many people find probabilistic ideas more difficult than deterministic ones, so we have drawn a clear separation between the two. The first three parts describe deterministic methods, and the last two parts cover problems with uncertainty. The whole book gives a solid foundation for understanding quantitative methods and their use in business.

Format

Each chapter uses a consistent format which includes:

- a list of chapter contents
- an outline of material covered and a list of things you should be able to do after finishing the chapter
- the main material of the chapter divided into coherent sections
- worked examples to illustrate methods
- 'ideas in practice' to show how the methods are actually used
- short review questions throughout the text to make sure you understand the material (with solutions in Appendix A)
- key terms highlighted in the chapter, with a glossary at the end of the book
- a chapter review listing the material that has been covered
- a case study based on material in the chapter
- problems (with solutions given on the Companion Website at **www.pearsoned.co.uk/waters**)
- research projects, which allow you to look deeper into a topic
- sources of information, including references, suggestions for further reading and useful websites.

To summarise

This is a book on quantitative methods for business and management. The book:

- is an introductory text that assumes no previous knowledge of business, management or quantitative methods
- takes a broad view and is useful for students doing a wide range of courses, or people studying by themselves
- covers a lot of material, concentrating on the most widely-used methods
- develops the contents in a logical order
- presents ideas in a straightforward, reader-friendly style
- avoids abstract discussion, mathematical proofs and derivations
- illustrates principles by examples from a wide range of applications
- uses spreadsheets and other software to illustrate calculations
- includes a range of learning features to help you understand the material.

Companion Website

The Companion Website for the book is **www.pearsoned.co.uk/waters**. This contains valuable teaching and learning information including:

For students:
- Study material designed to help your understanding
- Data sets for problems, examples and cases in the book
- Spreadsheet templates for calculations
- Additional material to extend the coverage of key topics
- Proofs and derivations of formulae
- Answers to problems
- Additional worked examples and case studies

For lecturers adopting the book for courses:
- A secure password-protected site with teaching material
- A review of key aims and points for each chapter
- Worked solutions to problems
- Comments on case studies
- Copies of figures and artwork from the book
- Additional worked examples and case studies.

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A lot of software is available for quantitative methods. The following list gives packages that are mentioned in the book, with their developers. You can find more information about products from company websites.

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PART ONE

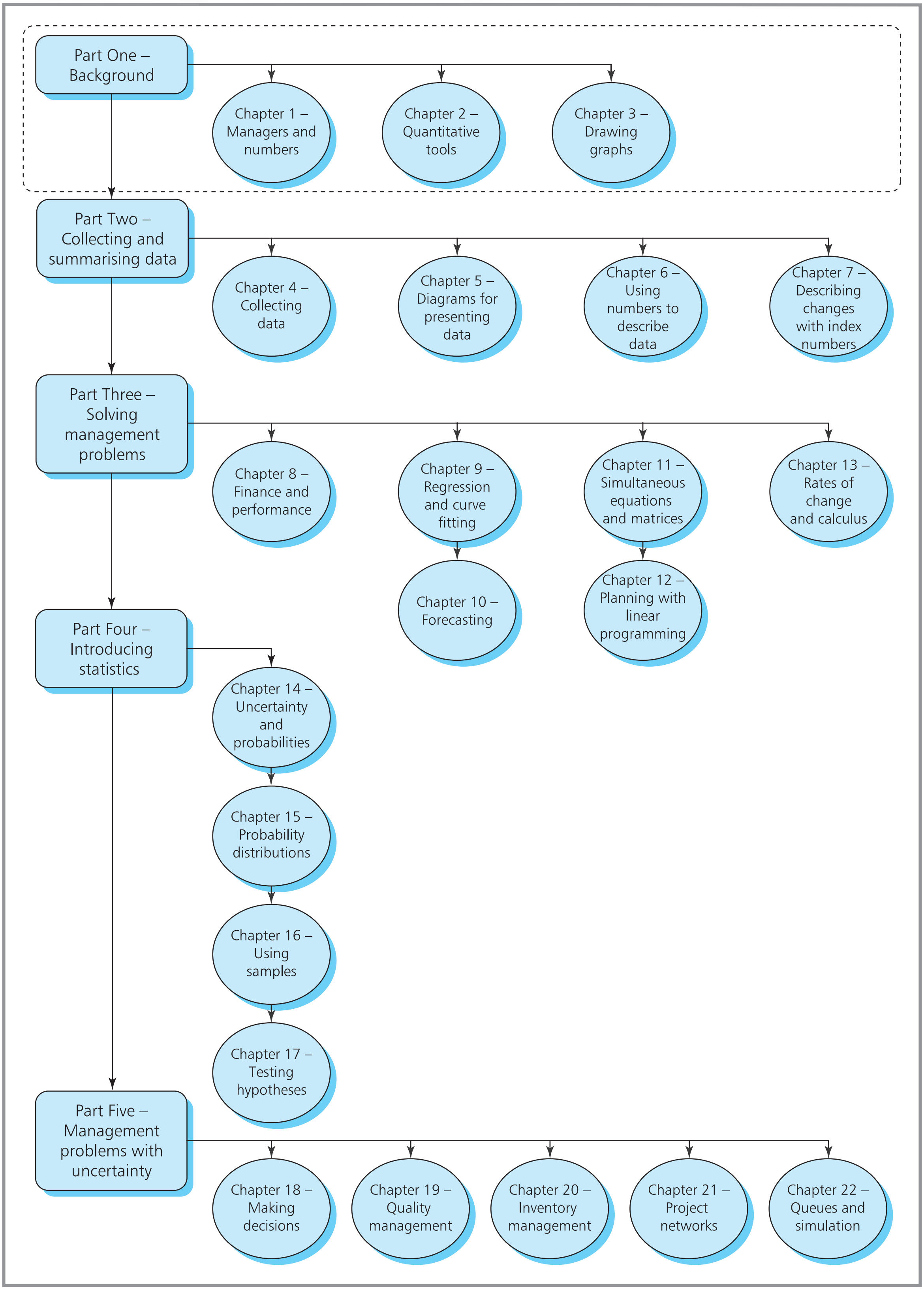
Background

Managers are the people who run their organisations. They need many skills, with key ones being the ability to analyse problems and make the best decisions to solve them. Each problem is in some way distinct, but they all share common features – and, in particular, they generally have some quantitative features. This book describes the quantitative methods that managers use most often to analyse and solve their problems.

The book is divided into five parts, each of which covers a different aspect of quantitative methods. This first part describes the underlying concepts of quantitative methods, setting the context for the rest of the book. The second part shows how to collect and summarise data, and the third part uses this data to solve some common management problems. The fourth part introduces some statistics, and the fifth part uses these to solve problems with uncertainty.

There are three chapters in this first part. Chapter 1 shows that managers constantly use numbers, and they must understand a range of quantitative ideas. The rest of the book describes key methods. Before you look at them in detail, you have to be familiar with some basic quantitative tools. Chapters 2 and 3 review these tools – with Chapter 2 describing numerical skills and algebra, and Chapter 3 showing how to draw graphs.

Chapters in the book follow a logical path through the material, so it is best to take each one in turn. However, you can be flexible, as the map overleaf shows the relationships between chapters.



Map 1 Map of chapters – Part One

CHAPTER 1

Managers and numbers

Contents

Chapter outline

Managers analyse problems and make the best decisions to solve them. Their problems invariably have some numerical features, so managers must understand and use a range of quantitative methods. This chapter introduces the underlying ideas of quantitative methods. It discusses the importance of numerical information, the general approach of quantitative methods, and the way that quantitative models are used to solve problems.

After finishing this chapter you should be able to:

- Appreciate the importance and benefits of numbers
- Say why quantitative methods are particularly useful for managers
- Understand the use of models
- Describe a general approach to solving problems
- Use computers for calculations.

Why use numbers?

On an ordinary day, you might notice that the temperature is 17°C, petrol costs 92 pence per litre, 1.3 million people are unemployed, house prices rose by 12% last year, some people want a pay rise of £1.50 an hour, a football team has won its last seven games, 78% of people want shops to open longer hours, your telephone bill is £95, and a candidate won 32,487 votes in an election. These numbers give essential information. They have the benefit of giving a clear, precise and objective **measure**. When the temperature is 30 degrees, you know exactly how hot it is; when a bar contains 450 grams of chocolate, you know exactly how big it is; and your bank manager can say exactly how much money is in your account. On the other hand, when you cannot measure something it is much more difficult to describe and understand. When you get a pain in your stomach it is very difficult to describe the kind of pain, how bad it is, or how it makes you feel. When you read a book it is difficult to say how good the book is or to describe the pleasure it gave you.

A second benefit of numbers is that you can use them in **calculations**. If you buy three bars of chocolate that cost 30 pence each, you know the total cost is 90 pence; if you pay for these with a £5 note you expect £4.10 in change. If you start a 120 km journey at 12:00 and travel at 60 km an hour, you expect to arrive at 14:00.

- Any reasoning that uses numbers is **quantitative**.
- Any reasoning that does not use numbers, but is based on judgement and opinions, is **qualitative**.

WORKED EXAMPLE 1.1

An automatic ticket machine only accepts pound coins. The numbers of tickets it gives are:

£1 – 1 ticket, £2 – 3 tickets, £3 – 4 tickets,
£4 – 5 tickets, £5 – 7 tickets

How can you get the cheapest tickets?

Solution

You can do a simple calculation to find the best value for money. You know that:

- £1 gives 1 ticket, so each ticket costs £1/1 = £1
- £2 gives 3 tickets, so each ticket costs £2/3 = £0.67
- £3 gives 4 tickets, so each ticket costs £3/4 = £0.75
- £4 gives 5 tickets, so each ticket costs £4/5 = £0.80
- £5 gives 7 tickets, so each ticket costs £5/7 = £0.71

Buying three tickets for £2 clearly gives the lowest cost per ticket.

Numbers increase our understanding of a situation – and it is impossible to lead a normal life without them. This does not mean that we all have to be mathematical whiz-kids – but it does mean that we have to understand some numerical reasoning and know how to work with numbers.

Often we do not need precise answers, but are happy with rough estimates. If you can read a page a minute, you know that you can finish a 57-page report in about an hour. If you see a car for sale, you do not know exactly how much it costs to run, but a rough calculation shows whether you can afford it; if you get a bill from a builder you can quickly check that it seems reasonable; before you go into a restaurant you can get an idea of how much a meal will cost.

Numbers and management

Managers have to understand quantitative reasoning, as their decisions are almost invariably based on calculations. When they want to increase profits, they measure the current profits and set numerical targets for improvement. And they continually measure performance, including return on investment, turnover, share price, capacity, output, productivity, sales, market share, number of customers, costs, and so on. Annual accounts review overall performance, and these are largely quantitative. In reality, it is difficult to find any aspect of a manager's work that does not involve some quantitative methods.

> **Quantitative methods** form a broad range of numerical approaches for analysing and solving problems.

You should not be surprised that managers rely on quantitative reasoning, as this is a routine part of many jobs. Civil engineers do calculations when they design bridges; doctors prescribe measured amounts of medicines; telephone companies monitor the traffic on their networks; accountants give a quantitative view of performance. Some people imagine that managers do not need formal analyses but can somehow guess the right decisions using their intuition and judgement. We want to overcome this rather strange idea – but this does not mean that we expect managers to do all the analyses themselves. They can use experts to get results – in the same way that they use experts in communications, information processing, accounting, law and all the other specialised areas. However, we do expect managers to be aware of the analyses available, to understand the underlying principles, to recognise the limitations, to have intelligent discussions with experts, and to interpret the results.

Of course, not all aspects of a problem are quantitative. Judgement, intuition, experience and other human skills are important in many areas – such as industrial relations, negotiations, recruitment, setting strategic goals, and personal relations. But even here managers should consider all available information before reaching their decisions, and quantitative methods can still be useful. Figure 1.1 shows the usual approach to decisions, where managers identify a problem, do quantitative and qualitative analyses, evaluate the results, make their decisions, and implement them.

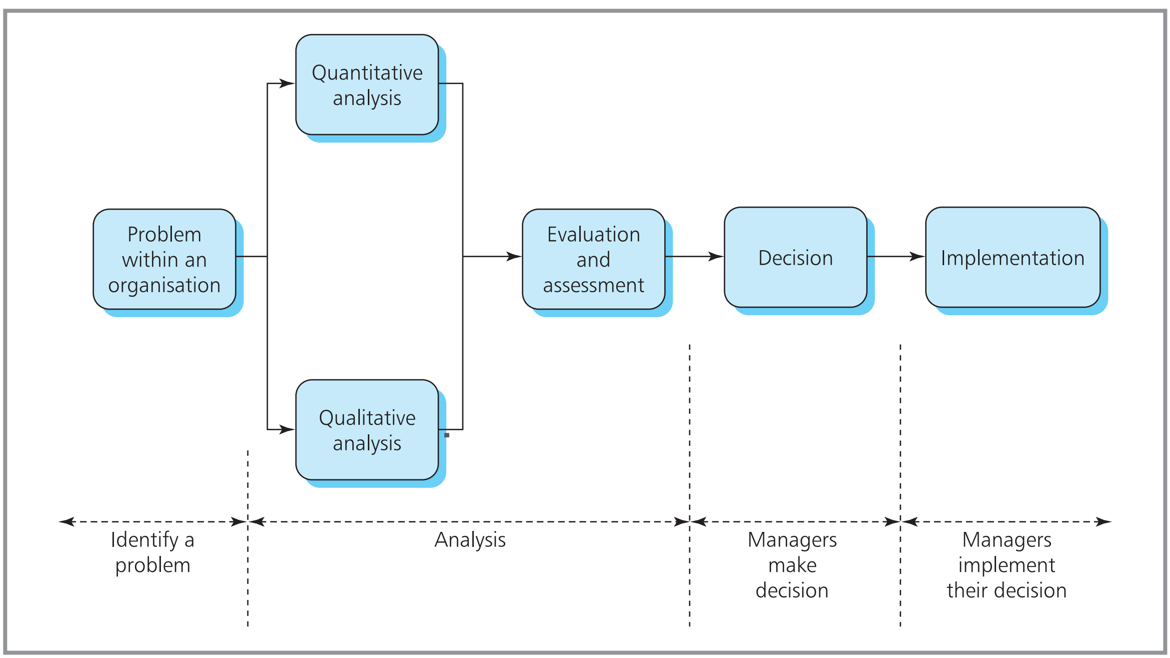


Figure 1.1 Usual approach to making a decision

WORKED EXAMPLE 1.2

The policy of Benchmark Global Consultants is to employ one consultant for every 10 clients on their books. Last month they had 125 clients. How many consultants should they employ?

Solution

A purely quantitative analysis suggests employing 125/10 = 12.5 consultants. They could employ part-time staff, but this may not be feasible, particularly if the number of clients keeps changing. Realistically the company could round the number of consultants to either 12 or 13. The best decision depends on a range of qualitative factors – such as expected changes to client numbers, amount of work sent by each client, attitudes of consultants, type of business, planned staff departures, recruitment, training, seasonal trends, and so on. Managers must review all the available information – both quantitative and qualitative – before making their decision.

Review questions

(Appendix A at the end of the book gives answers to all the review questions.)

1.1 What are the benefits of quantitative methods?

1.2 Do quantitative analyses make the best decisions?

1.3 Managers must be good mathematicians. Do you think this is true?

1.4 Why has the use of quantitative methods by managers increased in the past 20 years?

IDEAS IN PRACTICE RPF Global

Patrick Chua is the senior vice-president of RPF Global, a firm of financial consultants with offices in major cities around the Pacific Rim. He outlines his use of quantitative ideas as follows.

> 'Most of my work is communicating with managers in companies and government offices. I am certainly not a mathematician, and am often confused by figures – but I use quantitative ideas all the time. When I talk to a board of directors, they won't be impressed if I say, "This project is quite good; if all goes well you should make a profit at some point in the future." They want me to spell things out clearly and say, "You can expect a 20% return over the next two years."
>
> My clients look for a competitive advantage in a fast-moving world. They make difficult decisions. Quantitative methods help us make better decisions – and they help explain and communicate these decisions. Quantitative methods allow us to:
>
> - look logically and objectively at a problem;
> - measure key variables and the results in calculations;
> - analyse a problem and look for practical solutions;
> - compare alternative solutions and identify the best;
> - compare performance across different operations, companies and times;
> - explain the options and alternatives;
> - support or defend a particular decision;
> - overcome subjective and biased opinions.
>
> Quantitative methods are an essential part of any business. Without them, we just could not survive!'

Source: Chua P., personal correspondence, 2006; Chua P., talk to Eastern Business Forum, Hong Kong, 2005.

Solving problems

Building a model

'Quantitative methods' is a broad subject that includes many different approaches – but they all start with a **model** of a problem. In this sense, a 'model' is a simplified representation of reality, and we are not talking about toys or games. The main features of a model are as follows:

- It is a representation of reality.
- It is simplified, with only relevant details included.
- Properties in reality are represented by other properties in the model.

There are several types of model, but the most widely used by managers are **symbolic models**. These have properties in reality represented by some kind of symbol. Then a symbolic model for the amount of value added tax payable is:

$$\text{VAT} = \text{rate} \times \text{sales}$$

where the symbol 'VAT' in the model represents the amount of tax paid in reality, and the symbols 'rate' and 'sales' represent the actual rate of VAT and value of sales.

If a company sells a product for £300 a unit, a model of its income is:

$$\begin{aligned}\text{income} &= \text{number of units sold} \times \text{selling price}\\ &= \text{number of units sold} \times 300\end{aligned}$$

We can extend the model by finding the profit when it costs £200 to make each unit:

$$\text{profit} = \text{number of units sold} \times (\text{selling price} - \text{cost})$$

or

$$\begin{aligned}\text{profit} &= \text{number of units sold} \times (300 - 200)\\ &= \text{number of units sold} \times 100\end{aligned}$$

This equation is our model. Now we can do some experiments with it, perhaps seeing how the profit changes with the selling price or number of units sold. The important point is that if we did not have the model, we would have to experiment with real operations, getting the company to change its selling price and then measuring the change in profit. This has the obvious disadvantages of being difficult, time consuming, disruptive and expensive, and possibly causing permanent damage. It may also be impossible – for example, a wholesaler cannot find the best location for a new warehouse by experimentally trying all possible locations and keeping the best. Experimenting with real **operations** (which include all the organisation's activities) is at best expensive and at worst impossible, so the only feasible alternative is to build a model and experiment with this.

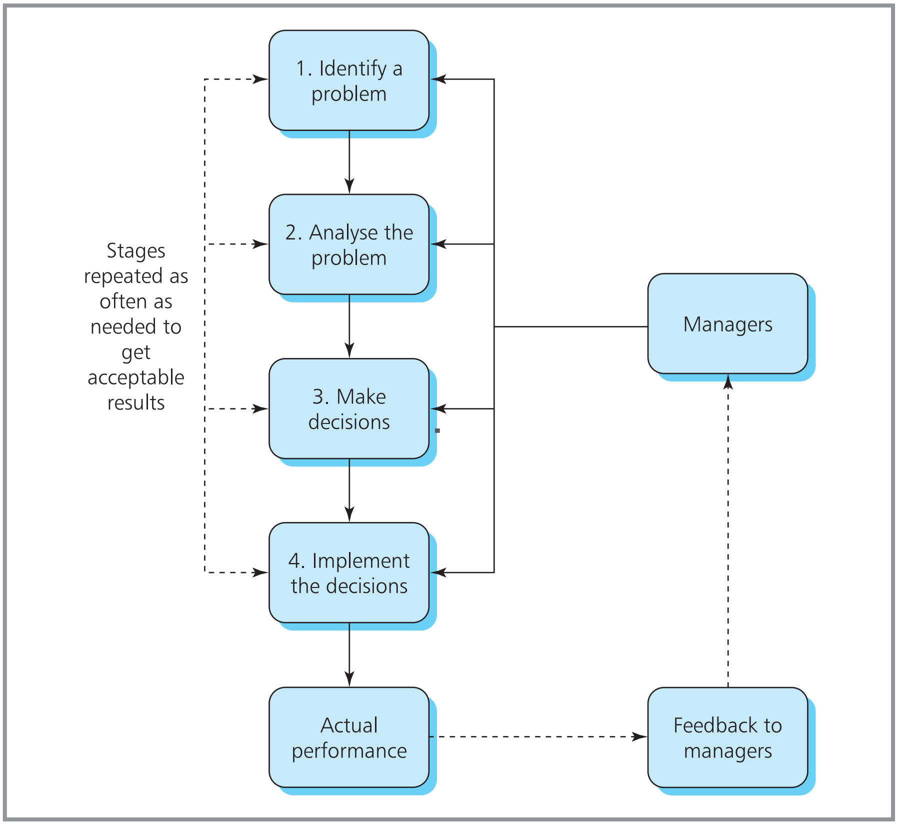


Figure 1.2 Stages in decision making

Stages in problem solving

Models form a key part of the broader decision-making process. You can see this by expanding the four stages for tackling a problem that we mentioned earlier – identify a problem, analyse it, make decisions, and implement the results (see Figure 1.2).

1 *Identify a problem*. Describe the details of the problem and its context. At the end of this stage, managers should have a clear understanding of the problem they are tackling and the requirements of their solution. This might include:
   (a) Initial investigation – to look at operations, identify difficulties and recognise that there is a problem.
   (b) Defining the problem – to add details to the initial investigation, saying exactly what the problem is (and not just its symptoms), its context, scope, boundaries and any other relevant details.
   (c) Setting objectives – to identify the decision makers, their aims, improvements they want, effects on the organisation, and measures of success.
   (d) Identifying variables, possible alternatives and courses of action.
   (e) Planning the work – showing how to tackle the problem, schedule activities, design timetables and check resources.

2 *Analyse the problem*. At the end of this stage, managers should have a clear understanding of their options and the consequences. For this they might:
   (a) Consider different approaches to solving the problem.
   (b) Check work done on similar problems and see if they can use the same approach.
   (c) Study the problem more closely and refine the details.
   (d) Identify the key variables and relationships between them.
   (e) Build a model of the problem and test its accuracy.
   (f) Collect data needed by the model and analyse it.
   (g) Run more tests on the model and data to make sure that they are working properly, are accurate and describe the real conditions.
   (h) Experiment with the model to find results in different circumstances and under different conditions.
   (i) Analyse the results, making sure that they are accurate and consistent.
3 *Make decisions*. This is where managers consider the results from analyses, review all the circumstances and make their decisions. This has three steps to:
   (a) Compare solutions, looking at all aspects of their performance.
   (b) Find solutions that best meet the decision makers' objectives.
   (c) Identify and agree the best overall solution.
4 *Implement the decisions*. At this point managers turn ideas into practice, moving from 'we should do this' to actually doing it. For this they:
   (a) Check that the proposed solution really works and is an improvement.
   (b) Plan details of the implementation.
   (c) Change operations to introduce new ways of doing things.

After implementing their decisions, managers still have to monitor performance to make sure that predicted results actually appear. For this they use **feedback**, which allows them to compare actual performance with plans, and make any necessary adjustments to the operations.

In reality, taking the four stages in strict sequence gives too simple a view, as managers often hit problems and have to return to an earlier point. For example, when making a decision in stage 3 they may find that they do not have enough information and return to stage 2 for more analysis. So they keep returning to earlier stages as often as needed – or until time for the decision runs out.

Figure 1.3 shows a more detailed view of decision making when the elements of quantitative analysis are added to stage 2.

People use different terms for the stages in decision making, such as Finlay and King's[1] description of conceptualisation, verbalisation, symbolisation, manipulation and representation. Waters[2] describes observation, modelling, experimentation and implementation, and a classic work by Ackoff[3] describes six stages of defining a problem, building a model, testing the model, getting a solution to the problem, implementing the solution, and controlling the solution.

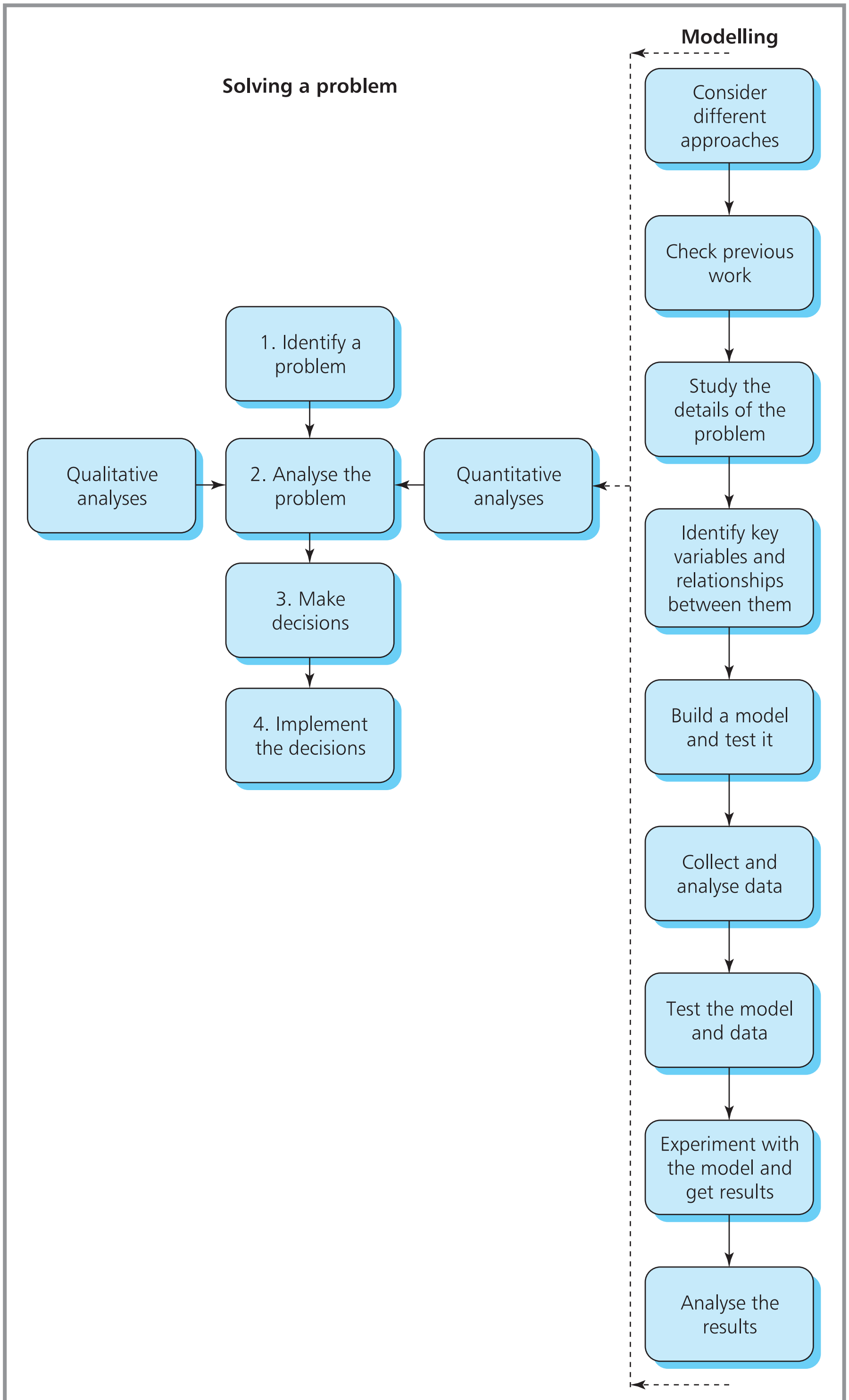


Figure 1.3 The role of modelling in solving a problem

IDEAS IN PRACTICE BG Group

BG Group (BGG) is an international energy group with a turnover of around $6 billion a year. Its main business is the supply of natural gas. This is a 'clean' fuel, and because the worldwide demand for energy is growing, sales are expected to rise significantly over the next 10 years. To meet this demand BGG has to continually find and develop new reserves.

National governments generally regard gas fields as a vital strategic resource, so they keep tight control over them. To develop a field, governments divide it into blocks and invite energy companies to bid for exploration rights. BGG, along with every other energy company, has to decide whether to bid for exploration rights in available blocks, and how much to bid. These are important decisions that are characterised by high costs (typically hundreds of millions of dollars), long lead times (typically five years before a project starts earning money), limited lifetime (as there is a finite amount of gas available) and complex tax and contractual arrangements.

BGG considers many factors in each decision. Firstly, there are qualitative factors, particularly BGG's rigorous ethical guidelines and business principles. These are important in showing how BG Group does business and what it stands for – and how it deals with issues such as gas finds in sensitive environments, conflict zones, or areas where indigenous peoples are contesting land rights. Other qualitative questions concern the availability of alternative projects, structure of the company's long-term portfolio of fields, partnerships, effect on share value, and so on.

Secondly, there are quantitative factors. These focus on two issues:

1. Risks – where geologists look at the chances of finding gas and the likely size of discoveries, engineers look at potential difficulties with production, health and safety look at safety and environmental risks, and economists look at likely demand, prices and commercial risks.
2. Return from the project, starting with the basic formula:

net cash flow = revenue – costs – taxes

Managers review the results from both qualitative and quantitative analyses before making any decision.

Sources: BG Annual Reports and websites www.bg-group.com and www.thetimes100.co.uk.

Review questions

1.5 Why do managers use models?

1.6 What are the stages in solving a problem?

1.7 Where do quantitative models fit into this approach?

1.8 Is there only one correct way to tackle a problem?

Useful software

An obvious problem with using numbers is that we make mistakes with even the simplest calculations. Thankfully, we can use calculators for simple arithmetic and computers for anything more ambitious. In this book, we assume that you use computers for all the routine arithmetic. Then you might think that there is no point in doing any of the calculations – but when you leave everything to the computer, you get no insight into the calculations or 'feel' for the numbers. You need at least some contact with the calculations to say whether the results make sense, or whether they are absurd. If your computer says that a company made £14 million profit last year, this might be good news; alternatively, you might have some feel for the calculations and realise that there is a mistake. If your computer calculates an answer of 15 km, this may be good – but it was nonsense when a study quoted this as the diameter needed for a sewer pipe in Karachi.[4] So it is always useful to do some calculations – if only to see what is happening and check the results.

Spreadsheets give a particularly useful format for calculations. They consist of a grid of related cells, with the rows numbered 1, 2, 3, etc., and the columns labelled A, B, C, etc. Then each cell in the grid has a unique address such as A1, A2, B1, C6 or F92. Each cell can contain three types of data:

1 a simple number – for example, we can set cell B1 to equal 21, and B2 to 12
2 a relationship between cells – so we can set cell B3 to equal the sum of cells B1 and B2
3 a label – so we can set cell A3 to contain the word 'Total'.

	A	B
1		21
2		12
3	**Total**	**33**

Figure 1.4 Example of a spreadsheet calculation

You can see the result in Figure 1.4. The benefit of this format is that you can change the value of any cell (such as B1 or B2) and the spreadsheet will automatically do the calculations.

The most widely used spreadsheet is Microsoft Excel, but alternatives include Ability Office Spreadsheet, Lotus 1-2-3, OpenOffice Calc, Quattro Pro, and StarOffice. You can use any appropriate package for the calculations, and the general guidance is to use the software that you are happiest with. If you want lessons or revision in spreadsheets, there are some suggestions in the sources of information at the end of the chapter.

Spreadsheets are easy to use and have a standard format – but they have limitations and cannot do every type of analysis. Sometimes it is better to use a specialised package that is better at handling data, uses the best method to solve a particular problem, includes special procedures, gives results in the best format – and may be easier to use. But specialised software can be more complicated and more expensive, so you have to balance the benefits with the extra effort of learning to use it.

WORKED EXAMPLE 1.3

Earlier in the chapter we described an automatic ticket machine that only accepts pound coins and gives out:

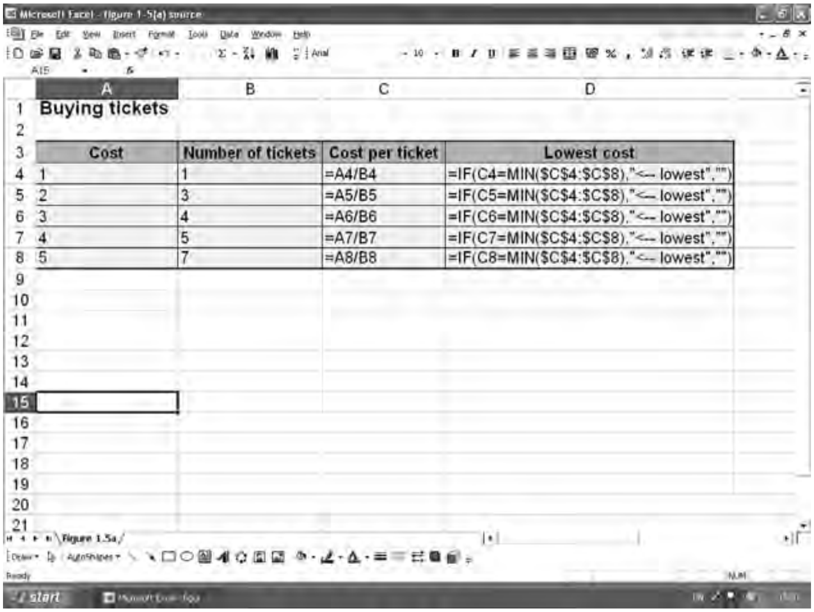
> 1 ticket for £1, 3 tickets for £2, 4 tickets for £3, 5 tickets for £4, and 7 tickets for £5

Use a spreadsheet to find the best value from the machine.

Solution

Figure 1.5(a) shows the calculations for this, while Figure 1.5(b) shows the results. If you do not understand these results, it is worth getting some practice with spreadsheets. The key point is that each cell can contain text, a value, or a calculation. An equals sign shows that it contains a calculation – such as '=A4/B4', where the cell contains the result of dividing the value in cell A4 by the value in cell B4. The calculations can contain standard functions, such as 'SUM' (adding the values in a range of cells), 'MAX' (finding the maximum value), 'MIN' (finding the minimum value), and the conditional 'IF'.

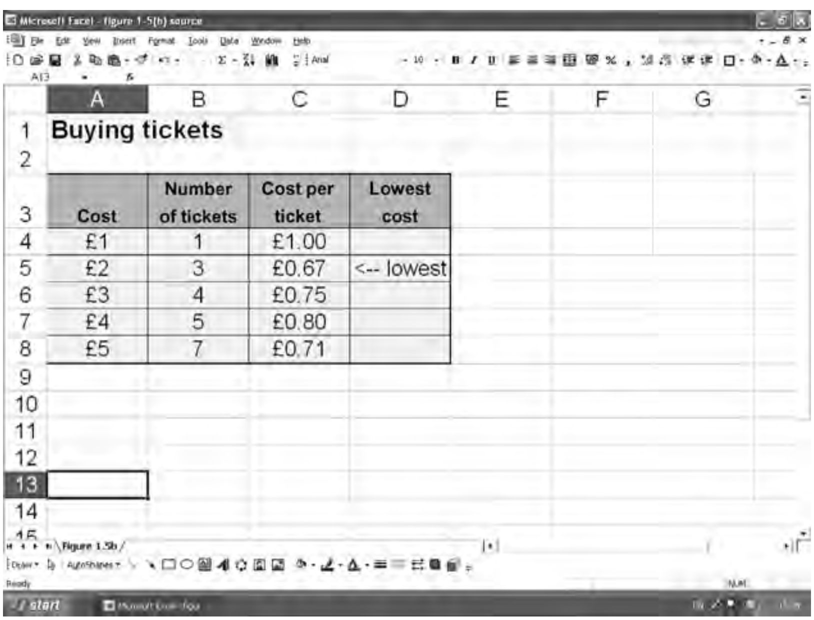
Worked example 1.3 continued



Buying tickets

Cost	Number of tickets	Cost per ticket	Lowest cost
1	1	=A4/B4	=IF(C4=MIN($C$4:$C$8),"<-- lowest","")
2	3	=A5/B5	=IF(C5=MIN($C$4:$C$8),"<-- lowest","")
3	4	=A6/B6	=IF(C6=MIN($C$4:$C$8),"<-- lowest","")
4	5	=A7/B7	=IF(C7=MIN($C$4:$C$8),"<-- lowest","")
5	7	=A8/B8	=IF(C8=MIN($C$4:$C$8),"<-- lowest","")

Figure 1.5(a) Spreadsheet calculations for ticket machine



Buying tickets

Cost	Number of tickets	Cost per ticket	Lowest cost
£1	1	£1.00	
£2	3	£0.67	<-- lowest
£3	4	£0.75	
£4	5	£0.80	
£5	7	£0.71	

Figure 1.5(b) Results from the calculations

Review questions

1.9 'To get a feel for a calculation you should do it by hand first, and then use a computer to check the result.' Do you think this is true?

1.10 Why would you use general-purpose software like spreadsheets when there are specialised packages for most problems?

CHAPTER REVIEW

This chapter introduced the idea of quantitative analysis and the use of numbers in everyday life.

- Numbers have two major advantages. Firstly, they give a clear, concise and objective measure of a feature; secondly, we can use them in calculations.
- Business problems almost invariably have some numerical features. To deal with these, managers need some appreciation of quantitative methods. This does not mean that they have to be expert mathematicians, but they must have a basic understanding of the principles.
- Quantitative methods normally use symbolic models, which represent real features by symbols. In particular, they use equations to describe real problems.
- A general approach to problem solving has four stages: to identify a problem, analyse it, make decisions and implement the results. Quantitative methods form a key part of the analysis stage.
- Computers do the routine arithmetic for quantitative methods using standard software, particularly spreadsheets – but you still need some feel for the calculations and results.

CASE STUDY Hamerson and Partners

Albert Hamerson is Managing Director of his family firm of builders' merchants. He is the third generation to run the company, and is keen for his daughter, Georgina, to join him when she leaves university. Georgina is also keen to join the company, but she is not sure what kind of job she wants.

Hamerson and Partners is essentially a wholesaler. They buy 17,000 different products from 1,100 manufacturers and importers, including all the usual materials needed by builders. Their main customers are small building firms, but they have some long-term contracts with bigger organisations, and many one-off and DIY customers. The company works from four sites around Dublin and Cork and employs over 300 people.

Georgina feels that the company is getting behind the times. She assumed that computers would reduce the amount of paperwork, but when she goes into company offices she is surprised at the amount of paper. For instance, she thought that most orders would be collected automatically through the company's website, but she saw that they were also written on scraps of paper, printed forms, faxes, and downloaded emails. When she walks around the stores, things still seem to be organised in the way they were 20 years ago.

Georgina has several ideas for improvements – often emerging from her university studies in mathematics and business. She wants to develop these ideas, and imagines herself as an 'internal consultant' looking around the company, finding areas for improvement, and doing projects to make operations more efficient. One problem is that her father has had little contact with quantitative analyses beyond reading the company accounts. He makes decisions based on experience gained through 35 years of work with the company, and discussions with staff. He is not sure that Georgina's mathematical training will be of any practical value.

After some discussion, Georgina agreed to write a report describing the type of problem that she

Case study continued

could tackle. She will outline her approach to these problems and the benefits the company could expect. Then she will spend some time in her next vacation looking in more detail at one of these problems.

Question

- **If you were in Georgina's position, what would you put in your report? What benefits do you think that she could bring to the company?**

PROBLEMS

(The answers to these problems are given on the Companion Website **www.pearsoned.co.uk/waters**)

1.1 At last year's Southern Florida Amateur Tennis Championships there were 1,947 entries in the women's singles. This is a standard knockout tournament, so how many matches did the organisers have to arrange?

1.2 European coins have denominations of 1, 2, 5, 10, 20 and 50 cents, and 1 and 2 euros. What is the smallest number of coins needed to pay exactly a bill of €127.87?

1.3 Sally was pleased when a meal at the Golden Orient restaurant appeared to cost $28 for food and $16 for drinks. Unfortunately, her final bill added 15% alcohol duty, 10% service charge, 12% federal tax, 6% state tax and 2% city tax. How much did she pay in tax, and what was her final bill?

1.4 A family of three is having grilled steak for dinner, and they like to grill their steaks for 10 minutes on each side. Unfortunately, the family's grill pan is only big enough to grill one side of two steaks at a time. How long will it take to cook dinner?

1.5 A shopkeeper buys an article for £25 and then sells it to a customer for £35. The customer pays with a £50 note. The shopkeeper does not have enough change, so he goes to a neighbour and changes the £50 note. A week later the neighbour tells him that the £50 note was a forgery, so he immediately repays the £50. How much does the shopkeeper lose in this transaction?

1.6 Devise a scheme for doctors to see how bad a stomach pain is.

1.7 Design a fair system for electing Parliamentary candidates.

RESEARCH PROJECTS

1.1 It might seem an exaggeration to say that every problem that managers tackle has a quantitative aspect. Try doing a review of the types of decisions made by managers and see if you can find examples of problems that are purely qualitative.

1.2 You can use computers to do any of the arithmetic described in this book. Check the computers and software that are available. It would be particularly useful to have a spreadsheet with good graphics. Make sure that you are familiar with the resources available, and know how to use them.

1.3 The following table shows the number of units of a product sold each month by a shop, the amount the shop paid for each unit, and the selling price. Use a spreadsheet to find the total values of sales, costs, income and profit. What other analyses can you do?

Month	Year 1 Units sold	Year 1 Unit cost to the shop	Year 1 Selling price	Year 2 Units sold	Year 2 Unit cost to the shop	Year 2 Selling price
January	56	120	135	61	121	161
February	58	122	138	60	121	161
March	55	121	145	49	122	162
April	60	117	145	57	120	155
May	54	110	140	62	115	150
June	62	106	135	66	109	155
July	70	98	130	68	103	156
August	72	110	132	71	105	157
September	43	119	149	48	113	161
October	36	127	155	39	120	161
November	21	133	161	32	126	160
December	22	130	161	25	130	160

Remember that the data sets used in the book are all given in the resources of the Companion Website **www.pearsoned.co.uk/waters**.

Sources of information

References

1 Finlay P.N. and King M., Examples to help management students to love mathematical modelling, *Teaching Mathematics and its Applications*, vol. 5(2), pages 78–93, 1986.

2 Waters D., *A Practical Introduction to Management Science*, Addison Wesley Longman, Harlow, 1998.

3 Ackoff R.L., *Scientific Method*, John Wiley, New York, 1962.

4 Mian H.M., personal correspondence, 1986.

Further reading

There are several general books on quantitative methods for business, with the following giving a good starting point:

Bancroft G. and O'Sullivan G., *Foundations of Quantitative Business Techniques*, McGraw-Hill, Maidenhead, 2000.

Curwin J. and Slater R., *Quantitative Methods for Business Decisions* (5th edition), Thomson Learning, London, 2001.

Morris C., *Quantitative Approaches in Business Studies* (6th edition), FT Prentice Hall, Harlow, 2003.

Oakshot L., *Essential Quantitative Methods for Business Management and Finance* (3rd edition), Palgrave, Basingstoke, 2006.

Swift L., *Quantitative Methods for Business, Management and Finance* (2nd edition), Palgrave, Basingstoke, 2005.

Waters D., *Essential Quantitative Methods*, Addison Wesley, Harlow, 1998.

Wisniewski M., *Foundations of Quantitative Methods for Business* (4th edition), FT Prentice Hall, Harlow, 2005.

Some more specialised books on models and decision making include:

Clemen R.T. and Reilly T., *Making Hard Decisions with Decision Tool Suite*, Duxbury Press, Boston, MA, 2000.

Goodwin P. and Wright G., *Decision Analysis for Management Judgement* (2nd edition), John Wiley, Chichester, 1998.

Hammond J.S., Keeney R.L. and Raiffa H., *Smart Choices: a Practical Guide to Making Better Decisions*, Broadway Books, New York, 2002.

Heller R., *Making Decisions*, Dorling Kindersley, London, 1998.

Koomey J.G., *Turning Numbers into Knowledge*, Analytics Press, Oakland, CA, 2004.

Waters D., *A Practical Introduction to Management Science* (2nd edition), Addison Wesley, Harlow, 1998.

Many books describe how to use spreadsheets, with a small sample including:

Albright S., *Data Analysis and Decision Making with Microsoft Excel*, Duxbury Press, Boston, MA, 2002.

Barlow J.F., *Excel Models for Business and Management* (2nd edition), John Wiley, Chichester, 2005.

Hesse R., *Managerial Spreadsheet Modelling and Analysis*, Richard D. Irwin, Homewood, IL, 1996.

Moore J.H. and Weatherford L.R., *Decision Modelling with Microsoft Excel* (6th edition), Prentice Hall, Upper Saddle River, NJ, 2001.

Nelson S.L., *Excel Data Analysis for Dummies*, Hungry Minds, Inc., New York, 2002.

Vazsonyi A., Weida N. and Richardson R., *Quantitative Management Using Microsoft Excel*, Duxbury Press, Boston, MA, 1999.

Whigham D., *Quantitative Business Methods using Excel*, Oxford University Press, Oxford, 1998.

Winston W., *Data Analysis and Business Modelling with Excel*, Microsoft Press, Redmond, WA, 2004.

Useful websites

The general website to accompany this book is at **www.pearsoned.co.uk/waters**. This also includes a list of other useful websites.

You can find details of software from suppliers' sites, such as www.microsoft.com. There is a huge amount of information on the Web, and it is best to start with a search engine, such as those available at www.altavista.com, www.excite.com, www.google.com, www.infoseek.com, www.lycos.com, www.webcrawler.com and www.yahoo.com.

CHAPTER 2

Quantitative tools

Contents

Chapter outline

In the following chapters we describe some quantitative methods that are widely used by managers. Before looking at these in detail, you have to understand some basic tools of arithmetic. You have probably met most of these before, and can move through the chapter fairly quickly. But you might find some new material and want to spend more time on it.

All this material is used in the following chapters, so it is important that you understand it. If you have any difficulties, it is worth spending the time to sort them out. If you want more information, you might find some of the further reading at the end of the chapter useful.

After finishing this chapter you should be able to:

- Understand the basic operations of arithmetic
- Work with integers, fractions, decimals and percentages
- Round numbers to decimal places and significant figures
- Understand the principles of algebra
- Solve an equation to find the value of a constant or variable
- Appreciate the use of inequalities
- Work with powers and roots
- Describe numbers in scientific notation
- Use logarithms.

Working with numbers

The following chapters describe some widely used quantitative analyses. All of these use the basic tools of arithmetic – so our main assumption is that you are familiar with numbers and arithmetic. You can see that:

- If you buy 10 loaves of bread costing 92 pence a loaf, the bill is £9.20.
- If you drive a car at 80 kilometres an hour, it will take 5 hours to travel 400 kilometres.
- If you spend €500 a month on housing, €200 a month on food and entertainment, and €300 a month on other things, your total expenditure is €1,000 a month, which is the same as €12,000 a year or €230.77 a week.
- If a company has a gross income of $2 million a year and costs of $1.6 million a year, it makes a profit of $400,000 a year.

Arithmetic

There are four basic operations in **arithmetic**: addition, subtraction, multiplication and division. We describe these with the notation:

- $+$ addition e.g. $2 + 7 = 9$
- $-$ subtraction e.g. $15 - 7 = 8$
- $\times$ multiplication e.g. $4 \times 5 = 20$
- $/$ division e.g. $12 / 4 = 3$

There are variations on this notation, and you can also see:

- division written as $12 \div 4 = 3$ or $\frac{12}{4} = 3$
- multiplication written as $4 \cdot 5 = 20$ or $4(5) = 20$.

Calculations are always done in the same order, with multiplication and division done before addition and subtraction. If you see $3 + 4 \times 5$, you do the multiplication first to give $3 + 20 = 23$. Whenever there is any doubt about the order of arithmetic, you can put brackets around the parts of the calculation that are done together. Then:

$$(3 + 4) \times 5 = 7 \times 5 = 35$$

while

$$3 + (4 \times 5) = 3 + 20 = 23$$

Calculations in brackets are always done first, so the general order of calculation is as follows:

1. Calculations inside brackets, starting from the inside set and working outwards
2. Raising to powers (which we mention later in the chapter)
3. Multiplication and division in the order they appear
4. Addition and subtraction in the order they appear

Brackets can change the order of calculation, so that:

$$12 \times 2 + 4 + 2 = 24 + 4 + 2 = 30$$

while

$$12 \times (2 + 4 + 2) = 12 \times 8 = 96$$

and

$$12 \times (2 + 4) + 4 = 12 \times 6 + 4 = 72 + 4 = 76$$

If one set of brackets is not enough, you can 'nest' more sets inside others. Then you start calculations with the inside set of brackets and work outwards. So

$$((32 / 2) + (6 / 3)) - 1 = (16 + 2) - 1 = 18 - 1 = 17$$

while

$$(32 / (2 + 6)) / (3 - 1) = (32 / 8) / 2 = 4 / 2 = 2$$

Calculations with a lot of brackets look messy, but you should have no problems if you always take them in the standard order.

Numbers are either **positive** when they are above zero, or **negative** when they are below zero. So +10 is positive (and the positive sign is usually implicit, so we write this as 10), and –10 is negative. You should remember three things about negative numbers. Firstly, adding a negative number is the same as subtracting a positive number, so:

$$8 + (-5) = 8 - (+5) = 8 - 5 = 3$$

Secondly, when you multiply or divide a positive number by a negative number, the result is negative:

$$4 \times (-2) = -8 \quad \text{and} \quad 15 / (-5) = -3$$

Thirdly, when you multiply or divide two negative numbers, the result is positive:

$$(-4) \times (-2) = 8 \quad \text{and} \quad (-15) / (-5) = 3$$

WORKED EXAMPLE 2.1

What are the values of:

(a) $(10 + 20) - (3 \times 7)$?
(b) $((-2 \times 4) \times (15 - 17)) \times (-3)$?
(c) $(20 - 5) \times (30 / (2 + 1))$?

Solution

(a) $(10 + 20) - (3 \times 7) = 30 - 21 = 9$
(b) $((-2 \times 4) \times (15 - 17)) \times (-3) = ((-8) \times (-2)) \times (-3) = 16 \times (-3) = -48$
(c) $(20 - 5) \times (30 / (2 + 1)) = 15 \times (30 / 3) = 15 \times 10 = 150$

Fractions and decimals

The numbers in worked example 2.1 are **integers**, which means that they are whole numbers, such as 20, 9 or 150. To make long numbers easier to read, we usually divide them into groups of three digits separated by commas, such as 1,234,567. Some people prefer spaces to commas, so you also see 1 234 567.

When we divide integers into smaller parts, we get **fractions**. For example, when two people share a bar of chocolate they get a half each. We can describe fractions as either:

- **common fractions** – such as $\frac{1}{2}$ or $\frac{1}{4}$, or
- **decimal fractions** – such as 0.5 or 0.25.

Decimal fractions are more widely used, but common fractions – which are invariably abbreviated to 'fractions' – can save a lot of effort. The top line of a fraction is the **numerator**, while the bottom line is the **denominator**.

$$\text{fraction} = \frac{\text{numerator}}{\text{denominator}}$$

If you multiply or divide both the numerator and the denominator by the same amount, the fraction keeps the same value. So 5/10 is the same as 1/2 (dividing both the numerator and the denominator by 5) or 20/40 (multiplying both by 4).

To change fractions into decimal fractions, you divide the numerator by the denominator, so that $\frac{1}{4}$ is 1 divided by 4, which is 0.25. To change a decimal fraction with one digit to a fraction, you put the number after the decimal point over 10, so 0.6 = 6/10 = 3/5. If there are two digits after the decimal point you put them over 100, if there are three digits you put them over 1000, and so on.

WORKED EXAMPLE 2.2

Describe as decimal fractions: (a) 5/4, (b) 38/8, (c) –12/16.
Describe as common fractions: (d) 0.4, (e) 0.75, (f) 0.125.

Solution

Using long division, you can see that:

(a) 5/4 = 1.25
(b) 38/8 = 19/4 = 4.75
(c) –12/16 = –3/4 = –0.75

Expanding the decimal fraction gives:

(d) 0.4 = 4/10 = 2/5
(e) 0.75 = 75/100 = 3/4
(f) 0.125 = 125/1000 = 1/8

To multiply fractions together, you multiply all the numerators together to give the new numerator, and you multiply all the denominators together to give the new denominator.

WORKED EXAMPLE 2.3

Find the values of: (a) $1/2 \times 1/5$, (b) $1/4 \times 2/3$, (c) $-1/4 \times 2/3 \times 1/2$.

Solution

Multiplying the numerators together and the denominators together gives:

(a) $1/2 \times 1/5 = (1 \times 1) / (2 \times 5) = 1/10$

(b) $1/4 \times 2/3 = (1 \times 2) / (4 \times 3) = 2/12 = 1/6$

(c) $-1/4 \times 2/3 \times 1/2 = (-1 \times 2 \times 1) / (4 \times 3 \times 2) = -2/24 = -1/12$

To divide one fraction by another, you invert the fraction that is dividing and then multiply the two together. (This might seem rather strange until you work out what is actually happening.)

WORKED EXAMPLE 2.4

Find the values of: (a) $(3/5) \div (4/5)$, (b) $3/6 \times 2/5 \div 3/7$, (c) $2/5 \div 16/4$.

Solution

Inverting the dividing fraction and then multiplying gives:

(a) $(3/5) \div (4/5) = 3/5 \times 5/4 = (3 \times 5) / (5 \times 4) = 15/20 = 3/4$

(b) $3/6 \times 2/5 \div 3/7 = 3/6 \times 2/5 \times 7/3 = (3 \times 2 \times 7) / (6 \times 5 \times 3) = 42/90 = 7/15$

(c) $2/5 \div 16/4 = 2/5 \times 4/16 = (2 \times 4) / (5 \times 16) = 8/80 = 1/10$

To add or subtract fractions, you have to get all the denominators the same. So you take each fraction in turn, and multiply the top and bottom by the number that gives this common denominator – and then you add or subtract the numerators. This soon becomes very messy, so if you are doing a lot of arithmetic it is easier to work with decimal fractions.

WORKED EXAMPLE 2.5

Find the values of: (a) $1/2 + 1/4$, (b) $1/2 + 4/5$, (c) $3/4 - 1/6$.

Solution

(a) We have to get the same denominator for both fractions. The easiest value is 4, and to get this we multiply the top and bottom of the first fraction by 2. Then:

$$1/2 + 1/4 = 2/4 + 1/4 = (2 + 1)/4 = 3/4 = 0.75$$

(b) This time the easiest denominator for both fractions is 10, which we get by multiplying the top and bottom of the first fraction by 5 and the second fraction by 2. Then:

$$1/2 + 4/5 = 5/10 + 8/10 = 13/10 = 1.3$$

(c) Here the easiest denominator is 12, which we get by multiplying the top and bottom of the first fraction by 3 and the second fraction by 2, giving:

$$3/4 - 1/6 = 9/12 - 2/12 = 7/12 = 0.583$$

WORKED EXAMPLE 2.6

A Canadian visitor to Britain wants to change \$350 into pounds. The exchange rate is \$2.28 to the pound and the bank charges a fee of £10 for the conversion. How many pounds does the visitor get?

Solution

\$350 is equivalent to 350/2.28 = £153.51. Then the bank takes its fee of £10 to give the visitor £143.51.

Percentages give another way of describing fractions. These are fractions where the bottom line is 100, and the '/100' has been replaced by the abbreviation '%'. If you hear that '60% of the electorate voted in the last election', you know that 60/100 or 60 people out of each 100 voted. We can represent this as any of the following:

- common fraction: 60/100 = 3/5
- decimal fraction: 0.6
- percentage: 60%.

To describe one figure as a percentage of a second, you divide the first figure by the second and multiply by 100. So to describe 15 as a percentage of 20, you calculate 15/20 × 100 = 75%. To find a given percentage of a number, you multiply the number by the percentage and divide by 100. So to find 45% of 80, you calculate 80 × 45/100 = 36.

WORKED EXAMPLE 2.7

Find: (a) 17/20 as a percentage, (b) 80% as a fraction, (c) 35% as a decimal fraction, (d) 40% of 120, (e) 36 as a percentage of 80.

Solution

(a) Multiplying the top and bottom of the fraction by 5 shows that 17/20 = 85/100 = 85%.
(b) 80% = 80/100 = 4/5
(c) 35% = 35/100 = 0.35
(d) 40% of 120 = 120 × 40 / 100 = 4800 / 100 = 48
(e) 36/80 × 100 = 45%

WORKED EXAMPLE 2.8

If you multiply 20% of 50 by 1/4 of 60 and divide the result by 0.25 of 80, what answer do you get?

Solution

Doing this in stages:

20% of 50 = 50 × 20 / 100 = 10

1/4 of 60 = 1/4 × 60 = 15

0.25 of 80 = 0.25 × 80 = 20

Then the calculation is:

(10 × 15) / 20 = 150/20 = 7.5

Rounding numbers

If you calculate 4/3 as a decimal fraction, the answer is 1.333333333 . . . where the dots represent an unending row of 3s. For convenience, we **round** such numbers to one of the following:

- A certain number of **decimal places**, showing only a reasonable number of digits after the decimal point. Rounding to two decimal places gives 1.33.
- A certain number of **significant figures**, showing only the most important digits to the left. Rounding to four significant figures gives 1.333.

By convention, when rounding to, say, two decimal places and the digit in the third decimal place is 0, 1, 2, 3 or 4, we round the result *down*; when the digit in the third decimal place is 5, 6, 7, 8 or 9, we round the result *up*. Then 1.64 becomes 1.6 to one decimal place, while 1.65 becomes 1.7; similarly, 12.344 becomes 12.34 to four significant figures, while 12.346 becomes 12.35.

The purpose of rounding is to give enough information, but not to overwhelm us with too much detail. There is no rule for choosing the number of decimal places or significant figures, except the rather vague advice to use the number that best suits your purpose. When people ask how tall you are, you probably give an answer to the nearest centimetre or inch; when people talk about house prices, they generally round to the nearest thousand pounds; when governments talk about populations, they generally round to the nearest million. We can make two other suggestions for rounding:

- Give only the number of decimal places or significant figures that is useful. For instance, it never makes sense to give a figure like £952.347826596 and we generally round this to £952.35 – or £952, £950 or £1,000, depending on the circumstances.
- Results from calculation are only as accurate as the data used to get them. If you multiply a demand of 32.63 units by a unit cost of €17.19, you can quote the total cost to only two decimal places. So you should not describe the result as 32.63 × 17.19 = €560.9097, but you should describe it as €560.91 (or €561, €560 or €600, again depending on the circumstances).

WORKED EXAMPLE 2.9

What is 1374.3414812 to (a) four decimal places, (b) two decimal places, (c) four significant figures, (d) two significant figures?

Solution

(a) 1374.3415 when rounded to four decimal places
(b) 1374.34 to two decimal places
(c) 1374 to four significant figures
(d) 1400 to two significant figures

IDEAS IN PRACTICE T.D. Hughes Ltd

T.D. Hughes Ltd are retailers of high quality furniture. They sell a particular dining room table for £800. Last summer they had some difficulty with supplies and raised the price of the table by 20%. When supplies returned to normal, they reduced the higher price by 20% and advertised this as part of their January sale.

Some customers were not happy with this deal, saying that the company had only returned the table to its original price. But company managers pointed out that increasing the original price by 20% raised it to 120% of £800, which is (120/100) × 800 = £960. Then reducing the higher price by 20% took it down to 80% of £960 which is (80/100) × 960 = £768. There was a genuine reduction of £32 or a proportion of 32/800 = 4%. If they had wanted to return the table to its original price, they would have reduced it by the proportion 160/960, which equals 16.67/100 or 16.67%.

Review questions

2.1 Why should you do any calculations by hand, when computers can do them more easily and accurately?

2.2 What is the value of: (a) (–12) / (–3), (b) (24/5) ÷ (3/7), (c) ((2 – 4) × (3 – 6)) / (7 – 5)?

2.3 What is the difference between 75%, 3/4, 15/20 and 0.75? Which is the best format?

2.4 What is 1,745,800.36237 rounded to three decimal places and to three significant figures?

Changing numbers to letters

Suppose you want to see how the cost of running a car depends on the distance you travel. In one month you might find that you drive 6,000 km at a total cost of £2,400. You can find the cost per kilometre by dividing the cost by the number of kilometres travelled, giving 2,400 / 6,000 = £0.40 per kilometre. In general, this calculation is:

$$\text{cost per kilometre} = \frac{\text{total cost}}{\text{number of kilometres travelled}}$$

Rather than writing the equation in full, you can save time by using some abbreviations. You can abbreviate the total cost to T, which is simple and easy to remember. And you can abbreviate the cost per kilometre to C and the number of kilometres travelled to K. Putting these abbreviations into the general equation gives:

$$C = \frac{T}{K} = T/K$$

Now you have an **equation** relating the variables, in a general, concise and accurate form. The only difference from the original equation is that you have used letters to represent numbers or quantities. This is the basis of **algebra** – which uses symbols to represent variables and to describe the relationships between them.

In practice, the cost per kilometre of running a car is fixed and a driver cannot change it – so C is constant. The number of kilometres travelled, K, and the total cost, T, can both vary. So equations describe the relationships between:

- **constants** – which have fixed values, and
- **variables** – which can take different values.

We chose the abbreviations C, T and K to remind us of what they stand for. We could have chosen any other names, perhaps giving:

$$c = t/k$$

$$y = x/z$$

COST = TOTAL / KILOM

COSTPERKM = TOTALCOST / KILOMETRES

Provided the meaning is clear, the names are not important. But if you want to save time, it makes sense to use short names, such as a, x and N. The only thing you have to be careful about is the assumption in algebra that adjacent variables are multiplied together – so $a \times b$ is written as ab, $4 \times a \times b \times c$ is written as $4abc$, $a \times (b + c)$ is written as $a(b + c)$, and $(a + b) \times (x - y)$ is written as $(a + b)(x - y)$. This causes no problems with single-letter abbreviations but can be misleading with longer names. If you abbreviate the total unit cost to TUC, it makes no sense to write an equation:

TUC = NTUC

when you really mean

TUC = $N \times T \times$ UC

With algebra we are only replacing specific numbers by general names, so all aspects of the calculations remain the same. In particular, there are still the four basic operations (addition, subtraction, multiplication and division) and they are done in the same order (things inside brackets, raising to powers, multiplication and division, and then addition and subtraction).

WORKED EXAMPLE 2.10

How would you calculate: (a) $w(x + y)$, (b) $p - (s - 2t)$, (c) $(a + b)(c + d)$?

Solution

(a) Here we have to multiply everything inside the brackets by w, so:

$$w(x + y) = wx + wy$$

You can always check your results by substituting test values, such as $w = 2$, $x = 3$ and $y = 4$. Then:

$$w(x + y) = 2(3 + 4) = 2 \times 7 = 14$$

$$wx + wy = (2 \times 3) + (2 \times 4) = 6 + 8 = 14$$

(b) The minus sign before the brackets means that we effectively have to multiply everything inside the brackets by –1, giving:

$$p - (s - 2t) = p + (-1) \times (s - 2t) = p - s + 2t$$

(c) Here the two expressions inside brackets are multiplied together, so we have to multiply everything inside the first bracket by

Worked example 2.10 continued

everything inside the second bracket. The easiest way to arrange this is:

$$\begin{aligned}(a + b)(c + d) &= (a + b) \times (c + d)\\ &= a \times (c + d) + b \times (c + d)\\ &= (ac + ad) + (bc + bd)\\ &= ac + ad + bc + bd\end{aligned}$$

Remember that you can always check the results of your algebra by substituting trial values.

WORKED EXAMPLE 2.11

What is the equation for the percentage of people who voted in an election?

Solution

Here you have to take the number of people who actually voted, divide this by the number of people who could have voted, and then multiply the result by 100. This is rather long-winded, so we can use some abbreviations, starting by defining:

v = the number of people who actually voted
n = the number of people who could have voted
p = the percentage of people who actually voted

Then the calculation is:

$$p = v/n \times 100$$

Solving equations

An equation shows the relationship between a set of constants and variables – saying that the value of one expression equals the value of a second expression. We can use an equation to find the value of a previously unknown constant or variable. This is called **solving an equation**.

To solve an equation, you arrange it so that the unknown value is on one side of the equals sign, and all known values are on the other side. Here you have to remember that an equation remains true when you do the same thing to both sides – so you can add a number to both sides, or multiply both sides by a constant, and the equation still remains true. Returning to the equation for the cost of running a car, $C = T/K$, we can multiply both sides of the equation by K and get:

$$C \times K = \frac{T}{K} \times K$$

Here the right-hand side has K/K. When you divide anything by itself you get 1, so we can cancel – or delete – the K/K to give $T = C \times K$. Dividing both sides of this new equation by C gives:

$$\frac{T}{C} = \frac{C \times K}{C} \quad \text{or} \quad K = \frac{T}{C}$$

All three forms are simply rearrangements of the first equation.

Suppose you know that $a + b = c + d$, when $a = 2$, $b = 6$ and $c = 3$. You can solve the equation to find the unknown value of d. For this you rearrange the equation to get d on one side of the equals sign and all the other variables on the other side (in effect subtracting c from both sides):

$$d = a + b - c$$

and substituting the known values gives:

$$d = 2 + 6 - 3 = 5$$

WORKED EXAMPLE 2.12

Rearrange the following equation to find the value of y when $a = 2$, $b = 3$ and $c = 4$:

$$\frac{(2a - 7)}{6y} = \frac{(3b - 5)}{2c}$$

Solution

The unknown variable is y, so we have to rearrange the equation to put y on one side and all the known values on the other side. We can start by multiplying both sides of the equation by $(6y \times 2c)$ to get everything on one line:

$$\frac{(2a - 7)(6y \times 2c)}{6y} = \frac{(3b - 5)(6y \times 2c)}{2c}$$

Cancelling the $6y$ from the left-hand side and the $2c$ from the right-hand side gives:

$$(2a - 7) \times 2c = (3b - 5) \times 6y$$

or

$$2c(2a - 7) = 6y(3b - 5)$$

Then we can separate out the y by dividing both sides by $6(3b - 5)$:

$$y = \frac{2c(2a - 7)}{6(3b - 5)}$$

Substituting $a = 2$, $b = 3$ and $c = 4$:

$$y = \frac{(2 \times 4)(2 \times 2 - 7)}{6(3 \times 3 - 5)} = \frac{8 \times (-3)}{24} = -1$$

Notice that you can find only *one* unknown value from a single equation. So if an equation has two unknowns, say, $x + y = 10$, you cannot find values for both x and y. If you have several unknowns, you need several equations to find all of them (which we discuss in Chapter 11). And remember that an equation works only if the units are consistent, so you have to be careful to use the same units (say hours, tonnes and dollars) in all parts of the equation.

WORKED EXAMPLE 2.13

Last October the Porth Merrion Hotel paid £1,800 for heat and power, with heat costing £300 more than double the cost of power. How much did power cost?

Solution

If we call the cost of power in the month P, then the cost of heat is $2P + 300$. The hotel's total cost is P for power plus $2P + 300$ for heat, and we know that this came to £1,800, so:

$$P + 2P + 300 = 3P + 300 = 1{,}800$$

Subtracting 300 from both sides gives:

$$3P = 1{,}500$$

or

$$P = £500$$

Power cost £500 and heat cost $2P + 300 = 2 \times 500 + 300 = £1{,}300$.

This last example shows that the steps in solving an equation are as follows:

1. Define the relevant constants and variables.
2. Develop an equation to describe the relationship between them – that is, build a model of the problem.
3. Rearrange the equation separating the unknown value from the known values.
4. Substitute the known values to solve the equation.

WORKED EXAMPLE 2.14

Sempervig Securitas employs 10 people with costs of €500,000 a year. This cost includes fixed overheads of €100,000, and a variable cost for each person employed. What is the variable cost? What is the total cost if Sempervig expands to employ 20 people?

Solution

If we let t = total cost per year, o = annual overheads, v = variable cost per employee, and n = number of people employed, then the total cost is:

$$\text{total cost} = \text{overheads} + (\text{variable cost} \times \text{number employed})$$

or

$$t = o + vn$$

We know the values for t, o and n, and we want to find v. We can rearrange the equation by subtracting o from both sides to give $t - o = nv$, and then dividing both sides by n gives:

$$v = \frac{t - o}{n}$$

Substituting the known values:

$$v = \frac{500{,}000 - 100{,}000}{10}$$
$$= €40{,}000 \text{ a year for each employee}$$

If the company expands, we can find the new total cost, t, by substituting known values for v, o and n in the original equation:

$$t = o + vn = 100{,}000 + 40{,}000 \times 20$$
$$= €900{,}000 \text{ a year}$$

WORKED EXAMPLE 2.15

1,200 parts arrived from a manufacturer in two batches. Each unit of the first batch cost \$37, while each unit of the second batch cost \$35. If the total cost is \$43,600, how many units were in each batch?

Solution

If we let f be the number of units in the first batch, the number of units in the second batch is $(1{,}200 - f)$ and the total cost is:

$$37f + 35(1{,}200 - f) = 43{,}600$$

So

$$37f + 42{,}000 - 35f = 43{,}600$$
$$37f - 35f = 43{,}600 - 42{,}000$$
$$2f = 1600$$
$$f = 800$$

So the first batch had 800 units and the second batch had $1{,}200 - 800 = 400$ units.

Inequalities

Sometimes we do not have enough information to write an equation, but we can still describe some sort of relationship. For example, we might not know the rate of inflation exactly, but we know that it is less than 4%. We can describe this as an **inequality**. There are five types of inequality, which we write as:

$a < b$ means that a is less than b

$a \le b$ means that a is less than or equal to b

$a > b$ means that a is greater than b

$a \ge b$ means that a is greater than or equal to b

$a \ne b$ means that a is not equal to b.

Then we can write inflation < 4%, to show that inflation is less than 4%; profit > 0 to show that profit is always positive; and 1,000 ≤ cost ≤ 2,000 shows that the cost is between 1,000 and 2,000.

As with equations, inequalities remain valid if you do exactly the same thing to both sides. So you can multiply, divide, add and subtract anything, provided you do it to both sides. If you take a basic inequality $x \ge y$, add 20 to both sides, multiply the result by 2, and divide by a, you still get valid inequalities:

$$x + 20 \ge y + 20$$
$$2x + 40 \ge 2y + 40$$
$$(2x + 40)/a \ge (2y + 40)/a$$

But there is one exception to this rule: when you multiply or divide both sides by a negative number, you have to change the direction of the inequality. We can demonstrate this with the obvious statement that 3 > 2. If you multiply both sides by −1, you get the false statement that −3 > −2, but changing the direction of the inequality gives the correct version of −3 < −2.

WORKED EXAMPLE 2.16

A department has an annual budget of \$200,000, which it divides between capital expenditure (C), running costs (R) and overheads (O). How can you describe its expenditure?

Solution

Total expenditure is $C + R + O$ and this must be less than or equal to the budget, so:

$$C + R + O \le 200{,}000$$

WORKED EXAMPLE 2.17

What can you say if (a) $6x - 4 \ge 4x + 3$, (b) $y / (-3) > 4$?

Solution

(a) Rearranging this in the usual way gives:

$$6x - 4 \ge 4x + 3 \quad \text{or} \quad 6x - 4x \ge 3 + 4$$

so

$$2x \ge 7 \quad \text{or} \quad x \ge 3.5$$

(b) We have to be a little more careful here. We can multiply both sides by −3, but must remember to change the direction of the sign:

$$y / (-3) > 4 \quad \text{means that} \quad y < 4 \times (-3)$$

or

$$y < -12$$

Review questions

2.5 Why do managers use algebra?

2.6 Is the order of doing calculations always the same?

2.7 Can you solve an equation of the form $y = 4x + 3$, where both x and y are unknown?

2.8 What is the difference between a constant and a variable?

2.9 Is it better to write an equation in the form (a) speed = distance/time, (b) SPD = DST/TME, or (c) $S = D/T$?

2.10 If you know values for p, q and r, how could you solve the following equations?
(a) $4r/(33 - 3x) = q/2p$
(b) $(q - 4x)/2q - 7p/r = 0$

2.11 What is an inequality?

Powers and roots

When you multiply a number by itself one or more times, the convention is to use a superscript – or **power** – to show how many times you have done the multiplication. When you multiply a variable b by itself you get b^2, which is described as 'b to the power 2' or 'b squared'. When you multiply b by itself three times you get b^3, which is described as 'b to the power 3' or 'b cubed'. Then:

b to the power 1 $= b = b^1$
b squared $= b \times b = b^2$
b cubed $= b \times b \times b = b^3$
b to the fourth $= b \times b \times b \times b = b^4$

and in general

b to the power $n = b \times b \times b \times \ldots$ (n times) $= b^n$

Taking a specific value for b, say 3, we have:

3 to the power 1 $= 3^1 = 3$
3 squared $= 3 \times 3 = 3^2 = 9$
3 cubed $= 3 \times 3 \times 3 = 3^3 = 27$
3 to the fourth $= 3 \times 3 \times 3 \times 3 = 3^4 = 81$

and in general

3 to the power $n = 3 \times 3 \times 3 \times \ldots$ (n times) $= 3^n$

You can find the result of multiplying two powers together, such as $b^2 \times b^3$, by writing the calculation in full:

$$b^2 \times b^3 = (b \times b) \times (b \times b \times b) = b \times b \times b \times b \times b = b^5$$

This illustrates the general rule that:

> When multiplying, add the powers:
>
> $$b^m \times b^n = b^{m+n}$$

For example, $4^2 \times 4^4 = 4^6$, which you can confirm by expanding the calculation to $4^2 \times 4^4 = 16 \times 256 = 4{,}096 = 4^6$. In passing we should mention two common errors, and emphasise that:

- $b^m + b^n$ does *not* equal b^{m+n} ✗
- $a^n + b^n$ does *not* equal $(a + b)^n$ ✗

You can check this by substituting, say, $a = 4$, $b = 3$, $m = 2$ and $n = 1$.

To do a division, such as b^5/b^2, you can again find the result by writing the calculation in full:

$$\frac{b^5}{b^2} = \frac{b \times b \times b \times b \times b}{b \times b} = b \times b \times b = b^3$$

This illustrates the general rule that:

> When dividing, subtract the powers:
>
> $$\frac{b^m}{b^n} = b^{m-n}$$

For example, $5^4/5^3 = 5^{4-3} = 5^1$, which you can confirm by expanding the calculation to $5^4/5^3 = 625/125 = 5 = 5^1$.

An interesting result comes when $m = n$. For example, when $m = n = 3$, then:

$$\frac{b^3}{b^3} = \frac{b \times b \times b}{b \times b \times b} = 1$$

but:

$$\frac{b^3}{b^3} = b^{3-3} = b^0$$

So $b^0 = 1$. This is a general rule that anything raised to the power 0 equals 1.

WORKED EXAMPLE 2.18

What are the values of (a) $b^4 \times b^2$, (b) $b^6 \div b^2$, (c) $b^6 - b^3$, (d) $2^3 \times 2^2$, (e) $3^4 \div 3^2$, (f) $x(1 + x)$, (g) $(b^m)^n$?

Solution

Using the rules above gives:

(a) $b^4 \times b^2 = b^{4+2} = b^6$

(b) $b^6 \div b^2 = b^{6-2} = b^4$

(c) Trick question! You cannot simplify $b^6 - b^3$ (and it is *not* b^{6-3}, which would be b^6/b^3).

(d) $2^3 \times 2^2 = 2^{3+2} = 2^5 = 32$

(e) $3^4 \div 3^2 = 3^{4-2} = 3^2 = 9$

(f) We have to multiply the first x by everything inside the brackets, giving:

$$x(1 + x) = (x \times 1) + (x \times x) = x + x^2$$

(h) To raise b^m to the power n, we multiply b^m by itself n times. But b^m is b multiplied by itself m times, so now we are multiplying b by itself mn times. This gives the general rule that $(b^m)^n = b^{mn}$.

WORKED EXAMPLE 2.19

Expand the expressions (a) $(1 + b)^2$, (b) $(y - 1)(y + 4)$, (c) $(1 + a)(1 - a)$.

Solution

(a) The square applies to the whole bracket, so:

$$(1 + b)^2 = (1 + b) \times (1 + b)$$

which we can expand to give:

$$1(1 + b) + b(1 + b) = (1 + b) + (b + b^2)$$
$$= 1 + 2b + b^2$$

(b) Now we have to multiply everything inside the first brackets by everything inside the second brackets, giving:

$$(y - 1)(y + 4) = y(y + 4) - 1(y + 4)$$
$$= y^2 + 4y - y - 4 = y^2 + 3y - 4$$

(c) Again we multiply everything inside the first brackets by everything inside the second brackets:

$$(1 + a)(1 - a) = 1(1 - a) + a(1 - a)$$
$$= 1 - a + a - a^2 = 1 - a^2$$

Negative and fractional powers

The rule of division says that $b^m / b^n = b^{m-n}$, but what happens when n is bigger than m? If, say, $n = 4$ and $m = 2$, we get $b^m / b^n = b^2 / b^4 = b^{2-4} = b^{-2}$ and we have to interpret a negative power. To do this we can expand the calculation:

$$\frac{b^2}{b^4} = \frac{b \times b}{b \times b \times b \times b} = \frac{1}{b^2}$$

So $b^{-2} = 1/b^2$, which illustrates the general rule:

$$b^{-n} = \frac{1}{b^n}$$

One final point about raising to powers concerns fractional powers, such as $b^{1/2}$. You can see how to interpret this from the work that we have already done. If you square $b^{1/2}$ you get:

$$b^{1/2} \times b^{1/2} = b^{1/2+1/2} = b^1 = b$$

When you multiply $b^{1/2}$ by itself you get b – but by definition, the number that you multiply by itself to give b is the square root of b. So $b^{1/2}$ must be the square root of b, which we write as $\sqrt{b}$. Now we have:

$$b^{0.5} = b^{1/2} = \sqrt{b}$$

In the same way we can show that:

- $b^{0.33} = b^{1/3}$ which is the cube root of b (the number that gives b when multiplied by itself three times)
- $b^{0.25} = b^{1/4}$ which is the fourth root of b (the number that gives b when multiplied by itself four times)
- and so on.

We can extend this reasoning to more complex fractional powers. For example:

$$b^{1.5} = b^{3/2} = (b^{1/2})^3 = (\sqrt{b})^3$$
$$b^{2.5} = b^{5/2} = (b^{1/2})^5 = (\sqrt{b})^5$$

WORKED EXAMPLE 2.20

What are the values of (a) $1/b^4$, (b) $b^5 \times b^{1/2}$, (c) $25^{1/2}$, (d) $9^{1.5}$, (e) $8^{0.67}$?

Solution

Using the standard rules:

(a) $1/b^4 = b^{-4}$

(b) $b^5 \times b^{1/2} = b^{5+1/2} = b^{5.5} = b^{11/2} = (b^{1/2})^{11} = (\sqrt{b})^{11}$

(c) $25^{1/2} = \sqrt{25} = 5$

(d) $9^{1.5} = 9^{3/2} = (9^{1/2})^3 = (\sqrt{9})^3 = 3^3 = 27$

(e) $8^{0.67} = 8^{2/3} = (8^{1/3})^2 = 2^2 = 4$

WORKED EXAMPLE 2.21

If you leave £100 in the bank earning 6% interest, at the end of n years you will have £100 × 1.06^n (we discuss this in Chapter 8). How much will you have at the end of each of the next 10 years?

Solution

We could do these calculations separately:

- at the end of the first year you have 100×1.06^1 = £106,
- at the end of the second year you have $100 \times 1.06^2 = 100 \times 1.1236$ = £112.36,

and so on. However, it is much easier to use a spreadsheet for this kind of repeated calculation. Figure 2.1(a) shows the results for this example, while Figure 2.1(b) shows the formulae used in the calculations. Notice that calculations in spreadsheets do not use superscripts, so you have to use the symbol ^ to raise something to a power. Then

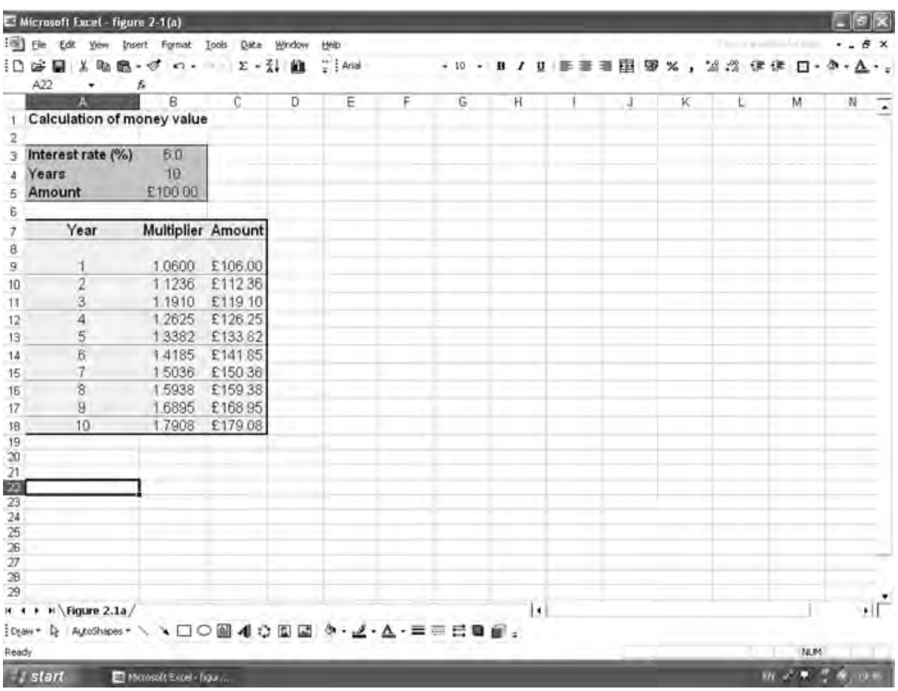


Figure 2.1(a) Spreadsheet giving the results for worked example 2.21

Worked example 2.21 continued

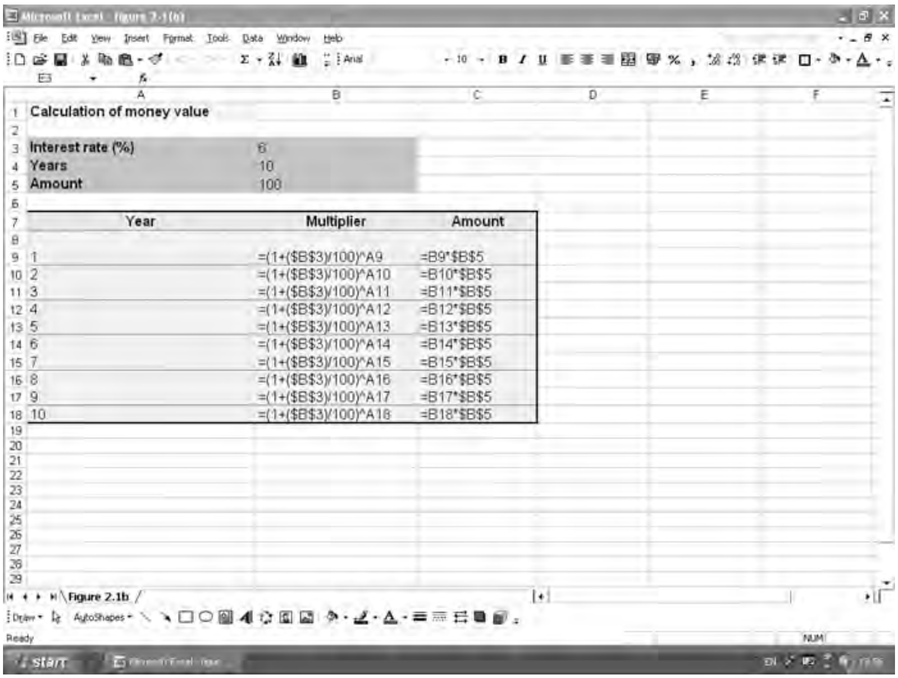


Figure 2.1(b) Formulae used for the calculations in worked example 2.21

a^b becomes a^b, 3^2 becomes 3^2, and 5 raised to the power –2 becomes 5^–2. (Alternatively, you can use the standard function POWER(a,b) which finds a^b.)

The calculation in, say, cell B10 finds 1.06^2. For this it takes the interest rate in cell B3, divides it by 100 and adds 1 to get 1.06; raising this to the power in cell A10 gives 1.06^2 and multiplying this by the £100 in cell B5 gives the value at the end of the second year in cell C10.

Scientific notation

We can use powers to give a convenient notation for very large and very small numbers. This **scientific notation** describes any number in the format:

$$a \times 10^b$$

where

a is a number between 1 and 10 (or –1 and –10 for negative numbers)
b is a power of 10.

This notation uses the fact that $10^1 = 10$, $10^2 = 100$, $10^3 = 1000$, and so on. Then we can describe a number like 120 as 1.2×100 which is 1.2×10^2. Similarly:

- $12 = 1.2 \times 10 = 1.2 \times 10^1$
- $1{,}200 = 1.2 \times 1{,}000 = 1.2 \times 10^3$

- $12{,}000 = 1.2 \times 10{,}000 = 1.2 \times 10^4$
- $1{,}380{,}197.892$ is about 1.38×10^6
- The UK's annual exports are about £1.5×10^{11}.

You find b, the power of 10, by counting the number of places to the left that the decimal point has to move. For example, with 15,762 the decimal point moves four places to the left, so 10 is raised to the power 4, giving 1.5762×10^4.

We can use the same notation to describe very small numbers, using the fact that $10^{-1} = 0.1$, $10^{-2} = 0.01$, $10^{-3} = 0.001$, and so on. Then we can take a number like 0.0012 and describe it as 1.2×0.001, which is 1.2×10^{-3}. Similarly:

- $0.12 = 1.2 \times 0.1 = 1.2 \times 10^{-1}$
- $0.012 = 1.2 \times 0.01 = 1.2 \times 10^{-2}$
- $0.000012 = 1.2 \times 0.00001 = 1.2 \times 10^{-5}$
- $0.00000429 = 4.29 \times 0.000001 = 4.29 \times 10^{-6}$.

Here you find the power of 10 by counting the number of places to the *right* that the decimal point has to move. For example, with 0.0057 the decimal point moves three places to the right, so the 10 is raised to the power -3, giving 5.7×10^{-3}.

WORKED EXAMPLE 2.22

Use scientific notation for (a) 123,000, (b) two million, (c) 0.05, (d) 0.000123.

Solution

The key thing is to count the number of places the decimal point has to move – if it moves to the left the power is positive; if it moves to the right the power is negative. Then:

(a) $123{,}000 = 1.23 \times 10^5$
(b) two million $= 2{,}000{,}000 = 2 \times 10^6$
(c) $0.05 = 5 \times 10^{-2}$
(d) $0.000123 = 1.23 \times 10^{-4}$

Again there is a problem with computers not using superscripts, so for scientific notation they use a slightly different convention. In particular, the '×10' is replaced by the letter E, and this is followed by the power that 10 is raised to. So 2×10^4 becomes 2E+04, 1.23×10^5 appears as 1.23E+05, and 4.56×10^{-4} becomes 4.56E–04.

Logarithms and the exponential function

We will describe one other format for numbers, which uses logarithms. When we write $8 = 2^3$ we are representing one number (8) by a second number (2) raised to a power (3). This is a surprisingly useful format, with the general structure:

$$n = b^p$$

The second number, b, is called the **base**, and the power, p, is called the **logarithm** (usually abbreviated to 'log'). So the logarithm of a number is the power to which you raise the base to give the number. This is a messy statement, but you can see how it works from the following examples.

- $16 = 2^4$, so the logarithm to the base 2 of 16 is 4, which means that you raise the base 2 to the power 4 to give 16. We normally write this as $\log_2 16 = 4$.
- $9 = 3^2$, so the logarithm to the base 3 of 9 is 2, which means that we raise the base 3 to the power 2 to give 9, and we write this as $\log_3 9 = 2$.
- $10^3 = 1{,}000$, so the logarithm to the base 10 of 1,000 is 3, which we write as $\log_{10} 1{,}000 = 3$.

When:

$$n = b^p$$

then:

$$p = \log_b n$$

You probably think this notation is rather obscure, but in the past logarithms were the easiest way of doing multiplication and division. Now we have computers and do not need them for this – but they can still simplify some calculations and solve problems where numbers are raised to unknown powers.

Only two types of logarithm are widely used:

1 **Common logarithms** use the base 10, so that:

$$y = \log x \quad \text{means that} \quad x = 10^y$$

If you do not explicitly put in the base of the logarithm, it is assumed to be 10.

2 **Natural logarithms** use the base **e**, which is the **exponential constant** and equals 2.7182818. This seems a very strange number – which you can calculate from $(1 + 1/n)^n$, where n is a large number. However, it appears surprisingly often when, for example, describing random events or exponential growth. We meet it several times in later chapters, but for now we will simply say that natural logarithms are written as:

$$y = \ln x \quad \text{meaning that} \quad x = e^y$$

Again you do not need to specify the base of the logarithm, as this is understood.

WORKED EXAMPLE 2.23

What are the values of (a) $\log_2 32$, (b) log 1,000, (c) ln 2?

Solution

(a) $\log_2 32$ is the power to which you raise the base 2 to give 32. As $2^5 = 32$, this power is 5 and $\log_2 32 = 5$. You can confirm this using the LOG function in Excel; this has the format LOG(number,base), so LOG(32,2) returns the value 5.

(b) log 1,000 is a common logarithm, and is the power to which you raise 10 to give 1,000. As $10^3 = 1{,}000$, this is 3 and log 1,000 = 3. You can confirm this using the LOG10 function in Excel; this has the format LOG10(number), so LOG10(1000) and LOG(1000,10) both return the value 3.

(c) ln 2 is a natural logarithm, and is the power to which we raise e to give 2. There is no easy way to calculate this, but you can find it from the LN(number) function in Excel; this has the format LN(number), so LN(2) and LOG(2, 2.7182818) both return the value 0.6931, meaning that $e^{0.6931} = 2$.

WORKED EXAMPLE 2.24

If $2^x = 32$, use logarithms to find the value of x.

Solution

To solve this equation we have to rearrange it so that the x appears on one side of the equals sign and everything else appears on the other side. But no matter how you try, you are always left with something raised to the power x. The only way of getting around this is to use logarithms. When $n = b^p$, then $p = \log_b n$, so we can rewrite $2^x = 32$ as $x = \log_2 32$. But in the last worked example we found that $\log_2 32 = 5$, so the result is $x = 5$, which you can check by doing the calculation.

Review questions

2.12 Rank in order of size $4^{1/2}$, 4^{-1}, 4^1, 1^4, $(\frac{1}{2})^{-4}$ and $(\frac{1}{2})^4$.

2.13 What is the value of $x^{1.5} / y^{2.5}$ when $x = 9$ and $y = 4$?

2.14 What is the value of 41.1635^0?

2.15 Write 1,230,000,000 and 0.000000253 in scientific notation.

2.16 What is a logarithm and when would you use one?

IDEAS IN PRACTICE Canada Revenue Agency

Benjamin Franklin said, 'In this world nothing can be said to be certain, except death and taxes'.[1] Governments in most countries use similar calculations to assess income tax, and we can illustrate this with the calculations in Canada. Every Canadian citizen completes an income tax return for each financial year. In principle, the calculations are fairly straightforward, but most people find them both arduous and traumatic. Canada Revenue Agency describes the steps as follows:

1. Calculate total income for the tax year – which includes most income but with some exceptions such as child tax credit, veterans' disability allowance, lottery winnings and welfare payments.
2. Calculate taxable income – by subtracting payments for pension plans and other allowed expenses from the total income.
3. Find the gross federal tax – which is taxable income × tax rate. There are higher tax rates for higher incomes, so this calculation has to include amounts paid in each tax band.
4. Calculate tax credits – which are individual tax credits that give a personal allowance, and further allowances for children, education fees, medical expenses, and other allowed expenses.
5. Find the basic federal tax – by subtracting the tax credits from the gross federal tax and adding any adjustments for foreign earnings.
6. Add federal surtax – when the basic federal tax is above a certain limit, a percentage of the amount over this limit is added as a federal surtax.
7. Add the provincial tax – which is a proportion of the federal tax.
8. Add provincial surtax – when the provincial tax is above a certain limit, a percentage of the amount over this limit is added as a provincial surtax.
9. Subtract tax that has already been paid – usually from pay cheques or advances.
10. This gives the total amount payable or to be refunded.

This is clearly a simplified description, but you can see that – like many aspects of business and life in general – it depends on a lot of calculations. Not surprisingly, there is a thriving business for accountants who help people fill in their returns.

Sources: Websites at www.cra-arc.gc.ca and www.statcan.ca. Federal and Provincial General Tax Guide and Returns, CRA, Ottawa.

CHAPTER REVIEW

This chapter reviewed some of the basic numerical tools that are needed to understand the following chapters.

- All quantitative reasoning is based on numbers. These appear in different forms, including integers, decimals, fractions and percentages.
- Numbers are used in arithmetic, where there are standard rules for raising to powers, multiplication, division, addition and subtraction.
- Algebra gives names to constants and variables, and uses these in equations to give precise and general descriptions of relationships.
- You solve an equation by using known values to find a previously unknown value for a constant or variable. To do this, you rearrange the equation until the unknown value is on one side of the equals sign, and all the known values are on the other side.
- Inequalities give less precise descriptions of relationships, typically of the form $a < b$. You handle these in the same way as equations, but sometimes you have to be careful with the direction of the sign.
- Superscripts show that a value is raised to a particular power – or multiplied by itself this number of times. There are standard rules for manipulating powers.
- Scientific notation describes a number in the format $a \times 10^b$.
- A logarithm is defined as the power to which a base is raised to equal a number. Common logarithms use the base 10, while natural logarithms use the base e.

If you want to learn more about any point mentioned in this chapter, you can look at a more detailed book on mathematics. Many of these are available, with some listed in the further reading at the end of the chapter.

CASE STUDY The Crown and Anchor

Tina Jones runs the Crown and Anchor pub in Middleton, along with her husband and staff of eight. The riverside pub has a core of local customers, but half of its business depends on tourists and passing trade. In recent years, the pub has expanded its sales of food, and this has become an increasingly important source of income. Now Tina wants to make some improvements to the dining room and hopes that the bank will lend her the money. She is sure that the improvements will increase profits and wants to make a good case to her bank manager.

Tina has kept a record of the average number of meals served each day over the past two years, and the daily income from food. Now she wants to do some work on these figures and present them in a convincing way. Of course, her sales depend on a number of factors. Some of these are under her control, such as the menu, quantity of food, quality of cooking and serving, etc. Some are outside her control, such as the trends in eating out, national income, and local unemployment. She wants to include all of these factors in a report to the bank manager.

Question

- **What do you think Tina should put in her report? How can she use the figures that she has collected – or other figures that are publicly available? What other information should she collect?**

Case study continued

	Year 1			Year 2		
	Dinners	Lunches	Income (£)	Dinners	Lunches	Income (£)
January	25	6	180	32	30	441
February	23	6	178	30	25	405
March	24	8	196	31	24	415
April	26	9	216	32	26	440
May	27	9	230	35	30	463
June	42	32	525	45	35	572
July	48	36	605	51	38	590
August	48	37	603	50	45	638
September	35	34	498	38	41	580
October	31	30	451	35	36	579
November	30	31	464	32	35	508
December	37	38	592	48	54	776

PROBLEMS

2.1 What are the values of (a) -12×8, (b) $-35 / (-7)$, (c) $(24 - 8) \times (4 + 5)$, (d) $(18 - 4) / (3 + 9 - 5)$, (e) $(22/11) \times (-18 + 3) / (12/4)$?

2.2 Simplify the common fractions (a) $3/5 + 1/2$, (b) $3/4 \times 1/6$, (c) $3/4 - 1/8$, (d) $-18/5 \div 6/25$, (e) $(3/8 - 1/6) \div 4/7$.

2.3 What are the answers to Problem 2.2 as decimal fractions?

2.4 What is (a) 23/40 as a percentage, (b) 65% as a fraction, (c) 17% as a decimal, (d) 12% of 74, (e) 27 as a percentage of 85?

2.5 What is 1,037/14 to (a) three decimal places, (b) one decimal place, (c) two significant figures, (d) one significant figure?

2.6 In one exam 64 people passed and 23 failed; in a second exam 163 people passed and 43 failed. How could you compare the pass rates?

2.7 A car travels 240 kilometres in three hours. What is its average speed? What is the equation for the average speed of a car on any journey?

2.8 Shopkeepers buy an item from a wholesaler and sell it to customers. How would you build a model to describe their profit?

2.9 Logan Bay School has £1,515 to spend on footballs. Match balls cost £35 each, and practice balls cost £22 each. The school must buy 60 balls each year, so how many of each type should it buy to exactly match the budget?

2.10 Sanderson BhP finds that 30% of its costs are direct labour. Each week raw materials cost €2,000 more than twice this amount, and there is an overhead of 20% of direct labour costs. What are the company's weekly costs?

2.11 Lun Ho Stadium sells 2,200 tickets for a concert. It sells a quarter of them at a 20% discount and a further 10% at a 40% discount. How much must it charge for tickets if it wants to generate an income of $40,000?

2.12 What can you say if (a) $3x + 4 \geq 6x - 3$, (b) $2x + 7 > 13 > 3x - 4$?

2.13 Mario bought a round of five beers and three glasses of wine in a bar. He paid with a €20 note and noticed that his change contained at least one euro coin. He thought that each beer costs more than €2, so what can he say about the price of a glass of wine?

2.14 What are the values of (a) $x^{1/2} \times x^{1/4}$, (b) $(x^{1/3})^3$, (c) $9^{0.5}$, (d) $4^{2.5}$, (e) $7^{3.2}$, (f) $4^{1.5} \times 6^{3.7}/6^{1.7}$? Can you use this result to find some general rules for arithmetic with logarithms?

2.15 If log $a = 0.3010$, log $b = 0.4771$ and log $c = 0.6021$, what is the value of log (ab/c)?

2.16 If $3{,}000 = 1{,}500 \times 1.1^n$, what is the value of n?

RESEARCH PROJECTS

2.1 Companies' annual reports show a lot of quantitative information. This usually goes beyond basic accounts and includes operational, environmental, social and competitive performance. Examine the report of a major company and describe the quantitative analyses that it contains. (You can find some useful information in company websites.)

2.2 Jefferson Young & Co. is a manufacturer of automotive components. Over the past 14 months, they have collected information about production, income and costs. They keep this information in a simple spreadsheet, with the format shown in Figure 2.2. Describe how this spreadsheet works and what it shows. What else could they do with the data? What other features do you think they should add to the spreadsheet?

2.3 A lot of websites give tutorials on various topics of mathematics that are useful for managers. These are produced by universities, institutions, publishers, training companies, software providers, tutoring services, consultants, and so on. Do some searches on the Web to find sites that are useful for this course.

Sources of information

Reference

1 Letter to Jean Baptiste le Roy, 13th November 1789, published in Works of Benjamin Franklin, 1817.

Further reading

Many books introduce the ideas of mathematics, ranging from the trivial through to the very difficult. If you want some further information, the following list gives some useful ideas.

Amdahl K. and Loats J., *Algebra Unplugged*, Clearwater Publishing, Broomfield, CO, 1996.

Bradley T. and Patton P., *Essential Mathematics for Economics and Business*, John Wiley, Chichester, 2002.

Curwin J. and Slater R., *Improve Your Mathematics: a Refresher Course*, Thomson Learning, London, 1999.

Economist, *Numbers Guide: Essential Business Numeracy*, Economist Books, London, 2003.

Francis A., *Business Maths and Statistics*, Thomson Learning, London, 2004.

Gough L., *The Financial Times Guide to Business Numeracy*, FT Prentice Hall, Basingstoke, 1994.

Jacques I., *Mathematics for Economics and Business* (5th edition), FT Prentice Hall, Basingstoke, 2006.

Lerner J. and Don E., *Schaum's Outline of Basic Business Mathematics*, McGraw-Hill, New York, 2000.

Rowe N., *Refresher in Basic Maths* (2nd edition), Thomson Learning, London, 2001.

Selby P.H. and Slavin S., *Practical Algebra* (2nd edition), John Wiley, Chichester, 1991.

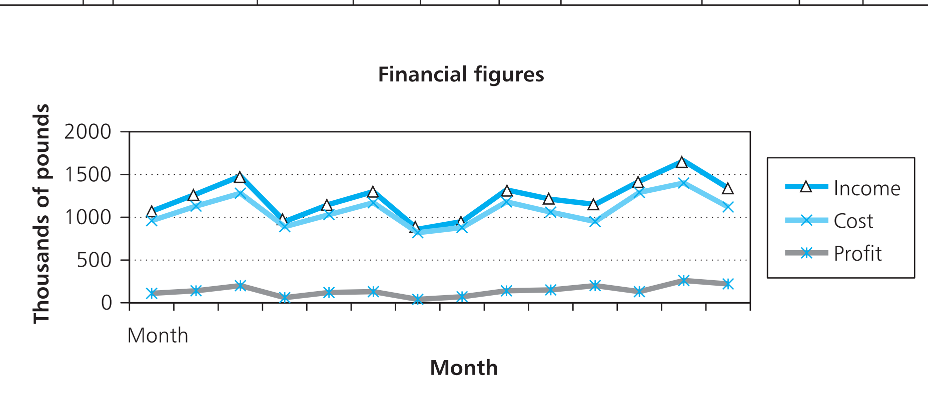
Soper J., *Mathematics for Economics and Business* (2nd edition), Blackwell, Oxford, 2004.

Winston W., *Data Analysis and Business Modelling with Excel*, Microsoft Press, Redmond, WA, 2004.

Useful websites

www.pearsoned.co.uk/waters – the Companion Website for this book. This contains a list of useful websites.

	A	B	C	D	E	F	G	H	I	J	K
1	**Jefferson Young & Co**										
2											
3			**Raw data**					**Sorted**			
4	**Month**		**Production**	**Income**	**Cost**	**Profit**		**Production**	**Income**	**Cost**	**Profit**
5	1		120	1070	960	110		100	860	820	40
6	2		150	1270	1130	140		110	950	890	60
7	3		180	1480	1280	200		110	950	880	70
8	4		110	950	890	60		120	1070	960	110
9	5		130	1150	1030	120		120	1150	950	200
10	6		160	1300	1170	130		130	1150	1030	120
11	7		100	860	820	40		140	1210	1060	150
12	8		110	950	880	70		150	1270	1130	140
13	9		160	1320	1180	140		150	1340	1120	220
14	10		140	1210	1060	150		160	1300	1170	130
15	11		120	1150	950	200		160	1320	1180	140
16	12		160	1420	1290	130		160	1420	1290	130
17	13		170	1660	1400	260		170	1660	1400	260
18	14		150	1340	1120	220		180	1480	1280	200
19											



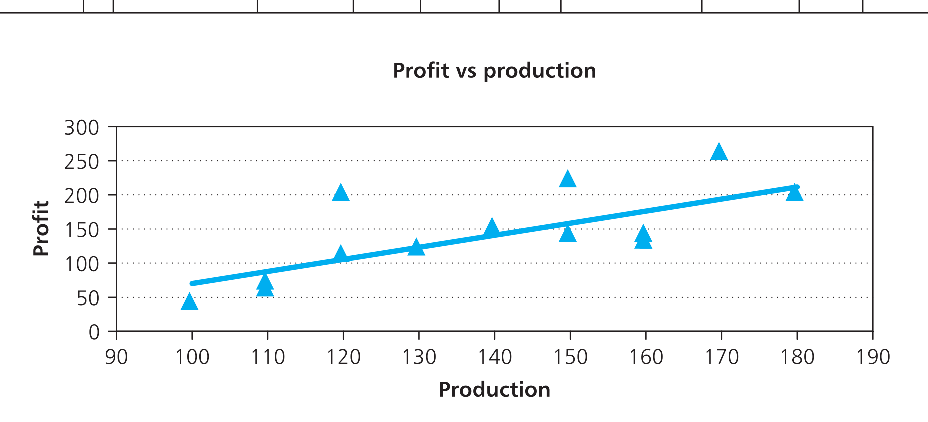


Figure 2.2 Spreadsheet for Jefferson Young & Co.

CHAPTER 3

Drawing graphs

Contents

Chapter outline

Graphs give a lot of information in a simple and effective format. This chapter shows how to draw line graphs on Cartesian co-ordinates. We start with simple linear graphs, and then build up to more complicated functions, including quadratic equations, higher polynomials and exponential curves. We return to this theme in Chapter 5, when we discuss other types of diagram for presenting information.

After finishing this chapter you should be able to:

- Appreciate the benefits of graphs
- Use Cartesian co-ordinates to draw graphs
- Draw straight line graphs and interpret the results
- Draw graphs of quadratic equations and calculate the roots
- Draw graphs of more complicated curves, including polynomials and exponential curves.

Graphs on Cartesian co-ordinates

The last chapter showed how to build an algebraic model of a situation – but most people find it difficult to understand what is happening, or to follow the logic of a set of equations. Diagrams are much better at presenting information, and you can look at a well-drawn diagram and quickly see its main features. We develop this theme in Chapter 5, but here we introduce the **line graph** or **graph** to show the relationship between two variables.

Cartesian axes

The most common type of graph has two rectangular (or Cartesian) **axes**. The horizontal axis is traditionally labelled x and the vertical axis is labelled y (as shown in Figure 3.1). The x is the **independent variable**, which is the one that we can set or control, and y is the **dependent variable**, whose value is set by x. Then x might be the amount we spend on advertising, and y is the resulting sales; x might be the interest rate we charge for lending money, and y is the corresponding amount borrowed; x might be the price we charge for a service, and y is the resulting demand.

When we talk about dependent and independent variables, we do not assume any cause and effect. There might be a clear relationship between two variables, but this does not necessarily mean that a change in one actually causes a change in the other. For example, a department store might find that when it reduces the price of overcoats the sales of ice cream rise. There might be a clear relationship between these two, but one does not cause the other – and both are likely to be a result of hot weather. Unfortunately, people do not always recognise this, and they imagine ridiculous causes-and-effects (which we discuss in Chapter 9).

The point where the two axes cross is the **origin**. This is the point where both x and y have the value zero. At any point above the origin, y is positive, and at any point below it, y is negative; at any point to the right of the origin,

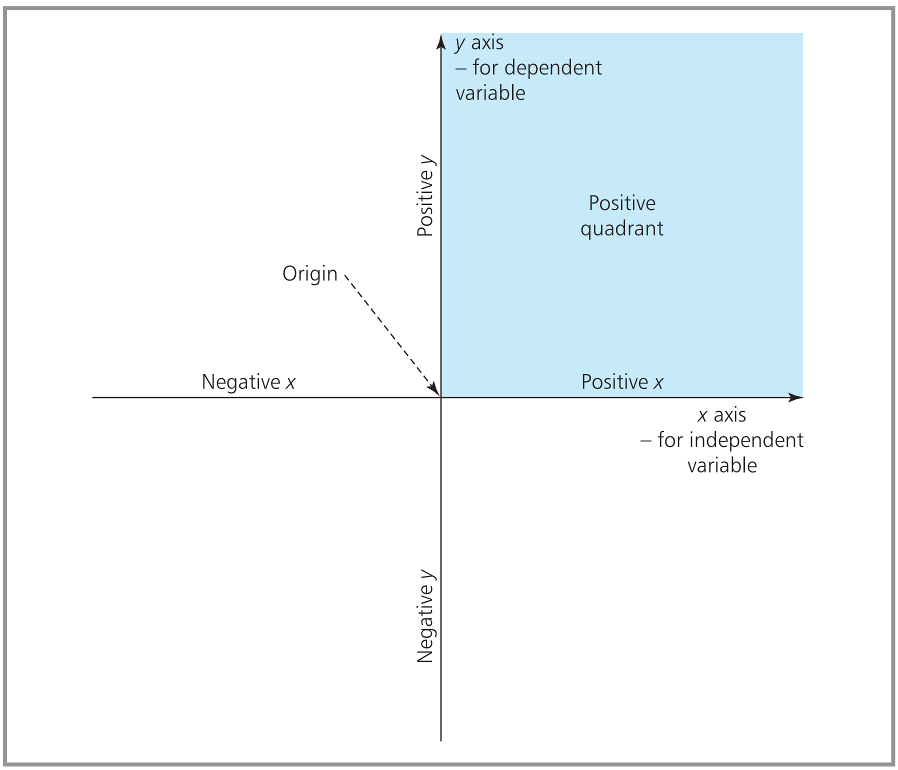


Figure 3.1 Cartesian axes

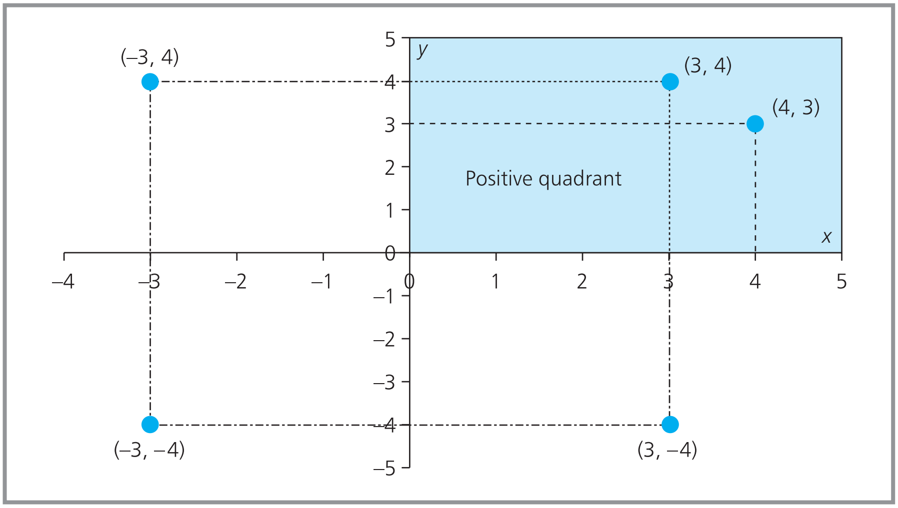


Figure 3.2 Locating points with Cartesian co-ordinates

x is positive, and at any point to the left of it, x is negative. Often, we are only interested in positive values of x and y – perhaps with a graph of income against sales. Then we show only the top right-hand corner of the graph, which is the **positive quadrant**.

We can describe any point on a graph by two numbers called **co-ordinates**. The first number gives the distance along the x axis from the origin, and the second number gives the distance up the y axis. For example, the point $x = 3$, $y = 4$ is situated three units along the x axis and four units up the y axis. A standard notation describes co-ordinates as (x, y), so this is point (3, 4). The only thing you have to be careful about is that (3, 4) is not the same as (4, 3), as you can see in Figure 3.2. And these points are some way from (–3, 4), (3, –4) and (–3, –4).

Points on the x axis have co-ordinates $(x, 0)$ and points on the y axis have co-ordinates $(0, y)$. The origin is the point where the axes cross; it has co-ordinates (0, 0).

WORKED EXAMPLE 3.1

Plot the following points on a graph.

x	2	5	7	10	12	15
y	7	20	22	28	41	48

Solution

As all the numbers are positive, we need draw only the positive quadrant. Then the first point, (2, 7), is two units along the x axis and seven units up the y axis, and is shown as point A in Figure 3.3. The second point, (5, 20), is five units along the x axis and 20 units up the y axis, and is shown by point B. Adding the other points in the same way gives the result shown in Figure 3.3.

There is a clear relationship between x and y, and we can emphasise this by joining the points. For this either we can draw a line connecting all the points to show the details, or we can ignore the details and draw a line of the general trend (shown in Figure 3.4).

Worked example 3.1 continued

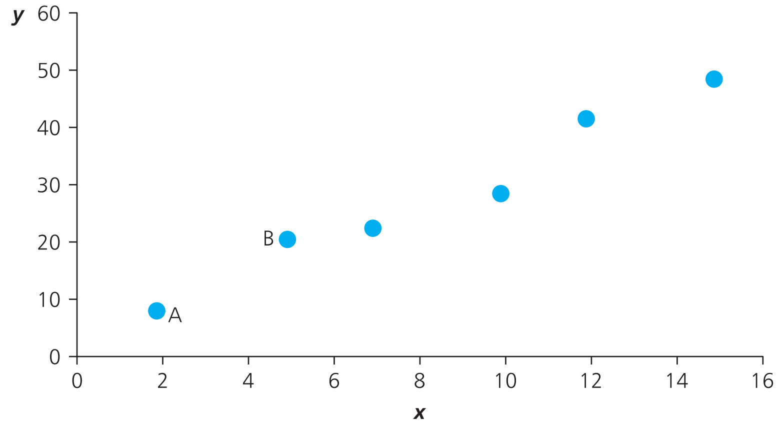


Figure 3.3 Graph of points for worked example 3.1

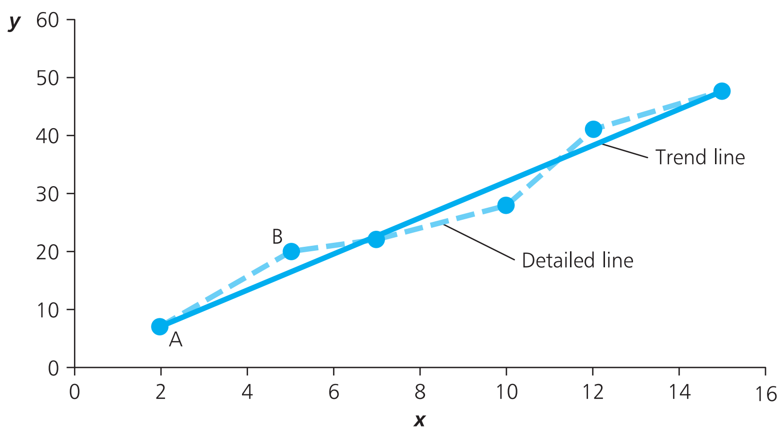


Figure 3.4 Connecting points to emphasise the relationship

Of course, you can draw graphs in the traditional way, by hand on graph paper, but the easiest and most reliable way uses specialised graphics packages. Many of these are available, such as ConceptDraw, CorelDraw, DrawPlus, Freelance Graphics, Harvard Graphics, Sigmaplot, SmartDraw and Visio. Many other packages also have graphics functions, such as presentation packages, desktop publishing, design packages and picture editors. Excel has a graphics function, and Figure 3.5 shows an example of the results when you press the 'chart wizard' button.

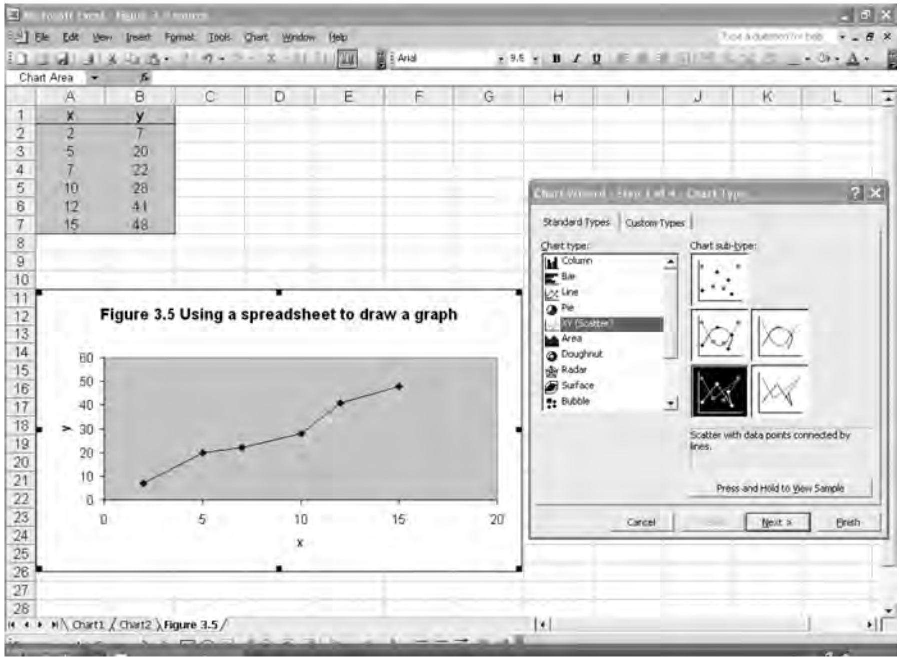


Figure 3.5 Using a spreadsheet to draw a graph

Drawing straight line graphs

The usual way of drawing a graph is to plot a series of points (x, y) and then draw a line through them. For complicated graphs we need a lot of points to get the right shape, but for straightforward graphs we need only a few points – and in the simplest case we can draw a straight line graph through only two points. You can see this in a simple case, where y always has a constant value whatever the value of x. For example, if $y = 10$ for all values of x, then we can take two arbitrary values of x, say 2 and 14, and get two points (2, 10) and (14, 10). Then we can draw a straight line through these that is 10 units above the y axis and parallel to it (as shown in Figure 3.6).

In general, a graph of $y = c$, where c is any constant, is a straight line that is parallel to the x axis and c units above it. This line divides the area of the graph into three zones:

- At any point *on* the line, y is equal to the constant, so $y = c$.
- At any point *above* the line, y is greater than c, so $y > c$.
- At any point *below* the line, y is less than c, so $y < c$.

We could equally say:

- At any point *on or above* the line, y is greater than or equal to c, so $y \geq c$.
- At any point *on or below* the line, y is less than or equal to c, so $y \leq c$.

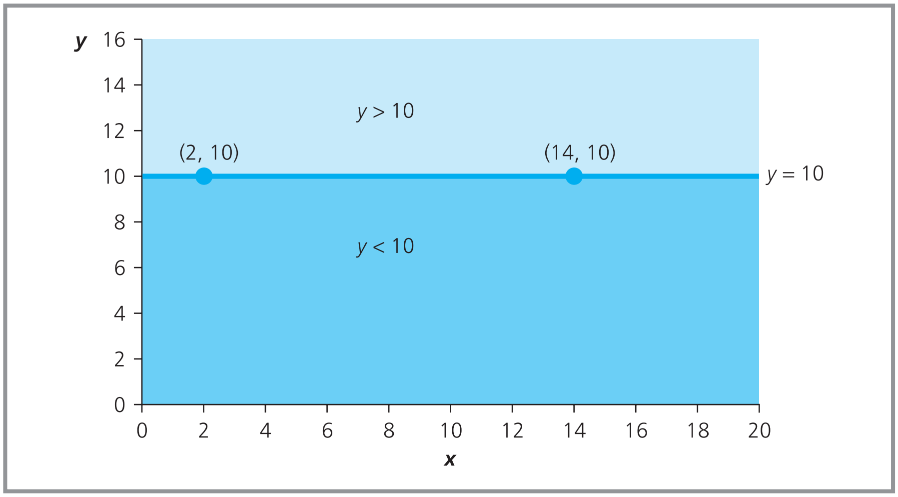


Figure 3.6 Straight line graph of $y = 10$

The graph of $y = c$ is an example of a straight line graph. Not surprisingly, any relationship that gives a straight line graph is called a **linear relationship**.

> Linear relationships have the general form:
>
> $y = ax + b$
>
> where:
>
> x and y are the independent and dependent variables
>
> a and b are constants.

WORKED EXAMPLE 3.2

Draw a graph of $y = 10x + 50$.

Solution

This is a straight line graph of the standard form $y = ax + b$, with $a = 10$ and $b = 50$. We need only two points to draw the line and can take any convenient ones. Here we will arbitrarily take the points where $x = 0$ and $x = 20$.

- When $x = 0$, $y = 10x + 50 = 10 \times 0 + 50 = 50$, which defines the point (0, 50).
- When $x = 20$, $y = 10x + 50 = 10 \times 20 + 50 = 250$, which defines the point (20, 250).

Plotting these points and drawing a line through them gives the graph shown in Figure 3.7.

Worked example 3.2 continued

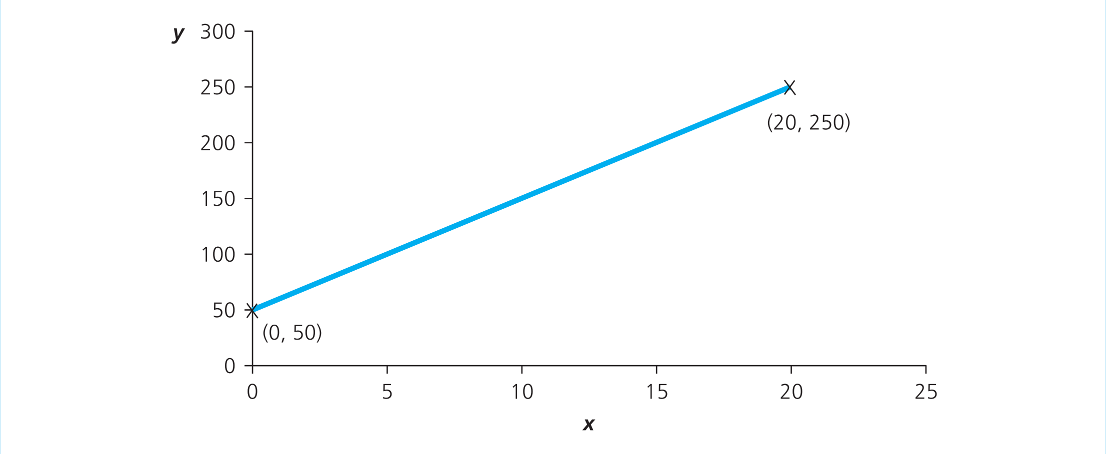


Figure 3.7 Straight line graph of $y = 10x + 50$

When you look at a straight line graph, there are two obvious features:

- the **intercept**, which shows where the line crosses the y axis;
- the **gradient**, which shows how steep the line is.

When the line crosses the y axis, x has the value 0. And if we substitute $x = 0$ into the equation $y = ax + b$, you see that $ax = 0$, so $y = b$. In other words, the constant b is the intercept of the line.

The gradient of a line shows how quickly it is rising, and is defined as the increase in y for a unit increase in x. The gradient is clearly the same at every point on a straight line, so we can find the increase in y when x increases from, say, n to $n + 1$:

- When $x = n$, $y = ax + b = an + b$.
- When $x = n + 1$, $y = ax + b = a(n + 1) + b = an + a + b$.

As you can see, the difference between these two is a, and this shows that an increase of 1 in x always gives an increase of a in y. So the constant a is the gradient, meaning that the general equation for a straight line is:

$y = \text{gradient} \times x + \text{intercept}$

WORKED EXAMPLE 3.3

Describe the graph of the equation $y = 4x + 20$.

Solution

This is a straight line graph with intercept of 20 and gradient of 4, as shown in Figure 3.8.

You can also see the following:

- For any point actually on the line, $y = 4x + 20$.
- For any point above the line, $y > 4x + 20$.
- For any point below the line, $y < 4x + 20$.

Worked example 3.3 continued

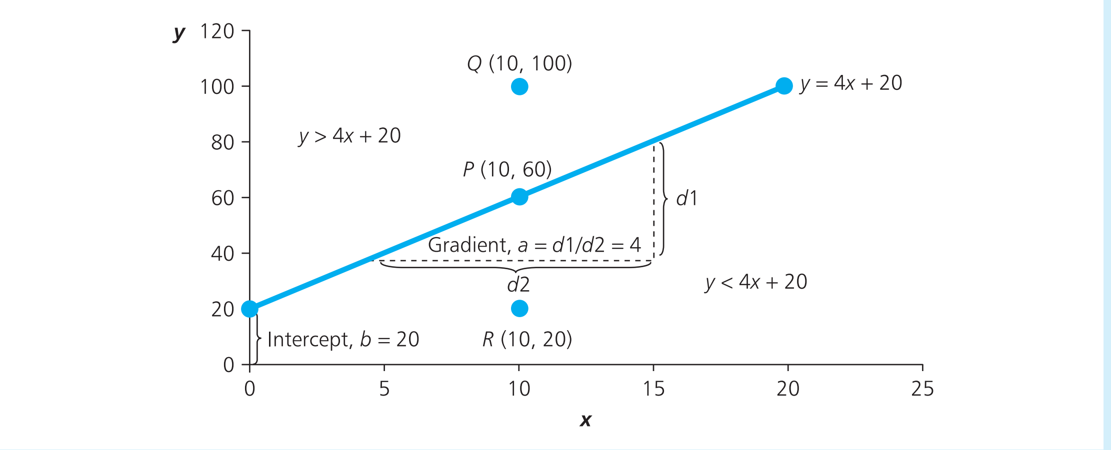


Figure 3.8 Straight line graph of $y = 4x + 20$

You can check this by taking any arbitrary points. For example, when $x = 10$ the corresponding value of y on the line is $y = 4 \times 10 + 20 = 60$, giving the point P at (10, 60). The point Q has co-ordinates (10, 100), is above the line, and y is clearly greater than $4x + 20$; the point R has co-ordinates (10, 20), is below the line, and y is clearly less than $4x + 20$.

WORKED EXAMPLE 3.4

Anita notices that the sales of a product vary with its price, so that:

sales = 100 − 5 × price

What are her sales when the price is 6?

Solution

Substituting the price, 6, into the equation gives sales of $100 - 5 \times 6 = 70$. But we can find more information from a graph. The relationship is a straight line with the equation $y = ax + b$, where y is the sales, x is the price, a is the gradient of −5, and b is the intercept of 100. The negative gradient shows that y decreases as x increases – and with every unit increase in price, sales fall by 5. To draw the graph (shown in Figure 3.9) we take two arbitrary points, say $x = 0$ and $x = 10$:

- When $x = 0$, $y = 100 - 5x = 100 - 5 \times 0 = 100$, giving the point (0, 100).
- When $x = 10$, $y = 100 - 5x = 100 - 5 \times 10 = 50$, giving the point (10, 50).

There is an upper limit on sales, given by the intercept – and even when the price is reduced to zero the expected sales are 100. Any point above the line shows that sales are higher than expected, while any point below shows that they are lower.

Worked example 3.4 continued

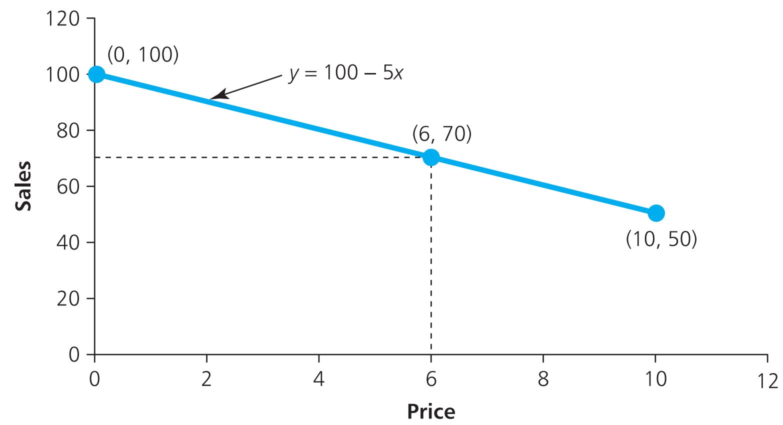


Figure 3.9 Graph of Anita's sales

Review questions

3.1 What is meant by a dependent variable?

3.2 Graphs show the changes in y caused by changes in x. Is this true?

3.3 With Cartesian co-ordinates, what are the co-ordinates of the origin?

3.4 Is there a difference between the points (3, −4) and (−3, 4)?

3.5 Describe the graph of the equation $y = -2x - 4$.

3.6 What are the gradients of the lines (a) $y = 10$, (b) $y = x$, (c) $y = 10 - 6x$?

3.7 If $y = 3x + 5$, what can you say about all the points above the graph of this line?

Quadratic equations

Any relationship between variables that is not linear is – not surprisingly – called a **non-linear relationship**. These have more complicated graphs, but we can draw them using the same principles as straight lines. The only concern is that we need more points to show the exact shape of the curve. The easiest way of getting these is to take a series of convenient values for x, and substitute them into the equation to find corresponding values for y. Then plot the resulting points (x, y) and draw a line through them.

WORKED EXAMPLE 3.5

Draw a graph of the equation $y = 2x^2 + 3x - 3$, between $x = -6$ and $x = 5$.

Solution

We are interested in values of x between –6 and +5, so take a series of points within this range and substitute them to find corresponding values for y. We can start with:

- $x = -6$, and substitution gives $y = 2x^2 + 3x - 3 = 2 \times (-6)^2 + 3 \times (-6) - 3 = 51$
- $x = -5$, and substitution gives $y = 2x^2 + 3x - 3 = 2 \times (-5)^2 + 3 \times (-5) - 3 = 32$

and so on, to give the following table.

x	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5
y	51	32	17	6	–1	–4	–3	2	11	24	41	62

Plotting these points on Cartesian axes and drawing a curved line through them gives the graph in Figure 3.10.

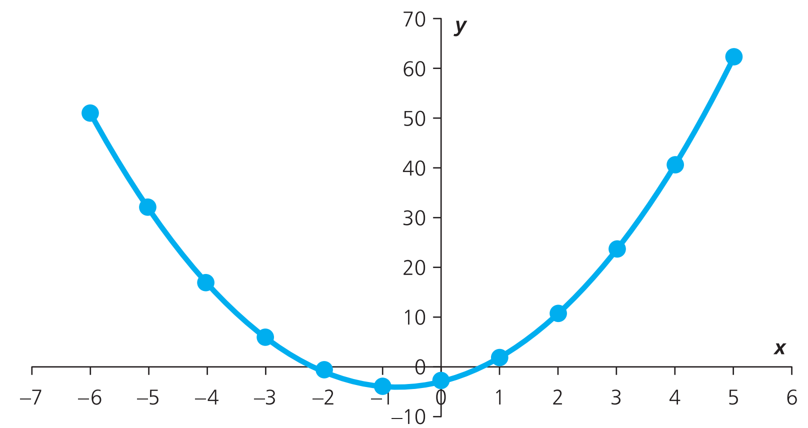


Figure 3.10 Graph of $y = 2x^2 + 3x - 3$

The last example showed a **quadratic equation**. These have the general form $y = ax^2 + bx + c$ where a, b and c are constants. Their graphs are always U-shaped – but when a is negative the graph is inverted and it looks like a hill rather than a valley. The top of the hill, or bottom of the valley, is called a **turning point**, where the graph changes direction and the gradient changes sign.

Quadratic equations are quite common, as we can show from an example of production costs. Suppose economies of scale and other effects mean that the average cost of making a product changes with the number of units made. The basic cost of making one unit might be €200, and this falls by €5 for every unit of weekly production. Then the unit cost is $200 - 5x$, where x is the weekly production. If there are fixed overheads of €2,000 a week:

$$\begin{aligned}\text{total weekly cost} &= \text{overheads} + \text{number of units made in the week} \\ &\quad \times \text{unit cost} \\ &= 2{,}000 + x \times (200 - 5x) \\ &= 2{,}000 + 200x - 5x^2\end{aligned}$$

WORKED EXAMPLE 3.6

Sonja Thorsen bought shares worth €10,000 in her employer's profit-sharing scheme. When the share price rose by €10, she kept 1,000 shares and sold the rest for €11,000. How can you describe her share purchases?

Solution

If Sonja originally bought x shares, the price of each was €10,000/x. When this rose to (10,000/x + 10) she sold (x – 1,000) shares for €11,000. So:

$$\begin{aligned} 11{,}000 &= \text{number of shares sold} \times \text{selling price} \\ &= (x - 1{,}000) \times (10{,}000/x + 10) \end{aligned}$$

Rearranging this equation gives:

$$\begin{aligned} 11{,}000 &= (x - 1{,}000) \times \left(\frac{10{,}000}{x} + 10\right) \\ &= 10{,}000 + 10x - \frac{10{,}000{,}000}{x} - 10{,}000 \end{aligned}$$

or

$$11{,}000x = 10{,}000x + 10x^2 - 10{,}000{,}000 - 10{,}000x$$

i.e.

$$10x^2 - 11{,}000x - 10{,}000{,}000 = 0$$

In the next section we show how to solve this equation, and find that Sonja originally bought 1,691 shares. You can check this by seeing that she bought 1,691 shares at 10,000/1,691 = €5.91 each. When the shares rose to €15.91, she sold 691 of them for 691 × 15.91 = €11,000, and kept the remainder with a value of 1,000 × 15.91 = €15,910.

WORKED EXAMPLE 3.7

Draw a graph of $y = 15 + 12x - 3x^2$ for values of x between –2 and 6. Where does this curve cross the x axis?

Solution

We can take a range of values for x and substitute these to get corresponding values for y, as follows.

x	–2	–1	0	1	2	3	4	5	6
y	–21	0	15	24	27	24	15	0	–21

Plotting these points and joining them together gives the results in Figure 3.11. As a has a negative value of –3, the graph is an inverted U, and you can see that it crosses the x axis at $x = -1$ and $x = 5$.

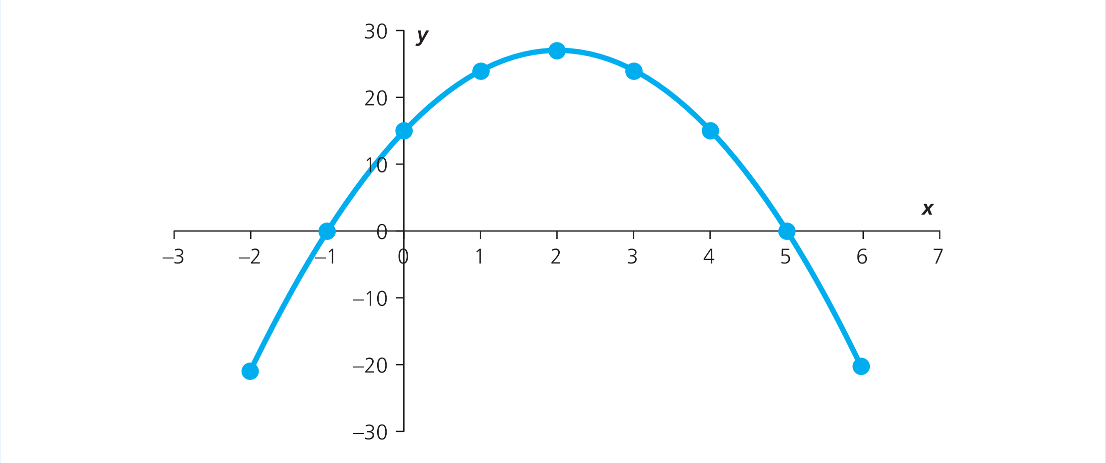


Figure 3.11 Graph of $y = 15 + 12x - 3x^2$

In the last example, we found the points where the curve crossed the x axis. By definition, these are the two points where $ax^2 + bx + c = 0$, and they are called the **roots** of the quadratic. You can estimate these from a graph, but there is a standard calculation for finding them (whose derivation you can find in the Companion Website **www.pearsoned.co.uk/waters**). This shows that the two points where $y = 0$ correspond to the values of x where:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

In the worked example above, $a = -3$, $b = 12$ and $c = 15$, and we can substitute these values to get:

$$x = \frac{-12 + \sqrt{12^2 - 4 \times (-3) \times 15}}{2 \times (-3)} \quad \text{and} \quad x = \frac{-12 - \sqrt{12^2 - 4 \times (-3) \times 15}}{2 \times (-3)}$$

$$= \frac{-12 + \sqrt{(144 + 180)}}{-6} \qquad = \frac{-12 - \sqrt{(144 + 180)}}{-6}$$

$$= (-12 + 18) / (-6) \qquad = (-12 - 18) / (-6)$$

$$= -1 \qquad = 5$$

This confirms our findings from the graph, that the curve crosses the x axis at the points $(-1, 0)$ and $(5, 0)$.

WORKED EXAMPLE 3.8

Find the roots of the equation $2x^2 + 3x - 2 = 0$.

Solution

This is a quadratic with $a = 2$, $b = 3$ and $c = -2$. Substituting these values into the standard equations gives the two roots:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 + \sqrt{3^2 - 4 \times 2 \times (-2)}}{2 \times 2}$$

$$= (-3 + \sqrt{25}) / 4$$

$$= 0.5$$

and

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3 - \sqrt{3^2 - 4 \times 2 \times (-2)}}{2 \times 2}$$

$$= (-3 - \sqrt{25}) / 4$$

$$= -2$$

You can check these values by substituting them in the original equation:

$$2 \times 0.5^2 + 3 \times 0.5 - 2 = 0$$

and

$$2 \times (-2)^2 + 3 \times (-2) - 2 = 0$$

The only problem with calculating the roots comes when $4ac$ is greater than b^2. Then $b^2 - 4ac$ is negative, and we have to find the square root of a negative number. This is not defined in real arithmetic, so we conclude that there are no real roots and they are both imaginary.

Review questions

3.8 Are the graphs of all quadratic equations exactly the same shape?

3.9 What are the roots of a quadratic equation?

3.10 What can you say about the roots of $y = x^2 + 2x + 3$?

3.11 Why is it better to calculate the roots of a quadratic equation than to read them from a graph?

IDEAS IN PRACTICE Emjit Chandrasaika

In his spare time, Emjit Chandrasaika sells computer software through his website. Because he does this from home, and considers it a mixture of business and pleasure, he does not keep a tight check on his accounts. He thinks that he gets a basic income of £12 for every unit he sells, but economies of scale mean that this increases by £2 for every unit. He estimates the fixed costs of his website, advertising and time is £1,000 a month.

Emjit's income per unit is $12 + 2x$, where x is the number of units that he sells per month. Then:

$$\begin{aligned}\text{profit} &= \text{number of units sold per month} \\ &\quad \times \text{unit income} - \text{overheads} \\ &= x(12 + 2x) - 1{,}000 \\ &= 2x^2 + 12x - 1{,}000\end{aligned}$$

When this equals zero his income just covers his costs, and this happens when:

$$\begin{aligned}x &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-12 + \sqrt{12^2 - 4 \times 2 \times (-1{,}000)}}{2 \times 2} \\ &= 19.6\end{aligned}$$

or

$$\begin{aligned}x &= \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-12 - \sqrt{12^2 - 4 \times 2 \times (-1{,}000)}}{2 \times 2} \\ &= -25.6\end{aligned}$$

Obviously, he cannot sell a negative number of units, so he must sell 20 units a month to make a profit. His actual sales are much more than this, and rising by 50% a year. This analysis is encouraging Emjit to consider a move into full-time web sales.

Drawing other graphs

When we can express one variable, y, in terms of another, x, we say that 'y is a function of x', and write this as $y = f(x)$. With a straight line, y is a linear function of x, which means that $y = f(x)$ and $f(x) = ax + b$; with a quadratic $y = f(x)$, and $f(x) = ax^2 + bx + c$. This is just a convenient shorthand that can save time explaining relationships. You can draw a graph of any relationship where y is a function of x, with $y = f(x)$.

Polynomials

We have drawn graphs of straight lines (where $y = ax + b$) and quadratic equations (where $y = ax^2 + bx + c$). These are two examples of **polynomials** – which is the general term for equations that contain a variable, x, raised to some power. For straight lines we raised x to the power 1, for quadratics we raised x to the power 2 – and for more complicated polynomials we raise x

to higher powers. Cubic equations contain x raised to the power 3, with the form $y = ax^3 + bx^2 + cx + d$. The constants a, b, c, d are called **coefficients** of the polynomial. Higher polynomials have more complex curves, and when drawing graphs you have to plot enough points to show the details.

WORKED EXAMPLE 3.9

Draw a graph of the function $y = x^3 - 1.5x^2 - 18x$ between $x = -5$ and $x = +6$.

Solution

Figure 3.12 shows a spreadsheet of the results. The top part shows a series of values of y calculated for x between -5 and 6; then the Chart Wizard actually draws the graph. This has a trough around $x = 3$ and a peak around $x = -2$. Cubic equations always have this general shape, with two turning points, but they vary in detail; some are the other way around, some have the two turning points merged into one, and so on.

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Graph of cubic equation**												
2													
3	x	–5	–4	–3	–2	–1	0	1	2	3	4	5	6
4	y	–72.5	–16	13.5	22	15.5	0	–18.5	–34	–40.5	–32	–2.5	54
5													
6													
7													
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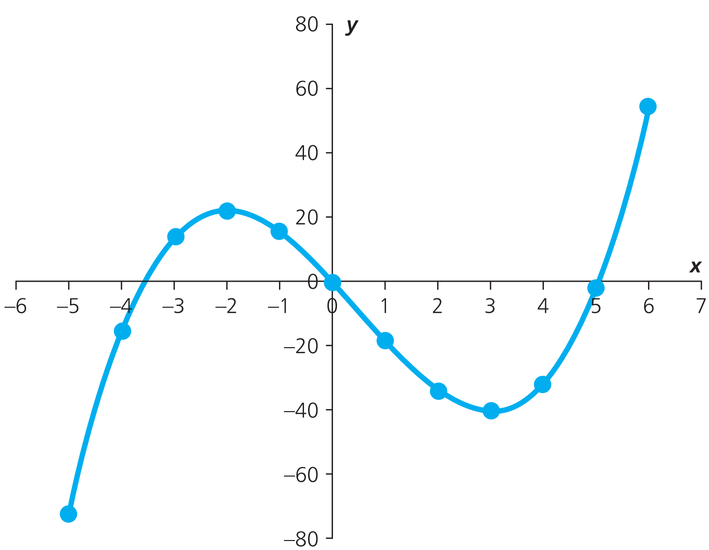


Figure 3.12 Graph of $y = x^3 - 1.5x^2 - 18x$

Exponential curves

We mentioned earlier the exponential constant, e, which is defined as e = 2.7182818. . . . This strange number is useful for describing functions that rise or fall at an accelerating rate. Exponential curves have the general form $y = ne^{mx}$, where n and m are constants. The exact shape depends on the values of n and m, but when m is positive there is an accelerating rise – described as exponential growth – and when m is negative there is a decreasing fall – described as exponential decline.

WORKED EXAMPLE 3.10

Draw graphs of $y = e^x$ and $y = e^{0.9x}$ for values of x between 0 and 10.

Solution

Figure 3.13 shows these results in a spreadsheet. When e is raised to a positive power, the characteristic exponential curves rise very quickly with x.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Graphs of exponential growth**											
2												
3	**x**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
4	**$e^{0.9x}$**	1	2	6	15	37	90	221	545	1339	3294	8103
5	**e^x**	1	3	7	20	55	148	403	1097	2981	8103	22026

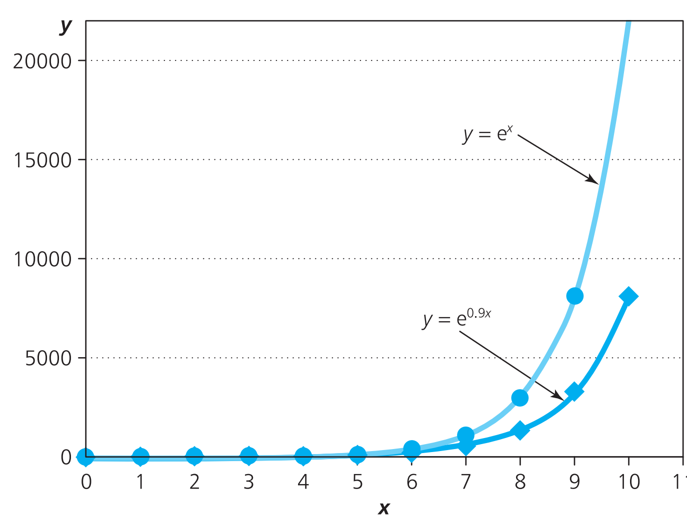


Figure 3.13 Graphs of exponential growth

WORKED EXAMPLE 3.11

Draw the graph of $y = 1000e^{-0.5x}$ between $x = 0$ and $x = 10$.

Solution

Figure 3.14 shows the calculations and graph on a spreadsheet. When e is raised to a negative power, the exponential curve falls quickly towards zero and then flattens out with increasing x.

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Graph of exponential decline**											
2												
3	***x***	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
4	**$1000e^{-0.5x}$**	1000	607	368	223	135	82	50	30	18	11	7
5												
6												

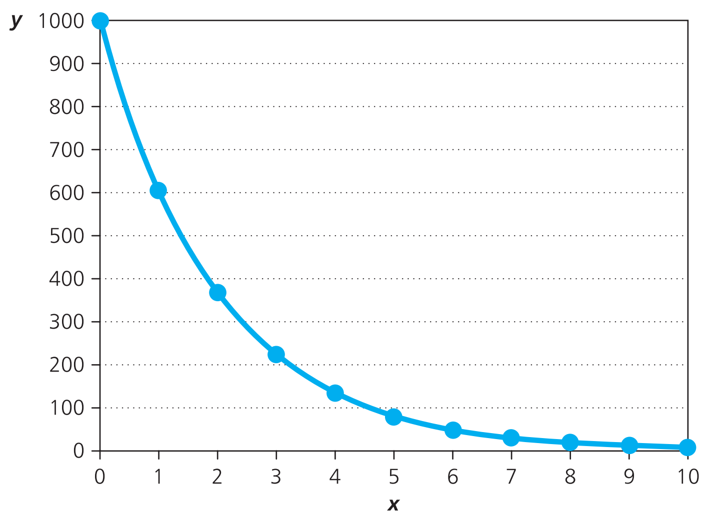


Figure 3.14 Graph of exponential decline

Review questions

3.12 What is a polynomial?

3.13 What is a turning point in a graph?

3.14 How can you draw a graph of exponential growth?

3.15 Can you draw a graph of $y = 12x + 7z$, where both x and z are variables?

IDEAS IN PRACTICE Konrad Schimmer

You can find examples of graphs in almost any newspaper or magazine. Many of these are time series, which show a series of observations taken at regular intervals of time – such as monthly unemployment figures, daily rainfall, weekly demand for a product, and annual profit. Financial analysts use many types of graph, as they are the best way of showing trends and underlying patterns.

Konrad Schimmer is a financial analyst of the Frankfurt Stock Exchange, and he plots graphs for every aspect of companies' performance. Typically he plots the quarterly profit for the past six years, monthly sales for the past three years, or closing share price over the past year. Figure 3.15 shows one of his graphs for comparing two companies, with the closing share prices at the end of each week for the past year. Konrad studies the details of such graphs, looking for trends, unusual patterns, possible causes, and how the company is likely to perform in the future. He has used this approach to amass considerable wealth.

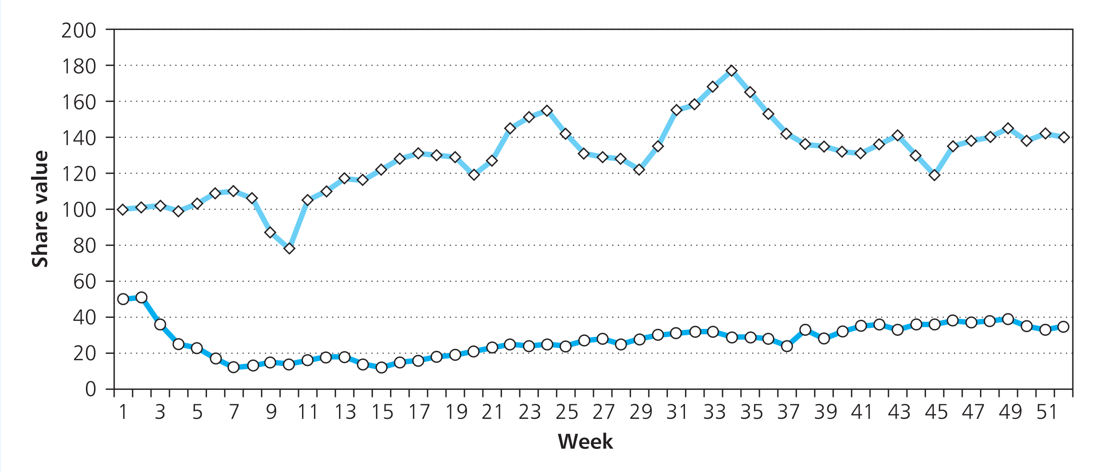


Figure 3.15 Comparison of closing weekly share prices

CHAPTER REVIEW

This chapter showed how to draw different types of graph.

- Diagrams give an easy and efficient way of presenting information. People can look at a well-drawn diagram and quickly see the main features and patterns.
- Graphs are one of the most useful types of diagram, and usually use Cartesian co-ordinates to show a relationship between two variables.
- Straight line graphs have the form $y = ax + b$, where a is the gradient and b is the intercept. You can draw straight line graphs by plotting two points and drawing the line through them.
- You can extend this method to more complicated curves, such as quadratic equations. These have the general form $y = ax^2 + bx + c$, and are U-shaped – or inverted when a is negative.

- The roots of a quadratic equation are the points where the curve crosses the x axis, and there is a standard calculation to identify these points.
- You can use the standard method for drawing graphs for any relationship where y is a function of x, meaning that $y = f(x)$, including polynomials and exponential curves.

CASE STUDY McFarlane & Sons

John McFarlane works for his family company, which has sold traditional clothing from four shops in the east of Scotland since 1886. He wants to compare the performance of each shop, and has collected some detailed information for the past year. Now he wants a convenient format to present this to the company Board of Directors.

John's problem is that he has a huge amount of data. The following table shows the number of units of five products sold each month in each of the shops. John has this kind of information for several hundred products, along with costs, profit margins, advertising expenditure – and many other figures.

Month	Shop	Product A	Product B	Product C	Product D	Product E
January						
	1	15	87	2	21	65
	2	12	42	0	15	32
	3	8	21	3	33	40
	4	7	9	3	10	22
February						
	1	16	80	1	22	67
	2	16	43	2	12	34
	3	8	24	5	31	41
	4	8	8	2	9	21
March						
	1	18	78	6	15	70
	2	16	45	6	8	30
	3	8	21	8	23	44
	4	10	7	2	8	19
April						
	1	21	83	11	16	71
	2	17	46	13	7	30
	3	11	19	9	25	47
	4	9	8	4	9	21
May						
	1	24	86	2	25	66
	2	20	49	7	16	32
	3	14	23	3	37	46
	4	10	6	3	13	22
June						
	1	27	91	3	33	65
	2	23	52	1	17	33
	3	15	20	0	51	47
	4	12	9	2	17	10
July						
	1	27	88	2	38	65
	2	22	55	0	20	38
	3	16	20	2	58	46
	4	9	8	1	19	20

Case study continued

Month	Shop	Product A	Product B	Product C	Product D	Product E
August						
	1	20	90	1	37	68
	2	21	57	0	24	35
	3	11	23	1	60	40
	4	10	8	0	20	18
September						
	1	17	84	7	26	65
	2	17	63	8	17	31
	3	10	21	4	39	46
	4	6	7	9	12	19
October						
	1	17	85	24	19	70
	2	14	61	23	13	33
	3	11	21	21	30	39
	4	9	7	19	11	21
November						
	1	15	85	37	11	69
	2	13	55	36	10	33
	3	9	22	28	15	44
	4	9	9	19	5	21
December						
	1	15	88	81	17	68
	2	12	54	65	14	34
	3	7	18	67	24	40
	4	8	7	53	8	22

Question

- **How could John McFarlane use graphs to present information to the company Board of Directors?**

PROBLEMS

3.1 Draw a graph of the points (2, 12), (4, 16), (7, 22), (10, 28) and (15, 38). How would you describe this graph?

3.2 Draw a graph of the following points. What can you say about the results?

x	1	3	6	8	9	10	13	14	17	18	21	25	26	29
y	22	24	31	38	41	44	52	55	61	64	69	76	81	83

3.3 The number of people employed in a chain of workshops is related to the size (in consistent units) by the equation:

employees = size / 1,000 + 3

Draw a graph of this equation and use it to find the number of employees in a workshop of size 50,000 units.

3.4 Draw graphs of (a) $y = 10$, (b) $y = x + 10$, (c) $y = x^2 + x + 10$, and (d) $y = x^3 + x^2 + x + 10$.

3.5 What are the roots of (a) $x^2 - 6x + 8$, (b) $3x^2 - 2x - 5$, and (c) $x^2 + x + 1$?

3.6 Deng Chow Chan found that the basic income generated by his main product is £10 a unit, but this increases by £1 for every unit he makes. If he has to cover fixed costs of £100, how many units must he sell to cover all his costs?

3.7 The output, y, from an assembly line is related to one of the settings, x, by the equation

$$y = -5x^2 + 2{,}500x - 12{,}500$$

What is the maximum output from the line, and the corresponding value for x?

3.8 Martha Berryman finds that the unit cost of using production equipment is:

$$\text{cost} = 1.5x^2 - 120x + 4{,}000$$

where x is the number of units produced. Draw a graph to find the lowest unit cost. What production level does this correspond to?

3.9 Compare the graphs of $y = 2^x$ and $y = 3^x$.

3.10 Draw a graph of $y = 1/x$ for x between -5 and $+5$.

3.11 If you leave £100 in the bank earning 6% interest, at the end of n years you will have 100×1.06^n. Draw a graph of this amount over the next 20 years.

RESEARCH PROJECTS

3.1 Spreadsheets are a convenient way of drawing graphs, but there are more specialised graphics packages. What additional features do these specialised packages have? Do a small survey of packages, comparing their graphics features. What other features would you like?

3.2 Find some examples of graphs presented in newspapers and magazines. Describe some that are particularly good, and others that are particularly bad. What can the bad ones learn from the good ones?

3.3 You can monitor the trends in share prices using various indices, such as the London Stock Exchange FTSE 100 or FTSE 250 indices. Similar indices are calculated for other stock exchanges, such as the Nikkei in Tokyo, Dow-Jones in New York, Hang Seng in Hong Kong, Dax in Frankfurt, and CAC in Paris. Collect some figures for a specific company over some period and draw graphs to compare its performance with the broader stock market. Can you find any obvious trends? What do you expect to happen in the future?

Sources of information

Further reading

Most of the books on mathematics mentioned in Chapter 2 include sections on graphs. Some other useful books on graphs include:

Few S., *Show Me the Numbers*, Analytics Press, 2004.

Hyggett R., *Graphs and Charts*, Palgrave Macmillan, Basingstoke, 1990.

Robbins N.B., *Creating More Effective Graphs*, John Wiley, Chichester, 2005.

Walkenbach J., *Excel Charts*, Hungry Minds, Inc., New York, 2002.

Zelazny G., *Say it with Charts* (4th edition), McGraw-Hill, New York, 2001.

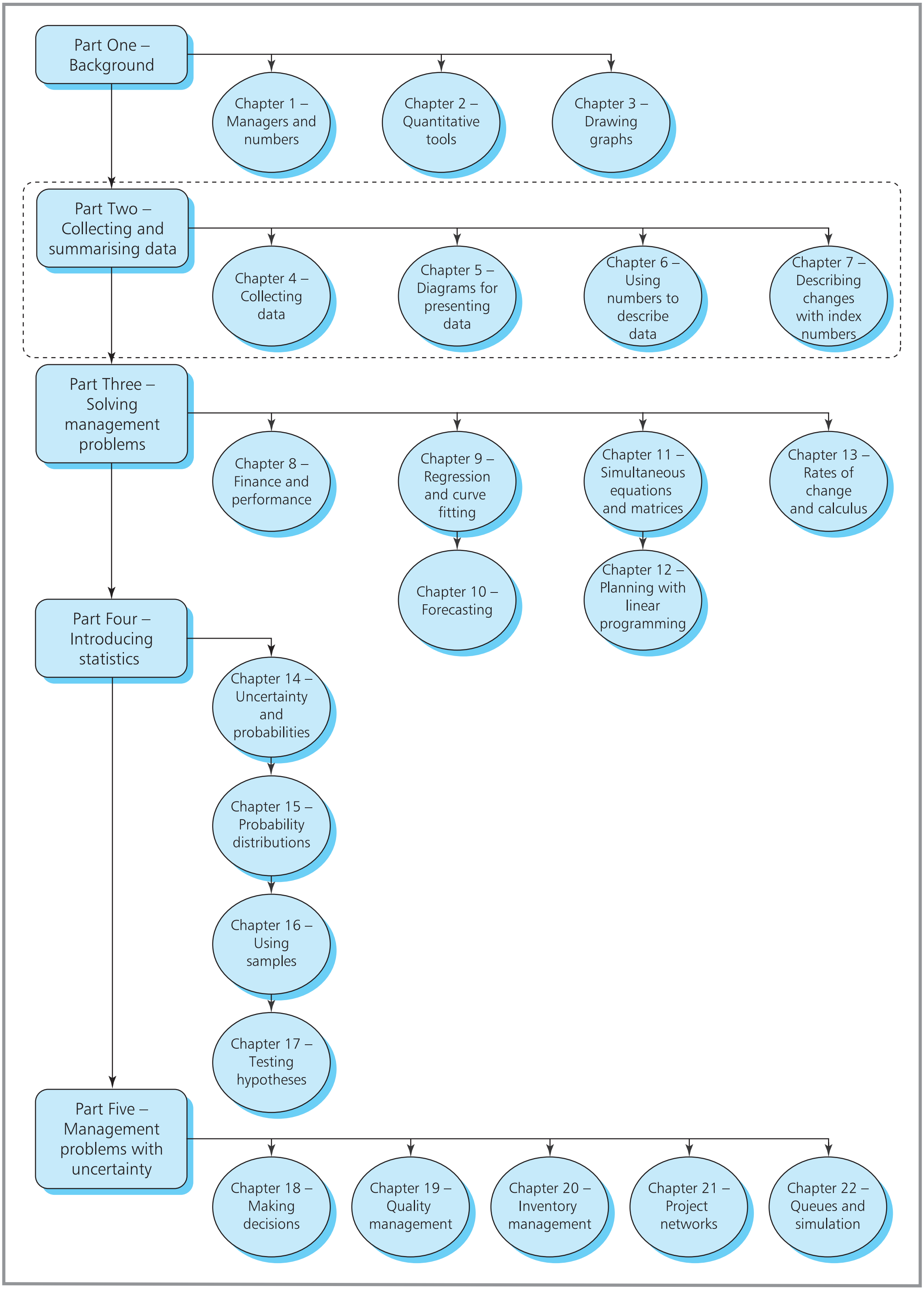
PART TWO

Collecting and summarising data

This book is divided into five parts. The first part looked at the background and context for quantitative methods. This second part shows how to collect and summarise data. The third part looks at ways of solving specific types of problem. Then the fourth part describes some useful statistics, and the fifth part uses these to solve problems with uncertainty.

There are four chapters in this part. Chapter 4 shows how to collect the data that managers need for their decisions. The raw data often has too much detail, so we have to summarise it and present it in ways that highlight its important features. Chapter 5 shows how to do this with different types of diagrams. Chapter 6 continues this theme by looking at numerical descriptions of data. Chapter 7 describes index numbers, which monitor changing values over time.

Map 2 shows how these chapters fit into the rest of the book.



Map 2 Map of chapters – Part Two

CHAPTER 4

Collecting data

Contents

Chapter outline

All quantitative analyses need reliable data. To make reasoned decisions, managers have to collect relevant data from a variety of sources. But there are many types of data, and each can be collected, analysed and presented in different ways. This chapter discusses the most common ways of collecting data. In practice, this usually means sampling, where data is collected from a representative sample of possible sources. Following chapters discuss the analysis and presentation of this data.

After finishing this chapter you should be able to:

- Appreciate the importance of data collection
- Discuss the amount of data to be collected
- Classify data in different ways
- Identify sources of data
- Understand the concept of populations and samples
- Discuss the reasons for using samples
- Describe and use different types of sample
- Consider different ways of collecting data from samples
- Design questionnaires.

Data and information

There is a difference between data and information.

Data are the raw numbers or facts that we process to give useful **information**.

Then 78, 64, 36, 70 and 52 are data that we process to give the information that the average exam mark of five students is 60%. A government census collects data from individuals, and processes this to give information about the population as a whole. You can collect data about new businesses and process this to give information about their performance. The principle is that data consists of raw numbers and facts, while information gives some useful knowledge.

Managers need relevant information before they can make decisions. To get this information, they start with **data collection**. This data is then processed to give information, and the results are presented in the best formats (as shown in Figure 4.1). This shows why data collection is essential in every organisation, as it starts the process that gives managers the information they need for their decisions. Without proper data collection, managers cannot make informed decisions.

The three main steps in preparing information are:

1 Data collection
2 Processing to give information
3 Presentation.

In this chapter we concentrate on data collection, while the following chapters look at processing and presentation. This seems a sensible approach, as it follows the natural timing – but things are not this simple, and how you use data can affect the way that you collect it. Suppose you want some data about the city of Malaga. If you are going there on holiday, you might use your phone or teletext to get weather forecasts for the next week; if you want information about local companies, you might look at their websites; if you want details of the city's history, you might look in an encyclopaedia; if you want to know the attitude towards business, you might send out a questionnaire; if you want to see how busy the streets are, you might do a survey. So the first step is really to define the purpose of the data and how it will be used. The second step is to decide which data is needed to achieve this

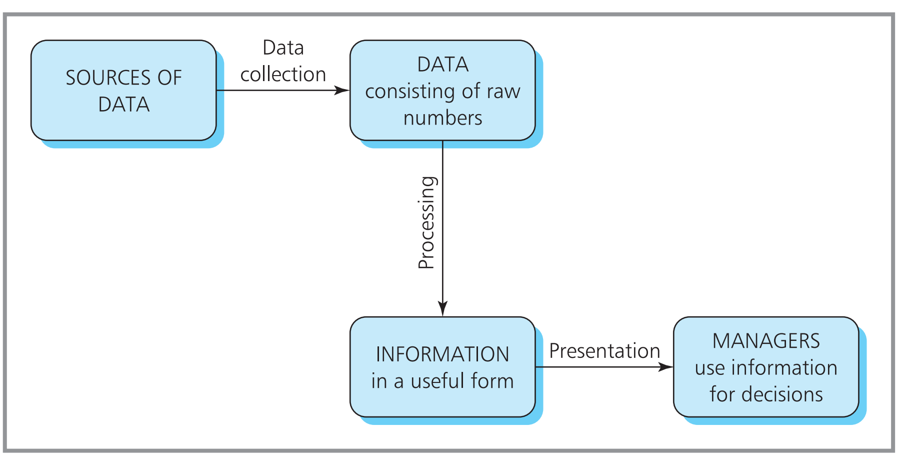


Figure 4.1 Processing data to help with decisions

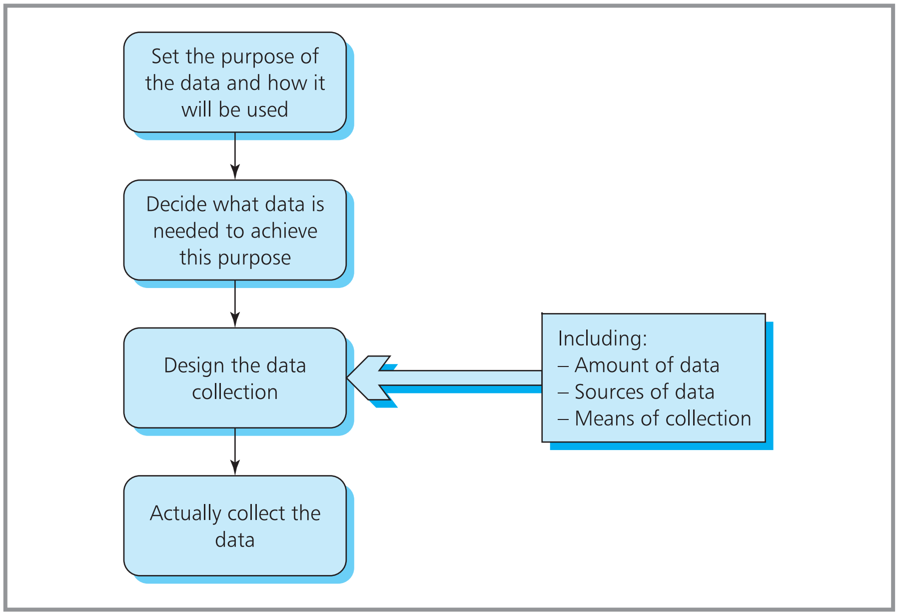


Figure 4.2 Planning data collection

purpose. Then the third step is to design the data collection and actually set about collecting it (as shown in Figure 4.2).

At first, it seems easy to collect data. After all, you can use a search engine on the Web to find a huge amount of data about almost anything. Unfortunately, you soon find that most of this is irrelevant, and you are swamped by details that are of no interest. If you want data that is relevant for your needs, accurate and reliable, you have to plan the collection more carefully.

Amount of data

Three important questions for data collection include the amount of data, its source, and the means of collection. Starting with the amount of data, managers want enough to allow good decisions, but not so much that they are swamped by irrelevant detail. This balance is difficult. There is often a huge amount of data they could collect, and that might be useful. But all data collection and analysis costs money, so they must resist the temptation to go on a spree and collect everything available. Imagine that you want a group of people's answers to five questions. You will probably use a questionnaire to collect these – and as you are sending a questionnaire you might as well add some extra questions to get a few more details. But if you end up with, say, 20 questions, you have the extra costs of collecting and analysing 15 questions that do not tell you anything useful – and you irritate people who have to spend more time completing the questionnaire.

In principle, we can define a **marginal cost** as the extra cost of collecting one more bit of data, and this rises with the amount of data collected. You can find some general data about, say, the Burlington Northern Railroad very easily (it runs trains, employs staff, etc.); for more detailed data you need a trip to a specialised transport library (perhaps finding what kinds of engines it has, or staff at different grades); for yet more detailed data you need to search the company's own records; for yet more detailed data you need a special survey of employees. At each stage, the more data you want, the more it costs to collect.

On the other hand, the **marginal benefit** of data – which is the benefit from the last bit collected – falls with the amount collected. The fact that Burlington Northern Railroad runs a rail service is very useful, but as you continue collecting more details, the value of each bit of data gets progressively smaller.

Figure 4.3 summarises these effects, and shows how to find the optimal amount of data for any specific purpose. This occurs at the point where the marginal cost equals the marginal benefit. If you collect less than this, you lose potential benefit as the cost of collection is less than the benefit; if you collect more data than this, you waste resources as the cost of collection is more than the benefit.

In reality, it is virtually impossible to find convincing values for the marginal costs and benefits, so most people simply collect the amount that their experience and judgement suggest is reasonable. An important factor in this decision is the time available. Some methods of data collection are very fast (such as searching websites) but other methods need a lot of time (such as running consumer surveys). There is always pressure on managers' time, so they prefer fast methods – commonly arguing that when data collection takes too long, the results become obsolete and irrelevant. Unfortunately, when managers do not allow enough time for proper data collection, they

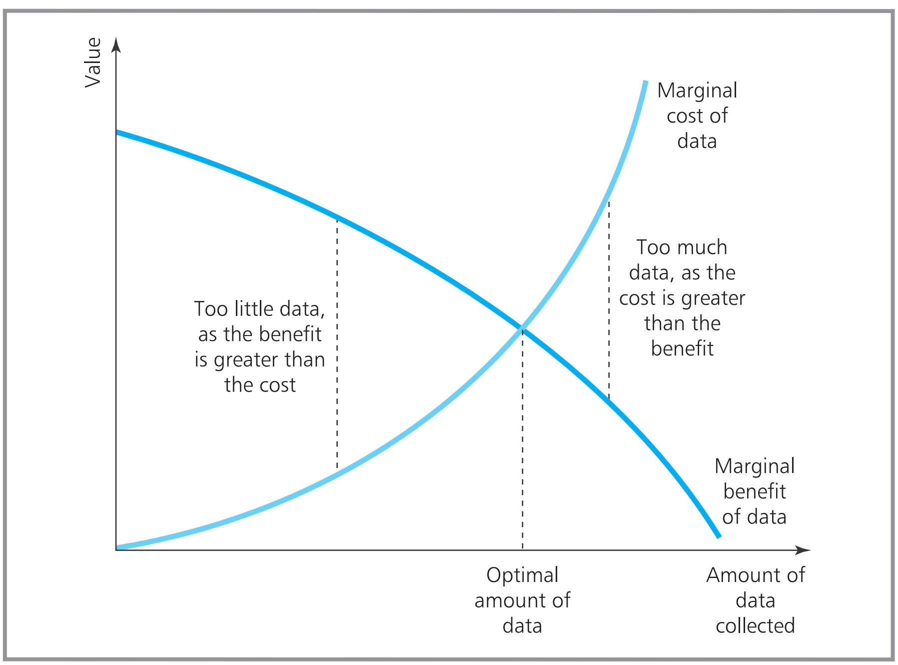


Figure 4.3 Finding the amount of data to collect

are tempted to take shortcuts and assume that any data – even if slightly inaccurate – is better than no data at all. Sometimes this is true. If a company does not have time for a full market survey, it can still get useful information from a limited study; and when you buy a car it is better to ask a salesmen for some details, even if their replies do not tell the whole story. Often, though, wrong data can be worse than no data at all. A limited market survey might give misleading results that encourage a company to start a hopeless venture, and car salesmen might underestimate the running costs so that you get into debt trying to meet the expenses. Inaccurate data can lead to bad decisions – so the clear message is that managers need accurate data, and this needs careful planning.

Review questions

4.1 What is the difference between data and information?

4.2 Why is data collection important to an organisation?

4.3 'It is always best to collect as much data as possible.' Do you think this is true?

Types of data

We can classify data in several ways. One way that we have already used describes data as either quantitative (based on numbers) or qualitative (where there are no numbers). Quantitative data is much easier to collect, analyse and describe, so you should use it whenever possible. You can even transform data that is essentially qualitative into a quantitative form. For example, when people have different opinions about some issue, you cannot measure their opinions, but you can see whether they agree with some statement. Then you might say, '70% of people agree with the statement that . . .' Sometimes we can add a notional scale. When doctors want to know how bad a patient's pain is, they ask them to rank it on a scale of 1 to 10 – and questionnaires often ask respondents to rate the strength of their opinion on a scale of 1 to 5.

We cannot transform all data into a convincing quantitative form, and when we hear Browning ask, 'How do I love thee? Let me count the ways . . .'[1] we know this is more for effect than for realism. A useful classification of data describes it as nominal, ordinal or cardinal, depending on how easy it is to measure.

- **Nominal data** is the kind that we really cannot quantify with any meaningful units. The facts that a person is an accountant, or a country operates a market economy, or a cake has cream in it, or a car is blue, are examples of nominal data, as there are no real measures for the features. The usual analysis for nominal data is to define a number of distinct categories and say how many observations fall into each – which is why it is also called categorical or descriptive data. A survey of companies in a town might give the nominal data that seven are manufacturers, 16 are service companies and five are in primary industries. A key point about nominal data is that the order in which the categories are listed does not matter, as you can see from the example in Figure 4.4.

(a) Nominal data

Percentage of respondents who would vote for political party X	35%
Percentage of respondents who would vote for political party Y	40%
Percentage of respondents who would vote for political party Z	20%
Percentage of respondents who do not know who they would vote for	5%

(b) Ordinal data

Percentage of people who feel 'very strongly' in favour of a proposal	8%
Percentage of people who feel 'strongly' in favour of a proposal	14%
Percentage of people who feel 'neutral' about a proposal	49%
Percentage of people who feel 'strongly' against a proposal	22%
Percentage of people who feel 'very strongly' against a proposal	7%

(c) Cardinal data

Percentage of people in a club who are less than 20 years old	12%
Percentage of people in a club who are between 20 and 35 years old	18%
Percentage of people in a club who are between 35 and 50 years old	27%
Percentage of people in a club who are between 50 and 65 years old	29%
Percentage of people in a club who are more than 65 years old	14%

Figure 4.4 Typical analyses for nominal, ordinal and cardinal data

- **Ordinal data** is one step more quantitative than nominal data. Here we can rank the categories of observations into some meaningful order. For example, we can describe sweaters as large, medium or small. The order of these categories is important, as we know that 'medium' comes between 'large' and 'small' – but this is all that we can say about ordinal data. Other examples of ordinal data are the strength of people's opinions on a scale of 1 to 5, sociological descriptions of employees as A, B1, B2, C1, etc., and exam results as distinction, pass or fail. The key point is that the order of the categories is important, which is why ordinal data is sometimes called ordered or ranked.
- **Cardinal data** has some attribute that can be directly measured. For example, we can weigh a sack of potatoes, measure the time taken to finish a job, find the temperature in an office, and record the time of deliveries. These measures give a precise description, and are clearly the most relevant to quantitative methods.

  We can divide cardinal data into two types depending on whether it is **discrete** or **continuous**. Data is discrete if it takes only integer values. The number of children in a family is discrete data, as is the number of cars owned, machines operated, shops opened and people employed. Continuous data can take any value and is not restricted to integers. The weight of a bag of biscuits is continuous, because it can take a non-integer value like 256.312 grams – as are the time taken to serve a customer, the volume of oil delivered, the area covered in carpet, and the length of a pipeline.

Sometimes there is a mismatch in data types. For example, the lengths of people's feet are continuous data, but shoes come in a range of discrete sizes that are good enough for most needs; people's heights are continuous, but most people describe their height to the nearest centimetre or inch. If the units of measurement are small, the distinction between discrete and continuous data begins to disappear. Salaries are discrete as they are multiples of a penny or cent, but the units are so small that it is reasonable to describe them as continuous.

Primary and secondary data

Another important classification of data describes the way that it is collected. When you want some data you can either collect it yourself (giving primary data) or use data that someone else has already collected (secondary data).

- **Primary data** is new data collected by an organisation itself for a specific purpose.
- **Secondary data** is existing data collected by other organisations or for other purposes.

Only data collected by the user for a particular purpose is primary, and this has the benefits of fitting the needs exactly, being up to date, and being reliable. Secondary data might be published by other organisations, available from research studies, published by the government, already available within an organisation, and so on. This has the advantages of being much cheaper, faster and easier to collect. It also has the benefit of using sources that are not generally available, as firms are willing to give information to impartial bodies, such as governments, international organisations, universities, industry representatives, trade unions and professional institutions.

If there is reasonable secondary data, you should use it. There is no point in spending time and effort in duplicating data that someone already has. Unfortunately, secondary data is often not good enough for a particular purpose, is in the wrong form, or is out of date. Then you have to balance the benefits of having primary data with the cost and effort of collecting it. For major decisions such as the launch of a new product, it is worth running a market survey to collect primary data about customer reactions; for broader issues, such as future economic conditions, it is better to use secondary data prepared by the government.

In practice, the best option is often a combination of primary and secondary data – perhaps with secondary data giving the overall picture and primary data adding the details. For example, a UK logistics company might get a broad view of industrial prospects from secondary data collected by the government and the European Union; more details come from secondary data collected by, say, the Road Haulage Association and the Chartered Institute for Transport and Logistics; then the company can collect specific primary data from its customers.

Review questions

4.4 Why is it useful to classify data?

4.5 How can you classify data?

4.6 What is the difference between discrete and continuous data?

4.7 Give examples of nominal, ordinal and cardinal data.

4.8 'Primary data is always better than secondary data.' Do you agree?

IDEAS IN PRACTICE Finding secondary data

There are many sources of secondary data. For example, the UK government's Statistical Service publishes broad reviews in a *Monthly Digest of Statistics*[2] and an *Annual Abstract of Statistics*.[3] Their *Guide to Official Statistics*[4] lists the more specialised figures they publish. Other countries have similar publications, and the results are summarised by international bodies such as the United Nations, the European Union, the World Bank and the International Monetary Fund. Most of this data is available on official websites. In addition to government information, a huge amount of data is published by individual companies and organisations – as well as information provided by services such as Reuters, CNN, BBC, the *Financial Times*, etc., or survey companies, such as Gallup, Nielsen and Mori.

Using samples to collect data

When there is no appropriate secondary data, you have to collect your own primary data. You collect this from the relevant **population** that can supply data. Here we are using population in the statistical sense of all people or items that share some common characteristic. When the Post Office wants to see how long it takes to deliver first-class letters, the population is all letters that are posted first-class; a consumer organisation testing the quality of Whirlpool dishwashers would define the population as all the dishwashers made by Whirlpool; a toy manufacturer getting reactions to a new game might define the population of potential customers as all girls between the ages of 6 and 11.

Obviously, it is important to identify the right population, as a mistake here makes all the subsequent data collection and analysis pointless. But this is not as easy as it seems. For a survey of student opinion, the population is clearly students – but does this mean only full-time students, or does it include part-time, day-release and distance-learning students? What about students who are doing a period of work experience, school students, and those studying but not enrolled in courses? If the population is 'imported cars', does this include those where components are imported but assembly is done in this country, or those where almost-finished cars are imported but finishing is done here, or those where components are exported for assembly and the finished car is then brought back?

Even when we can identify a population in principle, there can be difficulties translating this into actual sources of data. If the population is houses with telephones, you can get a list of these from a telephone directory (being careful with houses that have ex-directory numbers, or where entries are missing for some other reason). A complete list of a population is a **sampling frame**. Some common sources of sampling frames include electoral registers, memberships of organisations (such as the Automobile Association), lists of employees, customer loyalty cards, website addresses, and credit rating agencies. But suppose your population is people who bought an imported television set within the last five years, or people who use a particular supermarket, or people who caught a cold last winter. How could you get a list of such people? The only real way is to ask a very large number, and then ignore everyone who does not have the features you want. However, this only works with people, and there are real problems when the population is, say, basking sharks that visited the Cornish coast last year.

Even when you have a sampling frame it can still be difficult to collect data. The most common problem is the size of the population. The sampling frame of, say, 'households that use electricity in Germany' has over 60 million entries. Then there are two alternatives:

- a **census**, which collects data from every entry in the sampling frame – which is the whole population, or
- a **sample**, which collects data only from a representative sample of entries in the sampling frame.

WORKED EXAMPLE 4.1

What does it really mean when an advertisement says that 'eight out of ten dogs prefer' a particular brand of dog food?

Solution

It probably means that in a particular, relatively small test – and under certain conditions – eight out of ten dog owners who expressed an opinion said that their dog seemed to show some preference for this food over some alternative they were offered.

Types of sample

When the population is small and the results are important, it is worth doing a census and collecting data from every member of the population. A company might run a census of every person working in a department to get their views on a proposed reorganisation; a service might ask all of its users to comment on its quality; a housing association might ask all of its tenants about some proposal.

A census clearly gives the most accurate results – but it is never completely accurate, as there are inevitably errors, misunderstandings and omissions. A census of people living in Copenhagen, for example, will always find some who are ill, are on holiday, are travelling, cannot answer, or simply refuse to answer. So a census is difficult, time-consuming, expensive – and still not entirely reliable.

Sampling might be inherently less accurate – but it is also easier, cheaper and faster. The difficult part is to identify a sample that gives a fair representation of the whole population. And this gives the inherent weakness of samples, as there is always uncertainty and the results can never give a completely reliable picture of the population. When your breakfast cereal contains 20% fruit, you do not expect every spoonful to contain exactly this amount. However, we can choose samples carefully to make them more reliable. In particular, the samples should be big enough to give a fair representation of the population, but small enough to be practical and cost effective. We return to this problem of sample size in Chapter 16, but here we consider the different ways of choosing a representative sample (illustrated in Figure 4.5).

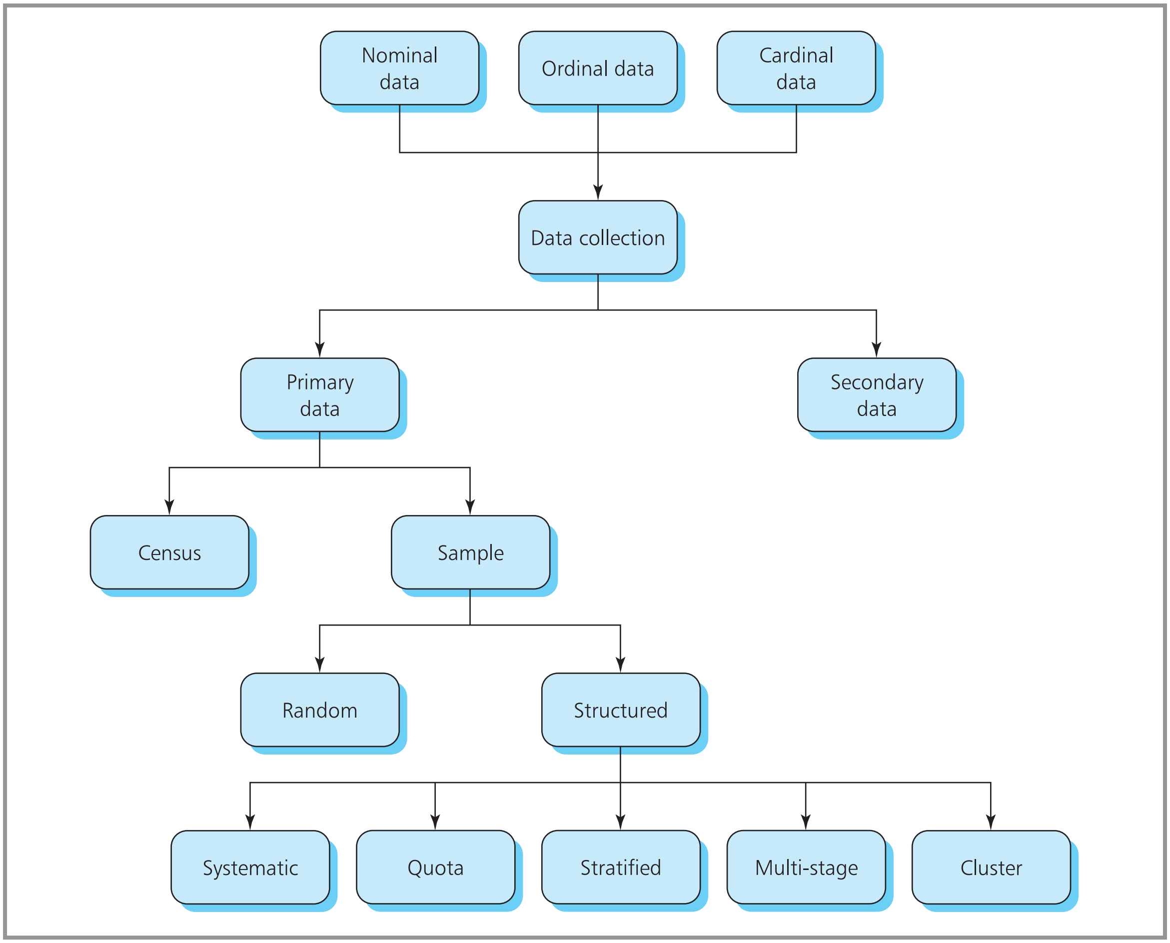


Figure 4.5 Types of samples

Random sample

The **random sample** is the most common type of sample, and has the essential feature that every member of the population has exactly the same chance of being chosen to supply data. If you randomly choose one member of a football team, it means that all 11 members form the population and they all have exactly the same chance of being chosen.

Remember that a random sample is not disorganised or haphazard. If you want some data about tinned soup, you could go to a supermarket and buy the first dozen tins of soup you see. This is haphazard – but it is certainly not random, as tins of soup in every other shop have no chance of being chosen. When a television programme asks people to phone in and give their views, only certain types of people bother to respond, and they form nothing like a random sample of viewers.

With a raffle you can take a genuinely random sample by putting numbers into a hat and choosing one without looking. On a bigger scale, national lotteries use some mechanism to select numbered balls at random. In business, a more common approach uses **random numbers**, which form a stream of random digits – such as 5 8 6 4 5 3 0 1 1 7 2. . . . You can get these from the RAND function in a spreadsheet, which generates a random number between 0 and 1, or RANDBETWEEN (lower, upper) which generates a random integer between lower and upper. (Actually, computers generate 'pseudo-random' numbers but they are good enough for most purposes.)

Suppose that you want to collect data from a random sample of people visiting an office. Using the string of random digits above, you could stand by the door and interview the fifth person to pass, then the eighth person after that, then the sixth after that, then the fourth after that, and so on. The result is a completely random sample, which should give a fair representation of the population. If a sample does not exactly reflect the population, it is said to be **biased**. Suppose that you take a sample of shops in a town centre and find that 25% of them opened within the past two years – but you know that only 10% of shops really opened in this time. Then your sample is biased in favour of newer shops.

It is often difficult to avoid bias. If you decide to save time by simply writing down a series of digits that looks random, you will always introduce bias – perhaps reflecting your preferences for even numbers, sequences that are easy to type on a keyboard, or numbers that are under stronger fingers on a number pad. Similarly, if you ask interviewers to select people at random they will give a biased sample, as they approach people they find attractive – and avoid people that they find unattractive, very tall people, people in an obvious hurry, people in groups, and so on.

WORKED EXAMPLE 4.2

J.T. Eriksson received 10,000 invoices in the last financial year. Their auditors do not have time to examine all of these, so they take a random sample of 200. How could they organise the sample?

Solution

The auditors should start by forming the sampling frame by listing the invoices and numbering them 0000 to 9999. Then they can generate a set of 200 four-digit random numbers, such as 4271 6845 2246 9715 4415 0330 8837 etc. Selecting invoices numbered 4271, 6845, 2246, etc., gives a completely random sample.

Even a well-organised random sample can be affected by a few atypical results. A survey of the amount people spend on transport is biased if one randomly chosen person is a film star who just bought a Boeing 747. We can avoid problems like this by adding some structure to the sample. The results are not entirely random, but they keep significant random elements – and they aim at giving results of equivalent accuracy, but with a much smaller and more convenient sample.

Systematic sample

An easy way of organising a non-random sample is to collect data at regular intervals by means of a **systematic sample**. For example, you might interview every tenth person using a service, weigh every twentieth unit from a production line, or count the people in every sixth car passing. Clearly this is not a random sample, as every member of the population does not have the same chance of being chosen. If you interview every tenth person using a service, then members 11, 12, 13 and so on have no chance of being selected.

Sometimes a systematic sample is near enough to random – or at least pseudo-random – but the regularity often introduces bias. Checking the contents of every twentieth bottle filled by a bottling machine may be unreliable if every twentieth bottle is filled by the same head on the machine; collecting data from every thirtieth person leaving a bus station may introduce bias if buses carry about 30 people, and you are always interviewing the people who get off last.

WORKED EXAMPLE 4.3

A production line makes 5,000 units a day. How can the Quality Control Department take a systematic sample of 2% of these?

Solution

Quality Control check 2% of 5,000 units, which is $5,000/100 \times 2 = 100$ units a day. A systematic sample checks every $5,000/100 = 50$th unit, which is units numbered 50, 100, 150, 200 and so on.

Stratified samples

When there are distinct groups or strata in the population, it is a good idea to make sure that members from each stratum are represented in the sample. So we divide the population into strata, and then take a random sample from each, with the number chosen from each stratum ensuring that the overall sample contains the right mix. This is called a **stratified sample**. For example, 60% of people working in a company might be women. To get a stratified sample of views, you divide the population of employees into two strata – women and men – and randomly select 60% of your sample from women and 40% from men.

WORKED EXAMPLE 4.4

In Westmorefield, companies are classified as manufacturers (20%), transport operators (5%), retailers (30%), utilities (10%), and other services (35%). How would you select a stratified sample of 40 companies?

Solution

The strata are the types of company, so you divide the population into these strata and randomly select the appropriate number from each. The population has 20% manufacturers, so you randomly select $40/100 \times 0.2 = 8$ of these for the sample. Similarly, you randomly select 2 transport operators, 12 retailers, 4 utilities and 14 other services.

A problem with stratified samples appears with small groups. In worked example 4.4, if Westmorefield had very few transport operators, the strata sample would not have suggested collecting data from any of them – but their views might still be important. We could get around this by increasing the sample size, but the sample becomes very large if we include every possible stratum. An alternative is simply to collect views from all strata, even if they are very small. Then the small groups are over-represented, so the sample is biased – but it does include contributions from all parts.

Quota samples

Quota samples extend the idea of stratified sampling by adding a more rigid structure. It looks at the characteristics of the population, and then specifies the characteristics needed in the sample to match this exactly. Suppose you want to see how people will vote in an election. For a quota sample you choose a sample that contains exactly the same proportions of people with different characteristics as the population of people eligible to vote. If the population consists of 4% of men who are over 60, retired from manual work and living alone, then the sample is chosen to also have this proportion.

Quota sampling usually has interviewers who are given a quota of people with different characteristics to interview. Each interviewer has to fill their quota, but they still choose the actual people, so there is still a significant random element. But the process is not random, as interviewers who have filled their quota of one category do not interview any more people in that category, and they have no chance of being chosen.

WORKED EXAMPLE 4.5

Census records of 56,300 people in a town show the following features.

Age	18 to 25	16%
	26 to 35	27%
	36 to 45	22%
	46 to 55	18%
	56 to 65	12%
	66 and over	5%
Sex	Female	53%
	Male	47%
Social class	A	13%
	B	27%
	C1	22%
	C2	15%
	D	23%

How could you organise a quota sample of 1,200 people?

Solution

The sample should contain exactly the same proportion in each category as the population. 16%, or 192 people, should be aged 18 to 25. Of these 192 people, 53%, or 102, should be women. Of these 102 women, 13%, or 13, should be in social class A. Similarly, 5%, or 60 people, should be at least 66 years old; 47%, or 28 of these, should be male; and 23% of these, or 6 people, should be in social class D. Repeating these calculations for all other combinations gives the following quotas.

Age		18 to 25	26 to 35	36 to 45	46 to 55	56 to 65	66 and over
Female	A	13	22	18	15	10	4
	B	27	46	38	31	21	9
	C1	22	38	31	25	17	7
	C2	15	26	21	17	11	5
	D	23	40	32	26	18	7
Male	A	12	20	16	13	9	4
	B	24	41	34	27	18	8
	C1	20	34	27	22	15	6
	C2	14	23	19	15	10	4
	D	21	35	29	23	16	6

Rounding to integers introduces small errors in the quotas, but these make little difference with reasonably large samples.

Multi-stage samples

Suppose that you want a sample of people who subscribe to a particular magazine. If you take a random sample, you will probably find that they are spread over a wide geographical area, and it is inconvenient and expensive to travel and interview them. A cheaper option is to use **multi-stage sampling**, which makes sure that a sample is confined to a smaller geographical area.

The usual approach is to divide the country into a number of geographical regions, such as television or local radio regions. Then select some of these regions at random, and divide them into smaller subdivisions, perhaps parliamentary constituencies or local government areas. Then select some of these subdivisions at random and again divide them into smaller areas, perhaps towns or parliamentary wards. Continue in this way until you have small enough areas, and then identify a sample of individuals from within these areas.

WORKED EXAMPLE 4.6

How would you set about choosing a stratified sample of 1,000 people in Scotland?

Solution

One approach is to randomly select two regions, then randomly select three parliamentary constituencies in each of these, then randomly select three wards in each of these, and so on. The following table shows an outline plan that gives a sample size of $2 \times 3 \times 3 \times 4 \times 15 = 1,080$. There are obviously other alternatives.

Stage	Area	Number selected
1	Region	2
2	Parliamentary constituency	3
3	Ward	3
4	Street	4
5	Individuals	15

Cluster sampling

Cluster sampling chooses the members in a sample not individually, but in clusters. If you want views from people living in a county, it is much easier to visit a cluster of people in a single town rather than visit people spread over the whole county. Similarly, one file of invoices might be representative of all invoices, so it is easier to sample from this than look through all files. The approach is to divide the whole population into a number of groups or clusters, choose a number of these clusters at random, and then take a random sample or census from each cluster.

Cluster sampling has the benefits of reducing costs and being convenient to organise. It is especially useful when surveying people working in a particular industry when individual companies form the clusters.

Review questions

4.9 Why would you use a sample to collect data?

4.10 It is always difficult to go through the steps of defining the population, designing a sampling frame, identifying actual individuals in this frame, and then collecting data from them. Do you agree with this?

4.11 Why is it important to identify the correct population for a survey?

4.12 What types of sampling can you use?

4.13 What is the key feature of a random sample?

Organising data collection

After identifying an appropriate sample, the next stage is actually to collect the data. There are two ways of doing this. Firstly, you can use direct observation to see what is happening, and secondly you can ask people questions.

Observation

When the population consists of machines, animals, files, documents or any other inanimate objects, the only way to collect data is by direct observation. Then an observer watches some activity and records what happens, typically counting a number of events, taking some measurements, or seeing how something works. But the observers do not have to be human, as automatic recorders are better for simple tasks like counting the number of people who enter a shop. Similarly, courier services automatically track parcels using bar codes, magnetic stripes or radio frequency identification tags (RFIDs), supermarket check-outs collect data about purchases, telephone services record communications data, computers analyse CCTV images, and so on.

Observation is usually more reliable than asking for data – but human observers get tired, make mistakes, get distracted and misunderstand, while automatic observers break down and develop faults.

Questionnaires

When you cannot collect data by observation, you have to ask people to supply it. This means interviewing the sample and eliciting data by a series of questions. According to the Gallup organisation,[5] such interviews can find:

- whether a respondent is aware of an issue ('Do you know of any plans to develop . . . ?'),
- general feelings for an issue ('Do you think this development is beneficial . . . ?'),
- views about specific points in an issue ('Do you think this development will affect . . . ?'),
- reasons for a respondent's views ('Are you against this development because . . . ?'),
- how strongly these views are held ('On a scale of 1 to 5, how strong are your feelings about this development . . . ?').

A major problem with asking questions is reliability, as people tend to give the answers they feel they ought to give – or the answer the interviewer wants – rather than the true answer. For example, fewer people say that they use their mobile phone while driving than is found from direct observation – and more people claim to wash their hands after using a public lavatory, eat healthier food, do exercise, give more to charity, read books, and so on. There are also problems with emotional responses, so asking customers how they liked the food in a restaurant is likely to get replies based on the whole experience, including who they were with, how they felt, what the weather was like, how attractive the servers were, and so on.

The usual way of asking questions is to present them in a **questionnaire** – which is an ordered list of questions. There are several arrangements for administering a questionnaire, including personal interview, telephone interview, postal survey, email survey, panel survey and longitudinal survey. Sometimes the questionnaire is given to people to complete themselves (particularly by post or email), and sometimes it is completed by an interviewer.

Personal interviews

These can be the most reliable way of getting detailed data. They get a high response rate, as only about 10% of people refuse to answer on principle, but this depends on circumstances and few people will agree to a long, complicated or inconvenient interview. Quota sampling needs some assessment and selection of the people questioned, so it must use personal interviews.

In principle, collecting data by personal interviews is easy, as it only needs someone to ask questions and record the answers. The reality is more complicated and depends on skilled interviewers. For instance, they must be careful not to direct respondents to a particular answer by their expression, tone of voice, or comments. And they should help sort out questions that are unclear and ask follow-up questions – but not explain the questions or offer any help, as this would introduce bias to the answers.

Usually the main drawback with personal interviews is the high cost. Each interviewer has to be trained, taken to the right place, given somewhere to work, fed, given overnight accommodation, and so on. Typically, an interviewer spends 40% of their time in travel, 25% in preparation and administration, and only 35% in asking questions.

Telephone interviews

These can be used for the 95% of people who own a telephone. This is a popular way of organising surveys, as it is cheap and easy, involves no travel and gets a high response rate. On the other hand, it has the disadvantage of bias, as it uses only people with telephones who accept anonymous calls and are willing to give honest answers over the phone. Other weaknesses are that observers cannot see the respondents, and phone calls annoy people who object to the intrusion.

The usual procedure for telephone interviews has a computer selecting a number at random from a directory listing. Then an interviewer asks the questions presented on a computer screen and types in the answers. This allows the computer to analyse answers interactively and choose an appropriate set of questions – and it prevents errors during the transfer of data.

Postal surveys

These send a questionnaire through the post, and ask people to complete it and return the answers. This works best with a series of short questions asking for factual – preferably numerical – data. Postal surveys are cheap and easy to organise, and are suitable for very large samples. But there are numerous drawbacks, such as the lack of opportunity to observe people or clarify points, and the difficulty of getting a questionnaire to the right people. There are also problems with bias as only certain types of people bother to reply, and those who have had bad experiences are more likely to respond than those who have had good experiences.

The major problem with postal surveys is the low response rate, which is usually less than 20% and can approach zero. This might be raised by making the questionnaire short and easy to complete, sending it to a named person (or at least a title), enclosing a covering letter to explain the purpose of the survey, including a pre-paid return envelope, promising anonymity of replies, using a follow-up letter or telephone call if replies are slow, and promising a summary of results. Often people try to increase the response rate by offering some reward – typically a small gift, discount on a future purchase, or entry to a prize draw – but this introduces more bias, as respondents now feel more kindly towards the questionnaire.

Email surveys

These are an extension to postal surveys, and they can contact a huge number of people at almost no cost. But there are obvious problems with bias, as they are limited to people who regularly use email, publish their address, accept unsolicited messages, and want to reply. Spam is a huge problem on the Internet, and most people use filters that would not allow random questionnaires through. An alternative is to open a website and ask people to visit and give their views. These get replies that are in no way representative, but a lot of data is routinely collected from surveys on websites such as www.yougov.com.

Panel surveys

These assemble a representative panel of respondents who are monitored to see how their opinions change over time. For example, you could monitor the political views of a panel during the lifetime of a government, or their awareness of a product during an advertising campaign. Panel surveys are expensive and difficult to administer, so they use very small samples.

One interesting problem is that members of the panel can become so involved in the issues that they change their views and behaviour. For instance, a panel that is looking at the effects of a healthy eating campaign might become more interested in health issues and change their own habits. Another problem is that some panel members inevitably have to leave, and the remainder become less representative of the population.

Longitudinal surveys

These are an extension of panel surveys that monitor a group of respondents over a long period. For example, studies routinely monitor the effects of lifestyles on health over many decades – and find that exercise reduces heart disease, alcohol increases liver disease, smoking reduces life expectancy, and so on. One television company has been monitoring the progress of a group of children – then adults – over the past 50 years. These studies need a lot of resources, and they are generally limited to studies of sociological, health and physical changes.

IDEAS IN PRACTICE Mareco

For 50 years after the Second World War Poland had a centrally planned economy with the government controlling most businesses. However, the economy was reformed in 1990, and newly privatised companies began to approach their customers to find exactly what they wanted. The market research industry grew from nothing in 1990 to $50 million in 1998. This trend continued as the economy evolved and Poland joined the European Union in 2004. There are now nine major research companies, with Mareco as the largest with up to 90% of some markets.

Many Polish companies still have little interest in market research, maintaining their traditional view that it is a waste of money. However, foreign companies investing in Poland do not know the country well, and they want to learn about the new market. Eighty per cent of Mareco's clients are foreign companies starting new operations in Poland.

Mareco aims to conduct research as quickly and accurately as possible, 'to provide the best insights into our clients' markets'. They organise operations from a head office in Warsaw, which has three separate departments.

- *Opinion Polls and Market Research Department* – works at the start of a project, forming relations with customers, preparing research offers, scheduling work, designing questionnaires, selecting samples, and so on. It also works at the end of projects, analysing results of surveys and writing reports for clients.
- *Field Research Department* – actually collects the data using a network of 24 co-ordinators and 200 interviewers throughout Poland.
- *Data Processing Department* – takes the data collected in the field research, analyses it and creates databases that are passed back to the Opinion Polls and Market Research Department.

Mareco joined Gallup International Association in 1994, allowing it to introduce new ideas and use the experience of other Gallup companies. Mareco's main problem comes from interviewers, who are often students who want short-term employment. They can do simple data collection, but lack the skills for in-depth interviews, or other more demanding jobs.

Sources: website at www.mareco.pl and company reports, Mareco, Warsaw.

Design of questionnaires

Designing a good questionnaire is far from easy – and many surveys fail because they asked the wrong questions, or asked the right questions in the wrong way. Even subtle differences in wording and layout can have unexpected effects. A lot of research into the design of questionnaires has led to useful guidelines, illustrated in the following list. Many of these are common sense, but they are often overlooked.

- A questionnaire should ask a series of related questions, and should follow a logical sequence.
- Make the questionnaire as short as possible. People will not answer long or poorly presented questionnaires, and unnecessary questions cost more to collect and analyse.
- Questions should be short, simple, unambiguous, easy to understand and phrased in everyday terms: if people do not understand a question they will give any response.
- Even simple changes to phrasing can give very different results. For example, people look more favourably on a medical treatment described as having a 60% success rate than on the same treatment described as having a 40% failure rate.

- People are not always objective, so asking 'Do you think that prison sentences would deter speeding drivers?' gets a different response from 'If you are caught driving too fast, should you go to prison?'
- Avoid leading questions such as 'Do you agree with the common view that NBC news is more objective than Fox news?' Such questions encourage conformity rather than truthful answers.
- Use phrases that are as neutral as possible – rephrasing 'Do you like this cake?' to 'How do you rate the taste of this cake on a scale of 1 to 5?'
- Phrase all personal questions carefully – with 'Have you retired from paid work?' being more sensitive than 'Are you an old age pensioner?'
- Do not give warnings, as a question that starts 'We understand if you do not want to answer this, but . . .' will discourage everyone from answering.
- Avoid vague questions like 'Do you usually buy more meat than vegetables?' This raises a series of questions – what does 'usually' mean? What is 'more'? Do frozen meals count as meat or vegetables?
- Ask positive questions like 'Did you buy a Sunday newspaper last week?' rather than the less definite 'Has the number of Sunday newspapers you buy changed?'
- Avoid hypothetical questions such as 'How much would you spend on life insurance if you won a million dollars on a lottery?' Any answer is speculative and probably not based on any real thought.
- Avoid asking two or more questions in one, such as 'Do you think this development should go ahead because it will increase employment in the area and improve facilities?' This will get confused answers from people who think the development should not go ahead, or those who think it will increase employment but not improve facilities.
- Open questions – such as 'Have you any other comments?' – allow general comments, but they favour the articulate and quick thinking, and are difficult to analyse.
- Ask questions with pre-coded answers, with respondents choosing the most appropriate answer from a set of alternatives. There are many formats for these, with examples in Figure 4.6.
- Be prepared for unexpected effects, such as sensitivity to the colour and format of the questionnaire, or different types of interviewer getting different responses.
- Always run a pilot survey before starting the whole survey. This is the only way to identify problems and improve the questionnaire design.

Non-responses

Even the best questionnaire will not get a response from everyone in the sample. There are several reasons for this, including the following:

- People are unable to answer the questions, perhaps because of language difficulties or ill health – or they simply do not know the answer.
- They are out when the interviewer called – but careful timing of calls, appointments and repeat visits can reduce this problem.
- They are away for some period, with holiday and business commitments making surveys in the summer particularly difficult.

1 Will you be joining an evening class this term? YES/NO

2 How many children do you have? (please circle answer)

0 1 2 3 4 more

3 Do you use a computer in your office? (please tick a box)

Yes ☐ No ☐

4 Do you think the government should spend more on education? (please tick a box)

Strongly agree	Agree	Do not know	Disagree	Strongly disagree

5 A proposal has been made to ban all traffic from the town centre. Please circle the number which most accurately reflects your view on this.

Strongly approve	1	2	3	4 neutral	5	6	7	Strongly disapprove

6 Why are you taking this course? (please circle any appropriate answers)
(a) out of interest
(b) to get a qualification
(c) to help in work
(d) friends are taking it
(e) to resit a course failed last year
(f) other reason, please specify ....

7 How old are you? (please tick one answer)

☐ less than 18 ☐ 50 to 70
☐ 18 to 35 ☐ more than 70
☐ 35 to 50

Figure 4.6 Examples of precoded questions

- They have moved and are no longer at the given address – in which case it is rarely worth following up a new address.
- They refuse to answer – about 10% of people refuse to answer on principle, and nothing can be done about these.

You might be tempted to ignore non-responses. But then you are assuming that the non-respondents make no difference – meaning that actual respondents fairly represent the sample, and this in turn fairly represents the population. This is not necessarily true, and there can be a systematic reason for the non-responses. Suppose you run a survey and start with the question 'Does your company have any strategic alliances?' Companies that would answer 'No' to this question are unlikely to be interested enough to complete the rest of the questionnaire, so replies are biased towards companies that actually have alliances. One extreme case used a postal questionnaire to ask about literacy skills (in the way that people with reading difficulties can pick up information packs when visiting their local library).

To avoid these effects you should always follow up non-respondents. Another well-timed visit, telephone call or letter might encourage non-respondents to reply, but realistically you will get limited success. Then the only option is to examine non-respondents to make sure they do not share some common characteristic that introduces bias.

Summary of data collection

The important point about data collection is that it does not happen by chance, but needs careful planning. This typically involves the following steps.

1 Define the purpose of the data.
2 Describe the data you need to achieve this purpose.
3 Check available secondary data and see how useful it is.
4 Define the population and sampling frame to give primary data.
5 Choose the best sampling method and sample size.
6 Identify an appropriate sample.
7 Design a questionnaire or other method of data collection.
8 Run a pilot study and check for problems.
9 Train interviewers, observers or experimenters.
10 Do the main data collection.
11 Do follow-up, such as contacting non-respondents.
12 Analyse and present the results.

This seems rather complicated, and you may be tempted to take short-cuts. Remember, though, that every single decision in an organisation depends on available information – and this, in turn, depends on reliable data collection. Unfortunately, even careful planning cannot eliminate all errors, and typical problems arise from the following factors:

- Failure to identify the right population
- Choosing a sample that does not represent this population
- Mistakes in contacting members of the sample
- Mistakes in collecting data from the sample
- Introducing bias from non-respondents
- Mistakes during data analysis
- Drawing invalid conclusions from the analysis.

Review questions

4.14 What method of data collection is best for:
  (a) asking how companies use their websites
  (b) asking colleagues for their views on a proposed change to working conditions
  (c) testing the effect of exercise on heart disease
  (d) testing the accuracy of invoices?

4.15 What is wrong with the following questions in a survey?
  (a) 'Most people want higher retirement pensions. Do you agree with them?'
  (b) 'Does watching too much television affect children's school work?'
  (c) 'Should the UK destroy its nuclear arms, reduce spending on conventional arms and increase expenditure on education?'
  (d) 'What is the most likely effect of a single European currency on pensions?'

4.16 What can you do about non-responses in a postal survey?

4.17 Why are non-responses irrelevant for quota sampling?

4.18 'It is best to get some data quickly so that you can start planning the analyses.' Do you agree with this?

IDEAS IN PRACTICE PhD research

In 2003 David Grant was awarded a PhD by the University of Edinburgh, for work on 'A study of customer service, customer satisfaction and service quality in the logistics function of the UK food processing industry'. This won a prestigious award from the Institute for Logistics and Transport. The research needed a lot of data collection, which David undertook in the following stages.

1 Initial review of published material, interviews and discussions to identify a suitable research project
2 Literature review of published material to find other work in the area and assess its importance
3 Collection of secondary data to find comparisons for the research
4 Design and initial testing of a questionnaire for a postal survey, with interviews to gauge reactions to the survey
5 Pilot study with the questionnaire sent to 380 companies, with follow-up of those who did not initially respond, giving a response rate of 28%
6 Interviews with respondents to clarify results and refine the questionnaire
7 Main study with the questionnaire sent to 1,215 companies, with follow-up of those that did not initially reply, giving a response rate of 17%.

The details of every research project vary, but this illustrates a common approach. Initial data collection sets the scene for the research; secondary data identifies work that has already been done; primary data extends the research into new areas.

Sources: Grant D., A study of customer service, customer satisfaction and service quality in the logistics function of the UK food processing industry, unpublished PhD thesis, University of Edinburgh, 2003, and private correspondence.

CHAPTER REVIEW

This chapter described different ways of collecting data.

- Managers need reliable information to make decisions, and they get this through data collection, analysis and presentation. Data collection is an essential requirement for management in every organisation.
- Data collection does not just happen, but it needs careful planning. This starts by defining the purpose of the data. In principle, there is an optimal amount of data to collect for any purpose.
- We can classify data in several ways, including quantitative/qualitative, nominal/ordinal/cardinal, and primary/secondary. Data of different types – and with different uses – is collected in different ways.
- The population consists of all people or items that could supply data. These are listed in a sampling frame. It can be difficult to choose the right population and find a sampling frame.
- It is usually too expensive, time-consuming and difficult to collect data from the whole population – giving a census. The alternative collects data from a representative sample of the population and uses this to estimate values for the whole population. There are several types of sample, including random, systematic, stratified, quota, multi-stage and cluster samples.
- The two alternatives for collecting data from the sample are observation and questionnaires. Questionnaires can be administered through personal interview, telephone interview, the Internet, postal survey, panel survey or longitudinal survey.

- There are useful guidelines for designing questionnaires. You should always run a pilot survey to sort out any problems, and examine non-respondents to make sure they do not introduce bias.

CASE STUDY Natural Wholemeal Biscuits

Natural Wholemeal Biscuits (NWB) make a range of foods that they sell to health food shops around eastern Canada. They divide the country into 13 geographical regions based around major cities. The following table shows the number of shops stocking their goods and annual sales in each region.

Region	Shops	Sales ($'000)
Toronto	94	240
Montreal	18	51
Hamilton	8	24
Sudbury	9	18
Windsor	7	23
Québec	12	35
Halifax	8	17
Niagara	6	8
London	5	4
Ottawa	17	66
St John's	8	32
Moncton	4	15
Trois-Rivières	4	25

NWB are about to introduce a 'Vegan Veggie Bar' that is made from a combination of nuts, seeds and dried fruit, and is guaranteed to contain no animal products. The company wants to find likely sales of the bar and is considering a market survey. They already sell 300,000 similar bars a year at an average price of $1.80, and with an average contribution to profit of 36 cents. An initial survey of 120 customers in three shops gave the following characteristics of customers for these bars.

Sex	Female	64%
	Male	36%
Age	Less than 20	16%
	20 to 30	43%
	30 to 40	28%
	40 to 60	9%
	More than 60	4%
Social class	A	6%
	B	48%
	C1	33%
	C2	10%
	D	3%
Vegetarian	Yes	36% (5% vegan)
	No	60%
	Other response	4%
Reason for buying	Like the taste	35%
	For fibre content	17%
	Never tried before	11%
	Help diet	8%
	Other response	29%
Regular buyer of bar	Yes	32%
	No	31%
	Other response	37%

The company wants as much information as possible, but must limit costs to reasonable levels. Experience suggests that it costs $24 to interview a customer personally, while a postal or telephone survey costs $8 per response. The Management Information Group at NWB can analyse the data relatively cheaply, but management time for reviews and discussion is expensive.

Questions

- **How can NWB collect data on potential sales of its Vegan Veggie Bar? What secondary data is available? What primary data do they need, and how should they collect it?**
- **Design a plan for NWB's data collection, including timing, costs and assumptions.**

PROBLEMS

4.1 How would you describe the following data?
(a) Weights of books posted to a bookshop
(b) Numbers of pages in books
(c) Position of football teams in a league
(d) Opinions about a new novel

4.2 What is the appropriate population to give data about the following?
(a) Likely sales of a computer game
(b) Problems facing small shopkeepers
(c) Parking near a new shopping mall
(d) Proposals to close a shopping area to all vehicles

4.3 Describe a sampling procedure to find reliable data about house values around the country.

4.4 Auditors want to select a sample of 300 invoices from 9,000 available. How could they do this?

4.5 Use a computer to generate a set of random numbers. Use these to design a sampling scheme to find the views of passengers using a low-cost airline.

4.6 The readership of a Sunday newspaper has the following characteristics.

Age	16 to 25	12%
	26 to 35	22%
	36 to 45	24%
	46 to 55	18%
	56 to 65	12%
	66 to 75	8%
	76 and over	4%
Sex	Female	38%
	Male	62%
Social class	A	24%
	B	36%
	C1	24%
	C2	12%
	D	4%

What are the quotas for a sample of 2,000 readers? Design a spreadsheet to find the quotas in each category for different sample sizes.

4.7 Give some examples of poor questions used in a survey. How could you improve these?

4.8 Give some examples of bias in a survey. What caused the bias and how could it have been avoided?

4.9 Design a questionnaire to collect data on the closure of a shopping area to all vehicles.

4.10 Describe the data collection for a recent survey by the Consumers' Association or an equivalent organisation. How does this compare with data collection at www.yougov.com?

4.11 How could you collect data about the following?
(a) The consumption of cakes and biscuits in a particular country last year
(b) The amount of time people spend on the telephone
(c) Potential sales of a new format for PC data storage
(d) Likely membership of a new running club
(e) Opinions about traffic congestion charges
(f) Satisfaction with government policies
(g) Views on local government expenditure and financing

RESEARCH PROJECTS

4.1 Design a survey of the way that companies use the Web in your area. What kind of companies form your population? How can you select an unbiased sample from these? What data would you want and how would you collect it? What problems might you meet?

4.2 Use government statistics to see how the gross national product has changed over the past 30 years. How does this compare with other countries? What other international comparisons can you make?

4.3 Run a survey of the way people travel in your area. You might ask basic questions such as how often they make journeys, how far they go and what transport they use. Then you can collect opinions about the quality of their journeys, problems and possible improvements. To make sure your sample is representative, you might add some demographic questions about age, gender, occupation, etc.

Sources of information

References

1 Browning E.B., *Sonnets from the Portuguese*, Number 1, 1850.

2 Statistical Service, *Monthly Digest of Statistics*, Office for National Statistics, London, 2006.

3 Statistical Service, *Annual Abstract of Statistics*, Office for National Statistics, London, 2006.

4 Statistical Service, *Guide to Official Statistics*, Office for National Statistics, London, 2006.

5 Website at www.gallup.com.

Further reading

Books on sampling often focus on market surveys or statistical analysis. The following give more general material.

Barnett V., *Sample Surveys* (3rd edition), Edward Arnold, London, 2002.

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CHAPTER 5

Diagrams for presenting data

Contents

Chapter outline

The amount of detail in raw data obscures the underlying patterns. Data reduction and presentation clears away this detail and highlights the overall features and patterns. It gives a view of the data that is concise, but still accurate. There are two approaches to summarising data. In this chapter we describe some diagrams, and Chapter 6 continues the theme by looking at numerical summaries.

After finishing this chapter you should be able to:

- Discuss the aims of data reduction and presentation
- Design tables of numerical data
- Draw frequency distributions of data
- Use graphs to show the relationship between two variables
- Design pie charts
- Draw different kinds of bar chart
- Consider pictograms and other formats
- Draw histograms for continuous data
- Draw ogives and Lorenz curves for cumulative data.

Data reduction and presentation

In the last chapter we saw that data are the basic numbers and facts that we process to give useful information. So 78, 64, 36, 70 and 52 are data that we process to give the information that the average mark of five students sitting an exam is 60%.

Most people can deal with small amounts of numerical data. We happily say, 'this building is 60 metres tall', 'a car travels 15 kilometres on a litre of petrol', and '16% of people use a particular product'. But we have problems when there is a lot of data. For instance, the weekly sales of a product from a website over the past year are:

51 60 58 56 62 69 58 76 80 82 68 90 72
84 91 82 78 76 75 66 57 78 65 50 61 54
49 44 41 45 38 28 37 40 42 22 25 26 21
30 32 30 32 31 29 30 41 45 44 47 53 54

This gives the raw data – which you probably skipped over with hardly a glance. If you put such figures in a report, people would find it boring and skip to something more interesting – even though the figures could be very important. To make them less daunting we could try putting them in the text, starting with 'In the first week sales were 51 units, and they rose by nine units in the second week, but in the third week they fell back to 58 units, and fell another two units in the fourth week . . .'. Clearly, this does not work with so many numbers, and we need a more convenient format.

The problem is that raw data swamps us with detail, obscuring the overall patterns and shape of the data – we cannot see the wood for the trees. Usually we are not interested in the minute detail, but really want only the overall picture. So imagine that you have put a lot of effort into collecting data and now want to show it to other people. You have two jobs – data reduction to reduce the amount of detail, and data presentation to give the results in a useful format.

- **Data reduction** gives a simplified and accurate view of the data, showing the underlying patterns but not overwhelming us with detail.
- **Data presentation** shows clearly and accurately the characteristics of a set of data and highlights the patterns.

Now we have the sequence of activities for analysing data, which starts with data collection, moves to data reduction and then data presentation. In practice, there is no clear distinction between data reduction and data presentation, and we usually combine them into a single activity. This combined activity of summarising – or more broadly processing or managing – data has the advantages of:

- showing results in a compact form
- using formats that are easy to understand
- allowing diagrams, graphs or pictures
- highlighting underlying patterns
- allowing comparisons of different sets of data
- using quantitative measures.

On the other hand, summarising data has the major disadvantage that it loses details of the original data and is irreversible.

Diagrams for presenting data

When you look around, there are countless examples of diagrams giving information. A newspaper article adds a diagram to summarise its story; an advertisement uses a picture to get across its message; a company's financial performance is summarised in a graph. Diagrams attract people's attention, and we are more likely to look at them than read the accompanying text – hence the saying, 'One picture is worth a thousand words'. Good diagrams are attractive, they make information more interesting, give a clear summary of data, emphasise underlying patterns, and allow us to extract a lot of information in a short time. But they do not happen by chance, and need careful planning.

If you look at a diagram and cannot understand what is happening, it means that the presentation is poor – and the fault is with the presenter rather than the viewer. Sometimes there is a more subtle problem – when you look at a diagram quickly and immediately see one pattern, but then look more closely and see that your initial impression was wrong. To be generous, this might be a simple mistake in presenting the data poorly, but the truth is that people often make a deliberate decision to present data in a form that is misleading and dishonest. Advertisements are notorious for presenting data in a way that gives the desired impression, rather than accurately reflecting a situation, and politicians might be more concerned with appearance than with truth. Huff[1] developed this theme in the 1950s with his classic descriptions of 'How to lie with statistics' and this has been followed by similar descriptions, such as those of Kimble in the 1970s[2] and more recently Wainer.[3,4] The problem is that diagrams are a powerful means of presenting data, but they give only a summary – and this summary can easily be misleading. In this chapter we show how to use diagrams to present information properly, giving a fair and honest summary of the raw data.

There are many types of diagram for presenting data, with the most common including:

- tables of numerical data and frequency distributions
- graphs to show relationships between variables
- pie charts, bar charts and pictograms showing relative frequencies
- histograms that show relative frequencies of continuous data.

The choice of best format is often a matter of personal judgement and preference. But remember that we want to give people information as fairly and efficiently as possible – and we are not just looking for the prettiest picture. Some guidelines for choosing the type of diagram include the following, where appropriate:

- Choose the most suitable format for the purpose
- Always present data fairly and honestly
- Make sure any diagram is clear and easy to understand
- State the source of data
- Use consistent units and say what these are
- Include totals, sub-totals and any other useful summaries
- Give each diagram a title
- Add notes to highlight assumptions and reasons for unusual or atypical values.

Review questions

5.1 What is the difference between data and information?

5.2 What is the purpose of data reduction?

5.3 'Data presentation always gives a clear, detailed and accurate view.' Is this true?

5.4 What are the two main methods of presenting data?

Tables of numerical data

Tables are probably the most common way of summarising data. We have already used several in this book, and you can see more whenever you pick up a newspaper, a magazine, a books or a report. Table 5.1 shows the weekly sales of the product mentioned above, and this gives the general format for tables.

Table 5.1 Weekly sales of product

Week	Quarter 1	Quarter 2	Quarter 3	Quarter 4	Total
1	51	84	49	30	214
2	60	91	44	32	227
3	58	82	41	30	211
4	56	78	45	32	211
5	62	76	38	31	207
6	69	75	28	29	201
7	58	66	37	30	191
8	76	57	40	41	214
9	80	78	42	45	245
10	82	65	22	44	213
11	68	50	25	47	190
12	90	61	26	53	230
13	72	54	21	54	201
Totals	882	917	458	498	2,755

Now you can see that sales are higher in the first two quarters and lower in the second two. But the table is still only a presentation of the raw data – and it does not really give a feel for a typical week's sales, it is difficult to find the minimum or maximum sales, and patterns are not clear. We can emphasise the underlying patterns by reducing the data. To start with, we can find that minimum sales are 21, and then count the number of weeks with sales in a range of, say, 20 to 29. There are six weeks in this range, and counting the number of weeks with sales in other ranges gives the summary shown in Table 5.2.

Tables that show the number of values in different ranges are called **frequency tables**, and the 'ranges' are called **classes**. Then we can talk about the 'class of 20 to 29', where 20 is the lower class limit, 29 is the upper class limit, and the class width is 29 – 20 = 9. We arbitrarily chose classes of 20 to 29, 30 to 39, and so on, but could have used any other reasonable values.

Table 5.2 Frequency table of sales

Range of sales	Number of weeks
20 to 29	6
30 to 39	8
40 to 49	10
50 to 59	9
60 to 69	7
70 to 79	6
80 to 89	4
90 to 99	2

This is largely a choice that is guided by the structure of the data and the use of the table. Two guidelines are as follows:

- The classes should all be the same width.
- There should be enough classes to make patterns clear, but not too many to obscure them; this usually suggests a minimum of four classes, and a maximum around ten.

If the eight classes in Table 5.2 seem too many, we could divide the data into, say, four classes and add a note about the source to give the final result shown in Table 5.3.

Table 5.3 Frequency table of weekly sales

Range	Number of weeks
20 to 39	14
40 to 59	19
60 to 79	13
80 to 99	6

Source: Company Weekly Sales Reports.

Tables 5.1 to 5.3 show an inevitable effect of data reduction – the more data is summarised, the more detail is lost. For instance, Table 5.3 shows the distribution of weekly sales, but it gives no idea of the seasonal variations. We can accept this loss of detail if the result still shows all the information we want and is easier to understand – but not if the details are important. If we want to plan the number of seasonal employees, we could not use Table 5.3 but would have to return to Table 5.1.

You can obviously present a set of data in many different tables – and you always have to compromise between making them too long (when they show lots of detail, but are complicated and obscure underlying patterns) and too short (when they show patterns clearly, but lose most of the detail). Another guideline says that if you repeatedly present data over some period, you should always keep the same format to allow direct comparisons (and you see this effect in company reports and government publications).

WORKED EXAMPLE 5.1

Carlson Industries has collected the following monthly performance indicators over the past five years. How would you summarise these in a different table?

	Year 1	2	3	4	5
January	136	135	141	138	143
February	109	112	121	117	118
March	92	100	104	105	121
April	107	116	116	121	135
May	128	127	135	133	136
June	145	132	138	154	147
July	138	146	159	136	150
August	127	130	131	135	144
September	135	127	129	140	140
October	141	156	137	134	142
November	147	136	149	148	147
December	135	141	144	140	147

Solution

There are many different ways of summarising these, with some options shown in the spreadsheet of Figure 5.1.

	A	B	C	D	E	F	G	H
1	**Carlson Industries**							
2								
3	**Range**	**Frequency**		**Monthly averages**			**Annual averages**	
4								
5	< 99	1		January	138.6		Year 1	128.3
6	100–109	5		February	115.4		Year 2	129.8
7	110–119	5		March	104.4		Year 3	133.7
8	120–129	8		April	119		Year 4	133.4
9	130–139	19		May	131.8		Year 5	139.2
10	140–149	18		June	143.2			
11	> 150	4		July	145.8			
12				August	133.4			
13				September	134.2			
14				October	142			
15				November	145.4			
16				December	141.4			

Figure 5.1 Table of results for Carlson Industries

Frequency distributions

The results shown in a frequency table form a **frequency distribution**. For example, the following table shows the frequency distribution for the number of weekly deliveries to a logistics centre during a year.

Number of deliveries	20 to 39	40 to 59	60 to 79	80 to 99
Number of weeks	14	19	13	6

As you can see, there are six observations in the highest class of deliveries, 80 to 99. But suppose that there had been one unusual week with 140 deliveries. In this table it would be in a class of its own some distance away from the others.

Number of deliveries	20 to 39	40 to 59	60 to 79	80 to 99	100 to 119	120 to 139	140 to 169
Number of weeks	14	19	13	5	0	0	1

Sometimes it is important to highlight outlying values – but usually it is just confusing, so we define the highest class to include all outliers. Here we do this by defining the top class as '80 or more'. Similarly, it is better to replace the precise '20 to 39' for the bottom class by the less precise '39 or fewer'.

An obvious point is that you have to define boundaries between classes so that there is no doubt about which class an observation is in. Here you could not define adjacent classes of '20 to 30' and '30 to 40', as a value of 30 could be in either one. Using '20 to 29' and '30 to 39' avoids this problem for discrete data, but fails with continuous data. For instance, you could not classify people's ages as '20 to 29' and '30 to 39', as this would leave no place for people who are 29.5. Instead you have to use more precise – but rather messy – phrases like 'aged 20 or more and less than 30'.

WORKED EXAMPLE 5.2

The weights of materials (in kilograms) needed for 30 projects are as follows. Draw a frequency distribution of this data.

202 457 310 176 480 277 87 391 325 120 554
94 362 221 274 145 240 437 404 398 361 144
429 216 282 153 470 303 338 209

Solution

The first decision is the best number of classes. The range is between 87 kg and 554 kg, so a reasonable solution is to use six classes of 'less than 100 kg', '100 kg or more and less than 200 kg', '200 kg or more and less than 300 kg', and so on. Notice that we are careful not to phrase these as 'more than 100 kg and less than 200 kg', as a project needing exactly 100 kg would not fit into any class. Adding the number of observations in each class gives the frequency distribution in Figure 5.2.

	A	B	C
1	**Frequency distribution**		
2			
3	**Class**	**Frequency**	**Percentage frequency**
4	**less than 100 kg**	2	6.7
5	**100 kg or more, and less than 200 kg**	5	16.7
6	**200 kg or more, and less than 300 kg**	8	26.7
7	**300 kg or more, and less than 400 kg**	9	30.0
8	**400 kg or more, and less than 500 kg**	5	16.7
9	**500 kg or more**	1	3.3
10	**Totals**	**30**	**100.0**

Figure 5.2 Frequency distribution for worked example 5.2

	A	B	C	D	E
1	**Different types of frequency distribution**				
2					
3	**Class**	**Frequency**	**Cumulative frequency**	**Percentage frequency**	**Cumulative percentage frequency**
4	**less than 100 kg**	2	2	6.7	6.7
5	**100 kg or more, and less than 200 kg**	5	7	16.7	23.3
6	**200 kg or more, and less than 300 kg**	8	15	26.7	50.0
7	**300 kg or more, and less than 400 kg**	9	24	30.0	80.0
8	**400 kg or more, and less than 500 kg**	5	29	16.7	96.7
9	**500 kg or more**	1	30	3.3	100.0
10	**Total**	**30**		**100.0**	

Figure 5.3 Different types of frequency distribution

Frequency distributions show the number of observations in each class, but in Figure 5.2 we also calculated a **percentage frequency distribution**, which shows the percentage of observations in each class. Another useful extension shows the cumulative frequencies. Instead of recording the number of observations in a class, **cumulative frequency distributions** add all observations in lower classes. In Figure 5.2 there were two observations in the first class, five in the second class and eight in the third. The cumulative frequency distribution shows two observations in the first class, $2 + 5 = 7$ in the second class, and $2 + 5 + 8 = 15$ in the third. In the same way, we can also draw a **cumulative percentage frequency distribution**, as shown in Figure 5.3.

Review questions

5.5 What are the advantages of using tables of data?

5.6 What is a frequency distribution?

5.7 What is the best number of classes for a table of data?

5.8 Tables of data can seem very dull – so why are they so widely used?

IDEAS IN PRACTICE UK cereal production

Tables range from the very simple to the very complex. For example, we can show the percentages of wheat, barley, oats and other cereals grown in the UK in the following simple table:

Cereal	Percentage of cereal-growing land
Wheat	64%
Barley	32%
Oats	3%
Others	1%

Or we can add a lot more detail to get the result in Table 5.4 – and we could continue adding more data until the tables become very complex.

Ideas in practice continued

Table 5.4 Main cereal crops grown in the United Kingdom

	1990	1995	2000	2005
Wheat				
Area ('000 hectares)	2,014 (55.0)	1,859 (58.4)	2,086 (62.3)	1,869 (63.8)
Harvest ('000 tonnes)	14,033 (62.1)	14,312 (65.4)	16,708 (69.7)	14,877 (70.6)
Yield (tonnes per hectare)	7.0	7.7	8.0	8.0
Barley				
Area ('000 hectares)	1,517 (41.4)	1,193 (37.5)	1,128 (33.7)	944 (32.2)
Harvest ('000 tonnes)	7,911 (35.0)	6,842 (31.3)	6,492 (27.1)	5,533 (26.3)
Yield (tonnes per hectare)	5.2	5.7	5.8	5.9
Oats				
Area ('000 hectares)	107 (2.9)	112 (3.5)	109 (3.3)	91 (3.1)
Harvest ('000 tonnes)	530 (2.3)	617 (2.8)	640 (2.7)	534 (2.5)
Yield (tonnes per hectare)	5.0	5.5	5.9	5.9
Totals				
Area ('000 hectares)	3,660	3,182	3,348	2,928
Harvest ('000 tonnes)	22,582	21,870	23,988	21,060

Sources: adapted from Department for the Environment, Farming and Rural Affairs, *Agriculture in the UK Annual Report 2005*, London, and website at www.defra.gov.uk.

Notes: figures in brackets are percentages of annual totals, 2005 figures are current estimates. Rounding means that percentages may not add to 100%.

Diagrams of data

Tables are good at presenting a lot of information, but it can still be difficult to identify underlying patterns. We can see these more clearly in other kinds of diagram, such as the graphs we described in Chapter 3. These graphs show the relationship between two variables on a pair of Cartesian axes, with the x-axis showing the independent variable and the y-axis showing corresponding values of a dependent variable. If we are plotting sales of ice cream against temperature, there is clearly an independent variable (the temperature) and a dependent variable (the consequent sales of ice cream). But if we are plotting sales of ice cream against sales of sausages, there is no clear relationship and we can choose to draw the axes either way round.

Formats for graphs

As with tables, there are many different formats for graphs. Returning to the weekly sales of a product described earlier, we can start by plotting sales (the dependent variable) against the week (the independent variable). Then the simplest graph shows the individual points in a scatter diagram, shown in Figure 5.4(a).

You can see the general pattern here, but this becomes even clearer when we join the points, as shown in Figure 5.4(b). The sales clearly follow a

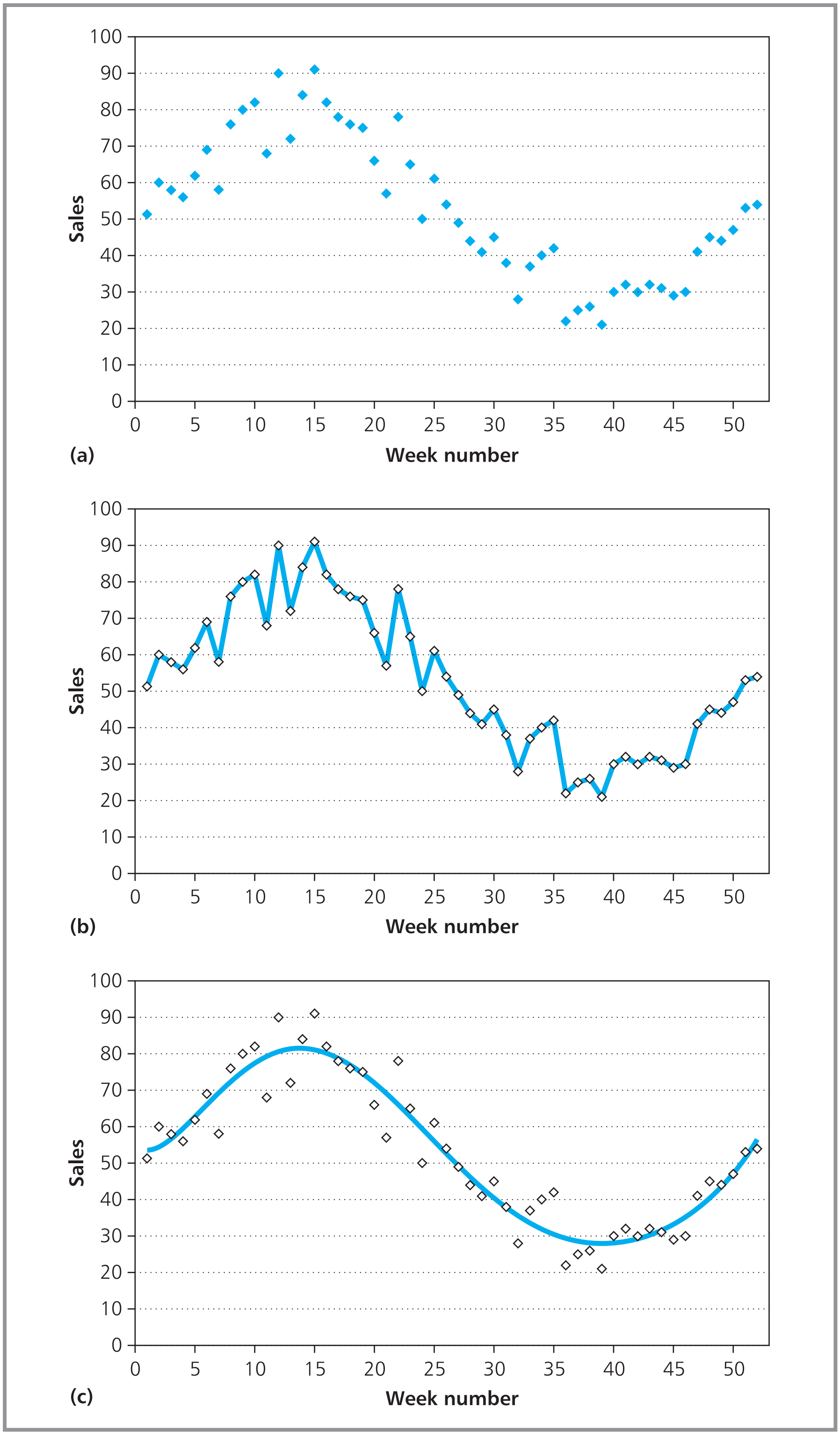

Figure 5.4 Graph of sales figures: (a) scatter diagram, (b) connecting the points to emphasise the pattern, (c) showing the underlying pattern

seasonal cycle, with a peak around week 12 and a trough around week 38. There are small random variations away from this overall pattern, so the graph is not a smooth curve but is rather jagged. We are usually more interested in the underlying patterns than the random variations, so we can emphasise this by drawing a smooth trend line through the individual points, as shown in Figure 5.4(c).

The most common difficulty with graphs is choosing the scale for the y-axis. We could redraw the graphs in Figure 5.4 with different scales for the y-axis, and give completely different views. Figure 5.5(a) has a long scale for the y-axis, so

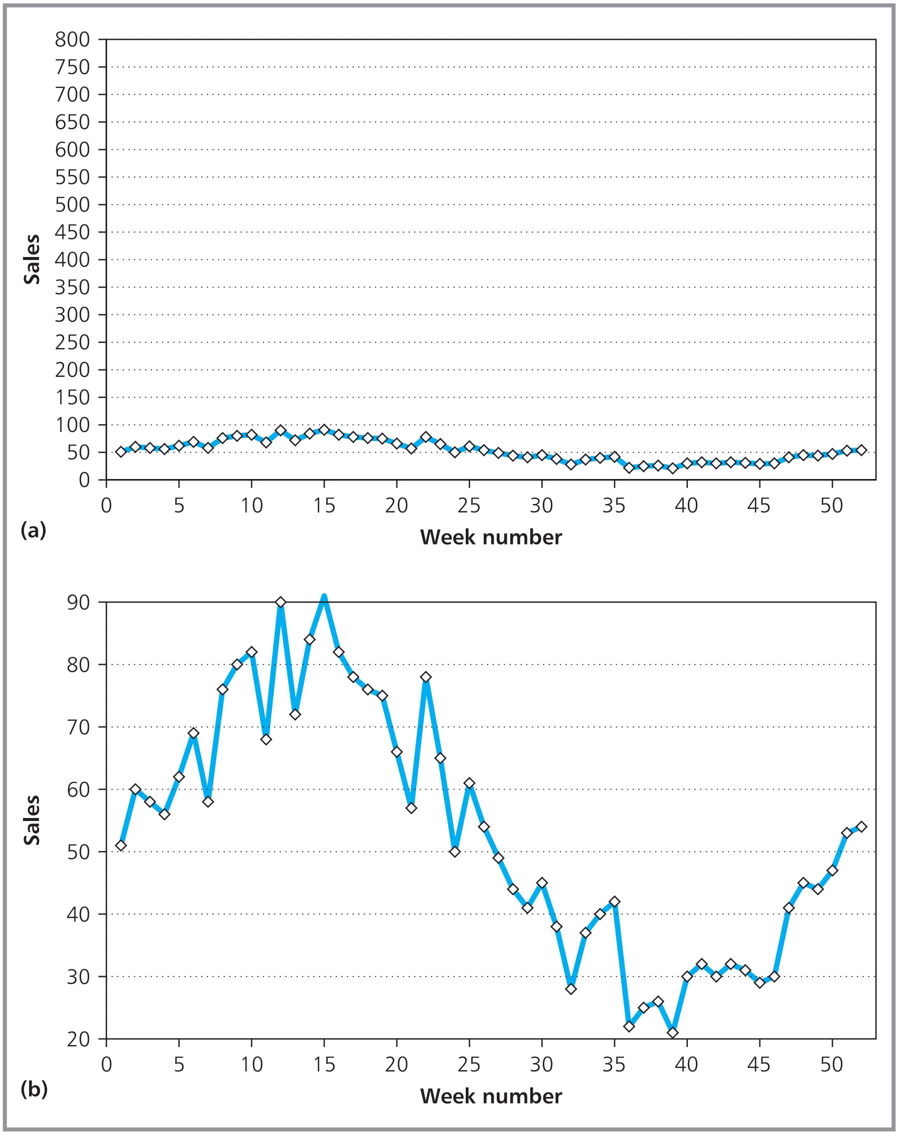

Figure 5.5 (a) Showing poorly drawn graphs: a vertical scale that is too long hides the patterns, (b) part of the vertical scale is omitted, giving a false impression

the graph appears to show stable sales with only small variations; Figure 5.5(b) has a broken scale (omitting values 0 to 20), so the graph suggests high sales in the first half and almost no sales in the second half. Both of these views are misleading.

Graphs have a very strong initial impact, so it is important to choose the right scales, and some guidelines for good practice include the following:

- Always label both axes clearly and accurately
- Show the scales on both axes
- The maximum of the scale should be slightly above the maximum observation
- Wherever possible, the scale on axes should be continuous from zero; if this is too difficult, or hides patterns, show any break clearly in the scale
- Where appropriate, give the source of data
- Where appropriate, give the graph a title.

Drawing several graphs on the same axes makes it easy to compare different sets of data. For example, we can plot the price of electricity, gas, oil and coal on the same axes to see how they have varied over the past year – or we could compare the price of a single commodity over different years. Figure 5.6 shows the average monthly price of a commodity over five years. As the price differences are small, we have highlighted the pattern by plotting only the relevant part of the y-axis.

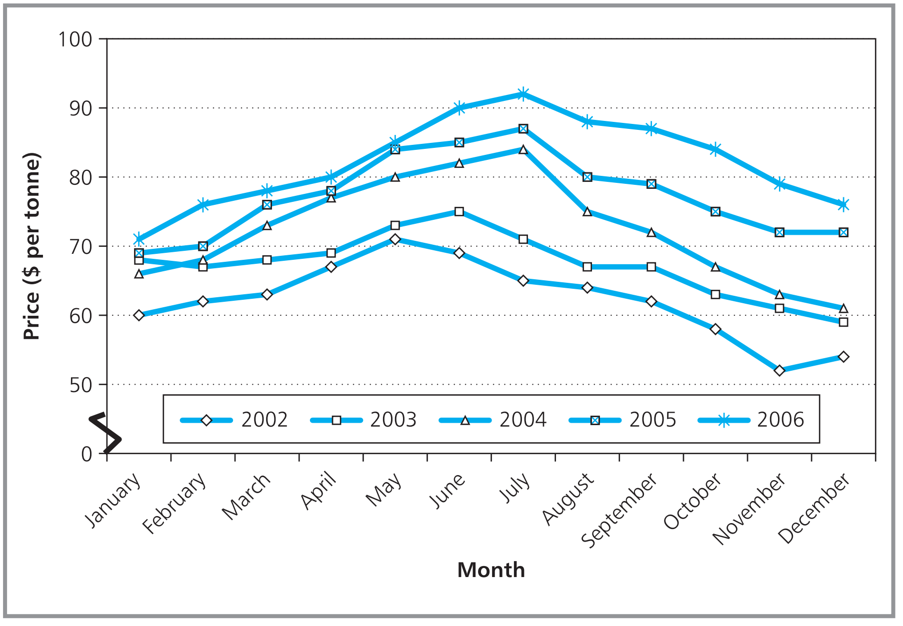

Figure 5.6 Average monthly price of a commodity over five years

WORKED EXAMPLE 5.3

Table 5.5 shows the profit reported by Majestica, Inc. and the corresponding share price. Draw a graph of this data.

Solution

Here there are three variables – quarter, profit and share price – but we can plot only two of these on a graph. There are several options for graphs, such as the variation in share price (or profit) against quarter. The most interesting is the relationship between profit (as the independent variable) and share price (as the dependent variable) shown in Figure 5.7. Here we have chosen the scales to highlight the main areas of interest, and drawn a trend line to suggest the underlying relationship.

Table 5.5 Quarterly company profit and average share price

	Year 1				Year 2				Year 3			
Quarter	1	2	3	4	1	2	3	4	1	2	3	4
Profit	12.1	12.2	11.6	10.8	13.0	13.6	11.9	11.7	14.2	14.5	12.5	13.0
Share price	122	129	89	92	132	135	101	104	154	156	125	136

Source: Company financial reports, New York Stock Exchange and the *Wall Street Journal*.

Note: Profits are in millions of dollars and share prices are in cents.

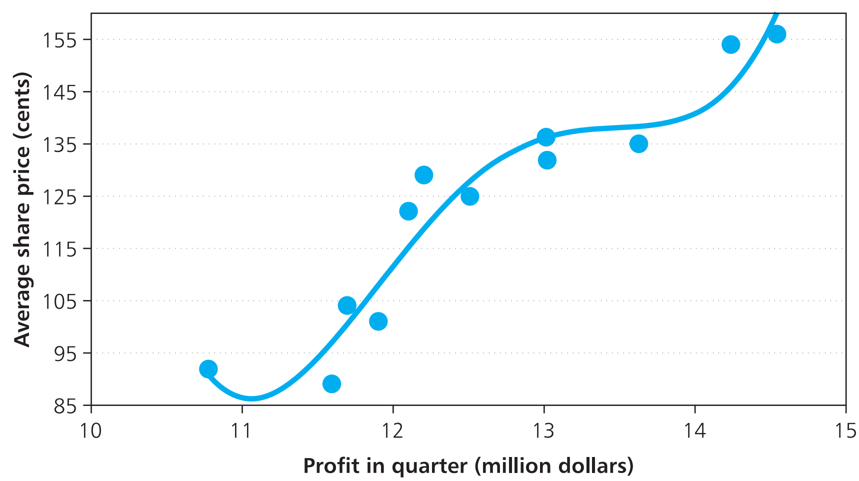

Figure 5.7 Graph of share price against profit for worked example 5.3

Pie charts

Pie charts are simple diagrams that give a summary of categorical data. To draw a pie chart you draw a circle – the pie – and divide this into slices, each of which represents one category. The area of each slice – and hence the angle at the centre of the circle – is proportional to the number of observations in the category.

WORKED EXAMPLE 5.4

Hambro GmbH has operations in four regions of Europe, with annual sales in millions of euros given in the following table. Draw a pie chart to represent these.

Region	North	South	East	West	Total
Sales	25	10	35	45	115

Solution

There are 360° in a circle, and these represent 115 observations. So each observation is represented by an angle of 360/115 = 3.13° at the centre of the circle. Then the sales in the North region are represented by a slice with an angle of $25 \times 3.13 = 78.3°$ at the centre; sales in the South region are represented by a slice with an angle of $10 \times 3.13 = 31.3°$ at the centre, and so on. Figure 5.8(a) shows a basic chart for this data. Of course, you do not really have to do these calculations, as many standard packages draw pie charts automatically. They can also improve the presentation, and Figure 5.8(b) shows the same data when it is sorted into order, rotated to put the biggest slice at the back, labelled and given a three-dimensional effect.

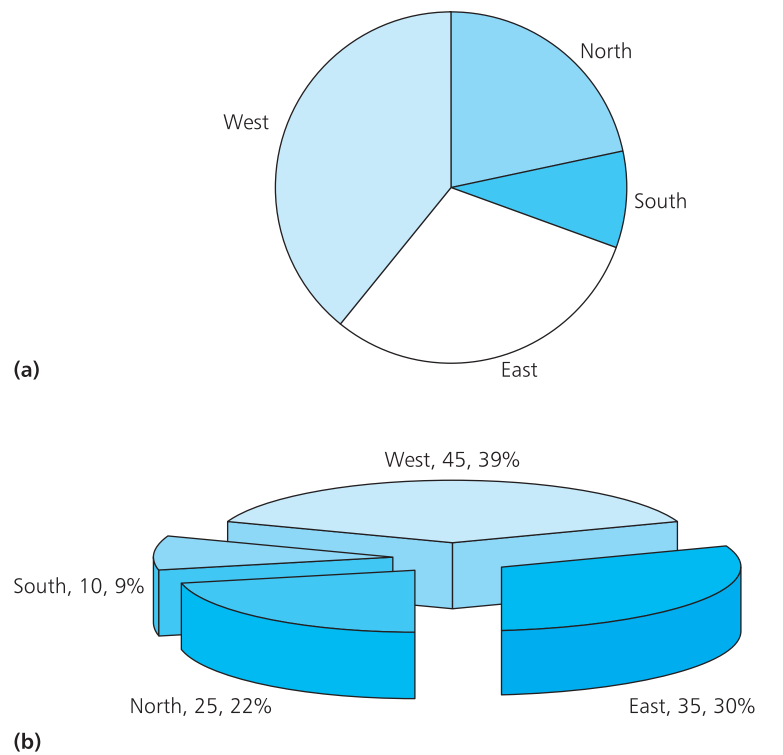

Figure 5.8 (a) Basic pie chart for sales of Hambro GmbH, (b) Adding more features to the pie

Pie charts are very simple and can make an impact, but they show only very small amounts of data. When there are more than, say, six or seven slices they become too complicated and confusing. There is also some concern about whether people really understand data presented in this format or whether it gives a misleading view.[5,6]

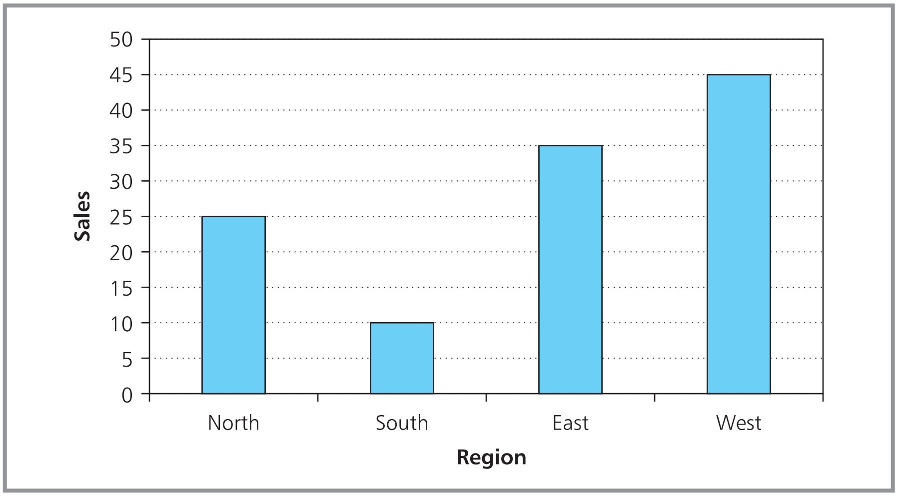

Figure 5.9 Bar chart of results for Hambro GmbH

Bar charts

Like pie charts, **bar charts** show the number of observations in different categories. But here each category is represented by its own line or bar, and the length of this bar is proportional to the number of observations. Figure 5.9 shows a bar chart for the data from Hambro GmbH given in worked example 5.4. Here the bars are vertical, but they could equally be horizontal and – as with pie charts – we can add many variations to enhance the appearance. One constant rule, though, is that you should always start the scale for the bars at zero, and never be tempted to save space by omitting the lower parts of bars. This is sometimes unavoidable in graphs, but in bar charts the result is simply confusing.

WORKED EXAMPLE 5.5

South Middleton Health District has five hospitals, with the following numbers of beds in each. How could you represent this data in bar charts?

	Hospital				
	Foothills	General	Southern	Heathview	St John
Maternity	24	38	6	0	0
Surgical	86	85	45	30	24
Medical	82	55	30	30	35
Psychiatric	25	22	30	65	76

Solution

There are many possible formats here. One shows the number of surgical beds in each hospital – illustrated in Figure 5.10(a). A particular strength of bar charts is that we can show several sets of data in the same diagram to make direct comparisons. For example, Figure 5.10(b) compares the number of beds of each type in each hospital. If we want to highlight the relative sizes of the hospitals, we could combine the bars by 'stacking' them, as shown in Figure 5.10(c). If we want to emphasise type of beds in each hospital, we could describe the percentages of beds, as shown in Figure 5.10(d).

Worked example 5.5 continued

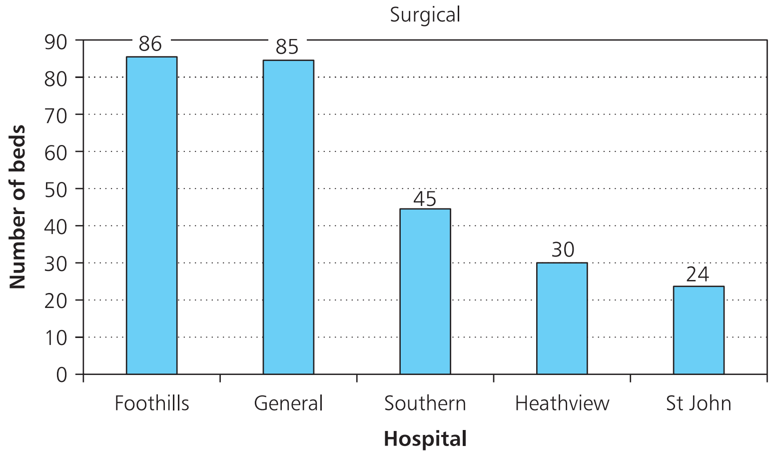

Figure 5.10(a) Bar chart for South Middleton Health District hospitals: number of surgical beds

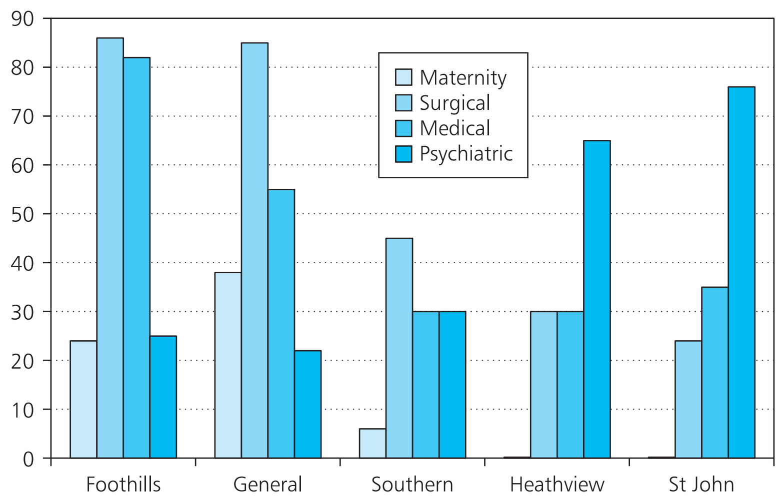

Figure 5.10(b) Comparison of the number of beds in each hospital

Worked example 5.5 continued

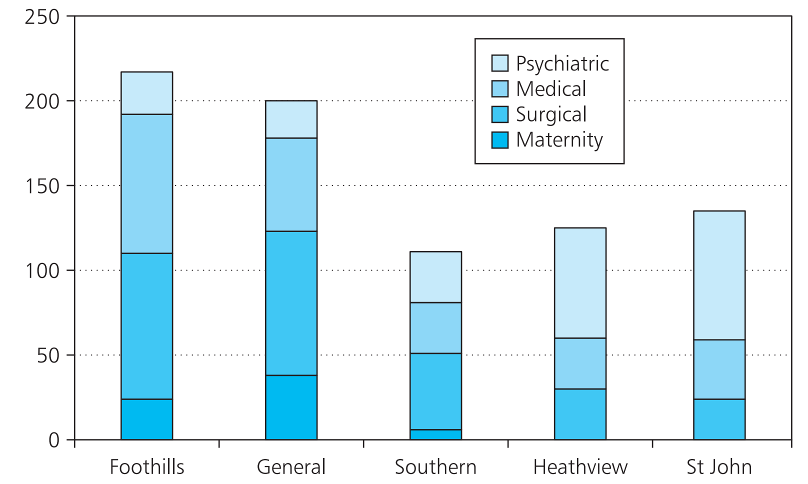

Figure 5.10(c) Stacked bars to emphasise the relative sizes

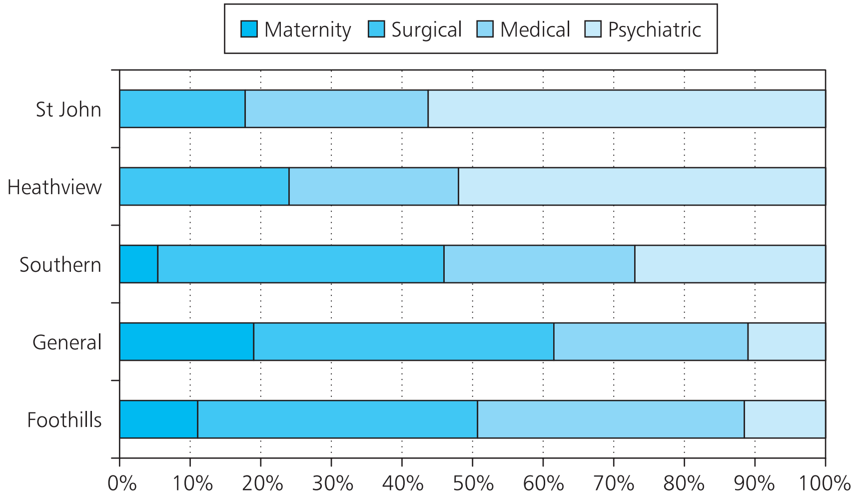

Figure 5.10(d) Percentages of beds of each type

WORKED EXAMPLE 5.6

Draw a frequency distribution for the following discrete data:

150	141	158	147	132	153	176	162	180	165
174	133	129	119	133	188	190	165	157	146
161	130	122	169	159	152	173	148	154	171
136	155	141	153	147	168	150	140	161	185

Solution

This illustrates one of the main uses of bar charts, which is to show frequency distributions. We start by defining suitable classes. The values range from 119 to 190, so a class width of 10 gives nine classes. As the values are discrete, we can arbitrarily use 110 to 119, 120 to 129, 130 to 139, and so on. Figure 5.11 shows a spreadsheet calculating a frequency distribution, percentage frequency, and cumulative frequencies – and then drawing these in a bar chart.

	A	B	C	D	E	F
1	**Frequency distributions**					
2						
3	**Class**	**Frequency**	**Cumulative frequency**	**Percentage frequency**	**Cumulative percentage frequency**	
4	110 to 119	1	1	2.5	2.5	
5	120 to 129	2	3	5	7.5	
6	130 to 139	5	8	12.5	20	
7	140 to 149	7	15	17.5	37.5	
8	150 to 159	10	25	25	62.5	
9	160 to 169	7	32	17.5	80	
10	170 to 179	4	36	10	90	
11	180 to 189	3	39	7.5	97.5	
12	190 to 199	1	40	2.5	100	
13	**Totals**	**40**		**100**		

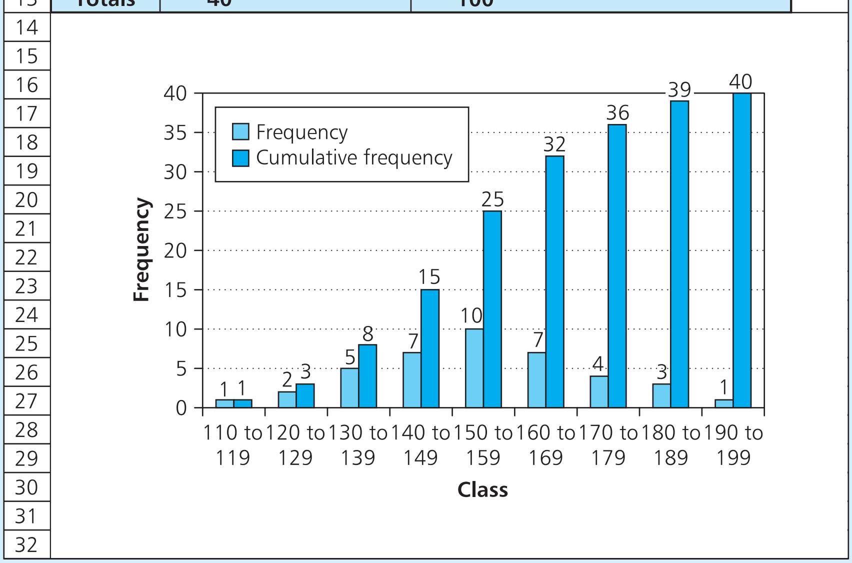

Figure 5.11 Frequency distribution with a spreadsheet

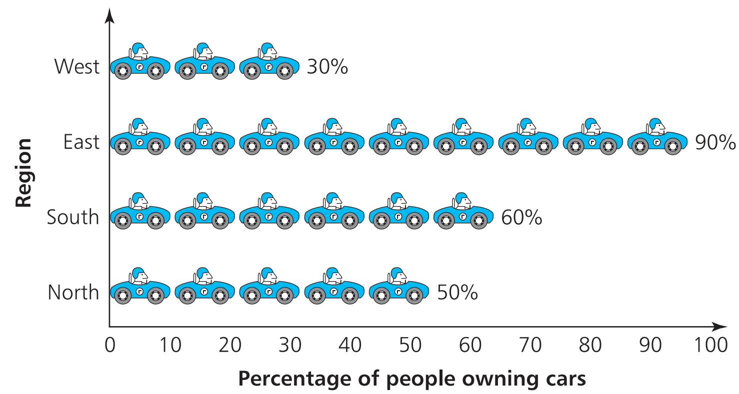

Figure 5.12 Pictogram of percentages of people owning cars in different regions

Pictograms and other images

Basic diagrams can have a considerable impact, but people often want to enhance this by adding even more features. One way of doing this is through **pictograms**, which are similar to bar charts, except the bars are replaced by sketches of the things being described. For example, Figure 5.12 shows the percentage of people owning cars in different regions by a pictogram. Instead of plain bars, we have used pictures of cars, each of which represents 10% of people.

Pictograms are very eye-catching and good at conveying general impressions, but they not always accurate. An obvious problem comes with fractional values: if 53% of people owned cars in one region of Figure 5.12, a line of 5.3 cars would be neither clear not particularly attractive (although it would still give the right impression of 'just over 50%'). A more serious problem comes when the pictures, images and added effects become more important than the charts themselves. Imagine a mushroom farmer who uses the pictogram in Figure 5.13 to show that sales have doubled in the past year. Rather than draw a row of small mushrooms, the farmer uses sketches of single mushrooms. The problem is that we should be concentrating on the height of the sketches, where one is quite rightly twice as high as the other – but it is the area that has immediate impact, and doubling the number of observations increases the area by a factor of four.

Unfortunately, the meaning of many diagrams is hidden by poor design or too much artwork. The unnecessary decoration is sometimes called 'chartjunk' and people refer to the 'ink ratio' – which compares the amount of ink used to describe the data with the amount used in decoration. Remember that the aim of data presentation is not to draw the prettiest picture, but to give the best view of the data. Some useful guidelines for this refer to 'graphical excellence'[6] which has:

- a well-designed presentation of data that combines significant content, statistical description, and design
- clarity, giving results that are easy to understand

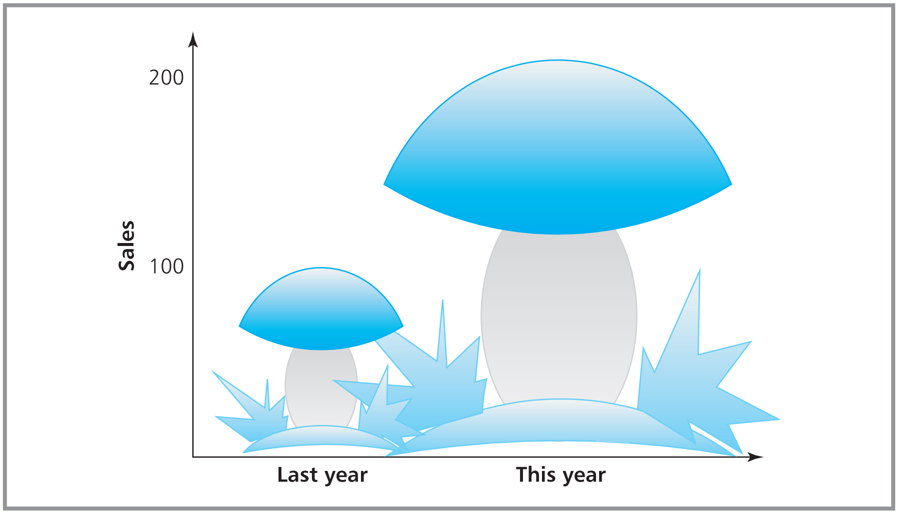

Figure 5.13 Misleading pictogram of increasing mushroom sales

- efficiency, with a precise presentation transmitting the largest number of ideas in the shortest possible time
- accuracy, giving a fair and honest representation of the data.

Review questions

5.9 You should always try to find the best diagram for presenting data. Do you think this is true?

5.10 Why must you label the axes of graphs?

5.11 'There is only one bar chart that accurately describes a set of data.' Is this true?

5.12 If you wanted to make an impact with some data, what format would you consider?

5.13 What are the problems with pictograms?

5.14 Give some examples of diagrams where the added artwork has hidden or distorted the results.

Continuous data

Bar charts are easy to use for discrete data, but with continuous data we have already mentioned the messiness of defining classes as '20 units or more and less than 30 units'. This affects the way we draw frequency distributions of continuous data – and it is sometimes easier to use histograms.

Histograms

Histograms are frequency distributions for continuous data. They look similar to bar charts, but there are important differences. The most important is that histograms are used only for continuous data, so the classes are joined and form a continuous scale. When we draw bars on this scale, their width – as well as

	A	B	C	D	E	F
1	**Histogram**					
2						
3	**Class**	**Percentage**				
4	betwwen –20 and –10 (inclusive)	4				
5	between –10 and zero (inclusive)	16				
6	between zero and 10 (inclusive)	27				
7	between 10 and 20 (inclusive)	35				
8	between 20 and 30 (inclusive)	10				
9	between 30 and 40 (inclusive)	6				
10	between 40 and 50 (inclusive)	2				
11						

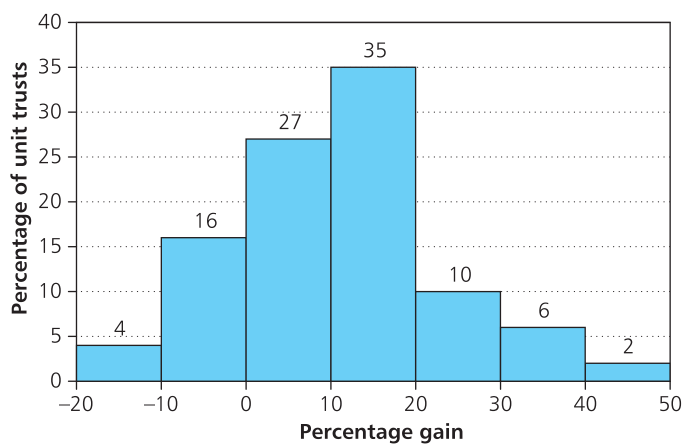

Figure 5.14 Histogram of the percentage gain in value of unit trusts

their length – has a definite meaning. The width shows the class size, and the area of the bar shows the frequency.

Figure 5.14 shows a frequency distribution for the percentage gain in value of certain unit trusts over the past five years, and the associated histogram. Here there is a continuous scale along the x-axis for the percentage gain, and each class is the same width, so both the heights of the bars and their areas represent the frequencies.

WORKED EXAMPLE 5.7

Draw a histogram of the following continuous data.

Class	Frequency
Less than 10	8
10 or more, and less than 20	10
20 or more, and less than 30	16
30 or more, and less than 40	15
40 or more, and less than 50	11
50 or more, and less than 60	4
60 or more, and less than 70	2
70 or more, and less than 80	1
80 or more, and less than 90	1

Worked example 5.7 continued

Solution

Figure 5.15(a) shows this distribution as a histogram. This has a rather long tail with only eight observations in the last four classes, and you might be tempted to combine these into one class with eight observations. But you have to be careful here. You cannot change the scale of the *x*-axis as it is continuous, so the single last class will be four times as wide as the other classes. Then you have to remember that histograms use the area of the bar to represent the frequency, and not the height. So you want to show eight observations in an area four units wide – which means that it must be two units high, as shown in Figure 5.15(b). This gives the histogram an even more extended tail.

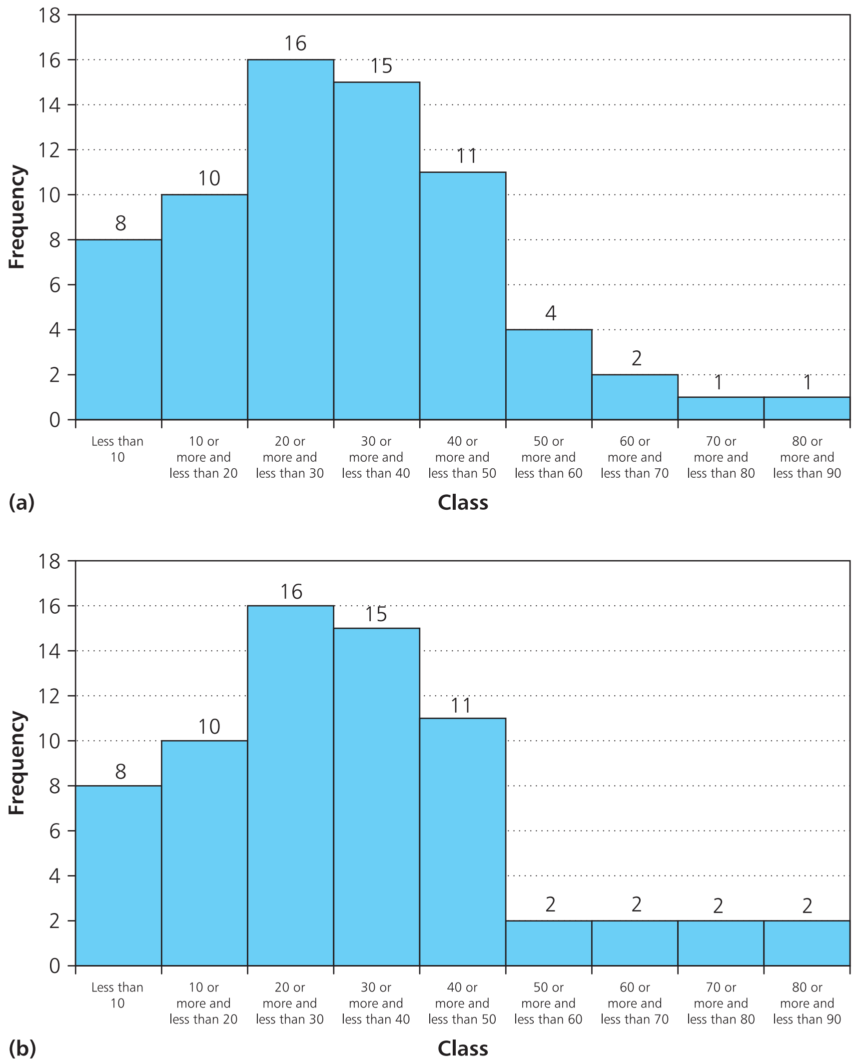

Figure 5.15 (a) Histogram for worked example 5.7 with nine classes, (b) the last four classes combined into one

You can see from the last example that you have to be careful when drawing histograms. A consistent problem is that the shape of a histogram depends on the way that you define classes. Another problem comes with open-ended classes, where there is no obvious way of dealing with a class like 'greater than 20'. One answer is to avoid such definitions wherever possible. Another is to make assumptions about limits, so that we might reasonably interpret 'greater than 20' as 'greater than 20 and less than 24'.

It can be difficult to draw histograms properly – and many people do not realise that they are different from bar charts. Although bar charts might be less precise, they often give better-looking results with less effort – so some people suggest sticking to bar charts.

Ogive

An **ogive** is a graph that shows the relationship between class (on the x-axis) and cumulative frequency (on the y-axis) for continuous data. With the cumulative frequency distribution in the following table, you can start drawing an ogive by plotting the point (100, 12) to show that 12 observations are in the class '100 or less'. Then you can plot the point (150, 38) which shows that 38 observations are 150 or less; then the point (200, 104) shows that 104 observations are 200 or less, then (250, 207), and so on. Plotting all of these points and joining them gives the result shown in Figure 5.16. Ogives are always drawn vertically, and they have this characteristic elongated 'S'-shape.

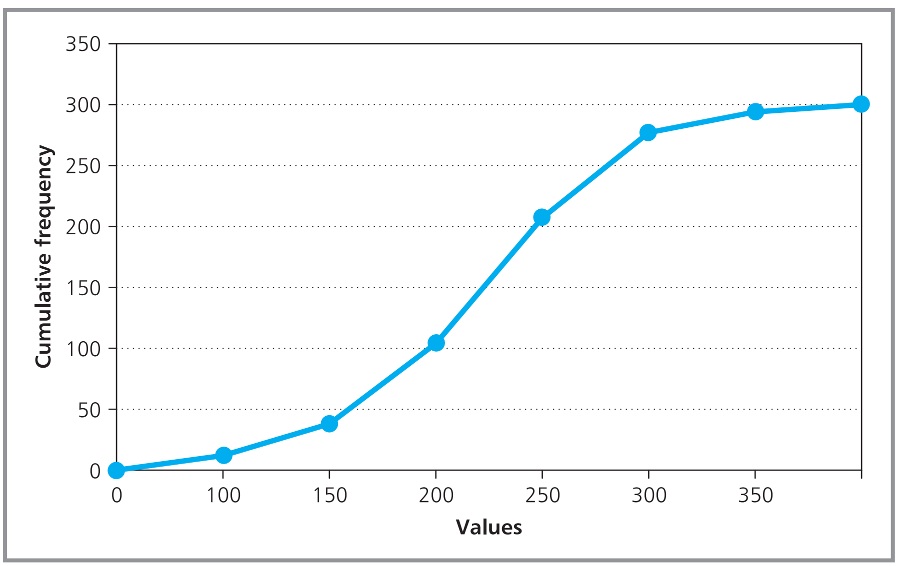

Figure 5.16 Characteristic shape of an ogive

Class	Frequency	Cumulative frequency
100 or less	12	12
More than 100 and less than or equal to 150	26	38
More than 150 and less than or equal to 200	66	104
More than 200 and less than or equal to 250	103	207
More than 250 and less than or equal to 300	70	277
More than 300 and less than or equal to 350	17	294
More than 350 and less than or equal to 400	6	300

A Lorenz curve is an extension of the ogive that is used in economics to show the distribution of income or wealth among a population. It is a graph of cumulative percentage wealth, income or some other measure of wealth, against cumulative percentage of the population. Because a few people have most of the wealth, this is not a standard 'S'-shape, as you can see in the following example.

WORKED EXAMPLE 5.8

Tax offices in Chu Nang County calculate the following percentages of total wealth – before and after tax – owned by various percentages of the population. Draw a Lorenz curve of this data.

Percentage of population	Percentage of wealth before tax	Percentage of wealth after tax
45	5	15
20	10	15
15	12	15
10	13	15
5	15	15
3	25	15
2	20	10

Solution

A Lorenz curve shows the cumulative percentage of wealth against the cumulative percentage of population. Figure 5.17 shows these calculations in a spreadsheet, followed by the Lorenz curves. Starting with a graph of the cumulative percentage of wealth before tax, the first point is (45, 5), followed by (65, 15), (80, 27), and so on. Similarly, with a graph of the cumulative percentage of wealth after tax, the first point is (45, 15), followed by (65, 30), (80, 45) and so on.

If the distribution of wealth is perfectly equitable, a Lorenz curve would be a straight line connecting the origin to the point (100, 100). If the graph is significantly below this, the distribution of wealth is unequal, and the further from the straight line the less equitable is the distribution. Here the Lorenz curve for after-tax wealth is considerably closer to the diagonal, and this shows that taxes have had an effect in redistributing wealth.

Worked example 5.8 continued

	A	B	C	D	E	F
1	**Lorenz curves**					
2						
3	**Percentage of population**	**Cumulative percentage of populaton**	**Percentage of wealth before tax**	**Cumulative percentage of wealth before tax**	**Percentage of wealth after tax**	**Cumulative percentage of wealth after tax**
4	0	0	0	0	0	0
5	45	45	5	5	15	15
6	20	65	10	15	15	30
7	15	80	12	27	15	45
8	10	90	13	40	15	60
9	5	95	15	55	15	75
10	3	98	25	80	15	90
11	2	100	20	100	10	100
12						

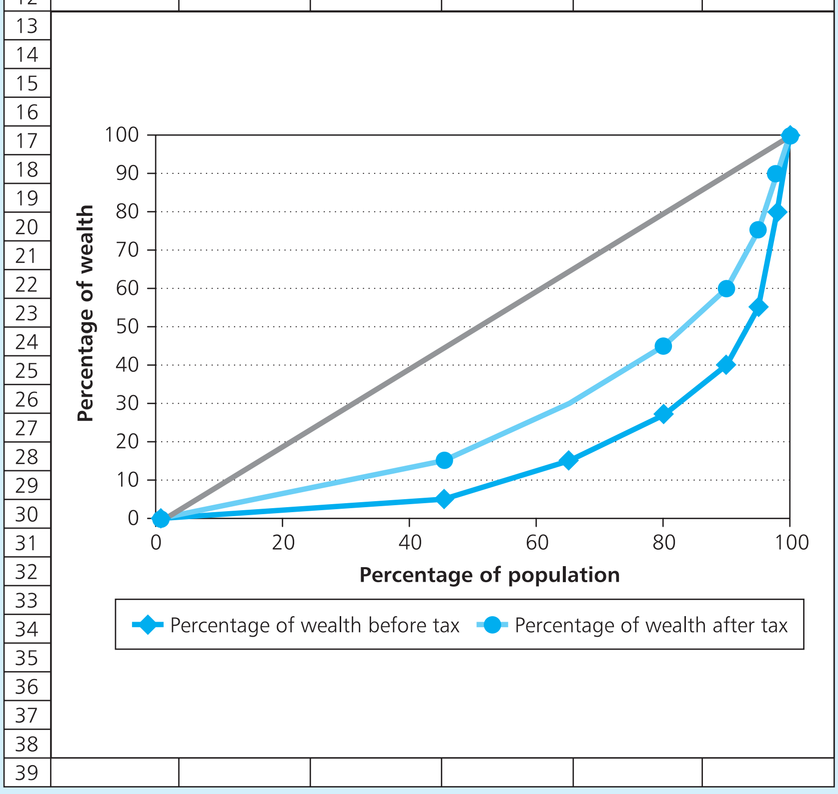

Figure 5.17 Lorenz curves before and after tax

Review questions

5.15 'In bar charts and histograms the height of the bar shows the number of observations in each class.' Is this true?

5.16 If two classes of equal width are combined into one for a histogram, how high is the resulting bar?

5.17 Why would you draw histograms when bar charts are easier and can have more impact?

5.18 What is the purpose of an ogive?

5.19 'A fair Lorenz curve should be a straight line connecting points (0, 0) and (100, 100).' Do you think this is true?

IDEAS IN PRACTICE Software for drawing diagrams

There is a lot of software for drawing diagrams, ranging from simple programs that come free with computer magazines to specialised graphics packages used by commercial artists. We mentioned some of these in Chapter 3, and can again mention ConceptDraw, CorelDraw, DrawPlus, Freelance Graphics, Harvard Graphics, PowerPoint, Sigmaplot, SmartDraw, Visio, and so on.

Other standard packages include drawing functions – particularly spreadsheets. Excel has a 'chart wizard' that easily turns spreadsheets into diagrams. Figure 5.18 shows some of the formats it offers.

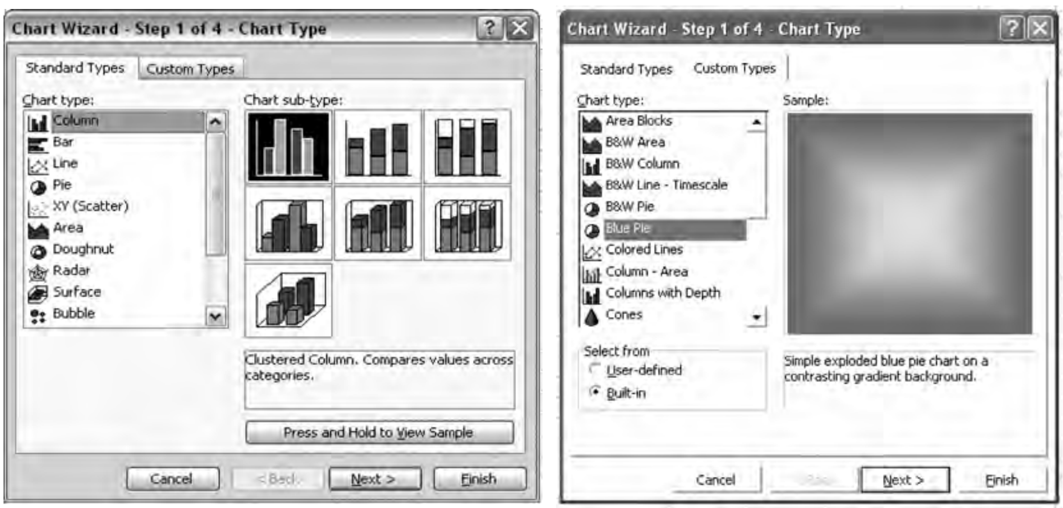

Figure 5.18 Some chart formats offered in Excel

CHAPTER REVIEW

This chapter showed how to summarise data in different types of diagrams.

- After collecting data, you have to process it into useful information. This starts with data reduction to remove the details and focus on the underlying patterns. Data presentation then shows the results in the best format. In practice, there is no clear distinction between these two.

- Diagrams can have a considerable impact, but you have to design them carefully to give an accurate and fair view. There are many types of diagram, and the choice is often a matter of personal preference.
- Tables are the most widely used method of summarising numerical data. They can show a lot of information and be tailored to specific needs. They are particularly useful for showing frequency distributions, which describe the number of observations in different classes. Associated calculations show percentage frequency distributions and cumulative distributions.
- Graphs show relationships between two variables, and highlight the underlying patterns.
- Pie charts describe categorical data, representing the relative frequency of observations by the sectors of a circle. Bar charts give more flexible presentations for categorical data, with the length of each bar proportional to the number of observations in the category. Bar charts can be drawn as pictograms, but you have to be careful not to divert attention away from, or obscure, the important figures.
- Histograms are often confused with bar charts, but they show frequency distributions for continuous data and represent the frequency by the area of a bar. These can be extended to show ogives and Lorenz curves.

CASE STUDY High Acclaim Trading

High Acclaim Trading is based in Delhi, from where it has rapidly increased international operations in recent years. A group of influential shareholders recently asked the finance director to review this international business. In turn, he asked Hari Chandrasan from the audit department to collect some data from company records for his presentation, stressing that he wanted to make an impact with his talk.

At first Hari was worried by the amount of detail available. The company seemed to keep enormous amounts of data on all aspects of its operations. This ranged from transaction records for the movement of virtually every product handled by the company, to subjective management views that nobody ever formally recorded. Often, there seemed no reason for keeping the data and it was rarely summarised or analysed.

Hari did a conscientious job of collecting and summarising data and felt that he had made considerable progress when he approached the finance director and handed over the results in the following table. He explained that 'This table shows some of our most important trading results. We trade in four main regions, and I have recorded eight key facts about the movements between them. Each element in the table shows the number of units shipped (in hundreds), the average income per unit (in dollars), the percentage gross profit, the percentage return on investment, a measure (between 1 and 5) of trading difficulty, potential for growth (again on a scale of 1 to 5), the number of finance administrators employed in each area, and the number of agents. I think this gives a good summary of our operations, and should give a useful focus for your presentation.'

Case study continued

		To			
		Africa	America	Asia	Europe
From	Africa	105, 45, 12, 4, 4, 1, 15, 4	85, 75, 14, 7, 3, 2, 20, 3	25, 60, 15, 8, 3, 2, 12, 2	160, 80, 13, 7, 2, 2, 25, 4
	America	45, 75, 12, 3, 4, 1, 15, 3	255, 120, 15, 9, 1, 3, 45, 5	60, 95, 8, 2, 2, 3, 35, 6	345, 115, 10, 7, 1, 4, 65, 5
	Asia	85, 70, 8, 4, 5, 2, 20, 4	334, 145, 10, 5, 2, 4, 55, 6	265, 85, 8, 3, 2, 4, 65, 7	405, 125, 8, 3, 2, 5, 70, 8
	Europe	100, 80, 10, 5, 4, 2, 30, 3	425, 120, 12, 8, 1, 4, 70, 7	380, 105, 9, 4, 2, 3, 45, 5	555, 140, 10, 6, 4, 1, 10, 8

The finance director looked at the figures for a few minutes and then asked for some details on how trade had changed over the past 10 years. Hari replied that in general terms the volume of trade had risen by 1.5, 3, 2.5, 2.5, 1, 1, 2.5, 3.5, 3 and 2.5% respectively in each of the last 10 years, while the average price had risen by 4, 4.5, 5.5, 7, 3.5, 4.5, 6, 5.5, 5 and 5% respectively.

The finance director looked up from the figures and said, 'To be honest I had hoped for something with a bit more impact. Could you work these into something more forceful within the next couple of days?'

Question

- **If you were Hari Chandrasan how would you put the figures into a suitable format for the presentation?**

PROBLEMS

5.1 Find some recent trade statistics published by the government and present these in different ways to emphasise different features. Discuss which formats are fairest and which are most misleading.

5.2 A question in a survey gets the answer 'Yes' from 47% of men and 38% of women, 'No' from 32% of men and 53% of women, and 'Do not know' from the remainder. How could you present this effectively?

5.3 The number of students taking a course in the past 10 years is summarised in the following table. Use a selection of graphical methods to summarise this data. Which do you think is the best?

Year	1	2	3	4	5	6	7	8	9	10
Male	21	22	20	18	28	26	29	30	32	29
Female	4	6	3	5	12	16	14	19	17	25

5.4 The following table shows the quarterly profit in millions of dollars reported by the Lebal Corporation, and the corresponding closing share quoted in cents on the Toronto Stock Exchange. Design suitable formats for presenting this data.

	Year 1				Year 2				Year 3			
Quarter	1	2	3	4	1	2	3	4	1	2	3	4
Profit	36	45	56	55	48	55	62	68	65	65	69	74
Share price	137	145	160	162	160	163	166	172	165	170	175	182

5.5 The following table shows the number of people employed by Testel Electronics over the past 10 years. How can you present this data?

Year	1	2	3	4	5	6	7	8	9	10
Number	24	27	29	34	38	42	46	51	60	67

5.6 Four regions of Yorkshire classify companies according to primary, manufacturing, transport, retail and service. The number of companies operating in each region in each category is shown in the following table. Show these figures in a number of different bar charts.

	Industry type				
	Primary	Manufacturing	Transport	Retail	Service
Daleside	143	38	10	87	46
Twendale	134	89	15	73	39
Underhill	72	67	11	165	55
Perithorp	54	41	23	287	89

5.7 Jefferson Chang recorded the average wages of 45 people as follows:

221	254	83	320	367	450	292	161	216	410
380	355	502	144	362	112	387	324	576	156
295	77	391	324	126	154	94	350	239	263
276	232	467	413	472	361	132	429	310	272
408	480	253	338	217					

Draw a frequency table, histogram, percentage frequency and cumulative frequency table of this data.

5.8 Draw a histogram of the following data.

Class	Frequency
Less than 100	120
100 or more, and less than 200	185
200 or more, and less than 300	285
300 or more, and less than 400	260
400 or more, and less than 500	205
500 or more, and less than 600	150
600 or more, and less than 700	75
700 or more, and less than 800	35
800 or more, and less than 900	15

5.9 Draw an ogive of the data in Problem 5.8.

5.10 The wealth of a population is described in the following frequency distribution. Draw Lorenz curves and other diagrams to represent this data.

Percentage of people	5	10	15	20	20	15	10	5
Percentage of wealth before tax	1	3	6	15	20	20	15	20
Percentage of wealth after tax	3	6	10	16	20	20	10	15

5.11 The following table shows last year's total production and profits (in consistent units) from six factories. Use a graphics package to explore the ways that you can present this data.

Factory	A	B	C	D	E	F
Production	125	53	227	36	215	163
Profit	202	93	501	57	413	296

RESEARCH PROJECTS

5.1 Do a small survey of graphics packages and find one that you prefer. Why do you find this better than the others? Explore the different formats that it can produce for diagrams. Compare this with Excel, which has 30 chart types and many variations.

5.2 Jan Schwartzkopf has collected the following set of data. Explore ways of reducing, manipulating and presenting this data in diagrams.

245 487 123 012 159 751 222 035 487 655
197 655 458 766 123 453 493 444 123 537
254 514 324 215 367 557 330 204 506 804
941 354 226 870 652 458 425 248 560 510
234 542 671 874 710 702 701 540 360 654
323 410 405 531 489 695 409 375 521 624
357 678 809 901 567 481 246 027 310 679
548 227 150 600 845 521 777 304 286 220
667 111 485 266 472 700 705 466 591 398
367 331 458 466 571 489 257 100 874 577
037 682 395 421 233 577 802 190 721 320
444 690 511 103 242 386 400 532 621 144

5.3 Governments collect huge amounts of data and present it in long series of tables. Find some figures for transport over the past 20 years and present these in useful formats. Prepare a presentation of your findings suitable for transport managers, general business people and government transport planners.

Sources of information

References

1 Huff D., *How to Lie with Statistics*, Victor Gollancz, New York, 1954.

2 Kimble G.A., *How to Use (and Misuse) Statistics*, Prentice Hall, Englewood Cliffs, NJ, 1978.

3 Wainer H., How to display data badly, *The American Statistician*, volume 38, pages 137–147, May 1984.

4 Wainer H., *Visual Revelations*, Copernicus/Springer-Verlag, New York, 1997.

5 Cleveland W.S. and McGill R., Graphical perception; theory, experimentation and application to the development of graphical methods, *Journal of the American Statistical Association*, volume 79, pages 531–554, 1984.

6 Tufte E.R., *The Visual Display of Quantitative Information* (2nd edition), Graphics Press, Cheshire, CT, 2001.

Further reading

Most of the books on mathematics mentioned at the end of Chapter 2 also refer to graphs. Some other books include:

Chapman M. and Wykes C., *Plain Figures* (2nd edition), HMSO, London, 1996.

Few S., *Show Me the Numbers*, Analytics Press, Oakland, CA, 2004.

Harris R.L., *Information Graphics*, Oxford University Press, Oxford, 1999.

Hyggett R., *Graphs and Charts*, Palgrave Macmillan, Basingstoke, 1990.

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CHAPTER 6

Using numbers to describe data

Contents

Chapter outline

The amount of detail in raw data obscures the underlying patterns. We use data reduction and presentation to clear away the detail and highlight the important features. There are two ways of doing this. The last chapter described diagrams for summarising data; in this chapter we continue the theme by looking at numerical descriptions. The most important of these describe the average and spread.

After finishing this chapter you should be able to:

- Appreciate the need for numerical measures of data
- Understand measures of location
- Find the arithmetic mean, median and mode of data
- Understand measures of data spread
- Find the range and quartile deviation of data
- Calculate mean absolute deviations, variances and standard deviations
- Use coefficients of variation and skewness.

Measuring data

We are usually more interested in the overall patterns in data rather than the minute detail, so we use data reduction and presentation to get summaries. The last chapter described some diagrams for this. They can have considerable impact, but they are better at giving overall impressions and a 'feel' for the data rather than objective measures. We really need some objective ways of describing and summarising data – and for this we need numerical measures.

Location and spread

Suppose that you have the following set of data, perhaps for weekly sales:

32 33 36 38 37 35 35 34 33 35 34 36 35 34 36 35 37 34 35 33

There are 20 values here, but what measures can you use to describe the data and differentiate it from the following set?

2 8 21 10 17 24 18 12 1 16 12 3 7 8 9 10 9 21 19 6

You could start by drawing frequency distributions, shown in Figure 6.1.

Each set of data has 20 values, but there are two clear differences:

- The second set is lower than the first set, with values centred around 12 rather than 35.
- The second set is more spread out than the first set, ranging from 1 to 24 rather than 32 to 38.

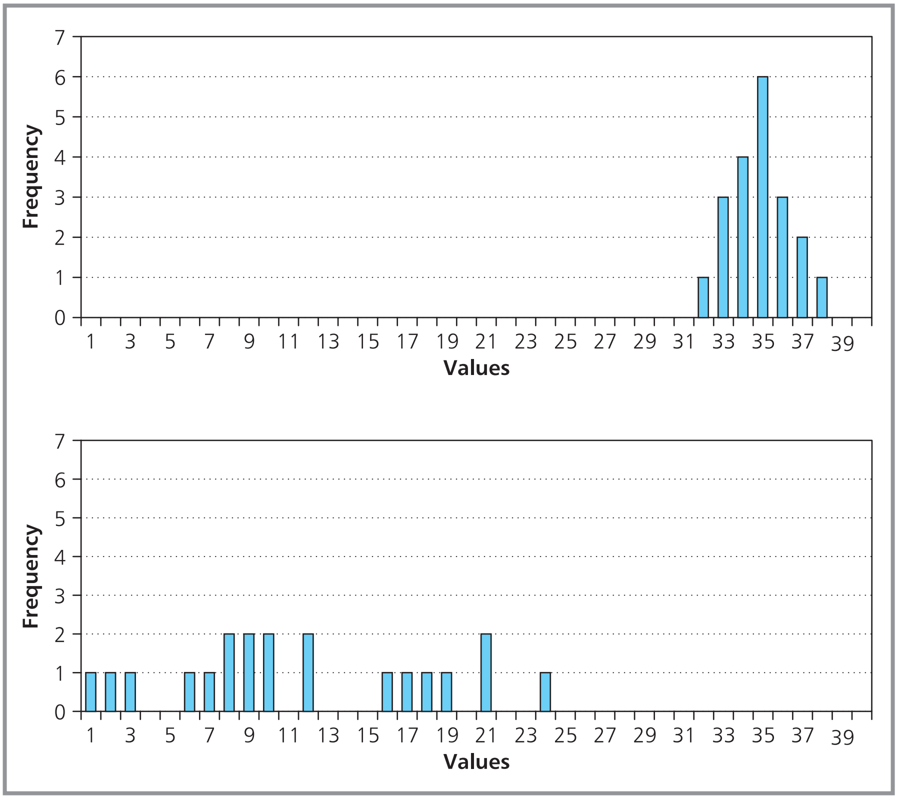


Figure 6.1 Frequency distributions for two sets of data

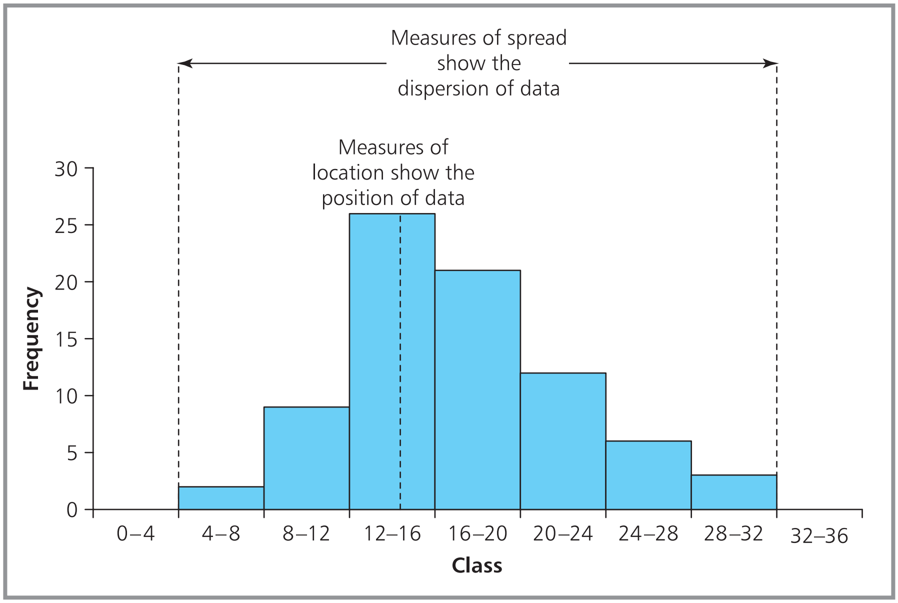


Figure 6.2 Describing the location and spread of data

This suggests two useful measures for data:

- a **measure of location** to show where the centre of the data is, giving some kind of typical or average value
- a **measure of spread** to show how spread out the data is around this centre, giving an idea of the range of values.

In a bar chart or histogram, like Figure 6.2, measures of location show where the data lies on the x-axis, while measures of spread show how dispersed the data is along the axis.

Review questions

6.1 What is the main weakness of using diagrams to describe data?

6.2 What do we mean by the location of data?

6.3 'You need to measure only location and spread to get a complete description of data.' Do you think this is true?

Measures of location

Most people are happy to use an average as a typical value – and this is certainly the most common measure of location. If the average age of students in a night class is 45, you have some feel for what the class looks like; if the average income of a group of people is £140,000 a year, you know they are prosperous; if houses in a village have an average of six bedrooms, you know they are large. However, the simple average can be misleading. For example, the group of people with an average income of £140,000 a year might consist of 10 people, nine of whom have an income of £20,000 a year and one of whom has an income of £1,220,000. The village where houses have an average of six bedrooms might have 99 houses with two bedrooms each, and a stately home with 402 bedrooms. In both of these examples the quoted average is accurate, but it does not represent a typical value or give any real feeling for the data. To get around this problem, we can define different types of average, the three most important alternatives being:

- **arithmetic mean**, or simple average
- **median**, which is the middle value
- **mode**, which is the most frequent value.

Arithmetic mean

If you ask a group of people to find the average of 2, 4 and 6, they will usually say 4. This average is the most widely used measure of location. It is technically called the arithmetic mean – usually abbreviated **mean** (there are other types of mean, but they are rarely used).

To find the mean of a set of values you:

- add all the values together to get the sum
- divide this sum by the number of values to get the mean.

To find the mean of 2, 4 and 6 you add them together to get $2 + 4 + 6 = 12$, and then divide this sum by the number of values, 3, to calculate the mean as $12/3 = 4$.

At this point we can introduce a notation that uses subscripts to describe these calculations much more efficiently. If we have a set of values, we can call the whole set x, and identify each individual value by a subscript. Then x_1 is the first value, x_2 is the second value, x_3 is the third value, and x_n is the nth value. The advantage of this notation is that we can refer to a general value as x_i. Then when $i = 5$, x_i is x_5. At first sight this might not seem very useful, but in practice it saves a lot of effort. For instance, suppose you have four values, x_1, x_2, x_3 and x_4, and want to add them together. You could write an expression for this:

$$y = x_1 + x_2 + x_3 + x_4$$

Alternatively, you could get the same result by writing:

$$y = \text{sum of } x_i \text{ when } i = 1, 2, 3 \text{ and } 4$$

A standard abbreviation replaces 'the sum of' by the Greek capital letter sigma, Σ. Then we get:

$$y = \Sigma x_i \text{ when } i = 1, 2, 3 \text{ and } 4$$

And then we put the values of i around the Σ to give the standard form:

$$y = \sum_{i=1}^{4} x_i$$

The '$i = 1$' below the Σ gives the name of the variable, i, and the initial value, 1. The '4' above the Σ gives the final value for i. The steps between the initial and final values are always assumed to be 1.

WORKED EXAMPLE 6.1

(a) If you have a set of values, x, how would you describe the sum of the first 10?
(b) How would you describe the sum of values numbered 18 to 35 in a set of data?
(c) If you have the following set of data, p, what is the value of $\sum_{i=4}^{8} p_i$?

5 14 13 6 8 10 3 0 5 1 15 8 0

Solution

(a) We want the sum of x_i when $i = 1$ to 10, which we can write as $\sum_{i=1}^{10} x_i$.
(b) Now we have a set of values, say a, and want the sum of a_i from $i = 18$ to 35. We can write this as $\sum_{i=18}^{35} a_i$.
(c) We want to calculate $p_4 + p_5 + p_6 + p_7 + p_8$. Reading the list of data, p_4 is the fourth number, 6, p_5 is the fifth number, 8, and so on. Then the calculation is $6 + 8 + 10 + 3 + 0 = 27$.

We can use this subscript notation to give a formal definition of the mean of a set of data. For some reason this mean is called $\bar{x}$, which is pronounced 'x bar', and is defined as:

$$\text{mean} = \bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum x}{n}$$

Notice that we have also used the abbreviation $\sum x$ for the summation. When there can be no misunderstanding, we can replace the rather cumbersome $\sum_{i=1}^{n} x_i$ by the simpler $\sum x$, and assume that the sum includes all values of x_i from $i = 1$ to n. The fuller notation is more precise, but it makes even simple equations appear rather daunting.

WORKED EXAMPLE 6.2

James Wong found the times taken to answer six telephone enquiries as 3, 4, 1, 5, 7 and 1 minutes. What is the mean?

Solution

You find the mean by adding the values, x_i, and dividing by the number of values, n:

$$\text{mean} = \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\sum x}{n}$$

$$= \frac{3 + 4 + 1 + 5 + 7 + 1}{6}$$

$$= \frac{21}{6} = 3.5 \text{ minutes}$$

This example shows that the mean of a set of integers is often not an integer itself – for example, an average family might have 1.7 children. So the mean gives an objective calculation for the location of data, but it obviously does not give a typical result. Another problem, which we saw in the examples at the beginning of the chapter, is that the mean is affected by a few extreme values and can be some distance away from most values. When you hear that the average mark of five students in an exam is 50% you would expect this to represent a typical value – but if the actual marks are 100%, 40%, 40%, 35% and 35%, four results are below the mean and only one is above.

The mean gives the same weight to every value, and although this seems reasonable, it can cause problems. When the three owner/directors of Henderson Associates had to decide how much of their annual profits to retain for future growth, each voted to retain 3%, 7% and 11% of the profits. Initially it seems that a reasonable compromise takes the mean of the three values, which is 7%. However, the three directors actually hold 10, 10 and 1,000 shares respectively, so this result no longer seems fair. The views of the third director should really be given more weight, and we can do this with a **weighted mean**.

$$\text{weighted mean} = \frac{\sum w_i x_i}{\sum w_i}$$

where: x_i = value i
w_i = weight given to value i.

With Henderson Associates it makes sense to assign weights in proportion to the number of shares each director holds, giving the following result:

$$\text{weighted mean} = \frac{\sum wx}{\sum w} = \frac{10 \times 3 + 10 \times 7 + 1{,}000 \times 11}{10 + 10 + 1{,}000} = 10.88$$

Usually the weights given to each value are not as clear as this, and they need some discussion and agreement. But this negotiation adds subjectivity to the calculations, and we no longer have a purely objective measure. Largely for this reason, the weighted mean is not widely used. However, we can extend its reasoning to estimate the mean of data that has already had some processing – typically with the raw data already summarised in a frequency distribution. Then we have **grouped data** where we do not know the actual values, but know the number of values in each class. Because we do not have the actual values we cannot find the true mean – but we can get a reasonable approximation by assuming that all values in a class lie at the midpoint of the class. If we have 10 values in a class 20 to 29, we assume that all 10 have the value (20 + 29)/2 = 24.5. Then we calculate the mean in the usual way.

When we have a frequency distribution of n values, there are:

- f_i values in class i, and
- x_i is the midpoint of class i.

Then the sum of all values is $\sum f_i x_i$ (usually abbreviated to $\sum fx$) and the number of values is $\sum f$. Then the mean of grouped data is:

$$\text{mean} = \bar{x} = \frac{\sum fx}{\sum f} = \frac{\sum fx}{n}$$

WORKED EXAMPLE 6.3

Estimate the mean of the data in the following discrete frequency distribution.

Class	1–3	4–6	7–9	10–12	13–15	16–18	19–21	22–24
Frequency	1	4	8	13	9	5	2	1

Solution

Remember that x_i is the midpoint of class i, so x_1 is the midpoint of the first class which is $(1 + 3)/2 = 2$, x_2 is the midpoint of the second class which is $(4 + 6)/2 = 5$, and so on. Figure 6.3 shows a spreadsheet with the calculations. As you can see, $\sum f = 43$ and $\sum fx = 503$. So the estimated mean is $503/43 = 11.7$.

	A	B	C	D	E
1	**Frequency distribution**				
2					
3	**Class**		**Midpoint**	**Frequency**	
4	**From**	**to**	***x***	***f***	***fx***
5	1	3	2	1	2
6	4	6	5	4	20
7	7	9	8	8	64
8	10	12	11	13	143
9	13	15	14	9	126
10	16	18	17	5	85
11	19	21	20	2	40
12	22	24	23	1	23
13	**Totals**			**43**	**503**
14	**Mean**				**11.70**

Figure 6.3 Calculating the arithmetic mean of grouped data

The arithmetic mean usually gives a reasonable measure for location and has the advantages of being:

- objective
- easy to calculate
- familiar and easy to understand
- calculated from all the data
- usually capable of giving a reasonable summary of the data
- useful in a number of other analyses.

However, we have seen that it has weaknesses, as it:

- works only with cardinal data
- is affected by outlying values
- can be some distance from most values
- gives fractional values, even for discrete data
- may not give an accurate view.

We really need some other measures to overcome these weaknesses, and the two most common are the median and mode.

Median

When a set of data is arranged in order of increasing size, the **median** is defined as the middle value. With five values – 10, 20, 30, 40 and 50 – the median is the middle or third value, which is 30. This does not really need any calculation, but we find it by:

- arranging the values in order of size
- finding the number of values
- identifying the middle value – which is the median.

With n values, the median is value number $(n + 1)/2$ when they are sorted into order. It follows that half the values are smaller than the median, and half are bigger.

WORKED EXAMPLE 6.4

The annualised returns from a set of low-risk bonds over the past four years have been

4.4 5.3 6.1 7.9 5.6 2.9 2.3 3.0 3.3 4.5 2.1 7.1 6.8 5.0 3.6 4.9 5.4

What is the median?

Solution

We start by sorting the data into ascending order:

Position	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Value	2.1	2.3	2.9	3.0	3.3	3.6	4.4	4.5	4.9	5.0	5.3	5.4	5.6	6.1	6.8	7.1	7.9

There are 17 values, so the median is number $(17 + 1)/2 = 9$. This is 4.9, with eight values above it and eight below.

In this last example we deliberately chose an odd number of values, so that we could identify a middle one – but what happens when there is an even number? If the example had one more value of 8.1, then the middle point of the 18 values would be number $(18 + 1)/2 = 9.5$, which is midway between the ninth and tenth. The usual convention is to take the median as the average of these two. The ninth value is 4.9 and the tenth is 5.0, so we describe the median as $(4.9 + 5.0)/2 = 4.95$. Although this gives a value that did not actually occur, it is the best approximation we can get.

When data comes in a frequency distribution, finding the median is a bit more complicated. We start by seeing which class the median is in, and then finding how far up this class it is.

WORKED EXAMPLE 6.5

Find the median of the following continuous frequency distribution.

Class	0–0.99	1.00–1.99	2.00–2.99	3.00–3.99	4.00–4.99	5.00–5.99
Frequency	1	4	8	6	3	1

Solution

There are 23 values, so when they are sorted into order the median is number $(n + 1)/2 = (23 + 1)/2 = 12$. There is one value in the first class, four in the second class, and eight in the third class – so the median is the seventh value in the third class (2.00–2.99). As there are eight values in this class, it

Worked example 6.5 continued

is reasonable to assume that the median is seven-eighths of the way up the class. In other words:

median = lower limit of third class
+ 7/8 × class width
= 2.00 + 7/8 × (2.99 − 2.00) = 2.87

This calculation is equivalent to drawing an ogive (remember from the last chapter that this plots the cumulative number of values against value) and finding the point on the *x*-axis that corresponds to the 12th value (as shown in Figure 6.4).

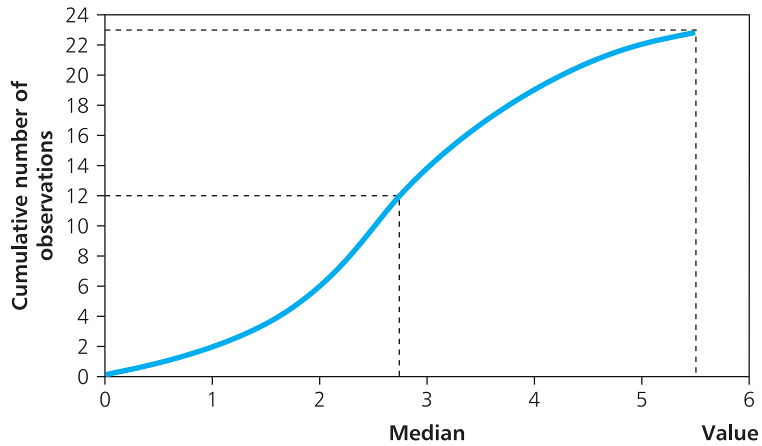


Figure 6.4 Identifying the median from an ogive

The median has the advantages of:

- being easy to understand
- giving a value that actually occurred (except with grouped data)
- sometimes giving a more reliable measure than the mean
- not being affected by outlying values
- needing no calculation (except for grouped data).

On the other hand it has weaknesses, in that it:

- can be used only with cardinal data
- does not really consider data that is not at the median
- can give values for grouped data that have not actually occurred
- is not so easy to use in other analyses.

Mode

The **mode** is the value that occurs most often. If we have four values, 5, 7, 7 and 9, the value that occurs most often is 7, so this is the mode. Like the median, the mode relies more on observation than calculation, and we find it by:

- drawing a frequency distribution of the data
- identifying the most frequent value – which is the mode.

WORKED EXAMPLE 6.6

Maria Panelli recorded the number of goals that her local football team scored in the last 12 matches as 3, 4, 3, 1, 5, 2, 3, 3, 2, 4, 3 and 2. What is the mode of the goals?

Solution

The following table shows the frequency distribution for these 12 values. As you can see, the most frequent value is 3, so this is the mode. This is shown in Figure 6.5(a). This compares with a mean of 2.9 and a median of 3.

Class	Frequency
1	1
2	3
3	**5**
4	2
5	1

Unfortunately, data is often not as convenient as in the last example. If the numbers that Maria Panelli recorded were:

3, 5, 3, 7, 6, 7, 4, 3, 7, 6, 7, 3, 2, 3, 2, 4, 6, 7, 8

you can see that the most common values are 3 and 7, which both appear five times. Then the data has two modes – or it is bimodal – at 3 and 7, as shown in Figure 6.5(b). Data commonly has several modes, making it multi-modal. On the other hand, if you draw a frequency distribution of:

3, 5, 4, 3, 5, 2, 2, 1, 2, 5, 4, 1, 4, 1, 3

you see that each value occurs three times, so there is no mode, as shown in Figure 6.5(c).

It is a bit more difficult to find the mode of data that is grouped in a frequency distribution. We start by identifying the modal class, which is the class with most values. This gives the range within which the mode lies, but we still have to identify an actual value. The easiest way of doing this is to draw two crossing lines, shown in the histogram in Figure 6.6. The point where these two lines cross is the mode. In practice, it is debatable whether this adds much to our understanding of the data, so it is rarely used.

The mode has the advantages of:

- being an actual value (except for grouped data)
- showing the most frequent value, and arguably the most typical
- needing no calculation (except for grouped data)
- not being affected by outlying values.

On the other hand its weaknesses include:

- there can be several modes, or none
- it ignores all data that is not at the mode
- it cannot be used in further analyses.

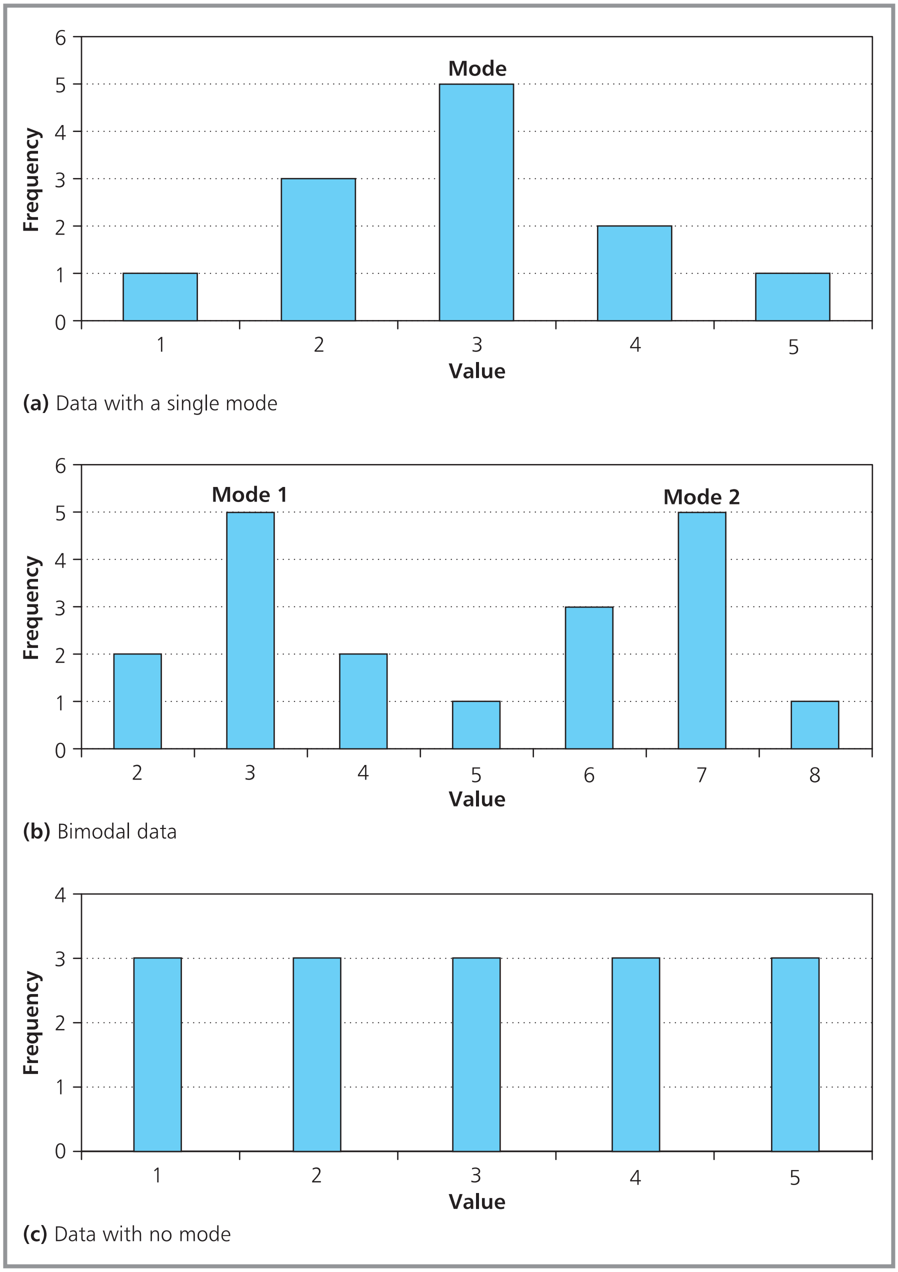


Figure 6.5 Different patterns for the mode

Choice of measure

Each of these three measures for location gives a different view:

- The mean is the simple average
- The median is the middle value
- The mode is the most frequent value.

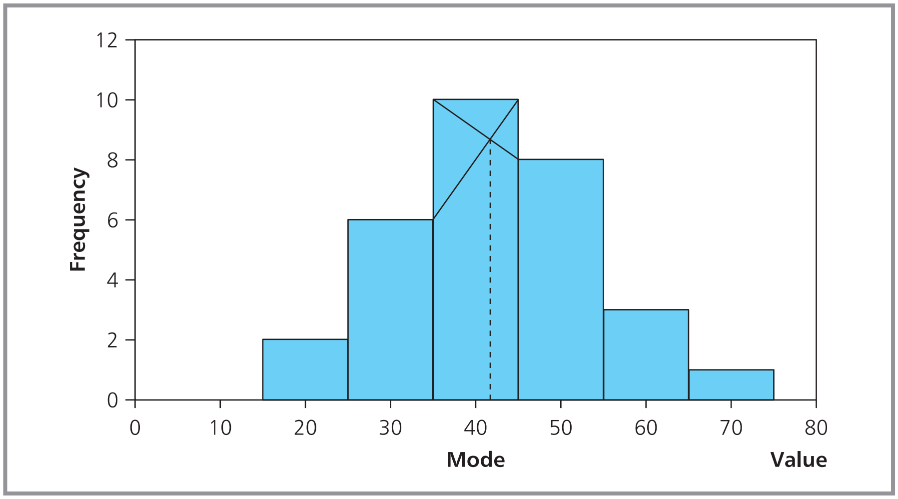


Figure 6.6 Identifying the mode of grouped data

Figure 6.7 shows the typical relationship between these in a histogram. Usually the measures are quite close to each other – and when the histogram is symmetrical they coincide. Sometimes, though, the histogram is very asymmetrical and the measures are some distance apart. The mean is certainly the most widely used, but the median often gives a fairer picture. As with diagrams, the choice of best is often a matter of opinion.

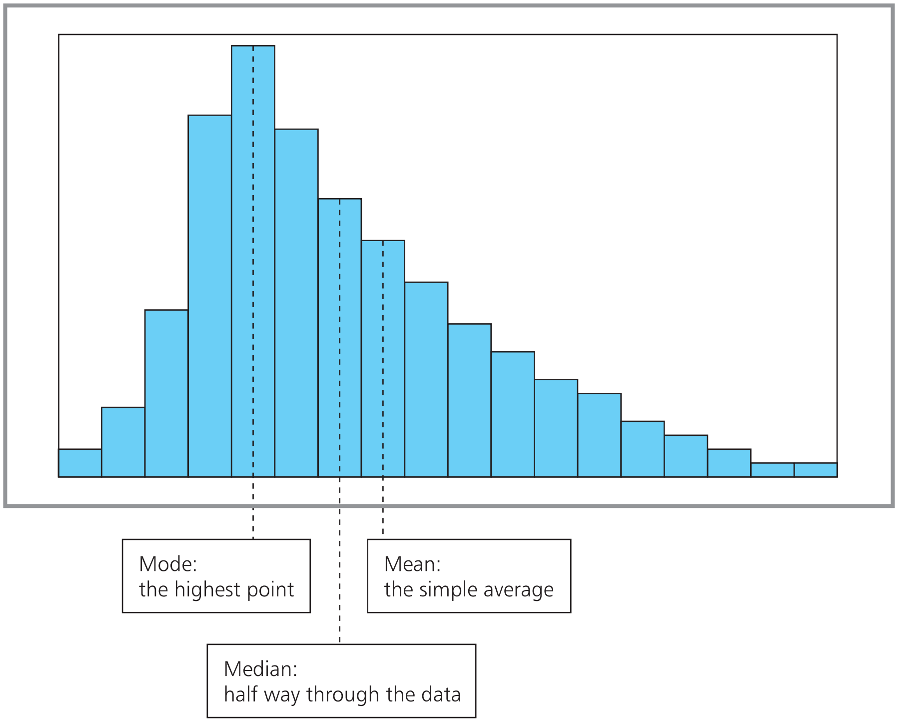


Figure 6.7 Relationship between mean, median and mode

WORKED EXAMPLE 6.7

Taranangama Village Health Centre employs two doctors, one clinical technician, three nurses and four receptionists. Last year the centre published their rates of pay, shown in Figure 6.8. What do these show?

Solution

The figures show the gross pay for each person in the centre, a bar chart of the distribution, and summaries of the mean, median and mode. The calculations were done using the spreadsheet's standard functions AVERAGE, MEDIAN and MODE. The mean is $58,800 – but only two people earn more than this, while eight earn less. Only the technician is within $13,800 of the mean. The median is $42,000, which gives a better view of typical pay at the centre. The mode is $16,000 – but, again, this is not a typical value and simply shows that two receptionists are the only people paid the same amount.

	A	B	C	D	E	F	G	H
1	**Taranangama Village Health Centre**							
2								
3		**Gross pay**						
4	Doctor 1	$168,000						
5	Doctor 2	$162,000						
6	Technician	$58,000						
7	Nurse 1	$45,000						
8	Nurse 2	$43,000						
9	Nurse 3	$41,000						
10	Receptionist 1	$21,000						
11	Receptionist 2	$18,000						
12	Receptionist 3	$16,000						
13	Receptionist 4	$16,000						
14								
15								
16	Mean	$58,800						
17	Median	$42,000						
18	Mode	$16,000						

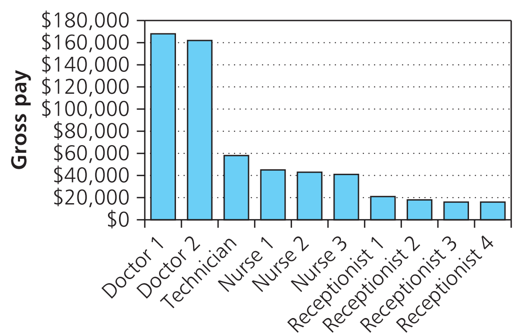


Figure 6.8 Gross pay at Taranangama Village Health Centre

Review questions

6.4 What is a 'measure of location'?

6.5 'The average of a set of data has a clear meaning that accurately describes the data.' Do you think this is true?

6.6 Define three measures for the location of a set of data.

6.7 If the mean of 10 values is 34, and the mean of an additional five values is 37, what is the mean of all 15 values?

6.8 What functions on a spreadsheet describe the location of data?

IDEAS IN PRACTICE Tax on house purchase

The UK government wants more people to own their own houses. In the past, they encouraged this by giving tax relief on mortgage interest, and in the mid-1990s returned almost £3 billion to people who were buying their own homes. However, government policies change. They abolished tax relief on mortgages, and the Council for Mortgage Lenders argued that the increasing effects of inheritance tax (paid on inherited property) and stamp duty (paid when buying property) significantly increased the tax burden on homeowners.[1] By 2005, payments in inheritance tax had reached £3 billion (largely because of rising property values) and payments in stamp duty rose to £5 billion.[2,3]

The overall effect was an average increase in tax of £550 a year for each homeowner.[4] However, this average came from dividing the total increase in tax collected by the number of houses. In reality, three groups of people were affected: people with mortgages no longer had tax relief on the interest and paid extra tax of about 1.5% of their mortgage value per year; people who bought houses costing more than £120,000 paid 1% of the value in stamp duty; and inheritance tax started on estates valued at more than £275,000 and rose to 40% of the value. The real picture is more complicated than the headline suggests – with many people not affected at all, and a few paying a lot.

Measures of spread

The mean, median and mode are measures for the location of a set of data, but they give no idea of its spread or dispersion. The mean age of students in a night class might be 45 – but this does not say whether they are all around the same age, or whether their ages range from 5 to 95. The amount of dispersion is often important. A library might have an average of 300 visitors a day, but it is much easier for staff to deal with small variations (from say 290 on quiet days to 310 on busy ones) than large variations (between 0 and 1,000).

Range and quartiles

The simplest measure of spread is the **range**, which is the difference between the largest and smallest values in a set of data. Clearly, the broader the range the more spread out the data.

> range = largest value − smallest value

This is usually an easy calculation, but there is a warning for grouped data. If you simply take the range as the difference between the top of the largest class and the bottom of the smallest one, the result depends on the definition of classes rather than on actual values.

Another problem is that one or two extreme values can affect the range, making it artificially wide. An obvious way of avoiding this is to ignore extreme values that are a long way from the centre. We can do this using **quartiles**. When data is sorted into ascending size, quartiles are defined as the values that divide the values into quarters. In particular:

- The first quartile, Q_1, is the value a quarter of the way through the data with 25% of values smaller and 75% bigger. It is value number $(n + 1)/4$.

- The second quartile, Q_2, is the value halfway through the data with 50% of values smaller and 50% bigger. This is the median, which is value number $(n + 1)/2$.
- The third quartile, Q_3, is the value three-quarters of the way through the data with 75% of values smaller and 25% bigger. It is value number $3(n + 1)/4$.

With 11 ordered values:

12, 14, 17, 21, 24, 30, 39, 43, 45, 56, 58

the first quartile is value number (11 + 1)/4 = 3, which is 17. The second quartile, or median, is value number (11 + 1)/2 = 6, which is 30. The third quartile is value number 3 × (11 + 1)/4 = 9, which is 45. Figure 6.9 shows these results in a 'box plot' or 'box-and-whisker diagram'. This shows the range between the first and third quartiles by a box, with two whiskers showing the extreme values.

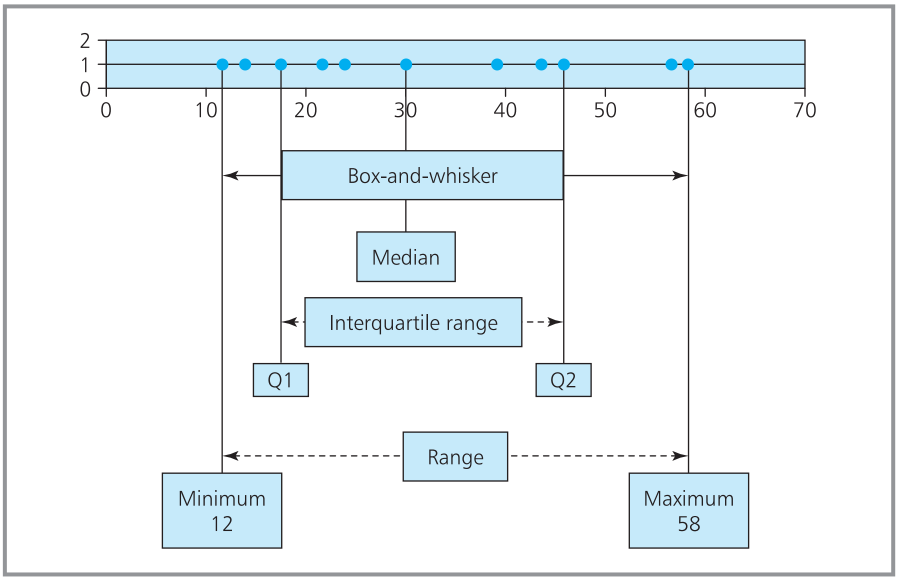


Figure 6.9 Box plot diagram showing the spread of data

Obviously, we chose 11 values so that the quartiles were easy to find. But what happens if there are, say, 200 values, where the first quartile is value number (200 + 1)/4 = 50.25? When there are many values, the usual convention is simply to round to the nearest integer. If you want the 50.25th value, you simply round down and approximate it by the 50th; if you want the 50.75th value, you round this up and approximate it by the 51st. And if you want the 50.5th value you might take the average of the 50th and 51st values. In practice, the difference should be small with a reasonable number of values.

You can use the quartiles to define a narrower range $Q_3 - Q_1$ that contains 50% of values – giving the **interquartile range**. Then the **quartile deviation** or **semi-interquartile range** is defined as:

interquartile range = $Q_3 - Q_1$

quartile deviation = $\dfrac{\text{interquartile range}}{2} = \dfrac{Q_3 - Q_1}{2}$

WORKED EXAMPLE 6.8

Find the quartile deviation of the following continuous frequency distribution.

Class	0–9.9	10–19.9	20–29.9	30–39.9	40–49.9	50–59.9	60–69.9
Values	5	19	38	43	34	17	4

Solution

There are 160 values, or 40 in each quarter. As this number is fairly large, we can approximate the first quartile by the 40th value, the median by the 80th, and the third quartile by the 120th.

- There are 24 values in the first two classes, so the first quartile, Q_1, is the 16th value out of 38 in the class 20–29.9. A reasonable estimate has the quartile 16/38 of the way through this class, so:

  $Q_1 = 20 + (16/38) \times (29.9 - 20) = 24.2$

- There are 62 values in the first three classes so the median, Q_2, is the 18th value out of 43 in the class 30–39.9. A reasonable estimate puts this 18/43 of the way through this class, so:

  $Q_2 = 30 + (18/43) \times (39.9 - 30) = 34.1$

- There are 105 values in the first four classes, so the third quartile, Q_3, is the 15th value out of 34 in the class of 40–49.9. A reasonable estimate for this is:

  $Q_3 = 40 + (15/34) \times (49.9 - 40) = 44.4$

Then the quartile deviation is:

$(Q_3 - Q_1)/2 = (44.4 - 24.2)/2 = 10.1$

Several variations on the quartile deviation are based on percentiles. For example, the 5th percentile is defined as the value with 5% of values below it, and the 95th percentile is defined as the value with 95% of values below it. A common measure finds the range between the 5th and 95th percentiles, as this still includes most of the values but ignores any outlying ones.

Mean absolute deviation

The range and quartile deviation focus on a few values and are clearly related to the median. Other measures of spread include more values, and are related to the mean. In particular, they consider the distance that each value is away from the mean, which is called the **deviation**.

$$\text{deviation} = \text{value} - \text{mean value} = x_i - \bar{x}$$

Each value has a deviation, so the mean of these deviations gives a measure of spread. Unfortunately, the mean deviation has the major disadvantage of allowing positive and negative deviations to cancel. If we have the three values 3, 4 and 8, the mean is 5 and the mean deviation is:

$$\text{mean deviation} = \frac{\sum(x - \bar{x})}{n} = \frac{(3-5) + (4-5) + (8-5)}{3} = 0$$

Even dispersed data has a mean deviation of zero, which is why this measure is never used. A more useful alternative is the **mean absolute deviation** (MAD), which simply takes the absolute values of deviations. In other words, it ignores negative signs and adds all deviations as if they are positive. The result is a measure of the mean distance of values from the mean – so the larger the mean absolute deviation, the more dispersed the data.

$$\text{mean absolute deviation} = \frac{\sum \text{ABS}(x - \bar{x})}{n}$$

$$\text{MAD} = \frac{\sum |x - \bar{x}|}{n}$$

where: x = the values
$\bar{x}$ = mean value
n = number of values
$\text{ABS}(x - \bar{x})$ = the absolute value of $x - \bar{x}$ (that is, ignoring the sign) which is also written as $|x - \bar{x}|$.

WORKED EXAMPLE 6.9

What is the mean absolute deviation of 4, 7, 6, 10 and 8?

Solution

The calculation for the mean absolute deviation starts by finding the mean of the numbers, which is:

$$\bar{x} = \frac{4 + 7 + 6 + 10 + 8}{5} = 7$$

Then the mean absolute deviation is:

$$\text{MAD} = \frac{\sum |x - \bar{x}|}{n}$$

$$= \frac{|4 - 7| + |7 - 7| + |6 - 7| + |10 - 7| + |8 - 7|}{5}$$

$$= \frac{|-3| + |0| + |-1| + |3| + |1|}{5} = \frac{3 + 0 + 1 + 3 + 1}{5}$$

$$= 1.6$$

This shows that on average the values are 1.6 units away from the mean. In practice you will normally use a standard function like AVEDEV in Excel for this calculation.

Calculating the MAD for grouped data is a bit more awkward. To find the mean of grouped data, we took the midpoint of each class and multiplied this by the number of values in the class. Using the same approach to calculate a mean absolute deviation, we approximate the absolute deviation of each class by the difference between its midpoint and the mean of the data. Then the calculation for the mean absolute deviation for grouped data is:

$$\text{mean absolute deviation} = \frac{\sum |x - \bar{x}| f}{\sum f} = \frac{\sum |x - \bar{x}| f}{n}$$

where: x = midpoint of a class
f = number of values in the class
$\bar{x}$ = mean value
n = total number of values.

WORKED EXAMPLE 6.10

Find the mean absolute deviation of the following data.

Class	0–4.9	5–9.9	10–14.9	15–19.9
Frequency	3	5	9	7
Class	20–24.9	25–29.9	30–34.9	35–39.9
Frequency	4	2	1	1

Solution

Figure 6.10 shows the calculations in a spreadsheet (the details are given in full, but obviously you never really have to be this explicit).

There are 32 values with a mean of 15.4. The deviation of each class is the distance its midpoint is away from this mean. Then we find the mean absolute deviation by taking the absolute deviations, multiplying by the frequency, adding the results, and dividing by the number of values. The result is 6.6, which shows that values are, on average, 6.6 away from the mean.

	A	B	C	D	E	F	G	H
1	**Mean absolute deviation**							
2								
3	**Class**		**Midpoint**	**Frequency**	**Product**	**Deviation**	**Absolute deviation**	**Product**
4	**From**	**To**	***x***	***f***	***fx***	**(x – mean)**	**|x – mean|**	**f|x – mean|**
5	0	4.9	2.5	3	7.4	–13.0	13.0	38.9
6	5	9.9	7.5	5	37.3	–8.0	8.0	39.8
7	10	14.9	12.5	9	112.1	–3.0	3.0	26.7
8	15	19.9	17.5	7	122.2	2.0	2.0	14.2
9	20	24.9	22.5	4	89.8	7.0	7.0	28.1
10	25	29.9	27.5	2	54.9	12.0	12.0	24.1
11	30	34.9	32.5	1	32.5	17.0	17.0	17.0
12	35	39.9	37.5	1	37.5	22.0	22.0	22.0
13	**Sums**			**32**	**493.4**			**210.9**
14	**Means**				**15.4**			**6.6**
15								
16	**Mean absolute deviation =**				**6.6**			

Figure 6.10 Calculation of the mean absolute deviation

The MAD is easy to calculate, uses all the data, and has a clear meaning. However, it also has weaknesses. For instance, it gives equal weight to all values, and can be affected by a few outlying numbers. Perhaps a more fundamental problem is the difficulty of using it in other analyses. This limits its use, and a more widely used alternative is the variance.

Variance and standard deviation

The mean absolute deviation stops positive and negative deviations from cancelling by taking their absolute values. An alternative is to square the deviations and calculate a **mean squared deviation** – which is always referred to as the **variance**.

$$\text{variance} = \frac{\sum (x - \bar{x})^2}{n}$$

This has all the benefits of MAD, but overcomes some of its limitations – but with one obvious problem, that the units are the square of the units of the original values. If the values are measured in tonnes, the variance has the meaningless units of tonnes squared; if the values are in dollars, the variance is in dollars squared. To return units to normal, we simply take the square root of the variance. This gives the most widely used measure of spread, which is the **standard deviation**.

$$\text{standard deviation} = \sqrt{\frac{\sum (x - \bar{x})^2}{n}} = \sqrt{\text{variance}}$$

WORKED EXAMPLE 6.11

What are the variance and standard deviation of 2, 3, 7, 8 and 10?

Solution

Again, the calculation starts by finding the mean of the numbers, $\bar{x}$, which is $(2 + 3 + 7 + 8 + 10)/5 = 6$. The variance is the mean squared deviation, which is:

$$\text{variance} = \frac{\sum (x - \bar{x})^2}{n}$$

$$= \frac{(2-6)^2 + (3-6)^2 + (7-6)^2 + (8-6)^2 + (10-6)^2}{5}$$

$$= \frac{(-4)^2 + (-3)^2 + 1^2 + 2^2 + 4^2}{5}$$

$$= \frac{16 + 9 + 1 + 4 + 16}{5} = \frac{46}{5} = 9.2$$

The standard deviation is the square root of the variance:

$$\text{standard deviation} = \sqrt{9.2} = 3.03$$

Again, in practice you are more likely to use a standard spreadsheet function for these calculations, such as VARP and STDEVP.

Again, we can extend the calculations for variance and standard deviation to grouped data, using the same approach as for the MAD, approximating values by the midpoints of classes. Then:

$$\text{variance} = \frac{\sum (x - \bar{x})^2 f}{\sum f} = \frac{\sum (x - \bar{x})^2 f}{n}$$

$$\text{standard deviation} = \sqrt{\text{variance}} = \sqrt{\frac{\sum (x - \bar{x})^2 f}{\sum f}} = \sqrt{\frac{\sum (x - \bar{x})^2 f}{n}}$$

where: x = midpoint of a class
f = number of values in the class
$\bar{x}$ = mean value
n = total number of values.

WORKED EXAMPLE 6.12

Find the variance and standard deviation of the following data.

Class	0–9.9	10–19.9	20–29.9	30–39.9
Frequency	1	4	8	13
Class	40–49.9	50–59.9	60–69.9	70–79.9
Frequency	11	9	5	2

Solution

Figure 6.11 shows the calculations in the same spreadsheet format as Figure 6.10. As you can see, there are 53 values with a mean of 41.2. The deviation of each class is the distance its midpoint is away from the mean. Then we find the variance by taking the square of the deviations, multiplying by the frequency, adding the results, and dividing by the number of values. This gives a value for the variance of 257.5. Taking the square root of this gives the standard deviation of 16.0.

	A	B	C	D	E	F	G	H
1								
2								
3								
4	**Class**		**Midpoint**	**Frequency**	**Product**	**Deviation**	**Squared deviation**	**Product**
5	**From**	**To**	***x***	***f***	***fx***	**(*x* – mean)**	**(*x* – mean) ^ 2**	***f*(*x* – mean) ^ 2**
6	0	9.9	5.0	1	5.0	–36.2	1312.4	1312.4
7	10	19.9	15.0	4	59.8	–26.2	687.8	2751.3
8	20	29.9	25.0	8	199.6	–16.2	263.3	2106.4
9	30	39.9	35.0	13	454.4	–6.2	38.8	504.0
10	40	49.9	45.0	11	494.5	3.8	14.2	156.6
11	50	59.9	55.0	9	494.6	13.8	189.7	1707.4
12	60	69.9	65.0	5	324.8	23.8	565.2	2825.9
13	70	79.9	75.0	2	149.9	33.8	1140.7	2281.3
14	**Sums**			**53**	**2182.4**			**13645.3**
15	**Means**				**41.2**			**257.5**
16								
17	**Variance =**			**257.5**				
18	**Standard deviation =**			**6.0**				

Figure 6.11 Calculation of variance and standard deviation for grouped data

Unfortunately, the variance and standard deviation do not have such a clear meaning as the mean absolute deviation. A large variance shows more spread than a smaller one, so data with a variance of 42.5 is less spread out than equivalent data with a variance of 22.5, but we cannot say much more than this. However, they are useful in a variety of other analyses, and this makes them the most widely used measures of dispersion. For instance, a crucial observation is that a known proportion of values is within a specified number of standard deviations from the mean. Chebyshev first did this analysis, and found that for any data with a standard deviation of *s*:

- It is possible that no values will fall within one standard deviation of the mean – which is within the range $(\bar{x} + s)$ to $(\bar{x} - s)$.
- At least three-quarters of values will fall within two standard deviations of the mean – which is within the range $(\bar{x} + 2s)$ to $(\bar{x} - 2s)$.
- At least eight-ninths of values will fall within three standard deviations of the mean – which is within the range $(\bar{x} + 3s)$ to $(\bar{x} - 3s)$.
- In general, at least $(1 - 1/k^2)$ values will fall within k standard deviations of the mean – which is within the range $(\bar{x} + ks)$ to $(\bar{x} - ks)$.

This rule is actually quite conservative, and empirical evidence suggests that for a frequency distribution with a single mode, 68% of values usually fall within one standard deviation of the mean, 95% of values within two standard deviations and almost all values within three standard deviations.

Another important point is that you can sometimes add variances. Provided two sets of values are completely unrelated (which is technically described as their covariance being zero), the variance of the sum of data is equal to the sum of the variances of each set. For example, if the daily demand for an item has a variance of 4, while the daily demand for a second item has a variance of 5, the variance of total demand for both items is 4 + 5 = 9. You can never add standard deviations in this way.

WORKED EXAMPLE 6.13

The mean weight and standard deviation of airline passengers are 72 kg and 6 kg respectively. What is the mean weight and standard deviation of total passenger weight in a 200-seat aeroplane?

Solution

You find the total mean weight of passengers by multiplying the mean weight of each passenger by the number of passengers:

mean = 200 × 72 = 14,400 kg

You cannot add the standard deviations like this, but you can add the variances. So the variance in weight of 200 passengers is the variance in weight of each passenger multiplied by the number of passengers:

variance = 200×6^2 = 7,200 kg^2

The standard deviation in total weight is $\sqrt{7{,}200}$ = 84.85 kg.

Review questions

6.9 List four measures for data spread. Are there any other measures?

6.10 Why is the mean deviation not used to measure data dispersion?

6.11 If the mean of a set of values is 10.37 metres, what are the units of the variance and standard deviation?

6.12 Why is the standard deviation so widely used, when its practical meaning is unclear?

6.13 The number of cars entering a shopping mall car park per hour has a mean of 120 and standard deviation of 10. In one hour an observer reports 210 cars entering. What can you say about this?

6.14 What functions in a spreadsheet find the dispersion of data?

Other measures of data

One reason why the standard deviation is important is that it is used in other analyses, including the coefficient of variation and the coefficient of skewness.

Coefficient of variation

The measures of spread that we have described give absolute values – so they describe a particular set of data, but it is difficult to use them for comparisons. It would be useful to have a measure of relative spread that considers both the amount of spread and its location. The usual measure for this is the **coefficient of variation**, which is defined as the ratio of standard deviation over the mean.

$$\text{coefficient of variation} = \frac{\text{standard deviation}}{\text{mean}}$$

The higher the coefficient of variation, the more dispersed the data. If the cost of operating various facilities in one year has a coefficient of variation of 0.8 and this rises to 0.9 in the following year, it means that the variation in cost has increased, regardless of how the cost has changed in absolute terms.

WORKED EXAMPLE 6.14

Ambrose Financial classify shares in the energy sector as low, medium or high risk. In recent years, these have had mean annual returns of 9.2%, 17.0% and 14.8% respectively. The standard deviations have been 3.9%, 9.8% and 13.6% respectively. What does this tell you?

Solution

The coefficients of variation for share returns are:

- low risk:

  mean = 9.2%, standard deviation = 3.9%
  coefficient of variation = 3.9/9.2 = 0.42

- medium risk:

  mean = 17.0%, standard deviation = 9.8%
  coefficient of variation = 9.8/17.0 = 0.58

- high risk:

  mean = 14.8%, standard deviation = 13.6%
  coefficient of variation = 13.6/14.8 = 0.92

The returns from high-risk shares are more spread out than from lower-risk ones – which is almost a definition of risk. Medium-risk shares had the highest returns, and the relatively low coefficient of variation suggests a comparatively stable performance.

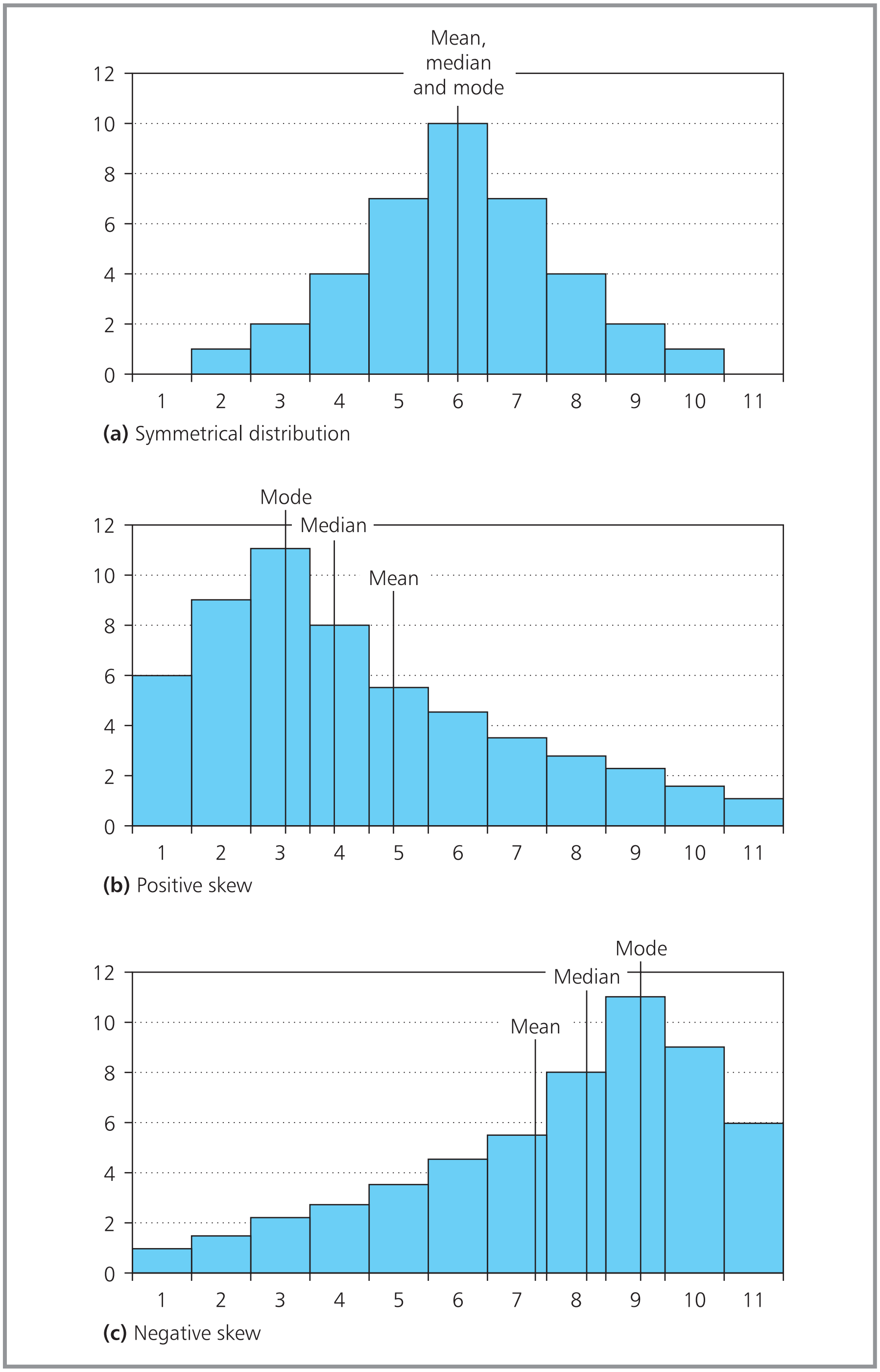


Figure 6.12 Skewness in frequency distributions

Coefficient of skewness

The coefficient of skewness describes the 'shape' of a set of data. A frequency distribution may be symmetrical about its mean, or it may be skewed. A negative or left-skewed distribution has a longer tail to the left (as shown in Figure 6.12(c)); a positive or right-skewed distribution has a longer tail to the right (as shown in Figure 6.12(b)).

In a symmetrical distribution the mean, median and mode all have the same value (Figure 6.12(a)). A positive skew means that the mean is bigger than the median, while a negative skew means that the median is bigger than the mean. A formal measure for the amount of skewness comes from Pearson's coefficient of skewness. This has the rather unusual definition of:

$$\text{coefficient of skewness} = \frac{3 \times (\text{mean} - \text{median})}{\text{standard deviation}}$$

This automatically gives the correct sign of the skew, but its precise interpretation is rather difficult. Values around +1 or −1 are generally considered highly skewed.

Review questions

6.15 Why would you use the coefficient of variation?

6.16 What does the coefficient of skewness measure?

6.17 Two sets of data have means 10.2 and 33.4 and variances 4.3 and 18.2. What does this tell you?

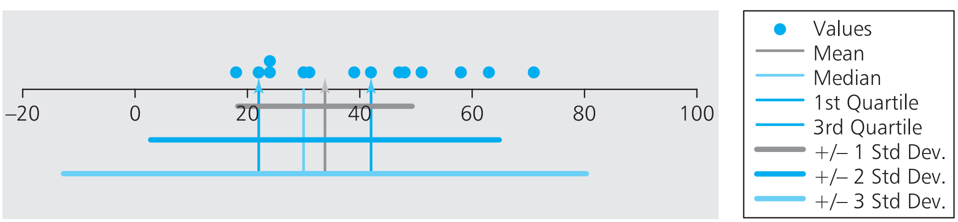
IDEAS IN PRACTICE Prinseptia

Prinseptia is a diversified international company operating largely in southern Europe. In 2005 they bought an art auction house in Tuscany. A year later they reviewed operations to see how their investment was developing. At this point they had 44 weeks of contract information and produced their first progress report and planning document.

One part of this document included the figures shown in Figure 6.13 with the aim of giving – when viewed with other information – a review of weekly contract value.

Ideas in practice continued

	A	B	C	D	E	F	G	H
1	**Prinseptia contract value**							
2								
3	**Weekly contract values**					**Data description**		
4	42	18	33	31		Count	44	
5	51	24	17	42		Sum	1487	
6	47	31	22	71				
7	30	56	18	35		Mean	33.80	
8	22	27	24	17		Median	30	
9	39	54	30	34		Mode	18	
10	48	62	41	29				
11	63	30	26	19		Minimum	12	
12	58	33	20	17		Maximum	71	
13	71	26	18	12		Range	59	
14	24	26	28	21		Variance	235.12	
15						Stan dev	15.33	
16						Coef of variation	0.45	
17								
18						Quartile 1	22	
19						Quartile 2	30	
20						Quartile 3	42	
21								
22						Coef of skewness	0.93	
23								



	A	B
35	**Class**	**Frequency**
36	0–9	0
37	10–19	8
38	20–29	13
39	30–39	10
40	40–49	5
41	50–59	4
42	60–69	2
43	70–79	2

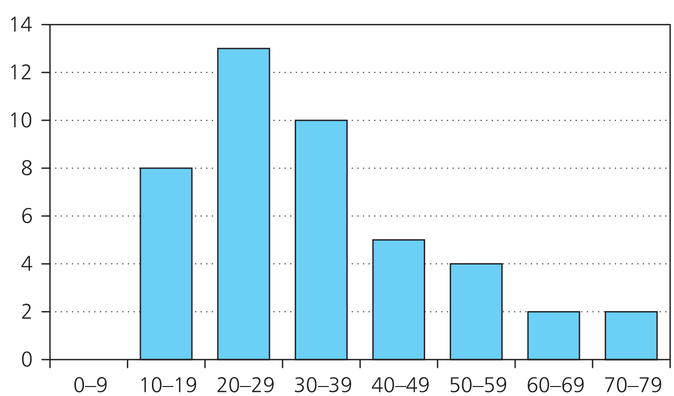


Figure 6.13 Summary of contract value in Prinseptia

CHAPTER REVIEW

This chapter described a number of numerical measures of data.

- The last chapter described some diagrams for summarising data, and this chapter showed how numerical measures give more objective and accurate descriptions. Two key measures describe the location and spread of data.
- Measures of location find the centre of data or a typical value. The (arithmetic) mean is the most widely used measure, giving an average value. Alternatives are the median (which is the middle value, when they are ranked in order of size) and the mode (which is the most frequently occurring value).
- Other measures are needed for the spread of data. The obvious measure is range, but this can be affected by a few outlying results. More reliable values come from the interquartile range or quartile deviation.
- The deviation is the difference between a value and the mean. A basic measure gives the mean absolute deviation. Alternatively, we can square the deviations and calculate the mean squared deviation – or the variance.
- The square root of the variance is the standard deviation, which is the most widely used measure of spread. It is used for other analyses, such as the coefficient of variation (which gives a relative view of spread) and the coefficient of skewness (which describes the shape of a distribution).

CASE STUDY Consumer Advice Office

When people buy things, they have a number of statutory rights. A basic right is that the product should be of adequate quality and fit for the purpose intended. When customers think these rights have been infringed, they might contact their local government's trading standards service.

Mary Lomamanu has been working in Auckland as a consumer advice officer for the past 14 months, where her job is to advise people who have complaints against traders. She listens to the complaints, assesses the problem and then takes the follow-up action she thinks is needed. Often she can deal with a client's problem quite quickly – when she thinks the trader has done nothing wrong, or when she advises customers to go back to the place they bought a product and complain to the manager. But some cases are more difficult and need a lot of follow-up, including legal work and appearances in court.

The local government is always looking for ways to reduce costs and improve their service, so it is important for Mary to show that she is doing a good job. She is particularly keen to show that her increasing experience and response to pressures means that she is more efficient and deals with more clients. To help with this, she has kept records of the number of clients she dealt with during her first eight weeks at work, and during the same eight weeks this year.

- Number of customers dealt with each working day in the first eight weeks:

  6 18 22 9 10 14 22 15 28 9 30 26 17 9 11 25
  31 17 25 30 32 17 27 34 15 9 7 10 28 10 31 12
  16 26 21 37 25 7 36 29

- Number of customers dealt with each working day in the last eight weeks:

  30 26 40 19 26 31 28 41 18 27 29 30 33 43 19
  20 44 37 29 22 41 39 15 9 22 26 30 35 38 26
  19 25 33 39 31 30 20 34 43 45

During the past year she estimates that her working hours have increased by an average of two hours a week, which is unpaid overtime. Her wages increased by 3% after allowing for inflation.

Question

- **Mary needs a way of presenting these figures to her employers in a form that they will understand. How do you think she should set about this?**

PROBLEMS

6.1 When Winnie Mbecu was in hospital for two weeks, the number of visitors she received on consecutive days were 4, 2, 1, 5, 1, 3, 3, 5, 2, 1, 6, 4, 1 and 4. How would you describe this data?

6.2 Find the mean, median and mode of the following numbers. What other measures can you use?

24 26 23 24 23 24 27 26 28 25 21 22 25 23 26 29 27 24 25 24 24 25

6.3 What measures can you use for the following discrete frequency distribution?

Class	0–5	6–10	11–15	16–20
Frequency	1	5	8	11
Class	21–25	26–30	31–35	36–40
Frequency	7	4	2	1

6.4 What measures can you use for the following continuous frequency distribution?

Class	1.00–2.99	3.00–4.99	5.00–6.99	7.00–8.99
Frequency	2	6	15	22
Class	9.00–10.99	11.00–12.99	13.00–14.99	15.00–16.99
Frequency	13	9	5	2

6.5 How would you describe the following data?

3 45 28 83 62 44 60 18 73 44 59 67 78 32
74 28 67 97 34 44 23 66 25 12 58 9 34 58
29 45 37 91 73 50 10 71 72 19 18 27 41 91
90 23 23 33

6.6 The Langborne Hotel is concerned about the number of people who book rooms by telephone but do not actually turn up. The following table shows the numbers of people who have done this over the past few weeks. How can they summarise this data?

Day	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
No-shows	4	5	2	3	3	2	1	4	7	2	0	3	1	4	5

Day	16	17	18	19	20	21	22	23	24	25	26
No-shows	2	6	2	3	3	4	2	5	5	2	4
Day	27	28	29	30	31	32	33	34	35	36	37
No-shows	3	3	1	4	5	3	6	4	3	1	4
Day	38	39	40	41	42	43	44	45			
No-shows	5	6	3	3	2	4	3	4			

6.7 In the last chapter (research project 5.2) we described a set of data that had been collected by Jan Schwartzkopf. What numerical summaries can you use for this data?

6.8 Describe the distributions of incomes in a number of different countries.

RESEARCH PROJECTS

6.1 Spreadsheets have procedures for automatically describing data, such as the 'Data Analysis' tool in Excel (if this is missing you have to load the Analysis ToolPac as an add-in). An option in this is 'Descriptive Statistics' which automatically finds 13 measures for a set of data (illustrated in Figure 6.14). Explore the analyses done by these procedures.

6.2 Spreadsheets are not really designed for statistical analysis, but there are many specialised packages. Perhaps the best known is Minitab, with others including SPSS, S-plus, Systat and JMP. Do a small survey of packages that include statistical measures. Compare their features and say which package you think is most useful.

6.3 Find a set of data about the performance of sports teams, such as last year's results from a football league. Describe the performance of the teams, both numerically and graphically.

	A	B	C	D
1	**Data description**			
2				
3	**Data**		***Data Description***	
4	45			
5	35		Mean	45.41
6	63		Standard Error	3.92
7	21		Median	45.00
8	34		Mode	45.00
9	45		Standard Deviation	16.15
10	60		Sample Variance	260.76
11	19		Kurtosis	–0.75
12	72		Skewness	–0.21
13	54		Range	53.00
14	42		Minimum	19.00
15	67		Maximum	72.00
16	20		Sum	772
17	48		Count	17
18	51			
19	39			
20	57			

Figure 6.14 Data description with Excel

Include these in a report to the directors of the league to review the year's performance.

6.4 Most organisations try to present data fairly, but some presentations are criticised as giving the wrong impression. Perhaps the data is skewed and the median would give a fairer view than the mean; perhaps there are outlying values and the quartile deviation would be fairer than the range. People presenting the data usually argue that they have used objective measures and readers have interpreted these in the wrong ways. Have a look for summaries of data that you feel are misleading. Say why they are misleading and how you would improve them.

Sources of information

References

1 HM Revenue and Customs, *Income Tax Statistics and Distributions*, HMSO, London, 2006.

2 Websites at www.hmrc.gov.uk and www.statistics.gov.uk.

3 Council for Mortgage Lenders, Inheritance tax and home ownership, *CML News and Views*, London, 24th January 2006.

4 Murray-West R., Home-owners are £550 a year poorer under Labour, *The Daily Telegraph*, 25th January 2006.

Further reading

Most statistics books cover the material in this chapter, and you might start looking through the following (some more general statistics books are given in Chapter 14):

Clarke G.M. and Cooke D., *A Basic Course in Statistics* (5th edition), Hodder Arnold, London, 2004.

Levine D.M., Stephan D., Krehbiel T.C. and Berenson M.L., *Statistics for Managers* (3rd edition), Prentice Hall, Upper Saddle River, NJ, 2001.

Wonnacott R.J. and Wonnacott T.H., *Business Statistics* (5th edition), John Wiley, Chichester, 1999.

CHAPTER 7

Describing changes with index numbers

Contents

Chapter outline

Managers often have to monitor the way in which some value changes over time – perhaps the price of raw materials, monthly sales, company share price, number of customers, and so on. Index numbers give a way of monitoring such changes. In particular, they show how a variable changes over time in relation to its value at some fixed point.

After finishing this chapter you should be able to:

- Understand the purpose of index numbers
- Calculate indices for changes in the value of a variable
- Change the base of an index
- Use simple aggregate and mean price relative indices
- Calculate aggregate indices using base-weighting and current-weighting
- Appreciate the use of the Retail Price Index.

Measuring change

The last two chapters showed how to summarise data with diagrams and numerical measures. Both of these take a snapshot of data and describe its features at a specific time. But the values of most variables in business change over time – such as income, sales, profit, share price, productivity, number of customers, and so on. It would be useful to have a way of describing these changes, and this is given by index numbers. So this chapter continues the theme of data presentation, using index numbers to show how values change over time.

Indices

Suppose that you want to show how the number of crimes committed in an area has changed over time. In the first year you might find that there were 127 crimes, then 142 crimes in the second year, 116 crimes in the third year, and 124 in the fourth year. You could say the number of crimes rose by 11.8% between years 1 and 2, and then fell by 18.3% between years 2 and 3, and rose again by 6.9% between years 3 and 4. Although accurate, this description has the disadvantages of being messy and not giving direct comparisons of the number of crimes in, say, years 1 and 4. You could plot a graph of the crimes each year and this would certainly show the pattern – but it would not give a measure of the changes. You really need some way of measuring the changes – and this is given by an **index** or **index number**.

An index is a number that compares the value of a variable at any point in time with its value at another fixed reference point. We call the fixed reference point the **base period**, and the value of the variable at this point the **base value**. Then:

$$\text{index for the time} = \frac{\text{value at the time}}{\text{value in base period}} = \frac{\text{value at the time}}{\text{base value}}$$

With the crime figures above, we could use the first year as a base period, giving a base value of 127. Then the calculation of each year's index is shown in the following table.

Year	Value	Calculation	Index
1	127	127/127	1.00
2	142	142/127	1.12
3	116	116/127	0.91
4	124	124/127	0.98

The index in the base period is 1.00. The index of 1.12 in the second year shows that the number of crimes is 12% higher than the base value, the index of 0.91 in the third year shows that the number of crimes is 9% lower than the base value, and the index of 0.98 in the fourth year shows that the number of crimes is 2% lower than the base value.

We chose the first year as the base period, but this was an arbitrary choice and we could have used any other year. The choice depends on the information we want to present – with the base year chosen as a fixed reference point. If we want to compare the number of crimes in the fourth year with numbers in previous years, we would take year 4 as the base year. Then the base value is 124, and the calculation of each year's index is shown in the following table.

Year	Value	Calculation	Index
1	127	127/124	1.02
2	142	142/124	1.15
3	116	116/124	0.94
4	124	124/124	1.00

You can use indices to monitor the value of any variable that changes over time, but one of the most common uses shows how the price of a product varies. There are many reasons why prices change – changing costs of raw materials, new suppliers, changes in operations, variable supply (such as seasonal vegetables), variable demand (such as package holidays), changing financial conditions (such as exchange rates), inflation which causes prices to drift upwards – and a wide range of other factors. The overall effect is monitored by a price index.

Calculations with indices

You can set an index to 1.00 in the base period, but for convenience it is usually multiplied by 100 to give an index in the base period of 100. Then subsequent indices are defined as the ratio of the current value over the base value multiplied by 100.

$$\text{index in period } N = \frac{\text{value in period } N}{\text{base value}} \times 100$$

If the base price of a product is €5 and this rises to €7, the price index is $7/5 \times 100 = 140$. This shows that the price has risen by 40% since the base period. If the price in the next period is €4, the price index is $4/5 \times 100 = 80$, which is a decrease of 20% since the base period. As well as monitoring changes in one product's price, indices compare price changes in different products, and if the price indices of two products are 125 and 150, you know that the price of the second product has risen twice as quickly as the price of the first (assuming the same base period is used).

WORKED EXAMPLE 7.1

A shop sells an item for £20 in January, £22 in February, £25 in March, and £26 in April. What is the price index in each month using January as the base month?

Solution

The base price is £20 and the price indices for each month are:

- January: $\frac{20}{20} \times 100 = 100$ (as expected in the base period)
- February: $\frac{22}{20} \times 100 = 110$
- March: $\frac{25}{20} \times 100 = 125$
- April: $\frac{26}{20} \times 100 = 130$

The price index of 110 in February shows the price has risen by 10% over the base level, an index of 125 in March shows a rise of 25% over the base level, and so on. Changes in indices between periods are referred to as **percentage point** changes. Thus between February and March the index shows an increase of $125 - 110 = 15$ percentage points. Between March and April the price index rose from 125 to 130, giving a rise of 5 percentage points.

Remember that percentage point changes are not the same as percentage changes, so a rise of 15 percentage points is not the same as a rise of 15%. Here, there is a price rise of 15 percentage points between February and March, but the percentage rise is $(25 - 22)/22 \times 100 = 13.6\%$. Percentage point changes always refer back to the base price and not the current price.

WORKED EXAMPLE 7.2

Amil Gupta's car showroom is giving a special offer on one model. Their advertised price for this model in four consecutive quarters was £10,450, £10,800, £11,450 and £9,999. How would you describe the changes in price?

Solution

Taking price indices based on the fourth quarter:

$$\text{price index in quarter} = \frac{\text{price in quarter} \times 100}{\text{price in fourth quarter}}$$

Figure 7.1 shows these calculations, along with the percentage point rise in prices, which is:

$$\text{percentage point price rise} \\ = \text{index this quarter} - \text{index last quarter}$$

The percentage price rise in each quarter is:

$$\text{percentage price rise} \\ = \frac{\text{price this quarter} - \text{price last quarter}}{\text{price last quarter}} \times 100$$

	A	B	C	D	E	F
1	**Price indices**					
2						
3	**Period**	**Price**	**Index**	**Price rise**	**Percentage price rise**	**Percentage point price rise**
4	1	10450	104.5	0.0	0.0	0.0
5	2	10800	108.0	350.0	3.3	3.5
6	3	11450	114.5	650.0	6.0	6.5
7	4	9999	100.0	–1451.0	–12.7	–14.5

Figure 7.1 Price indices for Amil Gupta

If you rearrange the equation for an index in any period n:

$$\text{index for period } n = \frac{\text{value in period } n \times 100}{\text{base value}}$$

you get:

$$\frac{\text{base value}}{100} = \frac{\text{value in period } n}{\text{index for period } n}$$

And as the base value is constant, you can take any other period, m, and say that:

$$\frac{\text{base value}}{100} = \frac{\text{value in period } n}{\text{index for period } n} = \frac{\text{value in period } m}{\text{index for period } m}$$

You can use this result to compare values at different times.

WORKED EXAMPLE 7.3

The following table shows the monthly index for sales of an item.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Index	121	112	98	81	63	57	89	109	131	147	132	126

(a) If sales in month 3 are 240 units, what are sales in month 8?
(b) If sales in month 10 are 1,200 units, what are sales in month 2?

Solution

(a) Using the ratios:

$$\frac{\text{sales in month 8}}{\text{index in month 8}} = \frac{\text{sales in month 3}}{\text{index in month 3}}$$

$$\frac{\text{sales in month 8}}{109} = \frac{240}{98}$$

so

$$\text{sales in month 8} = 240 \times 109 / 98 = 267$$

(b) Again you can use the indices directly to give:

$$\frac{\text{sales in month 2}}{\text{index in month 2}} = \frac{\text{sales in month 10}}{\text{index in month 10}}$$

or

$$\text{sales in month 2} = 1{,}200 \times 112 / 147 = 914$$

We have described a standard format for indices, but remember the following:

- You can use an index to measure the way in which any variable – not just price – changes over time.
- The usual base value is 100, but this is only for convenience and you can use any other value.
- You can choose the base period as any appropriate point for comparisons. It is usually a typical period with no unusual circumstances – or it might be a period that you are particularly interested in, such as the first period of a financial review.
- You can calculate an index with any convenient frequency, such as monthly indices for unemployment, daily indices for stock market prices, annual indices for GNP, and so on.

Review questions

7.1 What is the purpose of an index?

7.2 Indices always use a base value of 100. Why is this?

7.3 What is the difference between a rise of 10% and a rise of 10 percentage points?

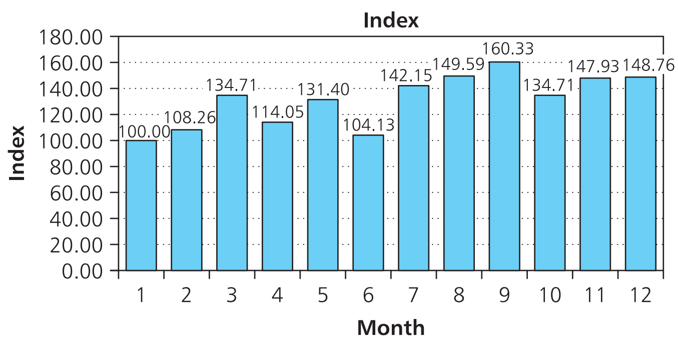
IDEAS IN PRACTICE Mohan Dass and Partners

In 2006 Mohan Dass bought out the other partners in a company that distributes medical supplies around the Middle East. He immediately started a programme of improvement and hopes to see the results during the period 2008 to 2013. In particular, he wants the company to expand rapidly, with turnover increasing by 100% a year for the next five years. To achieve this he is focusing on sales through the company website, introducing generic brands, improving logistics flows, expanding the product range, moving into new geographical areas, forming partnerships with major suppliers and customers, and raising the company profile with health service providers.

To monitor his progress, Mohan collects information about operations, illustrated in Figure 7.2 which shows the index of sales over the past year. Mohan continually monitors a set of 82 measures of this kind to show different aspects of company performance.

Ideas in practice continued

	A	B	C	D	E
1	**Indices for Mohan Dass and Partners**				
2					
3	**Month**	**Sales**	**Index**	**Percentage change**	**Percentage point change**
4	1	12.1	100.00	0	0
5	2	13.1	108.26	8.26	8.26
6	3	16.3	134.71	24.43	26.45
7	4	13.8	114.05	−15.34	−20.66
8	5	15.9	131.40	15.22	17.36
9	6	12.6	104.13	−20.75	−27.27
10	7	17.2	142.15	36.51	38.02
11	8	18.1	149.59	5.23	7.44
12	9	19.4	160.33	7.18	10.74
13	10	16.3	134.71	−15.98	−25.62
14	11	17.9	147.93	9.82	13.22
15	12	18	148.76	0.56	0.83



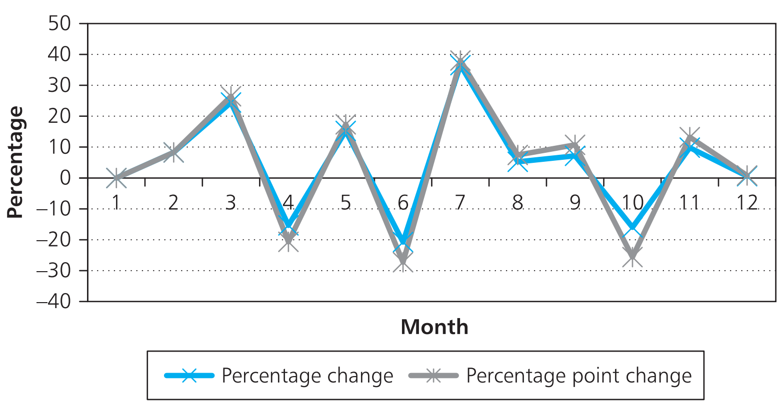


Figure 7.2 Index of sales at Mohan Dass and Partners

Source: Richmond E., Internal Report 147/06, Richmond, Parkes and Wright, Cairo, 2006.

Changing the base period

An index can use any convenient base period, but rather than keep the same one for a long time it is best to update it periodically. There are two reasons for this:

- *Changing circumstances*. You should reset an index whenever there are significant changes that make comparisons with earlier periods meaningless. A service provider might use an index to monitor the number of customers, but should change the base year whenever there are significant changes in the service offered.
- *An index becomes too large*. When an index rises to, say, 5,000 a 10% increase raises it by 500 points, and this seems a much more significant change than a jump from 100 to 110. So it is generally better to reset an index when it becomes too big.

Changing the base of an index introduces a discontinuity that makes comparisons over long periods more difficult. This is why people often keep the same base even when it becomes very high (like the Nikkei index of the Tokyo stock exchange which is around 20,000). But you can overcome this by converting an existing index to a new one. When you have an old index that is calculated from an old base value, the old index for period M is:

$$\text{old index} = \frac{\text{value in period } M}{\text{old base value}} \times 100$$

Now calculating a new index for period M using a new base period gives:

$$\text{new index} = \frac{\text{value in period } M}{\text{new base value}} \times 100$$

Rearranging these equations gives:

$$\text{value in period } M \times 100 = \text{old index} \times \text{old base value}$$
$$\text{value in period } M \times 100 = \text{new index} \times \text{new base value}$$

or

$$\text{new index} = \text{old index} \times \frac{\text{old base value}}{\text{new base value}}$$

As both the old and new base values are fixed, you can find the new index by multiplying the old index by a constant. For example, if the old base value was 200 and the new base value is 500, you find the new index for any period by multiplying the old index by 200/500.

WORKED EXAMPLE 7.4

The following indices monitor the annual profits of J.R. Hartman and Associates.

Year	1	2	3	4	5	6	7	8
Index 1	100	138	162	196	220			
Index 2					100	125	140	165

(a) What are the base years for the indices?
(b) If the company had not changed to Index 2, what values would Index 1 have in years 6 to 8?
(c) What values does Index 2 have in years 1 to 4?
(d) If the company made a profit of €4.86 million in year 3, how much did it make in the other years?

Solution

(a) Indices generally have a value of 100 in base periods, so Index 1 uses the base year 1 and Index 2 uses the base year 5.
(b) You find Index 1 by multiplying Index 2 by a constant amount. You can find this constant from year 5, when Index 1 is 220 and Index 2 is 100 – so to convert from Index 1 to Index 2 you multiply by 220/100. Then Index 1 for year 6 is $125 \times 220/100 = 275$, and so on, as shown in Figure 7.3.
(c) Similarly, you change from Index 2 to Index 1 by multiplying by 100/220. Index 2 for year 4 is $196 \times 100/220 = 89.09$, and so on, as shown in Figure 7.3.

	A	B	C	D	E	F	G	H	I
1	**Indices for profit at J.R. Hartman and Associates**								
2									
3	**Year**	1	2	3	4	5	6	7	8
4	**Index 1**	100	138	162	196	220	275	308	363
5	**Index 2**	45.45	62.73	73.64	89.09	100.00	125.00	140.00	165.00
6	**Profit**	3	4.14	4.86	5.88	6.6	8.25	9.24	10.89

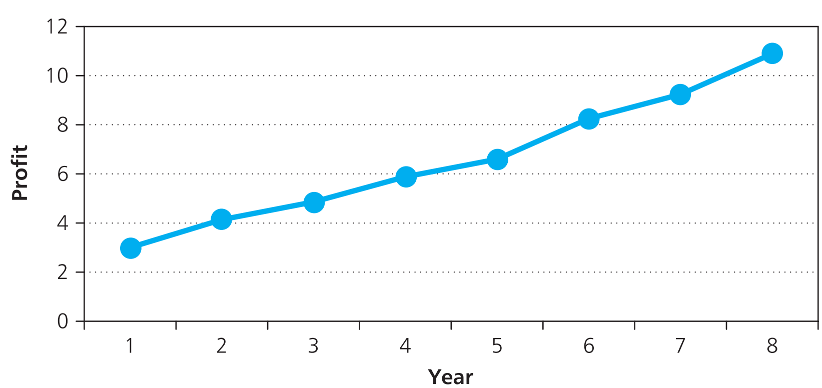


Figure 7.3 Indices for profit at J.R. Hartman and Associates

Worked example 7.4 continued

(d) If the company made a profit of €4.86 million in year 3, you find the profit in any other year from:

$$\frac{\text{profit in year } n}{\text{profit in year } m} = \frac{\text{index in year } n}{\text{index in year } m}$$

Rearranging this and setting year 3 as year n:

$$\text{profit in year } m = \frac{\text{profit in year 3} \times \text{Index 1 in year } m}{\text{Index 1 in year 3}} = \frac{4.86 \times \text{Index 1 in year } m}{162}$$

Then:

$$\text{profit in year 4} = \frac{4.86 \times \text{Index 1 in year 4}}{162} = \frac{4.86 \times 196}{162}$$

$= 5.88$ or €5.88 million

Here we used Index 1, but you can confirm the result using Index 2:

$$\text{profit in year 4} = \frac{4.86 \times 89.09}{73.64} = 5.88$$

Figure 7.3 shows the profits for other years.

Review questions

7.4 When should you change the base period?

7.5 The old price index for a period is 345 while a new price index is 125. In the following period, the new price index is 132. What would the old index have been?

Indices for more than one variable

Often you do not want to monitor the changes in a single variable, but in a combination of different variables. For instance, a car owner might want to monitor the separate costs of fuel, tax, insurance and maintenance; a company might want to monitor changes in sales of different types of products; a charity might monitor its donations to different types of causes. Indices that measure changes in a number of variables are called **aggregate indices**.

Aggregate indices

For simplicity we will talk about aggregate price indices, but remember that you can use the same reasoning for any other type of index. There are two obvious ways of defining an aggregate price index:

1 *The mean of the separate indices for each item.* The price of an item at any time divided by the base price is called the **price relative**, so the mean of separate indices is called the **mean price relative index**:

$$\text{mean price relative index for period } n = \frac{\text{sum of separate indices}}{\text{number of indices}}$$

$$= \frac{\text{sum of all price relatives for period } n}{\text{number of indices}} \times 100$$

2 *Add all prices together and calculate an index using the total price*. This is a **simple aggregate index** or **simple composite index**:

$$\text{simple aggregate index for period } n = \frac{\text{sum of prices in period } n}{\text{sum of prices in base period}} \times 100$$

WORKED EXAMPLE 7.5

Last year the price of coffee, tea and hot chocolate in a café were 55 pence, 28 pence and 72 pence respectively. This year the same items cost 62 pence, 32 pence and 74 pence. What are the mean price relative index and simple aggregate index for this year based on last year?

Solution

- The mean price relative index uses the price relatives for each item, which are:

  coffee: 62/55 = 1.127

  tea: 32/28 = 1.143

  hot chocolate: 74/72 = 1.028

  Taking the mean of these and multiplying by 100 gives:

  mean price relative index
  = 100 × (1.127 + 1.143 + 1.028)/3 = 109.9

- For the simple aggregate index we add all the prices:

  sum of base prices = 55 + 28 + 72 = 155

  sum of current prices = 62 + 32 + 74 = 168

Then:

simple aggregate index

$$= \frac{\text{sum of current prices}}{\text{sum of base prices}} \times 100$$

= 168/155 × 100 = 108.4

These two indices are easy to use, but they do not really give good measures. An obvious criticism – particularly of the simple aggregate index – is that it depends on the units used for each index. An aggregate index that includes the price of, say, butter per kilogram gives a different index from one that includes the price per pound – and if we use the price of butter per tonne, this is so high that it swamps the other costs and effectively ignores them. For example, if the price of a loaf of bread rises from €1 to €1.40 and the price of a tonne of butter rises from €2,684 to €2,713, it makes no sense to calculate a simple aggregate index of (2,713 + 1.40)/(2,684 + 1) × 100 = 101.09.

Another weakness of the two indices is that they do not consider the relative importance of each product. If people in the café buy more tea than hot chocolate, the index should reflect this. Imagine a service company that spent $1,000 on raw materials and $1 million on wages in the base year, and this year it spends $2,000 on raw materials and $1 million on wages. Again, it makes no sense to say that the price relative for raw materials is 2 and for wages is 1, so the mean price relative index is (2 + 1)/2 × 100 = 150.

A reasonable aggregate index must take into account two factors:

- the price paid for each unit of a product
- the number of units of each product used.

There are several ways of combining these into a **weighted index**. Suppose that you want to measure changes in the amount a family pays for food. The easiest way of doing this is to look at each week's shopping basket and find the total cost – which depends on both the price of each item and the number of items they buy. Then we can define a weighted price index as:

$$\text{weighted price index} = \frac{\text{current cost of a week's shopping basket}}{\text{cost of the shopping basket in a base period}}$$

At first this seems reasonable, but we soon hit a problem. When the price of, say, cake increases with respect to the price of biscuits, a family may reduce the number of cakes it buys and increase the number of biscuits. Changes in relative price clearly change the amounts that a family buys. Two alternatives allow for this:

- **Base-period weighted index** or **base-weighted index** assumes that quantities purchased do not change from the base period.
- **Current-period weighted index** or **current-weighted index** assumes that the current shopping basket was used in the base period.

Base-weighted index

Suppose that in the base period a family's shopping basket contained quantities Q_0 of different items at prices P_0. The total cost of the basket is the sum of all the quantities multiplied by the prices.

$$\text{total cost in base period} = \text{sum of quantities} \times \text{price} = \Sigma Q_0 P_0$$

In another period, n, the prices changed to P_n, but we assume the quantities bought remain unchanged, so the total cost is now $\Sigma Q_0 P_n$. Then the base-weighted index is the ratio of these two costs.

base-weighted index

$$= \frac{\text{cost of base period quantities at current prices}}{\text{cost of base period quantities at base period prices}} \times 100 = \frac{\Sigma Q_0 P_n}{\Sigma Q_0 P_0} \times 100$$

This is sometimes called the **Laspeyre index** after its inventor. It has the advantage of giving a direct comparison of costs and reacting to actual price rises. On the other hand, it has the disadvantage of assuming that amounts bought do not change over time, and it does not respond to general trends in buying habits and specific responses to changes in price. Base-weighted indices do not notice that people substitute cheaper items for ones whose price is rising, so they tend to be too high.

Current-weighted index

Suppose that in a period, n, a family's shopping basket contains quantities Q_n of different items at prices P_n and the total cost is $\Sigma Q_n P_n$. We can compare this cost with the cost of the same products in the base period, which would have been $\Sigma Q_n P_0$. Then the current-weighted index is the ratio of these costs.

current-weighted index

$$= \frac{\text{cost of current quantities at current prices}}{\text{cost of current quantities at base period prices}} \times 100 = \frac{\Sigma Q_n P_n}{\Sigma Q_n P_0} \times 100$$

This is sometimes called the **Paasche index**. It has the advantage of giving an accurate measure of changes in the costs of current purchases, but it changes the calculation each period, so it does not give a direct comparison over time. It also needs more effort to update, as it relies on constant monitoring of purchasing habits to find the amounts currently purchased. A Paasche index introduces new products that are relatively cheaper than they were in the base period, so it tends to be too low.

WORKED EXAMPLE 7.6

A company buys four products with the following features.

	Number of units bought		Price paid per unit	
Item	Year 1	Year 2	Year 1	Year 2
A	20	24	10	11
B	55	51	23	25
C	63	84	17	17
D	28	34	19	20

(a) What are the price indices for each product in year 2 using year 1 as the base year?
(b) Calculate a base-weighted index for the products.
(c) Calculate a current-weighted index.

Solution

(a) Simple price indices look only at the prices and do not take into account usage of a product, so the values are:

Product A: $11/10 \times 100 = 110$
Product B: $25/23 \times 100 = 108.7$
Product C: $17/17 \times 100 = 100$
Product D: $20/19 \times 100 = 105.3$

(b) A base-weighted index compares prices for the basket of items bought in the base period:

$$\text{base-weighted index} = \frac{\sum Q_0 P_n}{\sum Q_0 P_0} \times 100$$

$$= \frac{20\times11 + 55\times25 + 63\times17 + 28\times20}{20\times10 + 55\times23 + 63\times17 + 28\times19} \times 100$$

$$= \frac{3{,}226}{3{,}068} \times 100 = 105.15$$

(c) A current-weighted index compares prices for the basket of items bought in the current period:

$$\text{current-weighted index} = \frac{\sum Q_n P_n}{\sum Q_n P_0} \times 100$$

$$= \frac{24\times11 + 51\times25 + 84\times17 + 34\times20}{24\times10 + 51\times23 + 84\times17 + 34\times19} \times 100$$

$$= \frac{3{,}647}{3{,}487} \times 100 = 104.59$$

Other weighted indices

Base-weighting and current-weighting indices both assign weights to prices according to the quantities bought. But sometimes it is better to use other weighting. For instance, suppose that you are looking at the cost of journeys on public transport. The two indices would consider the costs of travel and the number of journeys – but it would make more sense to include some measure of the distances travelled. We can assign other weights, w, to the prices to reflect some other measure of importance, and define a weighted index as:

$$\text{weighted index} = \frac{\sum w P_n / P_0}{\sum w} \times 100$$

In principle the weights can take any values, but they are usually related to total expenditure, time, typical value, general importance, and so on.

WORKED EXAMPLE 7.7

Francesca Birtolli has designed a spreadsheet for calculating indices for the materials that her company buys. Figure 7.4 shows an outline of this. Can you say what the spreadsheet is doing?

Solution

The spreadsheet calculates price indices for four years. The raw data appears at the top of the table as a set of quantities and costs for 10 products. The spreadsheet calculates individual indices for quantities of each product purchased and the prices. Then it calculates four aggregate indices – simple aggregate, mean price relative, base-weighted and current-weighted. You can see that these aggregate indices give quite different results. The first two do not consider the amounts bought, so they give general impressions but are not too reliable. The second two indices are more reliable, but the base-weighted index tends to be high, while the current-weighted index tends to be low.

	A	B	C	D	E	F	G	H	I
1	**Francesca Birtolli – index calculations**								
2									
3	**Purchases by year**								
4		**Year 1**		**Year 2**		**Year 3**		**Year 4**	
5	**Product**	**Quantity**	**Cost**	**Quantity**	**Cost**	**Quantity**	**Cost**	**Quantity**	**Cost**
6	1	24	16	26	16	30	15	35	15
7	2	3	21	5	21	8	21	10	21
8	3	11	20	11	21	10	22	8	24
9	4	15	9	10	11	5	14	2	16
10	5	8	22	12	21	14	21	16	20
11	6	2	40	2	41	2	40	2	40
12	7	1	36	1	37	1	37	1	37
13	8	1	5	2	7	1	8	1	10
14	9	8	16	6	17	4	19	2	19
15	10	20	12	19	13	15	14	10	15
16									
17	**Simple indices**								
18		**Year 1**		**Year 2**		**Year 2**		**Year 4**	
19	**Product**	**Quantity**	**Cost**	**Quantity**	**Cost**	**Quantity**	**Cost**	**Quantity**	**Cost**
20	1	100	100	108.3	100.0	125.0	93.8	145.8	93.8
21	2	100	100	166.7	100.0	266.7	100.0	333.3	100.0
22	3	100	100	100.0	105.0	90.9	110.0	72.7	120.0
23	4	100	100	66.7	122.2	33.3	155.6	13.3	177.8
24	5	100	100	150.0	95.5	175.0	95.5	200.0	90.9
25	6	100	100	100.0	102.5	100.0	100.0	100.0	100.0
26	7	100	100	100.0	102.8	100.0	102.8	100.0	102.8
27	8	100	100	200.0	140.0	100.0	160.0	100.0	200.0
28	9	100	100	75.0	106.3	50.0	118.8	25.0	118.8
29	10	100	100	95.0	108.3	75.0	116.7	50.0	125.0
30									
31	**Aggregate indices**								
32				**Year 1**	**Year 2**	**Year 3**	**Year 4**		
33	**Simple aggregate**			100	104.1	107.1	110.2		
34	**Mean price relative**			100	108.3	115.3	122.9		
35									
36	**Base weighted**			100	104.5	109.1	113.6		
37	**Current weighted**			100	100.8	100.3	101.3		

Figure 7.4 Indices calculated by Francesca Birtolli

IDEAS IN PRACTICE Retail Price Index

Every month since 1914 the UK government has published figures for the annual rate of inflation. It uses several indices to monitor this, but the most important is the Retail Price Index (RPI). This aims to show how the amount spent by a typical household changes over time. This calculation needs two sets of data – the items that a typical household buys, and the prices that it pays.

To find the items a family buys, the government runs an Expenditure and Food Survey for which 6,500 families around the country keep a record of their purchases. This identifies 350 major products and services in 14 groups, with the weights used in 2005 shown in the following table.

Food	110	Clothing and footwear	48
Catering	49	Personal goods and services	41
Alcoholic drink	67	Motoring expenditure	136
Tobacco	29	Fares and travel costs	19
Housing	224	Leisure goods	46
Fuel and light	31	Leisure services	68
Household goods	71		
Household services	61		

Prices are monitored by collecting 120,000 prices on the Tuesday nearest the middle of each month. Some of these are collected centrally (from websites, catalogues, advertisements, etc.) but accurate figures have to allow for price variations around the country, so around 110,000 are collected by personal visits in 150 representative shopping areas.

The weights and current values are used to calculate a broad aggregate index of prices that is used for many purposes, including wage bargaining, calculating index-linked benefits, raising insurance values, and adjusting pensions. But it is not a perfect answer, and it does not represent the true rate of inflation felt by certain groups whose buying habits are not 'typical'. The government takes some of these into account with special indices for pensioners and very prosperous households. It also publishes specific indices for each group of items, such as food and transport. In practice, these effects are surprisingly small and the RPI is widely accepted as a reasonable measure of changing prices. It has important implications. For instance, if someone's pay doubled between 1974 and 1979 they would expect to be much better off – but this was a period of high inflation and the RPI in 1979, with 1974 as the base year, was 206.

Review questions

7.6 What are the mean price relative index and simple aggregate index?

7.7 What are the weaknesses in these measures?

7.8 What is the difference between base-period weighting and current-period weighting for aggregate indices?

7.9 Why does base-weighting give a higher index than current-weighting?

7.10 Is it possible to use a weighting other than base period or current period?

7.11 'The retail price index gives an accurate measure of the cost of living.' Do you think this is true?

CHAPTER REVIEW

This chapter showed how indices can monitor changing values over time.

- The values of most variables – like prices, output, employment, sales, rainfall, etc. – change over time. You can use an index to monitor these changes.
- An index is defined as the ratio of the current value of a variable over its base value – which is its value in the base period. This is normally multiplied

by 100 to give a more convenient figure. Then changes in the index show percentage point changes.

- The base period of an index can be any convenient point, but it should be revised periodically. To calculate a new index you multiply the old index by a constant.
- As well as measuring changes in a single variable, you can also monitor changes in a combination of related variables using aggregate indices. Two basic aggregate indices are the simple aggregate index and the mean price relative index. But both of these have weaknesses, and in particular, they do not reflect the quantities bought.
- Better options use base-period weighting (or the Laspeyre index) and current-period weighting (or the Paasche index).
- The Retail Price Index is a widely accepted measure of price increase based on the expenditure of a typical family.

CASE STUDY Heinz Muller Engineering

In 1999 Heinz Muller Engineering had some problems with industrial relations and productivity. By 2006 it tried hard to overcome these and made a series of changes in the way that employees were rewarded and involved in decision making. Some of these changes included profit sharing, quality circles, reducing the number of layers of management from 13 to six, more flexible working practices, improved communications, and the same basic pay rise for all employees.

As part of these changes the company negotiates an annual basic pay rise, which is proposed by a committee of representatives from all parts of the company, along with a number of independent members. The independent members give an impartial view in a process which, by its nature, generates strong feelings. Turek Camalli is one of these independent members, which means that he cannot be connected with Heinz Muller in any way. He is an accountant working at the head office of a major bank, and his employers have no connection with Heinz Muller or with engineering work.

Recently Turek has started preparing for the first meeting to set this year's annual wage rise. He has some data about Heinz Muller for the past 10 years, shown in the following table, but unfortunately he does not know how well the company is doing at the moment, nor how well it is likely to do next year. He has to work out some initial ideas, based only on this limited data.

Year	Average weekly earnings	Average hours worked	Company revenue (€million)	Gross company profit (€'000)	Index of industry wages	Retail Price Index
1	80.45	44	24.0	2,410	85.5	84.5
2	104.32	43	30.2	2,900	100.0	100.0
3	124.21	45	34.6	3,300	115.6	113.5
4	140.56	46	41.6	3,840	130.2	126.4
5	152.80	46	43.2	4,300	141.1	139.8
6	182.90	45	44.6	4,580	158.3	156.2
7	214.33	44	58.6	5,900	168.1	168.8
8	242.75	43	69.0	4,420	182.5	185.6
9	254.16	43	85.2	5,780	190.7	198.9
10	264.34	42	89.0	7,740	201.3	218.4

Questions

- **If you were Turek Camalli, how would you start thinking about this problem?**
- **What other data would you like to see and how can this be collected?**

PROBLEMS

7.1 The price of an item in consecutive months has been £106, £108, £111, £112, £118, £125, £130 and £132. Use an index based on the first month to describe these changes. How would this compare with an index based on the last month?

7.2 The following numbers of fishing boats have operated from Porto Novapietro over the past 10 years:

325 321 316 294 263 241 197 148 102 70

Describe these changes by indices based on the first and last year's figures.

7.3 The number of people employed by Westbury Cladding over the past 12 months is as follows. Use an index to describe these figures.

Month	1	2	3	4	5	6	7	8	9	10	11	12
Number	121	115	97	112	127	135	152	155	161	147	133	131

7.4 The annual output of a company is described by the following indices.

Year	1	2	3	4	5	6	7	8
Index 1	100	125	153	167				
Index 2				100	109	125	140	165

If the company made 23,850 units in year 2, how many did it make in the other years? What is the percentage increase in output each year?

7.5 ARP insurance company uses an index to describe the number of agents working for it. This index was revised five years ago, and had the following values over the past 10 years.

Year	1	2	3	4	5	6	7	8	9	10
Index 1	106	129	154	173	195	231				
Index 2						100	113	126	153	172

If the company had 645 agents in year 4, how many did it have in the other years?

7.6 Employees in a company are put into four wage groups. During a three-year period the numbers employed in each group and the average weekly wage are as follows.

	Year 1		Year 2		Year 3	
Group	Number	Wage	Number	Wage	Number	Wage
1	45	125	55	133	60	143
2	122	205	125	211	132	224
3	63	245	66	268	71	293
4	7	408	9	473	13	521

Use different indices to describe changes in wages paid and numbers employed.

7.7 The following table shows the price of drinks served in The Lion Inn. How would you describe the price changes?

	Wine	Spirits	Beer	Soft drinks
Year 1	91	95	78	35
Year 1	97	105	85	39
Year 3	102	112	88	42
Year 4	107	125	93	47

7.8 A company buys four products with the following characteristics.

	Number of units bought		Price paid per unit	
Product	Year 1	Year 2	Year 1	Year 2
A	121	141	9	10
B	149	163	21	23
C	173	182	26	27
D	194	103	31	33

Calculate a base-weighted index and a current-weighted index for the products.

7.9 The average prices for four items over four years are as follows.

Item	Year 1	Year 2	Year 3	Year 4
A	25	26	30	32
B	56	61	67	74
C	20	25	30	36
D	110	115	130	150

A company annually bought 400, 300, 800 and 200 units of each item respectively. Calculate weighted price indices for years 2 to 4, taking year 1 as the base year.

7.10 Calculate appropriate indices for the data in the table below.

	Year 1		Year 2		Year 3		Year 4	
Item	Price	Quantity	Price	Quantity	Price	Quantity	Price	Quantity
AL403	142	27	147	26	155	32	165	32
ML127	54	284	58	295	65	306	75	285
FE872	1,026	5	1,026	8	1,250	2	1,250	3
KP332	687	25	699	25	749	20	735	55
KP333	29	1,045	31	1,024	32	1,125	36	1,254
CG196	58	754	64	788	72	798	81	801
CG197	529	102	599	110	675	120	750	108
CG404	254	306	275	310	289	305	329	299
CG405	109	58	115	62	130	59	140	57
NA112	86	257	83	350	85	366	90	360
QF016	220	86	220	86	225	86	225	86
QT195	850	10	899	9	949	12	999	16
LJ878	336	29	359	38	499	11	499	25

RESEARCH PROJECTS

7.1 The following table shows the UK's Retail Price Index from 1970 to 2005. As you can see, the index was reset to 100 in January 1974 and again in January 1987.

Year	Index	Year	Index	Year	Index	Year	Index
1970	140	1979	224	1988	107	1997	158
1971	153	1980	264	1989	115	1998	163
1972	164	1981	295	1990	126	1999	165
1973	179	1982	320	1991	134	2000	170
1974	109	1983	335	1992	139	2001	173
1975	135	1984	352	1993	141	2002	176
1976	157	1985	373	1994	144	2003	181
1977	182	1986	386	1995	149	2004	187
1978	197	1987	102	1996	153	2005	192

What is the annual rate of inflation for each year? The UK government publishes several values for inflation, each of which is calculated in a different way. How do the figures above compare with other published results? Why are there differences? How do these figures compare with those from other countries?

7.2 In 1998 the American Retail Consortium sponsored an investigation into the main trends in retail purchasing. Part of this study included a breakdown of retail sales by major geographic region, as shown in the following table:

Year	USA	NE	Midwest	South	West
1970	375	94	108	107	67
1971	414	99	117	119	73
1972	459	105	128	133	83
1973	512	114	145	153	92
1974	542	119	157	163	98
1975	588	126	171	177	110
1976	656	136	191	199	125
1977	722	152	196	228	137
1978	804	165	223	252	164
1979	897	181	243	285	188
1980	957	195	247	312	203
1981	1,039	209	263	345	221
1982	1,069	219	269	356	225
1983	1,170	241	290	394	246
1984	1,287	266	320	434	267
1985	1,375	286	342	461	287
1986	1,450	313	354	482	300
1987	1,541	333	369	516	322
1988	1,656	366	395	547	347
1989	1,759	381	418	577	383
1990	1,845	386	437	610	412
1991	1,856	379	447	619	410
1992	1,952	389	472	663	427
1993	2,074	407	503	716	447
1994	2,230	427	539	783	482
1995	2,329	438	567	822	502
1996	2,461	463	598	870	530
1997	2,566	485	624	902	552

What indices could they use to describe these figures? Collect information to show how these would compare with an equivalent study in Europe.

Sources of information

Further reading

There are not really any books exclusively about indices, but you can find them discussed in most statistics books. Some suggested titles are given in Chapter 14.

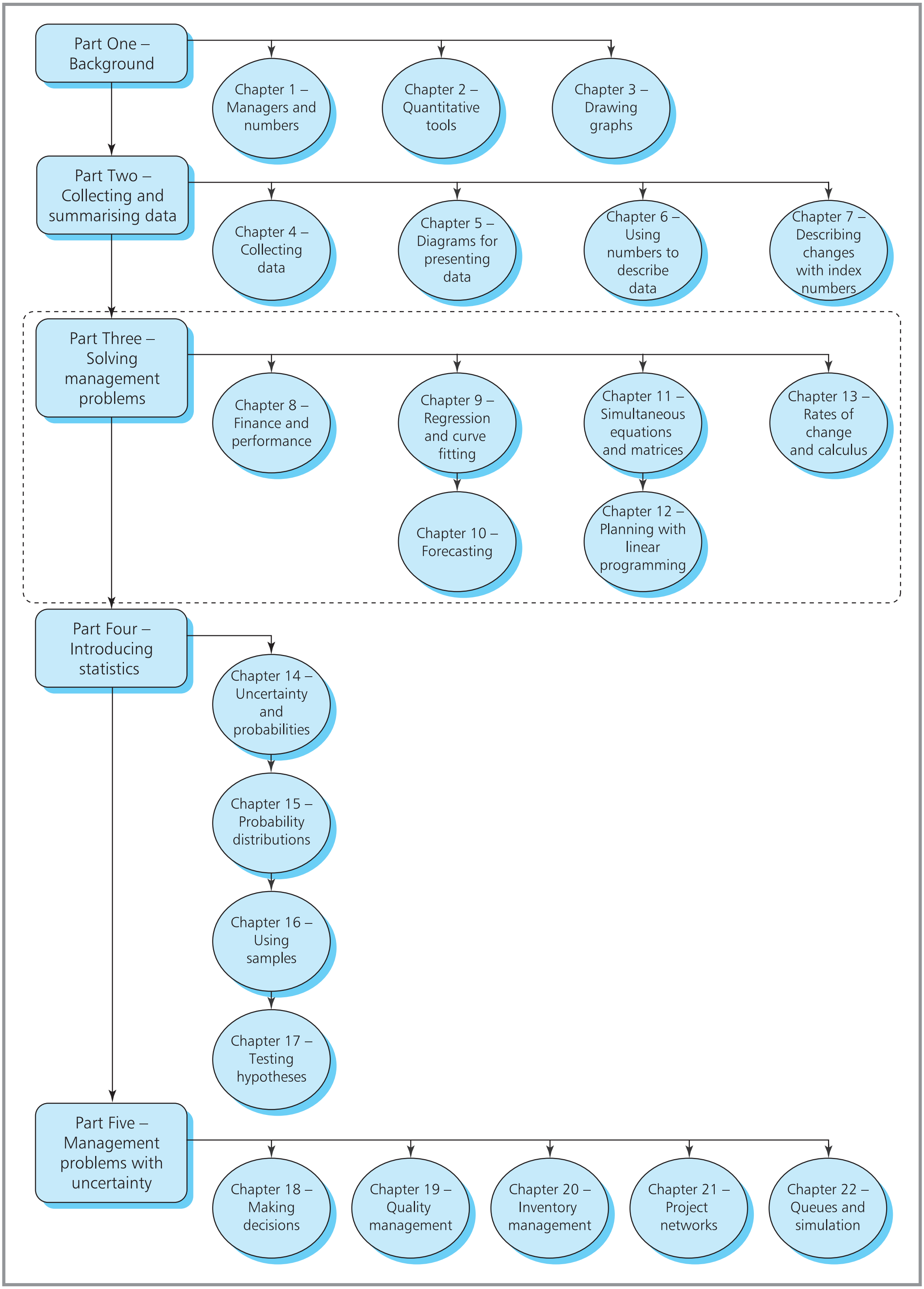
PART THREE

Solving management problems

This book is divided into five parts. The first part looked at the background and context for quantitative methods. The second part showed how to collect, summarise and present data. These laid the foundation, and we now have a set of tools for tackling management problems. The next step is to use these to solve some common – and even universal – problems. This is the third part of the book, which shows how to use quantitative methods for solving different types of management problem. The problems tackled here are deterministic, which means we are dealing with conditions of certainty. The fourth part of the book introduces probabilities and statistical analyses. The last part shows how to solve some management problems that include uncertainty.

There are six chapters in this part. Chapter 8 describes some calculations for finance and performance. Chapter 9 uses regression to describe the relationship between variables, and Chapter 10 extends these ideas in forecasting. Chapter 11 shows how to use matrices, and Chapter 12 uses these for linear programming. Chapter 13 reviews some uses of calculus.

Map 3 shows how the chapters in this part fit into the rest of the book.



Map 3 Map of chapters – Part Three

CHAPTER 8

Finance and performance

Contents

Chapter outline

Managers use a range of different measures to monitor the performance of their organisations. These measures show how well the organisation is doing, how it compares with competitors, how performance has changed in the past, and whether they can meet future targets. Many measures consider finance, so this chapter describes a range of calculations that managers use for money and performance.

After finishing this chapter you should be able to:

- Appreciate the importance of measuring performance
- Calculate a number of performance ratios
- Find break-even points
- Understand the reasons for economies of scale
- Do calculations for compound interest
- Discount amounts of money to their present value
- Calculate net present values and internal rates of return
- Depreciate the value of assets
- Calculate the payments for sinking funds, mortgages and annuities.

Measures of performance

Managers must know how well their organisations are performing, and they use many different measures, such as sales, profit, output, number of customers, growth, share price, and so on. Without these measures, managers have no idea whether they are meeting targets, improving performance, keeping up with competitors, investing in the right areas, giving areas enough attention, or a host of other questions.

A basic measure of performance is **capacity**, which sets the maximum output from a process in a specified time. The capacity of a bottling plant is the maximum output of 1,000 bottles an hour, the capacity of a call centre is 2,000 calls a day, and the capacity of a theatre is 1,200 people.

At first it seems strange to describe capacity as a measure of performance rather than a fixed constraint – but you have to remember the following:

- The capacity of a process depends on the way in which resources are organised and managed, so that two organisations can use identical resources in different ways and have different capacities.
- Capacity varies over time. Imagine a team of people who are shovelling sand: at eight o'clock in the morning they are fresh and working hard, but by six o'clock in the evening they are tired and working more slowly. So their capacity has changed, even though the operations are the same.

Even an apparently simple measure like capacity depends on assumptions, approximations and opinions – and the same is true of most other measures of performance, like output, income, sales, profit, customers served, messages processed, time available, people employed, space available, and so on.

Another problem is that absolute measures do not really say much. When you hear that a company made a profit of €1 million last year, this does not say much about its performance. It would certainly be an excellent performance for Jane's Bespoke Software – but it would be a disaster for Microsoft. We really need more information about the context of a measure, and the easiest way of getting this uses a **performance ratio**.

Performance ratios

A performance ratio takes a direct measure of performance, and divides this by another reference value that sets the broader context. For example, you can take the profit and put this into context by dividing it by sales volume – or you can find the ratio of sales over assets employed, output over machine time used, production over the number of employees, and so on.

One of the most widely used performance ratios is **utilisation**, which shows how much of the available capacity is actually used. If a process has a capacity of 100 units a week but makes only 60 units, then:

$$\text{utilisation} = \frac{\text{amount of capacity used}}{\text{available capacity}} = \frac{60}{100} = 0.6 \text{ or } 60\%$$

Managers like to use resources as fully as possible and generally aim for high utilisation. There are only two ways of achieving this: either by raising the top line (the level of performance) or by reducing the bottom line (the standard for comparison). RyanAir's utilisation of seats is defined as:

$$\text{utilisation} = \frac{\text{number of seats used}}{\text{number of seats available}}$$

The company makes more profit by having high utilisation, and the only ways of raising this are to get more passengers sitting on seats (adjusting the demand) or to reduce the number of seats available (adjusting the supply).

Another widely used performance ratio gives the **productivity**. People often assume that this is the amount produced per person, but it is more general than this and measures the amount of output for each unit of resource used. In its broadest sense the **total productivity** relates output to *all* resources used.

$$\text{total productivity} = \frac{\text{total output}}{\text{total resources used}}$$

But this is very difficult to calculate, so most organisations use **partial productivity**, which finds the volume of products made for each unit of a specified resource.

$$\text{partial productivity} = \frac{\text{amount of products made}}{\text{units of a single resource used}}$$

If a process uses 25 hours of machine time to make 50 units, the productivity is two units per machine-hour; if it employs five people the productivity is 10 units per person; if it uses 50 tonnes of raw material the productivity is one unit per tonne. There are four main types of partial productivity:

- *equipment productivity* – such as the number of units made per machine
- *labour productivity* – typically the output from each per employee
- *capital productivity* – such as the production for each pound invested
- *energy productivity* – such as the amount produced from each barrel of oil.

WORKED EXAMPLE 8.1

Peter Keller collected the following data for a process over two consecutive years. What can you say about performance?

	2005	2006
Number of units made	1,000	1,200
Selling price	£100	£100
Raw materials used	5,100 kg	5,800 kg
Cost of raw materials	£20,500	£25,500
Hours worked	4,300	4,500
Direct labour costs	£52,000	£58,000
Energy used	10,000 kWh	14,000 kWh
Energy cost	£1,000	£1,500
Other costs	£10,000	£10,000

Solution

You can consider various ratios, such as the units of output per kilogram of raw material. In 2005 this was 1,000/5,100 = 0.196, and in 2006 it had risen to 1,200/5,800 = 0.207. Some other measures are:

	2005	2006	Percentage increase
Units / kg of raw material	0.196	0.207	5.6
Units / £ of raw material	0.049	0.047	–4.1
Units / hour	0.233	0.267	14.6
Units / £ of labour	0.019	0.021	10.5
Units / kWh	0.100	0.086	–14.0
Units / £ of energy	1.000	0.800	–20

Worked example 8.1 continued

In general, labour productivity has risen, raw materials productivity has stayed about the same, and energy productivity has fallen.

You can also estimate the total productivity as the value of the output (number of units multiplied by selling price) divided by the value of inputs (the sum of the costs of all inputs). Then the total productivity in 2005 is:

$$\frac{\text{total output}}{\text{total input}} = \frac{100 \times 1{,}000}{20{,}500 + 52{,}000 + 1{,}000 + 10{,}000} = 1.2$$

By 2006 this had risen to 120,000/95,000 = 1.26, an increase of 5%.

Financial ratios

A key performance figure for many organisations is the profit. If you subtract all the costs of running a business from the income generated by sales, you are left with the profit. If the income is less than the costs, the organisation makes a loss.

$$\text{profit} = \text{revenue} - \text{costs}$$

This seems straightforward, but remember that any financial data depends on accounting conventions and does not necessarily give an objective view (as illustrated by some well-known examples of 'financial irregularities'[1–3]). Again, absolute measures do not really say much, and managers usually calculate different kinds of ratio. Some widely used financial ratios include the following.

- **Profit margin** – the profit before tax and interest as a percentage of sales:

$$\text{profit margin} = \frac{\text{profit before tax and interest}}{\text{sales}} \times 100$$

- **Return on assets (ROA)** – profit as a percentage of the organisation's assets:

$$\text{return on assets} = \frac{\text{profit before interest and tax}}{\text{fix assets} + \text{current assets}} \times 100$$

  This is arguably the most comprehensive measure of business performance. From a purely financial point of view, the ROA should be as high as possible – but remember that different types of organisation need very different amounts of assets. An advertising agency needs few assets and should have a much higher ROA than, say, a power station or a car assembly plant.

- **Acid test** – the ratio of liquid assets and liabilities:

$$\text{acid test} = \frac{\text{liquid assets (cash and readily saleable assets)}}{\text{current liabilities}}$$

Some other ratios that are particularly important for investors include:

$$\text{return on equity} = \frac{\text{profit after tax}}{\text{shareholders' money}} \times 100$$

$$\text{gearing} = \frac{\text{borrowed money}}{\text{shareholders' money}}$$

$$\text{earnings per share} = \frac{\text{profit after tax}}{\text{number of shares}}$$

$$\text{dividends per share} = \frac{\text{amount distributed as dividends}}{\text{number of shares}}$$

$$\text{price–earnings ratio} = \frac{\text{share price}}{\text{earnings per share}}$$

$$\text{dividend cover} = \frac{\text{profit after tax}}{\text{profit distributed to shareholders}}$$

$$\text{yield (as a percentage)} = \frac{\text{dividend}}{\text{share price}} \times 100$$

IDEAS IN PRACTICE AstraZeneca

AstraZeneca is one of the world's major pharmaceutical companies, with sales around $20 billion. Its operations range from basic research to find new medicines, through manufacturing and distribution, to after-sales service and social health. The company uses thousands of measures for different aspects of its performance. The following table shows some calculations it might include, based on figures for 2004.

Direct measures	$ million	
Sales	21,426	
Other income	630	
Operating costs	16,971	
Profit before tax	21,426 + 630 − 16,971	= 5,085
tax paid		1,254
Profit after tax	5,085 − 1,254	= 3,831
minority interests		18
Net profit	3,831 − 18	= 3,813
dividends paid		1,555
Retained earnings	3,813 − 1,555	= 2,258
Total assets	25,616	
Number of ordinary shares	1,673 million	
Share price at year end	$18.89	

Financial ratios

Earnings per share	$2.28	($3,813 million / 1,673 million)
Dividend per share	$0.93	($1,555 million / 1,673 million)
Gross return on sales	23.7%	($5,085 million / $21,426 million)
Gross return on assets	19.9%	($5,085 million / $25,616 million)
Share price to earnings	8.3	($18.89 / $2.28)
Yield	4.9%	($0.93 / $18.89)

Sources: AstraZeneca annual report 2005; www.astrazeneca.com; www.uk.finance.yahoo.com; www.lse.co.uk.

Review questions

8.1 Why is it often better to use ratios rather than absolute measures?

8.2 What is the difference between total and partial productivity?

8.3 Is it possible for some measures of performance to rise while others fall?

8.4 Are profit-making companies the only ones concerned with their finances?

Break-even point

When an organisation sells a product – which might be either services or goods – a key piece of information is the **break-even point**. This is the number of units it must sell to cover all costs and start to make a profit. You calculate this by comparing the revenue and total cost of production.

The revenue from a product is:

$$\text{revenue} = \text{price charged per unit} \times \text{number of units sold}$$

The total production costs are a bit more awkward, as some vary with the number of units made, and others are fixed. For instance, when a company leases a machine to make a product, the cost of leasing is fixed regardless of the number of units made, while the cost of raw materials varies with production. You see the same effect with a car, where some costs are fixed (repayment of purchase loan, road tax, insurance, etc.) and others vary with the distance travelled (petrol, oil, tyres, depreciation, etc.). Then we have:

$$\begin{aligned}\text{total cost} &= \text{fixed cost} + \text{variable cost}\\ &= \text{fixed cost} + (\text{cost per unit} \times \text{number of units made})\end{aligned}$$

The break-even point is defined as the point where revenue covers the total cost, so that:

$$\text{revenue} = \text{total cost}$$

$$\begin{matrix}\text{price per} \\ \text{unit}\end{matrix} \times \begin{matrix}\text{number of} \\ \text{units sold}\end{matrix} = \text{fixed cost} + \begin{matrix}\text{cost per} \\ \text{unit}\end{matrix} \times \begin{matrix}\text{number of} \\ \text{units made}\end{matrix}$$

You can see that both the revenue and total cost rise linearly with the number of units, so we can plot the relationship in Figure 8.1.

If we let P = price charged per unit, C = production cost per unit, F = fixed cost and N = number of units sold (assuming this is the same as the number of units made), the break-even point has:

$$PN = F + CN$$

Rearranging this gives:

$$N(P - C) = F$$

or:

$$\text{break-even point} = N = \frac{F}{P - C}$$

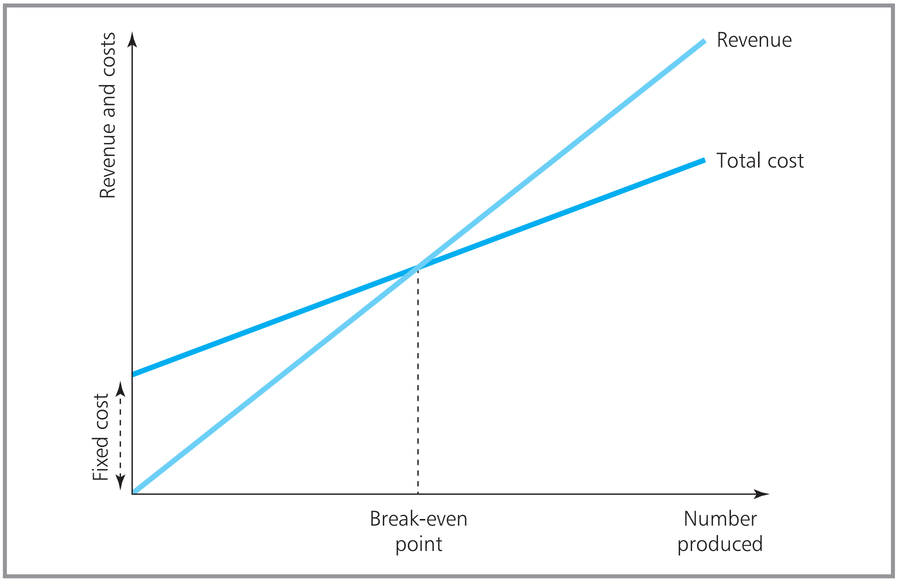


Figure 8.1 The break-even point

Suppose a company spends £500,000 on research, development, equipment and other fixed costs before it starts making a new product. If each unit of the product costs £30 to make and sells for £50, the break-even point is:

$$\begin{aligned} N &= F/(P - C) \\ &= 500{,}000/(50 - 30) \\ &= 25{,}000 \text{ units} \end{aligned}$$

You can see how this occurs, as the company makes a profit only when it recovers this initial investment, and as each unit contributes £50 – £30 = £20 the company has to sell 25,000 units before this happens. Before this point the revenue does not cover fixed costs; after this point the excess revenue gives a profit.

We can add a few more details to Figure 8.1, such as the following.

- When the number of units sold is higher than the break-even point, revenue is greater than total cost and there is a profit (shown in Figure 8.2):

$$\text{profit} = N(P - C) - F$$

- When the number of units sold equals the break-even point, revenue equals total cost:

$$N(P - C) = F$$

- When the number of units sold is less than the break-even point, total cost is higher than revenue and there is a loss:

$$\text{loss} = F - N(P - C)$$

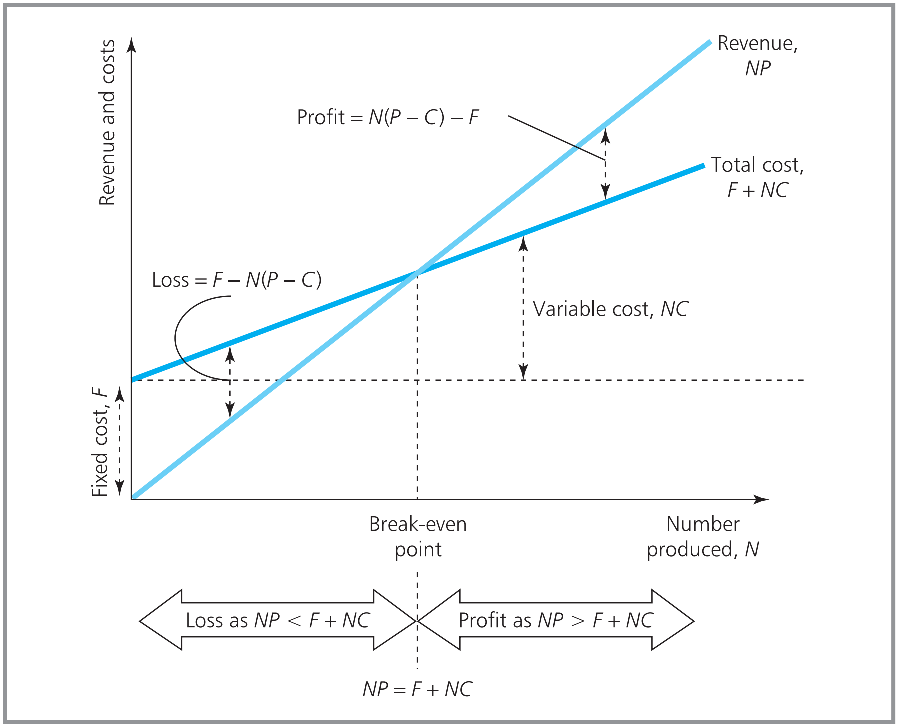


Figure 8.2 Profit and loss around the break-even point

WORKED EXAMPLE 8.2

A company sells 200 units of a product every week; the fixed costs for buildings, machines and employees are €12,000 a week, while raw material and other variable costs are €50 a unit.

(a) What is the profit if the selling price is €130 a unit?
(b) What is the profit if the selling price is €80 a unit?
(c) What is the profit if the selling price is fixed at €80 but sales rise to 450 units a week?

Solution

(a) We know that:

$N = 200$ units = number of units sold each week
$F = €12{,}000$ a week = fixed cost each week
$C = €50$ a unit = variable cost per unit

With a selling price, P, of €130 the break-even point is:

$$N = \frac{F}{P - C} = \frac{12{,}000}{130 - 50} = 150 \text{ units}$$

Actual sales are more than this, so the product makes a profit of:

$$\begin{aligned}\text{profit} &= N(P - C) - F = 200 \times (130 - 50) - 12{,}000 \\ &= €4{,}000 \text{ a week}\end{aligned}$$

(b) With a selling price, P, of €80 the break-even point is:

$$N = \frac{F}{P - C} = \frac{12{,}000}{80 - 50} = 400 \text{ units}$$

Actual sales are less than this, so the product makes a loss of:

$$\begin{aligned}\text{loss} &= F - N(P - C) = 12{,}000 - 200 \times (80 - 50) \\ &= €6{,}000 \text{ a week}\end{aligned}$$

(c) With a selling price of €80 we know that the break-even point is 400 units. If sales increase to 450 units a week, the product makes a profit of:

$$\begin{aligned}\text{profit} &= N(P - C) - F = 450 \times (80 - 50) - 12{,}000 \\ &= €1{,}500 \text{ a week}\end{aligned}$$

The company can still make a profit with a lower selling price, provided that sales are high enough.

Break-even analyses are useful in a variety of circumstances, such as the choice between buying or leasing equipment, setting the capacity of new equipment, deciding whether to buy an item or make it within the company, comparing competitive tenders, and so on. But it is worth mentioning the most common difficulty of finding break-even points, which is assigning a reasonable proportion of overheads to the fixed cost of each product. This depends on the accounting conventions used – with the problem becoming worse when the product mix is continually changing, and a changing amount of overheads is assigned to each product. Then the costs of making a particular product can apparently change, even though there is no change in the product itself or the way it is made.

WORKED EXAMPLE 8.3

NorElec offers two prices to domestic consumers. The normal rate has a standing charge of £18.20 a quarter, and each unit of electricity used costs £0.142. A special economy rate has a standing charge of £22.70 a quarter, with each unit of electricity used during the day costing £0.162, but each unit used during the night costing only £0.082. What pattern of consumption makes it cheaper to use the economy rate?

Solution

If a consumer uses an average of D units a quarter during the day and N units a quarter during the night, their costs are:

normal rate: $18.20 + 0.142 \times (D + N)$
economy rate: $22.70 + 0.162 \times D + 0.082 \times N$

It is cheaper to use the economy rate when:

$$22.7 + 0.162D + 0.082N < 18.2 + 0.142(D + N)$$

i.e.

$$4.5 < 0.06N - 0.02D$$

or

$$D < 3N - 225$$

When consumption during the day is less than three times consumption during the night minus 225 units, it is cheaper to use the economy rate: otherwise it is cheaper to use the standard rate (as shown in Figure 8.3).

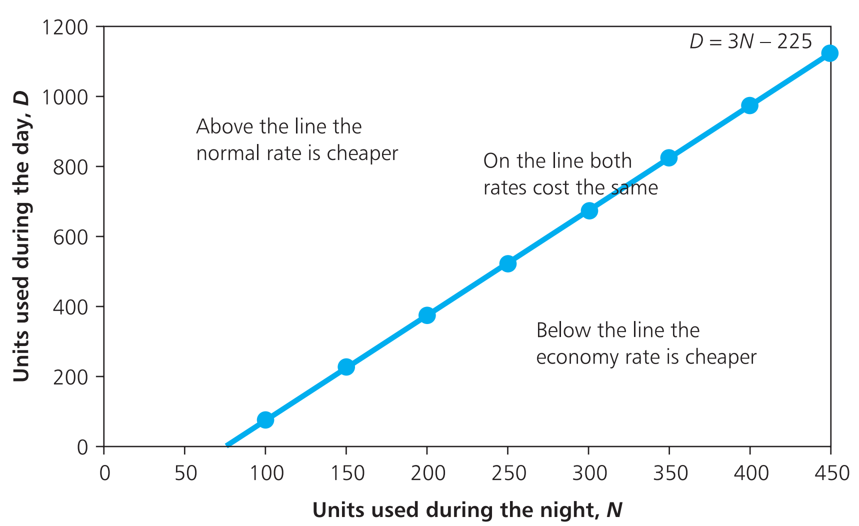


Figure 8.3 Identifying the cheapest options with NorElec

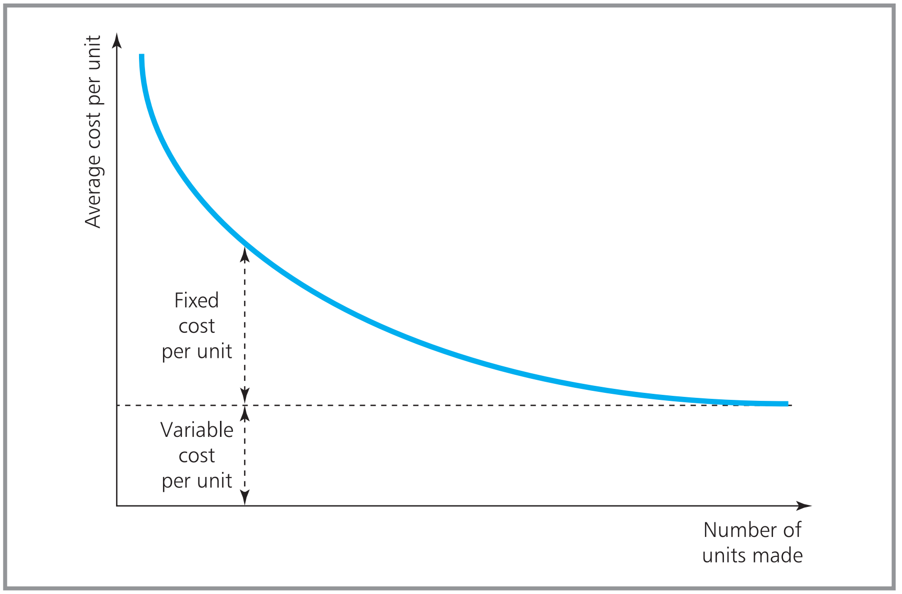


Figure 8.4 Decreasing average cost with increasing production

Economies of scale

Economies of scale mean that the average cost per unit declines as the number of units produced increases. You can see one reason for this from the break-even analysis, which said that:

total cost = fixed cost + variable cost = $F + NC$

Dividing this total cost by the number of units made, N, gives the average cost per unit:

average cost per unit = $(F + NC)/N = F/N + C$

As N increases the value of F/N decreases – meaning that the proportion of the fixed cost recovered by each unit falls, and the average cost per unit also falls (as shown in Figure 8.4).

WORKED EXAMPLE 8.4

Jane's Seafood Diner serves 200 meals a day at an average price of €20. The variable cost of each meal is €10, and the fixed costs of running the restaurant are €1,750 a day.

(a) What profit does the restaurant make?
(b) What is the average cost of a meal?
(c) By how much would the average cost of a meal fall if the number of meals served rose to 250 a day?

Solution

(a) The break-even point is:

$$N = F/(P - C) = 1{,}750/(20 - 10) = 175 \text{ meals}$$

Actual sales are above this, so there is a profit of:

$$\begin{aligned} \text{profit} &= N(P - C) - F \\ &= 200 \times (20 - 10) - 1{,}750 \\ &= €250 \text{ a day} \end{aligned}$$

Worked example 8.4 continued

(b) The average cost of a meal is:

average cost = (fixed cost + variable cost) / number of meals
= (1,750 + 200 × 10) / 200
= €18.75 a meal

(c) Serving 250 meals a day would give:

average cost = (1,750 + 250 × 10) / 250
= €17 a meal

Spreading the fixed costs over more units is only one reason for economies of scale. The unit cost can also fall because operations become more efficient, people are more familiar with the work and take less time, problems are sorted out, disruptions are eliminated, planning becomes routine, and so on. These effects seem to suggest that facilities should always be as big as possible. This is certainly the reason why mobile phone companies, banks and oil companies have become so big. But there can also be **diseconomies of scale**. Here the benefits of larger operations are more than offset by the problems, which include more bureaucracy, difficulties of communication, more complex management hierarchies, increased costs of supervision, and perceived reduction in importance of individuals. These effects usually lead to economies of scale up to an optimal size, and then diseconomies of scale, as shown in Figure 8.5.

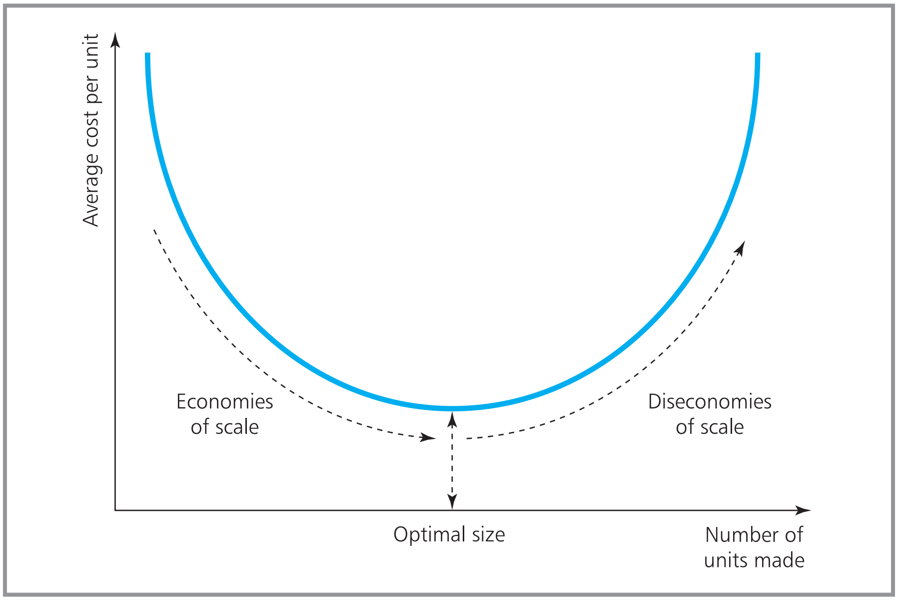


Figure 8.5 Finding the optimal size of facilities

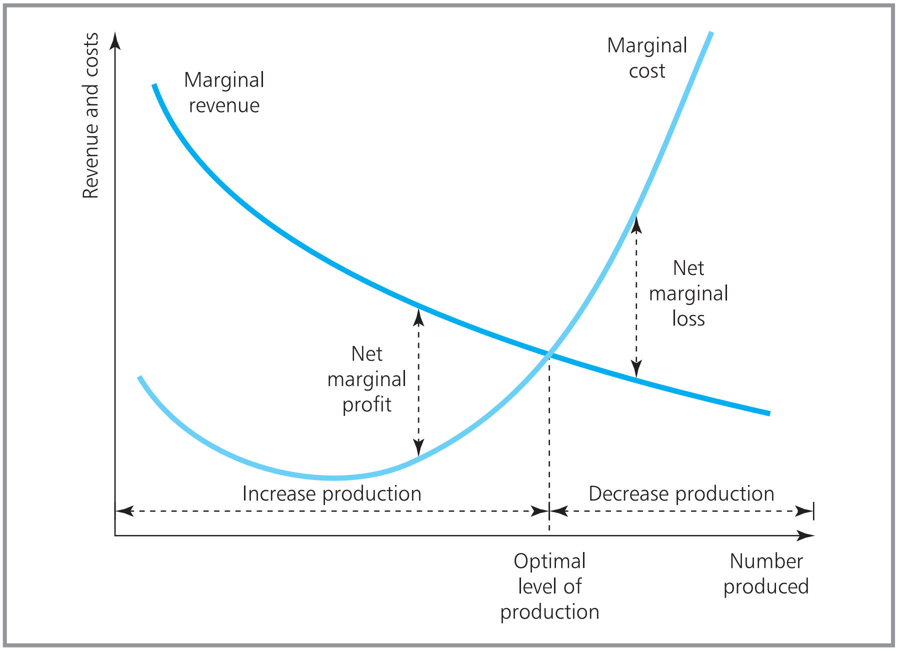


Figure 8.6 Marginal analysis finds the best level of production

Marginal values

The break-even model assumes that the variable cost is constant, regardless of the number of units made. But we have just said that larger operations can be more efficient – suggesting that the variable cost can fall with increasing production. This effect is described by a **marginal cost** – which is the cost of making one extra unit of a product. The marginal cost is generally higher when small numbers are produced, but falls with higher production. But again there comes a point where diseconomies of scale come into play, making the marginal cost rise as shown in Figure 8.6.

We can also define a **marginal revenue**, which is the revenue generated by selling one more unit of a product. The break-even analysis again assumes that customers pay a constant price, regardless of sales. But the price they are willing to pay really varies with production, and they generally expect to pay less for higher levels of output. In other words, the marginal revenue falls with increasing production.

You can see an important pattern in Figure 8.6. With low production levels, the marginal cost is less than the marginal revenue, so there is a net profit on every extra unit produced. This encourages the company to increase production. As production increases, the marginal revenue declines, but after an initial fall the marginal cost begins to rise. Then at high production levels, the marginal cost is more than the marginal revenue, and there is a net loss on every extra unit produced. This encourages the company to reduce production. The result is that organisations always move towards the point where the marginal revenue exactly matches the marginal cost, which defines their optimal production level. Below this they are missing out on potential profits, and above it they are making unnecessary losses (we return to this theme in Chapter 13).

Review questions

8.5 What does the variable cost vary with?

8.6 What exactly is the break-even point?

8.7 Because of economies of scale, it is always better to have a single large office than a number of smaller ones. Is this true?

8.8 What is the significance of marginal cost and revenue?

Value of money over time

If you want to buy a house, you can try saving enough money to pay cash – but experience suggests that house prices rise a lot faster than savings. A better option is to save a deposit and then borrow the rest of the money as a mortgage. Then you repay the loan over a long period, typically around 25 years. If you add up all of your repayments, they come to several times the amount you originally borrowed. The additional payments are for **interest**.

Interest

When someone borrows money, the amount they borrow is the **principal**. The borrower agrees both to repay the loan over some period, and to pay an additional amount of interest. Interest is the lender's reward for lending money – and the borrower's penalty – and is usually quoted as a percentage of the principal, so you might pay interest of 7% a year.

Suppose you have some money to spare and put it into a bank account – effectively lending your money to the bank. If you leave £1,000 in an account offering interest of 8% a year, it earns $1000 \times 8/100 =$ £80 interest at the end of the year. If you take the interest earned out of the account, the initial deposit stays unchanged at £1,000. This is the principle of **simple interest**, which pays interest only on the initial deposit, and the amount of interest paid each year remains the same. If the original investment is A_0 and the interest rate is I, the amount of interest paid each year is $A_0 I/100$. It is easier to do calculations with the interest rate described as a decimal fraction, i, rather than a percentage, I, so we use the definition that $i = I/100$. Then an interest rate of 10% means that $i = 0.1$, an interest rate of 15% has $i = 0.15$, and so on. The amount of interest paid each year is $A_0 i$.

In practice, loans rarely use simple interest, and almost invariably offer **compound interest**. This pays interest both on the original investment and on interest earned previously and left in the account. If you put an amount of money A_0 into a bank account and leave it untouched for a year, earning interest at an annual rate i, at the end of the year you have an amount A_1, where:

$$A_1 = A_0 \times (1 + i)$$

In general we can call the amount of money in the account after n years, A_n. Then if you leave the amount A_1 untouched for a second year, it will earn interest not only on the initial amount deposited, but also on the interest earned in the first year. This gives:

$$A_2 = A_1 \times (1 + i)$$

Here we can substitute the value for A_1 to give:

$$A_2 = [A_0 \times (1 + i)] \times (1 + i) = A_0 \times (1 + i)^2$$

If you leave the amount A_2 untouched for a third year, you have:

$$A_3 = A_2 \times (1 + i)$$

and substituting for A_2:

$$A_3 = A_0 \times (1 + i)^3$$

Your money increases in this compound way, and at the end of n years you have A_n, where:

> for compound interest
>
> $$A_n = A_0 \times (1 + i)^n$$

The longer you leave money in the account, the greater the annual interest it earns. Figure 8.7 shows this cumulative effect on the value of €1 invested over time with interest rates between 3% and 25%.

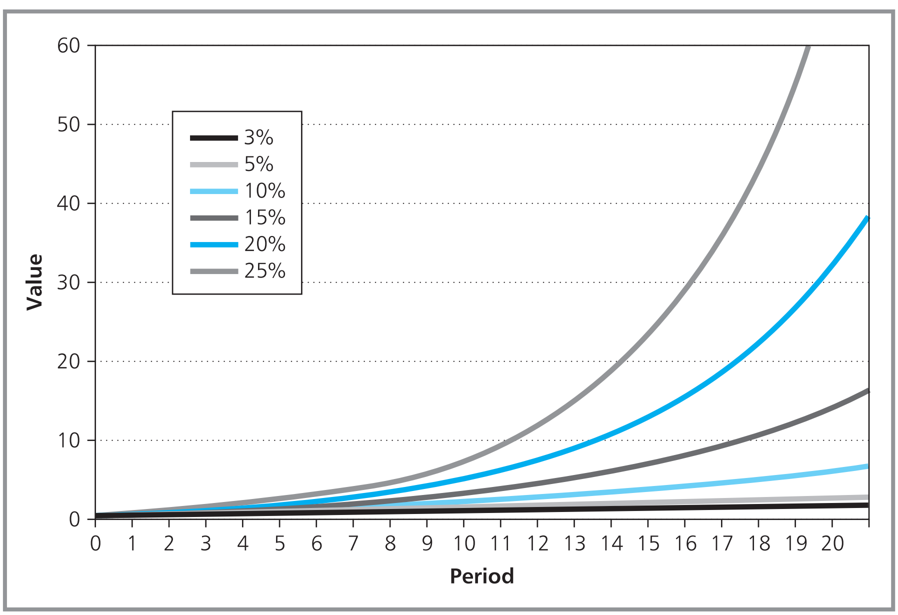


Figure 8.7 The increasing value of €1 invested at different interest rates

WORKED EXAMPLE 8.5

If you leave £1,000 in a bank account earning 5% compound interest a year, how much will be in your account at the end of five years? How much will there be at the end of 20 years?

Solution

We know that:

$A_0 = £1{,}000$

$i = 0.05$

With compound interest the amount in the account is:

$A_n = A_0 \times (1 + i)^n$

At the end of five years you will have:

$A_5 = 1{,}000 \times (1 + 0.05)^5 = 1{,}000 \times 1.2763$
$= £1{,}276$

At the end of 20 years you will have:

$A_{20} = 1{,}000 \times (1 + 0.05)^{20} = 1{,}000 \times 2.6533$
$= £2{,}653$

Suppose that you borrow £100 with interest of 2% payable at the end of each month. You might do a quick calculation and assume that this is equal to $2 \times 12 = 24\%$ a year – but this is wrong, as you can see from the following comparison:

- Borrowing £100 at 24% a year raises the debt to $100 \times (1 + 0.24) = £124$ at the end of the year.
- Borrowing £100 at 2% a month, and using compound interest to calculate the debt at the end of 12 months, gives:

$$A_n = A_0 \times (1 + i)^n = 100 \times (1 + 0.02)^{12} = £126.82$$

This effect becomes confusing when credit cards quote a daily interest rate, and some loans simply give the amounts payable every month. To avoid this confusion, many countries have a legal requirement to quote an effective or real annual interest rate. This is the true cost of borrowing, which we will call the **annual equivalent rate** (AER) or **annual percentage rate** (APR). When a lender offers an APR of 12%, this means that by the end of the year interest payments will be exactly 12% of the principal.

WORKED EXAMPLE 8.6

If you invest €2,000 how much will you have after three years if the interest rate is 12% a year? How does this differ from rates of 3% a quarter, 1% a month, or (12/52 =) 0.23% a week?

Solution

- With annual payments the interest rate, i, is 0.12 and after three years you have: $2{,}000 \times (1 + 0.12)^3 = €2{,}809.86$.
- With quarterly payments the interest rate, i, is 0.03 and after 12 quarters you have: $2{,}000 \times (1 + 0.03)^{12} = €2{,}851.52$.
- With monthly payments the interest rate, i, is 0.01 and after 36 months you have: $2{,}000 \times (1 + 0.01)^{36} = €2{,}861.54$.
- With weekly payments the interest rate, i, is $0.12/52 = 0.0023$ and after 156 weeks you have: $2{,}000 \times (1 + 0.0023)^{156} = €2{,}862.04$.

The differences may be small, but they do accumulate. You can also see that shorter times between interest payments give larger returns, because interest already earned is added more quickly to the principal, and starts earning its own interest sooner.

WORKED EXAMPLE 8.7

How much will an initial investment of £10,000 earning nominal interest of 0.19% a week be worth at the end of 10 years?

Solution

With $i = 0.0019$, the calculation is:

$$A_{10} = A_0 \times (1 + i)^n = 10{,}000 \times (1 + 0.0019)^{10\times 52}$$
$$= £26{,}833.41$$

As you can see, this kind of arithmetic can get rather messy. If the interest here is paid daily, i would be $0.0019/7 = 0.0002714$ and you would have to calculate $10{,}000 \times (1.0002714)^{365\times 10} = £26{,}928$.

It is easy to get errors in these calculations (or even forget to allow for leap years), so it can be more convenient to use a simple approximation. For large values of n and short time periods, the value of $(1 + i)^n$ is very close to the value of e^{in}, where e is the exponential constant equal to 2.71828. This approximates the discrete payments by a continuous value. Here we calculate:

$$A_{20} = A_0 \times e^{in} = 10{,}000 \times e^{(0.0019\times 10\times 52)} = £26{,}858.57$$

The approximation gives a slightly different result, but the effect is small and the approximation becomes better with larger values of n.

Review questions

8.9 Would you rather have £1,000 now or in five years' time?

8.10 If you leave an amount of money in a bank account, why does its value not rise linearly?

8.11 Is an interest rate of 12% a year the same as 1% a month?

Discounting to present value

An amount of money A_0 invested now will earn interest and have a value of $A_n = A_0 \times (1 + i)^n$ at a point n periods in the future. So £1,000 invested now at 8% a year will be worth £1,469 in five years' time. We can turn this the other way around and say that £1,469 in five years' time is worth £1,000 now. And in general, an amount, A_n, n periods in the future has a **present value** of A_0, where:

$$A_0 = A_n / (1 + i)^n = A_n \times (1 + i)^{-n}$$

Calculating the present value of an amount in the future is called **discounting to present value**. Then the value of i becomes a **discount rate** and $(1 + i)^{-n}$ is the **discount factor**. You can use this result to compare amounts of money that become available at different points in the future by finding the present value of each amount.

WORKED EXAMPLE 8.8

Rockwall Trust is thinking of investing in a new technology company. There are two possible investments, whose profits can be summarised as follows:

- Option 1 gives a profit of €300,000 in five years' time.
- Option 2 gives a profit of €500,000 in 10 years' time.

Which option should the company choose if it uses a discount rate of 20% a year for future profits?

Worked example 8.8 continued

Solution

Rockwall has to compare amounts of money generated at different times, and can do this by comparing the present value of each.

- Option 1 has $i = 0.2$, $n = 5$ and $A_5 = 300{,}000$. Then:

$$A_0 = A_n \times (1+i)^{-n} = A_5 \times (1+i)^{-5}$$
$$= 300{,}000 \times (1+0.2)^{-5} = €120{,}563$$

- Option 2 has $i = 0.2$, $n = 10$ and $A_{10} = 500{,}000$. Then:

$$A_0 = A_n \times (1+i)^{-n} = A_{10} \times (1+i)^{-10}$$
$$= 500{,}000 \times (1+0.2)^{-10} = €80{,}753$$

Option 1 clearly has the higher present value and on this evidence is the better alternative.

Discounting to present value is particularly useful with large projects that have payments and incomes spread over varying periods. Then you can compare all the costs and incomes by discounting them to their present value – and subtracting the present value of all costs from the present value of all revenues gives a **net present value**.

net present value = sum of discounted revenues – sum of discounted costs

If the net present value (NPV) is negative, a project will make a loss and should not be started; if alternative projects all have positive net present values, the best is the highest.

WORKED EXAMPLE 8.9

FHP Construction is considering three alternative projects with initial costs and projected revenues (each in thousands of dollars) over the next five years shown in the following table. If the company has enough resources to start only one project, use a discount rate of 10% to suggest the best.

		Initial cost	Net revenue generated in each year: 1	2	3	4	5
Project	A	1,000	500	400	300	200	100
	B	1,000	200	200	300	400	400
	C	500	50	200	200	100	50

Solution

The revenues for each project vary over time, with A offering more in the early years and B offering more later on. To get a valid comparison we can transform all amounts to present values and compare the net present value of each project. So for project A:

- 500 in year 1 has a present value of $500/1.1 = 454.545$
- 400 in year 2 has a present value of $400/1.1^2 = 330.579$
- 300 in year 3 has a present value of $300/1.1^3 = 225.394$, and so on.

Figure 8.8 shows the details of these calculations. Adding the present values of revenues and then subtracting the costs (in this case the single initial project cost) gives the net present values.

Project A has the highest NPV and is the one that FHP should choose (all things being equal). Project C has a negative NPV, showing a loss, so the company should clearly avoid this one. Another consideration is that the revenues from A are declining, suggesting the project has a limited life span of around five years; revenues from project B are rising, implying a longer potential life.

Worked example 8.9 continued

	A	B	C	D	E	F	G	H
1	**Net present value**							
2								
3	**Discount rate**	0.1						
4								
5	**Year**	**Discount factor**	**Project A**		**Project B**		**Project C**	
6			**Revenue**	**Present value**	**Revenue**	**Present value**	**Revenue**	**Present value**
7	**1**	1.1	$500.00	$454.55	$200.00	$181.82	$50.00	$45.45
8	**2**	1.21	$400.00	$330.58	$200.00	$165.29	$200.00	$165.29
9	**3**	1.331	$300.00	$225.39	$300.00	$225.39	$200.00	$150.26
10	**4**	1.4641	$200.00	$136.60	$400.00	$273.21	$100.00	$68.30
11	**5**	1.61051	$100.00	$62.09	$400.00	$248.37	$50.00	$31.05
12	**Totals**		$1,500.00	$1,209.21	$1,500.00	$1,094.08	$600.00	$460.35
13								
14	**Present values**							
15		**Revenues**		$1,209.21		$1,094.08		$460.35
16		**Costs**		$1,000.00		$1,000.00		$500.00
17								
18	**Net present value**			**$209.21**		**$94.08**		**–$39.65**

Figure 8.8 Calculation of net present values for the three projects in worked example 8.9

WORKED EXAMPLE 8.10

Use an annual discount rate of 15% to find the NPV of a project with the following returns (in thousands of euros) at the end of each year.

Year	1	2	3	4	5	6	7	8	9	10	11
Revenue	–70	–30	5	15	25	40	60	50	40	30	10

Solution

Spreadsheets have standard functions to calculate net present values – such as Excel's NPV or XNPV. These functions use slightly different assumptions about the time of payments, so you have to be a bit careful. Figure 8.9 shows two ways of doing the calculations. The first uses the standard function NPV to find the net present value of €17,870. The second does the full calculations to check this value.

Worked example 8.10 continued

	A	B	C	D	E	F	G	H	I	J	K	L
1	**Net present value**											
2												
3	**Discount rate**		0.15									
4												
5	**Year**	1	2	3	4	5	6	7	8	9	10	11
6	**Revenue**	–70	–30	5	15	25	40	60	50	40	30	10
7												
8	**Function, NPV**											
9	**Net present value**	**€17.87**										
10												
11	**Calculation to check**											
12	**Discount factor**	1.15	1.32	1.52	1.75	2.01	2.31	2.66	3.06	3.52	4.05	4.65
13	**Discounted revenue**	–60.87	–22.68	–3.29	8.58	12.43	17.29	22.56	16.35	11.37	7.42	2.15
14	**Net present value**	**€17.87**										

Figure 8.9 Using the function NPV to calculate net present value in worked example 8.10

Traditional accounting uses two other measures to evaluate projects – but neither is very reliable. The first is an average rate of return, which is the average annual revenue as a percentage of the initial investment. In worked example 8.10 there was an initial investment of €100,000 in the first two years, followed by average revenues in the next nine years of €30,556, so the average rate of return is 30,566/100,000 = 0.31 or 31%. This makes the project seem more attractive than the net present value, because it does not discount future values to reflect their lower value.

The second measure is the payback period, which shows the time before the project will make a net profit. Here the initial investment of €100,000 is repaid sometime in year 7 – but again this does not take into account the reducing value of future income.

Internal rate of return

An obvious problem with the net present value is setting a realistic discount rate that takes into account interest, inflation, taxes, opportunity costs, target returns, exchange rates, risk, competition, and all the other factors that affect the future value of money. If the rate is set too high, good long-term projects have future incomes heavily discounted and become less attractive; if the rate is set too low, risky projects with speculative benefits far into the future seem unrealistically attractive.

However, there is an alternative to setting a discount rate. Rather than using a fixed discount rate to find the net present value of each project, we can use a variable discount rate that leads to a specified net present value. In other words, we keep the same net present value for each project and then calculate different discount rates to achieve this. In practice, the target net present value is almost invariably set to zero, and the discount rate that achieves this is the **internal rate of return**.

The **internal rate of return** (IRR) is the discount rate that gives a net present value of zero.

Projects with better financial performance have higher internal rates of return – so to compare projects we find the internal rate of return for each, and choose the one with the highest IRR. Unfortunately, there is no straightforward equation for calculating the IRR, but you can use iterative calculations, repeatedly testing values and homing in on the answer. In practice, you are more likely to use standard functions like Excel's IRR, MIRR or XIRR.

WORKED EXAMPLE 8.11

What is the internal rate of return of a project with the following cash flows?

Year	1	2	3	4	5	6	7	8	9
Net cash flow (£)	−1,800	−500	−200	800	1,800	1,600	1,500	200	100

Solution

Figure 8.10 illustrates an iterative approach to this. With a discount rate of 15% the net present value is £627; with a discount rate of 20% the NPV is £167; and with 25% it is −£164. A discount rate of 20% gives a positive NPV, and a discount rate of 25% gives a negative NPV. We want the discount rate that gives an NPV of zero, so it must be somewhere between these two, in the range 20–25%. If we try 22% the net present value is £21 and we are getting closer; 22.5% gives an NPV of −£12; and the answer must be close to 22.3%, which gives an NPV of £1.

Of course, you do not really have to do this repetitive calculation, and cell D10 uses the standard function IRR to give the actual internal rate of return as just over 22.32%.

	A	B	C	D	E	F	G	H	I	J	K
1	**Internal rate of return**										
2											
3	**Year**		1	2	3	4	5	6	7	8	9
4	**Net revenue**		−£1,800	−£500	−£200	£800	£1,800	£1,600	£1,500	£200	£100
5											
6	**Iterative calculation**										
7	**Discount rate**		0.15	0.2	0.25	0.22	0.225	0.223			
8	**NPV**		£626.96	£166.57	−£163.93	£21.22	−£12.22	£1.02			
9											
10	**Standard function IRR =**			22.32%							

Figure 8.10 Iterative calculation of IRR, and standard function, in worked example 8.11

IDEAS IN PRACTICE Melchior Trust 'E'

Melchior Trust 'E' is based in New York and funds new companies in the Balkan countries with the aim of helping them to expand and contribute to their national economies. Part of its 2006 round of investment decisions considered five alternative companies in Croatia. Figure 8.11 shows the estimated returns from each company in thousands of dollars a year. The internal rates of return are between –2% and 14%, with the highest value from company 3. Melchior considered this – along with a lot of other information – before coming to their final decision.

	A	B	C	D	E	F
1	**Melchior Trust 'E'**					
2						
3	**Year**	**Company 1**	**Company 2**	**Company 3**	**Company 4**	**Company 5**
4	**1**	–80	–100	–50	–35	0
5	**2**	–30	–50	–40	–15	–10
6	**3**	–15	–20	–10	–5	–15
7	**4**	0	–10	0	15	10
8	**5**	10	0	20	15	10
9	**6**	25	10	40	–5	10
10	**7**	50	50	40	–15	–15
11	**8**	55	70	40	10	–15
12	**9**	60	90	40	20	10
13	**10**	60	100	40	10	20
14	**IRR**	**12%**	**9%**	**14%**	**–2%**	**4%**

Figure 8.11 IRRs of five projects considered by Melchior Trust 'E'

Depreciation

When people buy a car, they expect to drive it for a few years and then replace it. This is because maintenance and repair costs rise, the car breaks down more often, new cars are more efficient, they are more comfortable, and so on. Not surprisingly, the value of a car declines as it gets older. In the same way, a company buys a piece of equipment, uses it for some time and then replaces it – and the value of equipment declines over time. But equipment forms part of a company's assets and the balance sheet must always show a reasonable valuation. So organisations write-down the value of their assets each year, meaning that they reduce the book value by an amount of **depreciation**.

The two most widely used methods of calculating depreciation are straight-line and reducing-balance methods.

Straight-line depreciation

This reduces the value of equipment by a fixed amount each year. If we assume that equipment is bought, works for its expected life, and is then sold for scrap:

$$\text{annual depreciation} = \frac{\text{cost of equipment} - \text{scrap value}}{\text{life of equipment}}$$

Here the scrap value is normally the resale value and does not imply that the equipment is actually scrapped. Perhaps better terms are residual or resale value. Then a machine costing £20,000 with an estimated resale value of £5,000 after a useful life of five years has an annual depreciation of:

$$\text{annual depreciation} = \frac{20{,}000 - 5{,}000}{5} = £3{,}000$$

Straight-line depreciation is easy to calculate, but it does not reflect actual values. Most equipment loses a lot of value in the first years of operation, and it is actually worth less than its depreciated value.

Reducing-balance depreciation

This reduces the value of equipment by a fixed percentage of its residual value each year – so an organisation might write off 20% of book value each year. Then if a machine has a residual value of £2,000 at the end of a year, 20% of this is written off for the next year to give a new residual value of $2{,}000 \times 0.8 = £1{,}600$. This has the benefit of giving more depreciation in the first few years, and a more accurate view of equipment's value.

Calculations for the reducing-balance method are a simple extension of compound interest. With interest we know that an amount A_0 increases at a fixed rate, i, each period and after n periods has a value of:

$$A_n = A_0 \times (1 + i)^n$$

With depreciation the amount is decreasing at a fixed rate, so we simply subtract the rate i instead of adding it. Then for a depreciation rate of i, equipment whose initial cost is A_0 has a depreciated value after n periods of:

$$A_n = A_0 \times (1 - i)^n$$

WORKED EXAMPLE 8.12

David Krishnan bought a machine for €10,000 and now has to consider its depreciation.

(a) If the machine has an expected life of five years and a scrap value of €1,000, what is the annual rate of straight-line depreciation?
(b) What is the value of the machine after five years with the reducing-balance method and a depreciation rate of 30%?
(c) What depreciation rate would reduce the machine's value to €2,000 after three years?

Solution

(a) For straight-line depreciation:

annual depreciation

$$= \frac{\text{cost of equipment} - \text{scrap value}}{\text{life of equipment}}$$

$$= \frac{10{,}000 - 1{,}000}{5} = €1{,}800$$

Worked example 8.12 continued

(b) For reducing-balance depreciation:

$$A_n = A_0 \times (1 - i)^n$$

Then after five years:

$$A_5 = 10{,}000 \times (1 - 0.3)^5 = €1{,}681$$

(c) With straight-line depreciation is it easy to get a final value of €2,000, as you simply change the annual depreciation to:

$$\text{annual depreciation} = \frac{10{,}000 - 2{,}000}{3}$$

$$= €2{,}667$$

With reducing-balance we want A_3 to be €2,000, so:

$$A_3 = A_0 \times (1 - i)^3$$

or

$$2{,}000 = 10{,}000 \times (1 - i)^3$$

Then:

$$0.2 = (1 - i)^3$$
$$1 - i = 0.585$$
$$i = 0.415$$

giving a depreciation rate of 41.5%.

WORKED EXAMPLE 8.13

Hamil Leasing buys vans for €50,000 and expects to use them for five years. Then the suppliers buy them back for €10,000 and offer a replacement. If Hamil uses straight-line depreciation, what is the book value of a van each year? What depreciation rate should the company use with the reducing-balance method, and what is a van's book value each year? If Hamil discounts future amounts by 10% a year, what are the current values of all these amounts?

Solution

Figure 8.12 shows a spreadsheet of these calculations. Straight-line depreciation reduces the book value of the machine by (50,000 − 10,000)/5 = €8,000 a year (shown in cell D4), with the results shown in column D. For the reducing balance method, $50{,}000 \times (1 - i)^5 = 10{,}000$, so $i = 0.27522$ (calculated in cell F4) and this gives the results shown in column F. Discounting book values to present values in the usual way gives the results in columns E and G.

	A	B	C	D	E	F	G
1	**Depreciation**						
2							
3				**Straight line**		**Reducing balance**	
4	**Rate**	**10%**		**€ 8,000.00**		**0.2752203**	
5	**Year**	**Discount factor**		**Book value**	**Present value**	**Book value**	**Present value**
6	0	1		€50,000.00	€50,000.00	€50,000.00	€50,000.00
7	1	1.1		€42,000.00	€38,181.82	€36,238.99	€32,944.53
8	2	1.21		€34,000.00	€28,099.17	€26,265.28	€21,706.84
9	3	1.331		€26,000.00	€19,534.18	€19,036.54	€14,302.44
10	4	1.4641		€18,000.00	€12,294.24	€13,797.30	€9,423.74
11	5	1.61051		€10,000.00	€6,209.21	€10,000.00	€6,209.21

Figure 8.12 Depreciated value of vans at Hamil Leasing in worked example 8.13

Review questions

8.12 How could you compare the net benefits of two projects, one of which lasts for five years and the other for seven years?

8.13 What is a discount rate?

8.14 What is the difference between NPV and IRR?

8.15 What is the difference between the straight-line and reducing-balance methods of depreciation?

Mortgages, annuities and sinking funds

If you invest an initial amount A_0 at the end of n periods, you have $A_0 \times (1 + i)^n$. But suppose that you save an additional amount F at the end of each period, then a standard result gives the amount invested after n periods as:

$$A_n = A_0 \times (1 + i)^n + \frac{F \times (1 + i)^n - F}{i}$$

The first part of this equation shows the income from the original investment, and the second part shows the amount accumulated by regular payments (you can find the derivation of this on the Companion Website **www.pearsoned.co.uk/waters**).

WORKED EXAMPLE 8.14

Gaynor Johnson puts £1,000 into a building society account that earns 10% interest a year.

(a) How much will be in her account at the end of five years?
(b) How much will there be if she adds an extra £500 at the end of each year?

Solution

(a) Without additional payments the standard result is:

$$A_n = A_0 \times (1 + i)^n$$

Then substituting A_0 = £1,000, i = 0.1 and n = 5 gives:

$$A_5 = 1{,}000 \times (1 + 0.1)^5 = £1{,}611$$

(b) With additional payments the revised equation gives:

$$A_n = A_0 \times (1 + i)^n + \frac{F \times (1 + i)^n - F}{i}$$

Then with F = £500:

$$A_5 = 1{,}000 \times (1 + 0.1)^5 + \frac{500 \times (1 + 0.1)^5 - 500}{0.1}$$

$$= £4{,}663$$

The equation has five variables: i, n, A_0, A_n and F. If you know any four of them, you can find a value for the fifth. In the last example, we used known values for i, n, A_0 and F to find a value for A_n. Sometimes we want to do the calculations in other ways. For instance, managers might want to set aside regular payments to accumulate a certain amount at the end of a period, so they fix i, n, A_0 and A_n and find the amount of regular savings, F.

This forms a sinking fund, which is typically set up to replace equipment at the end of its life. If you want to replace your computer every three years, you might put regular payments into a sinking fund to accumulate enough for the replacement.

WORKED EXAMPLE 8.15

How much should you invest each year to get £40,000 in a sinking fund at the end of 10 years when expected interest rates are 12%? How would the payments differ if you could put an initial £5,000 into the fund?

Solution

The variables are:

- final value, $A_n = £40,000$
- no initial payment, so $A_0 = £0$
- interest rate, $i = 0.12$
- number of years, $n = 10$.

Substituting these values into the equation

$$A_n = A_0 \times (1 + i)^n + \frac{F \times (1 + i)^n - F}{i}$$

gives:

$$40{,}000 = 0 + \frac{F \times (1 + 0.12)^{10} - F}{0.12}$$

$$4{,}800 = F \times 3.106 - F$$

$$F = £2{,}280$$

If you invest £2,280 each year you actually pay £22,800 and this earns interest of £17,200 to give the total of £40,000 needed.

With an initial investment of $A_0 = £5{,}000$, the calculation becomes:

$$40{,}000 = 5{,}000 \times (1 + 0.12)^{10} + \frac{F \times (1 + 0.12)^{10} - F}{0.12}$$

or

$$4{,}800 = 1{,}863.51 + F \times (3.106 - 1)$$

$$F = £1{,}394$$

The initial payment of £5,000 reduces the annual payments by (2,280 − 1,394) = £886, saving £8,860 over the 10 years.

Spreadsheets have standard functions for these calculations, and Excel includes FV to find the future value of an investment, PV to find the present value of an investment, PMT to find the regular payments needed to accumulate an amount, and NPER to show how many periods it will take to accumulate some amount.

Repaying loans

Another variation of this calculation concerns loans instead of investments. The only difference is that you have to be careful with the positive and negative signs. We have assumed that all payments are positive, showing the benefits of investing, but if payments are negative they become loans rather than investments. $A_0 = £10$ shows that you invest some money; $A_0 = -£10$ shows that you borrow it.

For most people, their biggest debt comes from buying a house. These purchases are financed by a mortgage, which is repaid by regular payments over some extended period. Then the initial payment A_0 is negative, showing that you borrow money, and the value after n periods must be zero, showing that you have repaid it.

WORKED EXAMPLE 8.16

Hans Larsson borrowed £120,000 over 25 years at 8% annual interest to set up his own business. He repays this by regular instalments at the end of every year. How much is each instalment?

Solution

We know that:

$A_0 = -£120{,}000$

$A_{25} = £0$

$i = 0.08$

$n = 25$

and want to find F from the equation:

$$A_n = A_0 \times (1 + i)^n + \frac{F \times (1 + i)^n - F}{i}$$

Substituting the values we know:

$$0 = -120{,}000 \times (1 + 0.08)^{25} + \frac{F \times (1 + 0.08)^{25} - F}{0.08}$$

Then:

$$120{,}000 \times 6.848 = \frac{5.848 \times F}{0.08}$$

$$F = £11{,}241 \text{ per year}$$

After 25 annual payments of £11,241 the original debt is repaid. Notice that Hans has to pay a total of 25 × 11,241 = £281,025, which is 2.34 times the original loan.

An **annuity** is the reverse of a mortgage, and allows someone with a lump sum to invest it and receive regular income over some period in the future. This kind of arrangement is popular with retired people who convert their savings into a regular income.

WORKED EXAMPLE 8.17

Rohan Kalaran wants an annuity that will pay £10,000 a year for the next 10 years. If the prevailing interest rate is 12%, how much will this cost?

Solution

Rohan wants to find the initial payment, A_0, that gives $F = -£10{,}000$ (the negative sign showing a receipt rather than a payment) with $i = 0.12$. The arrangement lasts for 10 years, so $n = 10$, and after this the annuity has no value, so $A_{10} = 0$. Substituting into the standard equation

$$A_n = A_0 \times (1 + i)^n + \frac{F \times (1 + i)^n - F}{i}$$

gives:

$$A_{10} = 0$$

$$= A_0 \times (1 + 0.12)^{10} - \frac{10{,}000 \times (1 + 0.12)^{10} - 10{,}000}{0.12}$$

$$= A_0 \times 3.1059 - 175{,}487.35$$

or

$$A_0 = £56{,}502$$

Review questions

8.16 What is a sinking fund?

8.17 How would you calculate the payment worth making for an annuity?

8.18 The best value of i is the current interest rate. Do you think this is true?

CHAPTER REVIEW

This chapter described a range of common performance measures and financial calculations.

- Managers have to measure performance to see how well their organisation is working. They can use a huge number of measures for different aspects of performance.
- Absolute measures are the most straightforward, but they often give a limited view. More useful figures add some context, usually by calculating a performance ratio. There are many standard performance ratios, including productivity and utilisation. Others describe the financial performance, such as profit margins and return on assets.
- You can find a break-even point by comparing revenue with the costs of production. Extensions of this analysis consider economies of scale, average and marginal costs.
- People can invest (or borrow) money to earn (or pay) interest. An amount available now can earn interest and grow over time, usually by compound interest. This suggests that the value of money changes over time.
- You can compare amounts of money available at different times by discounting to their present values. Subtracting the present value of all costs from the present value of all revenues gives a net present value. It is difficult to set a reliable discount rate, so an alternative calculates an internal rate of return.
- Using similar reasoning you can depreciate the value of assets to give a reducing value over time. Other extensions consider sinking funds, annuities and mortgages.

CASE STUDY OnlineInkCartridges.com

Janet Simmons used to work from home and did a lot of printing from her computer. Over the years, the price of high-quality printers fell, but the replacement ink cartridges always seemed expensive. Ten years ago she formed OnlineInkCartridges.com to buy low-cost generic cartridges from China and sell them through the company website. Seven years ago, she added a recycling unit to refill customers' old cartridges.

At first the business made a steady loss, but now sales are climbing steadily by around 10% a year. The last financial year showed a gross profit of €80,000, giving a margin of 7% and a return on investment of almost 5%.

The long-term prospects for the company seem good, and Janet has to make some major decisions. Firstly, she can stay with the company and take it through a period of continuing growth. Her financial backers already own 35% of the shares, and her second option is to sell the rest of the company to them and either invest the money or start up another business. Her skills undoubtedly form part of the company's assets, and if she leaves, the remaining shareholders are likely to discount the company's value by about 50%. Her third option is a compromise, where she will sell some of the shares – perhaps 15–20%. This will have less effect on the share value, and still give her a lump sum to pay off her debts and invest for the future.

Janet's aim is to maximise the value of her assets over the next 10 or 15 years, by which time she will be ready to take early retirement. Her accountant is adamant that her best future lies in running the company. This has the disadvantages,

Case study continued

though, of putting all her assets in one place. Her bank's business advisor recommended the middle option of selling some shares to release money for other opportunities. She could add another €5,000 a year from her salary and build up a reasonable amount, perhaps using:

- a Saving Account which gives a return of 4.5% a year
- a Gold Account for the fixed sum, which gives a return of 6.5% but leaves the money tied up for at least a year; the additional savings could go into a Saving Account
- a Personal Accumulator which gives 5% interest on a minimum of €50,000, but 10% on any additional savings.

Janet also visited a building society manager who gave similar advice, but offered two other options. Firstly, she could put the money into an 'Inflation Fighter' account, which links the interest rate to the Retail Price Index and guarantees a return of 1% above inflation. Secondly, she could buy another house as an investment. The manager explained that, 'The rent-to-own market has been very unsettled lately. But if you take a long-term view, house prices have risen by 10% to 15% a year for the past 20 years, while inflation has become progressively lower. You can also generate income from rent – usually about 0.5% of the value of the house per month, a quarter of which is needed for repairs and maintenance.'

Janet thought about these alternatives, but found them all a bit boring. Perhaps she should go for the excitement of starting a new business and seeing it grow over time.

Questions

- **If Janet asks for your advice, how would you summarise her main options? What analyses would help her?**
- **Based on the information available, what recommendations would you make?**
- **What other information would you need for a reasoned decision?**

PROBLEMS

8.1 A family doctor sees patients for an average of 10 minutes each. There is an additional five minutes of paperwork for each visit, so she makes appointments at 15-minute intervals for five hours a day. During one surgery the doctor was called away for an emergency that lasted an hour and patients who had appointments during this time were told to come back later. How can you measure the doctor's performance in the surgery?

8.2 ABC Taxis has an average fixed cost of £9,000 a year for each car. Each kilometre driven has variable costs of 40 pence and collects fares of 60 pence. How many kilometres a year does each car have to travel before making a profit? Last year each car drove 160,000 kilometres. What does this tell you? How would the distance travelled have to change to reduce the average cost per kilometre by 5%?

8.3 Air Atlantic is considering a new service between Paris and Calgary. It can use existing aeroplanes, each of which has a capacity of 240 passengers, for one flight a week with fixed costs of $90,000 and variable costs amounting to 50% of ticket price. If the airline plans to sell tickets at $600 each, what can you say about their proposed service?

8.4 A company can introduce only one new product from three available. If it estimates the following data, which product would you recommend?

	Product A	Product B	Product C
Annual sales	600	900	1,200
Unit cost	680	900	1,200
Fixed cost	200,000	350,000	500,000
Product life	3 years	5 years	8 years
Selling price	760	1,000	1,290

8.5 How much will an initial investment of $1,000 earning interest of 8% a year be worth at the end of 20 years? How does this change if the interest is paid more frequently?

8.6 Several years ago John McGregor bought an endowment insurance policy that is about to mature. He has the option of receiving £20,000 now or £40,000 in 10 years' time. Because he has retired and pays no income tax, he could invest the money with a real interest rate expected to remain at 10% a year for the foreseeable future. Which option should he take?

8.7 Mitsushama Systems buys new development machines for ¥150,000 each, and these are used within the company for six years. If they have a resale value of ¥40,000, what is their value at the end of each year with straight-line depreciation? How does this compare with the values from reducing-balance depreciation at a rate of 25%? What depreciation rate would reduce the machine's value to ¥10,000 after four years?

8.8 A company makes fixed annual payments to a sinking fund to replace equipment in five years' time. The equipment is valued at £100,000 and interest rates are 12%. How much should each payment be? How would these payments change if the company could put an initial £10,000 into the fund?

8.9 How much would the monthly repayments be on a mortgage of €100,000 taken out for 25 years at an interest rate of 12% a year?

8.10 Suppose that you are about to buy a new car. You have decided on the model, which costs £12,000. The supplier gives you an option of either a five-year car loan at a reduced APR of 7%, or £1,250 in cash and a five-year car loan with an APR of 10%. Which choice is the better? If you depreciate the car at 20% a year, what is its value in 10 years' time?

8.11 Given the following cash flows for four projects, calculate the net present value using a discount rate of 12% a year. What are the results with continuous discounting? What are the internal rates of return for the projects?

	Project A		Project B		Project C		Project D	
Year	Income	Expenditure	Income	Expenditure	Income	Expenditure	Income	Expenditure
0	0	18,000	0	5,000	0	24,000	0	21,000
1	2,500	0	0	10,000	2,000	10,000	0	12,000
2	13,500	6,000	0	20,000	10,000	6,000	20,000	5,000
3	18,000	0	10,000	20,000	20,000	2,000	20,000	1,000
4	9,000	2,000	30,000	10,000	30,000	2,000	30,000	0
5	5,000	0	50,000	5,000	25,000	2,000	25,000	0
6	3,000	0	60,000	5,000	15,000	2,000	20,000	5,000
7	1,000	0	60,000	5,000	10,000	1,000	20,000	1,000

8.12 How does the net present value of the following net cash flows change with discount rate? What is the internal rate of return?

Year	1	2	3	4	5	6	7	8	9	10	11	12
Net cash flow	–6,000	–1,500	–500	600	1,800	2,000	1,800	1,300	900	500	300	100

RESEARCH PROJECTS

8.1 Spreadsheets have a range of standard functions and procedures for financial calculations. We have already mentioned some of these, including Excel's NPV, IRR, FV, PV, PMT, and NPER. Explore the financial functions that are available in a spreadsheet. Check the calculations in this chapter and describe the effects of changing parameter values. What assumptions do the functions make? What improvements would you like?

8.2 The following table shows the net cash flows for six projects over the next 15 years. How would you compare these projects?

Year	Project A	Project B	Project C	Project D	Project E	Project F
1	–140	–200	–80	0	–500	–50
2	–80	0	30	10	–200	50
3	–15	100	30	20	–100	100
4	15	80	30	30	50	50
5	35	60	30	20	100	–50
6	55	50	–40	15	150	60
7	65	40	30	–100	200	100
8	65	35	30	50	250	70
9	60	30	30	40	300	–50
10	50	30	30	30	300	70
11	40	25	–40	20	300	110
12	30	20	30	10	250	70
13	10	20	30	–100	150	–50
14	0	15	30	50	100	80
15	0	10	30	40	100	120

8.3 Imagine that you want a mortgage to buy a house. Many finance companies can lend you the money, but they quote widely differing terms and conditions. Collect information about offers currently advertised. How can you compare these? Which seems to be the best?

8.4 A lot of websites give tutorials on different types of quantitative problems faced by managers. These are produced by universities, institutions, publishers, training companies, software providers, tutoring services, consultants, and so on. Do some searches on the Web to find sites that are useful for this course.

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Further reading

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CHAPTER 9

Regression and curve fitting

Contents

Chapter outline

This chapter looks at the relationships between variables. It shows how observations often follow an underlying pattern, but with some unpredictable variations. We can use regression to identify the underlying pattern. In particular, linear regression draws a line of best fit through a set of data, and the amount of variation around this line shows how good the fit is.

After finishing this chapter you should be able to:

- Understand the purpose of regression
- See how the strength of a relationship is related to the amount of noise
- Measure the errors introduced by noise
- Use linear regression to find the line of best fit through a set of data
- Use this line of best fit for causal forecasting
- Calculate and interpret coefficients of determination and correlation
- Use Spearman's coefficient of rank correlation
- Understand the results of multiple regression
- Use curve fitting for more complex functions.

Measuring relationships

Chapter 3 showed how to draw a relationship between two variables as a graph. Now we are going to look at this idea in more detail, and consider relationships that are not perfect, so the observations do not all fall exactly on a curve but are somewhere close. Now there are differences between actual observations and expected ones. This raises two questions:

1 How do we find the equation of the best relationship, which is called **regression**?
2 How well does this relationship fit the data?

Both of these depend on the errors that appear in actual observations.

Errors

Suppose you have the following data (in consistent units) for the consumption of electricity in a region and the corresponding average daily temperature.

Temperature	0	2	5	7	10	12	15	17
Electricity	5	9	15	19	25	29	35	39

Figure 9.1 shows a graph of this, and you can see that there is a perfect relationship, with the consumption of electricity (the dependent variable) related perfectly to the temperature (the independent variable). In fact:

$$\text{consumption of electricity} = 2 \times \text{average temperature} + 5$$

In reality, you will rarely find such a perfect relationship, and there is usually some variation around the expected values. You are more likely to find the following pattern of electricity consumption, which was recorded in the USA in January 2006.

Temperature	0	2	5	7	10	12	15	17
Electricity	7	8	17	17	26	24	30	42

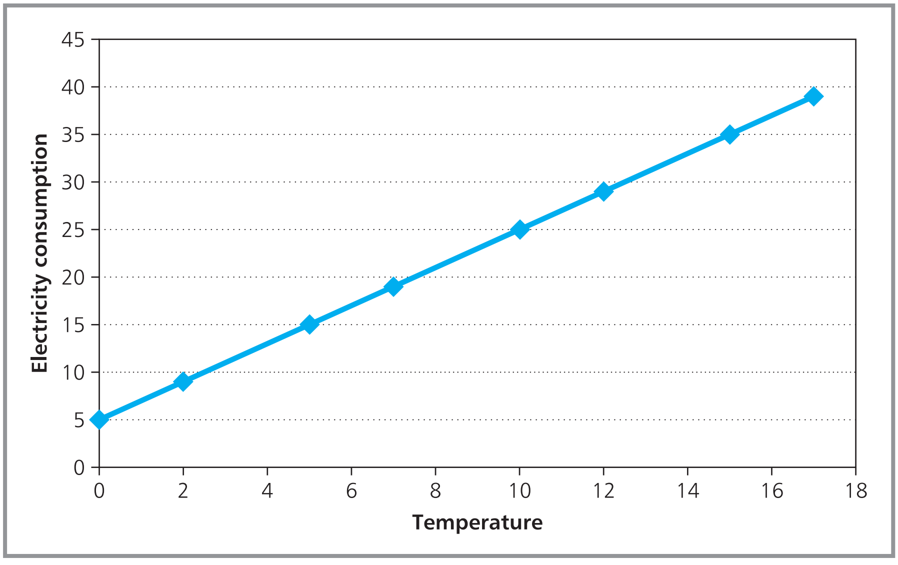

Figure 9.1 Relationship between electricity consumption and average daily temperature

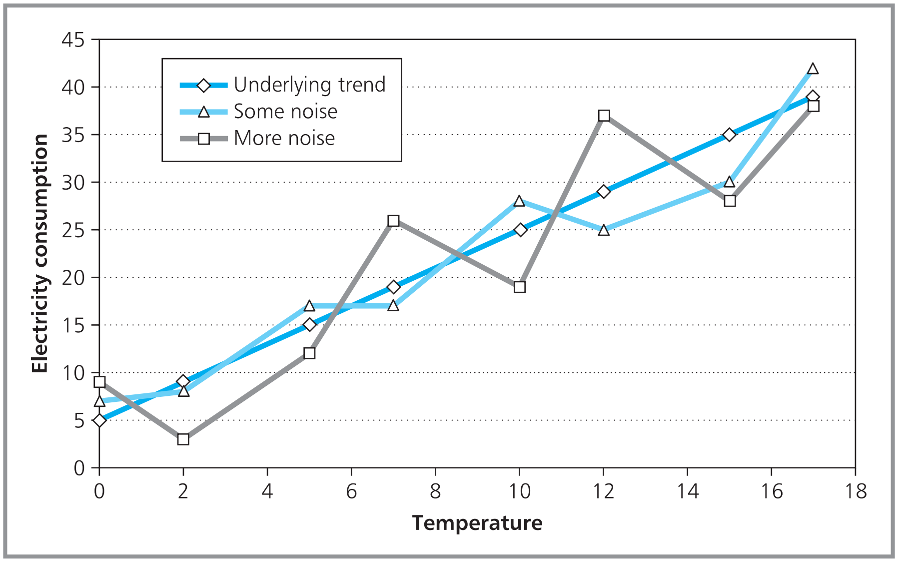

Figure 9.2 Relationships with different amounts of noise

There is still a clear linear relationship, but superimposed on this underlying pattern is a random variation called **noise**. Then we have:

> actual value = underlying pattern + random noise

There might be even more noise, with varying amounts shown in Figure 9.2.
The amount of random noise determines the strength of a relationship.

- When there is no noise – as in the first set of figures above – the relationship is perfect.
- When there is some noise the relationship is weaker.
- When there is a lot of noise, the relationship becomes even weaker and more difficult to identify.
- When the noise is overwhelming, it hides any underlying relationship and data appears to be random.

We really need some way of measuring the amount of noise and the strength of the relationship.

Measuring the noise

When there is a relationship between two variables, it means that each value of the independent variable has a corresponding value of the dependent variable. But when there is noise, there is a difference between the actual value of the dependent variable and the expected one. You can think of noise as an error in an observation.

> For each observation i,
>
> error, E_i = actual value − expected value from the relationship

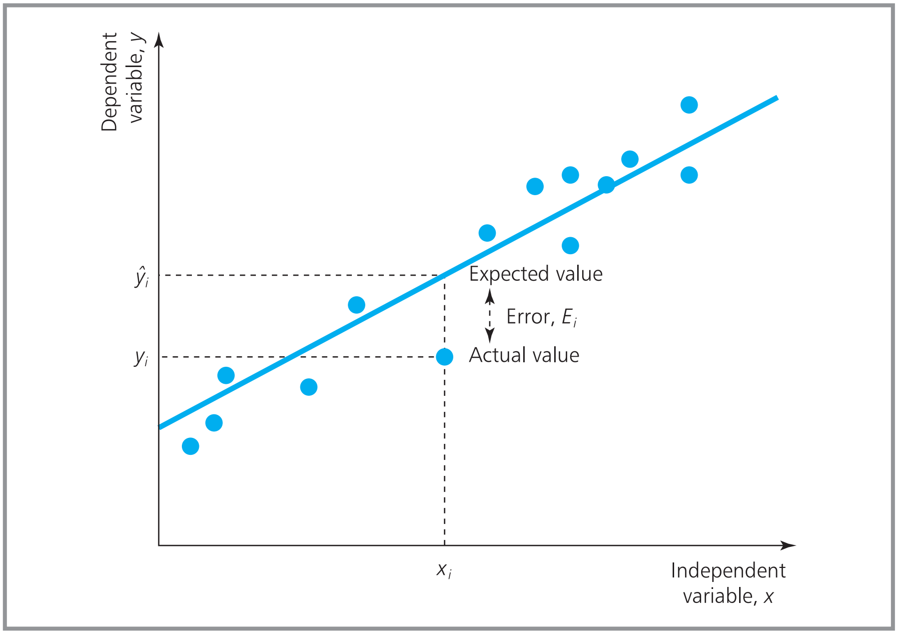

Figure 9.3 Noise introduces errors in observations

Figure 9.3 shows a linear relationship between two variables, with superimposed noise. The noise means that each observation has an error, which is its vertical distance from the line. Then:

$$E_i = y_i - \hat{y}_i$$

where: y_i = actual value
$\hat{y}_i$ = value suggested by the relationship (which is pronounced 'y hat').

Each observation has an error, so we can find the mean of these from:

$$\text{mean error} = \frac{\sum E_i}{n} = \frac{\sum(y_i - \hat{y}_i)}{n}$$

But the mean error has the major drawback (which we met with the variance in Chapter 6) of allowing positive and negative errors to cancel. So data with very large errors can still have a mean error of zero. The usual ways around this either take the absolute values of errors (and calculate the **mean absolute error**), or square the errors (and calculate the **mean squared error**).

$$\text{mean absolute error} = \frac{\sum|E_i|}{n} = \frac{\sum|y_i - \hat{y}_i|}{n}$$

$$\text{mean squared error} = \frac{\sum(E_i)^2}{n} = \frac{\sum(y_i - \hat{y}_i)^2}{n}$$

The mean absolute error has an obvious meaning; when it takes a value of 1.5 the actual value is on average 1.5 away from the expected value. The mean squared error has a less clear meaning, but is useful for other analyses. Whichever measure we use, smaller values show there is less noise in the observations and a stronger relationship between variables.

WORKED EXAMPLE 9.1

Sonja Prizniscz collects eight pairs of observations that she thinks are related by the equation $y = 3x + 3$. What are the errors in these observations?

x	3	6	10	15	8	4	1	12
y	10	24	29	48	25	12	5	41

Solution

When Sonja calculates the expected values of $\hat{y}_i$ by substituting values for x into the equation $y = 3x + 3$, the error in each observation is $E_i = y_i - \hat{y}_i$. For the first observation x is 3, so $y = 3x + 3 = 3 \times 3 + 3 = 12$. The error is $10 - 12 = -2$, the absolute error is 2 and the error squared is 4. Figure 9.4 shows all of the calculations in a spreadsheet. The mean error = –0.88 (showing that actual values are, on average, a bit lower than expected), the mean absolute error = 2.13 (showing that actual values are, on average, 2.13 away from expected ones) and the mean squared error = 5.88.

	A	B	C	D	E	F
1	**Measuring errors**					
2						
3	***x***	**Actual *y***	**Calculated *y***	**Error**	**Absolute error**	**Squared error**
4	3	10	12	–2	2	4
5	6	24	21	3	3	9
6	10	29	33	–4	4	16
7	15	48	48	0	0	0
8	8	25	27	–2	2	4
9	4	12	15	–3	3	9
10	1	5	6	–1	1	1
11	12	41	39	2	2	4
12	**Sums**			**–7**	**17**	**47**
13	**Means**			**–0.88**	**2.13**	**5.88**

Figure 9.4 Measuring average errors

Review questions

9.1 What is the 'noise' in a relationship?

9.2 Why do almost all relationships contain errors?

9.3 What is the mean error and why is it rarely used?

9.4 Define two other measures of error.

9.5 Two people suggest different equations for describing the relationship between two variables. How can you tell which is better?

Linear relationships

You can see many examples of relationships that consist of an underlying pattern with superimposed noise – such as the sales of a product falling with increasing price, demand for a service rising with advertising expenditure, productivity rising with bonus payments, borrowings falling with rising interest rates, and crop yield depending on the amount of fertiliser used. These are examples of causal relationships where changes in the first (dependent) variable are actually caused by changes in the second (independent) variable. People often assume that because there is a relationship there must be some cause and effect. But this is not true. Sales of ice-cream are directly related to sales of sunglasses, but there is no cause and effect, and the way to increase sales of ice-cream is not to increase the sales of sunglasses. Here the weather clearly affects the sales of both ice-cream and sunglasses. It is easy to spot ridiculous examples of assumed cause and effect – the number of lamp posts is related to prosecutions for drunken driving, the number of storks nesting in Sweden is related to the birth rate in Northern Europe, the number of people in higher education is related to life expectancy, and in the nineteenth century the number of asses in America was related to the number of PhD graduates.

Unfortunately, not all mistakes of this kind are as easy to spot. For example, the productivity of a coal mine declines with increasing investment (because of the age of the mine and increasing difficulty of extracting coal); economists say that high wages cause inflation (ignoring the fact that countries with the highest wages often have the lowest inflation); the revenue of a bus company is related to the fares charged (but increasing fares deters passengers and reduces long-term income).

So we are looking at relationships between variables, but are not implying any cause and effect. In particular, we start by looking at linear relationships, where the underlying pattern is a straight line. This process is called **linear regression**, which finds the straight line that best fits a set of data.

WORKED EXAMPLE 9.2

The following table shows the number of shifts worked each month and the production at Van Hofen, Inc. If the company plans 50 shifts for next month, what is their expected production?

Month	1	2	3	4	5	6	7	8	9
Shifts worked	50	70	25	55	20	60	40	25	35
Output	352	555	207	508	48	498	310	153	264

Solution

Figure 9.5 shows a scatter diagram of shifts worked (the independent variable, x) and production (the dependent variable, y). There is a clear linear relationship, and we can draw by eye a reasonable straight line through the data. This line shows that with 50 shifts worked, the output will be around 400 units.

Worked example 9.2 continued

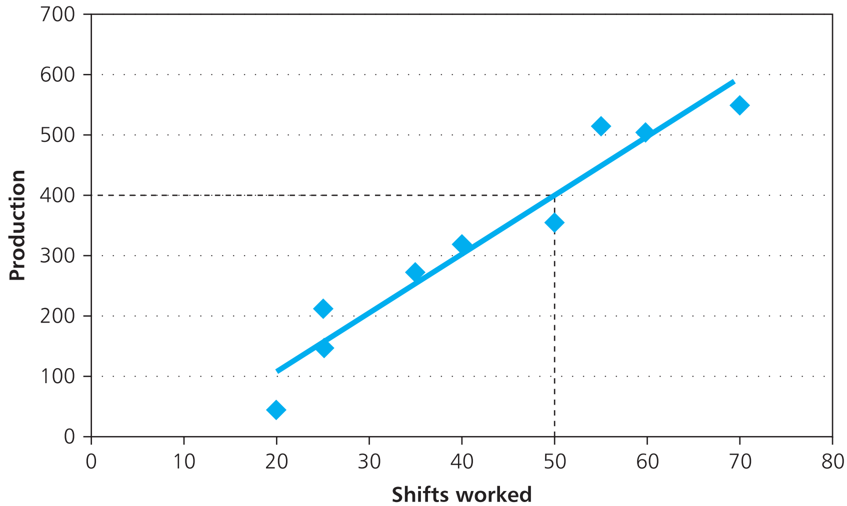

Figure 9.5 Linear relationship between output and shifts worked at Van Hofen, Inc.

The steps in the last worked example define the basic approach of linear regression, which:

1 draws a scatter diagram
2 identifies a linear relationship
3 draws a **line of best fit** through the data
4 uses this line to predict a value for the dependent variable from a known value of the independent variable.

We should make this process a bit more formal, particularly the steps for identifying a linear relationship and finding the line of best fit. For this we will use the equation of a straight line that we saw in Chapter 3:

$$y = a + bx$$

where x is the independent variable, y is the dependent variable, a is the intercept and b is the gradient. Noise means that even the best line is unlikely to fit the data perfectly, so there is an error at each point:

$$y_i = a + bx_i + E_i$$

The line of best fit is defined as the line that minimises some measure of this error. In practice, we always look for the line that minimises the mean squared error, and we find this using linear regression.

> **Linear regression** finds values for the constants a and b that define the line of best fit through a set of points.

A standard result (which is derived in the Companion Website at **www.pearsoned.co.uk/waters**) shows that the equation for the line of best fit is:

$$y = a + bx$$

where

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$a = \bar{y} - b\bar{x}$$

WORKED EXAMPLE 9.3

Find the line of best fit through the following data for an advertising budget (in thousands of euros) and units sold. Forecast the number of units sold with an advertising budget of €70,000.

Advertising budget	20	40	60	80	90	110
Units sold	110	150	230	230	300	360

Solution

Figure 9.6 shows that there is a clear linear relationship, with:

units sold (y) = $a + b \times$ advertising budget (x)

We can do the calculations in a number of ways. The equations are fairly messy, but Figure 9.7 shows the values of n, $\sum x$, $\sum y$, $\sum xy$ and $\sum x^2$. Substituting these into the standard equations gives:

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} = \frac{6 \times 107{,}000 - 400 \times 1{,}380}{6 \times 32{,}200 - 400 \times 400}$$

$$= 2.71$$

$$a = \bar{y} - b\bar{x} = 230 - 2.71 \times 66.67 = 49.28$$

So the line of best fit is:

units sold = 49.28 + 2.71 × advertising budget

With an advertising budget of €70,000, $x = 70$ and:

units sold = 49.28 + 2.71 × 70 = 239 units

Spreadsheets have standard functions for these calculations, and you can see these in the second

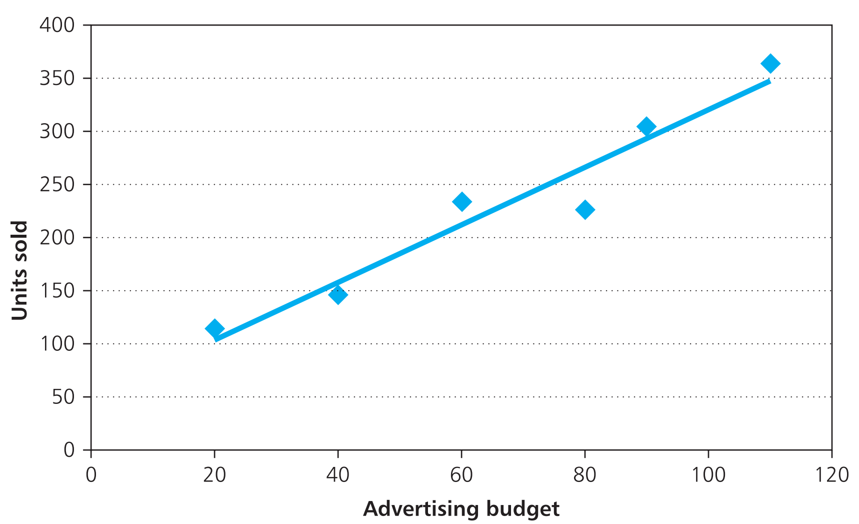

Figure 9.6 Relationship between units sold and advertising budget

Worked example 9.3 continued

part of Figure 9.7. Here Excel's INTERCEPT function gives the intercept of the line, *a*, the SLOPE function gives the gradient, *b*, and the FORECAST function substitutes these values and predicts the number of sales with €70,000 of advertising.

The third part of Figure 9.7 shows another option, which is to use a spreadsheet's Data Analysis ToolPak. Here the 'Regression' option automatically finds the line of best fit. These tools often give more information than we really want, so here we have given only the main results. The 'Regression statistics' show how well the line fits the data (which we discuss later in the chapter), the 'ANOVA' (analysis of variance) describes the errors, and the last table in row 37 shows the information we want. In particular, the 'intercept' value in cell B38 and 'X variable 1' value in cell B39 are the values of *a* and *b* respectively.

	A	B	C	D	E
1	**Linear regression**				
2					
3	**1. Calculation**				
4	**Data**			**Calculation**	
5		***x***	***y***	***xy***	***x*^2**
6		20	110	2200	400
7		40	150	6000	1600
8		60	230	13800	3600
9		80	230	18400	6400
10		90	300	27000	8100
11		110	360	39600	12100
12	**Sum**	**400**	**1380**	**107000**	**32200**
13	**Mean**	**66.67**	**230.00**	**17833.33**	**5366.67**
14	***n* =**	**6**			
15	**Substitution**	***a* =**	**49.28**	***b* =**	**2.71**
16	**Forecast**	***x* =**	**70**	***y* =**	**239.04**
17					
18	**2. Standard function**				
19	**Intercept**	**49.28**			
20	**Slope**	**2.71**			
21	**Forecast**	**239.04**			
22					
23	**3. Data analysis**				
24	Regression Statistics				
25	Multiple R	0.98			
26	R Square	0.95			
27	Adjust R Square	0.94			
28	Standard Error	22.01			
29	Observations	6			
30					
31	**ANOVA**				
32		df	SS	MS	
33	Regression	1	40662.65	40662.65	
34	Residual	4	1937.35	484.34	
35	Total	5	42600		
36					
37		Coefficients	Stand Error		
38	**Intercept**	**49.28**	21.67		
39	**X Variable 1**	**2.71**	0.30		

Figure 9.7 Three ways of doing the regression calculations in a spreadsheet

Using linear regression to forecast

The main purpose of linear regression is to predict the value of a dependent variable that corresponds to a known value of an independent variable. In the last worked example we found a relationship between advertising budget and sales, and then used this to forecast expected sales for a particular advertising budget. In worked example 9.4 we forecast the number of mistakes with a planned level of quality control. This approach is known as **causal forecasting**, even though changes in the independent variable may not actually cause changes in the dependent variable.

This last example shows that the line of best fit is valid only really within the range of x used to find it – and there is no evidence that the same relationship holds outside this range. Using a value of x outside the range to find a corresponding value of y is called **extrapolation**, and you cannot rely on the result. In practice, though, the results are generally acceptable provided the values of x are not too far outside the range. This is an important point, as linear regression is often used with time-period as the independent variable, using historical data to forecast values for the future. This is clearly extrapolation, but provided we do not forecast too far into the future the results are still fairly reliable.

WORKED EXAMPLE 9.4

Olfentia Travel arrange a large number of holidays, and in some of these they make administrative mistakes. They are about to change their quality control procedures, and have done some experiments to see how the number of mistakes varies with the number of checks. If the following table shows their findings, how many mistakes would Olfentia expect with six checks? How many would they expect with 20 checks?

Checks	0	1	2	3	4	5	6	7	8	9	10
Mistakes	92	86	81	72	67	59	53	43	32	24	12

Solution

The independent variable, x, is the number of checks and the dependent variable, y, is the consequent mistakes. Figure 9.8 shows that there is a clear linear relationship between these. If you do the calculations, you find that $n = 11$, $\sum x = 55$, $\sum y = 621$, $\sum xy = 2{,}238$ and $\sum x^2 = 385$. Substituting these values gives:

$$b = \frac{n\sum xy - \sum x \sum y}{n\sum x^2 - (\sum x)^2} = \frac{11 \times 2{,}238 - 55 \times 621}{11 \times 385 - 55 \times 55}$$

$$= -7.88$$

$$a = \bar{y} - b\bar{x} = 621/11 + 7.88 \times 55/11 = 95.86$$

This confirms the results given by the standard 'Regression' tool, that the line of best fit is:

number of mistakes = 95.86 − 7.88 × number of checks

With six inspections Olfentia would forecast 95.86 − 7.88 × 6 = 48.58 mistakes. With 20 inspections you have to be a bit more careful, as substitution gives 95.86 − 7.88 × 20 = −61.74. It is impossible to have a negative number of mistakes, so you simply forecast zero.

Worked example 9.4 continued

	A	B	C	D	E	F	G	H
1	**Quality control at Olfentia Travel**							
2								
3	**Checks**	**Mistakes**		**SUMMARY OUTPUT**				
4	0	92						
5	1	86		Regression Statistics				
6	2	81		Multiple R	0.994			
7	3	72		R Square	0.988			
8	4	67		Adjusted R Square	0.986			
9	5	59		Standard Error	3.077			
10	6	53		Observations	11			
11	7	43						
12	8	32		ANOVA				
13	9	24			*df*	*SS*	*MS*	
14	10	12		Regression	1	6833.536	6833.536	
15				Residual	9	85.191	9.466	
16				Total	10	6918.727		
17								
18					Coefficient	standard Error		
19				Intercept	95.864	1.735		
20				X Variable 1	–7.882	0.293		
21								
22								

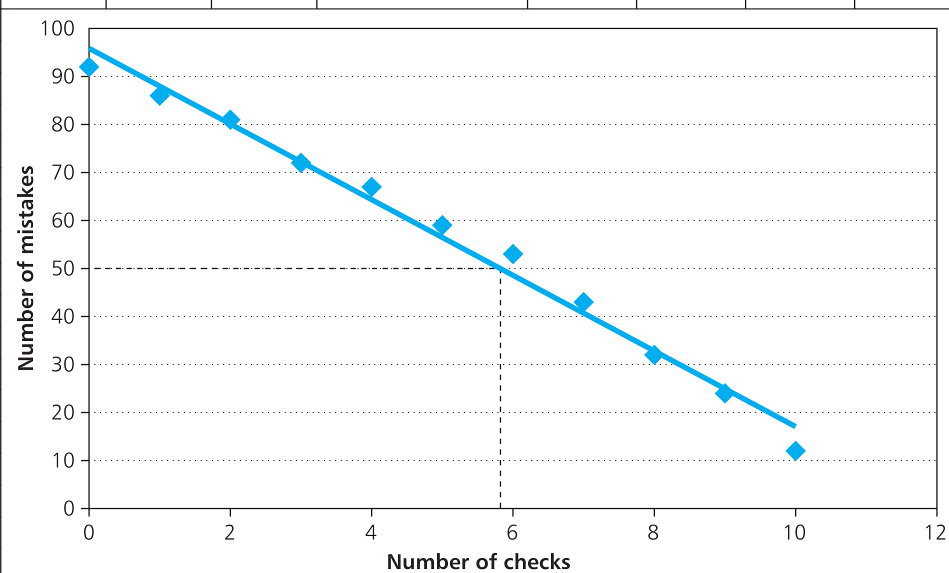

Figure 9.8 Linear relationship between the number of mistakes and checks at Olfentia Travel

WORKED EXAMPLE 9.5

Sales of a product over the last 10 weeks have been 17, 23, 41, 38, 42, 47, 51, 56, 63 and 71. Use linear regression to forecast demand for the next three weeks, and for week 30.

Solution

Here time – or week number – is the independent variable, and the dependent variable is sales. Figure 9.9 shows the line of best fit from the standard spreadsheet functions INTERCEPT and SLOPE:

sales = 15.4 + 5.36 × week

Substituting week numbers into this equation gives:

Week 11: sales = 15.4 + 5.36 × 11 = 74.4
Week 12: sales = 15.4 + 5.36 × 12 = 79.8
Week 13: sales = 15.4 + 5.36 × 13 = 85.1
Week 30: sales = 15.4 + 5.36 × 30 = 176.2

The relationship is valid only really for weeks 1 to 10, and we are fairly safe in extrapolating to week 13 – but must be far more cautious when extrapolating to week 30.

	A	B	C	D	E	F
1	**Linear regression**					
2						
3	**Week**	**Sales**	**Forecast**		**Intercept =**	**15.40**
4	1	17	20.8		**Gradient =**	**5.36**
5	2	23	26.1			
6	3	41	31.5			
7	4	38	36.9			
8	5	42	42.2			
9	6	47	47.6			
10	7	51	52.9			
11	8	56	58.3			
12	9	63	63.7			
13	10	71	69.0			
14	11		74.4			
15	12		79.8			
16	13		85.1			
17						
18	30		176.3			

Figure 9.9 Using linear regression with time as the independent variable

IDEAS IN PRACTICE Long Barrow Farm

Geoff Harris has been running a cereal farm for the past 15 years. His profit per hectare is affected by the amount he spends on fertiliser and pesticides. Although accurate data is very difficult to collect, he did a rough calculation to show the relationship in Figure 9.10. Here:

profit per hectare = −4.09 + 0.078 × cost of fertiliser

Geoff used this result to evaluate four options for expenditure next year. He wants to reduce his use of chemicals, but must make significant changes to operations before this becomes profitable. The financial analysis (which is omitted from the spreadsheet) shows the effects of changes to operations. Geoff concluded that over the next five years he could reduce expenditure on chemicals by 50%, and increase his profit margins by 20%.

Ideas in practice continued

	A	B	C	D	E	F	G	H	I	J	K	L	M
1	**Long Barrow Farm**												
2													
3	**Cost of fertiliser**	500	1000	1500	2000	2500	3000	3500	4000	4500	5000	5500	6000
4	**Profit per hectare**	35	70	110	160	200	220	260	310	350	380	430	460
5													
6	**Regression results**												
7	Intercept, a =	−4.09											
8	gradient, b =	0.078											
9													
10	**Options**	1	2	3	4								
11	**Cost of fertiliser**	0	1500	3000	4500								
12	**Profit per hectare**	−4	113	229	346								
13													
14	**Financial analysis**												

Figure 9.10 Start of financial analysis for Long Borrow Farm

Review questions

9.6 What is the purpose of linear regression?

9.7 Define each of the terms in the regression equation $y_i = a + b \times x_i + E_i$.

9.8 If you want to forecast future values, what is the most commonly used independent variable?

9.9 'Linear regression means that changes in the independent variable cause changes in the dependent variable.' Do you think this is true?

9.10 What are interpolation and extrapolation?

Measuring the strength of a relationship

Linear regression finds the line of best fit through a set of data – but we still have to measure how good the fit is. If observations are close to the line, the errors are small and the line is a good fit; but if observations are some way away from the line, errors are large and even the best line is not very good.

Coefficient of determination

Suppose you have a number of observations of y_i and calculate the mean, $\bar{y}$. Actual values vary around this mean, and you can measure the variation by the total sum of squared errors:

$$\text{total SSE} = \sum (y_i - \bar{y})^2$$

If you look carefully at this sum of squared errors (SSE) you can separate it into different components. When you build a regression model, you estimate values, $\hat{y}_i$, which show what the observations would be if there is no noise. So

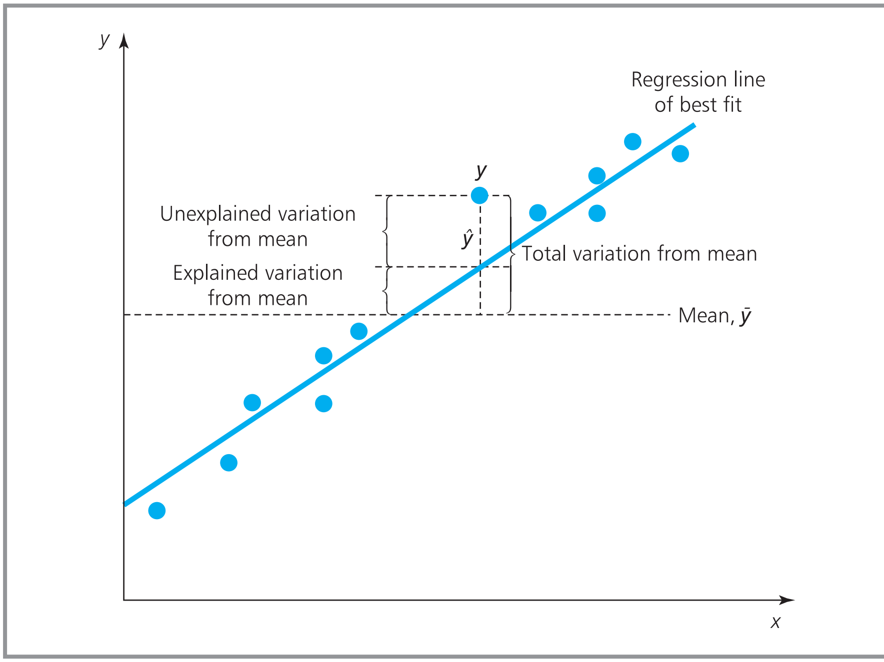

Figure 9.11 Explained and unexplained variation from the mean

the regression model explains some of the variation of actual observations from the mean.

$$\text{explained SSE} = \sum (\hat{y}_i - \bar{y})^2$$

But there is inevitably random noise, so the regression model does not explain all the variation and there is some residual left unexplained (shown in Figure 9.11).

$$\text{unexplained SSE} = \sum (y_i - \hat{y}_i)^2$$

With a bit of algebra you can find that:

$$\text{total SSE} = \text{explained SSE} + \text{unexplained SSE}$$

A measure of the goodness of fit is the proportion of total SSE that is explained by the regression model. This is the coefficient of determination.

$$\textbf{coefficient of determination} = \frac{\text{explained SSE}}{\text{total SSE}}$$

This measure has a value between 0 and 1. If it is near to 1, most of the variation is explained by the regression, there is little unexplained variation, and the line is a good fit for the data. If the value is near to 0, most of the variation is unexplained and the line is not a good fit.

You can calculate the coefficient of determination from the rather messy equation:

$$\text{coefficient of determination} = \left[\frac{n\sum xy - \sum x \sum y}{\sqrt{[n\sum x^2 - (\sum x)^2] \times [n\sum y^2 - (\sum y)^2]}}\right]^2$$

The coefficient of determination is usually called r^2. If you look back at Figure 9.5 you can see that this is one of the figures calculated by the regression tool, with the result given as 'R Square 0.95'. This is a high value and shows a strong linear relationship, with almost all the variation from the mean explained by the regression and only 0.05, or 5%, due to noise.

WORKED EXAMPLE 9.6

Calculate the coefficient of determination for the data from Long Barrow Farm. What does this tell you?

Solution

We know that:

coefficient of determination

$$= \left[\frac{n\sum xy - \sum x \sum y}{\sqrt{[n\sum x^2 - (\sum x)^2] \times [n\sum y^2 - (\sum y)^2]}}\right]^2$$

If you do the calculations, you find that $n = 12$, $\sum x = 39{,}000$, $\sum y = 2{,}985$, $\sum xy = 12{,}482{,}500$, $\sum x^2 = 162{,}500{,}000$ and $\sum y^2 = 959{,}325$. Then:

$$r^2 = \left[\frac{12 \times 12{,}482{,}500 - 39{,}000 \times 2{,}985}{\sqrt{[12 \times 162{,}500{,}000 - 39{,}000^2] \times [12 \times 959{,}325 - 2{,}985^2]}}\right]^2$$
$$= 0.998$$

This tells us two things. Firstly, this is very close to 1, so almost all the variation is explained by the regression and there is virtually no noise. There is a very strong linear relationship between the cost of fertiliser and the profit per hectare. Secondly, it shows that the arithmetic is messy, and it is always better to use a computer.

Normally any value for the coefficient of determination above about 0.5 is considered a good fit. If the coefficient of determination is lower, say closer to 0.2, then 80% of the variation is not explained by the regression and there is not a strong relationship. However, we should give a word of warning about outliers – which are single observations that are some distance away from the regression line. Including such points in the analysis lowers the coefficient of determination, so there is always a temptation to assume they are mistakes and simply ignore them. But you should not do this! You should ignore a point only when there is a genuine reason, like a mistake, or because the point is not strictly comparable with the rest of the data. Deciding to arbitrarily ignore some observations because they spoil a pattern is missing the whole point of the analysis – which is to see if there really is an underlying pattern and measure the strength of the relationship.

Coefficient of correlation

A second measure for regression is the **coefficient of correlation**, which answers the basic question 'are x and y linearly related?' The coefficients of correlation and determination obviously answer very similar questions, and a standard result shows that:

coefficient of correlation = √coefficient of determination

Now you can see that we refer to the coefficient of determination as r^2, so that we can refer to the coefficient of correlation as r. This correlation coefficient is also called **Pearson's coefficient** and it has a value between +1 and −1:

- A value of $r = 1$ shows that the two variables have a perfect linear relationship with no noise at all, and as one increases so does the other (shown in Figure 9.12).
- A lower positive value of r shows that the linear relationship is weaker.
- A value of $r = 0$ shows that there is no correlation at all between the two variables, and no linear relationship.
- A low negative value of r shows a weak linear relationship, and as one increases the other decreases.
- A lower value of r shows a stronger linear relationship.
- A value of $r = -1$ shows that the two variables have a perfect negative linear relationship.

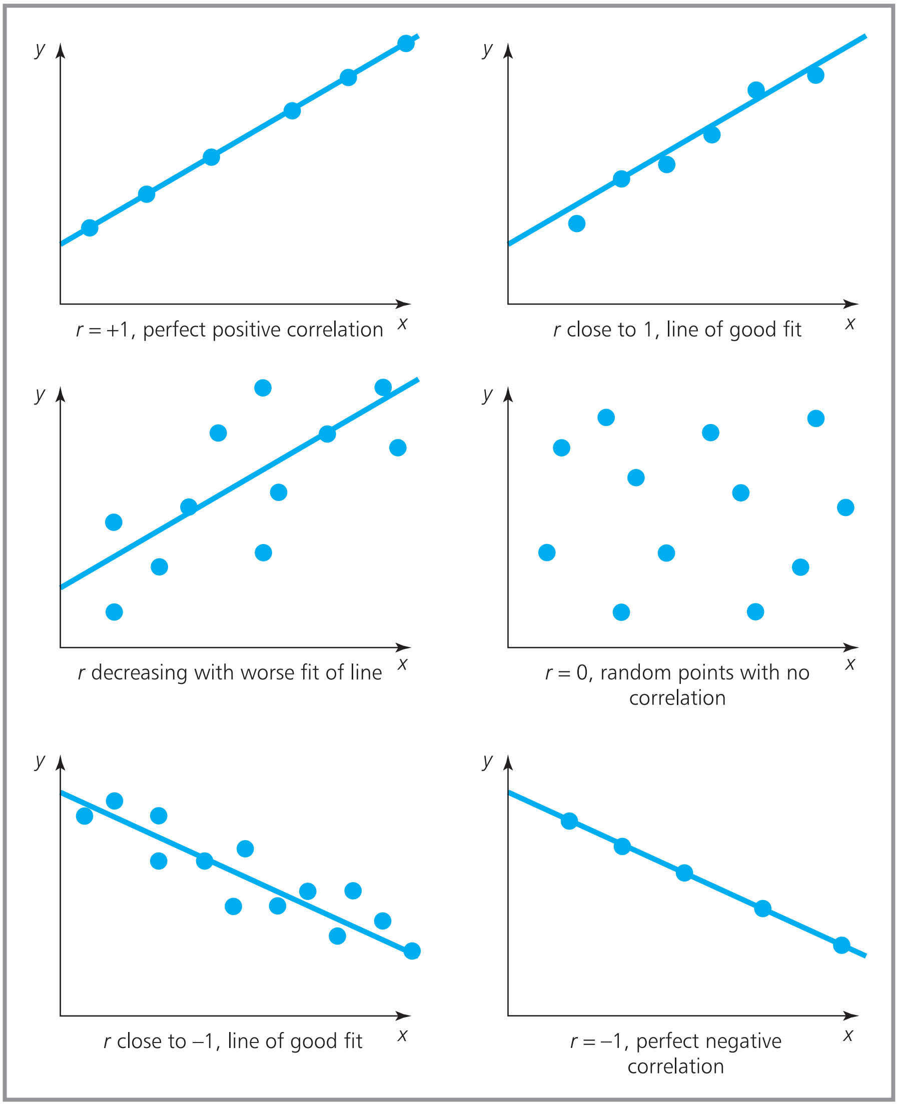

Figure 9.12 Interpreting the coefficient of correlation

With a correlation coefficient, r, near to +1 or −1 there is a strong linear relationship between the two variables. When r is between 0.7 and −0.7 the coefficient of determination, r^2, is less than 0.49, meaning that less than half the variation from the mean is explained by the regression model. This suggests, at best, a weak linear relationship.

WORKED EXAMPLE 9.7

Calculate the coefficients of correlation and determination for the following data. What conclusions can you draw from these? What is the line of best fit?

x	4	17	3	21	10	8	4	9	13	12	2	6	15	8	19
y	13	47	24	41	29	33	28	38	46	32	14	22	26	21	50

Solution

Figure 9.13 shows the results from the 'Regression' option in Excel's Data Analysis ToolPak (actually this gives a lot more detail, but we have focused on a limited part). The key points are as follows:

- There are 15 observations.
- The intercept = 15.376.
- The gradient = 1.545, so the line of best fit is:

  $y = 15.376 + 1.545x$

- The coefficient of correlation, described as 'Multiple R' = 0.797. This shows a reasonably strong linear relationship.
- The coefficient of determination, described as 'R Square' = 0.635. This is the square of the coefficient of correlation and shows that 63.5% of variation from the mean is explained by the linear relationship, and only 36.5% is unexplained.
- The coefficient of correlation is sometimes rather optimistic, especially when there are only a few observations. To overcome any bias the spreadsheet calculates an adjusted figure of 0.607. This can give a more realistic view, but it is usually close to the calculated value.

	A	B	C	D	E	F
1	**Correlation and determination**					
2						
3	*x*	*y*		**SUMMARY OUTPUT**		
4	4	13				
5	17	47		**Regression Statistics**		
6	3	24		Multiple R	0.797	
7	21	41		R Square	0.635	
8	10	29		Adjusted R Square	0.607	
9	8	33		Standard Error	7.261	
10	4	28		Observations	15	
11	9	38				
12	13	46		**ANOVA**		
13	12	32			*df*	*SS*
14	2	14		Regression	1	1191.630
15	6	22		Residual	13	685.304
16	15	26		Total	14	1876.933
17	8	21				
18	19	50			**Coefficients**	
19				Intercept	15.376	
20				X Variable 1	1.545	

Figure 9.13 Results from the 'Regression' data analysis tool

Worked example 9.7 continued

- The ANOVA – analysis of variance – shows the sum of squared errors from the mean, so total SSE = 1876.933.
- Of this, the regression explains 1191.630, giving the explained SSE. Dividing this by the total SSE gives the coefficient of determination, with 1191.630/1876.933 = 0.635.
- The remainder of unexplained SSE = 685.304, confirming that total SSE (1876.933) = explained SSE (1191.630) + unexplained SSE (685.304).
- The standard error = 7.261, and this gives a measure of the error in the predicted value of each *y*.

Rank correlation

Pearson's coefficient is the most widely used measure of correlation, but it works only for cardinal data (that is, numerical values). Sometimes we want to measure the strength of a relationship between ordinal data (data that is ranked but whose values are unknown). You can imagine this with a market survey that asks people to sort their choices into order of preference, in which you want to see whether there is a relationship between two sets of ranks. For example, a survey might rank the quality of service offered by different Internet Service Providers (ISPs) and the prices they charge, and then see whether there is a relationship between the quality and price. You can do this using **Spearman's coefficient of rank correlation**, which is called r_s.

$$\text{Spearman's coefficient} = r_s = 1 - \frac{6\sum D^2}{n(n^2 - 1)}$$

where: n = number of paired observations
D = difference in rankings
= first ranking – second ranking

WORKED EXAMPLE 9.8

A company offers five services and asks customers to rank them according to quality and cost. What can you find from the following results?

	Service				
	V	W	X	Y	Z
Quality ranking	2	5	1	3	4
Cost ranking	1	3	2	4	5

Solution

You can use Spearman's rank correlation coefficient to see whether there is a relationship between quality and cost. Then:

D = quality ranking – cost ranking

In this case there are five rankings, so $n = 5$ and the sum of D^2 is:

$$(2-1)^2 + (5-3)^2 + (1-2)^2 + (3-4)^2 + (4-5)^2 = 1 + 4 + 1 + 1 + 1 = 8$$

Spearman's coefficient is:

$$r_s = 1 - \frac{6\sum D^2}{n(n^2-1)} = 1 - \frac{6 \times 8}{5 \times (25-1)} = 0.6$$

Although it looks completely different, Spearman's coefficient is really derived from Pearson's coefficient, and you interpret it in exactly the same way. A value of 0.6 suggests some relationship between quality and cost – but not a very strong one.

It is worth remembering that ordinal data is far less precise than cardinal. This means that an item ranked first may be slightly better than the item ranked second, or it may be a lot better. It follows that the results of regressions are also less precise, and wherever possible you should use cardinal data and Pearson's coefficient.

WORKED EXAMPLE 9.9

Kipram Jansaporanam runs an apprentice scheme that judges the performance of trainees by a combination of interviews and job performance. Last year she had seven trainees and ranked them as follows.

Trainee	A	B	C	D	E	F	G
Interview	3	2	6	4	1	7	5
Job performance	1	3	5	2	4	6	7

Is there a link between the results of interviews and job performance?

Solution

For each trainee we can find D, the difference between each ranking. Then:

$$\begin{aligned}\sum D^2 &= (3-1)^2 + (2-3)^2 + (6-5)^2 + (4-2)^2 \\ &\quad + (1-4)^2 + (7-6)^2 + (5-7)^2 \\ &= 4+1+1+4+9+1+4 = 24\end{aligned}$$

Spearman's coefficient is:

$$r_s = 1 - \frac{6\sum D^2}{n(n^2-1)} = 1 - \frac{6 \times 24}{7(49-1)} = 0.57$$

This is not very high and does not suggest a strong relationship.

Review questions

9.11 What is measured by the coefficient of determination?

9.12 What values can the coefficient of correlation take, and how is it related to the coefficient of determination?

9.13 What is the difference between Pearson's and Spearman's coefficients of correlation?

9.14 'A coefficient of determination of 0.9 shows that 10% of variation in the dependent variable is caused by change in the independent variable.' Is this true?

Multiple regression

There are several extensions to linear regression, with the most common relating a dependent variable to more than one independent variable. You can imagine this with, say, the sales of a product that might be related to the advertising budget, price, unemployment rates, average incomes, competition, and so on. In other words, the dependent variable, y, is not set by a single independent variable, x, but by a number of separate independent variables x_i, meaning that:

$$y = a + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 + \ldots$$

or in our example:

$$\begin{aligned}\text{sales} = a &+ b_1 \times \text{advertising} + b_2 \times \text{price} + b_3 \times \text{unemployment rate} \\ &+ b_4 \times \text{income} + b_5 \times \text{competition}\end{aligned}$$

By adding more independent variables we are trying to get a more accurate model. Then we might find that advertising explains 60% of the variation in sales, but if we add another term for price this explains 75% of the variation, and if we add a third term for unemployment this explains 85% of the variation, and so on.

Because we are looking for a linear relationship between a dependent variable and a set of independent ones, we should really call this multiple linear regression – but it is always abbreviated to **multiple regression**.

- **Multiple regression** finds the line of best fit through a set of dependent variables.
- It finds the best values for a and b_i in the equation:

$$y = a + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 + \ldots$$

We have to calculate the coefficients a and b_i, but after seeing the arithmetic for linear regression you might guess, quite rightly, that the arithmetic is going to be even more messy. This is why multiple regression is never tackled by hand. Thankfully, there is a lot of standard software that includes multiple regression as a standard function.

WORKED EXAMPLE 9.10

The data section in Figure 9.14 shows sales, advertising costs and prices for a product at Soo Yueng Commercial. The rest of this figure shows some results when the 'Regression' function in Excel's Data Analysis ToolPak automatically does multiple regression. What do these figures show? What are the expected sales with a price of 60 and advertising of 200?

Solution

There is data for two independent variables – advertising and price – and one dependent variable – sales. So we are looking for a relationship of the form:

sales = a + b_1 × advertising + b_2 × price

Lines 26 to 29 of the spreadsheet show the value for the intercept and variables, with the line of best fit identified as:

sales = 585.96 + 9.92 × advertising + 19.11 × price

The coefficient of correlation $r = 0.996$. This shows a very strong linear relationship. This is confirmed by the coefficient of determination, r^2, which shows that 99.2% of the variation is explained by the relationship. Line 16 shows the adjusted r^2, which removes bias but is only slightly lower than the calculated value.

To find the expected sales we substitute values for advertising and price into the regression equation. With advertising of 200 and price of 60, the expected sales are:

sales = 585.96 + 9.92 × advertising + 19.11 × price
= 585.96 + 9.92 × 200 + 19.11 × 60
= 3,717

	A	B	C
1	**Multiple Regression**		
2			
3	**DATA**		
4	**Sales**	**Advertising**	**Price**
5	2450	100	50
6	3010	130	56
7	3090	160	45
8	3700	190	63
9	3550	210	48
10	4280	240	70
11			
12	**SUMMARY OUTPUT**		
13		Regression Statistics	
14	Multiple R	0.996	
15	R Square	0.992	
16	Adjusted R Square	0.986	
17	Standard Error	75.055	
18	Observations	6	
19			
20	ANOVA		
21		*df*	*SS*
22	Regression	2	2003633.452
23	Residual	3	16899.882
24	Total	5	2020533.333
25			
26		Coefficients	Standard Error
27	Intercept	585.96	195.57
28	X Variable 1	9.92	0.76
29	X Variable 2	19.11	4.13

Figure 9.14 Multiple regression results for worked example 9.10

With multiple regression, you have to take a few precautions to make sure the results are reliable. To start with, you have to make sure that there is enough data. In principle, you can draw a regression line with only two observations, but you need more data to get useful results. A rule of thumb suggests that there should be at least five observations for each variable fitted into the equation – so linear regression needs at least five observations, multiple regression with two variables needs at least 10 observations, and so on.

A second problem is that the method works properly only when there is no linear relationship between the independent variables. So in the last worked example there should be no relationship between the advertising costs and price. Obviously, if the two independent variables are related in some way, then they are not – by definition – independent. But these relationships often exist in real problems, and there might well be a relationship between advertising costs and price. In general, we accept the results if the relationships are not too strong. And we can measure the strengths of relationships between independent variables by using the coefficient of correlation. If the correlations are more than about 0.7 or less than about −0.7, we have to say that the relationships are too strong and we cannot rely on the multiple regression results. The technical term for a relationship between the independent variables is **multicollinearity**.

WORKED EXAMPLE 9.11

Are the advertising and price in worked example 9.10 really independent?

Solution

The coefficients of correlation between the variables are shown in the spreadsheet in Figure 9.15. These results come from the 'Correlation' tool in the 'Data Analysis ToolPak'. Not surprisingly, there is a perfect correlation between each variable and itself. We want a very high correlation between sales and each of the independent variables – so the correlations of 0.965 between sales and advertising, and 0.723 between sales and price, are both good. We want the correlation between

	A	B	C	D	E	F	G	H
1	**Coefficients of correlation**							
2								
3	**DATA**				**CORRELATIONS**			
4	**Sales**	**Advertising**	**Price**					
5	2450	100	50			**Sales**	**Advertising**	**Price**
6	3010	130	56		**Sales**	1		
7	3090	160	45		**Advertising**	0.965	1	
8	3700	190	63		**Price**	0.723	0.535	1
9	3550	210	48					
10	4280	240	70					

Figure 9.15 Correlations between variables in worked example 9.11 (from 9.10)

Worked example 9.11 continued

advertising and price to be low, and ideally less than its value of 0.535. Nonetheless, the results seem reasonably good.

These coefficients of correlation show that a simple linear regression model relating sales to advertising explains 93.2% of the variation in sales – as this is the value of r^2 when $r = 0.965$. But we saw in Figure 9.14 that adding price as a second variable increases this to 99.2%, showing an even better model. The result is now so good that we are unlikely to improve it any further. It is always tempting to keep adding another independent variable to see whether we can raise the coefficient of correlation a bit higher, but this soon becomes both pointless and misleading. Adding more independent variables can give small increases in correlation, even though the effects are not really significant. In general, it pays to be cautious, and add only variables that have an obvious effect.

Another problem with multiple regression is that it only really works when the errors are all independent. This might seem strange, but there can actually be a relationship between errors. For example, there might be regular seasonal variations that give a correlation between the error terms – perhaps with a low error in November always followed by a high error in December. When there is a relationship between the errors, it is called **autocorrelation**.

WORKED EXAMPLE 9.12

Elsom Service Corporation is trying to see how the number of shifts worked, bonus rates paid to employees, average hours of overtime, and staff morale affect production. They have collected the following data, using consistent units. What conclusions can they reach from this data?

Production	2,810	2,620	3,080	4,200	1,500	3,160	4,680	2,330	1,780	3,910
Shifts	6	3	3	4	1	2	2	7	1	8
Bonus	15	20	5	5	7	12	25	10	12	3
Overtime	8	10	22	31	9	22	30	5	7	20
Morale	5	6	3	2	8	10	7	7	5	3

Solution

Figure 9.16 shows the calculations for this problem. You can see from rows 38 to 42 that the intercept is 346.33, and the line of best fit is:

$$\text{production} = 346.33 + 181.80 \times \text{shifts} + 50.13 \times \text{bonus} + 96.17 \times \text{overtime} - 28.70 \times \text{morale}$$

This model fits the data very well, with a coefficient of correlation of 0.997, meaning that 99.5% of the variation in production is explained, and only 0.5% is unexplained. The separate coefficients of correlation between each pair of independent variables are low, so Elsom do not need to worry about multicollinearity. The coefficients of correlation between production and the independent variables also seem low – apart from the correlation between production and overtime – and there is surprisingly slight negative correlation between production and morale.

Elsom can use this model to forecast future production. For example, with five shifts, bonus of 10, overtime of 20 and morale of 6, their expected production is:

$$\begin{aligned}\text{production} &= 346.33 + 181.80 \times 5 + 50.13 \times 10 \\ &\quad + 96.17 \times 20 - 28.70 \times 6 \\ &= 3{,}508\end{aligned}$$

However, they should investigate the data to see whether there really is a significant relationship between, say, production and bonus.

Worked example 9.12 continued

	A	B	C	D	E	F
1	**Multiple regression**					
2						
3						
4	**Production**	**Shifts**	**Bonus**	**Overtime**	**Morale**	
5	2810	6	15	8	5	
6	2620	3	20	10	6	
7	3080	3	5	22	3	
8	4200	4	5	31	2	
9	1500	1	7	9	8	
10	3160	2	12	22	10	
11	4680	2	25	30	7	
12	2330	7	10	5	7	
13	1780	1	12	7	5	
14	3910	8	3	20	3	
15						
16	**Correlations**					
17		**Production**	**Shifts**	**Bonus**	**Overtime**	**Morale**
18	**Production**	1				
19	**Shifts**	0.262	1			
20	**Bonus**	0.153	−0.315	1		
21	**Overtime**	0.878	−0.108	−0.022	1	
22	**Morale**	−0.332	−0.395	0.451	−0.265	1
23						
24	**Regression**					
25	Multiple R	0.997				
26	R Square	0.995				
27	Adjusted R Square	0.990				
28	Standard Error	100.807				
29	Observations	10				
30						
31	**ANOVA**					
32		*df*	*SS*			
33	Regression	4	9439000			
34	Residual	5	50810			
35	Total	9	9489810			
36						
37		**Coefficients**	**Standard Error**			
38	Intercept	346.33	160.22			
39	X Variable 1	181.80	15.25			
40	X Variable 2	50.13	5.44			
41	X Variable 3	96.17	3.69			
42	X Variable 4	–28.70	16.76			

Figure 9.16 Multiple regression results for Elsom Service Corporation

Curve fitting

When you plot a set of data on a graph, there may be a clear pattern, but it may not necessarily be linear. For instance, there may be a clear quadratic relationship, or a constant rate of growth. To fit a more complicated function through a set of data we use **non-linear regression** – or more generally **curve fitting**. In principle, this is exactly the same as linear regression, and we look for an equation that minimises the errors. As you would expect, the arithmetic becomes more complicated, so we have two options:

- transform the data into a linear form
- use a computer package that automatically finds more complicated lines of best fit.

Sometimes we can use the first option, but it is generally difficult to transform data into a linear form. One occasion when this is possible is when the data has a growth curve of the form $y = bm^x$ where x is the independent variable, y is the dependent variable, and both b and m are constants. Taking the logarithm of both sides (have another look at Chapter 2 if you are not sure about this) gives:

$$\log y = \log b + x \log m$$

As both $\log b$ and $\log m$ are constants, we have a linear relationship between x and $\log y$.

WORKED EXAMPLE 9.13

Figure 9.17 shows Janet Curnow's local council tax over the past seven years. How much should she expect to pay next year?

Solution

The council tax is rising quickly, and we can try to fit a curve of the form:

$$y = bm^x \quad \text{or} \quad \text{tax} = bm^{\text{year}}$$

Then

$$\log(\text{tax}) = \log b + \text{year} \times \log m$$

and we have a linear relationship between log (tax) and year. We can use linear regression to find the line of best fit, as shown in Figure 9.17. Here rows 25 and 26 show the result:

$$\log(\text{tax}) = -159.432 + \text{year} \times 0.081$$

(These numbers have been rounded for display; the more exact values used in the calculation are –159.4323 and 0.0811909.) Substituting 2007 for the year gives:

$$\log(\text{tax}) = -159.4323 + 2{,}007 \times 0.0811909$$
$$= 3.5178$$

To turn this value back to the value we want, we have to remember the definition of a logarithm. When log (tax) = 3.5178 it means that tax = $10^{3.5178}$ = 3,294. You can see this calculation in cell D12.

Worked example 9.13 continued

	A	B	C	D	E	F	G
1	**Janet Curnow's Council Tax**						
2							
3	**Year**	**Tax**	**log(tax)**	**Forecast**			
4	1999	850	2.9294				
5	2000	923	2.9652				
6	2001	1015	3.0065				
7	2002	1130	3.0531				
8	2003	1370	3.1367				
9	2004	1856	3.2686				
10	2005	2334	3.3681				
11	2006	3102	3.4916				
12	2007		3.5178	3294			
13	2008		3.5990	3971			
14							
15	SUMMARY OUTPUT						
16							
17	Regression Statistics						
18	Multiple R	0.975					
19	R Square	0.951					
20	Adjusted R Square	0.942					
21	Standard Error	0.049					
22	Observations	8					
23							
24		Coefficients					
25	Intercept	–159.432					
26	X Variable 1	0.081					

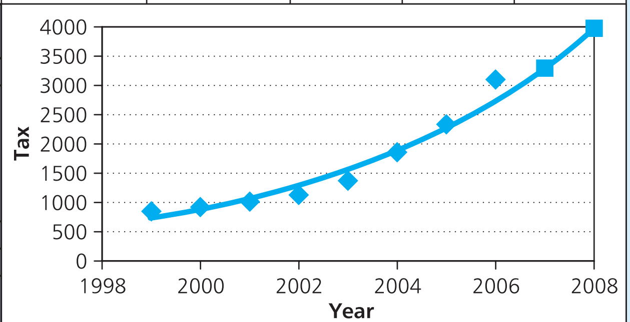

Figure 9.17 Calculations for Janet Curnow's council tax

Another time when we can transform curves into linear forms is with polynomials. Suppose you want to fit a quadratic equation through a set of points and are looking for a relationship of the form $y = a + b_1x + b_2x^2$. If you take one variable as x and a second variable as x^2 then you can use multiple regression to find the best values for a, b_1 and b_2.

WORKED EXAMPLE 9.14

Fit a quadratic curve through the points:

x	1	2	3	4	5	6	7	8	9	10
y	13	38	91	142	230	355	471	603	769	952

Solution

We can transform this into a multiple regression problem of the form:

$$y = a + b_1x_1 + b_2x_2$$

where one variable, x_1, is set as x and the second variable, x_2, is set as x^2. Then you can use multiple regression to find the values of b_1 and b_2, with the results shown in Figure 9.18. Results in rows 15 to 18 show that the line of best fit is:

$$y = 2.65 - 0.97x_1 + 9.59x_2$$

or

$$y = 2.65 - 0.97x + 9.59x^2$$

Worked example 9.14 continued

	A	B	C	D	E	F	G	H
1	**Curve fitting**							
2								
3	***x***	***x*^2**	***y***					
4	1	1	13					
5	2	4	38					
6	3	9	91					
7	4	16	142					
8	5	25	230					
9	6	36	355					
10	7	49	471					
11	8	64	603					
12	9	81	769					
13	10	100	952					
14								
15	**Multiple regression coefficients**							
16	**Intercept**	**2.65**						
17	**X Variable 1**	**–0.97**						
18	**X Variable 2**	**9.59**						

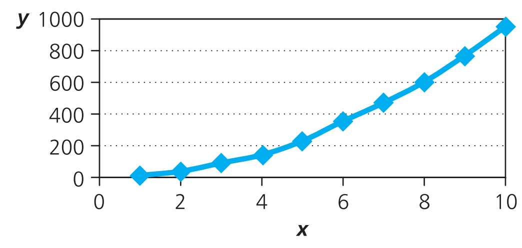

Figure 9.18 Using multiple regression for fitting a quadratic equation

It is usually difficult to transform relationships into simple linear forms, so you usually have to fit a standard curve through the data. Many packages have functions for this, including spreadsheets that typically fit:

- Linear models: $y = a + bx$
- Multiple linear: $y = a + b_1x_1 + b_2x_2 + b_3x_3 + \ldots$
- Polynomials: $y = a + bx + cx^2 + dx^3 + \ldots$
- Exponential: $y = ax^b$
- Growth curve: $y = ab^x$

WORKED EXAMPLE 9.15

John Mbulu is convinced that his accountant has raised prices by more than the cost of inflation. Over the past 11 years he has noticed that the cost of doing his accounts (in thousands of Rand) is as follows. What does this data show?

Year	1	2	3	4	5	6	7	8	9	10	11
Cost	0.8	1.0	1.3	1.7	2.0	2.4	2.9	3.8	4.7	6.2	7.5

Solution

You can see, without drawing a graph or calculating the correlation, that the data does not form a straight line. An alternative is to draw a growth curve through the data, with the standard form $y = bm^x$. In Excel, the function LOGEST does this automatically, with results shown in Figure 9.19. The line of best fit is:

$$y = bm^x \quad \text{or} \quad y = 0.6541 \times 1.2475^x$$

and substituting values for the year gives the predictions in column C. The value of 1.2475 for m suggests that the accountant's charges are rising by almost 25% a year.

Worked example 9.15 continued

	A	B	C	D	E	F	G	H	I
1	**Curve fitting**								
2									
3	**Year**	**Cost**	**Predictions**		**Line of fit**				
4	1	0.8	0.816		***y* = *bm* ^ *x***	***b* =**	0.6541	***m* =**	1.2475
5	2	1	1.018						
6	3	1.3	1.270						
7	4	1.7	1.584						
8	5	2	1.976						
9	6	2.4	2.465						
10	7	2.9	3.075						
11	8	3.8	3.836						
12	9	4.7	4.786						
13	10	6.2	5.970						
14	11	7.5	7.448						
15	12		9.291						
16	13		11.591						
17	14		14.459						
18									

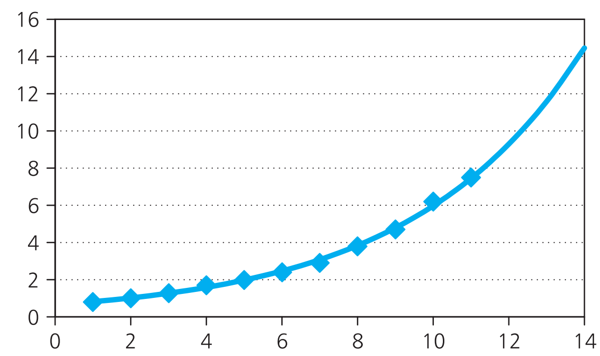

Figure 9.19 Curve fitting for John Mbulu's accountant's costs

Review questions

9.15 What are the most common extensions to linear regression?

9.16 'Multiple regression considers linear relationships between an independent variable and several dependent ones.' Is this true?

9.17 How can you tell whether multiple regression will find a better fit to a set of data than simple linear regression?

9.18 What is the difference between non-linear regression and curve fitting?

IDEAS IN PRACTICE Richmond, Parkes and Wright

Richmond, Parkes and Wright is a private company whose interests are in management research, analysis and education. They frequently use regression to describe the relationships between different variables, and they suggest a general approach with the following steps:

1. Collect and check relevant data.
2. Draw a graph of the data and see whether it suggests a linear relationship.
3. If there seems to be a linear relationship, find the line of best fit.
4. Calculate the coefficients of correlation and determination to see how well this line fits the data.
5. If there is a good fit, substitute appropriate values for the independent variable to predict corresponding values for the dependent variable.
6. If there is not a good fit – or there is some other problem – basic linear regression does not work.
7. Either look for some other approach, or refine the model to see whether multiple regression or non-linear regression gives better results.

CHAPTER REVIEW

This chapter has shown how to find and measure the relationships between variables.

- A relationship between two variables means that values of a dependent variable, y, are related to values of an independent variable, x. In practice, there is usually some random noise in the relationship, which means that there is a difference between the expected value and the observed one.
- The amount of noise determines the strength of a relationship – and we can consider the noise as an error. Stronger relationships have less noise. You can measure the error using the mean error, mean absolute error and mean squared error. The mean squared error is the most widely used.
- Linear regression finds the line of best fit through a set of data. This line is defined as the one that minimises the sum of squared errors. The main use of linear regression is to predict the value of a dependent variable for a known value of an independent variable.
- The coefficient of determination measures the proportion of the total variation from the mean explained by the regression line. A value close to 1 shows that the regression line gives a good fit, while a value close to zero shows a poor fit.
- Pearson's correlation coefficient shows how strong the linear relationship is between two variables. A value close to 1 or −1 shows a strong relationship, while a value close to zero shows a weak one. Spearman's coefficient shows the correlation for ranked data.
- Sometimes a dependent variable is related to a number of independent variables. Then you use multiple regression to find the best values of a and b_i. Many packages do these calculations automatically, but the interpretation of results can be difficult.
- Sometimes relationships are clearly not linear, and then you can use curve fitting to find more complex functions through data.

CASE STUDY Western General Hospital

Each term the Western General Hospital accepts a batch of 50 new student nurses. Their training lasts for several years before they become state registered or state enrolled. The hospital invests a lot of money in nurse training, and it wants to make sure that this is used efficiently.

A continuing problem is the number of nurses who fail exams and do not complete their training. One suggestion for reducing this number is to improve recruitment and select only students who are more likely to complete the course. For instance, the hospital might look for relationships between students' likely performance in nursing exams and their performance in school exams. But nurses come from a variety of backgrounds and start training at different ages, so their performance at school may not be relevant. Other possible factors are age and number of previous jobs.

The following table shows some results for last term's nurses. Grades in exams have been converted to numbers (A = 5, B = 4 and so on), and average marks are given.

Case study continued

Nurse	Year of birth	Nursing grade	School grade	Number of jobs	Nurse	Year of birth	Nursing grade	School grade	Number of jobs
1	82	2.3	3.2	0	26	70	4.1	3.7	4
2	75	3.2	4.5	1	27	84	2.6	2.3	1
3	82	2.8	2.1	1	28	84	2.3	2.7	1
4	72	4.1	1.6	4	29	82	1.8	1.9	2
5	80	4.0	3.7	2	30	81	3.1	1.0	0
6	83	3.7	2.0	1	31	72	4.8	1.2	3
7	75	3.5	1.5	0	32	78	2.3	3.0	1
8	73	4.8	3.6	0	33	80	3.1	2.1	5
9	83	2.8	3.4	2	34	81	2.2	4.0	2
10	84	1.9	1.2	1	35	82	3.0	4.5	3
11	84	2.3	4.8	2	36	72	4.3	3.3	0
12	83	2.5	4.5	0	37	82	2.4	3.1	1
13	76	2.8	1.0	0	38	78	3.2	2.9	0
14	69	4.5	2.2	3	39	84	1.1	2.5	0
15	84	2.0	3.0	1	40	69	4.2	1.9	2
16	80	3.4	4.0	0	41	78	2.0	1.2	1
17	78	3.0	3.9	2	42	84	1.0	4.1	0
18	78	2.5	2.9	2	43	77	3.0	3.0	0
19	79	2.8	2.0	1	44	80	2.0	2.2	0
20	81	2.8	2.1	1	45	76	2.3	2.0	2
21	78	2.7	3.8	0	46	76	3.7	3.7	4
22	71	4.5	1.4	3	47	68	4.7	4.0	5
23	75	3.7	1.8	2	48	75	4.0	1.9	2
24	80	3.0	2.4	6	49	75	3.8	3.1	0
25	81	2.9	3.0	0	50	79	2.5	4.6	1

The hospital collected data on the number of nurses who did not finish training in the past 10 terms, with the following results.

Term	1	2	3	4	5	6	7	8	9	10
Number	4	7	3	6	9	11	10	15	13	17

Questions

- **Having collected this data, how can the hospital present it in a useful format that is clear and easy to understand?**
- **Which factors can it use to predict nurses' grades? What other factors might be relevant?**
- **What do you think the hospital should do next?**

PROBLEMS

9.1 The productivity of a factory has been recorded over 10 months, together with forecasts made the previous month by the production manager, the foreman and the Management Services Department. Compare the accuracy of the three sets of forecasts.

Month	1	2	3	4	5	6	7	8	9	10
Productivity	22	24	28	27	23	24	20	18	20	23
Production manager	23	26	32	28	20	26	24	16	21	23
Foreman	22	28	29	29	24	26	21	21	24	25
Management Services	21	25	26	27	24	23	20	20	19	24

9.2 Find the line of best fit through the following data. How good is this fit?

x	10	19	29	42	51	60	73	79	90	101
y	69	114	163	231	272	299	361	411	483	522

9.3 Blaymount Amateur Dramatic Society is staging a play and wants to know how much to spend on advertising. Its objective is to attract as many people as possible, up to the hall capacity. For the past 11 productions their spending on advertising (in hundreds of pounds) and audience are shown in the following table. If the hall capacity is now 300 people, how much should Blaymount spend on advertising?

Spending	3	5	1	7	2	4	4	2	6	6	4
Audience	200	250	75	425	125	300	225	200	300	400	275

9.4 Ten experiments were done to find the effects of bonus rates paid to the sales team. What is the line of best fit through the following results? How good is the fit?

% Bonus	0	1	2	3	4	5	6	7	8	9
Sales ('00s)	3	4	8	10	15	18	20	22	27	28

9.5 Monthly sales for Sengler Marketing for the past year are:

6 21 41 75 98 132 153 189 211 243 267 301

Use linear regression to forecast sales for the next year. How reliable are these figures?

9.6 Jamie O'Connor appraises his employees using the views of two managers. In one department the two managers rank staff as follows. How reliable does this scheme seem?

Person	A	B	C	D	E	F	G	H	I	J	K	L
Rank 1	5	10	12	4	9	1	3	7	2	11	8	6
Rank 2	8	7	10	1	12	2	4	6	5	9	11	3

9.7 A food company wanted to know if the amount of taste enhancer added to one of its products has any effect. It ran a test by adding eight different amounts and asking a panel of tasters to rank the results. Does there seem to be a relationship between the amount of enhancer and taste?

Test	A	B	C	D	E	F	G	H
Amount of enhancer	22	17	67	35	68	10	37	50
Rank	3	2	8	5	7	1	4	6

9.8 Use multiple regression to find the line of best fit through the following data. What does this tell you?

y	420	520	860	740	510	630	650	760	590	680
a	1	2	3	4	5	6	7	8	9	10
b	3	7	9	3	1	6	2	9	6	6
c	23	15	64	52	13	40	36	20	19	24
d	109	121	160	155	175	90	132	145	97	107

9.9 What is the best line through the following data?

x	1	2	3	4	5	6	7	8	9	10
y	9	14	20	28	40	60	90	130	180	250

9.10 How could you fit a curve through the following points?

Time	1	2	3	4	5	6	7	8	9	10	11	12
Value	25	18	8	–6	–21	–31	–29	–24	–9	22	68	35

9.11 A company records sales of four products for a 10-month period. What can you say about these?

Month	1	2	3	4	5	6	7	8	9	10
P	24	36	45	52	61	72	80	94	105	110
Q	2,500	2,437	2,301	2,290	2,101	2,001	1,995	1,847	1,732	1,695
R	150	204	167	254	167	241	203	224	167	219
S	102	168	205	221	301	302	310	459	519	527

RESEARCH PROJECTS

9.1 The daily number of flights from Skorgaard Airport over a typical summer period are as follows.

24 23 25 24 27 29 32 30 35 34 34 39 41 40 38 46 41 51 48 46 41 57 56 62 61 62 68 74 80 81 76 80 93 82 88 91 95 99 97 98

Analyse these figures and forecast the future numbers of flights. How does this pattern compare with the numbers of flights from other airports?

9.2 Emilio Gaspin provides an information back-up service to industrial users. Over 19 typical months he records the following data. How useful would multiple regression be in analysing these results? In general, what problems are there likely to be with using multiple regression?

Month	Output	Shifts	Advertising	Bonuses	Faults
1	1,120	10	1,056	0	241
2	131	10	1,050	0	236
3	144	11	1,200	0	233
4	152	11	1,250	10	228
5	166	11	1,290	15	210
6	174	12	1,400	20	209
7	180	12	1,510	20	225
8	189	12	1,690	20	167
9	201	12	1,610	25	210
10	225	12	1,802	30	128
11	236	13	1,806	35	201
12	245	13	1,988	40	165
13	261	13	1,968	40	132
14	266	13	2,045	40	108
15	270	14	2,163	45	98
16	289	15	2,138	50	134
17	291	16	2,431	50	158
18	300	16	2,560	55	109
19	314	16	2,570	55	65

9.3 A Malaga tourist agency has been looking at the prices charged for hotel rooms in the city. They have collected the following data from a sample of hotels. What information can they get from this? What other data could they collect and analyse?

Cost (€)	Rating	Rooms	Location	Facilities	Meals	Staff
90	1	45	2	4	10	70
170	3	90	4	6	8	70
80	2	120	1	5	6	120
130	4	30	1	2	4	8
70	3	40	5	9	5	8
240	5	240	3	12	12	140
30	1	8	5	2	2	4
32	1	12	4	2	2	5
56	2	40	2	6	6	18
120	4	100	1	8	10	45
240	5	60	3	12	12	100
190	3	80	3	8	8	30
110	2	50	4	2	10	20
120	2	45	1	2	8	15
36	1	40	1	12	2	30
56	3	30	4	4	6	8

Sources of information

Further reading

There are a few books specifically about regression, but they soon get very complicated. It is generally better to look up regression in books on forecasting (with examples in the next chapter) or statistics (with examples in Chapter 14). Four books specifically on regression are:

Fox J., *Applied Regression Analysis, Linear Models, and Related Methods*, Sage Publications, London,1997.

Hair J., Tatham R. and Anderson R., *Multivariate Data Analysis* (6th edition), Prentice Hall, Englewood Cliffs, NJ, 2002.

Moore J.H. and Weatherford L.R., *Decision Modelling with Microsoft Excel* (6th edition), Prentice Hall, Upper Saddle River, NJ, 2001.

Weisberg S., *Applied Linear Regression*, John Wiley, Chichester, 2005.

CHAPTER 10

Forecasting

Contents

Chapter outline

All decisions become effective at some point in the future. So managers should not base their decisions on present circumstances, but on conditions as they will be when the decisions become effective. These conditions must be forecast, and this suggests that forecasting is a core concern of every organisation. Unfortunately, there is no single best way of forecasting, and managers have to choose the method that best suits their needs. This chapter describes a range of the most widely used approaches to forecasting.

After finishing this chapter you should be able to:

- Appreciate the importance of forecasting to every organisation
- List different types of forecasting method
- Discuss the characteristics of judgemental forecasting
- Use a variety of approaches to judgemental forecasting
- Describe the characteristics of projective forecasting
- Understand the importance of time series
- Calculate errors and a tracking signal for forecasts
- Forecast using simple averages, moving averages, and exponential smoothing
- Forecast time series with seasonality and trend.

Forecasting in organisations

In Chapter 8 we described a break-even analysis, which finds the number of units of a product that a firm must sell before it begins to make a profit. If sales of a new product are unlikely to reach the break-even point, the organisation should not bother making it. Unfortunately, it cannot know exactly what future sales will be – so its only option is to forecast likely sales. This means that a fundamental decision for the organisation depends on forecasts of likely future sales. If you continue thinking along these lines, it becomes clear that virtually every decision made by managers depends on forecasts of future conditions. All their decisions become effective at some point in the future – so managers should not base decisions on current circumstances, but on the circumstances prevailing when their decisions become effective. And this means that they must forecast future conditions.

The implication is that forecasting is an essential job in every organisation. If you have any doubts about this, try thinking of a decision that does not involve a forecast – or imagine the consequences when a forecast is wildly wrong!

A lot of the following discussion talks of 'forecasting demand', but this is only a convenient label. In practice, virtually everything has to be forecast – demand, costs, availability of resources, weather, staff turnover, competitors' actions, exchange rates, taxes, inflation, energy consumption, traffic levels, customer complaints – and just about anything else.

Methods of forecasting

It would be convenient to say that 'a lot of work has been done on forecasting and the best method is . . .'. Unfortunately, we cannot do this. Because of the wide range of things to be forecast and the different conditions in which forecasts are needed, there is no single best method. There are many different ways of forecasting – sometimes one method works best, and sometimes another method is better. Managers simply have to look at the methods available and choose the one that best suits their circumstances.

Unfortunately, even when they choose the best available method, the result is rarely entirely accurate, and there are differences between the forecast and actual results. If this were not true we could rely on weather forecasts, predict the winner of a horse race, become rich by speculating on the stock exchange, not buy too much food for a dinner party, and so on. But by preparing the forecasts carefully, we should get the most reliable information possible.

One classification of forecasting methods concerns the time in the future they cover. In particular:

- *Long-term forecasts* look ahead several years – the time typically needed to build a new factory.
- *Medium-term forecasts* look ahead between three months and two years – the time typically needed to replace an old product by a new one.
- *Short-term forecasts* cover the next few weeks – describing the continuing demand for a product.

The time horizon affects the choice of forecasting method because of the availability and relevance of historical data, time available to make the forecast, cost involved, seriousness of errors, effort considered worthwhile, and so on.

Another classification of methods draws a distinction between qualitative and quantitative approaches (as shown in Figure 10.1).

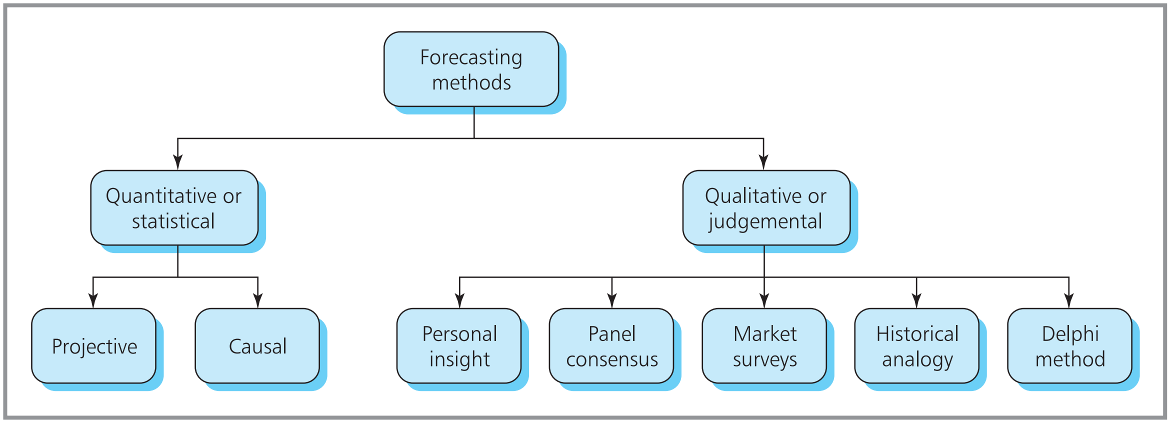

Figure 10.1 Classification of forecasting methods

When a company is already making a product, it has records of past sales and knows the factors that affect them. Then it can use a quantitative method to forecast future demand. There are two ways of doing this:

- **Causal methods** analyse the effects of outside influences and use these to produce forecasts. The demand for mortgages might depend on the interest rates charged, so lenders could use the proposed interest rate to forecast likely demand. This is the approach of linear regression that we described in the last chapter.
- **Projective methods** examine the pattern of past demand and extend this into the future. If demand in the past five weeks has been 10, 20, 30, 40 and 50, it seems reasonable to project this pattern into the future and suggest that demand in the next week will be 60.

Both of these methods need accurate, quantified data. But suppose that a company is introducing an entirely new product. There are no figures of past demand for the company to project forward, and they do not yet know the factors that affect demand for a causal forecast. So there is no quantitative data, which means that the company can use only a qualitative method. These methods are generally referred to as **judgemental**, and they rely on subjective assessments and opinions.

We described causal forecasting with regression in the last chapter, so here we concentrate on the other methods, starting with qualitative or judgemental methods.

Review questions

10.1 Why do managers use forecasts?

10.2 'Forecasting is a specialised function, where experts use mathematical techniques to project historical data.' Do you think this is true?

10.3 List three fundamentally different approaches to forecasting.

10.4 What factors should you consider when choosing a forecasting method?

Judgemental forecasts

Suppose a company is about to market an entirely new product, or a medical team is considering a new organ transplant, or a board of directors is considering plans for 25 years in the future. There is no historical data that the company can use for a quantitative forecast. This occurs either when there is simply no data available, or when there is some data but it is unreliable or irrelevant to the future. When there is no numerical data it is impossible to use a quantitative method, and the only alternative is a judgemental forecast.

The key feature of judgemental forecasts is that they use subjective opinions from informed people. The most widely used methods are:

- personal insight
- panel consensus
- market surveys
- historical analogy
- Delphi method.

Personal insight

This has a single expert who is familiar with the situation producing a forecast based on their own judgement. This is the most widely used forecasting method – and is the one that you should try to avoid. It relies entirely on one person's judgement – as well as their opinions, bias, objectives, prejudices, hidden agendas and ignorance. Sometimes it gives good forecasts, but more often it gives very bad ones and there are countless examples of experts being totally wrong. So its main weakness is unreliability. This may not matter for minor decisions, but when errors have serious consequences it is better to use a more reliable method.

Comparisons of forecasting methods clearly show that someone who is familiar with a situation, using their experience and knowledge, will consistently produce *worse* forecasts than someone who knows nothing about the situation but uses a more formal method.

Panel consensus

A single expert can easily make mistakes, but collecting together a panel of experts and allowing them to talk freely and exchange ideas should lead to a more reliable consensus. If the panel works well, with open discussions and no secrecy or hidden agendas, it can reach a genuine consensus. On the other hand, there can be difficulties in combining the views of different experts when they cannot reach a consensus.

Although it is more reliable than one person's insight, panel consensus still has the major weakness that even experts make mistakes. There are also problems of group working, where 'he who shouts loudest gets his way', everyone tries to please the boss, some people do not speak well in groups, and so on. Overall, panel consensus is an improvement on personal insight, but you should view results from either method with caution.

Market surveys

Even panels of experts may not have enough knowledge to make a convincing forecast. For instance, experts may have views about the likely success of a new product, but more useful information comes directly from potential customers. As we saw in Chapter 4, market surveys collect data from representative samples of customers, and analyse this to show likely behaviour of the whole population.

Market surveys can give useful information, but they tend to be expensive and time-consuming. They can also be wrong, as they rely on:

- identifying the right population
- choosing a sample of customers that accurately represents the whole population
- properly identifying and contacting the sample
- fair and unbiased data collection from the sample
- accurate analyses of the responses
- valid conclusions drawn from the analyses.

Historical analogy

When a company introduces a new product, it may have a similar product that it launched recently, and can assume that demand will follow the same pattern. For example, when a publisher introduces a new book, it forecasts likely sales from the demand for similar books that it published recently.

To use historical analogy, managers must have a product that is similar enough to the new one, that was launched recently, and for which they have reliable information. In practice, it is difficult to get all of these – but there is often enough data to give reasonable guidelines.

Delphi method

This is the most formal of the judgemental methods and has a well-defined procedure. A number of experts are contacted by post and each is given a questionnaire to complete – so data is collected from a group of experts without the problems of face-to-face discussions. The replies are analysed, and summaries passed back to the experts – with everything done anonymously to avoid undue influences of status, etc. Then each expert is asked to reconsider their original reply in the light of the replies from others, and perhaps to adjust their responses. This process of modifying responses in the light of replies made by the rest of the group is repeated several times – usually between three and six. By this time, the range of opinions should have narrowed enough to help with decisions.

We can illustrate this process by an example from offshore oil fields. A company wants to know when underwater inspections on platforms will be done entirely by robots rather than divers. To start the Delphi forecast the company contacts a number of experts from various backgrounds, including divers, technical staff from oil companies, ships' captains, maintenance engineers and robot designers. The overall problem is explained, and each expert is asked when they think robots will replace divers. The initial returns will probably give a wide range of dates from, say, 2012 to 2050 and these views are summarised and passed back to the group. Each expert is then asked whether they would like to reassess their answer in the light of other replies. After repeating this several times, views might converge so that 80% of replies suggest a date between 2015 and 2020, and this is enough to help with planning.

Review questions

10.5 What are judgemental forecasts?

10.6 List five types of judgemental forecast.

10.7 What are the main problems and benefits of judgemental forecasting?

IDEAS IN PRACTICE Forecasting oil prices

In March 1996 the California Energy Commission[1] published the results of their latest Delphi forecasts of oil prices. For this they used a panel of 21 experts from government, academia, consulting firms, industry and financial institutions. They asked seven questions about the likely price of oil up to 2016, and the factors that would affect this price. Starting from a base price of $15.96 a barrel in 1995, the Delphi forecasts gave an expected price (in 1996 dollars) of $19.93 by 2016, with a low estimate (the tenth percentile) of $13.33 and a high estimate (the 90th percentile) of $30.00.

In 2004 the State of Alaska forecast that the price of oil would reach $57.30 in 2006 and then fall back to $25.50 beyond 2008.[2]

Perhaps the most authoritative view of oil prices comes from the US Government's Energy Information Administration that uses huge statistical models to forecast energy prices. In 2006, they suggested that the price of oil in 2016 would be $43.39 (in 2004 dollars), rising to $49.99 by 2030 (compared with the 2004 price of $46).[3]

In 2006 World in Turmoil[4] suggested that world oil production had already peaked and would move into a rapid decline. This would cause oil shortages and a rapid increase in the price, moving beyond $100 by 2008 and rapidly higher afterwards.

There seems little agreement on even the price of the world's most important commodity. The actual price of crude oil reached $70 a barrel in 2006.

Projective forecasts

Projective forecasting takes historical observations and uses these to forecast future values. When the average cost of motor insurance in the past four years has been €300, €350, €400 and €450 we can project this pattern into the future and forecast the likely cost for next year as €500. This approach ignores any external influences and looks only at past values of demand to suggest future values. The four main methods of this type are:

- simple averages
- moving averages
- exponential smoothing
- models for seasonality and trend.

These generally produce forecasts for **time series**, which are series of observations taken at regular intervals.

Time series

Projective forecasts often work with time series, such as monthly unemployment figures, daily rainfall, weekly sales, quarterly profit, and annual fuel consumption. When you have a time series it is always useful to draw a graph, and a simple scatter diagram shows any underlying patterns. The three most common patterns in time series (shown in Figure 10.2) are:

- *constant series*, with observations taking roughly the same value over time, such as annual rainfall
- *series with a trend*, with values either rising or falling steadily, such as the gross national product per capita
- *seasonal series*, which have a cyclical component, such as the weekly sales of soft drinks.

If observations followed such simple patterns, there would be no problems with forecasting. Unfortunately, there are nearly always differences between actual observations and the underlying pattern. Random noise is superimposed on the underlying pattern (shown in Figure 10.3) so that a constant

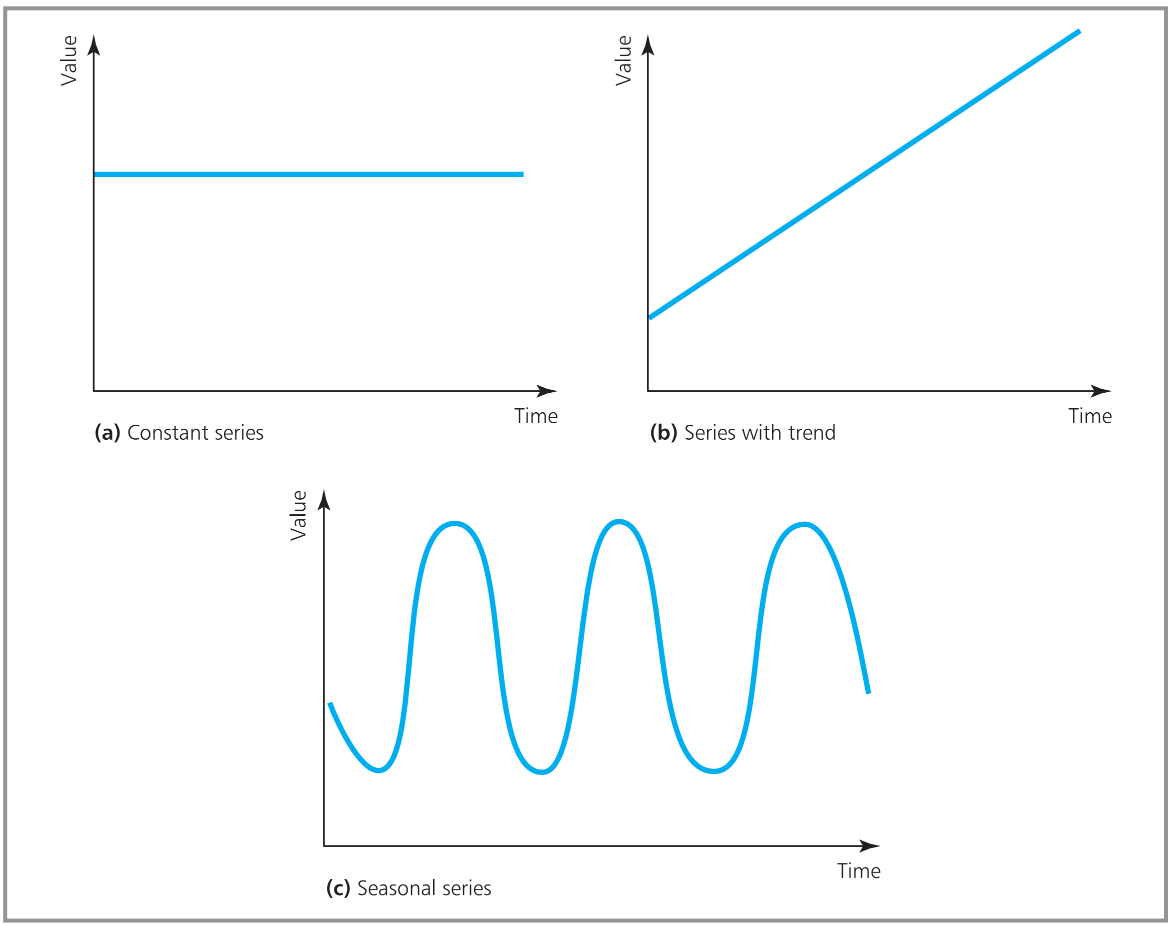

Figure 10.2 Common patterns in time series

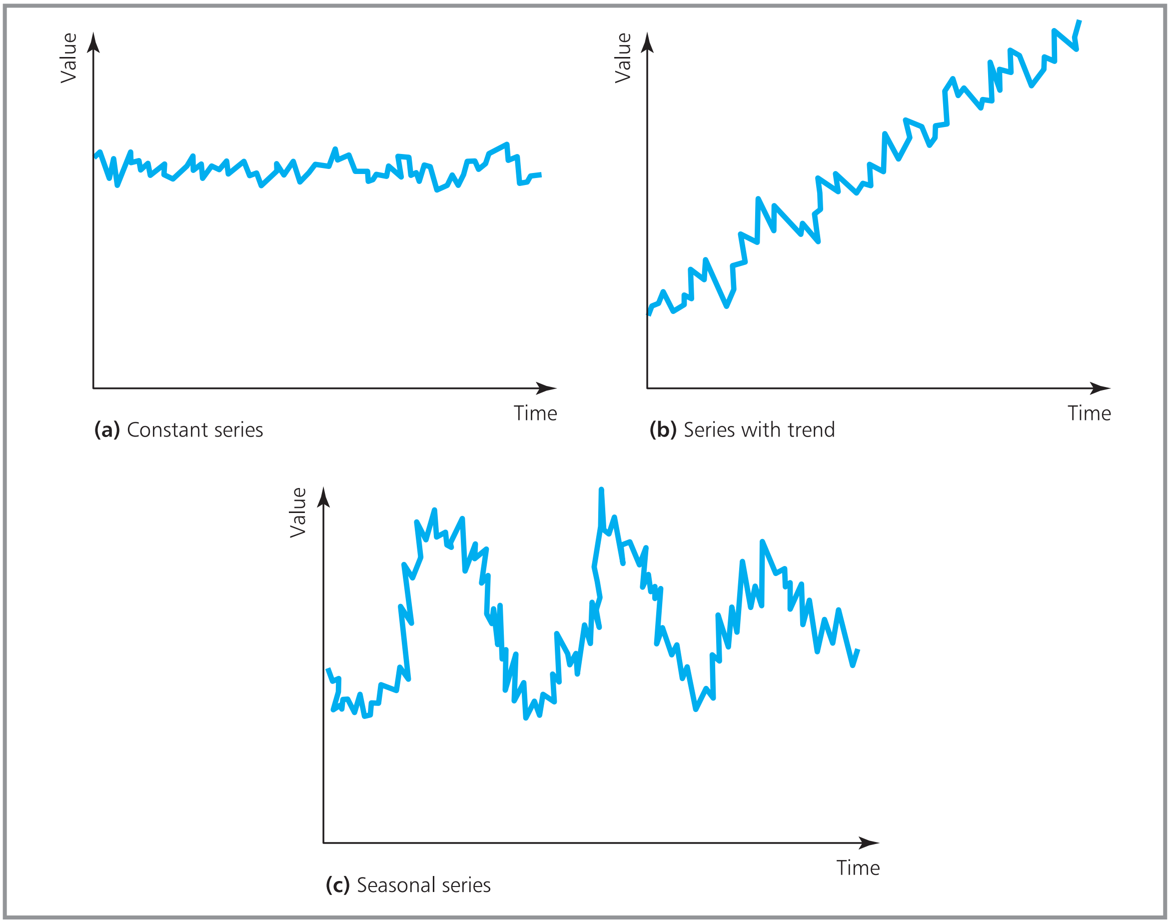

Figure 10.3 Patterns in time series including noise

series, for example, does not always take exactly the same value, but is somewhere close. Then:

200 205 194 195 208 203 200 193 201 198

is a constant series of 200 with superimposed noise.

actual value = underlying pattern + random noise

We met the idea of noise with regression in the last chapter, and it is these random effects that make forecasting so difficult. When there is little noise, forecasting is relatively easy and we can get good results, but when there is a lot of noise it hides the underlying pattern and forecasting becomes more difficult.

With regression we defined the difference between an actual observation and the expected value as an error. So when forecasting time series we get an error in each period.

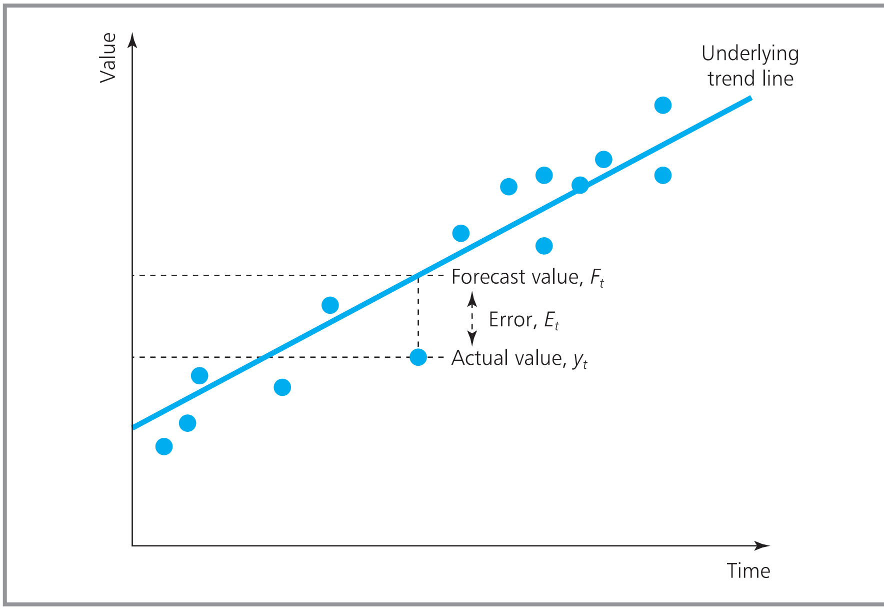

Figure 10.4 Errors in forecasts

E_t = error in the forecast in period t
= actual observation in period t – forecast value

Figure 10.4 shows this effect when there is an underlying trend, and the error in each observation is the vertical distance between the line and the actual observation.

If we define y_t as the actual observation in period t, and F_t as the forecast value, the error is:

$$E_t = y_t - F_t$$

Doing this calculation for each period allows us to calculate a mean error:

$$\text{mean error} = \frac{\sum E_t}{n} = \frac{\sum(y_t - E_t)}{n}$$

But we know that the mean error allows positive and negative errors to cancel, and data with very large errors can have zero mean error. In the following table the demand pattern is clear, and the forecasts are obviously very poor. Despite this, the mean error is zero. In reality, the mean error is not a reliable measure of forecast accuracy, but measures bias. If the mean error is positive, the forecast is consistently too low; if the mean error is negative, the forecast is consistently too high.

Period, t	1	2	3	4
Observation, y_t	100	200	300	400
Forecast, F_t	0	0	0	1,000

The two alternatives for measuring forecast errors are the mean absolute error and the mean squared error.

$$\textbf{mean absolute error} = \frac{\sum |E_t|}{n} = \frac{\sum |y_t - F_t|}{n}$$

$$\textbf{mean squared error} = \frac{\sum (E_t)^2}{n} = \frac{\sum (y_t - F_t)^2}{n}$$

WORKED EXAMPLE 10.1

Two forecasting methods give the following results for a time series. Which method is better?

t	1	2	3	4	5
y_t	20	22	26	19	14
F_t with method 1	17	23	24	22	17
F_t with method 2	15	20	22	24	19

Solution

Method 1 gives forecasts that are always nearer to actual demand than method 2, so in this case the decision is easy. We can confirm this by calculating the errors.

Method 1

t	1	2	3	4	5	Total	Mean
y_t	20	22	26	19	14	101	20.2
F_t with method 1	17	23	24	22	17	103	20.6
E_t	3	–1	2	–3	–3	–2	–0.4
$\|E_t\|$	3	1	2	3	3	12	2.4
$(E_t)^2$	9	1	4	9	9	32	6.4

- The mean error is –0.4, showing that the forecasts are slightly biased, being an average of 0.4 too high.
- The mean absolute error is 2.4, showing that forecasts are an average of 2.4 away from actual demand.
- The mean squared error is 6.4, which does not have such a clear meaning but is useful for other analyses.

Method 2

t	1	2	3	4	5	Total	Mean
y_t	20	22	26	19	14	101	20.2
F_t with method 2	15	20	22	24	19	100	20
E_t	5	2	4	–5	–5	1	0.2
$\|E_t\|$	5	2	4	5	5	21	4.2
$(E_t)^2$	25	4	16	25	25	95	19

- The mean error is 0.2, showing that each forecast is slightly biased, being an average of 0.2 too low.
- The mean absolute error is 4.2, so the forecast is an average of 4.2 away from actual demand.
- The mean squared error is 19.0.

The first method has lower mean absolute error and mean squared error, and is the better choice. The second method has slightly less bias, measured by the mean error.

Simple averages

Suppose you are going away on holiday and want to know the temperature at your destination. The easiest way of finding this is to look up records for past years and take an average. If your holiday is due to start on 1st July you could find the average temperature on 1st July over, say, the past 20 years. This is an example of forecasting using simple averages, where:

$$\text{forecast} = F_{t+1} = \frac{\sum y_t}{n}$$

where: t = time period
F_{t+1} = forecast for period $t + 1$
y_t = observation for period t
n = number of periods of historical data.

WORKED EXAMPLE 10.2

John Butler runs two dental surgeries, with the following numbers of patients visiting each over the past five weeks. Use simple averages to forecast the numbers of patients visiting in week six. How accurate are the forecasts? What are the forecasts for week 24?

Week	1	2	3	4	5
Surgery 1	98	100	98	104	100
Surgery 2	140	66	152	58	84

Solution

Calculating the simple averages:

- Surgery 1: $F_6 = (\sum y_t)/5 = 500/5 = 100$
- Surgery 2: $F_6 = (\sum y_t)/5 = 500/5 = 100$

Although the forecasts are the same, there is clearly less noise in the figures for Surgery 1 than for Surgery 2, so you should be more confident in the forecast for Surgery 1 and expect smaller errors.

Simple averages assume the underlying pattern is constant, so the forecasts for week 24 are the same as the forecasts for week 6, that is 100.

Simple averages can give good forecasts for constant series, and they are easy to use and understand. But they do not work well when the underlying pattern changes. The problem is that older data tends to swamp the latest figures and the forecast is very slow to follow the changing pattern. Suppose that demand for an item has been constant at 100 units a week for the past two years (104 weeks). Simple averages give a forecast demand for week 105 of 100 units. But if the actual demand in week 105 suddenly rises to 200 units, simple averages give a forecast for week 106 of:

$$F_{106} = (104 \times 100 + 200)/105 = 100.95$$

A rise in demand of 100 gives an increase of only 0.95 in the forecast. If demand continues at 200 units a week, following forecasts are:

$$F_{107} = 101.89\ F_{108} = 102.80,\ F_{109} = 103.70,\ F_{110} = 104.59,\ \ldots \text{ etc.}$$

The forecasts are rising but the response is very slow. Simple averages only really work for constant series. But very few time series are really stable over long periods, so this restriction makes them of limited value.

Moving averages

As patterns of demand tend to vary over time, only a certain amount of historical data is relevant to future forecasts. The problem with simple averages is that old, out-of-date data tends to swamp newer, more relevant data. A way around this is to ignore old data and use only a number of the most recent observations. This is the principle of **moving averages**.

If you decide that only the last n observations are relevant, and you can ignore all data older than this, your moving average forecast is:

$$F_{t+1} = \text{average of } n \text{ most recent observations}$$
$$= \frac{\text{latest demand} + \text{next latest} + \ldots + n\text{th latest}}{n}$$
$$= \frac{y_t + y_{t-1} + \ldots + y_{t-n+1}}{n}$$

WORKED EXAMPLE 10.3

Epsilan Court Co. has recorded the following numbers of customer complaints each month:

Month, t	1	2	3	4	5	6	7
Complaints, y_t	135	130	125	135	115	80	105

Continuously changing conditions mean that any data over three months old is no longer reliable. Use a moving average to forecast the number of complaints for the future.

Solution

Only data more recent than three months is reliable, so we can use a three-month moving average to forecast. Consider the situation at the end of month 3, when we can calculate the forecast for month 4 as:

$$F_4 = (y_1 + y_2 + y_3)/3 = (135 + 130 + 125)/3 = 130$$

At the end of month 4 we know the actual number is 135, so the forecast for month 5 is:

$$F_5 = (y_2 + y_3 + y_4)/3 = (130 + 125 + 135)/3 = 130$$

Similarly,

$$F_6 = (y_3 + y_4 + y_5)/3 = (125 + 135 + 115)/3 = 125$$
$$F_7 = (y_4 + y_5 + y_6)/3 = (135 + 115 + 80)/3 = 110$$
$$F_8 = (y_5 + y_6 + y_7)/3 = (115 + 80 + 105)/3 = 100$$

In this example, you can see that the forecast is clearly responding to changes, with a high number of complaints moving the forecast upwards, and a low number moving it downwards. This ability of a forecast to respond to changing demand is important. We want a forecast to respond to real changes – but not to follow random variations in the data. With most forecasting methods we can adjust the speed of response, or **sensitivity**. In a moving average we adjust the sensitivity by altering n, the number of periods averaged. A high value of n takes the average of a large number of observations and the forecast is unresponsive: it smoothes out random variations, but may not follow genuine changes. On the other hand, a low value of n takes the average of a few observations, giving a responsive forecast that follows genuine changes, but it may be too sensitive to random fluctuations. We need a compromise between these two, and this often means a value of n around six periods.

WORKED EXAMPLE 10.4

Column B in Figure 10.5 shows the monthly demand for a product. Use moving averages of three, six and nine months to give forecasts one month ahead.

Solution

Figure 10.5 also shows the calculations for moving average forecasts. With a three-month moving average (that is, $n = 3$), the earliest forecast we can make is for month 4, with $F_4 = (y_1 + y_2 + y_3)/3$. Similarly, the earliest forecasts for six- and nine-month moving averages are for F_7 and F_{10} respectively.

You can see from the graphs that for the first 10 months the pattern is fairly stable. All three forecasts do reasonably well here, smoothing out variations and following the underlying trends. The three-month moving average follows changes quite quickly, while the nine-month moving average is most stable. This is clearer after month 10 when there is a rising trend, and now the three-month moving average is much faster to respond, while the nine-month moving average is least responsive.

Worked example 10.4 continued

	A	B	C	D	E	F	G	H
1	**Moving Averages**							
2								
3	**Month**	**Demand**	**Forecasts**					
4			***n* = 3**	***n* = 6**	***n* = 9**			
5	1	16						
6	2	13						
7	3	11						
8	4	15	13.33					
9	5	17	13.00					
10	6	14	14.33					
11	7	12	15.33	14.33				
12	8	15	14.33	13.67				
13	9	10	13.67	14.00				
14	10	12	12.33	13.83	13.67			
15	11	18	12.33	13.33	13.22			
16	12	21	13.33	13.50	13.78			
17	13	25	17.00	14.67	14.89			
18	14	30	21.33	16.83	16.00			
19	15	28	25.33	19.33	17.44			
20	16	26	27.67	22.33	19.00			
21	17	34	28.00	24.67	20.56			
22	18	36	29.33	27.33	22.67			
23	19		32.00	29.83	25.56			
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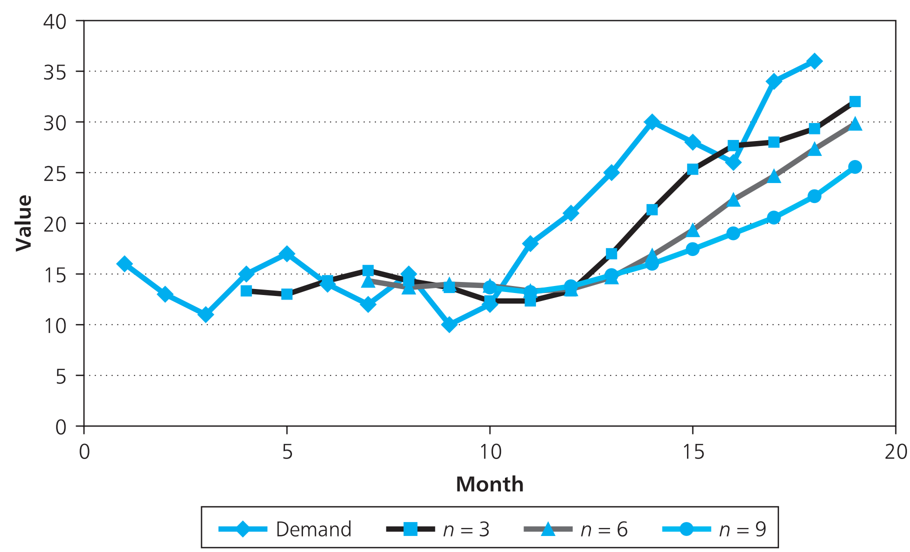

Figure 10.5 Moving average forecasts with different periods

You can see from the last example that moving averages give good results for stable patterns, but they tend to fall behind trends. However, they have a very useful property for data with strong seasonal variations; when you choose n equal to the number of periods in a season, a moving average will completely deseasonalise the data.

Although moving averages overcome some of the problems of simple averages, the method still has defects, including the following:

- It gives all observations the same weight.
- It works well only with constant time series (as we have seen, it lags behind trends and either removes seasonal factors or gets the timing wrong).
- It needs a lot of historical data to update the forecast.
- The choice of n is often arbitrary.

We can overcome the first of these problems by assigning different weights to observations. For example, a three-period moving average gives equal weight to the last three observations, so each is given a weight of 0.33. We can adjust these weights to put more emphasis on later results, perhaps using:

$$F_4 = 0.2 \times y_1 + 0.3 \times y_2 + 0.5 \times y_3$$

In practice, a more convenient way of changing the weights is to use exponential smoothing.

WORKED EXAMPLE 10.5

Use a moving average with two, four and six periods to calculate the forecasts one period ahead for the following data.

Quarter	1	2	3	4	5	6	7	8	9	10	11	12
Demand	100	50	20	150	110	55	25	140	95	45	30	145

Solution

This data has a clear seasonal pattern, with a peak in the fourth quarter of every year. Figure 10.6 shows the moving averages and you can clearly see the patterns. The moving averages with both $n = 2$ and $n = 6$ have responded to the peaks and troughs of demand, but neither has got the timing right: both forecasts lag behind demand. As you would expect, the two-period moving average is much more responsive than the six-period one. But the most interesting result is the four-period moving average, which has completely deseasonalised the data.

Worked example 10.5 continued

	A	B	C	D	E	F	G	H
1	**Moving Averages**							
2								
3								
4	**Year**	**Quarter**	**Demand**	**Forecasts**				
5				***n* = 2**	***n* = 4**	***n* = 6**		
6	1	1	100					
7		2	50					
8		3	20	75.0				
9		4	150	35.0				
10	2	1	110	85.0	80.00			
11		2	55	130.0	82.50			
12		3	25	82.5	83.75	80.83		
13		4	140	40.0	85.00	68.33		
14	3	1	95	82.5	82.50	83.33		
15		2	45	117.5	78.75	95.83		
16		3	30	70.0	76.25	78.33		
17		4	145	37.5	77.75	65.00		
18	4	1		73.3	78.75	80.00		

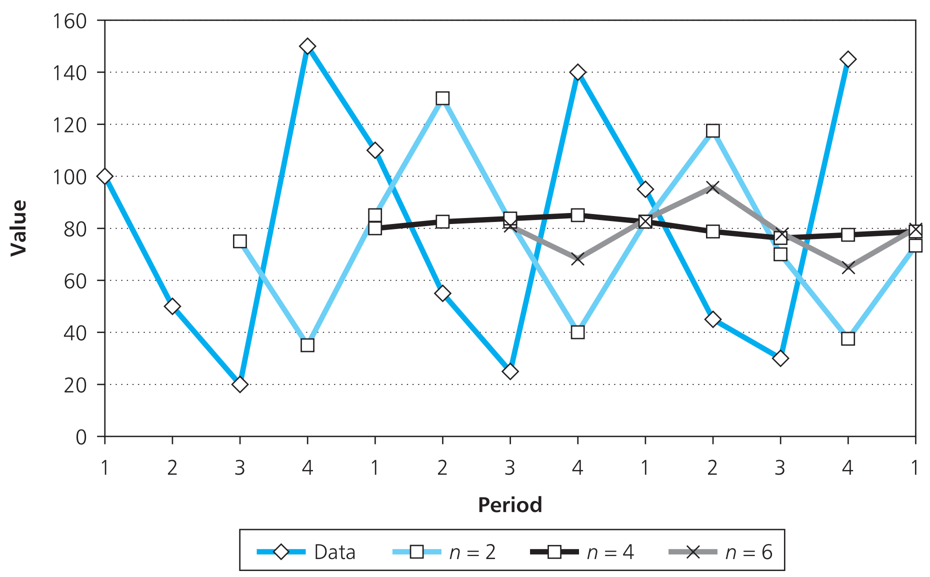

Figure 10.6 Using a moving average to deseasonalise data

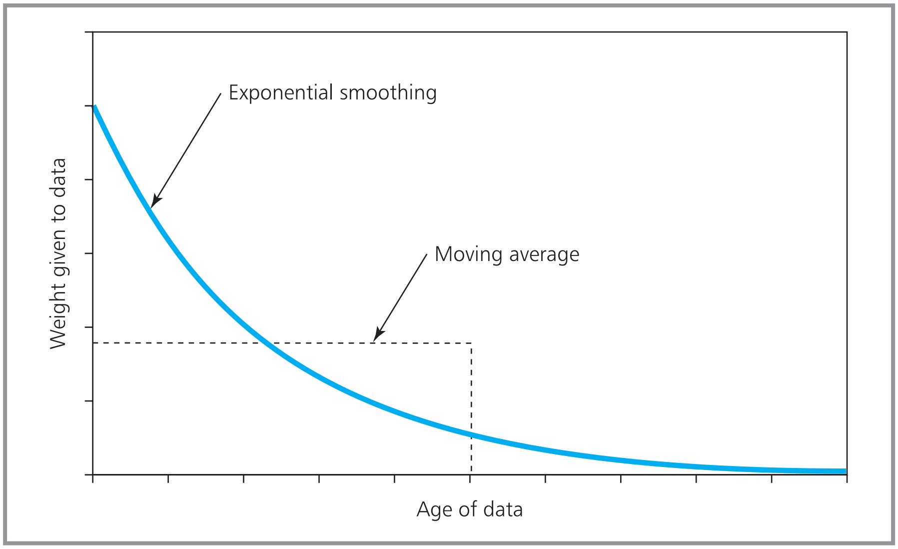

Figure 10.7 Weights given to data with exponential smoothing and moving average

Exponential smoothing

Exponential smoothing is based on the idea that as data gets older it becomes less relevant and should be given less weight. In particular, it gives an exponentially declining weight to observations (shown in Figure 10.7). It might seem difficult to organise this weighting, but in practice we can do it by using the latest observation to update a previous forecast. The calculation for this takes a proportion, α, of the latest observation and adds a proportion, $1 - \alpha$, of the previous forecast. (The Greek letter α is pronounced 'alpha'.)

> new forecast = $\alpha \times$ latest observation + $(1 - \alpha) \times$ last forecast
>
> $$F_{t+1} = \alpha x_t + (1 - \alpha)F_t$$

Here α is a **smoothing constant** that typically has a value between 0.1 and 0.2.

You can see how exponential smoothing adapts to changes with a simple example. Suppose a forecast is optimistic and suggests a value of 200 for an observation that actually turns out to be 180. Taking a value of $\alpha = 0.2$, the forecast for the next period is:

$$F_{t+1} = \alpha y_t + (1 - \alpha)F_t = 0.2 \times 180 + (1 - 0.2) \times 200 = 196$$

The method notices the optimistic forecast and adjusts the forecast for the next period downwards. You can see the reason for this adjustment by rearranging the exponential smoothing formula:

$$F_{t+1} = \alpha y_t + (1 - \alpha)F_t = F_t + \alpha(y_t - F_t)$$

But we define the error in a forecast as $E_t = y_t - F_t$, so the forecast is:

$$F_{t+1} = F_t + \alpha E_t$$

In other words, exponential smoothing takes the error in the last forecast, and adds a proportion of this to get the next forecast. The larger the error in the last forecast, the greater is the adjustment to the next forecast.

As exponential smoothing works by updating a previous forecast, it clearly needs an initial value to start. You can use a convenient value for this, such as the average demand in recent periods.

WORKED EXAMPLE 10.6

Use exponential smoothing with $\alpha = 0.2$ and an initial value of $F_1 = 170$ to get forecasts one period ahead for the following time series.

Month	1	2	3	4	5	6	7	8
Demand	178	180	156	150	162	158	154	132

Solution

We know that $F_1 = 170$ and $\alpha = 0.2$. Substituting these values gives a forecast for the second period:

$$F_2 = \alpha y_1 + (1 - \alpha)F_1 = 0.2 \times 178 + 0.8 \times 170 = 171.6$$

Then substituting for F_2 gives:

$$F_3 = \alpha y_2 + (1 - \alpha)F_2 = 0.2 \times 180 + 0.8 \times 171.6 = 173.3$$

$$F_4 = \alpha y_3 + (1 - \alpha)F_3 = 0.2 \times 156 + 0.8 \times 173.3 = 169.8$$

and so on, giving the following results.

Month, t	1	2	3	4	5	6	7	8	9
Demand, Y_t	178	180	156	150	162	158	154	132	
Forecast, F_t	170	171.6	173.3	169.8	165.8	165	163.6	161.7	155.8

From the exponential smoothing calculations it is probably not obvious that it actually does give less weight to data as it gets older. However, we can demonstrate this by taking an arbitrary value for α, say 0.2. Then we know that:

$$F_{t+1} = 0.2y_t + 0.8F_t$$

But substituting $t - 1$ for t gives:

$$F_t = 0.2y_{t-1} + 0.8F_{t-1}$$

and substituting this in the equation for F_{t+1} gives:

$$F_{t+1} = 0.2y_t + 0.8 \times (0.2y_{t-1} + 0.8F_{t-1}) = 0.2y_t + 0.16y_{t-1} + 0.64F_{t-1}$$

This includes both y_t and y_{t-1}. But we can go further, as we know that:

$$F_{t-1} = 0.2y_{t-2} + 0.8F_{t-2}$$

so

$$F_{t+1} = 0.2y_t + 0.16y_{t-1} + 0.64 \times (0.2y_{t-2} + 0.8F_{t-2})$$
$$= 0.2y_t + 0.16y_{t-1} + 0.128y_{t-2} + 0.512F_{t-2}$$

We could carry on with this, but it is clear that the equation actually includes all previous demands, and puts progressively less weight on each as it gets older. If you do the calculations you can find that with a smoothing constant of α equal to 0.2, the weights are:

Age of data	0	1	2	3	4	5	6	7
Weight	0	0.2	0.16	0.128	0.1024	0.08192	0.065536	0.0524288

The choice of α is important as it sets the balance between the previous forecast and the latest observation – and hence the sensitivity of the forecasts. A high value of α, say more than 0.3, gives a responsive forecast; a low value, say 0.05 to 0.1, gives a less responsive forecast. Again, we want a compromise between forecasts that are too responsive and follow random fluctuations, and ones that are not responsive enough and do not follow real patterns. A useful way of achieving this is to test several values for α over a trial period, and choose the one that gives smallest errors.

In the last worked example, the performance of the forecasts got worse when the demand pattern changed. It is useful to monitor forecasts and make sure that they continue giving reasonable results, and show when it is time to take some remedial action. For instance, you can monitor the mean absolute error and when it gets too big adjust the value of α. A more formal approach uses a **tracking signal**, with a common one defined as:

$$\text{tracking signal} = \frac{\text{sum of errors}}{\text{mean absolute error}}$$

On average a good forecast has as many positive errors as negative ones, so these should cancel, giving a sum around zero. While the tracking signal remains close to zero, the forecasts remain good – but if it increases to, say, 2.5 the errors are getting bigger and some remedial action is needed. This might require you to change to a more responsive value of α, or to make broader changes.

WORKED EXAMPLE 10.7

Remko van Rijn collected the demand figures shown in Figure 10.8. Use an initial forecast of 500 to compare exponential smoothing forecasts with different values of α.

Solution

You can start the calculations at the end of month 1, and taking a value of α = 0.1 gives:

$$F_2 = \alpha y_1 + (1 - \alpha)F_1 = 0.1 \times 470 + 0.9 \times 500 = 497$$

Then

$$F_3 = \alpha y_2 + (1 - \alpha)F_2 = 0.1 \times 510 + 0.9 \times 497 = 498.3$$

When α = 0.2 you get:

$$F_2 = \alpha y_1 + (1 - \alpha)F_1 = 0.2 \times 470 + 0.8 \times 500 = 494$$

Then

$$F_3 = \alpha y_2 + (1 - \alpha)F_2 = 0.2 \times 510 + 0.8 \times 494 = 497.2$$

Continuing these calculations gives the results in Figure 10.8 (in which α is shown as *a*).

You can see that demand is relatively stable for the first six months, and there is a sharp rise in month 7. The graph shows how differing values of α respond to this. All values follow the steady pattern well, and they all move upwards with the step in demand – but higher values of α make this adjustment more quickly and give a more responsive forecast. Eventually, all forecasts would home in on the new level of demand.

Worked example 10.7 continued

	A	B	C	D	E	F	G	H
1	**Exponential Smoothing**							
2								
3			**Forecast**					
4	**Month**	**Demand**	***a* = 0.1**	***a* = 0.2**	***a* = 0.3**	***a* = 0.4**		
5	1	470	500.0	500.0	500.0	500.0		
6	2	510	497.0	494.0	491.0	488.0		
7	3	460	498.3	497.2	496.7	496.8		
8	4	490	494.5	489.8	485.7	482.1		
9	5	520	494.0	489.8	487.0	485.2		
10	6	460	496.6	495.8	496.9	499.1		
11	7	1500	493.0	488.7	485.8	483.5		
12	8	1450	593.7	690.9	790.1	890.1		
13	9	1550	679.3	842.8	988.1	1114.1		
14	10	1500	766.4	984.2	1156.6	1288.4		
15	11	1480	839.7	1087.4	1259.6	1373.1		
16	12	1520	903.8	1165.9	1325.8	1415.8		
17	13	1500	965.4	1236.7	1384.0	1457.5		
18	14	1490	1018.8	1289.4	1418.8	1474.5		
19	15	1500	1066.0	1329.5	1440.2	1480.7		
20	16		1109.4	1363.6	1458.1	1488.4		
21								
22–43								
44								

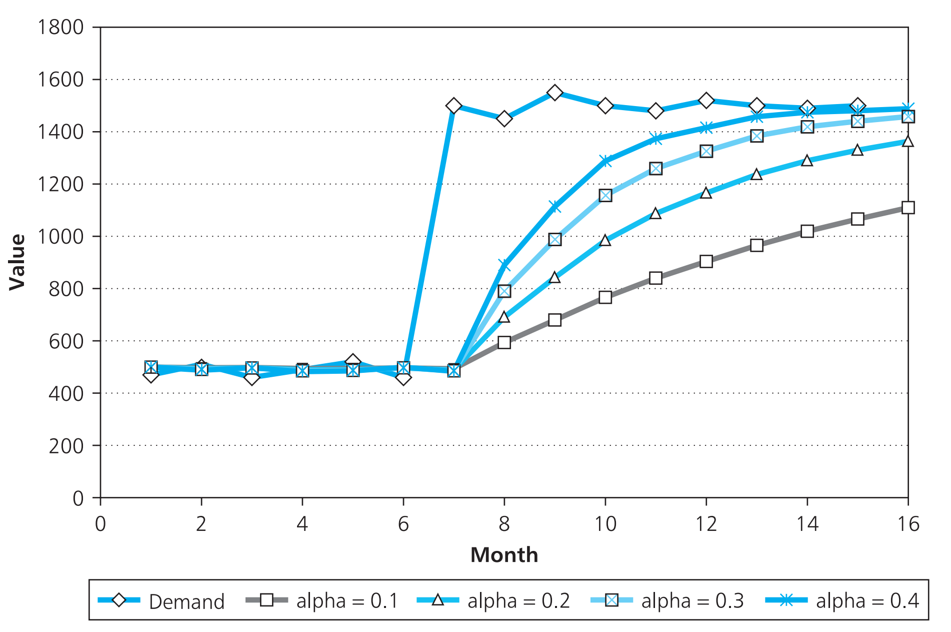

Figure 10.8 Exponential smoothing with varying values of alpha

Review questions

10.8 Why do virtually all forecasts contain errors?

10.9 How would you compare the results from two forecasting methods?

10.10 Why are simple averages of limited use for forecasting?

10.11 How can you make a moving average forecast more responsive?

10.12 What is the drawback with a responsive forecast?

10.13 How can you deseasonalise data?

10.14 Why is the forecasting method called 'exponential smoothing'?

10.15 How can you make exponential smoothing more responsive?

Forecasts with seasonality and trend

The methods we have described so far give good results for constant time series, but they need adjusting for other patterns. The easiest way of doing this is to divide the underlying pattern into separate components, and forecast each component separately. Then we get the final forecast by recombining the separate components. To be specific, we assume that an observation is made up of three components:

- *Trend* (T) is the long-term direction of a time series, typically a steady upward or downward movement.
- *Seasonal factor* (S) is the regular variation around the trend, which shows the variation in demand over a year, or some other period.
- *Residual* (R) is the random noise that we cannot properly explain.

Adding these three components gives an 'additive model' which assumes that an observation, y, is:

$$y = T + S + R$$

Then the seasonal factor, S, is an amount we add to the trend to allow for the season. If summer sales are 100 units higher than the trend, S has a value of 100; if winter sales are 100 units lower than the trend, S has a value of −100.

This additive model is easy to organise, but it can underestimate variations, particularly when there is a significant trend. Then it is better to use indices for seasonal variations, and put these into a 'multiplicative model' where:

$$y = T \times S \times R$$

If summer sales are 50% higher than the trend, S has a value of 1.5; if winter sales are 50% lower than the trend, S has a value of 0.5. As we do not know the random elements R, we cannot include this in forecasts, which become:

- Additive model: $F = T + S$
- Multiplicative model: $F = T \times S$

WORKED EXAMPLE 10.8

(a) What is the forecast for an additive model where the trend is 20 and the seasonal factor is 5?
(b) What is the forecast for a multiplicative model where the trend is 20 and the seasonal index is 1.25?

Solution

(a) The additive model forecasts by adding the factors, giving:

$$F = T + S = 20 + 5 = 25$$

(b) The multiplicative model forecasts by multiplying the factors, giving:

$$F = T \times S = 20 \times 1.25 = 25$$

The multiplicative model gives better results when there is a trend, so this is more widely used. We will describe the details of this, and remember that the additive model is very similar. Then we use historical data to:

- deseasonalise the data and find the underlying trend, T
- find the seasonal indices, S
- use the calculated trend and seasonal indices to forecast, using $F = T \times S$.

Finding the trend

There are two ways of finding the trend, T, both of which we have already met:

- linear regression with time as the independent variable
- moving averages with a period equal to the length of a season.

If the trend is clearly linear, regression is probably better as it gives more information; if the trend is not so clear, moving averages may be better.

WORKED EXAMPLE 10.9

Find the deseasonalised trend in the following set of observations using (a) linear regression, and (b) moving averages.

Period	1	2	3	4	5	6	7	8	9	10	11	12
Observation	291	320	142	198	389	412	271	305	492	518	363	388

Solution

(a) Figure 10.9 shows a spreadsheet that finds the line of best fit – which is the deseasonalised trend line – as:

$$\text{observation} = 223.41 + 18.05 \times \text{period}$$

You can confirm this result by doing the calculations: $n = 12$, $\sum x = 78$, $\sum y = 4{,}089$, $\sum x^2 = 650$, $\sum(xy) = 29{,}160$. Substituting these in the standard linear regression equations gives:

$$b = \frac{n\sum(xy) - \sum x \sum y}{n\sum x^2 - (\sum x)^2}$$

$$= \frac{12 \times 29{,}160 - 78 \times 4{,}089}{12 \times 650 - 78 \times 78} = 18.05$$

$$a = \bar{y} - b \times \bar{x} = 4{,}089/12 - 18.05 \times 78/12$$
$$= 223.41$$

Substituting values for the period in this equation gives the deseasonalised trend shown in column C.

One point about this regression line is that the coefficient of determination is only 0.35. The reason is obviously that a lot of the variation is explained not by the trend, but by the

Worked example 10.9 continued

	A	B	C	D	E	F	G
1	**Deseasonalised trend**						
2							
3	**Period**	**Observation**	**Deseasonalised trend**		SUMMARY OUTPUT		
4	1	291	241.46				
5	2	320	259.51		Regression Statistics		
6	3	142	277.57		Multiple R	0.59	
7	4	198	295.62		R Square	0.35	
8	5	389	313.67		Adjusted R Square	0.28	
9	6	412	331.72		Standard Error	93.14	
10	7	271	349.78		Observations	12	
11	8	305	367.83				
12	9	492	385.88				
13	10	518	403.93			Coefficients	
14	11	363	421.99		Intercept	223.41	
15	12	388	440.04		X Variable 1	18.05	
16							

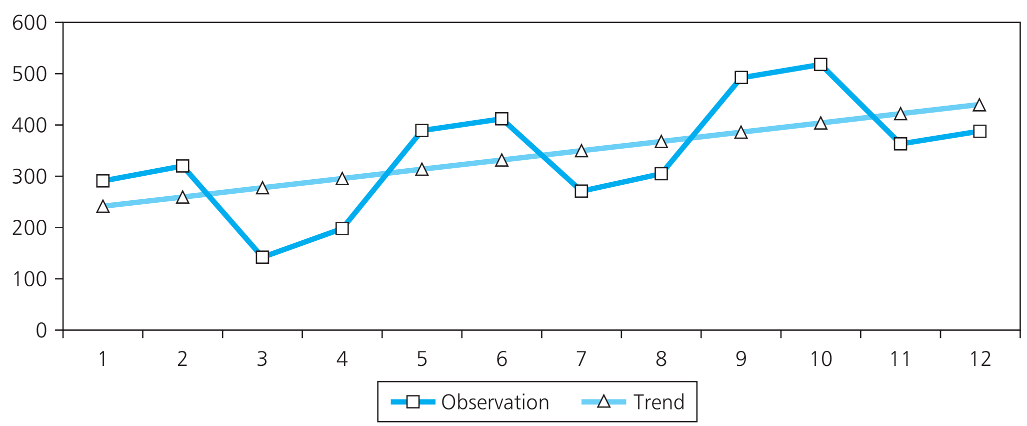

Figure 10.9 Deseasonalising data with linear regression

seasonality. When using linear regression to deseasonalise data, a low coefficient of determination does not necessarily mean that the results are poor.

(b) You can see from Figure 10.9 that the observations have a clear season of four periods, so we can deseasonalise them using a four-period moving average. But there is an immediate problem. The average values occur at average times, and taking the first four periods gives an average value of (291 + 320 + 142 + 198)/4 = 237.75, which occurs at the average time of (1 + 2 + 3 + 4)/4 = 2.5. In other words, it occurs halfway through a period. Whenever a season has an even number of periods, we have to work with 'half periods' (but obviously not when the season has an odd number of periods). Figure 10.10 shows a spreadsheet of the four-period moving averages, and the times at which they occur.

Worked example 10.9 continued

	A	B	C	D	E	F	G
1	**Deseasonalised trend**						
2							
3	**Period**	**Observation**	**Four period moving average**	**Deseasonalised values**			
4	1	291					
5	1.5						
6	2	320					
7	2.5		237.75				
8	3	142		250.00			
9	3.5		262.25				
10	4	198		273.75			
11	4.5		285.25				
12	5	389		301.38			
13	5.5		317.50				
14	6	412		330.88			
15	6.5		344.25				
16	7	271		357.13			
17	7.5		370.00				
18	8	305		383.25			
19	8.5		396.50				
20	9	492		408.00			
21	9.5		419.50				
22	10	518		429.88			
23	10.5		440.25				
24	11	363					
25	11.5						
26	12	388					
27							

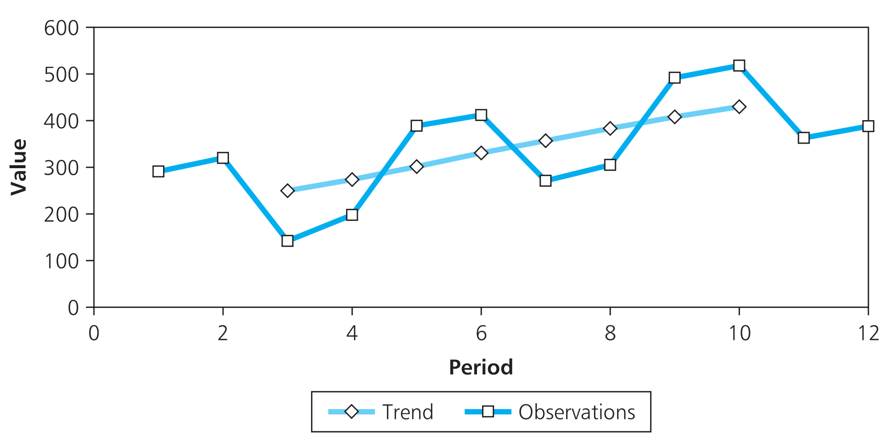

Figure 10.10 Deseasonalising data with moving averages

Worked example 10.9 continued

Now we have to return these deseasonalised values that occur halfway through periods to values for whole periods. The easiest way of doing this is to take the deseasonalised value for a period as the average of the two values on either side of it. Then the deseasonalised value for period 3 is the average of the deseasonalised values at times 2.5 and 3.5, or (237.75 + 262.25)/2 = 250. Repeating this calculation gives the deseasonalised values for periods 3 to 10, shown in column D.

Unfortunately, we now have deseasonalised data for only eight periods, rather than the original 12. This is just enough data to find the patterns, but it gives another reason why it is generally better to use regression. You can see from this example that the two methods give similar – but not identical – results.

Finding the seasonal indices

In multiplicative models seasonal variations are measured by **seasonal indices**, *S*, which are defined as the amounts by which deseasonalised values are multiplied to get seasonal values.

$$\text{seasonal index, } S = \frac{\text{seasonal value}}{\text{deseasonalised value}}$$

Suppose a newspaper sells an average of 1,000 copies a day in a town, but this rises to 2,000 copies on Saturday and falls to 500 copies on Monday and Tuesday. The deseasonalised value is 1,000, the seasonal index for Saturday is 2,000/1,000 = 2.0, the seasonal indices for Monday and Tuesday are 500/1,000 = 0.5, and the seasonal indices for other days are 1,000/1,000 = 1.0.

WORKED EXAMPLE 10.10

Worked example 10.9 found the deseasonalised trend using linear regression. What is the seasonal index for each period?

Solution

Figure 10.11 shows the actual observations and the deseasonalised trend values from the regression. To find the seasonal index for each period, you divide the actual observation by the trend value. For example, period 4 has an actual observation of 198 and a deseasonalised value of 295.62, so the seasonal index = 198/295.62 = 0.67. Repeating these calculations for other periods gives the indices in column D.

Each index is affected by noise in the data, so it is only an approximation. But if you take several complete seasons, you can find average indices that are more reliable. You can see from the graphs in Figures 10.9 and 10.10 that there are clearly four periods in a season – so you need to calculate four seasonal indices. Then periods 1, 5 and 9 are the first periods in consecutive seasons, and you can find an average index of (1.205 + 1.240 + 1.275)/3 = 1.240. Then periods 2, 6 and 10 are the second periods in consecutive seasons, and so on, so you can find the average indices for all periods in a season:

- first period in a season: (1.205 + 1.240 + 1.275)/3 = 1.240
- second period in a season: (1.233 + 1.242 + 1.282)/3 = 1.252
- third period in a season: (0.512 + 0.775 + 0.860)/3 = 0.716
- fourth period in a season: (0.670 + 0.829 + 0.882)/3 = 0.794

Worked example 10.10 continued

	A	B	C	D	E	F
1	**Seasonal indices**					
2						
3	**Period**	**Observation**	**Deseasonalised trend value**	**Seasonal index**	**Period in season**	**Average seasonal index**
4	1	291	241.46	1.205	1	1.240
5	2	320	259.51	1.233	2	1.252
6	3	142	277.57	0.512	3	0.716
7	4	198	295.62	0.670	4	0.794
8	5	389	313.67	1.240	1	1.240
9	6	412	331.72	1.242	2	1.252
10	7	271	349.78	0.775	3	0.716
11	8	305	367.83	0.829	4	0.794
12	9	492	385.88	1.275	1	1.240
13	10	518	403.93	1.282	2	1.252
14	11	363	421.99	0.860	3	0.716
15	12	388	440.04	0.882	4	0.794
16						

Figure 10.11 Calculating seasonal indices

Making forecasts

Now you have both the trend and seasonal indices, and can start forecasting. For this you:

- project the trend into the future to find the deseasonalised values
- multiply this by the appropriate seasonal index.

WORKED EXAMPLE 10.11

Forecast values for periods 13 to 17 for the time series in worked example 10.9.

Solution

We found the equation for the underlying trend to be:

value = 223.41 + 18.05 × period

Substituting values for future periods into this equation gives deseasonalised values. For period 13 the deseasonalised trend is 223.41 + 13 × 18.05 = 458.06. We also know that period 13 is the first period in a season, and the seasonal index is 1.240. Multiplying the deseasonalised trend by the seasonal index gives the forecast for period 13:

forecast = 458.06 × 1.240 = 568

Repeating this calculation for the other periods gives the following forecasts.

Worked example 10.11 continued

- *Period 14*

  deseasonalised trend = 223.41 + 18.05 × 14
  = 476.11
  seasonal index = 1.252 (second period in a season)
  forecast = 476.11 × 1.252 = 596

- *Period 15*

  deseasonalised trend = 223.41 + 18.05 × 15
  = 494.16
  seasonal index = 0.716 (third period in a season)
  forecast = 494.16 × 0.716 = 354

- *Period 16*

  deseasonalised trend = 223.41 + 18.05 × 16
  = 512.21
  seasonal index = 0.794 (fourth period in a season)
  forecast = 512.21 × 0.794 = 407

- *Period 17*

  deseasonalised trend = 223.41 + 18.05 × 17
  = 530.26
  seasonal index = 1.240 (first period in a season)
  forecast = 530.26 × 1.240 = 658

WORKED EXAMPLE 10.12

Forecast values for the next four periods of the following time series.

T	1	2	3	4	5	6	7	8
Y	986	1,245	902	704	812	1,048	706	514

Solution

Now you can do all the steps for forecasting together. Remember that you find the deseasonalised trend, calculate the seasonal index for each period, project the trend, and then use the indices to get a forecast.

If you draw a graph of the data (shown in Figure 10.12), you can see that there is a linear trend with a season of four periods. Linear regression (with results in rows 24 to 26) shows that the deseasonalised trend line is:

$$Y = 1{,}156.75 - 64.92T$$

This equation gives the deseasonalised values in column C. Dividing observations in column B by the corresponding values in column C gives the seasonal indices in column D. Taking average indices for each of the four periods in a season gives the results in column F of 0.939, 1.289, 0.971 and 0.796. The forecast for period 9 is:

deseasonalised value × seasonal index
= (1,156.75 − 64.92 × 9) × 0.939 = 538

Similarly the other forecasts are:

- Period 10: (1,156.75 − 64.92 × 10) × 1.289 = 654
- Period 11: (1,156.75 − 64.92 × 11) × 0.971 = 430
- Period 12: (1,156.75 − 64.92 × 12) × 0.796 = 301

Worked example 10.12 continued

	A	B	C	D	E	F	G	H
1	**Forecasting with seasonality and trend**							
2								
3	**Period (*T*)**	**Observation (*Y*)**	**Regression value**	**Seasonal index**	**Period in season**	**Average seasonal index**	**Forecast**	
4	1	986	1091.83	0.903	1	0.939		
5	2	1245	1026.92	1.212	2	1.289		
6	3	902	962.00	0.938	3	0.971		
7	4	704	897.08	0.785	4	0.796		
8	5	812	832.17	0.976	1	0.939		
9	6	1048	767.25	1.366	2	1.289		
10	7	706	702.33	1.005	3	0.971		
11	8	514	637.42	0.806	4	0.796		
12	9		572.50		1	0.939	538	
13	10		507.58		2	1.289	654	
14	11		442.67		3	0.971	430	
15	12		377.75		4	0.796	301	
16								
17	**SUMMARY OUTPUT**							
18	Regression Statistics							
19	Multiple R	0.691						
20	R Square	0.477						
21	Adjusted R Square	0.390						
22	Observation	8						
23								
24		Coefficients						
25	Intercept	1156.75						
26	X Variable 1	−64.92						
27								
28								
29								
30								

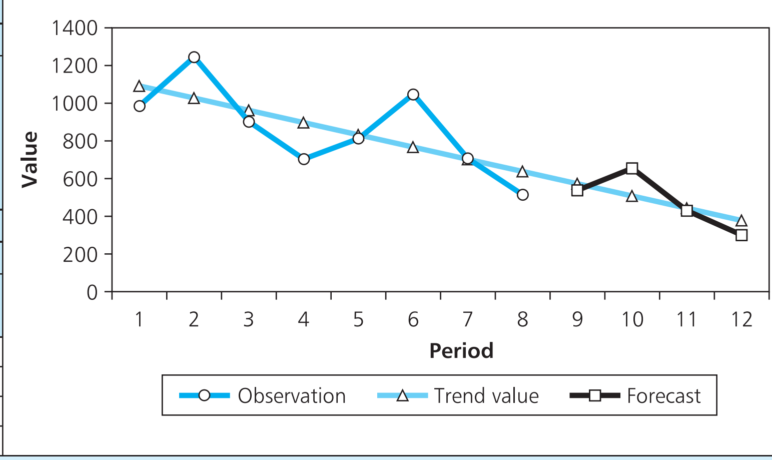

Figure 10.12 Calculations for worked example 10.12

WORKED EXAMPLE 10.13

See how Figure 10.13 does the calculations for an additive forecast for data with seasonality and trend.

Solution

Additive models work in virtually the same way as multiplicative models, except that seasonal adjustments are amounts added to the deseasonalised value (rather than an index to be multiplied). In Figure 10.13 the regression equation is 63.363 + 1.609 × period, and this gives the deseasonalised values in column C. Then there are seasonal adjustments, which are defined as:

$$\text{seasonal adjustment} = \text{observation} - \text{deseasonalised value}$$

Worked example 10.13 continued

	A	B	C	D	E	F	G
1	**Forecasting with seasonality and trend**						
2							
3	**Period**	**Observation**	**Regression value**	**Seasonal adjustment**	**Period in season**	**Average seasonal adjustment**	**Forecast**
4	1	42	64.97	–22.971	1	–18.102	
5	2	61	66.58	–5.580	2	–2.211	
6	3	83	68.19	14.811	3	22.680	
7	4	102	69.80	32.202	4	25.571	
8	5	79	71.41	7.593	5	8.463	
9	6	65	73.02	–8.015	6	–3.146	
10	7	39	74.62	–35.624	7	–33.255	
11	8	63	76.23	–13.233	1	–18.102	
12	9	79	77.84	1.158	2	–2.211	
13	10	110	79.45	30.549	3	22.680	
14	11	100	81.06	18.941	4	25.571	
15	12	92	82.67	9.332	5	8.463	
16	13	86	84.28	1.723	6	–3.146	
17	14	55	85.89	–30.886	7	–33.255	
18	15		87.49		1	–18.102	69
19	16		89.10		2	–2.211	87
20	17		90.71		3	22.680	113
21	18		92.32		4	25.571	118
22	19		93.93		5	8.463	102
23	20		95.54		6	–3.146	92
24	21		97.15		7	–33.255	64
25							

	A	B
26	SUMMARY OUTPUT	
27		
28	Regression Statistics	
29	Multiple R	0.306
30	R Square	0.093
31	Adjusted R Square	0.018
32	Standard Error	21.830
33	Observations	14
34	Intercept	63.363
35	X Variable 1	1.609
36		
37		
38		
39		
40		

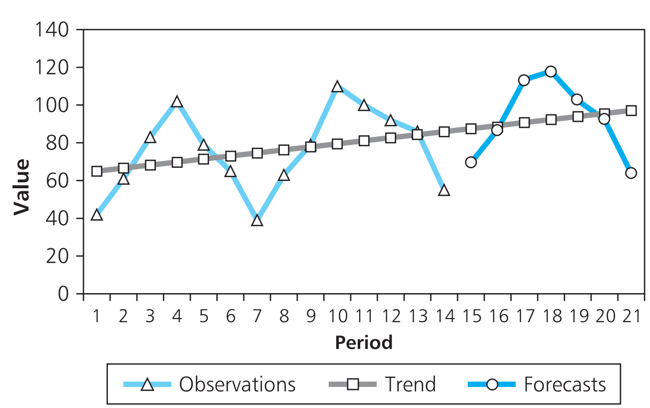

Figure 10.13 Forecasting with an additive model

Worked example 10.13 continued

Column D shows these results by subtracting entries in column C from corresponding observations in column B.

You can see from the graph that there are seven periods in a season, so this is probably weekly data. Then the average seasonal adjustment for the first period is $(-22.971 - 13.233)/2 = -18.102$. Adjustments for the other seasons are calculated in the same way, with the results in column F. The average adjustment for each period is shown in rows 18 to 24. Now we have the trend from the regression and the seasonal adjustments, and can do the forecasting from:

$$\text{forecast} = \text{deseasonalised trend} + \text{seasonal adjustment}$$

For period 15:

$$\text{forecast} = (63.363 + 1.609 \times 15) - 18.102 = 69.396$$

Repeating these calculations for the six periods gives the forecasts in column G.

Review questions

10.16 What are the steps in forecasting for data with seasonality and trend?

10.17 What is the difference between an additive and a multiplicative forecasting model?

10.18 Would you prefer to use regression or moving averages to deseasonalise data?

IDEAS IN PRACTICE BC Power Corp.

In practice, it is often very difficult to get good forecasts – as you can see with people trying to forecast the winner of a horse race, lottery numbers, price of oil, interest rates, or weather.

One of the most difficult problems of forecasting is the demand for electricity. Electricity cannot be stored – except in very small quantities using batteries – so the supply from power stations must exactly match the total demand. Any shortages in electricity generation give power cuts, which customers do not accept, while excess capacity wastes expensive resources.

In British Columbia, Canada, the long-term demand for electricity is rising steadily, so power companies must build enough generators to meet this increase. Planning and building a power station takes many years, so decisions are based on forecast demand 20 or more years in the future.

In the shorter term, demand for electricity follows an annual cycle, with demand higher in winter when people turn on their heating. There are also short, irregular periods of especially high demand during cold spells. There are cycles during the week, with lower demand at the weekends when industry is not working so intensely. On top of this are cycles during the day, with lighter demand during the night when most people are asleep. Finally, there are irregular peaks during the day, perhaps corresponding to breaks in television programmes when people turn on electric coffee pots and kettles.

Power stations need 'warming-up' before they start supplying electricity, so a stable demand would make operations much easier. In practice, though, they have to forecast demands with long-term trend, annual cycle, short-term peaks, weekly cycles, daily cycles, and short-term variations.

CHAPTER REVIEW

This chapter discussed methods of forecasting, which is an essential function in every organisation.

- There are many ways of forecasting, and we can classify them in several ways. There is no single best method, and the choice depends on circumstances. The three basic approaches are causal (discussed with regression in Chapter 9), judgemental and projective.
- Judgemental or qualitative forecasts are the only option when there is no relevant historical data. They rely on subjective views and opinions, as demonstrated by personal insight, panel consensus, market surveys, historical analogy and the Delphi method.
- It is always better to use quantitative forecasts when data is available. This often appears as time series, with observations taken at regular intervals. Then observations often follow an underlying pattern with superimposed noise.
- Projective forecasts look only at historical observations, and project the underlying patterns into the future. A basic form of projective forecasting uses simple averages, but this has limited practical use. Moving averages are more widely used. These forecast using the latest n observations, and ignore all older values.
- Exponential smoothing adds portions of the latest observation to the previous forecast. This reduces the weight given to data as it gets older.
- The easiest way to forecast time series with seasonality and trend is to divide underlying patterns into components, forecast each of these separately, and then combine the results into a final forecast.

CASE STUDY Workload planning

Maria Castigliani is head of the purchasing department of Ambrosiana Merceti, a medium-sized construction company. One morning she walked into the office and said, 'The main problem in this office is lack of planning. I have read a few articles about planning, and it seems that forecasting is the key to an efficient business. We have never done any forecasting, but simply rely on experience to guess our future workload. I think we should start using exponential smoothing to do some forecasting. Then we can foresee problems and schedule our time more efficiently.'

Unfortunately, the purchasing department was having a busy time and nobody in the office had time to develop Maria's ideas. A month later nothing had happened. Maria was not pleased and said that their current high workload was caused by lack of planning – and hence forecasting – and things would be much better if they organised their time more effectively. In particular, they could level their workload and would not be overwhelmed by periodic surges.

To make some progress with the forecasting, Maria seconded Piotr Zemlinski, a management trainee, to work on some figures. Piotr examined their work, and divided it into seven categories, including searching for business, preparing estimates, submitting tenders, finding suppliers, and so on. For each of these categories he added the number of distinct tasks the office had completed in each quarter of the past three years. Collecting the data took six weeks, and Piotr summarised it in the following table.

Case study continued

	Category						
Quarter	1	2	3	4	5	6	7
1,1	129	74	1,000	755	1,210	204	24
2,1	138	68	1,230	455	1,520	110	53
3,1	110	99	890	810	1,390	105	42
4,1	118	119	700	475	1,170	185	21
1,2	121	75	790	785	1,640	154	67
2,2	137	93	1,040	460	1,900	127	83
3,2	121	123	710	805	1,860	187	80
4,2	131	182	490	475	1,620	133	59
1,3	115	103	610	775	2,010	166	105
2,3	126	147	840	500	2,340	140	128
3,3	131	141	520	810	2,210	179	126
4,3	131	112	290	450	1,990	197	101

Now Piotr wants to forecast likely workload for the next two years. He knows a little about forecasting, and feels that exponential smoothing may not be the answer. He is not sure that the data is accurate enough, or that the results will be reliable. He feels that it would be better to link the forecasts directly to planning, overall workload, and capacity. To help with this, he has converted the effort involved with different tasks into 'standard work units'. After some discussion he allocated the following number of work units to a typical task in each category of work.

Category 1 2 work units
Category 2 1.5 work units
Category 3 1 work unit
Category 4 0.7 work units
Category 5 0.4 work units
Category 6 3 work units
Category 7 2.5 work units

Questions

- **What information has Piotr collected, and how useful is it? What other information does he need?**
- **How can Piotr forecast future workloads in the purchasing department? How reliable are the results? Do they suggest any patterns of workload?**
- **What are the implications of Piotr's work, and what should he do now?**

PROBLEMS

10.1 Use linear regression to forecast values for periods 11 to 13 for the following time series.

Period	1	2	3	4	5	6	7	8	9	10
Observation	121	133	142	150	159	167	185	187	192	208

10.2 Use simple averages to forecast values for the data in problem 10.1. Which method gives better results?

10.3 Use a four-period moving average to forecast values for the data in problem 10.1.

10.4 Find the two-, three- and four-period moving averages for the following time series, and use the errors to say which gives the best results.

T	1	2	3	4	5	6	7	8	9	10
Y	280	240	360	340	300	220	200	360	410	280

10.5 Use exponential smoothing with $\alpha = 0.1$ and 0.2 to forecast values for the data in problem 10.4. Which smoothing constant gives better forecasts? How would you monitor the results with a tracking signal?

10.6 Use exponential smoothing with α between 0.1 and 0.4 to get forecasts one period ahead for the following time series. Use an initial value of $F_1 = 208$ and say which value of α is best.

t	1	2	3	4	5	6	7	8	9	10
Demand	212	216	424	486	212	208	208	204	220	200

10.7 Balliol.com recorded their opening share price for 12 consecutive weeks. Deseasonalise their results and identify the underlying trend. Forecast values for the next six weeks.

Week	1	2	3	4	5	6	7	8	9	10	11	12
Share price (pence)	75	30	52	88	32	53	90	30	56	96	38	62

10.8 Use a multiplicative model to forecast values for the next six periods of the following time series.

t	1	2	3	4	5	6
y	100	160	95	140	115	170

10.9 Use a multiplicative model to forecast values for the next eight periods of the following time series.

t	1	2	3	4	5	6	7	8	9	10
y	101	125	121	110	145	165	160	154	186	210

10.10 Use additive models to forecast values for the time series in problems 10.8 and 10.9.

RESEARCH PROJECTS

10.1 In this chapter we have used spreadsheets for doing most of the calculations. Spreadsheets – or specialised add-ins – have standard functions for forecasting, typically including regression, simple averages, moving averages and exponential smoothing. Design a spreadsheet that uses these for a variety of forecasting methods. Specialised software may be better, so do a small survey of available forecasting packages. What extra features do these have?

10.2 Governments routinely collect huge amounts of data which they often present in extended time series – giving figures for populations, gross domestic product, employment, etc. Collect a reasonably long set of data, and see how well standard forecasting methods work. How can the forecasts be improved?

10.3 Energy consumption around the world is rising. Find some figures to describe this growth. Now forecast the future demand for electricity. How accurate are your results? What other factors should be taken into account? What are the implications of your findings?

Sources of information

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3 EIA, *Annual Energy Outlook*, Energy Information Administration, Washington, DC, 2006.

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Further reading

You can find material on forecasting in operations management and marketing books. The following list gives some more specialised books.

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CHAPTER 11

Simultaneous equations and matrices

Contents

Chapter outline

Chapter 2 showed how to solve an equation to find the value of a single unknown variable. To find values for more variables we need a set of simultaneous equations that relate the variables. Then we can solve these, either graphically or algebraically. Matrices are useful for this, as they give a convenient notation for describing certain types of problem.

After finishing this chapter you should be able to:

- Understand the principles of simultaneous equations
- Use algebraic methods to solve simultaneous equations
- Draw graphs to solve simultaneous equations
- Describe problems using matrices
- Do arithmetic with matrices
- Use matrices to solve simultaneous equations.

Simultaneous equations

You can solve an equation to find the value of one previously unknown variable. But when you want to find values for several unknown variables, you have to use more than one independent equation. Specifically, to find values for n variables, you need n **independent equations** relating them.

Suppose that you have two unknowns, x and y. If you know only that $x + y = 3$, you cannot find values for both x and y. But if you also know that $y - x = 1$, then you have two independent equations relating the variables and can find values for both of them (here $x = 1$ and $y = 2$).

In this sense 'independent' means that the two equations are not simply different ways of saying the same thing. For instance:

$$x + y = 10 \quad \text{and} \quad x - 10 = y$$

are not independent, as they are different forms of the same equation. Similarly:

$$x + y = 10 \quad \text{and} \quad 2x + 2y = 20$$

are not independent as, again, they are simply different forms of the same equation.

- Independent equations that show the relationship between a set of variables are called **simultaneous equations**.
- Solving simultaneous equations means that you find values for all the variables.

Solving simultaneous equations

Suppose that you have two unknown variables, x and y, related by two simultaneous equations:

$$x - y = 7 \quad (1)$$

$$x + 2y = 16 \quad (2)$$

You solve these to find the values of x and y. The easiest way of doing this is to put one equation in the form 'x = something', and then substitute this value for x into the second equation. Here we can write the first equation as

$$x = y + 7$$

and substituting this in the second equation gives:

$$x + 2y = 16 \quad \text{or} \quad (y + 7) + 2y = 16$$

Then

$$3y = 9 \quad \text{or} \quad y = 3$$

This gives one variable, and if you substitute $y = 3$ back into the first equation you get:

$$x - y = 7 \quad \text{so} \quad x - 3 = 7 \quad \text{or} \quad x = 10$$

giving the value of the second variable. You can check these values, $x = 10$ and $y = 3$, in the second equation, giving:

$$x + 2y = 10 + 2 \times 3 = 16 \quad ✓$$

which confirms the result.

Unfortunately, this method of substitution becomes very messy with more complicated equations. An alternative approach multiplies one equation by a number that allows the two equations to be added or subtracted to eliminate one of the variables. When one variable is eliminated, we are left with a single equation with one variable – which we can then solve. Then substituting this value into either of the original equations gives the value of the other variable. This sounds rather complicated, but it is easy to follow in an example.

WORKED EXAMPLE 11.1

Two variables, x and y, are related by the following equations:

$$3y = 4x + 2 \qquad (1)$$

$$y = -x + 10 \qquad (2)$$

What are the values of x and y?

Solution

If you multiply equation (2) by 3, you get the revised equations:

$$3y = 4x + 2 \quad \text{as before} \qquad (1)$$

$$3y = -3x + 30 \qquad (2)$$

Now subtracting equation (2) from equation (1):

$$3y - 3y = 4x - (-3x) + 2 - 30$$

so

$$0 = 7x - 28$$

$$x = 4$$

This gives one variable, which you substitute in one of the original equations, say (1), to give the value for the other variable:

$$3y = 4x + 2$$

so

$$3y = 4 \times 4 + 2$$

$$y = 6$$

You can check these answers in equation (2):

$$y = -x + 10$$

or

$$6 = -4 + 10 \quad \checkmark$$

which confirms the solution.

You have to be a bit careful with simultaneous equations, as there two circumstances where you cannot get solutions. In the first, the equations are not independent. Suppose that you have two equations:

$$2x + 3y = 6 \qquad (1)$$

$$6x + 9y = 18 \qquad (2)$$

Multiplying equation (1) by 3 immediately gives equation (2), so there is really only one equation and you cannot find two unknowns. The second problem is a contradiction. Suppose you have:

$$x + y = 7 \qquad (1)$$

$$2x + 2y = 12 \qquad (2)$$

Multiplying equation (1) by 2 gives $2x + 2y = 14$ which contradicts the second equation. Such a contradiction means that there is no feasible solution and you have to assume that there is a mistake in one of the equations.

WORKED EXAMPLE 11.2

What is the solution to the following set of simultaneous equations?

$$x + 2y = 7 \quad (1)$$

$$2x + y = 5 \quad (2)$$

Solution

If you multiply equation (1) by 2, you get the revised equations:

$$2x + 4y = 14 \quad (1)$$

$$2x + y = 5 \quad \text{as before} \quad (2)$$

Subtracting equation (2) from equation (1) gives:

$$3y = 9 \quad \text{or} \quad y = 3$$

You can substitute this into one of the original equations, say (1), to get the value for x:

$$x + 2y = 7$$

so

$$x + 6 = 7 \quad \text{or} \quad x = 1$$

Checking these answers in equation (2):

$$2x + y = 5 \quad \text{or} \quad 2 + 3 = 5 \quad ✓$$

which is correct and confirms the solution.

Graphs of simultaneous equations

With two variables you can draw a graph to show the relationship. Suppose you take the equations from worked example 11.1:

$$3y = 4x + 2 \quad (1)$$

$$y = -x + 10 \quad (2)$$

You can draw a graph of these equations, where each is a straight line, as shown in Figure 11.1.

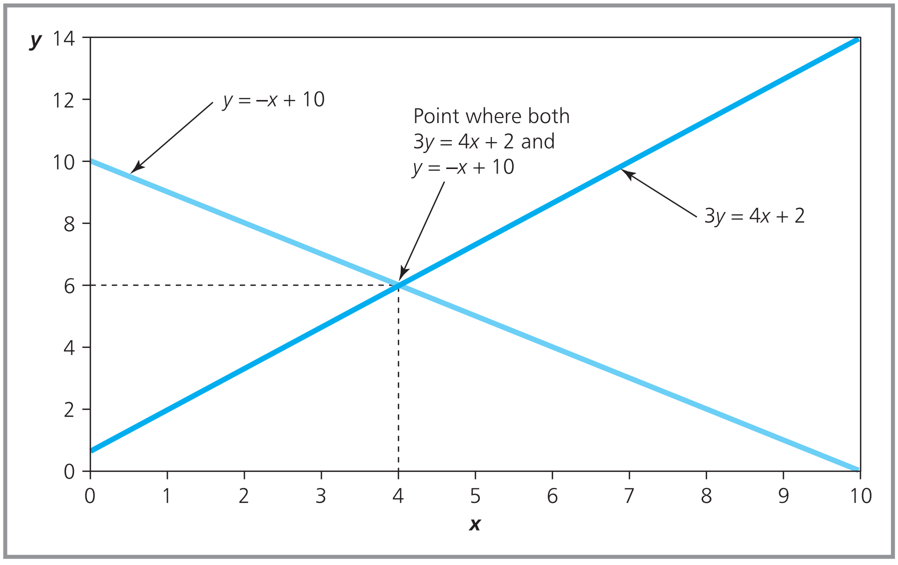

Figure 11.1 Graph to solve simultaneous equations

The first equation is true at any point on one line, and the second equation is true at any point on the second line. It follows that both equations are true at the point where the lines cross each other. You can see from Figure 11.1 that this is about the point where $x = 4$ and $y = 6$, and we already know that this is the solution from the worked example. So you can use a graph to solve simultaneous equations when there are two variables. An extension to the basic method replaces linear equations by more complicated ones, and sees where two curves intersect.

WORKED EXAMPLE 11.3

Use a graph to solve the simultaneous equations

$$y = 2x + 10 \quad (1)$$

$$2y = 5 - x \quad (2)$$

What happens if equation (2) is replaced by the quadratic equation $2x^2 + 3x - 2 = 0$?

Solution

Figure 11.2 shows a graph of these two equations, and you can see that the lines cross at about the point where $x = -3$ and $y = 4$. If you substitute these two values into the equations you get:

$$4 = 2 \times (-3) + 10 \quad ✓ \quad (1)$$

$$8 = 5 - (-3) \quad ✓ \quad (2)$$

Both of these are correct, which confirms the results from the graph.

If the second equation is replaced by the quadratic, you can tackle the problem in exactly the same way. The easiest way to draw the quadratic is to take a range of values, say from $x = -5$ to $x = +4$, and substitute them into $2x^2 + 3x - 2 = 0$ to give:

x	−5	−4	−3	−2	−1	0	1	2	3	4
y	33	18	7	0	−3	−2	3	12	25	42

The straight line of $y = 2x + 10$ crosses the quadratic curve at two points, at about (2, 14) and (−3, 5), as shown in Figure 11.3. At these two points, both equations are true. You can calculate the points more accurately by saying both equations are satisfied when:

$$y = 2x^2 + 3x - 2 \quad \text{and} \quad y = 2x + 10$$

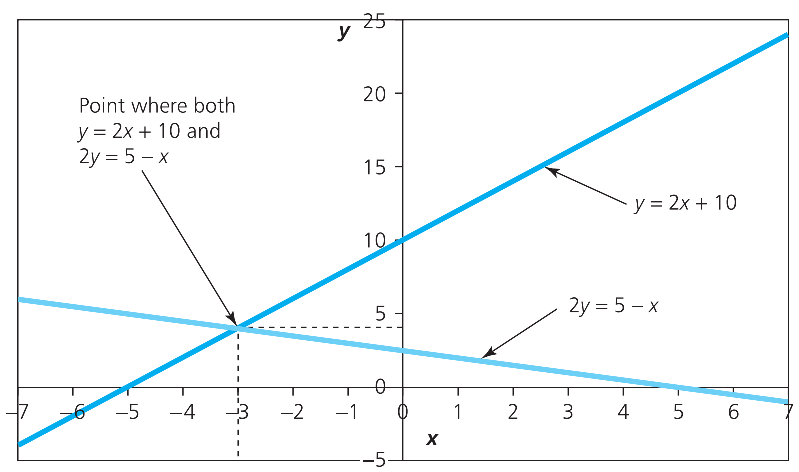

Figure 11.2 Graph of $y = 2x + 10$ and $2y = 5 - x$

Worked example 11.3 continued

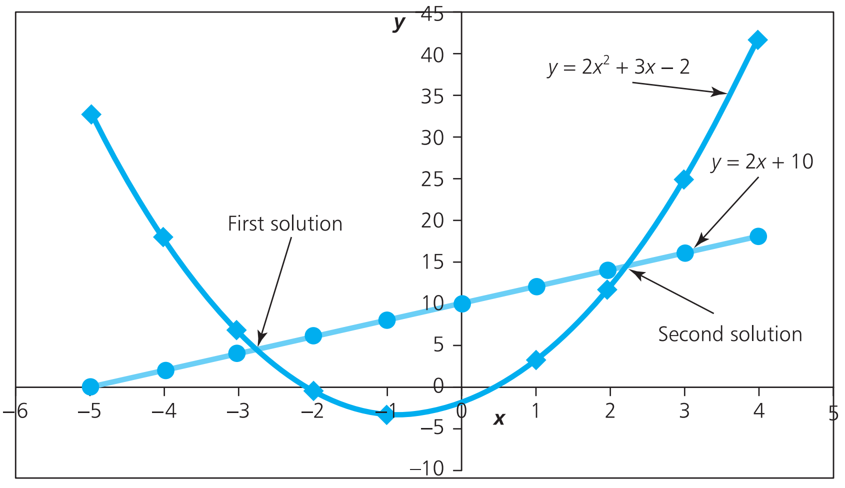

Figure 11.3 Replacing a linear equation by a quadratic

so when both equal y you have:

$$2x^2 + 3x - 2 = 2x + 10$$

and rearranging this gives:

$$2x^2 + x - 12 = 0$$

You can solve this using the standard equation described in Chapter 3, and find the solutions $x = 2.21$ and $x = -2.71$. Substituting these two values for x into one of the original equations, say $y = 2x + 10$, gives the corresponding values for y.

- When $x = 2.21$, $y = 2x + 10 = 2 \times 2.21 + 10 = 14.42$, giving the point (2.21, 14.42), which we estimated at (2, 14).
- When $x = -2.71$, $y = 2x + 10 = 2 \times (-2.71) + 10 = 4.58$, giving the point (−2.71, 4.58), which we estimated at (−3, 5).

Finding more variables

Graphical methods work for only two variables, and really show only patterns and give approximate answers. However, you can use the method of elimination for any number of variables. With, say, three variables, you can manipulate three simultaneous equations to remove one variable, giving two equations with two variables; then you can manipulate these to give one equation with one variable. Again, this sounds complicated, but is easier to see with an example.

WORKED EXAMPLE 11.4

Solve the simultaneous equations:

$$2x + y + 2z = 10 \quad (1)$$

$$x - 2y + 3z = 2 \quad (2)$$

$$-x + y + z = 0 \quad (3)$$

Solution

You can start by using equations (2) and (3) to eliminate the variable x from equation (1):

- Multiplying equation (2) by 2 gives:

$$2x - 4y + 6z = 4$$

Worked example 11.4 continued

Subtracting this from equation (1) gives:

$$5y - 4z = 6 \qquad (4)$$

- Multiplying equation (3) by 2 gives:

$$-2x + 2y + 2z = 0$$

Adding this to equation (1) gives:

$$3y + 4z = 10 \qquad (5)$$

Now you have two equations, (4) and (5), with two unknowns, y and z, and you can solve them as before. Adding equations (4) and (5) gives:

$$8y = 16 \quad \text{or} \quad y = 2$$

Substituting this value for y in equation (4) gives the value for Z:

$$10 - 4z = 6 \quad \text{or} \quad z = 1$$

And substituting these two values for y and z into equation (1) gives the value for x:

$$2x + 2 + 2 = 10 \quad \text{or} \quad x = 3$$

Now you have $x = 3$, $y = 2$ and $z = 1$, and can confirm these values by substituting them in equations (2) and (3):

$$3 - 4 + 3 = 2 \quad ✓ \qquad (2)$$

$$-3 + 2 + 1 = 0 \quad ✓ \qquad (3)$$

To solve simultaneous equations, we have tried straightforward substitution, and found that the manipulation soon gets messy. And the last worked example shows that the arithmetic with elimination also gets messy. Graphs are not very accurate and work with only two variables. So we cannot really use any of the three methods for larger problems. Thankfully, there is an easier format for the calculations, which uses matrices.

Review questions

11.1 What are simultaneous equations?

11.2 How many simultaneous equations would you need to find six unknown variables?

11.3 What does it mean when two graphs cross each other?

11.4 Why is it generally better to use algebraic rather than graphical methods to solve simultaneous equations?

Matrix notation

Imagine that you work for a company which runs two types of shop, wholesale and retail. These shops are in three geographical regions with 20 retail and 2 wholesale shops in the North, 60 retail and 6 wholesale shops in the Centre, and 30 retail and 4 wholesale shops in the South. You can show these figures in the following table.

		Type of shop	
		Retail	Wholesale
Region	North	20	2
	Centre	60	6
	South	30	4

If you want to do calculations with these numbers, it is rather awkward to keep drawing the table – so you can use a simplified format that keeps the same structure as the table, but focuses on the values. This format is a **matrix**, which shows the numbers in the body of the table, enclosed in square brackets, but ignores all other titles and lines. So the matrix of the figures above is written:

$$\mathbf{A} = \begin{bmatrix} 20 & 2 \\ 60 & 4 \\ 30 & 6 \end{bmatrix}$$

The numbers are arranged in rows and columns and each matrix is given an identifying name, which is usually a single, bold, capital letter. The size of matrix is described as:

(number of rows × number of columns)

Our matrix has three rows and two columns, so it is (3 × 2), which is said '3 by 2'. Then you can say that 'The number of shops is described by the (3 × 2) matrix **A**'.

In this notation each number in the matrix is called an **element** and is described by a lower-case letter with a double subscript to show its row and column. Then $a_{i,j}$ is the element in the ith row and jth column of matrix **A**. In the matrix above, $a_{1,2}$ is the element in the first row and second column, which is 2. So:

$$a_{1,1} = 20 \quad a_{1,2} = 2 \quad a_{2,1} = 60 \quad a_{2,2} = 4$$

and so on.

Sometimes a matrix has only one row, and then it is called a **row vector**. Sometimes it has only one column, and then it is called a **column vector**. So when:

$$\mathbf{C} = [5\ 7\ 3\ 1\ 5]$$

C is a row vector, and when:

$$\mathbf{D} = \begin{bmatrix} 5 \\ 4 \\ 1 \\ 6 \\ 4 \end{bmatrix}$$

D is a column vector.

WORKED EXAMPLE 11.5

A company makes three products from four raw materials. It uses 17, 22, 4 and 7 units respectively of materials to make one unit of product 1, it uses 12, 5, 22 and 6 units respectively to make a unit of product 2, and it uses 7, 3, 14 and 8 units respectively to make a unit of product 3. Show this data as a matrix, **M**. What are the values of $m_{2,3}$ and $m_{3,2}$?

Worked example 11.5 continued

Solution

Describing the problem in a table gives:

	Raw material 1	Raw material 2	Raw material 3	Raw material 4
Product 1	17	22	4	7
Product 2	12	5	22	6
Product 3	7	3	14	8

Putting the numbers into a matrix, **M**, gives:

$$\mathbf{M} = \begin{bmatrix} 17 & 22 & 4 & 7 \\ 12 & 5 & 22 & 6 \\ 7 & 3 & 14 & 8 \end{bmatrix}$$

Then $m_{2,3}$ is the element in the second row and third column which is 22, while $m_{3,2}$ is the element in the third row and second column which is 3.

The last worked example showed the products as the rows of the matrix and the raw materials as the columns. But there is no special significance in this and we could equally have drawn the matrix the other way around, giving:

$$\mathbf{M} = \begin{bmatrix} 17 & 12 & 7 \\ 22 & 5 & 3 \\ 4 & 22 & 14 \\ 7 & 6 & 8 \end{bmatrix}$$

This new matrix is described as the **transpose** of the original matrix. The transpose of a matrix **A** is written as $\mathbf{A}^t$ and is defined in the following way:

$$\mathbf{A} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad \text{and} \quad \mathbf{A}^t = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Although the orientation of a matrix is not significant, you must remember that each row and column has a distinct meaning, even though the labels are omitted.

Review questions

11.5 What is the purpose of a matrix?

11.6 Describe the following matrix. What is the value of $f_{1,3}$? What is the value of $f_{3,1}$?

$$\mathbf{F} = \begin{bmatrix} 3 & 1 & 4 \\ 4 & 1 & 1 \\ 6 & 3 & 1 \end{bmatrix}$$

11.7 What is a vector?

Matrix arithmetic

Matrices give a format for describing some problems, and they are useful for doing calculations with tables, but they do not give any fundamentally new methods. We can show this by describing the different types of matrix arithmetic.

Addition and subtraction

The basic arithmetic operations for matrices add or subtract matrices of the same size. You add one matrix to another by adding the corresponding elements.

> $\mathbf{C} = \mathbf{A} + \mathbf{B}$
>
> means that:
>
> $c_{i,j} = a_{i,j} + b_{i,j}$
>
> for all values of i and j.

This means that you can add matrices only if they are of the same size, and then it is clear that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$. Similarly, we can subtract one matrix from another of the same size by subtracting the corresponding elements.

> $\mathbf{C} = \mathbf{A} - \mathbf{B}$
>
> means that:
>
> $c_{i,j} = a_{i,j} - b_{i,j}$
>
> for all values of i and j.

WORKED EXAMPLE 11.6

If $\mathbf{A} = \begin{bmatrix} 6 & 5 \\ 1 & 8 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 3 & 5 \\ 6 & 4 \end{bmatrix}$, what is $\mathbf{A} + \mathbf{B}$? What is $\mathbf{A} - \mathbf{B}$?

Solution

Both **A** and **B** are (2×2) matrices, so you can add and subtract them. In particular, you find the answers by adding and subtracting corresponding elements.

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 6+3 & 5+5 \\ 1+6 & 8+4 \end{bmatrix} = \begin{bmatrix} 9 & 10 \\ 7 & 12 \end{bmatrix}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} 6-3 & 5-5 \\ 1-6 & 8-4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -5 & 4 \end{bmatrix}$$

Two matrices are equal if they are of the same size and each corresponding element is the same in both matrices. Then $\mathbf{A} = \mathbf{B}$ means that $a_{i,j} = b_{i,j}$ for all i and j. This leads to a special result when a matrix is subtracted from another equal matrix. Then $\mathbf{A} - \mathbf{B}$ defines a matrix that is the same size as **A** and **B**, but where every element is equal to zero. This is called a **zero matrix**, which serves the same purpose as a zero in ordinary arithmetic.

WORKED EXAMPLE 11.7

If $\mathbf{D} = \begin{bmatrix} 10 & 7 \\ 12 & 18 \\ 3 & 16 \end{bmatrix}$ and $\mathbf{E} = \mathbf{D}$, what is $e_{2,2}$? What is $\mathbf{D} - \mathbf{E}$?

Solution

$e_{2,2}$ is the element in the second row and second column of **E**. As $\mathbf{E} = \mathbf{D}$, $e_{2,2} = d_{2,2} = 18$. Similarly $e_{3,1} = d_{3,1} = 3$, $e_{1,2} = d_{1,2} = 7$, and so on.

Worked example 11.7 continued

We know that

$$\mathbf{D} = \begin{bmatrix} 10 & 7 \\ 12 & 18 \\ 3 & 16 \end{bmatrix} \text{ and } \mathbf{E} = \begin{bmatrix} 10 & 7 \\ 12 & 18 \\ 3 & 16 \end{bmatrix}$$

so

$$\mathbf{D} - \mathbf{E} = \begin{bmatrix} 10-10 & 7-7 \\ 12-12 & 18-18 \\ 3-3 & 16-16 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

WORKED EXAMPLE 11.8

Herschell Hardware stocks pots and pans in three different sizes. The numbers in stock are shown in the following table.

		Stock	
		Pots	Pans
Size	Large	10	13
	Medium	24	16
	Small	17	9

A delivery of pots and pans arrives, with the numbers shown in matrix **D** (with rows showing sizes and columns showing pots or pans). In the following two weeks the numbers of pots and pans sold are described by the matrices **W1** and **W2**. What are the stocks after the delivery arrives? What stocks remain at the end of the following weeks?

$$\mathbf{D} = \begin{bmatrix} 5 & 20 \\ 12 & 7 \\ 3 & 6 \end{bmatrix} \quad \mathbf{W1} = \begin{bmatrix} 3 & 16 \\ 10 & 15 \\ 4 & 2 \end{bmatrix} \quad \mathbf{W2} = \begin{bmatrix} 12 & 17 \\ 26 & 8 \\ 16 & 13 \end{bmatrix}$$

Solution

We can describe the current stocks by the (3 × 2) matrix **S**:

$$\mathbf{S} = \begin{bmatrix} 10 & 13 \\ 24 & 16 \\ 17 & 9 \end{bmatrix}$$

After the delivery, the stock rises to **S1**, where **S1** = **S** + **D**:

$$\mathbf{S1} = \begin{bmatrix} 10 & 13 \\ 24 & 16 \\ 17 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 20 \\ 12 & 7 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 33 \\ 36 & 23 \\ 20 & 15 \end{bmatrix}$$

After week 1 the stock falls to **S2**, where **S2** = **S1** – **W1**:

$$\mathbf{S2} = \begin{bmatrix} 15 & 33 \\ 36 & 23 \\ 20 & 15 \end{bmatrix} - \begin{bmatrix} 3 & 16 \\ 10 & 15 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 17 \\ 26 & 8 \\ 16 & 13 \end{bmatrix}$$

After week 2 the stock falls to **S3**, where **S3** = **S2** – **W2**:

$$\mathbf{S3} = \begin{bmatrix} 12 & 17 \\ 26 & 8 \\ 16 & 13 \end{bmatrix} - \begin{bmatrix} 12 & 17 \\ 26 & 8 \\ 16 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

All the elements in **S3** are zero, so this is a zero matrix, meaning that there is no stock left.

Multiplication of matrices

You can multiply a matrix by a single number, giving **scalar multiplication** where every element in the matrix is multiplied by the number. Then you can find D = $f \times$ E, where D and E are matrices, f is an ordinary number, and $d_{i,j} = f \times e_{i,j}$.

WORKED EXAMPLE 11.9

Every day Madog, Inc. serves different types of customers, with the average numbers described by the matrix:

$$\mathbf{N} = \begin{bmatrix} 10 & 6 & 20 \\ 4 & 12 & 6 \end{bmatrix}$$

How would you describe the number of customers served each week?

Solution

The number of customers served each week is simply seven times the number served each day, and you find this with scalar multiplication, where, $\mathbf{N1} = 7 \times \mathbf{N}$:

$$\mathbf{N1} = 7 \times \begin{bmatrix} 10 & 6 & 20 \\ 4 & 12 & 6 \end{bmatrix} = \begin{bmatrix} 7 \times 10 & 7 \times 6 & 7 \times 20 \\ 7 \times 4 & 7 \times 12 & 7 \times 6 \end{bmatrix}$$

$$= \begin{bmatrix} 70 & 42 & 140 \\ 28 & 84 & 42 \end{bmatrix}$$

Another type of multiplication is called **matrix multiplication**, where two matrices are multiplied together. Unfortunately, matrix multiplication is a bit more complicated than scalar multiplication. The first thing you have to do is make sure that the matrices are the right size to allow multiplication – and this means that the number of columns in the first matrix must equal the number of rows in the second. In other words, you can only multiply one matrix of size $(r \times c)$ by another matrix of size $(c \times n)$. Both r and n can be any numbers, and the result is an $(r \times n)$ matrix. So you can multiply a (2×4) matrix by a (4×3) matrix and the result is a (2×3) matrix.

The mechanics of matrix multiplication are a bit messy. When $\mathbf{D}$ equals $\mathbf{A} \times \mathbf{B}$ you have to calculate each element $d_{i,j}$ separately, and this uses the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$. To be specific:

$$d_{i,j} = a_{i,1} \times b_{1,j} + a_{i,2} \times b_{2,j} + a_{i,3} \times b_{3,j} + \ldots + a_{i,c} \times b_{c,j}$$

where c is the number of columns in $\mathbf{A}$ and rows in $\mathbf{B}$.

So you start by taking the first element in row i of $\mathbf{A}$ and multiplying it by the first element in column j of $\mathbf{B}$; then you take the second element in row i of $\mathbf{A}$ and multiply this by the second element in column j of $\mathbf{B}$; and you keep on doing this until you reach the end of the row. Then adding the separate numbers together gives the element $d_{i,j}$. But this gives only one element, and you have to repeat the calculation for all rows in matrix $\mathbf{A}$ to form the new column in matrix $\mathbf{D}$. Then you repeat the whole process for all columns in matrix $\mathbf{B}$ to form the remainder of the columns in $\mathbf{D}$. The calculations are:

$$d_{1,1} = a_{1,1} \times b_{1,1} + a_{1,2} \times b_{2,1} + a_{1,3} \times b_{3,1} + \ldots + a_{1,c} \times b_{c,1}$$

$$\begin{aligned}
d_{1,2} &= a_{1,1} \times b_{1,2} + \mathrm{a}_{1,2} \times b_{2,2} + a_{1,3} \times b_{3,2} + \ldots + a_{1,c} \times b_{c,2} \\
d_{1,3} &= a_{1,1} \times b_{1,3} + a_{1,2} \times b_{2,3} + a_{1,3} \times b_{3,3} + \ldots + a_{1,c} \times b_{c,3} \\
&. \quad . \quad . \quad . \quad . \\
d_{2,1} &= a_{2,1} \times b_{1,1} + a_{2,2} \times b_{2,1} + a_{2,3} \times b_{3,1} + \ldots + a_{2,c} \times b_{c,1} \\
d_{3,1} &= a_{3,1} \times b_{1,1} + a_{3,2} \times b_{2,1} + a_{3,3} \times b_{3,1} + \ldots + a_{3,c} \times b_{c,1} \\
&. \quad . \quad . \quad . \quad . \\
d_{2,2} &= a_{2,1} \times b_{1,2} + a_{2,2} \times b_{2,2} + a_{2,3} \times b_{3,2} + \ldots + a_{2,c} \times b_{c,2}
\end{aligned}$$

and so on.

WORKED EXAMPLE 11.10

If $\mathbf{A} = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 5 & 7 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$, what is $\mathbf{A} \times \mathbf{B}$?

Solution

You start by making sure that **A** and **B** are the right size to be multiplied. **A** is (3×2) and **B** is (2×3), so the number of columns in **A** equals the number of rows in **B** and the matrices can be multiplied to give a (3×3) matrix.

Setting $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, you start by calculating $c_{1,1}$:

$$c_{1,1} = a_{1,1} \times b_{1,1} + a_{1,1} \times b_{1,2} = 2 \times 4 + 3 \times 6 = 26$$

Then:

$$c_{1,1} = a_{1,1} \times b_{1,1} + a_{1,2} \times b_{2,1} = 2 \times 4 + 3 \times 6 = 26$$
$$c_{1,2} = a_{2,1} \times b_{1,1} + a_{2,2} \times b_{2,1} = 1 \times 4 + 6 \times 6 = 40$$
$$c_{1,2} = a_{1,1} \times b_{1,2} + a_{1,2} \times b_{2,2} = 2 \times 2 + 3 \times 1 = 7$$

and so on. Repeating these calculations for all values of $c_{i,j}$ gives:

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 1 & 6 \\ 5 & 7 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 3 \\ 6 & 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 3 \times 6 & 2 \times 2 + 3 \times 1 & 2 \times 3 + 3 \times 5 \\ 1 \times 4 + 6 \times 6 & 1 \times 2 + 6 \times 1 & 1 \times 3 + 6 \times 5 \\ 5 \times 4 + 7 \times 6 & 5 \times 2 + 7 \times 1 & 5 \times 3 + 7 \times 5 \end{bmatrix}$$

$$= \begin{bmatrix} 26 & 7 & 21 \\ 40 & 8 & 33 \\ 62 & 17 & 50 \end{bmatrix}$$

Of course, you can save a lot of effort by doing the calculations with a spreadsheet, particularly using a standard function such as Excel's 'MMULT' (as shown in Figure 11.4). The only thing to remember with spreadsheets is that you have to define the result as an 'array formula' – by pressing 'CTRL+SHIFT+ENTER' – or else it displays only the value for $c_{1,1}$.

	A	B	C	D	E	F	G	H	I	J
1	**Matrix multiplication**									
2										
3	**A**			**B**				**C = A × B**		
4	2	3		4	2	3		26	7	21
5	1	6		6	1	5		40	8	33
6	5	7						62	17	50

Figure 11.4 Matrix multiplication using MMULT function

With matrix multiplication it is clear that $\mathbf{A} \times \mathbf{B}$ does not equal $\mathbf{B} \times \mathbf{A}$. If **A** is a (4×2) matrix and **B** is (2×3), you can multiply **A** by **B** to get a (4×3) matrix $\mathbf{A} \times \mathbf{B}$. However, you cannot multiply **B** by **A** as their sizes do not match, meaning that the number of columns in **B** does not equal the number of rows in **A**. Here the value of $\mathbf{B} \times \mathbf{A}$ is not defined, but even when the matrices are of the right size, $\mathbf{A} \times \mathbf{B}$ does not equal $\mathbf{B} \times \mathbf{A}$, as you can see from the following example.

WORKED EXAMPLE 11.11

If $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$, what is $\mathbf{A} \times \mathbf{B}$? What is $\mathbf{B} \times \mathbf{A}$?

Solution

$$\mathbf{A} \times \mathbf{B} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \times 4 + 1 \times 2 & 2 \times 1 + 1 \times 7 \\ 3 \times 4 + 5 \times 2 & 3 \times 1 + 5 \times 7 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 22 & 38 \end{bmatrix}$$

$$\mathbf{B} \times \mathbf{A} = \begin{bmatrix} 4 & 1 \\ 2 & 7 \end{bmatrix} \times \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \times 2 + 1 \times 3 & 4 \times 1 + 1 \times 5 \\ 2 \times 2 + 7 \times 3 & 2 \times 1 + 7 \times 5 \end{bmatrix} = \begin{bmatrix} 11 & 9 \\ 25 & 37 \end{bmatrix}$$

WORKED EXAMPLE 11.12

If $\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & 6 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{C} = [1 \quad 1]$, what are $\mathbf{A} \times \mathbf{B}$ and $\mathbf{C} \times \mathbf{A}$?

Solution

Here

$$\mathbf{A} \times \mathbf{B} = \begin{bmatrix} 2 + 4 + 1 \\ 3 + 5 + 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 14 \end{bmatrix}$$

This is a useful result, as you can see that the multiplication has the effect of adding the rows. So to add the rows of a matrix, you multiply it by a column vector of 1's. Similarly, multiplying a row vector of 1's adds the columns, so:

$$\mathbf{C} \times \mathbf{B} = [1 \quad 1] \times \begin{bmatrix} 2 & 4 & 1 \\ 3 & 5 & 6 \end{bmatrix} = [5 \quad 9 \quad 7]$$

One other matrix that can be useful is an **identity matrix**. This is a matrix with 1's down the principal diagonal – which goes from the top left corner to the bottom right – and all other entries being zero. By definition an identity matrix is square, so the number of rows equals the number of columns, and it serves the same purpose as 1 in ordinary arithmetic. For example:

$$\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$$

When a matrix is multiplied by an identity matrix, **I**, the original matrix is unchanged. So:

$$\mathbf{A} \times \mathbf{I} = \mathbf{A} = \mathbf{I} \times \mathbf{A}$$

We have already mentioned the zero matrix, where all elements are zero. This serves the same purpose as zero in ordinary arithmetic, and when you multiply anything by a zero matrix you always get another zero matrix as the answer.

WORKED EXAMPLE 11.13

Lueng Cheng Hua's Clothing Emporium sells two styles of suit called Standard and Super. Each suit contains a jacket, trousers and a waistcoat. The following table shows the selling price of each part of a suit.

Selling price, $		Part		
		Jacket	Trousers	Waistcoat
Style	Standard	40	60	80
	Super	120	160	200

Three companies, X, Y and Z, place regular orders for their employees. The following table shows the number of suits ordered in one week.

Demand		Style	
		Standard	Super
Company	X	1	8
	Y	4	3
	Z	2	6

Use matrix multiplication to find the income from each company, the amount each company spent, and the total income.

Solution

You find the income from each company by multiplying the number of suits they bought by the prices and adding the results. This takes a bit of thinking about, but you start by putting the information into matrices. You can write the demand as a (3×2) matrix, **D**, and multiply this by the selling price, which must be a (2×3) matrix, **P**.

$$\mathbf{D} = \begin{bmatrix} 1 & 8 \\ 4 & 3 \\ 2 & 6 \end{bmatrix} \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} 40 & 60 & 80 \\ 120 & 160 & 200 \end{bmatrix}$$

When we multiply **D** by **P** you can see that the first element adds the cost of one standard jacket for X at $40 to the cost of eight super jackets at $120, and finds that X pays $1,000 for jackets. The second element in the row adds the cost of one pair of standard trousers for X to the cost of eight pairs of super trousers and finds that X pays $1,340 for trousers. Similarly, the first element in the second row adds the cost of four standard jackets for Y at $40 to the cost of three super jackets at $120, and finds that Y pays $520 for jackets. The second element in the row adds the cost of four pairs of standard trousers for Y to the cost of three pairs of super trousers and finds that Y pays $720 for trousers.

Completing the multiplication shows the amount each company (shown in the rows) spent on each part of the suit (shown in the columns). Then company X spent $1,000 on jackets, $1,340 on trousers and $1,680 on waistcoats; company Y spent $520 on jackets, $720 on trousers and $920 on waistcoats; and company Z spent $800 on jackets, $1,080 on trousers and $1,360 on waistcoats.

$$\begin{bmatrix} 1 & 8 \\ 4 & 3 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} 40 & 60 & 80 \\ 120 & 160 & 200 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 40 + 8 \times 120 & 1 \times 60 + 8 \times 160 & 1 \times 80 + 8 \times 200 \\ 4 \times 40 + 3 \times 120 & 4 \times 60 + 3 \times 160 & 4 \times 80 + 3 \times 200 \\ 2 \times 40 + 6 \times 120 & 2 \times 60 + 6 \times 160 & 2 \times 80 + 6 \times 200 \end{bmatrix}$$

$$= \begin{bmatrix} 1{,}000 & 1{,}340 & 1{,}680 \\ 520 & 720 & 920 \\ 800 & 1{,}080 & 1{,}360 \end{bmatrix}$$

To find the total spent by each company, you have to add the rows, and you do this by multiplying the sales matrix by a column vector consisting of 1's in each of three rows.

$$\begin{bmatrix} 1{,}000 & 1{,}340 & 1{,}680 \\ 520 & 720 & 920 \\ 800 & 1{,}080 & 1{,}360 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4{,}020 \\ 2{,}160 \\ 3{,}240 \end{bmatrix}$$

This shows that company X spent $4,020, Y spent $2,160 and Z spent $3,240. You find the total expenditure by adding these, and you do this by multiplying by a row vector consisting of 1's in each of three columns.

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \times \begin{bmatrix} 4{,}020 \\ 2{,}160 \\ 3{,}240 \end{bmatrix} = 9{,}420$$

The total expenditure is $9,420.

WORKED EXAMPLE 11.14

What happens if company Y doubles its order in the last example?

Solution

You could tackle this in the same way again and adjust the calculation, but it is much easier to do the calculation in a spreadsheet. Figure 11.5 shows the same calculations as in the last worked example, but with company Y now ordering eight standard suits and six super ones. As you can see, M1 gives a breakdown of the costs, M2 gives the expenditure by customer, and M3 gives the total expenditure of $11,580.

	A	B	C	D	E	F	G	H	I	J
1	**Calculations for Lueng Cheng Hua's Clothing Emporium**									
2										
3	Demand, D		Style			Selling price, P		Part		
4			Standard	Super				Jacket	Trousers	Waistcoat
5	Customer	X	1	8		Style	Standard	$ 40	$ 60	$ 80
6		Y	8	6			Super	$ 120	$ 160	$ 200
7		Z	2	6						
8										
9	M1 = D × P		Part				Vector, V1			
10			Jacket	Trousers	Waistcoat		1			
11	Customer	X	$1,000	$1,340	$1,680		1			
12		Y	$1,040	$1,440	$1,840		1			
13		Z	$800	$1,080	$1,360					
14										
15	M2 = M1 × V1						Vector, V2	1	1	1
16	Customer	X	$4,020							
17		Y	$4,320							
18		Z	$3,240							
19										
20	M3 = M2 × V2									
21	$ 11,580									

Figure 11.5 Spreadsheet of calculation for modified example of Lueng Cheng Hua

Matrix inversion

Now we have described matrix addition, subtraction and multiplication, it makes sense to move on to division. Unfortunately, division of matrices is not really defined. Instead we get the equivalent effect by multiplying by an **inverse matrix**. If we have a matrix **A**, its inverse is called $\mathbf{A}^{-1}$, and is defined so that:

$$\mathbf{A}^{-1} \times \mathbf{A} = \mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$$

As you can see, the inverse of a matrix is equivalent to the reciprocal in ordinary arithmetic.

- Matrix division **A/B** is not defined.
- An equivalent result comes from $\mathbf{B}^{-1} \times \mathbf{A}$ where $\mathbf{B}^{-1}$ is the inverse of **B**.

As the identity matrix **I** is square, a matrix has to be square to have an inverse, and then only some actually have inverses. Most matrices do not have inverses.

WORKED EXAMPLE 11.15

How can you confirm that the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}$ is $\mathbf{A}^{-1} = \begin{bmatrix} -1.5 & 1.25 \\ 0.5 & -0.25 \end{bmatrix}$?

Solution

You can do this by multiplying the two together and confirming that $\mathbf{A} \times \mathbf{A}^{-1}$ gives an identity matrix, **I**. Here you can confirm that:

$$\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix} \times \begin{bmatrix} -1.5 & 1.25 \\ 0.5 & -0.25 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \times 1.5 + 5 \times 0.5 & 1 \times 1.25 - 5 \times 0.25 \\ -2 \times 1.5 + 6 \times 0.5 & 2 \times 1.25 - 6 \times 0.25 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we need a way of finding the inverse, but the calculations are so messy that the only real option is to use standard procedures in a spreadsheet, such as ‘MINVERSE’ in Excel.

WORKED EXAMPLE 11.16

Find the inverse of the matrix $\mathbf{A} = \begin{bmatrix} 2 & 6 & 4 \\ 7 & 2 & 4 \\ 1 & 3 & 7 \end{bmatrix}$.

Solution

Figure 11.6 shows the result in a spreadsheet using the ‘MINVERSE’ function. This checks the result, using the ‘MMULT’ function to multiply the original matrix by its inverse. Remember that when you use matrix functions in spreadsheets you always set the results as an array formula by pressing ‘CTRL+SHIFT+ENTER’, or else only the first element is displayed.

	A	B	C	D	E	F	G
1	**Matrix inversion**						
2							
3	**Matrix, A**				**Inverse matrix**		
4	2	6	4		–0.011	0.158	–0.084
5	7	2	4		0.237	–0.053	–0.105
6	1	3	7		–0.100	0.000	0.200
7							
8	**Multiplication**						
9	1	0	0				
10	0	1	0				
11	0	0	1				

Figure 11.6 Illustrating the MINVERSE function

Review questions

11.8 What size is the resulting matrix when a (4×5) matrix is added to a (3×2) matrix?

11.9 What is the difference between scalar and matrix multiplication?

11.10 What size is the resulting matrix when you multiply a (4×5) matrix by a (5×6) matrix?

11.11 How can you add the rows of a matrix using matrix multiplication?

11.12 'Spreadsheets are the easiest way of finding the inverse of any matrix.' Do you think this is true?

Using matrices to solve simultaneous equations

In practice, the main use of matrices is to solve simultaneous equations. For this, you have to get the problem in the right format – which means defining the problem in terms of two known matrices **A** and **B**, which are related to an unknown matrix **X** by the equation:

$$\mathbf{A} \times \mathbf{X} = \mathbf{B}$$

Then you can find the unknown **X**, by multiplying both sides of the equation by the inverse of **A**, giving:

$$\mathbf{A}^{-1} \times \mathbf{A} \times \mathbf{X} = \mathbf{A}^{-1} \times \mathbf{B}$$

But $\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$, an identity matrix, and multiplying anything by an identity matrix leaves it unchanged, so you get:

$$\mathbf{X} = \mathbf{A}^{-1} \times \mathbf{B}$$

WORKED EXAMPLE 11.17

Use matrix inversion to solve the simultaneous equations

$$4x + y = 13$$

$$3x + 2y = 16$$

Solution

You have to start by writing the simultaneous equations in a matrix format. There are three parts to the equations – the unknown variables x and y (which form the matrix **X**), the known numbers on the left-hand side (which form the matrix **A**), and the known numbers on the right-hand side (which form the matrix **B**). So you can write the relationships in a matrix format as:

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

This is in the form we want, $\mathbf{A} \times \mathbf{X} = \mathbf{B}$, with:

$$\mathbf{A} = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 13 \\ 16 \end{bmatrix}$$

Now you find **X** by multiplying both sides of the equation by the inverse of **A**:

$$\mathbf{X} = \mathbf{A}^{-1} \times \mathbf{B}$$

so you need to find the inverse of **A**. This is shown in Figure 11.7 with:

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.6 & 0.8 \end{bmatrix}$$

You can also see in this spreadsheet that doing the multiplication $\mathbf{A}^{-1} \times \mathbf{B}$ gives:

$$\mathbf{X} = \mathbf{A}^{-1} \times \mathbf{B} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Worked example 11.7 continued

So the solution is $x = 2$ and $y = 5$. You can check this by substituting in the original equations.

$4x + y = 13$: $4 \times 2 + 5 = 13$ ✓
$3x + 2y = 16$: $3 \times 2 + 2 \times 5 = 16$ ✓

	A	B	C	D	E	F	G
1	**Solving simultaneous equations**						
2							
3	**Matrix, A**						
4	4	1					
5	3	2					
6							
7	**Inverse of matrix A**			**Matrix, B**		**Multiplication**	**Matrix X**
8	0.4	–0.2		13		2	x
9	–0.6	0.8		16		5	y

Figure 11.7 Using matrices to solve simultaneous equations

WORKED EXAMPLE 11.18

Mocha-to-Go blends two types of beans, American and Brazilian, to make two blends of coffee, Morning and Noon. The Morning blend uses 75% of the available American beans and 10% of the available Brazilian beans. The Noon blend uses 20% of available American beans and 60% of available Brazilian beans.

(a) If Mocha-to-Go buys 200 kg of American beans and 300 kg of Brazilian beans, how much of each blend can it make?
(b) If they want to make 400 kg of Morning blend and 600 kg of Noon blend, what beans should they buy?

Solution

(a) You start by defining A and B as the amounts of American and Brazilian beans that Mocha-to-Go buy, and M and N as the amounts of Morning and Noon blends that they make. Then the equations for production are:

$$0.75A + 0.1B = M$$

$$0.2A + 0.6B = N$$

In matrix notation, this becomes:

$$\begin{bmatrix} 0.75 & 0.1 \\ 0.2 & 0.6 \end{bmatrix} \times \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix}$$

If $A = 200$ and $B = 300$ you can find the values of M and N by multiplication:

$$\begin{bmatrix} 0.75 & 0.1 \\ 0.2 & 0.6 \end{bmatrix} \times \begin{bmatrix} 200 \\ 300 \end{bmatrix} = \begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} 180 \\ 220 \end{bmatrix}$$

The beans allow 180 kg of Morning blend and 220 kg of Noon blend.

(b) When $M = 400$ and $N = 600$ you have:

$$\begin{bmatrix} 0.75 & 0.1 \\ 0.2 & 0.6 \end{bmatrix} \times \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 400 \\ 600 \end{bmatrix}$$

To find the values of A and B you multiply both sides of this equation by the inverse of the blend matrix. You can find this inverse from a spreadsheet as:

$$\text{matrix} = \begin{bmatrix} 0.75 & 0.1 \\ 0.2 & 0.6 \end{bmatrix}$$

$$\text{inverse} = \begin{bmatrix} 1.395 & -0.233 \\ -0.465 & 1.744 \end{bmatrix}$$

Then:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1.395 & -0.233 \\ -0.465 & 1.744 \end{bmatrix} \times \begin{bmatrix} 400 \\ 600 \end{bmatrix} = \begin{bmatrix} 418.2 \\ 860.5 \end{bmatrix}$$

So the blender needs 418.2 kg of American beans and 860.5 kg of Brazilian beans.

WORKED EXAMPLE 11.19

Michigan Canners make four products (A, B, C and D) using four materials (w, x, y and z). Product A consists of 20% w, 30% x, 10% y and 40% z; product B consists of 10% w, 60% x and 30% z; product C consists of 30% w, 10% x, 50% y and 10% z; product D consists of 50% w, 20% x, 10% y and 20% z.

(a) Describe this in a matrix model.
(b) If the company want to make 100 tonnes of A, 50 tonnes of B, 40 tonnes of C and 60 tonnes of D, what materials does it need?
(c) If materials cost \$1,000, \$1,500, \$2,000 and \$1,600 a tonne respectively, what is the cost of the production plan?

Solution

(a) A matrix model of this problem has the form:

$$\mathbf{T} \times \mathbf{P} = \mathbf{M}$$

where: $\mathbf{T}$ = the tonnes of each product made
$\mathbf{P}$ = the proportion of materials in each product
$\mathbf{M}$ = the amounts of each ingredient needed.

We know the data for $\mathbf{P}$, and can define:

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.3 & 0.1 & 0.4 \\ 0.1 & 0.6 & 0.0 & 0.3 \\ 0.3 & 0.1 & 0.5 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.2 \end{bmatrix}$$

This shows for each product (in the rows) the proportion of each ingredient it uses (in the columns). The first row shows the proportion of materials in product A as 0.2 w, 0.3 x, 0.1 y and 0.4 z, and so on. It follows that the rows should add to 1.

(b) Now we have the amounts to be made in the form:

$$\mathbf{T} = [100\ 50\ 40\ 60]$$

The materials needed are $\mathbf{M} = \mathbf{T} \times \mathbf{P}$. Notice that we have been careful to define $\mathbf{T}$ as a row vector so that we can do the multiplication $\mathbf{T} \times \mathbf{P}$.

$$\mathbf{T} \times \mathbf{P} = [100 \quad 50 \quad 40 \quad 60] \times \begin{bmatrix} 0.2 & 0.3 & 0.1 & 0.4 \\ 0.1 & 0.6 & 0.0 & 0.3 \\ 0.3 & 0.1 & 0.5 & 0.1 \\ 0.5 & 0.2 & 0.1 & 0.2 \end{bmatrix} = [67 \quad 76 \quad 36 \quad 71]$$

The production plan needs 67 tonnes of w, 76 tonnes of x, 36 tonnes of y and 71 tonnes of z.

(c) Putting the costs into a matrix $\mathbf{C}$ gives the total cost as $\mathbf{M} \times \mathbf{C}$. Again we have been careful to define $\mathbf{C}$ as a column vector so that we can do the multiplication:

$$\mathbf{M} \times \mathbf{C} = [67 \quad 76 \quad 36 \quad 71] \times \begin{bmatrix} 1{,}000 \\ 1{,}500 \\ 2{,}000 \\ 1{,}600 \end{bmatrix} = [366{,}600]$$

The cost of materials is \$366,600.

Review questions

11.13 What is the main use of matrices?

11.14 'Matrices give an entirely new way of solving problems.' Do you think this is true?

IDEAS IN PRACTICE Economic input–output models

In the 1940s Leontief[1,2] used matrices to describe the inputs and outputs of industrial sectors. If you consider a particular sector, say the oil industry, it takes some inputs from initial suppliers (notably crude oil), some internal inputs from the oil industry, and other inputs from other industry sectors (such as finance, communications, transport, etc.). Then it sells products either to other industrial sectors, or internally to the oil sector, or to final customers. Now for any sector, the total outputs must equal the

Ideas in practice continued

total inputs. So if we take a simplified economy with only four industry sectors, and inputs and outputs measured in some consistent units, we might find that:

		Inputs to:					
		Sector A	Sector B	Sector C	Sector D	Final customers	Total output
Outputs from:	Sector A	30	20	50	40	60	200
	Sector B	10	60	120	30	80	300
	Sector C	40	60	20	30	100	250
	Sector D	20	10	30	60	80	200
	Initial suppliers	100	150	30	40		320
	Total inputs	200	300	250	200	320	1,270

If you look at Sector A, it produces 200 units of output, and for this it needs inputs of 30, 10, 40 and 20 units respectively from Sectors A, B, C and D, which account for proportions 0.15, 0.05, 0.2 and 0.1 respectively of inputs. Doing the same calculations for the other sectors gives a matrix of 'technical coefficients':

$$\mathbf{T} = \begin{bmatrix} 0.15 & 0.067 & 0.2 & 0.2 \\ 0.05 & 0.2 & 0.48 & 0.15 \\ 0.2 & 0.2 & 0.08 & 0.15 \\ 0.1 & 0.033 & 0.12 & 0.3 \end{bmatrix}$$

If you multiply this matrix by a column vector **S** that contains the outputs from each industry sector, you can see that $\mathbf{T} \times \mathbf{S}$ describes the outputs needed from each sector to support all industry sectors. Now if you add the demand from final customers, **D**, you get the total outputs required:

$$\mathbf{T} \times \mathbf{S} + \mathbf{D} = \mathbf{S}$$

i.e.

$$\begin{bmatrix} 0.15 & 0.067 & 0.2 & 0.2 \\ 0.05 & 0.2 & 0.48 & 0.15 \\ 0.2 & 0.2 & 0.08 & 0.15 \\ 0.1 & 0.033 & 0.12 & 0.3 \end{bmatrix} \times \begin{bmatrix} 200 \\ 300 \\ 250 \\ 200 \end{bmatrix} + \begin{bmatrix} 60 \\ 80 \\ 100 \\ 80 \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 250 \\ 200 \end{bmatrix}$$

So economists can set any pattern of final demand in **D**, assume that the matrix **T** remains constant for any final demand pattern, and calculate the outputs – and hence the capacity – needed for each industrial sector.

CHAPTER REVIEW

This chapter described the use of simultaneous equations and the use of matrices.

- You can solve an equation to find the value of a single unknown variable, but to find values for several variables you have to solve a set of simultaneous equations. These are independent equations relating the variables. To find n unknown variables you need n independent, simultaneous equations.
- You can solve simultaneous equations algebraically using either substitution or a process of elimination. You can also use graphs, but the results are not very accurate and work only with two variables.
- Matrices give a convenient format for describing certain types of problems and organising calculations – but they do not give any new means of solution.
- You can add or subtract matrices of the same size by adding or subtracting each element. Scalar multiplication multiplies all elements in a matrix by a

constant. With matrix multiplication a matrix **A** of size $(r \times c)$ is multiplied by a matrix **B** of size $(c \times n)$ to give a matrix of size $(r \times n)$.

- Division of matrices is defined only in terms of multiplication by an inverse. An inverse is defined only for some square matrices.
- The main use of matrices is to solve sets of simultaneous equations.

CASE STUDY Northern Feedstuffs

Northern Feedstuffs make a range of different feeds to supplement the diet of farm animals. In one area they make six products for horses (labelled A to F) using six ingredients (labelled g to l). The following table shows the composition of each product.

	g	h	i	j	k	l
A	20%	20%	10%	5%	20%	25%
B	10%	20%	15%	15%	15%	25%
C	0%	15%	20%	20%	25%	20%
D	10%	10%	25%	20%	15%	20%
E	15%	10%	20%	25%	15%	15%
F	15%	5%	25%	30%	15%	10%

Johann Sanderson is the Production Manager in charge of this area at Northern. Two years ago he set up a spreadsheet that uses matrix arithmetic to help with planning. Next month Northern want to make 200, 100, 80, 120, 80 and 150 tonnes respectively of each product. Johann can use his matrices to find the ingredients to buy, and he can simply update the figures the following month when they want to make 240, 120, 60, 100, 150 and 180 tonnes respectively of each product. By adding the costs of ingredients Johann can also see how much he will spend on raw materials (at the moment the prices of ingredients are €500, €300, €600, €400, €200 and €300 a tonne respectively).

Another part of Johann's spreadsheet allows him to change production plans at short notice. Last month, for example, bad weather affected the harvest in Canada and there were delays in getting ingredients. At the beginning of the month Johann estimated that he would get deliveries of only 120, 160, 100, 90, 130 and 150 tonnes respectively of each ingredient. He was able to calculate the products he could make and the total weight he would have available for customers.

Question

- **Northern want to extend the use of Johann's system into other areas of the company, and they want you to write a report showing how the system works. They are particularly interested in ways of extending the system to more complicated products and different types of analysis.**

PROBLEMS

11.1 Solve the following simultaneous equations:
(a) $a + b = 3$ and $a - b = 5$
(b) $2x + 3y = 27$ and $3x + 2y = 23$
(c) $2x + 2y + 4z = 24$ and $6x + 3y = 15$ and $y + 2z = 11$
(d) $x + y - 2z = -2$ and $2x - y + z = 9$ and $x + 3y + 2z = 4$
(e) $4r - 2s + 3t = 12$ and $r + 2s + t = -1$ and $3r - s - t = -5$

11.2 Sven Hendriksson finds that one of his productivity measures is related to the number of employees, e, and the production, n, by the equations:

$$10n + 3e = 45 \quad \text{and} \quad 2n + 5e = 31$$

What are the current values for e and n?

11.3 Draw graphs to solve the simultaneous equations in problem 11.1.

11.4 Where does the line $y = 20x + 15$ cross the line $y = 2x^2 - 4x + 1$?

11.5 Where does the line $y = e^{2x}$ cross the line $y = x^2 + 10$?

11.6 Three matrices are:

$$\mathbf{A} = \begin{bmatrix} 10 & 4 \\ 3 & 2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 6 & 10 \\ 7 & 4 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & 7 \\ 1 & 5 \end{bmatrix}$$

What are **A** + **B**, **B** + **A**, **A** − **B**, **B** − **A**? Find values for **A** + **B** + **C**, **A** + **B** − **C**, **B** + **C** − **A**, **C** − **B** − **A**, **C** + **B** + **A**.

11.7 Using the matrices defined in problem 11.6, what are **A** × **B** and **B** × **A**? What are **A** × **B** × **C** and **C** × **B** × **A**?

11.8 Use matrices to solve the simultaneous equations:

$$x + y - z = 4$$
$$2x + 3y + z = 13$$
$$3x - y + 2z = 9$$

11.9 Solve the simultaneous equations:

$$2a + b - c - d = 20$$
$$a + b + c + d = 20$$
$$3a - 4b - 2c + d = 3$$
$$a - b + c - d = 8$$

11.10 Indira used her last year's annual bonus to buy some shares. She bought 100 shares in company A, 200 shares in company B, 200 shares in company C and 300 shares in company D. The costs of a share in each company were €1.20, €3.15, €0.95 and €2.45 respectively. Use matrix arithmetic to describe the purchases.

11.11 A company makes three products A, B and C. To make each unit of A takes 3 units of component X and 2 units of component Y. To make each unit of B and C takes 5 and 7 units respectively of X, and 4 and 6 units respectively of Y. The company receives an order for 5 units of A, 10 of B and 4 of C. Use matrix arithmetic to find the components needed for this order. If each unit of X costs £10 and each unit of B costs £20, use matrices to find the total cost of the order.

11.12 What are the inverses of the matrices $\begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}$, $\begin{bmatrix} 7 & 4 \\ 5 & 3 \end{bmatrix}$ and $\begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$?

RESEARCH PROJECTS

11.1 An obvious problem with matrices is that it is difficult to follow exactly what is happening. You have to be very careful to use the matrices properly, so that they do the right arithmetic. Explore this problem by considering the detailed calculations in the following example.

A company makes four products (A, B, C and D) using four ingredients (w, x, y and z). Product A consists of 10% w, 20% x, 30% y and 40% z; product B consists of 40% w, 20% x, 10% y and 30% z; product C consists of 10% w, 20% x, 20% y and 50% z; product D consists of 20% w, 20% x, 10% y and 50% z. If the company wants to make 150 tonnes of A, 100 tonnes of B, 80 tonnes of C and 120 tonnes of D, what ingredients should it buy? If ingredients cost €100, €150, €200 and €160 a tonne respectively, what is the cost for the production plan? If the company gets a delivery of 180 tonnes of w, 180 tonnes of x, 250 tonnes of y and 200 tonnes of z, what products can it make?

This is a typical problem that can be tackled by matrices. Design a generalised spreadsheet for solving such problems.

11.2 Design a spreadsheet for doing routine matrix calculations. Use it to find the inverse of the following matrix, and check your results.

$$\begin{bmatrix} 12 & 25 & 12 & 3 & 45 & 15 & 41 \\ 52 & 54 & 33 & 16 & 16 & 18 & 15 \\ 45 & 2 & 51 & 14 & 37 & 18 & 24 \\ 65 & 9 & 57 & 16 & 23 & 33 & 50 \\ 21 & 24 & 24 & 26 & 15 & 47 & 15 \\ 9 & 11 & 24 & 5 & 8 & 52 & 14 \\ 22 & 6 & 7 & 27 & 19 & 48 & 32 \end{bmatrix}$$

Sources of information

References

1 Leontief W., *Structure of the American Economy 1919–1939*, Oxford University Press, Oxford, 1951.

2 Leontief W., *Input–output Economics*, Oxford University Press, Oxford, 1966.

Further reading

There are very few books specifically on matrices, but general books on mathematics – including those listed in Chapter 2 – often cover relevant material.

CHAPTER 12

Planning with linear programming

Contents

Chapter outline

Managers often face problems of allocating scarce resources in the best possible way to achieve their objectives. As there are constraints on the options available, these problems are described as 'constrained optimisation'. This chapter describes linear programming, which is a widely used method of solving problems of constrained optimisation. It builds on the ideas of simultaneous equations discussed in the last chapter, with 'programming' used in its broad sense of planning.

After finishing this chapter you should be able to:

- Appreciate the concept of constrained optimisation
- Describe the stages in solving a linear programme
- Formulate linear programmes and understand the assumptions
- Use graphs to solve linear programmes with two variables
- Calculate marginal values for resources
- Calculate the effect of changing an objective function
- Interpret printouts from computer packages.

Constrained optimisation

Managers search for the best possible solutions to problems, but there are inevitably constraints on their options. These constraints are set by the resources available – so an operations manager wants to maximise production, but has limited facilities; a marketing manager wants to maximise the impact of an advertising campaign, but cannot exceed a specified budget; a finance manager wants to maximise returns, but has limited funds; a construction manager wants to minimise the cost of a project, but has to finish within a specified time.

These problems are characterised by:

- an aim of optimising – that is either maximising or minimising – some objective
- a set of constraints that limit the possible solutions.

For this reason they are called problems of **constrained optimisation**.

Linear programming (LP) is a method of solving certain problems of constrained optimisation. We should say straight away that linear programming has nothing to do with computer programming, but its name comes from the more general meaning of planning.

> There are three distinct stages to solving a linear programme:
>
> - **formulation** – getting the problem in the right form
> - **solution** – finding an optimal solution to the problem
> - **sensitivity analysis** – seeing what happens when the problem is changed slightly.

Formulation is often the most difficult part of linear programming. It needs an accurate description of the problem and a lot of data – and the resulting model can be very bulky. But when a problem is in the right form, getting a solution can be relatively straightforward – because it needs a lot of repetitive calculation that is *always* done by computer. There are far too many calculations to consider doing them by hand. So we are going to demonstrate the principles with simple examples, showing how to approach a formulation, how computers get optimal solutions, and what the sensitivity analysis does.

Review questions

12.1 What is constrained optimisation?

12.2 What is linear programming?

Formulation

The first stage of solving a linear programme is to describe the problem in a standard format. It is easiest to illustrate this formulation with an example, and for this we use a problem from production planning.

Suppose a small factory makes two types of liquid fertiliser, Growbig and Thrive. It makes these by similar processes, using the same equipment for blending raw materials, distilling the mix and finishing (bottling, testing, weighing, etc.). Because the factory has a limited amount of equipment, there are constraints on the total time available for each process. In particular, there are only 40 hours of blending available in a week, 40 hours of distilling and 25 hours of finishing. We assume that these are the only constraints and there are none on, say, sales or availability of raw materials.

The fertilisers are made in batches, and each batch needs the following hours on each process.

	Growbig	Thrive
Blending	1	2
Distilling	2	1
Finishing	1	1

If the factory makes a net profit of €300 on each batch of Growbig and €200 on each batch of Thrive, how many batches of each should it make in a week?

This problem is clearly one of optimising an objective (maximising profit) subject to constraints (production capacity), as shown in Figure 12.1. The variables that the company can control are the number of batches of Growbig and Thrive they make, so these are the **decision variables**. We can define these as:

- G is the number of batches of Growbig made in a week
- T is the number of batches of Thrive made in a week.

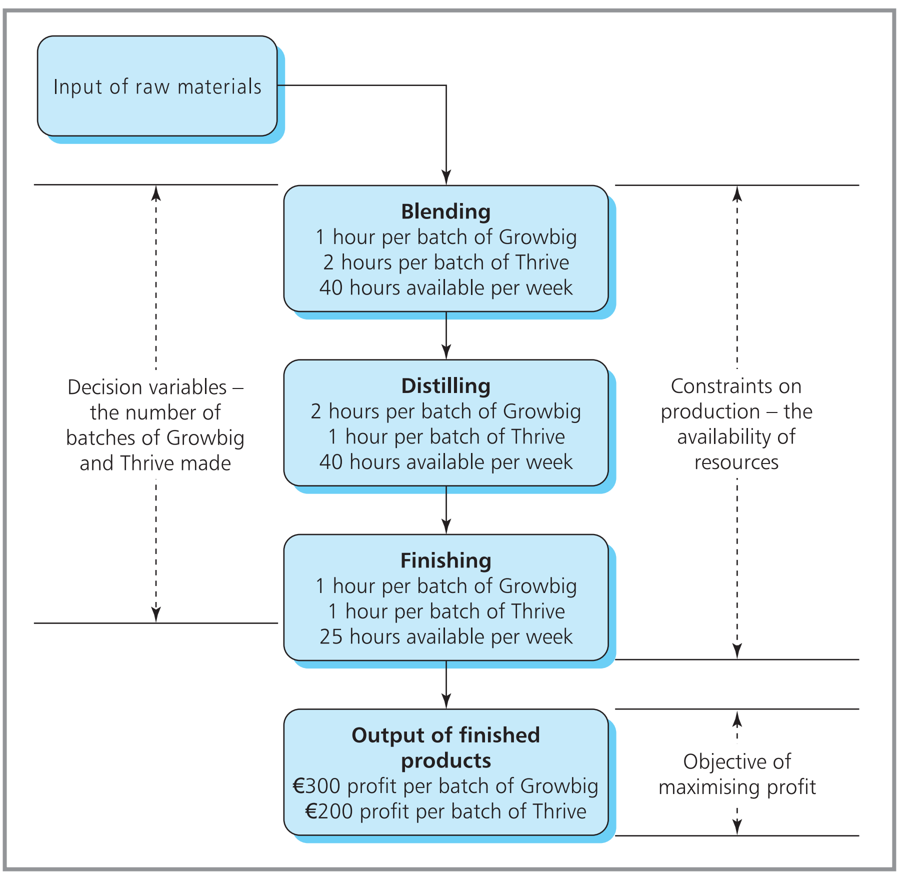

Figure 12.1 Production problem for Growbig and Thrive

Now consider the time available for blending. Each batch of Growbig uses one hour of blending, so G batches use G hours; each batch of Thrive uses two hours of blending, so T batches use $2T$ hours. Adding these together gives the total amount of blending used as $G + 2T$. The maximum amount of blending available is 40 hours, so the time used must be less than, or at worst equal to, this. So this gives the first constraint:

$G + 2T \leq 40$ (blending constraint)

(Remember that ≤ means 'less than or equal to'.)

Turning to the distilling constraint, each batch of Growbig uses two hours of distilling, so G batches use $2G$ hours; each batch of Thrive uses one hour of distilling, so T batches use T hours. Adding these together gives the total amount of distilling used and this must be less than, or at worst equal to, the amount of distilling available (40 hours). So this gives the second constraint:

$2G + T \leq 40$ (distilling constraint)

Now the finishing constraint has the total time used for finishing (G for batches of Growbig plus T for batches of Thrive) less than or equal to the time available (25 hours) to give:

$G + T \leq 25$ (finishing constraint)

These are the three constraints for the process – but there is another implicit constraint. The company cannot make a negative number of batches, so both G and T are positive. This **non-negativity constraint** is a standard feature of linear programmes.

$G \geq 0$ and $T \geq 0$ (non-negativity constraints)

Here the three problem constraints are all 'less than or equal to', but they can be of any type – less than, less than or equal to, equal to, greater than or equal to, or greater than.

Now we can turn to the objective, which is maximising the profit. The company makes €300 on each batch of Growbig, so with G batches the profit is $300G$; they make €200 on each batch of Thrive, so with T batches the profit is $200T$. Adding these gives the total profit that is to be maximised – this is the **objective function**.

Maximise $300G + 200T$ (objective function)

This objective is phrased in terms of maximising an objective. The alternative for LPs is to minimise an objective (typically phrased in terms of minimising costs).

This completes the linear programming formulation which we can summarise as:

Maximise:

$300G + 200T$ objective function

subject to:

$$\left.\begin{array}{l} G + 2T \leq 40 \\ 2G + T \leq 40 \\ G + T \leq 25 \end{array}\right\} \text{constraints}$$

with

$G \geq 0$ and $T \geq 0$ non-negativity constraints

This illustrates the features of all linear programming formulations, which consist of:

- decision variables
- an objective function
- a set of constraints
- a non-negativity constraint.

This formulation makes a number of assumptions that are implicit in all LPs. Most importantly, the objective function and constraints are all linear functions of the decision variables. This means that the use of resources is proportional to the quantity being produced and if, say, production is doubled, the use of resources is also doubled. This is usually a reasonable assumption, but it is not always true. For example, increasing production may give longer production runs that reduce set-up times and running-in problems. On the other hand, higher production may mean faster throughput with more units having faults and being scrapped.

A second assumption is that adding the resources used for each product gives the total amount of resources used. Again, this is not always true. For instance, a craft manufacturer will use the most skilled craftsmen for the most complex jobs – but if there are no complex jobs in one period, the skilled craftsmen do less complex jobs, and they do them better or faster than usual.

WORKED EXAMPLE 12.1

A political campaign wants to hire photocopiers to make leaflets for a local election. There are two suitable machines:

- ACTO costs £120 a month to rent, occupies 2.5 square metres of floor space and can produce 15,000 copies a day.
- ZENMAT costs £150 a month to rent, occupies 1.8 square metres of floor space and can produce 18,500 copies a day.

The campaign has allowed up to £1,200 a month for copying machines which will be put in a room of 19.2 square metres. Formulate this problem as a linear programme.

Solution

The problem variables are the things we can vary, which are the number of ACTO and ZENMAT machines rented. Let

- A be the number of ACTO machines rented
- Z be the number of ZENMAT machines rented.

The objective is to make as many copies as possible.

Maximise $15{,}000A + 18{,}500Z$ (objective function)

There are constraints on floor space and costs:

$120A + 150Z \le 1{,}200$ (cost constraint)

$2.5A + 1.8Z \le 19.2$ (space constraint)

with

$A \ge 0$ and $Z \ge 0$ (non-negativity constraint)

WORKED EXAMPLE 12.2

Foreshore Investment Trust has £1 million to invest. After consulting its financial advisers it considers six possible investments with the following characteristics.

Investment	% Risk	% Dividend	% Growth	Rating
1	18	4	22	4
2	6	5	7	10
3	10	9	12	2
4	4	7	8	10
5	12	6	15	4
6	8	8	8	6

The trust wants to invest the £1 million with minimum risk, but with a dividend of at least £70,000 a year, average growth of at least 12% and average rating of at least 7. Formulate this as a linear programme.

Solution

The decision variables are the amount of money put into each investment.

- Let X_1 be the amount of money put into investment 1.
- Let X_2 be the amount of money put into investment 2.

and so on, so that X_i is the amount of money put into investment i.

The objective is to minimise risk.

$$\text{Minimise} \quad 0.18X_1 + 0.06X_2 + 0.10X_3 + 0.04X_4 + 0.12X_5 + 0.08X_6$$

There are constraints on the amount of:

- money, as the total invested must equal £1 million:

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 = 1{,}000{,}000$$

- dividend, which must be at least 7% of £1 million:

$$0.04X_1 + 0.05X_2 + 0.09X_3 + 0.07X_4 + 0.06X_5 + 0.08X_6 \geq 70{,}000$$

- average growth, which must be at least 12% of £1 million:

$$0.22X_1 + 0.07X_2 + 0.12X_3 + 0.08X_4 + 0.15X_5 + 0.08X_6 \geq 120{,}000$$

- rating, where the average (weighted by the amount invested) must be at least 7:

$$4X_1 + 10X_2 + 2X_3 + 10X_4 + 4X_5 + 6X_6 \geq 7{,}000{,}000$$

The non-negativity constraints X_1, X_2, X_3, X_4, X_5 and $X_6 \geq 0$ complete the formulation.

WORKED EXAMPLE 12.3

StatFunt Oil makes two blends of fuel by mixing three oils. The costs and daily availability of the oils are:

Oil	Cost (€/litre)	Amount available (litres)
A	2.5	10,000
B	2.8	15,000
C	3.5	20,000

The blends of fuel contain:

Blend 1	at most 25% of A
	at least 30% of B
	at most 40% of C
Blend 2	at least 20% of A
	at most 50% of B
	at least 30% of C

Worked example 12.3 continued

Each litre of Blend 1 sells for €6 and each litre of Blend 2 sells for €7. Long-term contracts mean that at least 10,000 litres of each blend must be produced. The company has to decide the best mixture of oils for each blend. Formulate this as a linear programme.

Solution

The decision variables are the amount of each type of oil that the company puts into each blend:

- Let A_1 be the amount of oil A put into Blend 1.
- Let A_2 be the amount of oil A put into Blend 2.
- Let B_1 be the amount of oil B put into Blend 1, etc.

The total amounts of Blend 1 and Blend 2 produced are:

Blend 1: $A_1 + B_1 + C_1$
Blend 2: $A_2 + B_2 + C_2$

and the amounts of each oil used are:

oil A: $A_1 + A_2$
oil B: $B_1 + B_2$
oil C: $C_1 + C_2$

The objective is to maximise profit. We know that the income from selling blends is:

$$6 \times (A_1 + B_1 + C_1) + 7 \times (A_2 + B_2 + C_2)$$

while the cost of buying oil is:

$$2.5 \times (A_1 + A_2) + 2.8 \times (B_1 + B_2) + 3.5 \times (C_1 + C_2)$$

The profit is the difference between the income and the cost:

$$6A_1 + 6B_1 + 6C_1 + 7A_2 + 7B_2 + 7C_2 - 2.5A_1 - 2.5A_2 - 2.8B_1 - 2.8B_2 - 3.5C_1 - 3.5C_2$$

which we can rearrange to give the objective function:

$$\text{Maximise} \quad 0.35A_1 + 0.45A_2 + 0.32B_1 + 0.42B_2 + 0.25C_1 + 0.35C_2$$

There are constraints on the availability of oils:

$$A_1 + A_2 \leq 10{,}000$$
$$B_1 + B_2 \leq 15{,}000$$
$$C_1 + C_2 \leq 20{,}000$$

There are also six blending constraints. The first of these says that Blend 1 must be at most 25% of oil A. In other words:

$$A_1 \leq 0.25 \times (A_1 + B_1 + C_1) \quad \text{or}$$
$$0.75A_1 - 0.25B_1 - 0.25C_1 \leq 0$$

Similarly for the other blends:

$$B_1 \geq 0.3 \times (A_1 + B_1 + C_1) \quad \text{or}$$
$$0.3A_1 - 0.7B_1 + 0.3C_1 \leq 0$$

$$C_1 \leq 0.4 \times (A_1 + B_1 + C_1) \quad \text{or}$$
$$-0.4A_1 - 0.4B_1 + 0.6C_1 \leq 0$$

$$A_2 \geq 0.2 \times (A_2 + B_2 + C_2) \quad \text{or}$$
$$-0.8A_2 + 0.2B_2 + 0.2C_2 \leq 0$$

$$B_2 \leq 0.5 \times (A_2 + B_2 + C_2) \quad \text{or}$$
$$-0.5A_2 + 0.5B_2 - 0.5C_2 \leq 0$$

$$C_2 \geq 0.3 \times (A_2 + B_2 + C_2) \quad \text{or}$$
$$0.3A_2 + 0.3B_2 - 0.7C_2 \leq 0$$

Long-term contracts add the conditions that:

$$A_1 + B_1 + C_1 \geq 10{,}000$$
$$A_2 + B_2 + C_2 \geq 10{,}000$$

The non-negativity conditions that all variables, A_1, A_2, B_1, etc., are greater than or equal to 0 completes the formulation.

Review questions

12.3 What are the main assumptions of linear programming?

12.4 What happens when you formulate a linear programme?

12.5 What are the parts of an LP formulation?

IDEAS IN PRACTICE Argentia Life Assurance

In 2004 Argentia Life Assurance found that their reserves were not quite big enough to cover the life insurance policies they had issued. So they decided to transfer $20 million into their life policy reserves. The company wanted to maximise its returns on this new investment, but there were regulations and guidelines on the types of investment they could make. This was clearly a problem of constrained optimisation, and the company used linear programming to suggest the best investment.

The main constraint was on the amount invested. Other constraints concerned the types of investment available – or that Argentia wanted to use. They would not make any investment that had significant risk, so their choice was limited to government bonds, shares in blue-chip companies, property, mortgages and some unclassified investments. Guidelines prevented them from buying shares with a value greater than 25% of the total assets of the company. Similarly, property was limited to 15%, and unclassified investments to 2% of the total assets of the company. Other constraints limited the maximum amounts in each investment, spread of risk, and so on.

The final formulation had over 1,000 variables and 12,000 constraints. The solution gave the best options at a particular time. But financial markets change very quickly, and the data used in the model had to be updated frequently.

Source: Scarborough J., *Investment Policies*, Argentia Corp., New York, 2006.

Using graphs to solve linear programmes

In reality it needs many repetitive calculations to solve a linear programme, and these are always done on a computer. But we can illustrate the principles with a simple example, and for this we return to the previous Growbig and Thrive example.

Maximise:

$300G + 200T$ (objective function)

subject to:

$G + 2T \leq 40$ (blending constraint)

$2G + T \leq 40$ (distilling constraint)

$G + T \leq 25$ (finishing constraint)

and

$G \geq 0$ and $T \geq 0$ (non-negativity constraints)

Take the blending constraint, which is $G + 2T \leq 40$. We can draw the equation $G + 2T = 40$ as a straight line on a graph of G against T as shown in Figure 12.2. (If you are not sure about this, have another look at Chapter 3.) Remember that the easiest way to draw lines is to take two convenient points and draw a straight line through them. Here setting $G = 0$ gives $2T = 40$ or $T = 20$; and setting $T = 0$ gives $G = 40$. Then we can draw the line of the equation through the points (0, 20) and (40, 0). The non-negativity constraints mean that we have to consider this only in the positive quadrant – where both G and T are positive.

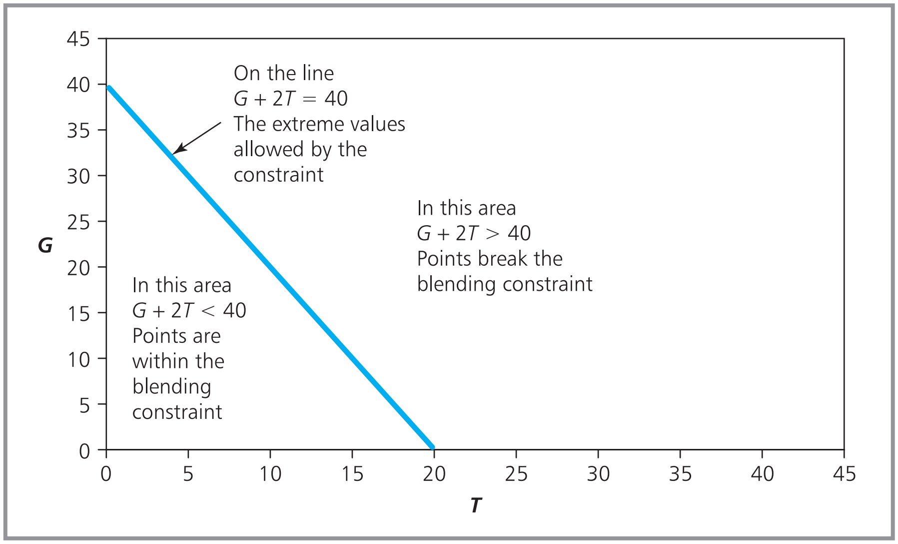

Figure 12.2 Graph of the blending constraint

The important point is that any point above this line breaks the blending constraint, while any point on or below the line does not break the constraint. You can check this by taking any points at random. For instance, the point $G = 10$, $T = 10$ is below the line and substituting into the constraint gives:

$$1 \times 10 + 2 \times 10 \leq 40 \quad ✓$$

which is true and the constraint is not broken. On the other hand, the point $G = 20$, $T = 20$ is above the line and substitution gives:

$$1 \times 20 + 2 \times 20 \leq 40 \quad ✗$$

which is not true and the constraint is broken. Points that are actually on the line satisfy the equality. For example, the point $G = 20$, $T = 10$ is on the line and substitution gives:

$$1 \times 20 + 2 \times 10 \leq 40 \quad ✓$$

which is true and shows the extreme values allowed by the constraint. So the line divides the graph into two areas: all points above the line break the constraint, while all points on or below the line do not break the constraint.

We can add the other two constraints in the same way (shown in Figure 12.3). The distilling constraint ($2G + T \leq 40$) is the straight line through $G = 20$, $T = 0$ and $G = 0$, $T = 40$. As before, any point above the line breaks the constraint, while any point on or below the line does not break the constraint. The finishing constraint ($G + T \leq 25$) is the straight line through the points $G = 0$, $T = 25$ and $G = 25$, $T = 0$, and again any point above the line breaks the constraint, while any point on or below the line does not break the constraint.

Any point that is below *all three* of the constraint lines represents a valid, feasible solution – but a point that is above *any* of the lines breaks at least

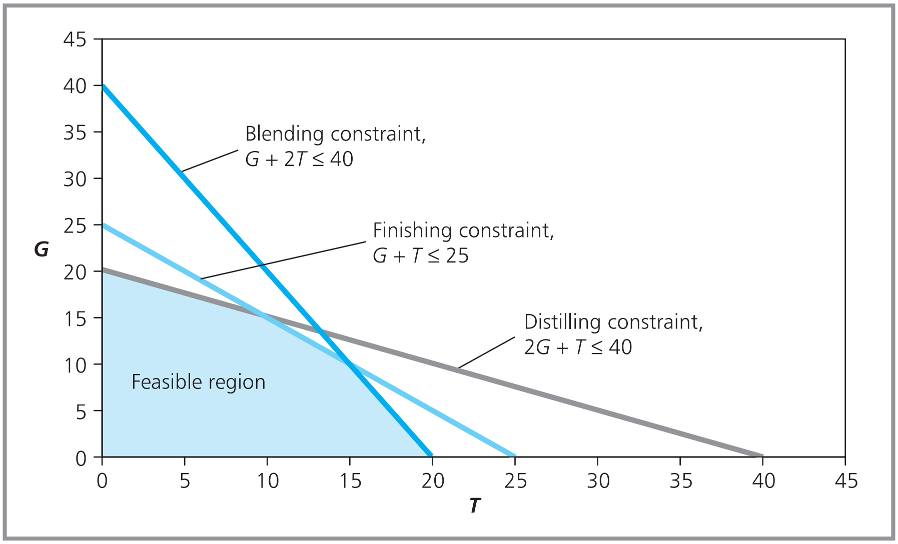

Figure 12.3 Graph of the three constraints defining a feasible region

one of the constraints and does not represent a feasible solution. So this defines a **feasible region**, which is the area in which all feasible solutions lie. Any point inside the feasible region represents a valid solution to the problem, while any point outside breaks at least one of the constraints.

Now we know the area in which feasible solutions lie, the next stage is to examine all feasible solutions and identify the best or optimal. For this we use the objective function, which is to maximise profit of $300G + 200T$. We can also draw this profit line on the graph of G against T. Although we do not know the optimal value of the profit, we can start looking at an arbitrary, trial value of, say, €6,000. Then we can draw the graph of $300G + 200T = 6{,}000$ through two convenient points, say $G = 0$, $T = 30$ and $G = 20$, $T = 0$. In the same way, we can draw a number of other arbitrary values for profit, with the results shown in Figure 12.4.

As you can see, the lines for different profits are all parallel. This is not surprising, as we can write the objective function in the standard form, $y = ax + b$:

$$G = \frac{-200T}{300} + \frac{\text{profit}}{300}$$

showing that the gradient of the line is constant at –200/300, and the line crosses the G axis at the point profit/300. Another interesting point is that the further the line is away from the origin, the higher is the value of the objective function. This suggests a way of finding the optimal solution. For this we superimpose an objective function line onto the graph of constraints, so that it passes through the feasible region (as shown in Figure 12.5). Then we move this line away from the origin, and the further we move it out, the higher is the profit. As the objective function line moves further out, there comes a point where it only just passes through the feasible region, and

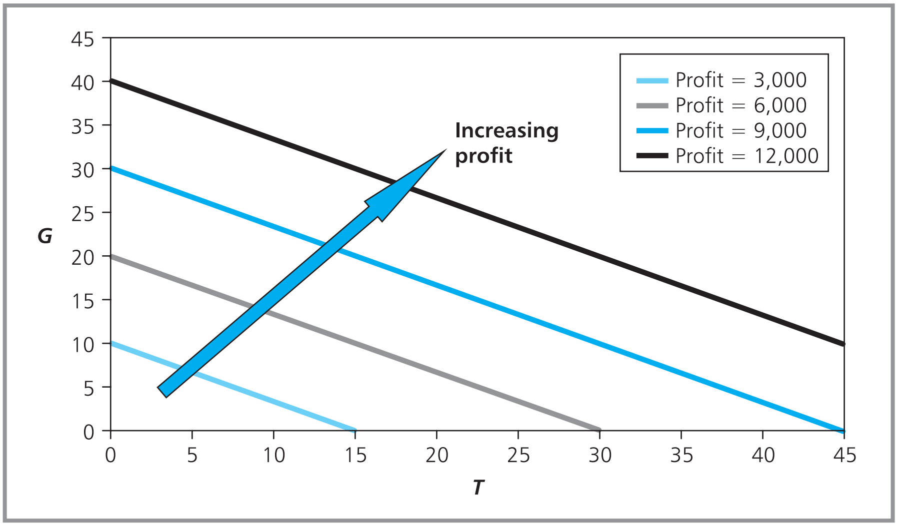

Figure 12.4 Profit lines for Growbig and Thrive

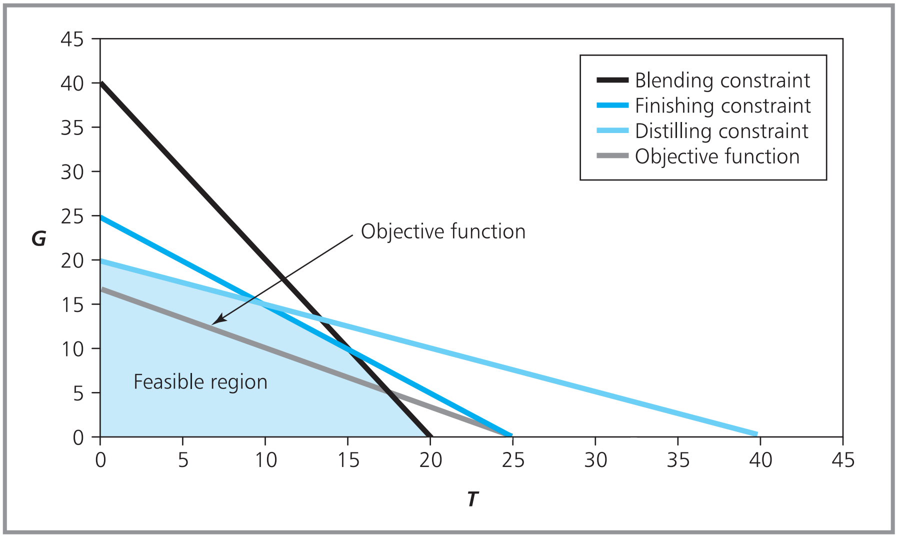

Figure 12.5 Superimposing the objective function on the feasible region

eventually it passes through only a single point (as shown in Figure 12.6). This single point is the optimal solution.

You can see from the graph that the optimal solution is at about the point $G = 15$, $T = 10$. To be more precise, it is at the point where the distilling constraint crosses the finishing constraint. These are the active constraints that limit production. In other words, there is no spare distilling or finishing capacity – but there is spare capacity in blending as this constraint does not limit production. We can find the optimal solution more accurately by solving the simultaneous equations of the limiting constraints.

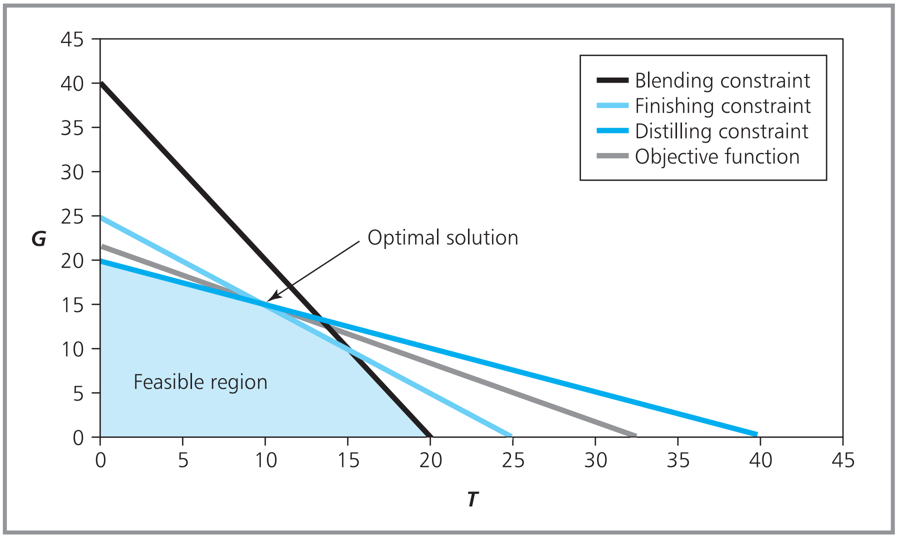

Figure 12.6 Moving the objective function line as far as possible away from the origin identifies the optimal solution

Limiting constraints are:

$2G + T = 40$ (distilling)
$G + T = 25$ (finishing)

which we can solve using the elimination process described in the last chapter. Then subtracting the finishing constraint from the distilling constraint gives $G = 15$, and substituting this value into the finishing constraint gives $T = 10$. This confirms the optimal solution as:

$$G = 15 \quad \text{and} \quad T = 10$$

Substituting these optimal values into the objective function gives the maximum profit:

$$300G + 200T = 300 \times 15 + 200 \times 10 = €6{,}500$$

We can find the resources used by substituting $G = 15$ and $T = 10$ into the constraints:

- Blending: time available = 40 hours
  time used = $G + 2T = 15 + 2 \times 10 = 35$
  spare capacity = 5 hours
- Distilling: time available = 40 hours
  time used = $2G + T = 2 \times 15 + 10 = 40$
  spare capacity = 0
- Finishing: time available = 25 hours
  time used = $G + T = 1 \times 15 + 1 \times 10 = 25$
  spare capacity = 0

WORKED EXAMPLE 12.4

Find the optimal solution to the following linear programme:

Minimise:

$2X + Y$

subject to:

$X + Y \leq 10$ (1)

$X - Y \leq 2$ (2)

$X \geq 4$ (3)

$Y \leq 5$ (4)

with X and Y greater than or equal to zero.

Solution

This problem has already been formulated, so we can immediately draw a graph (shown in Figure 12.7). Sometimes it is not obvious whether a constraint restricts solutions to points above the line or below it (constraint 2, for example). Then you simply take random points on either side of the line and see which ones break the constraint.

In this problem we want to *minimise* the objective function, so instead of moving it as far away from the origin as possible, we move it as close in as possible. As the line moves towards the origin the last point it passes through in the feasible region is the point where constraints (2) and (3) cross. These are the active constraints, and there must be some slack in the other constraints. Here:

$X - Y = 2$ (2)

$X = 4$ (3)

Solving these gives the optimal solution of $X = 4$ and $Y = 2$.

Substituting these optimal values into the objective function gives a minimum value of $2X + Y = 2 \times 4 + 1 \times 2 = 10$. Substituting the optimal values into the constraints gives:

1 $X + Y \leq 10$ $4 + 2 = 6$, giving spare capacity of $10 - 6 = 4$

2 $X - Y \leq 2$ $4 - 2 = 2$, giving no spare capacity and an active constraint

3 $X \geq 4$ 4, giving no spare capacity and an active constraint

4 $Y \leq 5$ 2, giving spare capacity of $5 - 2 = 3$

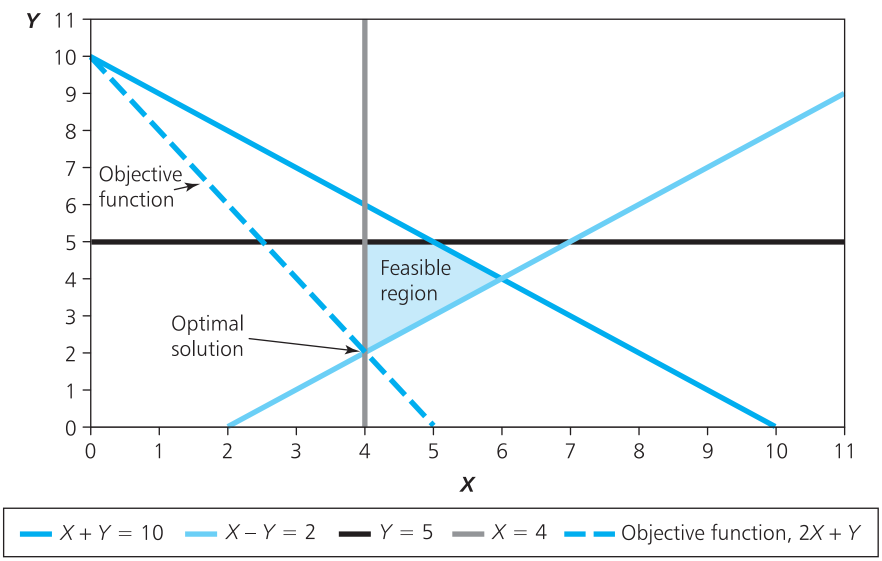

Figure 12.7 Identifying the optimal solution for worked example 12.4

You can see from these examples that the feasible region is always a polygon without any indentations, and the optimal solution is always at a corner or **extreme point**. This is not a coincidence but is a fundamental property of all linear programmes.

> If an optimal solution exists for a linear programme, it is at an extreme point of the feasible region.

This is a very useful property, as it shows how computers can tackle large problems. Essentially they identify the feasible region and then search the extreme points around the edge until they find an optimum.

Review questions

12.6 What is the feasible region for a problem?

12.7 What is the role of the objective function in an LP model?

12.8 What are the extreme points of a feasible region and why are they important?

12.9 How can you identify the optimal solution on a graph?

Sensitivity of solutions to changes

Linear programming finds an optimal solution, but managers might want to use a slightly different answer. For instance, they may want to take into account future conditions, use their experience with similar problems, allow for non-quantifiable factors, recognise that assumptions in the model are not entirely accurate – or simply adjust optimal solutions to give more convenient amounts. So it is important to know how sensitive the optimal solution is to changes. If an LP solution suggests a production quantity of 217 units, but managers feel that 250 units would be better, they need to know what effects this will have on profits. This is done in the third stage of solving an LP, which does a sensitivity analysis on the solution.

> Sensitivity analyses answer two important questions:
>
> - What happens when resources change?
> - What happens when the objective function changes?

Changes in resources

Returning to our original problem of Growbig and Thrive, the limiting constraints were distilling and finishing and we found the optimal solution by solving:

$$2G + T = 40 \quad \text{(distilling constraint)}$$

$$G + T = 25 \quad \text{(finishing constraint)}$$

Suppose that the company could buy an extra unit of distilling – how much is it worth? For small changes we can simply replace the original distilling constraint by a revised one with an extra unit available, and then find the new optimal solution from:

$2G + T = 41$ (new distilling constraint)
$G + T = 25$ (finishing constraint)

The solution here is $G = 16$ and $T = 9$, and substituting these values into the objective function gives a new maximum profit of $300 \times 16 + 200 \times 9 =$ €6,600. The extra hour of distilling has raised the profit from €6,500 to €6,600, showing that distilling has a marginal value of €100. In LP this marginal value is usually called a **shadow price**, and this is the maximum amount that you would pay for one extra unit of a resource. Conversely, if an hour of distilling is lost for any reason, the profit falls by €100.

The shadow price is valid only for relatively small changes. We found that an extra hour of distilling is worth €100, but there are limits and an extra 1,000 hours would certainly not be worth €100,000. The other two constraints would become active long before this, and they would limit production and leave spare distilling.

We can repeat this analysis to find a shadow price for finishing, by using a new finishing constraint:

$2G + T = 40$ (distilling constraint)
$G + T = 26$ (new finishing constraint)

Solving these equations gives $G = 14$ and $T = 12$, and substituting these values in the objective function gives a new maximum profit of $12 \times 200 + 14 \times 300 =$ €6,600. This is again an increase of €100 over the original profit, showing that the shadow price for finishing – which is the most you should pay for an extra hour of finishing – is €100. (It is simply coincidence that this is the same as the shadow price for distilling.) Again, this value holds for small changes, but if the capacity for finishing changes markedly, the other constraints become limiting.

Obviously, if a process already has spare capacity, there is no point in adding even more capacity – as this would just give more spare. It follows that shadow prices of non-limiting resources are zero. In this example, there is spare capacity in blending, so its shadow price is zero.

Now we have shadow prices for all three processes: €0 an hour for blending, €100 an hour for distilling, and €100 an hour for finishing. But it would be interesting to see what happens when several resources are increased at the same time. We can find the effect of an extra hour of both distilling and finishing by replacing the original constraints by:

$2G + T = 41$ (new distilling constraint)
$G + T = 26$ (new finishing constraint)

Solving these gives $G = 15$ and $T = 11$ and substitution in the objective function gives a maximum profit of €6,700. This is €200 more than the original solution – and is also the sum of the two individual shadow prices. In other words, for small changes the total benefit is the sum of the separate benefits of increasing each resource separately.

WORKED EXAMPLE 12.5

Suppose a new fertiliser, Vegup, can be made in addition to Growbig and Thrive. Vegup uses two hours of blending, two hours of distilling and two hours of packing for each batch and contributes €500 to profits. Should the company introduce this new product?

Solution

You can answer this by looking at the shadow prices. If the company makes a batch of Vegup, it must make fewer batches of Growbig and Thrive. You can use the shadow prices to see how much the profit will decline from fewer batches of Growbig and Thrive, and compare this with the extra profit from a batch of Vegup.

A batch of Vegup uses two hours of distilling with a shadow price of €100 an hour, so this cost €200. The batch also uses two hours of finishing with a shadow price of €100 an hour, so this also costs €200. The two hours of finishing has zero shadow price, so this does not cost anything. So making a batch of Vegup reduces the profit from Growbig and Thrive by a total of €400. But the batch of Vegup makes a profit of €500, so there is a net benefit of €100. It is clearly in the company's interest to make Vegup. The next obvious question is how much to make? Unfortunately, you cannot find this from the original solution, and have to add Vegup to the formulation and solve a new LP problem.

Changes in the objective function

The other aspect of sensitivity analysis considers changes to the objective function. How would changing the profit on batches of Growbig and Thrive affect the optimal solution? Provided the changes are small, the optimal solution, that is the numbers of batches of Growbig and Thrive made, does not change. If the profit on each batch of Growbig rises by €10 from €300 to €310, the optimal solution still makes 15 batches of Growbig, so the profit simply rises by $15 \times 10 =$ €150. But this argument is not valid for bigger changes. For example, raising the profit on each batch of Growbig from €300 to €600 would not raise the profit by $15 \times 300 =$ €4,500. What happens is that the gradient of the objective function changes and the optimal solution moves to another extreme point. We could calculate these effects in detail, but it is much easier to use a computer, as we shall see in the next section.

Review questions

12.10 What is the 'sensitivity analysis' in LP problems?

12.11 What is the shadow price of a resource?

12.12 Within what limits are the shadow prices valid?

Solving real problems

We can solve problems with two variables using a graph, but real problems commonly have hundreds or even thousands of variables. The way to solve these is to build matrices of the problem, and then do a lot of matrix arithmetic to get an optimal solution. A formal procedure for this is the 'simplex method', but it needs so much arithmetic that computers are always used. Many specialised programs are available for solving LPs, and Figure 12.8

-=*=- INFORMATION ENTERED -=*=-

PROBLEM NAME : GROWBIG AND THRIVE

NUMBER OF VARIABLES : 2
G = batches of Growbig
T = batches of Thrive

NUMBER OF <= CONSTRAINTS : 3
NUMBER OF = CONSTRAINTS : 0
NUMBER OF >= CONSTRAINTS : 0

MAXIMISE: Profit = 300 G + 200 T

SUBJECT TO:

Blending	1 G	+	2	T	<=	40
Distilling	2 G	+	1	T	<=	40
Finishing	1 G	+	1	T	<=	25

-=*=- RESULTS -=*=-

Optimal solution found after 3 iterations

OBJECTIVE FUNCTION VALUE: 6500

VARIABLE	OPTIMAL VALUE
G	15
T	10

CONSTRAINT	ORIGINAL RIGHT-HAND VALUE	USED	SLACK OR SURPLUS	SHADOW PRICE
Blending	40	35	5	0
Distilling	40	40	0	100
Finishing	25	25	0	100

-=*=- SENSITIVITY ANALYSIS -=*=-

OBJECTIVE FUNCTION COEFFICIENTS

VARIABLE	LOWER LIMIT	ORIGINAL COEFFICIENT	UPPER LIMIT
G	200	300	400
T	150	200	300

SHADOW PRICES VALID IN RHS RANGES

CONSTRAINT	LOWER LIMIT	ORIGINAL VALUE	UPPER LIMIT
Blending	35	40	NO LIMIT
Distilling	35	40	50
Finishing	20	25	26.667

-=*=- E N D O F A N A L Y S I S -=*=-

Figure 12.8 Printout for the Growbig and Thrive problem shows the results when the Growbig and Thrive problem is solved by a simple package. The output has three parts: the first part shows the data, to confirm that it was entered properly; the second part shows the main results, with optimal values, profits, spare resources, and shadow prices; and the third part shows a sensitivity analysis.

The results in this printout confirm our optimal solution and give more information about sensitivity. For example, they confirm that the shadow price for distilling is €100, but this is valid only while the amount of distilling is in the range 35 to 50 hours. Outside this range the shadow price changes. Similarly, the shadow price for finishing is €100, but this is valid only while the amount of finishing is in the range 20 to 26.67 hours. The shadow price for blending is zero, but if there is less than 35 hours available, it becomes limiting and the shadow price rises.

The sensitivity analysis also considers the objective function. The original profit on each batch of Growbig is €300 but this can vary between €200 and €400 without changing the location of the optimal solution (provided the profit on each batch of Thrive remains at €200). Similarly the profit on each batch of Thrive can vary between €150 and €300 without changing the location of the optimal solution (provided the profit on each batch of Growbig remains at €300).

Specialised computer programs are generally the best ways of solving LPs. You can use spreadsheets, but these are not really designed for such specialised use and are more difficult to use. In Excel you can use 'Solver' in the tools options, and Figure 12.9 shows how this works. Essentially you have to describe the details of a problem in the body of a spreadsheet, and then transfer information to the 'solver parameters' form.

- Here we define the cells D4 and E4 as containing the decision variables, meaning that they contain the values of G and T. Solver iteratively changes the values in these cells until it finds the optimal solution.
- We have explicitly written the objective function in terms of these cells in cell H8. Solver uses this as its 'target cell' whose value is to be, in this case, maximised.
- Now we add constraints – with the left-hand sides explicitly stated in cells H5 to H7 and the right-hand side limits given in cells G5 to G7.

This gives the structure of the problem, and now we can press 'solve' to get the answer. Solver gives three results – the optimal solution, a sensitivity analysis, and a limits report that shows the ranges within which the sensitivity analysis is valid.

You will probably need a bit of practice before you find Solver straightforward. In practice, specialised programs can be much easier to use.

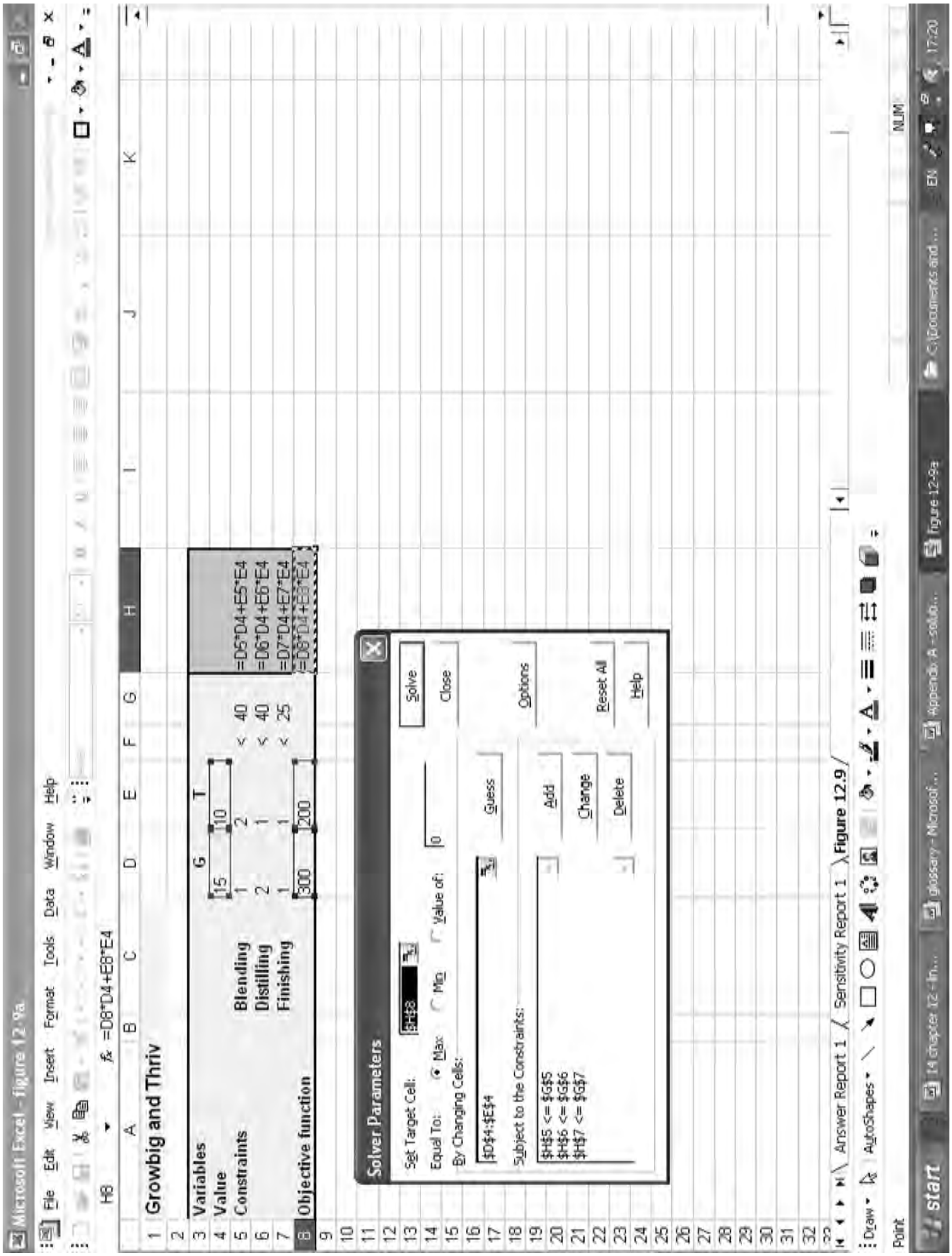

Figure 12.9 Using ‘Solver’ for a linear programme

WORKED EXAMPLE 12.6

Amalgamated Engineering makes two kinds of gearbox, manual and automatic. There are four stages in the production of these, with details of times needed and weekly availabilities given below. The company makes a profit of £64 on each Manual sold and £100 on each Automatic.

Stage in manufacture	Time needed (hours per unit)		Time available (hours per week)
	Manual	Automatic	
Foundry	3	5	7,500
Machine shop	5	4	10,000
Assembly	2	1	3,500
Testing	1	1	2,000

LINEAR PROGRAMMING SOLVER

PROBLEM ENTERED

VARIABLES M A

MAXIMISE 64 M + 100 A

CONSTRAINTS

1)	FOUNDRY	3 M	+	5 A	<=	7,500
2)	MACHINE SHOP	5 M	+	4 A	<=	10,000
3)	ASSEMBLY	2 M	+	1 A	<=	3,500
4)	TESTING	1 M	+	1 A	<=	2,000

SOLUTION

LP OPTIMUM FOUND AT STEP 2

OBJECTIVE FUNCTION VALUE
155,000

VARIABLE	VALUE
M	1,250.00
A	750.00

CONSTRAINT		SLACK OR SURPLUS	SHADOW PRICES
1)	FOUNDRY	0	18
2)	MACHINE SHOP	750	0
3)	ASSEMBLY	250	0
4)	TESTING	0	10

RANGES IN WHICH THE BASIS IS UNCHANGED:

OBJECTIVE FUNCTION COEFFICIENTS

VARIABLE	CURRENT VALUE	ALLOWABLE INCREASE	ALLOWABLE DECREASE
M	64	36	4
A	100	6.67	36

RIGHTHAND SIDE VALUES

CONSTRAINT		CURRENT RHS	ALLOWABLE INCREASE	ALLOWABLE DECREASE
1)	FOUNDRY	7,500	2,500	500
2)	MACHINE SHOP	10,000	INFINITY	750
3)	ASSEMBLY	3,500	INFINITY	250
4)	TESTING	2,000	71	500

Figure 12.10 Output from a LP package for worked example 12.6

Worked example 12.6 continued

(a) Formulate this problem as a linear programme.
(b) Find an optimal solution to the problem.
(c) Amalgamated can start making a new semi-automatic gearbox that needs 4, 4, 1 and 1 hour respectively in each manufacturing stage and gives a profit of £80 a unit. Should the company make this new gearbox?

Solution

(a) You start by defining the decision variables:
- M = number of manual gearboxes made a week
- A = number of automatic gearboxes made a week.

Then the formulation is:

Maximise:

$64M + 100A$

subject to

$3M + 5A \leq 7,500$	(foundry constraint)
$5M + 4A \leq 10,000$	(machine shop constraint)
$2M + A \leq 3,500$	(assembly constraint)
$M + A \leq 2,000$	(testing constraint)
$M \geq 0$ and $A \geq 0$	(non-negativity constraints)

(b) You can solve this problem using a graph, spreadsheet or LP package. Figure 12.10 shows the results from a typical package. This identifies the optimal solution as making 1,250 manual gearboxes and 750 automatic ones, with a profit of £155,000. All the foundry time is used and the shadow price is £18, which is valid between 700 and 10,000 hours (the current value of the right-hand side minus the allowable decrease, to the current value plus the allowable increase). There is spare capacity of 750 hours and 250 hours respectively in the machine shop and assembly. All testing time is used and this has a shadow price of £10 which is valid from 1,500 to 2,071 hours. The optimal solution does not change provided the profit on manual gearboxes stays between £60 and £100, and the profit on automatic gearboxes stays between £64 and £106.67.

Because there are only two decision variables, you can solve this problem using a graph, as shown in Figure 12.11. This shows the feasible region, with the optimal solution identified by the objective function as the extreme point where the foundry and testing constraints cross. If you solve these two simultaneous equations, and do the related calculations, you can confirm the results given by the computer.

(c) The new semi-automatic gearbox needs:
- 4 hours of foundry costing £18 an hour = £72
- 4 hours of machine shop costing £0 an hour = £0
- 1 hour of assembly costing £0 an hour = £0
- 1 hour of testing costing £10 an hour = £10

So the total cost of making a unit is £82, while the profit is £80. This means that there would be a loss of £2 on every unit made, and Amalgamated should not start making the semi-automatic gearbox.

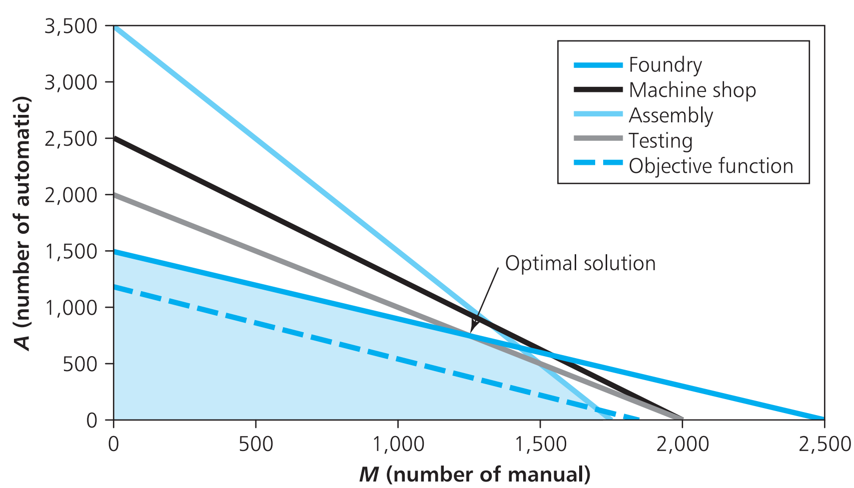

Figure 12.11 Graph of solution for Amalgamated Engineering, worked example 12.6

WORKED EXAMPLE 12.7

West Coast Wood Products Ltd make four types of pressed panels from pine and spruce. Each sheet of panel must be cut and pressed. The following table shows the hours needed to produce a batch of each type of panel and the hours available each week.

Panel type	Hours of cutting	Hours of pressing
Classic	1	1
Western	1	4
Nouveau	2	3
East Coast	2	2
Available	80	100

There is a limited amount of suitable wood available. The amounts needed for a batch of each type of panel and the maximum weekly availability are given in the following table.

	Classic	Western	Nouveau	East Coast	Availability
Pine	50	40	30	40	2,500
Spruce	20	30	50	20	2,000

The profit on each batch of panels is \$400 for Classic, \$1,100 for Western, \$750 for Nouveau and \$350 for East Coast.

(a) Formulate this as a linear programme.
(b) Explain the results given in the printout of Figure 12.12.

Solution

(a) We start by defining the decision variables as the number of batches of each type of panelling made a week (CLS, WST, NOU and EST). Then the formulation is:

Maximise:

400 CLS + 1,100 WST + 750 NOU + 350 EST

subject to

50 CLS + 40 WST + 30 NOU + 40 EST $\le$ 2,500 (pine)
20 CLS + 30 WST + 50 NOU + 20 EST $\le$ 2,000 (spruce)
1 CLS + 1 WST + 2 NOU + 2 EST $\le$ 80 (cutting)
1 CLS + 4 WST + 3 NOU + 2 EST $\le$ 100 (pressing)
with CLS, WST, NOU and EST $\ge$ 0.

(b) The computer package identifies the optimal solution and does a sensitivity analysis.

The optimal solution is to make 37.5 batches of Classic a week, 15.6 batches of Western and none of the others. This gives a profit of \$32,188. If a batch of Nouveau is made, it would reduce profit by \$75, so this is the amount the profit on a batch would have to increase before it becomes profitable. Similarly, making a batch of East Coast would reduce profit by \$263.

The limiting constraints are pine and pressing, with spare capacity in spruce (781 units) and cutting (27 hours). This remains true while the constraint on spruce remains over 1,219 units and the amount of cutting remains above 53 hours. The shadow price of pine is \$3 (valid between 1,000 and 3,933 units) and that of pressing is \$244 (valid for between 50 and 250 hours).

The profit for each batch of Classic could vary between \$275 and \$1,375 without changing the position of the optimal solution (provided the profits on the other panels remained unchanged). Similarly, the optimal solution remains at the same extreme point provided the profit on Western is between \$1,000 and \$1,600, Nouveau is below \$825 and East Coast is below \$613.

Worked example 12.7 continued

-=*=- INFORMATION ENTERED -=*=-

PROBLEM NAME : WEST COAST WOOD PRODUCTS LTD

NUMBER OF VARIABLES : 4
CLS = batches of classic
WST = batches of western
NOU = batches of nouveau
EST = batches of east coast

NUMBER OF <= CONSTRAINTS : 4
NUMBER OF = CONSTRAINTS : 0
NUMBER OF >= CONSTRAINTS : 0

MAXIMISE Profit = 400 CLS + 1,100 WST + 750 NOU + 350 EST

SUBJECT TO:

pine	50 CLS	+	40 WST	+	30 NOU	+	40 EST	<=	2,500
spruce	20 CLS	+	30 WST	+	50 NOU	+	20 EST	<=	2,000
cutting	1 CLS	+	1 WST	+	2 NOU	+	2 EST	<=	80
pressing	1 CLS	+	4 WST	+	3 NOU	+	2 EST	<=	100

-=*=- RESULTS -=*=-

Optimal solution found after 6 iterations

OBJECTIVE FUNCTION VALUE: 32188

VARIABLE	OPTIMAL VALUE	ORIGINAL PROFIT	INCREASE NEEDED
CLS	37.5	400	0
WST	15.6	1,100	0
NOU	0	750	75
EST	0	350	263

CONSTRAINT	ORIGINAL RIGHT-HAND VALUE	USED	SLACK OR SURPLUS	SHADOW PRICE
pine	2,500	2,500	0	3
spruce	2,000	1,219	781	0
cutting	80	53	27	0
pressing	100	100	0	244

-=*=- SENSITIVITY ANALYSIS -=*=-

OBJECTIVE FUNCTION COEFFICIENTS

VARIABLE	LOWER LIMIT	ORIGINAL COEFFICIENT	UPPER LIMIT
CLS	275	400	1,375
WST	100	1,100	1600
NOU	no limit	750	825
EST	no limit	350	613

SHADOW PRICES VALID IN RHS RANGES

CONSTRAINT	LOWER LIMIT	ORIGINAL VALUE	UPPER LIMIT
pine	1,000	2,500	3,933
spruce	1,219	2,000	no limit
cutting	53	80	no limit
pressing	50	100	250

-=*=- E N D O F A N A L Y S I S -=*=-

Figure 12.12 Printout for West Coast Wood Products, worked example 12.7

In practice, LP formulations can be very large and complex, and commonly have tens of thousands of constraints. It is very easy to make mistakes with such large models, and three particular concerns are:

- **Unbound solution** – which means the constraints do not limit the solution, and the feasible region effectively extends to infinity.
- **Infeasible solution** – which means the constraints are so tight that they have left no feasible region.
- **Degeneracy** – when there are many solutions that give the same optimal value.

If you get any of these, you should check the input data carefully for mistakes.

Review questions

12.13 Why are computers always used to solve LPs?

12.14 What information might an LP package give?

12.15 'Spreadsheets are the best way of getting solutions to LP problems.' Do you agree with this?

IDEAS IN PRACTICE Goolongon Smelting

In recent years the demand for lead has fallen, as substitutes are used in petrol, paint and construction. Prices have fallen, putting pressure on supplier profits. Goolongon Smelting are major producers of lead in Australia. They use linear programming in several ways to maximise profit, with one model looking at production planning.

Goolongon mine their own ore, and buy smaller amounts from other suppliers. The ores have variable composition and costs, but are processed in the same way. They are crushed and prepared before a sintering plant removes sulphur and replaces it with oxygen. Then a blast furnace removes the oxygen and other impurities to leave lead bullion. This bullion still contains metallic impurities that are removed in a refinery. Goolongon can sell the impurities – which include copper, silver, arsenic and bismuth – along with other by-products like sulphuric acid.

The model of production plans has three types of decision variables.

- There are 22 different grades of ore, and the first type of variable considers the amount of each ore used.
- There are 25 major products, and the second type of variable considers the amount of each product made.
- The third type of variable describes other conditions, such as the chemical composition of each ore.

The variables are combined into four types of constraint.

- The first type of constraint is on the preparation of the ores. There is always some variation in the incoming ores, but the smelters work best with fixed conditions – so the preparation of the ores includes blending to give consistent inputs.
- The second type of constraint looks at the availability of different types of ore. It makes sure that the amount of each ore used matches the supply.
- The third type of constraint looks at the different products. There is a varying demand for products, and Goolongon varies its supply to meet these.
- Finally, there are constraints to match the total composition of the input to the outputs – so that no materials are lost in the process.

Goolongon find their gross profit by subtracting the costs of all inputs from the revenues for all products. Their objective is to maximise this profit.

This basic production planning model has 214 decision variables and over 1,000 constraints. Goolongon have several versions of the model for different time periods and production assumptions. The larger models have several hundred variables and thousands of constraints.

CHAPTER REVIEW

This chapter has described linear programming (LP) as a way of solving some types of problem with constrained optimisation.

- With problems of constrained optimisation, managers want to get an optimal solution, but there are constraints on their choices.
- Linear programming is a method of tackling problems of constrained optimisation, where the objective function and constraints are linear functions of the decision variables. There are three stages in solving such problems.
- The first stage is to describe a problem in a standard form. The resulting formulation consists of decision variables, an objective function, problem constraints and non-negativity constraints.
- The constraints of a problem define a feasible region, which is a convex space surrounded by extreme points. The optimal solution is at one of the extreme points of the feasible region. The second stage of solving a linear programme is to identify the optimal solution. For small problems we can do this graphically, but real problems always need a computer.
- The third stage does a sensitivity analysis to examine the effects of small changes to the problem. In particular, it calculates shadow prices and looks at the effects of changing constraints and the objective function.
- The easiest way of solving linear programmes is to use a standard package, but you can use spreadsheets. All packages give similar results, but they need careful analysis and interpretation.

CASE STUDY Elemental Electronics

Elemental Electronics assemble microcomputers and act as wholesalers for some components. For the manufacturing business, they buy components from a number of suppliers and assemble them in a well-tried design. They do virtually no research and development, and are happy to use designs that have been tested by other manufacturers. They also spend little on advertising, preferring to sell computers through their website and a few specialised retailers. As a result they have very low overheads, and can sell their machines at a much lower cost than major competitors.

A typical component that Elemental buy is a standard motherboard. There are at least six suppliers of equivalent boards in America, Europe and the Far East. Elemental act as a wholesaler for two of these, one in the Far East and one in South America. The boards are delivered in bulk and Elemental test and repackage them to sell to a number of small manufacturers. Each board from the Far East takes two hours to test and two hours to repackage, while each board from South America takes three hours to test and one hour to repackage. Elemental has enough facilities to provide up to 8,000 hours a week for testing and 4,000 hours a week for repackaging. There are maximum sales of 1,500 a week for the board from the Far East and each board gives a profit of €20 when sold.

On the production side, Elemental manufactures four models of computer (A to D). Each of these has four stages in manufacturing: sub-assembly, main assembly, final assembly and finishing. The following table shows the times needed for each stage, total availability each week, and some related costs.

Case study continued

	Hours needed per unit				Number of machines	Hours available per machine per week	Time needed for maintenance
	Model A	Model B	Model C	Model D			
Sub-assembly	2	3	4	4	10	40	10%
Main assembly	1	2	2	3	6	36	16.7%
Final assembly	3	3	2	4	12	38	25%
Finishing	2	3	3	3	8	40	10%
Direct costs	€1,600	€1,800	€2,200	€2,500			
Selling price	€2,500	€2,800	€3,400	€4,000			

Fixed costs of production are €3 million a year and Elemental work a standard 48-hour week.

Question

- **Elemental work in a very competitive industry. Their Operations Manager has suggested that they should use linear programming more explicitly in their production planning. With the limited information about their operations, could you build a case to support the Operations Manager's position?**

PROBLEMS

12.1 Two additives, X1 and X2, can be used to increase the octane number of petrol. One kilogram of X1 in 5,000 litres of petrol increases the octane number by 10, while one kilogram of X2 in 5,000 litres increases the octane number by 20. The total additives must increase the octane number by at least 5, but a total of no more than 500 gram can be added to 5,000 litres, and the amount of X2 plus twice the amount of X1 must be at least 500 grams. If X1 costs €30 a kilogram and X2 costs €40 a kilogram, formulate this problem as a linear programme. Use a graphical method to find an optimal solution.

12.2 North Penberthy Housing Association is planning a number of blocks of flats. Five types of block have been designed, containing flats of four categories (senior citizens, single person, small family and large family). The number of flats in each block, and other relevant information, is as follows:

Type of block	Number of flats in category 1	2	3	4	Number of storeys	Plan area	Cost per block ($million)
A	1	2	4	0	3	5	2.08
B	0	3	6	0	6	5	3.2
C	2	2	2	4	2	8	3.0
D	0	6	0	8	8	6	4.8
E	0	0	10	5	3	4	4.8

The association wants to build a total of 500 flats with at least 40 in category 1 and 125 in each of the other categories. In the past, high-rise flats have proved unpopular and the association wants to limit the number of storeys in the development – with the average number of storeys at most five, and at least half the flats in blocks of three or fewer storeys. An area of 300 units has been set aside for the development and any spare land will be used as a park. Formulate this problem as a linear programme.

12.3 Jane MacFarlane wants a weekly schedule for two business services, X and Y. Each 'unit 'of X delivered to customers needs one service package, while each unit of Y uses two of the packages, and Jane has a maximum of 80 packages available a week. Each unit of X and Y needs 10 hours of subcontracted work, and Jane has signed agreements with subcontractors for a weekly minimum of 200 hours and a maximum of 600 hours. Jane knows from market surveys that demand for Y is high, and she will have no trouble selling any number of units. However, there is a maximum demand of 50 units of X, despite a long-term contract to supply 10 units to one customer. The net profit on each unit of X and Y is €2,000 and €3,000 respectively. Formulate this problem as a linear programme. Use a graphical method to find an optimal solution. Use a suitable program to check your results.

12.4 Novacook Ltd makes two types of cooker, one electric and one gas. There are four stages in the production of each of these, with the following features:

Manufacturing stage	Time needed (hours per unit)		Total time available (hours a week)
	Electric	Gas	
Forming	4	2	3,600
Machine shop	10	8	12,000
Assembly	6	4	6,000
Testing	2	2	2,800

Each electric cooker has a variable cost of £200 and a selling price of £300, while each gas cooker has a variable cost of £160 and a selling price of £240. Fixed overheads are £60,000 a week and the company works a 50-week year. The marketing department suggest maximum sales of 800 electric and 1,250 gas cookers a week.

(a) Formulate this as a linear programme.
(b) Find the optimal solution to the problem, and draw a graph to illustrate its features.
(c) A company offers testing services to Novacook. What price should they be prepared to pay for this service, and how much should they buy?
(d) A new cooker is planned that would use the manufacturing stages for 4, 6, 6 and 2 hours respectively. At what selling price should Novacook consider making this cooker if the other variable costs are £168 a unit?

12.5 Tarsands Oil make two blends of fuel by mixing three crude oils. The costs and daily availability of the oils are:

Oil	Cost ($/litre)	Amount available (litres)
A	0.33	5,000
B	0.40	10,000
C	0.48	15,000

The requirements of the blends of fuel are:

Blend 1	at least 30% of A
	at most 45% of B
	at least 25% of C
Blend 2	at most 35% of A
	at least 30% of B
	at most 40% of C

Tarsands sell each litre of Blend 1 for $1.00 and each litre of Blend 2 for $1.20. Long-term contracts require at least 10,000 litres of each blend to be produced. What should Tarsands do?

12.6 Figure 12.13 shows a printout from a linear programming package. Explain what these results show. How could the format of the results be improved?

-=*=- INFORMATION ENTERED -=*=-

PROBLEM NAME : Manheim Service

NUMBER OF VARIABLES : 3
PrA = batches of Service Product A
PrB = batches of Service Product B
PrC = batches of Service Product C

NUMBER OF <= CONSTRAINTS : 1
NUMBER OF = CONSTRAINTS : 0
NUMBER OF >= CONSTRAINTS : 2

MAXIMISE Profit = 10 PrA + 5 PrB + 3 PrC

SUBJECT TO:

Constraint 1	120 PrA	+	23 PrB	+	10 PrC	<=	1,545
Constraint 2	150 PrA	+	35 PrB	+	10 PrC	<=	550
Constraint 3	100 PrA	+	15 PrB	+	55 PrC	<=	675

-=*=- RESULTS -=*=-

Optimal solution found after 4 iterations

OBJECTIVE FUNCTION VALUE: 51.55

VARIABLE	OPTIMAL VALUE	ORIGINAL PROFIT	INCREASE NEEDED
PrA	3.24	10	0
PrB	0	5	3.07
PrC	6.38	3	0

CONSTRAINT	ORIGINAL RIGHT-HAND VALUE	USED	SLACK OR SURPLUS	SHADOW PRICE
Constraint 1	1,545	452.76	1,092.24	0
Constraint 2	550	550	0	0.034
Constraint 3	675	675	0	0.048

-=*=- SENSITIVITY ANALYSIS -=*=-

OBJECTIVE FUNCTION COEFFICIENTS

VARIABLE	LOWER LIMIT	ORIGINAL COEFFICIENT	UPPER LIMIT
PrA	5.46	10	22.54
PrB	1.93	5	no limit
PrC	0.67	3	5.5

SHADOW PRICES VALID IN RHS RANGES

CONSTRAINT	LOWER LIMIT	ORIGINAL VALUE	UPPER LIMIT
Constraint 1	452.76	1,545	no limit
Constraint 2	122.73	550	1,012.5
Constraint 3	366.67	675	3025

-=*=- E N D O F A N A L Y S I S -=*=-

Figure 12.13 Printout for problem 12.6

RESEARCH PROJECTS

12.1 Most spreadsheets have procedures – either as standard or as add-ins – that can solve linear programmes. The best known is Microsoft Excel's Solver. Unfortunately, these can be rather awkward to use. Have a look at the facilities offered by a spreadsheet and explore its features. How could it be made easier to use? What features do you think a reasonable spreadsheet function should have?

12.2 There are many specialised LP packages, ranging from the very basic (and free) to the very sophisticated (and expensive). See what packages you have access to and compare their features. You might find packages like LINDO, What'sBest, GAMS, SOPT (Smart Optimisation), XPRESS, ILOG or SAS/OR Software. Use a suitable package to check the results given in this chapter.

12.3 Leo Hamlich has forecast the numbers of a component that his company will need each month. Unfortunately, the cost of the component is rising, as shown in the following table.

Month	Jan	Feb	Mar	Apr	May	Jun
Demand	100	90	80	60	50	50
Cost (€)	200	200	205	205	210	210
Month	**Jul**	**Aug**	**Sep**	**Oct**	**Nov**	**Dec**
Demand	70	90	100	100	110	110
Cost (€)	210	220	220	230	230	240

Leo can purchase components ahead of time to avoid the price increases, but financial charges mean that there is a cost of €2 a unit for carrying stock from one month to the next. How can LP be used to find the best pattern of orders for the component? Are linear programmes widely used for such production decisions?

12.4 The demand in all branches of banks varies during the day. The AIBC International branch in Toronto has a peak demand for domestic transactions around lunchtime. When this is translated into the number of employees needed in the branch, it gives the following pattern.

Period	Number of employees	Period	Number of employees
0,900–1,000	20	1,400–1,500	85
1,000–1,100	35	1,500–1,600	70
1,100–1,200	40	1,600–1,700	45
1,200–1,300	55	1,700–1,800	20
1,300–1,400	75	1,800–1,900	10

This demand is met by a combination of normal work by full-time staff, overtime by full-time staff and part-time staff. Approximate hourly costs for these are \$20, \$25 and \$15 respectively. The bank aims to provide at least the number of people required at minimum cost. But there are a number of constraints. These include:

- The bank does not work outside the 0900 to 1900 time slot.
- Full-time staff are available for 35 hours a week, and anything beyond this is overtime.
- Full-time staff rarely like to work more than five hours of overtime a week.
- All full-time staff have an hour's lunch break at an agreed time between 1100 and 1300.
- Part-time staff work between 4 and 8 hours a day without a lunch break.
- A company policy limits part-time staff to a maximum of 40% of hours worked.

Can LP tackle this kind of scheduling problem?

Sources of information

Further reading

Linear programming appears in books on management science and operational research. The following list gives some more specific references.

Darst R.B., *Introduction to Linear Programming*, Marcel Dekker, New York, 1990.

Gass S., *Linear Programming* (5th edition), Dover Publications, New York, 2003.

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Moore J.H. and Weatherford L.R., *Decision Modelling with Microsoft Excel* (6th edition), Prentice Hall, Upper Saddle River, NJ, 2001.

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Schrage L., *LINDO: an Optimization Modelling System* (6th edition), Scientific Press, San Francisco, CA, 1998.

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Williams P., *Model Building in Mathematical Programming*, John Wiley, Chichester, 1999.

CHAPTER 13

Rates of change and calculus

Contents

Chapter outline

The average speed of a car is defined as the distance travelled divided by the time taken; any change in speed is described as either acceleration or deceleration. Calculus was originally developed to analyse this kind of physical change, but is now used for problems that involve any kind of change. This chapter discusses the application of calculus to business problems, such as the change of costs with production quantities, change of sales with price, and change of revenue with output. The chapter reviews the two aspects of calculus – differentiation and integration.

After finishing this chapter you should be able to:

- Understand the concept of differentiation
- Differentiate functions of the form $y = ax^n$
- Identify turning points of a function, and say whether these are maxima or minima
- Use differentiation for marginal analyses and price elasticity of demand
- Interpret integration as the reverse of differentiation and as a means of summation
- Integrate functions of the form $y = ax^n$.

Differentiation

The equation of a straight line is $y = ax + b$, where a is the gradient and b is the intercept where the line crosses the y axis. The gradient is defined as the rate of change of y with respect to x – in other words, the amount that y changes for every unit change in x.

WORKED EXAMPLE 13.1

Marcia Petrovski finds that the cost of running a car is £600 a year for fixed costs (road tax, insurance depreciation, etc.) plus a variable cost of 40 pence for every kilometre travelled (for petrol, oil, maintenance, etc.). What can you say about the total annual cost?

Solution

If you set y as the total annual cost (in pounds) of running the car, and x as the distance travelled (in kilometres) in a year, you have:

$$y = 0.4x + 600$$

This is a straight line graph whose gradient is the variable cost per kilometre – showing that for every unit increase in x there is an increase of 0.4 units in y (as illustrated in Figure 13.1). The gradient is constant, meaning that every additional kilometre travelled has the same variable cost, whether it is the fifth kilometre or the 5,000th.

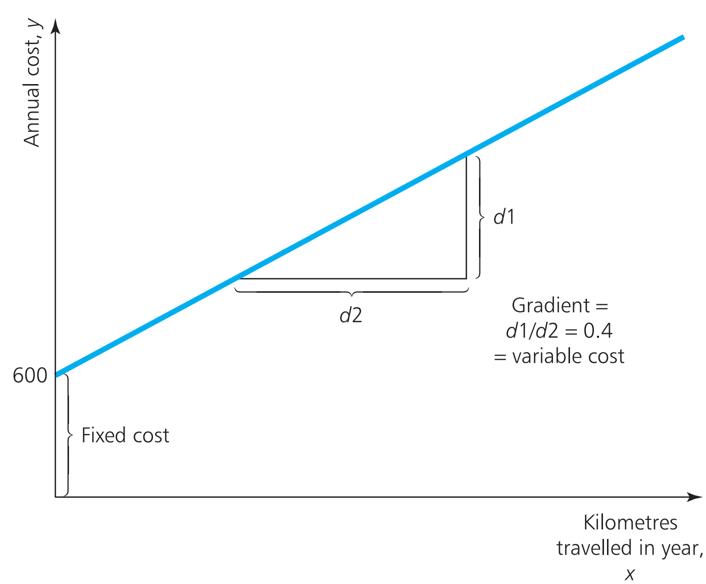


Figure 13.1 Constant gradient for a straight line graph

In this worked example it is easy to find the cost of travelling an extra kilometre, as it is always the same at £0.40. But suppose we look at the costs more closely and find that higher annual distances travelled increase depreciation, repair costs and insurance. Then the variable cost per kilometre actually rises with distance travelled, perhaps giving the total cost shown in Figure 13.2.

Now it is more difficult to find the cost of travelling a kilometre at any particular point. If the car has already travelled 5,000 kilometres this year,

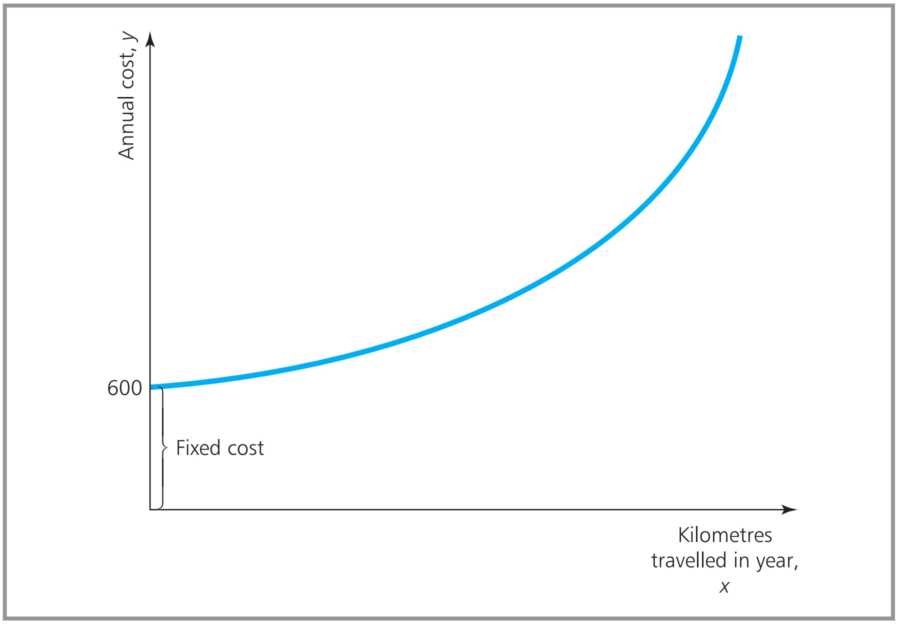


Figure 13.2 Variable cost rising with distance travelled

what is the cost of travelling the next kilometre? The worked example showed how to find this in principle – because the variable cost is the gradient of the total cost curve. This is fairly obvious, as the gradient is defined as the increase in total cost for a unit increase in distance – and this is also the variable cost. The problem with Figure 13.2 is that the gradient varies with x, and every point on the graph has a different gradient. So to find the cost of travelling the 5,000th kilometre we specifically need the gradient of the line when $x = 5{,}000$. Of course, we could draw a graph and carefully measure the gradient when $x = 5{,}000$, but this is time-consuming and not very accurate. A more useful approach is to calculate the gradient.

Calculating the gradient

To find the gradient of the curve at a particular point, x, we can start by approximating it to the average gradient near that point. Then we can take one point on either side of x – say (x_1, y_1) and (x_2, y_2) – and estimate:

$$\text{gradient at point } x = \text{approximately } \frac{y_2 - y_1}{x_2 - x_1}$$

You can see from Figure 13.3 that this gives the average gradient around x, but not the **instantaneous gradient** exactly at x. But if you move x_1 and x_2 closer together, you get closer to the instantaneous gradient. Then, if you make x_1 and x_2 very, very close to x, you get a value that is very, very close to the instantaneous gradient. In other words, as the distance between x_1 and x_2 approaches zero, you find the instantaneous gradient as the tangent to the curve at x (as shown in Figure 13.4).

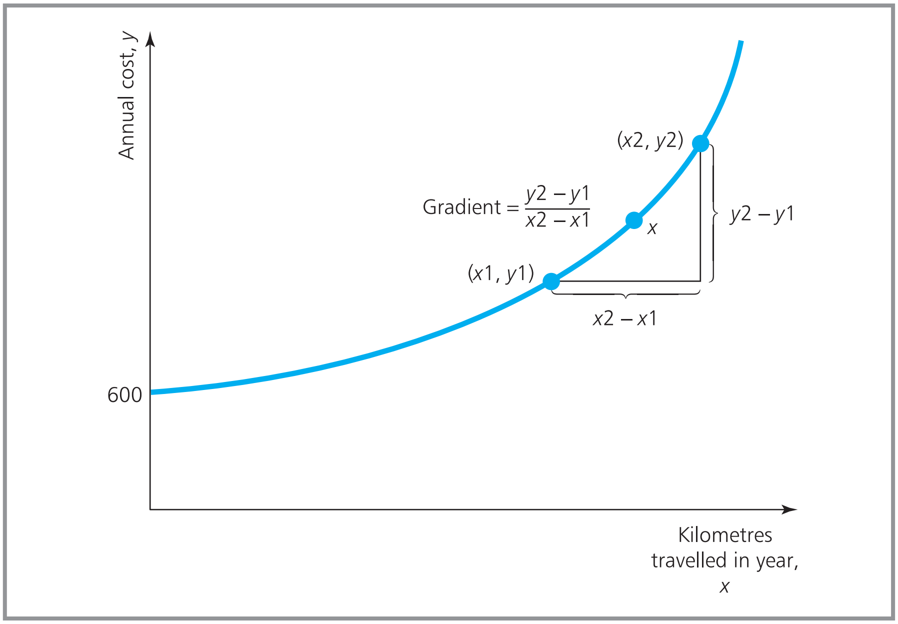


Figure 13.3 Approximate gradient at point *x*

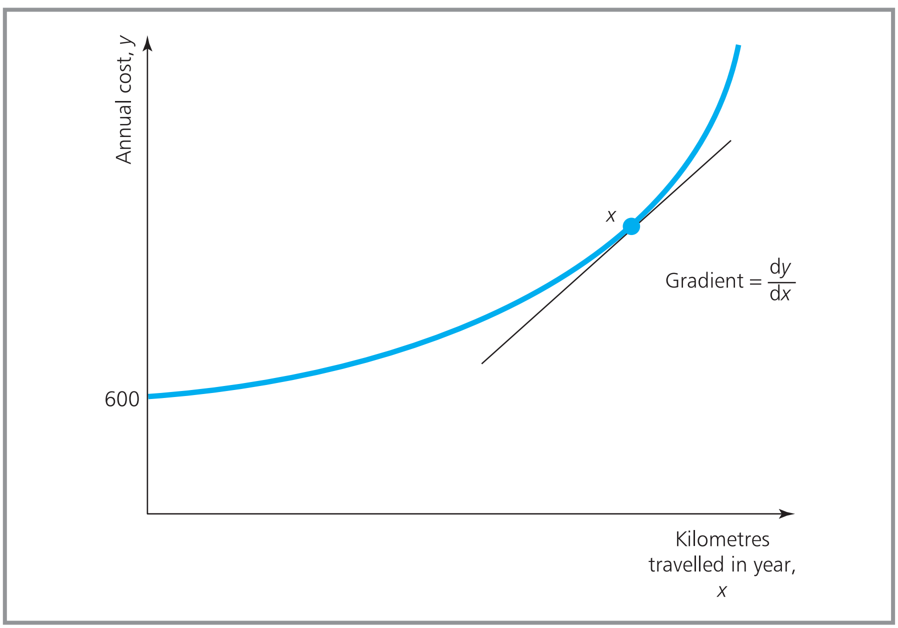


Figure 13.4 The tangent at *x* is the instantaneous gradient

We still need a way of calculating the gradient of the tangent at x, and for this we use **differentiation**.

- **Differentiation** calculates the instantaneous rate of change of y with respect to x.
- This instantaneous gradient is referred to as dy/dx (pronounced 'dee y by dee x').

Rules for differentiation

We can do a lot of differentiation by using two simple rules (the derivations of which are given in the Companion Website, **www.pearsoned.co.uk/waters**).

Differentiation Rule 1

If

$$y = ax^n$$

where a and n are constants, then

$$\frac{dy}{dx} = anx^{n-1}$$

WORKED EXAMPLE 13.2

Differentiate $y = 2x^3$. How quickly is y increasing when $x = 5$?

Solution

This has the general form $y = ax^n$ with $a = 2$ and $n = 3$. Substituting these into rule 1 gives:

$$\frac{dy}{dx} = anx^{n-1} = 2 \times 3 \times x^{3-1} = 6x^2$$

The instantaneous gradient of $2x^3$ at any point is $6x^2$. When $x = 5$, the gradient is $6 \times 5^2 = 150$.

WORKED EXAMPLE 13.3

The total cost of a service, y, is related to the number of people served each day, x, by the equation $y = 7x^2$. How quickly does the cost change when 100 people a day are served?

Solution

Differentiating the total cost gives the rate of change. Substituting $a = 7$ and $n = 2$ into the equation of rule 1 gives:

$$\frac{dy}{dx} = anx^{n-1} = 7 \times 2 \times x^{2-1} = 14x$$

When $x = 100$ the rate of change of cost is $14 \times 100 = 1{,}400$, so the cost is rising by 1,400 for each customer served.

You might interpret this result as the marginal cost, or the additional cost when the number of customers rises from 100 to 101. This would be a reasonable approximation, but it is not quite right. If you subtract the cost of serving 100 customers ($7 \times 100 \times 100 = 70{,}000$) from the cost of serving 101 customers ($7 \times 101 \times 101 = 71{,}407$), you see that it is actually 1,407. The difference is because the gradient of the total cost curve is 1,400 exactly at $x = 100$, but it changes slightly between $x = 100$ and 101.

A consequence of this rule is that when y is a constant, dy/dx equals zero. This is obvious when you remember that the line $y = a$ is a straight line parallel to the x axis, so it must have a gradient of zero.

A second rule for differentiation extends its scope to more complex functions.

Differentiation Rule 2

If

$$y = u + v$$

where both u and v are functions of x, then

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

To put it simply, this means that we can differentiate a function consisting of several terms, such as a polynomial, by applying the first rule to each term separately.

WORKED EXAMPLE 13.4

Differentiate $y = 3x^3 + 14x^2 - x - 12$ with respect to x.

Solution

Rule 2 says that we can apply rule 1 to each term in the equation separately.

- Differentiating the first term, $3x^3$, gives $3 \times 3 \times x^{3-1} = 9x^2$.
- Differentiating the second term, $14x^2$, gives $14 \times 2 \times x^{2-1} = 28x$.
- Differentiating the third term, $-x$, gives $-1 \times x^{1-1} = -1$.
- Differentiating the third term, -12, gives zero.

Adding the separate terms gives the solution – which is called the **derivative** or **first derivative** – as:

$$\frac{dy}{dx} = 9x^2 + 28x - 1$$

WORKED EXAMPLE 13.5

Subhendra Gita measures the output from a process as $y = 2x^3 - 3x$, where y is the total output up to time x. How fast is the output changing when (a) $x = 1$, (b) $x = 4$?

Solution

The gradient gives the rate at which the output is changing, so we need the derivative of y.

$$\frac{dy}{dx} = 3 \times 2 \times x^{3-1} - 1 \times 3 \times x^{1-1} = 6x^2 - 3$$

(a) Substituting $x = 1$ gives the instantaneous gradient when $x = 1$. This is $6x^2 - 3 = 6 \times 1^2 - 3 = 3$. So the output is rising by 3 units in each period.

(b) Substituting $x = 4$ gives the instantaneous gradient when $x = 4$. This is $6 \times 4^2 - 3 = 93$, showing that the output is now rising much faster.

Review questions

13.1 What is the purpose of differentiation?

13.2 What is dy/dx?

Finding the maximum and minimum

It would be very useful to find the minimum of a function so that we could find, for example, the point where the cost is lowest; or we might want to

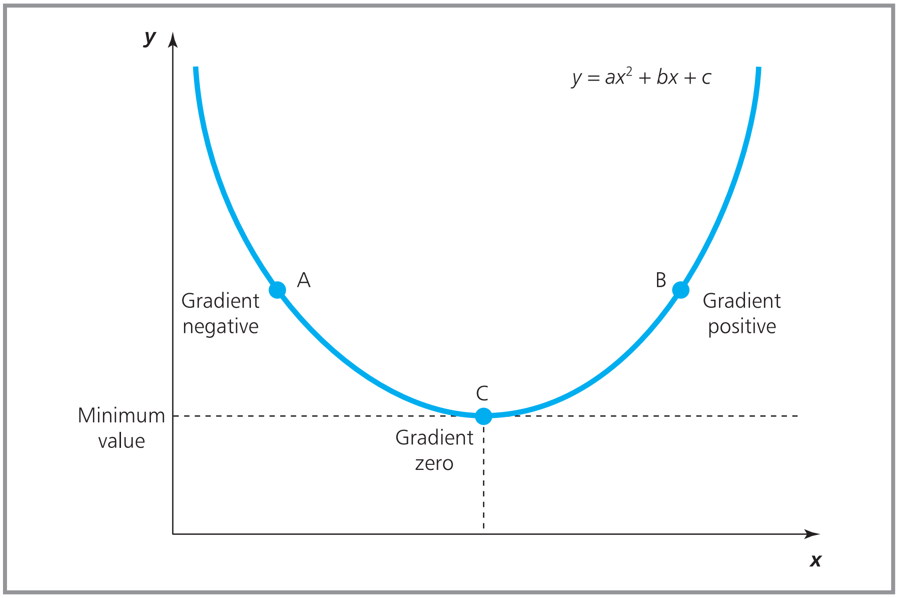


Figure 13.5 The minimum of a function has a gradient of zero

find the maximum, to find where, say, the profit is highest. Differentiation gives a way of finding such points.

Figure 13.5 shows a graph of the quadratic equation $y = ax^2 + bx + c$, which has a clear minimum. Now if you look below this to the point A, the gradient is clearly negative, showing that y is falling in value as x increases; and if you look above the minimum to the point B, the gradient is positive, showing that y is increasing in value as x increases. The most interesting point comes between these at point C, where the gradient is zero; in other words the tangent to the curve is parallel with the x axis. This happens at only one specific point, where the graph reaches its minimum.

This is a useful result, as it shows how to identify a minimum – as you find the gradient of a function, and the point where this is equal to zero identifies a minimum.

WORKED EXAMPLE 13.6

What can you say about the minimum of $y = 2x^2 - 4x + 10$?

Solution

You can differentiate $y = 2x^2 - 4x + 10$ to find the gradient at any point.

$$\frac{dy}{dx} = 2 \times 2 \times x - 4 = 4x - 4$$

For a minimum value of y this gradient is equal to zero, so:

$$4x - 4 = 0 \quad \text{or} \quad x = 1$$

The minimum point of the graph occurs when $x = 1$. Substituting $x = 1$ into the equation for y gives the minimum value of:

$$y = 2x^2 - 4x + 10 = 2 \times 1^2 - 4 \times 1 + 10 = 8$$

So the minimum value of the graph is $y = 8$ which occurs when $x = 1$.

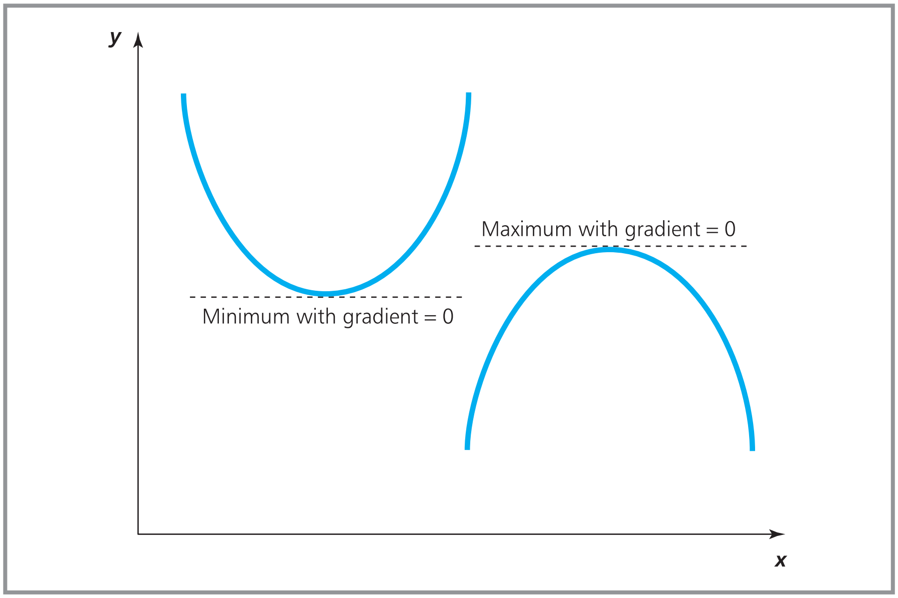


Figure 13.6 Both minima and maxima have gradients of zero

In this example, we assumed that the point where the gradient was zero was a minimum. But if you look at Figure 13.6, you can see that maximum values also have gradients of zero. Points on a graph where the gradient is zero are called **turning points** – which can be either maxima or minima. In reality, they are often **local optima** – which are maximum or minimum values within a restricted range – rather than **global optima**, as shown in Figure 13.7.

We clearly need more information about the shape of a curve and, in particular, whether a turning point is a minimum or a maximum. Of course, we could find this by drawing a graph, but we can get more information from calculations. Have a look at Figure 13.8, where the top part shows a curve with a minimum at point A and a maximum at point B. The gradient is clearly zero at these two points, but looking at the curve more closely shows some interesting points. At the left-hand side of the graph the gradient, dy/dx, is negative. Then at point A it goes through zero and becomes positive. Then at point B it goes through zero and becomes negative again. So we can draw a graph of the gradient at any point, and the bottom part of Figure 13.8 shows the graph of the derivative, dy/dx.

You can see from this that:

- for a minimum value of the function, y, the gradient dy/dx has the value 0 and is increasing with x
- for a maximum value of the function, y, the gradient dy/dx has the value 0 and is decreasing with x.

These observations show how we can identify a turning point as either a maximum or a minimum, without drawing the graph. All we need to do is find the gradient of dy/dx and see whether this is positive (showing that dy/dx is increasing and we have a minimum) or negative (showing that dy/dx is decreasing and we have a maximum).

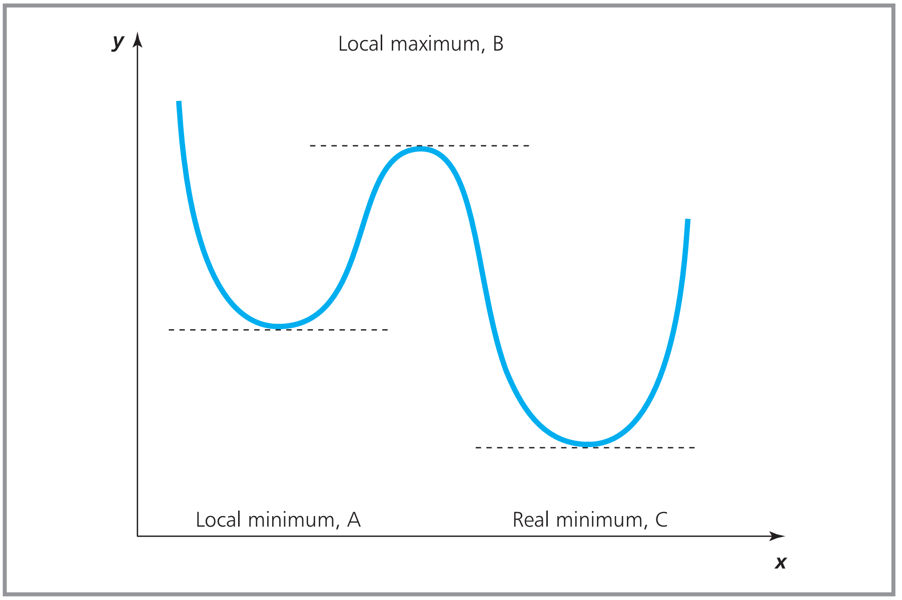


Figure 13.7 Local maxima and minima are turning points

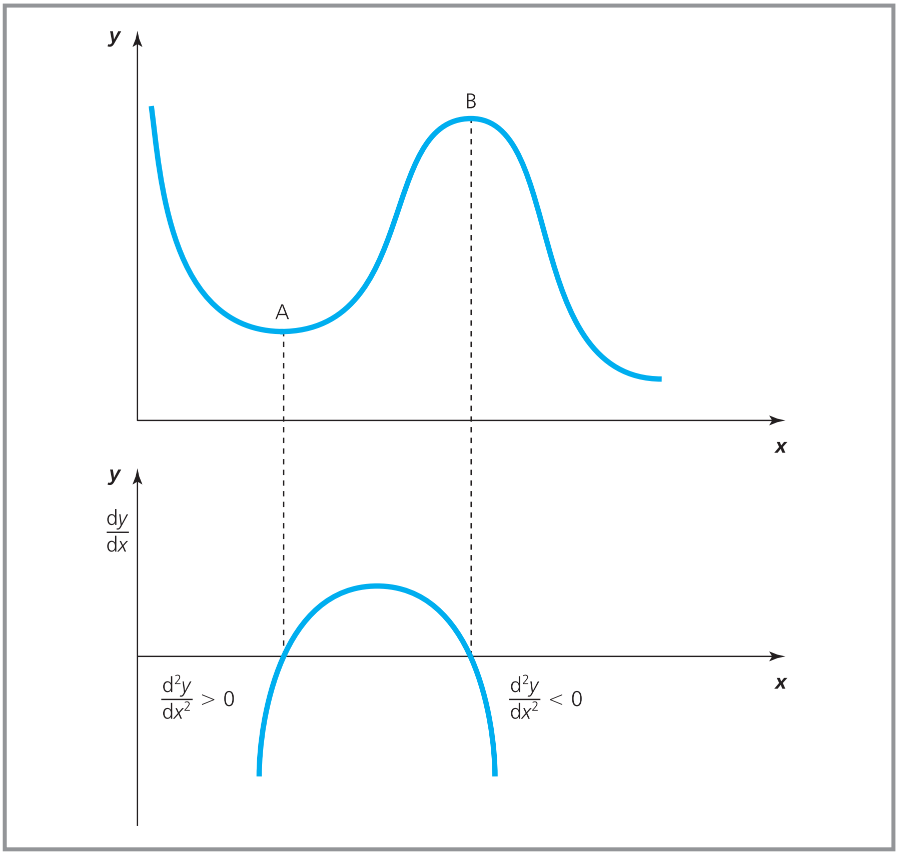


Figure 13.8 For optimal values of y, $dy/dx = 0$ and d^2y/dx^2 identifies maxima and minima

So how do we find whether dy/dx is increasing or decreasing? If we have a function y, we can differentiate it to find the gradient dy/dx. But now we have dy/dx and we can find its gradient by differentiation in exactly the same way. In other words, we differentiate dy/dx, to get a result called d^2y/dx^2 (pronounced 'dee two y by dee x squared' and technically called the **second derivative**). When this is negative, the gradient is decreasing and the original function has a maximum value; when this is positive, the gradient is increasing and the original function has a minimum value. This gives the general rule:

- The maximum of a function occurs when $dy/dx = 0$ and $d^2y/dx^2 < 0$.
- The minimum of a function occurs when $dy/dx = 0$ and $d^2y/dx^2 > 0$.

WORKED EXAMPLE 13.7

If $y = 4x^2 + 3x - 2$, what are dy/dx and d^2y/dx^2 and what do they mean?

Solution

With $y = 4x^2 + 3x - 2$, differentiating in the usual way gives:

$$\frac{dy}{dx} = 8x + 3$$

This is the derivative (or really the first derivative) which gives the gradient at any point on the curve. When this is zero the curve has a turning point. This occurs when $8x + 3 = 0$, or $x = -3/8$. Substituting this value into the equation gives:

$$y = 4x^2 + 3x - 2 = 4 \times (-3/8)^2 - 3 \times 3/8 - 2$$
$$= 0.5625 - 1.125 - 2 = -2.5625$$

Now looking at $dy/dx = 8x + 3$ and differentiating this in the usual way:

$$\frac{d^2y}{dx^2} = 8$$

This is the second derivative, which shows how the gradient is changing. This is positive, showing that the gradient is continually growing at a constant rate, and the value we identified is a minimum.

To summarise, the function $4x^2 + 3x - 2$ has a minimum value of -2.5625 when $x = -3/8$.

WORKED EXAMPLE 13.8

The total cost of providing a service is $3x^2 - 12x + 30$, where x is the number of people served each week (in some consistent units). What is the optimal size of the service?

Solution

With have only a limited amount of information, and have to assume that the best solution is to minimise the total cost. As $y = 3x^2 - 12x + 30$, we can differentiate this to find the gradient, which is $dy/dx = 6x - 12$. There is a turning point when this gradient is equal to zero – which means that $6x - 12 = 0$, or $x = 2$. Now we have to see whether this turning point is a maximum or a minimum. Differentiating dy/dx gives $d^2y/dx^2 = 6$. This is positive, so the turning point is a minimum.

At this minimum the function has the value $y = 3x^2 - 12x + 30 = 3 \times 2^2 - 12 \times 2 + 30 = 18$. So the optimal solution is to serve two customers a week, giving costs of 18.

WORKED EXAMPLE 13.9

The duration of a project can be altered by varying the resources available. Using an amount of resources x gives a net revenue of $x^3 - 3x^2$. This result is valid only in the range $x = 0$ to $x = 5$. What does the revenue function look like? What value of x gives the highest net revenue?

Solution

We can differentiate the revenue function to find the turning points. As the equation is a cubic, there are two turning points, one a maximum and the other a minimum. Then:

$$y = x^3 - 3x^2 \quad \text{so} \quad \frac{dy}{dx} = 3x^2 - 6x$$

The turning points occur when this is equal to zero, when:

$$3x^2 - 6x = 0 \quad \text{or} \quad 3x(x-2) = 0$$

giving

$$x = 0 \quad \text{and} \quad x = 2$$

Now differentiating the gradient gives:

$$\frac{d^2y}{dx^2} = 6x - 6$$

When $x = 0$, $d^2y/dx^2 = 6x - 6 = -6$, and as this is negative the point is a maximum; when $x = 2$, $d^2y/dx^2 = 6x - 6 = 6$, and as this is positive the point is a minimum.

From this information, it would be reasonable to assume that the maximum revenue occurs when $x = 0$. But if you draw a graph of the revenue (shown in Figure 13.9) you see that this is actually a local maximum. In the valid range of $x = 0$ to $x = 5$ the net revenue is essentially U-shaped, with a maximum value when $x = 5$. At this point revenue $= 5^3 - 3 \times 5^2 = 50$. Remember that you always have to be careful when analysing problems, and when you have some data or a function it is always worth drawing a graph to see what it looks like.

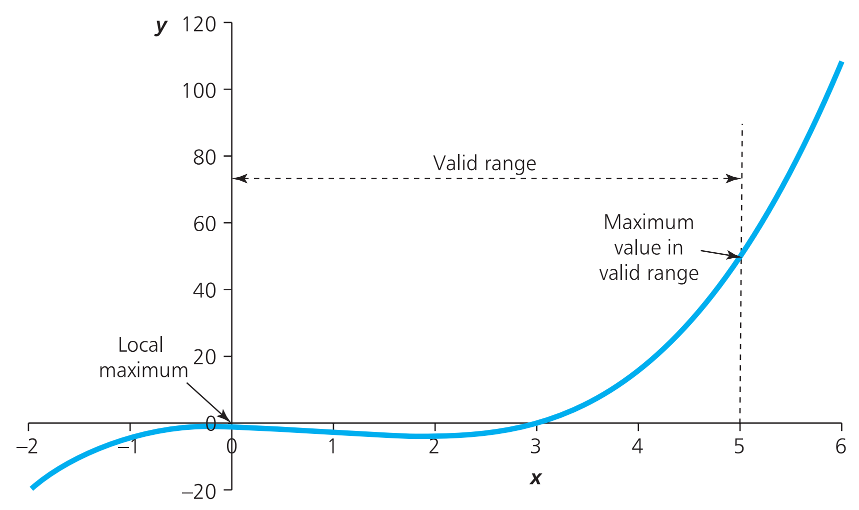


Figure 13.9 Optimal value for revenue in worked example 13.9

Review questions

13.3 How would you identify the minimum point of a curve?

13.4 If $p = q + r$ and both q and r are functions of c, how would you find dp/dc?

13.5 Explain in words what is meant by d^2y/dx^2.

Marginal analyses

Calculus is particularly useful for analysing the cost of a process. Suppose you know the total cost of producing a number of units of a product. You can easily find the average cost – but it is often more useful to look at the marginal cost, which is the cost of producing one extra unit. We saw one reason for this, when we showed that the optimal production level is the point where the marginal cost is equal to the marginal revenue.

When making one extra unit of a product, the total cost, TC, increases by the marginal cost, MC. In other words, the marginal cost is the rate at which the total cost is changing – so it is the gradient of the total cost curve. We have already seen that this is really an approximation, but it works well for high production numbers. And it means that when we know the equation for the total cost, we can differentiate it to get the marginal cost.

When

total cost, TC $= y$

then

$$\text{marginal cost, MC} = \frac{dy}{dx}$$

The same argument holds for revenues. When we sell one extra unit of a product the total revenue, TR, increases by the marginal revenue, MR. So the marginal revenue is the rate at which the total revenue is changing – or the gradient of the total revenue curve:

When

total revenue, TR $= y$

then

$$\text{marginal revenue, MR} = \frac{dy}{dx}$$

WORKED EXAMPLE 13.10

The total cost of making x units of a product is TC $= 2x^2 + 4x + 500$. What are the total, fixed, variable, marginal and average costs? What are the costs of making 500 units of the product?

Solution

- The total cost, TC, is given as TC $= 2x^2 + 4x + 500$.
- The fixed cost is the cost that occurs regardless of production quantity, so this is 500.
- The variable cost is the cost that changes with production quantity, so this is $2x^2 + 4x$.
- The marginal cost, MC, is the gradient of the total cost curve, which we find by differentiating TC, giving MC $= dy/dx = 4x + 4$.
- The average cost is the total cost divided by the number of units made, which is TC$/x = (2x^2 + 4x + 500)/x = 2x + 4 + 500/x$.

When x is 500:

- TC $= 2 \times 500^2 + 4 \times 500 + 500 = 502{,}500$.
- Fixed cost remains unchanged at 500.
- Variable cost is $2 \times 500^2 + 4 \times 500 = 502{,}000$.
- Marginal cost $= 4 \times 500 + 4 = 2{,}004$.
- Average cost $= 2 \times 500 + 4 + 500/500 = 1{,}005$.

The total profit, TP, of a process is the difference between total revenue, TR, and total cost, TC:

$$TP = TR - TC$$

We can differentiate this to find the point of maximum profit. But with a little thought, you can see that the profit is a maximum when the marginal revenue is equal to the marginal cost. The reasoning behind this is as follows:

- If the marginal cost is higher than the marginal revenue, each additional unit makes a net loss, so fewer units should be made.
- If the marginal revenue is higher than the marginal cost, each additional unit makes a net profit, so more units should be made.
- Only when the marginal cost is equal to the marginal revenue is there a point of stability that maximises profit.

WORKED EXAMPLE 13.11

The total revenue and total cost for a product are related to production, x, by:

$$TR = 14x - x^2 + 2{,}000$$
$$TC = x^3 - 15x^2 + 1{,}000$$

How many units should the company make to (a) maximise total revenue, (b) minimise total cost, and (c) maximise profit?

Solution

(a) You can find the rate of change of total revenue by differentiating TR with respect to x. This gives the marginal revenue, MR = $14 - 2x$. A turning point occurs when this is equal to zero, that is when $14 - 2x = 0$ or $x = 7$. Differentiating again gives the second derivative as -2, and as this is negative, it confirms that the turning point is a maximum. When $x = 7$ the total revenue reaches a maximum of $14 \times 7 - 7^2 + 2{,}000 = 2{,}049$.

(b) You can find the rate of change of total cost by differentiating TC with respect to x. This gives the marginal cost, MC = $3x^2 - 30x$. Turning points occur when this is equal to zero, meaning that $3x^2 - 30x = 0$. These occur when $x(3x - 30) = 0$, so either $x = 0$ or $x = 10$. Differentiating the marginal cost gives the second derivative as $6x - 30$. This is negative when $x = 0$, indicating a maximum; and it is positive when $x = 10$, indicating a minimum. So the total cost is minimised by making 10 units. At this point the total cost is $10^3 - 15 \times 10^2 + 1{,}000 = 500$.

(c) Total profit = total revenue – total cost

$$\begin{aligned} TP &= TR - TC \\ &= 14x - x^2 + 2{,}000 - (x^3 - 15x^2 + 1{,}000) \\ &= -x^3 + 14x^2 + 14x + 1{,}000 \end{aligned}$$

Differentiating this gives turning points when:

$$-3x^2 + 28x + 14 = 0$$

Using the standard equation to solve a quadratic equation (see Chapter 3), the positive root is 9.8 (which you can check by substitution). Differentiating again gives the second derivative as $-6x + 28$, which is negative when $x = 9.8$, confirming a maximum. At this point the maximum profit is $-x^3 + 14x^2 + 14x + 1{,}000 = -9.8^3 + 14 \times 9.8^2 + 14 \times 9.8 + 1{,}000 = 1{,}540.6$.

Price elasticity of demand

The **price elasticity of demand** is an important concept in economics, as it shows how the demand for a product changes when there is a change in its price. A formal definition says that:

$$\text{elasticity of demand} = \frac{\text{proportional change in demand}}{\text{proportional change in price}}$$

Suppose that a company sells a quantity q of a product at a price p, then the elasticity of demand shows the change in q that is caused by a change in p.

$$\text{elasticity of demand} = \frac{\text{change in } q/\text{original } q}{\text{change in } p/\text{original } p}$$

$$= \frac{\text{original } p}{\text{original } q} \times \frac{\text{change in } q}{\text{change in } p}$$

If the changes in price and quantity are small (in theory so small that they approach zero), then:

$$\frac{\text{change in } q}{\text{change in } p} = \text{gradient of } q \text{ with respect to } p = \frac{dq}{dp}$$

and:

$$\text{elasticity of demand} = \frac{p}{q} \times \frac{dq}{dp}$$

WORKED EXAMPLE 13.12

Suppose that the price and demand for a product are related by the equation:

$$p = 200 - q^2$$

What is the price elasticity of demand? What is the value of this when q is equal to 10?

Solution

To calculate the elasticity of demand we need to know dq/dp. We can differentiate p with respect to q to get:

$$p = 200 - q^2 \quad \text{so} \quad \frac{dp}{dq} = -2q$$

This gives dp/dq and we actually want dq/dp which we find from:

$$\frac{dq}{dp} = \frac{1}{dp/dq} = \frac{1}{-2q} = \frac{-1}{2q}$$

Now substituting the known values into the equation for elasticity gives:

$$\text{elasticity of demand} = \frac{p}{q} \times \frac{dq}{dp}$$

$$= \frac{200 - q^2}{q} \times \left(\frac{-1}{2q}\right) = \frac{-200 + q^2}{2q^2}$$

When $q = 10$:

$$\text{elasticity of demand} = \frac{-200 + q^2}{2q^2}$$

$$= \frac{-200 + 100}{200} = -0.5$$

A negative elasticity of demand is normal, as it means that any increase in price reduces demand. In this case, every unit increase in price decreases demand by 0.5 units.

Review questions

13.6 How would you define the marginal cost of production?

13.7 Given the total revenue function for a product, how could you find the average revenue and the marginal revenue?

13.8 What is the elasticity of demand?

13.9 What would a positive elasticity of demand mean?

IDEAS IN PRACTICE Novotnoya Chomskaya

Novotnoya Chomskaya (NC) is a small aviation company that runs a short-haul service to remote oil camps around the Arctic. This is a surprisingly competitive industry and the company often struggles to make a profit. In common with other forms of transport, when they try to raise prices the demand quickly falls, as customers move to other companies.

The company did an analysis of its costs, which is illustrated by a standard journey carrying freight to a work camp in Northern Siberia. By asking customers how much they would pay to get a one kilogram package delivered, they found a relationship that could be approximated by:

$$\text{demand} = 6{,}000 - 50 \times \text{price}$$

This was a useful result as:

$$\begin{aligned}\text{total revenue} &= \text{price} \times \text{demand}\\ &= 6{,}000 \times \text{price} - 50 \times \text{price}^2\end{aligned}$$

Differentiating this quadratic equation with respect to price shows that revenue is maximised when:

$$6{,}000 - 100 \times \text{price} = 0$$

or

$$\text{price} = 60$$

This rough calculation suggested that the company could maximise its revenue by charging \$60 for a one kilogram package. Then the demand is $6{,}000 - 50 \times 60 = 3{,}000$ and the total revenue is $3{,}000 \times 60 = 180{,}000$. When all the other data was reviewed in the light of similar results, the company adjusted its pricing policy and consolidated its presence in the market.

Source: Negeyovich P., *Pricing Policies*, Novotnoya Chomskaya, Moscow, 2006.

Integration

Differentiation takes a function and finds the instantaneous gradient at any point. Sometimes we want to work the other way around, so that we can start with an equation for the gradient and use this to find the function. This is the basis of **integration**, which you can imagine as the reverse of differentiation (as shown in Figure 13.10).

Suppose you have a function $y = x^2$. You can differentiate this to give the gradient of $dy/dx = 2x$. Taking this problem the other way around, if you have a function whose gradient is $2x$, you can integrate this to give the function as x^2. Unfortunately, this intuitive approach to integration immediately meets a problem. If you have a function $y = x^2 + 5$, you can also differentiate it to give a gradient of $2x$. So if you know the derivative is $2x$, you cannot tell whether the function is x^2 or $x^2 + 5$, or $x^2 + c$, where c is any constant.

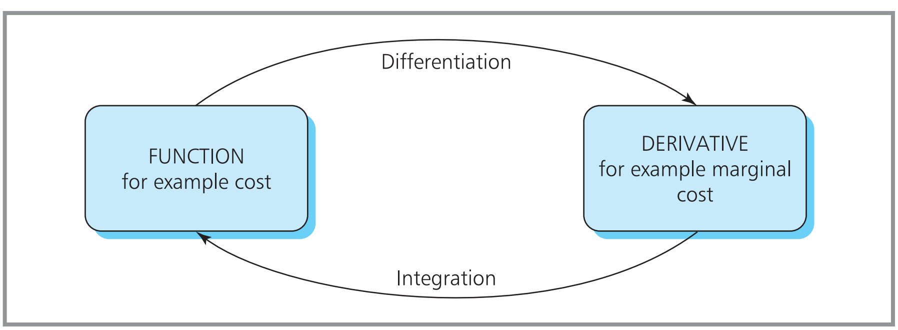


Figure 13.10 Relationship between differentiation and integration

The standard sign for integration is $\int$, which is an extended 's'. When we integrate with respect to x, we point this out by putting 'dx' at the end of the function being integrated. Then we write the integration as:

$$\int 2x \, dx = x^2 + c$$

where, unfortunately, we cannot give a value to c. So this leads to the first rule for integration, which is simply the reverse of rule 1 for differentiation:

Integration Rule 1

If

$$y = ax^n$$

then

$$\int y \, dx = \frac{ax^{n+1}}{n+1} + c$$

You can check this by differentiating $ax^{n+1}/(n + 1) + c$, which gives ax^n. This rule is true for all values of n except -1, which would lead to a division by zero. When $n = -1$ there is an unusual but standard result, which says that $\int x^{-1} \, dx = \int (1/x) \, dx = \log_e x + c$.

A second rule for integration is equivalent to rule 2 for differentiation:

Integration Rule 2

If u and v are both functions of x, then

$$\int (u + v) \, dx = \int u \, dx + \int v \, dx$$

WORKED EXAMPLE 13.13

What is the integral of $12x^5$?

Solution

This is in the form ax^n with $a = 12$ and $n = 5$. Rule 1 tells us that:

$$\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c$$

Substitution gives:

$$\int 12x^5 \, dx = \frac{12x^6}{6} + c = 2x^6 + c$$

You can check this by doing the differentiation to get back to the original function.

WORKED EXAMPLE 13.14

What is the integral of $x^2 + 3x - 6$ with respect to x?

Solution

Rule 2 tells us that we can apply rule 1 to each part of the function in turn, so:

$$\int (x^2 + 3x - 6)\, dx = \int x^2\, dx + \int 3x\, dx - \int 6\, dx$$

Applying rule 1 to each term:

- $\int x^2\, dx = \frac{x^3}{3}$ (substituting $a = 1$ and $n = 2$)
- $\int 3x\, dx = \frac{3x^2}{2}$ (substituting $a = 3$ and $n = 1$)
- $\int 6\, dx = 6x$ (substituting $a = 6$ and $n = 0$)

Adding the terms and the integration constant gives the solution:

$$\int (x^2 + 3x - 6)\, dx = x^3/3 + 3x^2/2 - 6x + c$$

Integration always leaves a constant, c, that we do not know. The only way of finding it is to have more information. For instance, you might know that $dy/dx = 2x$, and when $x = 2$, $y = 20$, so you could do a substitution and find the missing constant. Integrating $dy/dx = 2x$ gives $y = x^2 + c$, and you know that $y = 20$ when $x = 2$, so substitution gives $20 = 2^2 + c$, or $c = 16$.

WORKED EXAMPLE 13.15

If $dy/dx = x^2 - 2x + 4$, what is the value of y? What is the integration constant if $y = 30$ when $x = 3$?

Solution

To find y we integrate dy/dx, and from rule 2:

$$\int (x^2 - 2x + 4)\, dx = \int x^2\, dx - \int 2x\, dx + \int 4\, dx$$

Applying rule 1 to each term and adding the integration constant gives:

$$\int (x^2 - 2x + 4)\, dx = x^3/3 - x^2 + 4x + c$$

We know that $y = 30$ when $x = 3$, and substituting these values gives:

$$y = x^3/3 - x^2 + 4x + c$$

$$30 = 3^3/3 - 3^2 + 4 \times 3 + c = 9 - 9 + 12 + c$$

or

$$c = 18$$

The final solution is $y = x^3/3 - x^2 + 4x + 18$.

Definite integrals

The integrals we have described are **indefinite integrals**, which means the result is given as a function. A **definite integral** evaluates this function at two points and finds the difference. For example, the indefinite integral may tell us:

$$\int 2x\, dx = x^2 + c$$

The definite integral evaluates this at two points, say $x = 5$ and $x = 2$, and finds the difference. When $x = 5$, $x^2 + c = 25 + c$; when $x = 2$, $x^2 + c = 4 + c$. Then the difference between these is $25 + c - (4 + c) = 21$.

We can use a new notation to describe this:

$$\int_2^5 2x\ dx = |x^2 + c|_2^5 = (5^2 + c - 2^2 - c) = 21$$

In the first part, the 'limits of integration' – or the two values of x we want to substitute – are added around the integral sign. Then the second part shows the indefinite integral enclosed by vertical lines, with the limits put after the lines. By convention, the value of the integral at the bottom value is subtracted from the value at the top value.

This procedure might seem rather strange, but finding the definite integral has two benefits.

1 You do not have to worry about the integration constant, c, as this always disappears in the subtraction.
2 More importantly, it introduces the idea that integration can be viewed as a kind of summation.

We can illustrate the second of these features by returning to our analysis of the marginal cost. Remember that we found the marginal cost by differentiating the total cost function. Now reversing this shows that we can find the total cost by integrating the marginal cost function. This suggests that integration can be viewed as a means of summation: you find the total costs by adding – or integrating – the marginal cost. With this view, the definite integral makes more sense. In particular, when you find the definite integral of the marginal cost between two points x_1 and x_2, you get the total cost of making the units between x_1 and x_2.

WORKED EXAMPLE 13.16

The marginal cost of a process is $3x^2 + 6x - 10$. What is the cost of making units 10 to 20?

Solution

You find the total cost by integrating the marginal cost function:

$$TC = \int MC\ dx = \int (3x^2 + 6x - 10)\ dx$$
$$= x^3 + 3x^2 - 10x + c$$

This is the indefinite integral, and the total cost of making units 10 to 20 is the definite integral between the limits of 10 and 20.

$$\int_{10}^{20} (3x^2 + 6x - 10)\ dx = |x^3 + 3x^2 - 10x|_{10}^{20}$$
$$= [20^3 + 3 \times 20^2 - 10 \times 20] - [10^3 + 3 \times 10^2 - 10 \times 10]$$
$$= 9{,}000 - 1{,}200 = 7{,}800$$

Review questions

13.10 What is integration?

13.11 Explain the meaning of $\int y\ dx = f(x)$.

13.12 How can you find the integration constant?

13.13 What is the purpose of the definite integral?

CHAPTER REVIEW

This chapter introduced the ideas of calculus, describing the use of differentiation and integration.

- The gradient of a function describes its rate of change. You use calculus for calculations with this rate of change.
- Differentiating a function gives its gradient – or instantaneous rate of change – at any point. This is described by the first derivative, dy/dx. There are two basic rules for differentiation:
  - If $y = ax^n$ then $dy/dx = anx^{n-1}$, for any values of a and n.
  - If $y = u + v$ where u and v are both functions of x, then $dy/dx = du/dx + dv/dx$.
- When graphs have turning points – either maxima or minima – the gradient is zero. The turning point is a minimum when $dy/dx = 0$ and $d^2y/dx^2 > 0$; it is a maximum when $dy/dx = 0$ and $d^2y/dx^2 < 0$.
- If you know the equation for the total cost (or revenue) of a process, you can differentiate it to give the marginal cost (or revenue). A related calculation finds the price elasticity of demand.
- Integration is the reverse of differentiation, allowing us to define a function from its instantaneous gradient. You can also view it as a means of summation.

CASE STUDY Lundquist Transport

Lundquist Transport provides a specialist service, designing logistics systems for companies around Scandinavia. It also runs a fleet of long-distance trucks from its headquarters outside Copenhagen. The company wants to expand this fleet and take advantage of the growing trade between Scandinavia and central and Eastern Europe, where the amounts of goods moving to Poland, Hungary, the Czech Republic and beyond are increasing rapidly.

Lundquist recently examined their revenues and costs. Costs do not rise linearly with demand, as the company can get substantial economies of scale and can adjust operations to meet demand in different ways. Revenues are also not linear, as they give discounts for larger orders and longer distances. The actual figures are confidential, but to allow comparisons and discussion Lundquist illustrate them in terms of 'standard' values shown in the following table.

Volume of business	Revenues	Costs
0.25	3,250	3,375
0.5	4,500	3,000
0.75	4,750	2,875
0.85	4,570	2,895
1.0	4,000	3,000
1.1	3,420	3,120
1.25	2,250	3,375

The current volume of business is described as '1.0' and the operations managers think that this is putting too much pressure on them. In the long term, any expansion by Lundquist will need major changes in the way that they work – particularly buying new facilities and employing more people. In the short term, it may be better to keep the same operations and make adjustments to the volume of business. It is fairly easy to do this in such a competitive market, simply by changing the

Case study continued

price structure. Lundquist have done no detailed calculations on this, but the commonly accepted rule is that a 5% increase in price reduces demand by 10%.

Both of these options – long-term expansion and short-term adjustment – involve some risk. The expansion relies on forecasts of long-term trends in an area of continuing economic uncertainty. Short-term adjustments might harm their reputation and encourage competitors to develop new operations.

Question

- **What would you advise the senior managers of Lundquist to do? Prepare a report which describes the current situation, analyses available information, discusses alternative policies, and recommends a best option.**

PROBLEMS

13.1 Differentiate $y = 12x^7$ and $y = 7x^{12}$ with respect to x.

13.2 What is the derivative of $y = 6.2x^4 + 3.3x^3 - 7.1x^2 - 11.9x + 14.3$ with respect to x? Where are the turning points? What are the maximum and minimum values?

13.3 Differentiate $x = 2y^2 - 3y + 7$ with respect to y.

13.4 Jonas Steinway's product sells for €10 a unit. The total cost of making and selling x units of the product is $x^2 - 20x + 30$. What is the fixed cost of production? How is the profit related to the number of units made? What level of production maximises profit?

13.5 What are the first and second derivatives of the following functions?
(a) $7.2x^2 - 3.3x + 7.9$
(b) $2x^7 - 4x^4 - 3x^2$
(c) $x^{24} - 4x^{-1}$

13.6 Does the function $y = 2x^2 + 5x + 10$ have a turning point? Is this a minimum or a maximum? How does this compare with $y = x^3 + 6x^2 - 15x$?

13.7 The total cost, TC, and total revenue, TR, for a product are:

$$TR = 12x - 2x^2 + 5{,}000 \quad \text{and}$$
$$TC = x^3 - 10x^2 + 3{,}000$$

How many units will (a) maximise total revenue, (b) minimise total cost, (c) maximise profit?

13.8 Integrate $9x^3 - 12x^2 + 4x - 6$.

13.9 Integrate $6x^2 - 8$. If the integral is known to have a value of 300 when $x = 5$, what is the integration constant?

13.10 What is the definite integral of $4x^3 + 12x - 6$ between the limits 5 and 10?

RESEARCH PROJECTS

13.1 Calculus is one of the most difficult quantitative tools for managers to understand. Not surprisingly, this affects the amount it is used, even for areas like financial planning. Find some real examples where managers have used calculus. What benefits and problems have they found? In what other areas could it be used if managers understood it better?

13.2 There is a difference between the marginal cost (which is discrete) and the gradient of the total cost curve (which is continuous). This difference is small with high production rates. How could you demonstrate this? Run some experiments to see how differentiation works with various financial analyses.

Sources of information

Further reading

Books on calculus are often very mathematical, but you might look at the following.

Adams C.C., Hass J. and Thompson A., *How to Ace Calculus*, W.H. Freeman, New York, 1998.

Adams R.A., *Calculus; a Complete Course* (6th edition), Addison Wesley, Reading, MA, 2005.

Ayres F. and Mendelson E., *Schaum's Outline of Calculus*, McGraw-Hill, New York, 1999.

Ebersole D., Schattschneider D., Sevilla A. and Somers K., *A Companion to Calculus*, Thomson International, Belmont, CA, 2006.

Kelley W.M., *Complete Idiot's Guide to Calculus*, Penguin, London, 2002.

Ryan M., *Calculus for Dummies*, Hungry Minds, Inc., New York, 2003.

Van Glabek J., *Introduction to Calculus*, HarperCollins, New York, 2006.

PART 4

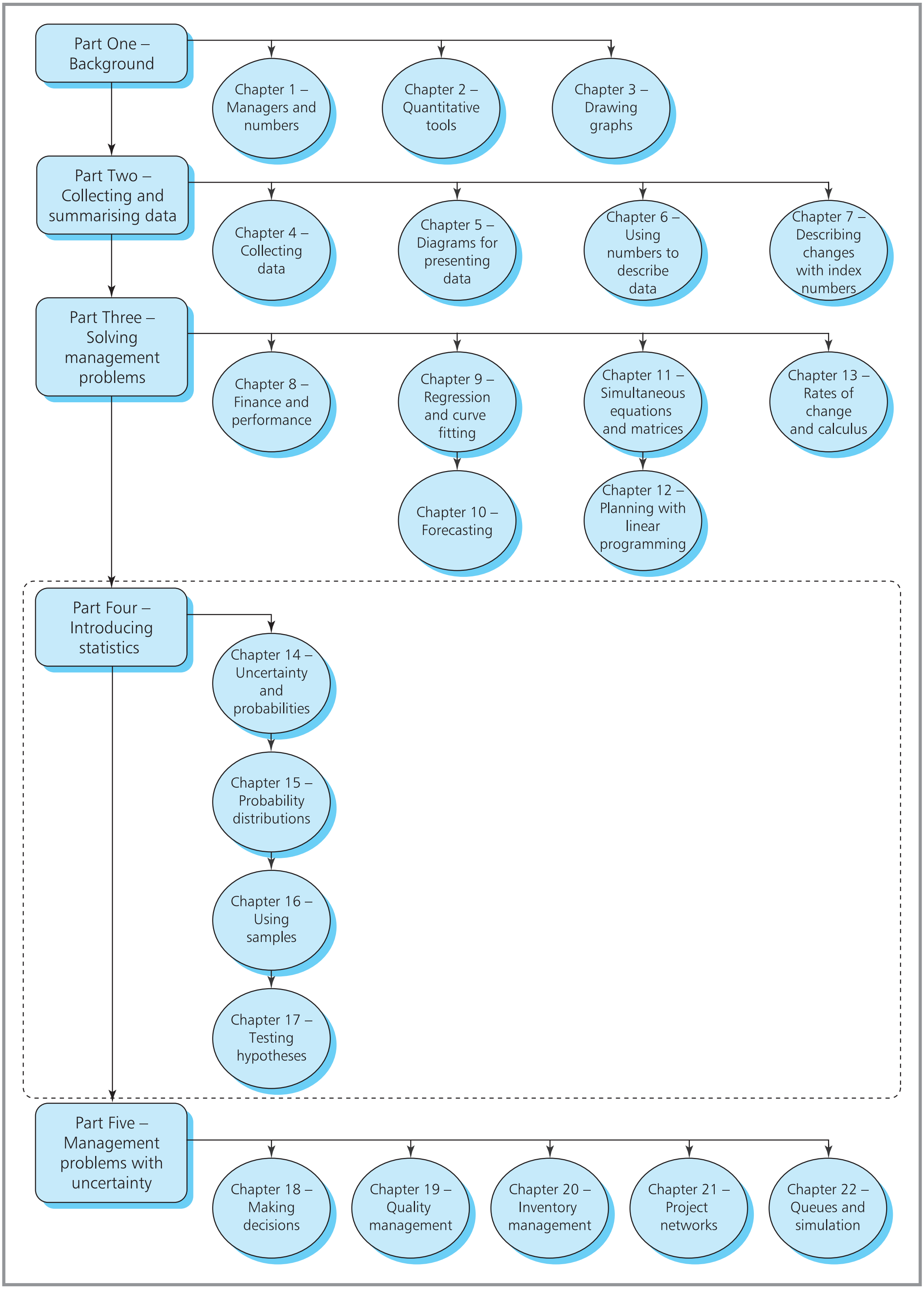
Introducing statistics

So far we have looked at problems where there is certainty – where we know exactly what will happen in the future, the values of variables, the effects of changes, and so on. In reality, most problems include a lot of uncertainty. We do not really know what customers will buy next year, how much oil will cost, what interest rates will be, or what competitors will do. In this part of the book, we look at ways of dealing with uncertainty.

The book is divided into five parts, each of which covers a different aspect of quantitative methods in business. The first part gave the background and context for the rest of the book. The second part showed how to collect and summarise data, and the third part used this data to solve some common business problems. These problems were deterministic, which means that they dealt with certainties. This is the fourth part of the book, which gives an introduction to probabilities and statistical methods. The final part uses these to solve problems with uncertainty.

There are four chapters in this fourth part. Chapter 14 introduces the ideas of probability as a way of measuring uncertainty. This is the core idea that is developed in the rest of the book. Chapter 15 looks at probability distributions, which describe some common patterns in uncertain data. Chapter 16 returns to the theme of using samples for collecting data, and Chapter 17 introduces the ideas of statistical testing, focusing on hypothesis testing.

Map 4 shows how these chapters fit into the rest of the book.



Map 4 Map of chapters – Part Four

CHAPTER 14

Uncertainty and probabilities

Contents

Chapter outline

Previous chapters have assumed that we can describe a problem with certainty. This chapter looks at situations where this is not true, and there is some level of uncertainty, which is measured by probabilities. The probability of an event is its likelihood or relative frequency.

This chapter introduces the underlying ideas of probabilities, laying the foundations for all following chapters. It is important that you understand this before moving on. If you have any difficulties, it is worth spending the time to sort them out. If you want more information, you might find some of the further reading at the end of the chapter useful.

After finishing this chapter you should be able to:

- appreciate the difference between deterministic and stochastic problems
- define probability and appreciate its importance
- calculate probabilities for independent events
- calculate probabilities for mutually exclusive events
- understand the concept of dependent events and conditional probabilities
- use Bayes' theorem to calculate conditional probabilities
- draw probability trees.

Measuring uncertainty

So far in this book we have assumed that we can describe a problem with certainty. We know the number of sales and the prices charged with certainty, and can say, 'PhoneLoft sells 1,000 units a year at €120 a unit'. When you look at production, you can say with certainty that 'Bionorm makes 120,000 units a year at an average cost of $8 a unit'; when you look at employees, you know how many are employed, their hours of work, the number of customers they serve, and so on. Such situations are called **deterministic**.

In reality, we can almost never be so confident. When you spin a coin, you do not know whether it will come down heads or tails; a company launching a new service does not know how many customers it will attract; someone selling a house does not know exactly how much a buyer will pay; a manufacturer does not know exactly how many units it will make. Each of these has some uncertainty, and such situations are described as **stochastic** or **probabilistic**.

Although stochastic problems contain uncertainty, this is not the same as ignorance. When you spin a coin the outcome is uncertain – but you know that it will come down either heads or tails, and you know that each of these outcomes is equally likely. When a company launches a new service, it does not know exactly how many customers it will attract, but market research can give some kind of estimate. So managers have to make decisions when they have some information – but there is a lot of uncertainty. In this chapter we discuss ways of measuring and dealing with uncertainty – and for this we use **probabilities**.

Defining probability

Probabilities give a measure of uncertainty. To be more precise, the probability of an event is a measure of its likelihood or relative frequency.

Experience leads us to believe that when we toss a fair coin it comes down heads half the time and tails half the time. From this observation we can say, 'the probability that a fair coin comes down heads is 0.5'. This uses a definition of the probability of an event as the proportion of times the event occurs.

$$\text{probability of an event} = \frac{\text{number of ways that the event can occur}}{\text{number of possible outcomes}}$$

When you spin a coin, there are two possible outcomes (heads and tails) and one of these is a head, so the probability of a head is 1/2. Similarly, there are 52 cards in a pack of cards and one ace of hearts, so the probability that a card chosen at random is the ace of hearts (or any other specified card) is 1/52. In the last 500 days the train to work has broken down 10 times, so the probability it broke down on any particular day is 10/500 or 0.02. For 200 of the last 240 trading days the New York Stock Exchange has had more advances than declines, so there is a probability of 200/240 or 0.83 that the stock exchange advanced on a particular day.

As probability measures the proportion of times an event occurs, its value is defined only in the range 0 to 1.

- Probability = 0 means the event will never occur.
- Probability = 1 means the event will always occur.
- Probability between 0 and 1 gives the relative frequency or likelihood.
- Probabilities outside the range 0 to 1 have no meaning.

An event with a probability of 0.8 is quite likely (it happens eight times out of 10); an event with a probability of 0.5 is equally likely to happen as not; an event with a probability of 0.2 is quite unlikely (it happens two times out of 10).

Rather than keep saying 'the probability of an event is 0.8', we can abbreviate this to $P(\text{event}) = 0.8$. Then when spinning a coin $P(\text{head}) = P(\text{tail}) = 0.5$.

WORKED EXAMPLE 14.1

The *Monthly Gazette* advertised a prize draw with one first prize, five second prizes, 100 third prizes and 1,000 fourth prizes. Prize winners were drawn at random from entries and after each draw the winning ticket was returned to the draw. By the closing date there were 10,000 entries, and at the draw no entry won more than one prize. What is the probability that a given ticket won first prize, or that it won any prize?

Solution

There were 10,000 entries and one first prize, so the probability that a given ticket wins first prize is 1/10,000.

There are five second prizes, so the probability of a given ticket winning one of these is 5/10,000. The probabilities of winning third or fourth prizes are 100/10,000 and 1,000/10,000 respectively.

There are a total of 1,106 prizes, so the probability of a ticket winning one of these is 1,106/10,000 = 0.1106. Conversely 8,894 tickets did not win a prize, so the probability of not winning a prize is 8,894/10,000 = 0.8894.

WORKED EXAMPLE 14.2

An office has the following types of employees.

	Female	Male
Administrators	25	15
Operators	35	25

If one person from the office is chosen at random, what is the probability that the person is (a) a male administrator, (b) a female operator, (c) male, (d) an operator?

Solution

(a) By adding them, you can see that there are 100 people in the office. Of these, 15 are male administrators, so:

$$P(\text{male administrator}) = 15/100 = 0.15$$

(b) 35 people in the office are female operators, so:

$$P(\text{female operator}) = 35/100 = 0.35$$

(c) A total of 40 people in the office are male, so:

$$P(\text{male}) = 40/100 = 0.4$$

(d) A total of 60 people in the office are operators, so:

$$P(\text{operator}) = 60/100 = 0.6$$

These two worked examples suggest two different ways of finding probabilities:

1 ***Calculation***: you can use your knowledge of a situation to calculate theoretical or *a priori* probabilities (called *a priori* because you calculate the probability of an event before it actually happens):

$$\text{probability of an event} = \frac{\text{number of ways that the event can occur}}{\text{number of possible outcomes}}$$

The probability that two people share the same birthday is 1/365 (ignoring leap years). This is an *a priori* probability calculated by saying that there are 365 days on which the second person can have a birthday and only one of these corresponds to the birthday of the first person.

2 ***Observation***: you can use historical data to see how often an event actually happened in the past, and use this to give experimental or empirical probabilities:

$$\text{probability of an event} = \frac{\text{number of times that the event occurred}}{\text{number of observations}}$$

In the last 100 matches that a football team has played at home, it attracted a crowd of more than 10,000 on 62 occasions. This gives an empirical probability of 62/100 = 0.62 that the team attracts a crowd of more than 10,000.

A problem with empirical values is that the historical data may not be typical. If a fair coin is tossed five times and comes down heads each time, this does not mean that it will always come down heads. Empirical values must be based on typical values, collected over a sufficiently long period.

There is a third way of getting probabilities, which is not generally recommended. This asks people to give their subjective views about likely probabilities. For instance, you might ask the Financial Director to give a probability that the company will make a profit next year. This is equivalent to judgemental forecasting, and has the same drawbacks of using personal opinions that are generally unreliable.

The probability measures the relative likelihood of an event. If you spin a coin 1,000 times, the probability of a head is 0.5, so you would expect $1,000 \times 0.5 = 500$ heads; if the probability of a bus arriving on time is 0.2, you would expect $5 \times 0.2 = 1$ to arrive on time in a working week; if the probability of an IT company making a loss is 0.1, you would expect a survey of 100 companies to find 10 making a loss.

Review questions

14.1 You cannot tell what is going to happen in the future, so all decisions contain uncertainty. Do you think this is true?

14.2 What is the probability of an event?

14.3 Does uncertainty mean the same as ignorance?

14.4 Is the probability of an event the same as its relative frequency?

14.5 If the probability of a new car having a fault is 0.01, how many defects would you expect in 10,000 cars?

IDEAS IN PRACTICE CIS Personal Pensions

Investing in the stock market is risky. Sometimes share prices rise quickly and lucky investors make a fortune; at other times, share prices plummet and unlucky investors lose their shirt. Unfortunately, there seems little underlying logic behind these variations. The 'value' of a company can collapse one day, and then soar the next day. A commentator for the Royal Bank of Canada looked at the wild fluctuations in the stock market and said that, 'Market volatility continued in the second

Ideas in practice continued

quarter of 2000 due to concerns about strong economic growth, inflationary pressures, currency fluctuations and valuations in general'.[1]

Financial managers often try to disguise the fact that they do not know what will happen by talking of uncertainty, difficult conditions, volatility, effects of distortions, cyclical factors, and so on. The sensible response of investors is to spread their risk by putting money into a wide range of options. There are several ways of organising this, including unit trusts, managed funds – and in the long term personal pensions. But finance companies have a fundamental problem, as they want their products to be attractive but they have little idea of long-term returns. Companies such as CIS can say how well they did in the past, and how much you will get if they achieve annual growth of 5%, 7% or 9% in the future, but they have to add cautionary notes such as:[2]

- 'These figures are only examples and are not guaranteed – they are not minimum or maximum amounts. What you get back depends on how your investment works.'
- 'You could get back more or less than this.'
- 'All insurance companies use the same rates of growth for illustrations but their charges vary.'
- 'Do not forget that inflation would reduce what you could buy in the future with the amounts shown.'

In practice, all organisations have to work with uncertainty. They do not know what will happen in the future and have to allow for uncertainty.

Calculations with probabilities

An important concept for events is independence. If the occurrence of one event does not affect the occurrence of a second event, the two events are said to be **independent**. The fact that a person works in a bank is independent of the fact that they are left-handed; the event that a company has a shipment of raw materials delayed is independent of the event that they increased their marketing budget. Using the notation:

$P(a)$ = the probability of event a
$P(a/b)$ = the probability of event a given that b has already occurred
$P(a/\cancel{b})$ = the probability of event a given that b has not occurred

the two events, a and b, are independent if:

$$P(a) = P(a/b) = P(a/\cancel{b})$$

The probability that a person buys a particular newspaper is independent of the probability that they suffer from hay fever, and:

$$\begin{aligned} P(\text{buys } \textit{Times}) &= P(\text{buys } \textit{Times}/\text{suffers from hay fever}) \\ &= P(\text{buys } \textit{Times}/\text{does not suffer from hay fever}) \end{aligned}$$

For independent events, you find the probability of several events happening by *multiplying* the probabilities of the separate events.

For independent events:
AND means that you **multiply** separate probabilities:

- $P(a \text{ AND } b) = P(a) \times P(b)$
- $P(a \text{ AND } b \text{ AND } c) = P(a) \times P(b) \times P(c)$
- $P(a \text{ AND } b \text{ AND } c \text{ AND } d) = P(a) \times P(b) \times P(c) \times P(d)$

etc.

WORKED EXAMPLE 14.3

Maxed Mail-order combines two parts – products and invoices – into a box for delivery. An average of 3% of products are defective and an average of 5% of invoices are faulty. What is the probability that a delivery has faults in both the product and the invoice?

Solution

We actually know four probabilities here:

P(defective product) $= 3/100 = 0.03$
P(good product) $= 97/100 = 0.97$
P(faulty invoice) $= 5/100 = 0.05$
P(good invoice) $= 95/100 = 0.95$

Assuming that defects in products and invoices are independent:

P(defective product AND faulty invoice)
$= P$(defective product) $\times$ P(faulty invoice)
$= 0.03 \times 0.05 = 0.0015$

Similarly:

P(defective product AND good invoice)
$= 0.03 \times 0.95 = 0.0285$
P(good product AND faulty invoice)
$= 0.97 \times 0.05 = 0.0485$
P(good product AND good invoice)
$= 0.97 \times 0.95 = 0.9215$

These are the only four possible combinations, so it is not surprising that the probabilities add to 1 – meaning that one of them is certain to happen.

WORKED EXAMPLE 14.4

A warehouse classifies stock into three categories A, B and C. On all category A items it promises a service level of 97% (in other words, there is a probability of 0.97 that the warehouse can meet demand from stock). On category B and C items it promises service levels of 94% and 90% respectively. What are the probabilities that the warehouse can immediately supply an order for:

(a) one item of category A and one item of category B
(b) one item from each category
(c) two different items from A, one from B and three from C
(d) three different items from each category?

Solution

(a) Assuming that the service levels of each item are independent, you can multiply the separate probabilities to get

P(one A AND one B) $= P$(one A) $\times$ P(one B)
$= 0.97 \times 0.94 = 0.912$

(b) Similarly:

P(one A AND one B AND one C)
$= P$(one A) $\times$ P(one B) $\times$ P(one C)
$= 0.97 \times 0.94 \times 0.90 = 0.821$

(c) P(two A AND one B AND three C)
$= P$(two A) $\times$ P(one B) $\times$ P(three C)

You have to break this down a bit further by noting that the probability of two items of category A being in stock is the probability that the first is there AND the probability that the second is there. In other words:

P(two A) $= P$(one A AND one A) $=$
P(one A) $\times$ P(one A) $= P$(one A)^2

Similarly:

P(three C) $= P$(one C)^3

Then the answer becomes:

P(one A)^2 $\times$ P(one B) $\times$ P(one C)^3 $=$
$0.97^2 \times 0.94 \times 0.9^3 = 0.645$

(d) Similarly:

P(three A AND three B AND three C)
$= P$(one A)^3 $\times$ P(one B)^3 $\times$ P(one C)^3
$= 0.97^3 \times 0.94^3 \times 0.9^3 = 0.553$

Mutually exclusive events

Another important idea for probabilities is **mutually exclusive events**. Two events are mutually exclusive when they cannot both happen – and if one of the events happens, then the other event cannot happen. When you toss a coin, having it come down heads is mutually exclusive with having it come down tails; the event that a company makes a profit is mutually exclusive with the event that it makes a loss; the event that sales increase is mutually exclusive with the event that sales decrease.

For mutually exclusive events, you can find the probabilities of one or another happening by *adding* the separate probabilities.

> **For mutually exclusive events:**
> **OR** means that you **add** separate probabilities:
>
> - $P(a \text{ OR } b) = P(a) + P(b)$
> - $P(a \text{ OR } b \text{ OR } c) = P(a) + P(b) + P(c)$
> - $P(a \text{ OR } b \text{ OR } c \text{ OR } d) = P(a) + P(b) + P(c) + P(d)$
>
> etc.

We have already met examples of this, for instance with the *Monthly Gazette*'s prize draw. There the probability that a particular ticket won a prize was 1,106/10,000, and the probability that it did not win was 8,894/10,000. Each ticket must either win or lose, and these two events are mutually exclusive, so:

$$P(\text{win OR lose}) = P(\text{win}) + P(\text{lose}) = 0.1106 + 0.8894 = 1$$

and

$$P(\text{lose}) = 1 - P(\text{win})$$

WORKED EXAMPLE 14.5

Santos Domestica make 40,000 washing machines a year. Of these 10,000 are for the home market, 12,000 are exported to the Americas, 8,000 to Europe, 4,000 to the Far East, and 6,000 to other markets.

(a) What are the probabilities that a particular machine is sold in each of the markets?
(b) What is the probability that a machine is exported?
(c) What is the probability that a machine is exported to either the Americas or Europe?
(d) What is the probability that a machine is sold in either the home or Far East markets?

Solution

(a) The probability that a machine is sold on the home market is:

$$P(\text{home}) = \frac{\text{number sold on home market}}{\text{total number sold}} = \frac{10{,}000}{40{,}000} = 0.25$$

Similarly for the other markets:

$P(\text{Americas}) = 12{,}000/40{,}000 = 0.3$

$P(\text{Europe}) = 8{,}000/40{,}000 = 0.2$

$P(\text{Far East}) = 4{,}000/40{,}000 = 0.1$

$P(\text{others}) = 6{,}000/40{,}000 = 0.15$

These are the only options, the events (that is, areas of sales) being mutually exclusive, so the probabilities add to 1, showing that one of them must happen.

(b) As the events are mutually exclusive, you can find the probability that a machine is exported

Worked example 14.5 continued

by adding the probabilities that it is sent to each export market:

$$\begin{aligned} P(\text{exported}) &= P(\text{Americas OR Europe OR Far East OR others}) \\ &= P(\text{Americas}) + P(\text{Europe}) + P(\text{Far East}) + P(\text{others}) \\ &= 0.3 + 0.2 + 0.1 + 0.15 \\ &= 0.75 \end{aligned}$$

Alternatively, you can say that all machines are sold somewhere, so the probability that a machine is sold is 1.0. It is either sold on the home market or exported, so:

$$\begin{aligned} P(\text{exported OR home}) &= 1 \\ &= P(\text{exported}) + P(\text{home}) \end{aligned}$$

and

$$P(\text{exported}) = 1 - P(\text{home}) = 1 - 0.25 = 0.75$$

(c) $P(\text{Americas OR Europe}) = P(\text{Americas}) + P(\text{Europe}) = 0.3 + 0.2 = 0.5$

(d) $P(\text{home OR Far East}) = P(\text{home}) + P(\text{Far East}) = 0.25 + 0.1 = 0.35$

Clearly, not all events are mutually exclusive. The event that someone has a blue car is not mutually exclusive with the event that they work in Paris; the event that a company makes a profit is not mutually exclusive with the event that they offer banking services. Suppose you pick a single card from a pack. What is the probability that it is an ace or a heart? Clearly these two are not mutually exclusive, as your card can be both an ace and a heart. Then you cannot say that:

$$P(\text{ace OR heart}) = P(\text{ace}) + P(\text{heart}) \quad ✗$$

Venn diagrams give a useful way of illustrating this point. These show the probabilities of events as circles. If two events are mutually exclusive, the Venn diagram shows completely separate circles, as shown in Figure 14.1.

If two events are *not* mutually exclusive, there is a probability they can both occur, as shown in Figure 14.2. The circles now overlap – with the overlap representing the probability that both events occur. If we simply add the probabilities of two events that are not mutually exclusive, we add the overlap twice. To correct this, we must subtract the probability of both events occurring.

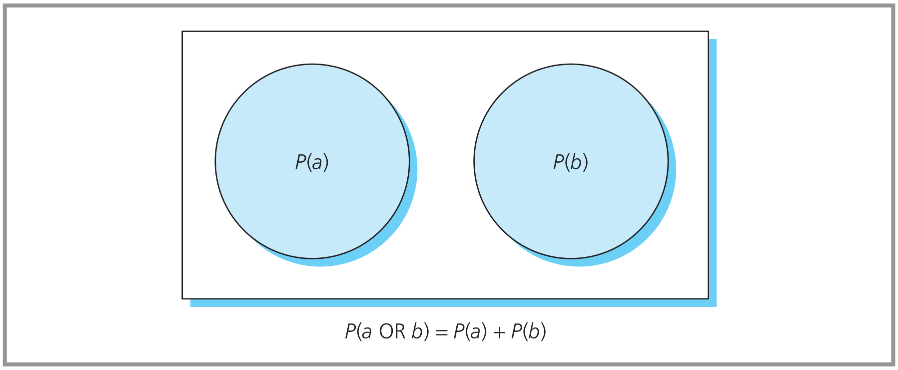


Figure 14.1 Venn diagram for mutually exclusive events

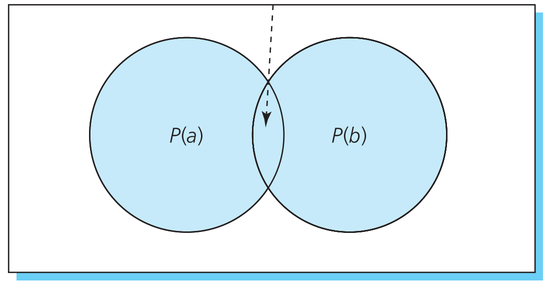


Figure 14.2 Venn diagram for non-mutually exclusive events

For events that are **not mutually exclusive**:

$P(a \text{ OR } b) = P(a) + P(b) - P(a \text{ AND } b)$

When you pick a single card from a pack, the probability that it is an ace or a heart is:

$$\begin{aligned} P(\text{ace OR heart}) &= P(\text{ace}) + P(\text{heart}) - P(\text{ace AND heart}) \\ &= 4/52 + 13/52 - 1/52 = 16/52 = 0.31 \end{aligned}$$

WORKED EXAMPLE 14.6

Sixty per cent of companies in an industrial estate own their premises, and 40% employ more than 30 people. What is the probability that a company owns its premises or employs more than 30 people?

Solution

We know that:

probability a company owns its premises $= P(\text{owns}) = 0.6$

probability a company employs more than 30 $= P(\text{employs}) = 0.4$

These are not mutually exclusive, so:

$$P(\text{owns OR employs}) = P(\text{owns}) + P(\text{employs}) - P(\text{owns AND employs})$$

Now we need $P(\text{owns AND employs})$, and assuming that these two are independent:

$$\begin{aligned} P(\text{owns AND employs}) &= P(\text{owns}) \times P(\text{employs}) \\ &= 0.6 \times 0.4 = 0.24 \end{aligned}$$

So:

$$P(\text{owns OR employs}) = 0.6 + 0.4 - 0.24 = 0.76$$

WORKED EXAMPLE 14.7

Every year Pieter Amundsen plc puts in a bid for an annual service contract. The probability that it wins the contract is 0.75. What is the probability that the company wins the contract at least once?

Solution

The probability that the company wins the contract in at least one year is:

$$1 - P(\text{loses in both years}) = 1 - P(\text{loses this year AND loses next year})$$

Worked example 14.7 continued

Assuming that the probability of losing the contract this year is independent of the probability of losing next year (for convenience rather than reality):

P(loses this year AND loses next year)
$= 0.25 \times 0.25 = 0.0625$

P(wins in at least one year) $= 1 - 0.0625$
$= 0.9375$

You could have got the same result by assuming that the events of winning this year and winning next year are not mutually exclusive:

P(wins in at least one year)
$= P$(wins this year OR wins next year)
$= P$(wins this year) $+ P$(wins next year)
$- P$(wins this year AND wins next year)

Now:

P(wins this year AND wins next year)
$= 0.75 \times 0.75 = 0.5625$

So:

P(wins in at least one year)
$= 0.75 + 0.75 - 0.5625 = 0.9375$

Review questions

14.6 What are independent events?

14.7 What are mutually exclusive events?

14.8 How would you find the probability that one of several mutually exclusive events occurs?

14.9 How would you find the probability that all of several independent events occur?

14.10 If one of events A, B and C is certain to happen, what is P(A)?

IDEAS IN PRACTICE Medical testing

Many medical conditions can be treated much more effectively when they are detected early. For example, when certain heart conditions are caught early, they can be treated with mild medicines – but if they are not detected until later, both the conditions and the treatment are much more severe. The problem is detecting conditions at an early stage, when there are no significant symptoms.

Many health services routinely screen populations to detect some conditions (such as breast cancer) as early as possible. Unfortunately, these tests do not give a simple result of 'problem' or 'no problem' that is guaranteed accurate.

In one area – and you do not need to worry about the details – it is estimated that 0.5% of the population will contract a certain type of cancer. A simple and established test for detecting this cancer is said to be 98% accurate. It is very good to have such a reliable test – but it is not used for routine screening. You can see the arguments for not using it with some calculations.

If a screening programme tests 10,000 people, 50 are likely to actually have the cancer. Of these 50, 98%, or 49, will give a correct positive response to the test, and one will be missed. Of the remaining 9,950 people, 2%, or 199, will give an incorrect positive response. So there is a total of 248 positive responses, of which 49 are correct. The remainder face a mixture of unnecessary anguish, further tests, and possible unneeded treatment.

Needless to say, the use of statistics in medicine is a very difficult area.

Conditional probabilities

Often events are not independent, and the occurrence of one event can directly affect the probability of another. For example, the fact that someone is employed in one of the professions is not independent of their having higher education; the probability that a machine breaks down this week is not independent of whether it was maintained last week; the probability that a train arrives late is not independent of the service operator.

For **dependent events**, the fact that one event has occurred or not changes the probability that a second event occurs. Again using:

$P(a)$ = the probability of event a

$P(a/b)$ = the probability of event a given that b has already occurred

$P(a/\cancel{b})$ = the probability of event a given that b has not occurred

two events, a and b, are dependent if:

$$P(a) \neq P(a/b) \neq P(a/\cancel{b})$$

(Remember that the symbol ≠ means 'is not equal to'.)

The probability that the price of a company's shares rises is dependent on whether the company announces a profit or a loss. Then:

$$\begin{aligned} P(\text{share price rises}) &\neq P(\text{share price rises / announce profit}) \\ &\neq P(\text{share price rises / announce loss}) \end{aligned}$$

Probabilities with the form $P(a/b)$ are called **conditional probabilities** – as the probability of a occurring is conditional on whether or not b has already occurred.

The most important rule for conditional probabilities is that the probability of two dependent events occurring is the probability of the first, multiplied by the conditional probability that the second occurs given that the first has already occurred. We can write this rather clumsy statement as:

$$P(a \text{ AND } b) = P(a) \times P(b/a)$$

With a bit of thought, you can extend this to the obvious result:

$$P(a \text{ AND } b) = P(a) \times P(b/a) = P(b) \times P(a/b)$$

Rearranging these gives the most important calculation for conditional probabilities, which is known as **Bayes' theorem**:

$$P(a/b) = \frac{P(b/a) \times P(a)}{P(b)} = \frac{P(a \text{ AND } b)}{P(b)}$$

To put it simply, when you have a conditional probability $P(b/a)$, Bayes' theorem lets you do the reverse calculation to find $P(a/b)$.

WORKED EXAMPLE 14.8

The students in a class can be described as follows:

	Home	Overseas
Male	66	29
Female	102	3

(a) If you choose a student from the class at random, what is the probability that the student is from overseas?
(b) If the student chosen is female, what is the probability that she is from overseas?
(c) If the student is from overseas, what is the probability that the student is female?

Solution

(a) There are 200 students and 32 of these are from overseas, so:

P(overseas) = number from overseas / number of students = 32/200 = 0.16

(b) We want P(overseas / female) and can calculate this from Bayes' theorem:

$$P(a/b) = \frac{P(a \text{ AND } b)}{P(b)}$$

so

P(overseas / female)

$$= \frac{P(\text{overseas AND female})}{P(\text{female})}$$

Now:

P(overseas AND female) = 3/200 = 0.015

P(female) = 105/200 = 0.525

so:

P(overseas / female) = 0.015/0.525 = 0.029

You can check this by considering the 105 female students, and see that three are from overseas, so:

P(overseas / female) = number of overseas females / number of females = 3/105 = 0.029

(c) Now we want P(female / overseas) and can again calculate this from Bayes' theorem:

$$P(a/b) = \frac{P(b/a) \times P(a)}{P(b)}$$

so

P(female / overseas)

$$= \frac{P(\text{overseas/female}) \times P(\text{female})}{P(\text{overseas})}$$

We have just calculated P(overseas/female) = 0.029, and we know that P(female) = 0.525 and P(overseas) = 0.16. So:

P(female / overseas) = 0.029 × 0.525/0.016
= 0.094

You can check this by considering the 32 overseas students and see that three of them are female, so:

P(female / overseas) = number of overseas females / number of overseas students
= 3/32 = 0.094

WORKED EXAMPLE 14.9

Two machines make identical parts that are combined on a production line. The older machine makes 40% of the parts, of which 85% are good and reach acceptable quality; the newer machine makes 60% of the parts, of which 92% are good. A random check further down the production line shows an unusual fault, which suggests the machine that made the part needs adjusting. What is the probability that the older machine made the part?

Solution

Figure 14.3 gives a summary of this problem. Using the abbreviations O for the older machine, N for the newer machine, G for good units and F for faulty ones, we have P(F/O), etc., and want to find P(O/F). For this we use Bayes' theorem:

$$P(a/b) = \frac{P(b/a) \times P(a)}{P(b)}$$

Worked example 14.9 continued

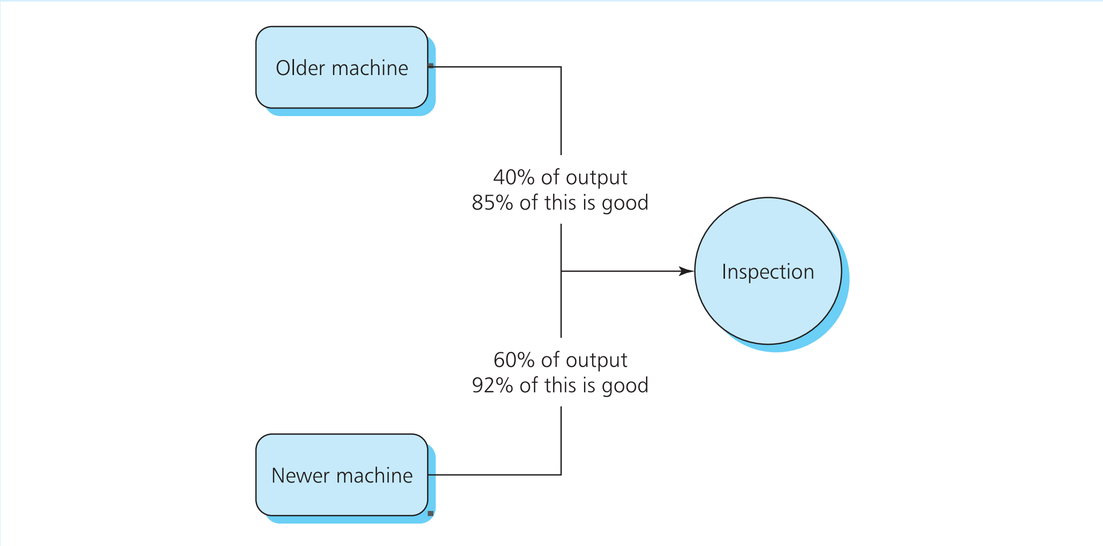


Figure 14.3 Process for worked example 14.9

so:

$$P(O/F) = \frac{P(F/O) \times P(O)}{P(F)}$$

We know that $P(O) = 0.4$ and $P(F/O) = 0.15$, so the remaining value we need is $P(F)$, the probability a unit is faulty. With a bit of thought you can see that:

probability a unit is faulty
= probability it is faulty from the old machine
OR it is faulty from the new machine

so:

$$P(F) = P(F \text{ AND } O) + P(F \text{ AND } N)$$

and:

$$P(F) = P(F/O) \times P(O) + P(F/N) \times P(N)$$

We know all of these, so:

$$P(F) = P(F/O) \times P(O) + P(F/N) \times P(N)$$
$$= 0.15 \times 0.4 + 0.08 \times 0.6 = 0.108$$

Then:

$$P(O/F) = \frac{P(F/O) \times P(O)}{P(F)} = \frac{0.15 \times 0.4}{0.108} = \frac{0.06}{0.108}$$
$$= 0.556$$

To check this result, we can also calculate the probability that the unit came from the new machine given that it is faulty:

$$P(N/F) = \frac{P(F/N) \times P(N)}{P(F)} = \frac{0.08 \times 0.6}{0.108} = 0.444$$

As the faulty unit must have come from either the older or the newer machine, the fact that $P(O/F) + P(N/F) = 1$ confirms the result.

Calculations for Bayes' theorem

The arithmetic for Bayes' theorem is straightforward, but it is messy for larger problems – so it makes sense to use a computer. There are two options for this. The first uses specialised statistical software that automatically does Bayesian analysis. Figure 14.4 shows a printout from a simple program for doing the calculations for the last worked example. In this printout you can see the figures that we calculated (together with some others that we explain below).

BAYESIAN ANALYSIS

-=*=- DATA ENTERED -=*=-

NUMBER OF STATES :	2
NUMBER OF ALTERNATIVES :	2

PRIOR PROBABILITIES

1 Older machine	0.4000
2 Newer machine	0.6000

CONDITIONAL PROBABILITIES

	1 Faulty	2 Good
1 Older machine	0.1500	0.8500
2 Newer machine	0.0800	0.9200

-=*=- RESULTS -=*=-

PREDICTIONS – JOINT PROBABILITIES

	1 Faulty	2 Good
1 Older Machine	0.0600	0.3400
2 Newer machine	0.0480	0.5520

PREDICTIONS – MARGINAL PROBABILITIES

1 Faulty	0.1080
2 Good	0.8920

PREDICTIONS – REVISED PROBABILITIES

	1 Faulty	2 Good
1 Older machine	0.5556	0.3812
2 Newer machine	0.4444	0.6188

---------- E N D O F A N A L Y S I S ----------

Figure 14.4 Printout for Bayesian analysis

The other option is to use a spreadsheet. Your spreadsheet may not have a standard function for this, but there are special add-ins – or you can do the calculations yourself. For this option, we can easily show how to do the calculations for the last worked example. This starts by putting the available information in a table.

	Conditional probabilities		
	Faulty	Good	Prior probabilities
Older machine	0.15	0.85	0.4
Newer machine	0.08	0.92	0.6

This box includes the 'conditional probabilities', which are the probabilities that units are good or faulty given that they come from each machine. So the conditional probabilities are P(F/O) = 0.15, P(F/N) = 0.08, etc. The entries to the right give the values of P(O) and P(N), which are called the 'prior probabilities'.

Now we form a third box by multiplying each conditional probability in the left-hand box by the prior probability on the same line. So 0.15 × 0.4 = 0.060, 0.08 × 0.6 = 0.048, etc. These results are called 'joint probabilities'.

	Conditional probabilities			Joint probabilities	
	Faulty	Good	Prior probabilities	Faulty	Good
Older machine	0.15	0.85	0.4	0.060	0.340
Newer machine	0.08	0.92	0.6	0.048	0.552
			Marginal probabilities	0.108	0.892
	Revised probabilities		Older machine	0.556	0.381
			Newer machine	0.444	0.619

Adding each column of joint probabilities gives a 'marginal probability'. Then 0.060 + 0.048 = 0.108, which is the probability that a unit is faulty; 0.340 + 0.552 = 0.892, which is the probability that a unit is good.

Finally, dividing each joint probability by the marginal probability in the same column gives a revised probability shown in a bottom box. Then 0.060/0.108 = 0.556, 0.340/0.892 = 0.381, etc. These revised probabilities are the results that we want, giving P(O/F), P(N/G), and so on.

At first this procedure seems strange, but if you look at the equation for Bayes' theorem you can see that we are simply repeating the calculations. The joint probabilities are the top lines of Bayes' theorem, the marginal probabilities are the bottom line, and dividing one by the other gives the final calculation.

WORKED EXAMPLE 14.10

The probabilities of two events X and Y are 0.3 and 0.7 respectively. Three events A, B and C can follow X and Y, with conditional probabilities given in the following table. What results do you get from using Bayes' theorem on these figures?

	A	B	C
X	0.1	0.5	0.4
Y	0.7	0.2	0.1

Solution

The table gives the conditional probabilities of P(A/X), P(B/X), etc. You know the prior probabilities of P(X) = 0.3 and P(Y) = 0.7. Then Figure 14.5 shows a spreadsheet of the results from the mechanical procedure for Bayes' theorem.

The marginal probabilities show P(A) = 0.52, P(B) = 0.29 and P(C) = 0.19. The predictions in the bottom boxes show the probability of P(X/A) = 0.058, P(X/B) = 0.517, P(Y/A) = 0.942, and so on.

Worked example 14.10 continued

	A	B	C	D	E	F	G	H
1	**Bayes' Theorem**							
2								
3		Conditional probabilities			Priors	Joint probabilities		
4		A	B	C		A	B	C
5	X	0.1	0.5	0.4	0.3	0.03	0.15	0.12
6	Y	0.7	0.2	0.1	0.7	0.49	0.14	0.07
7						0.52	0.29	0.19
8					X	0.058	0.517	0.632
9					Y	0.942	0.483	0.368

Figure 14.5 Calculations for Bayes' theorem in worked example 14.10

Probability trees

It is difficult to follow the calculations of complex probabilities, and it is easier to visualise them in a probability tree. These diagrams have branches representing sequences of possible events, with the probabilities given to each branch. Figure 14.6 shows a probability tree for the worked example of faulty parts produced by two machines.

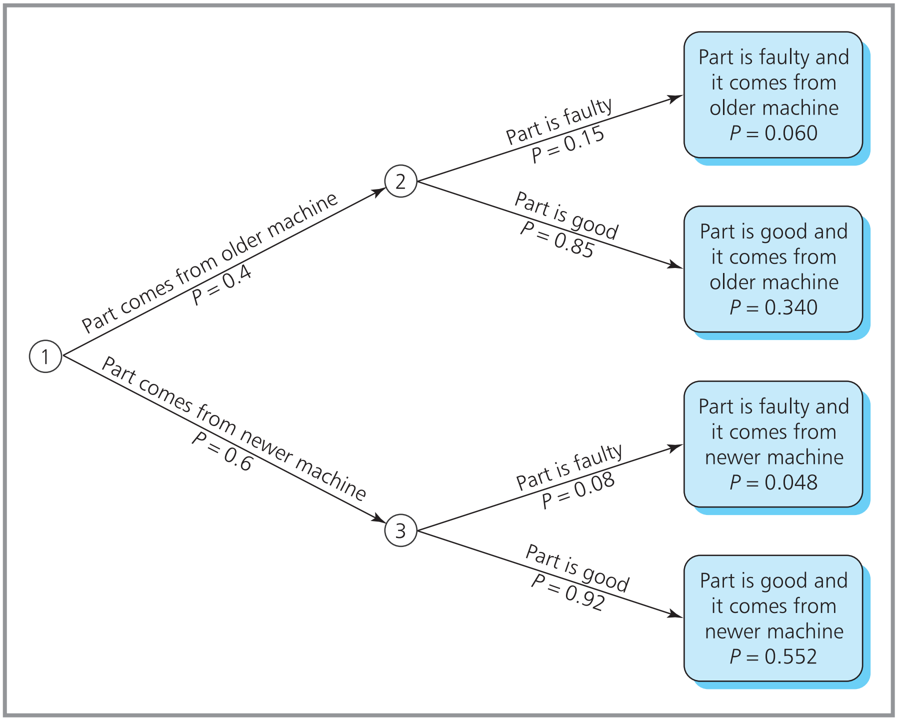


Figure 14.6 A probability tree of faulty parts for worked example 14.9

You move through the diagram from left to right, with each branch representing a possible event, and they emerge from nodes which are the circles. Node 1 is the starting point. From here there are two alternatives – either the part comes from the older machine or it comes from the newer machine, with probabilities 0.4 and 0.6 respectively. Then at both nodes 2 and 3 there are two possibilities, as a part is either faulty or good. Each branch is labelled with its probability – and as we include all possible events, the sum of probabilities on branches leaving any node should be 1. At the right of the tree are the terminal values which show the overall probability of reaching the points. We find these by multiplying the probabilities on all branches taken to reach that point. So the first terminal node shows the probability that a part is faulty and comes from the older machine, which is $0.4 \times 0.15 = 0.060$. The second terminal node shows the probability that a part is good and comes from the older machine, which is $0.4 \times 0.85 = 0.340$. Again, the sum of all terminal values should be 1.

WORKED EXAMPLE 14.11

Sixty per cent of second-hand cars can be classified as good buys, and the remainder are bad buys. Among good buys 70% have low oil consumption and 20% have medium oil consumption. Among bad buys 50% have high oil consumption and 30% have medium oil consumption. A test was done on a second-hand car and showed low oil consumption. What is the probability that this car is a good buy?

Solution

We start by defining the values that we know, using the abbreviations GB and BB for good buy and bad buy, and HOC, MOC and LOC for high, medium and low oil consumption. We know figures for *P*(LOC/GB), etc., and we are looking for *P*(GB/LOC) – and we find these by substituting into Bayes' theorem. Figure 14.7 shows these results in a spreadsheet, and you can see that the probability a car is a good buy, given that it has a low oil consumption, is 0.84. You can also see that the probability of a low oil consumption is 0.5. Figure 14.8 shows these results in a probability tree.

	A	B	C	D	E	F	G	H
1	**Bayes' Theorem**							
2								
3		Conditional probabilities			Priors	Joint probabilities		
4		HOC	MOC	LOC		HOC	MOC	LOC
5	GB	0.1	0.2	0.7	0.6	0.06	0.12	0.42
6	BB	0.5	0.3	0.2	0.4	0.20	0.12	0.08
7						0.26	0.24	0.50
8					GB	0.23	0.50	0.84
9					BB	0.77	0.50	0.16

Figure 14.7 Calculations for Bayes' theorem in worked example 14.11

Worked example 14.11 continued

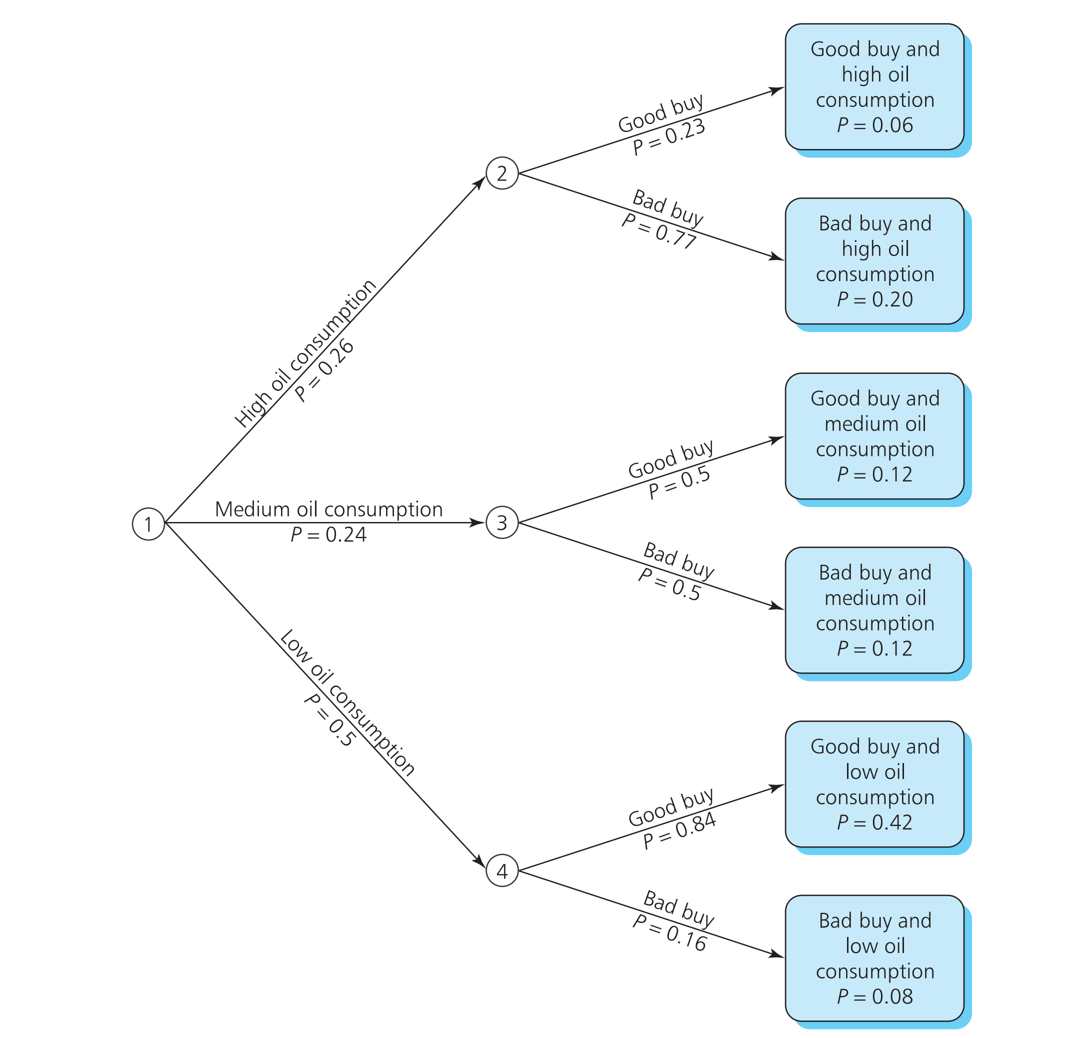


Figure 14.8 Probability tree for worked example 14.11

Review questions

14.11 What are dependent events?

14.12 What are conditional probabilities?

14.13 What is Bayes' theorem and when is it used?

14.14 What is the benefit of a probability tree?

IDEAS IN PRACTICE US Coast Guard

The US Coast Guard monitors and protects America's 20,000 km of coastline.[3] They use many measures of performance, but one considers the probability that they will intercept a target vessel. The probability that a target vessel is intercepted depends on the probability that it is detected. So a basic model has:

$$P(\mathrm{I}) = P(\mathrm{I/D}) \times P(\mathrm{D})$$

where: $P(\mathrm{I})$ = probability that a vessel is intercepted
$P(\mathrm{I/D})$ = probability that a vessel is intercepted, given that it is detected, which can be estimated from historical records.

A fuller model has:

$$P(\mathrm{I}) = P(\mathrm{I/D}) \times P(\mathrm{D/O}) \times P(\mathrm{O/A}) \times P(\mathrm{A})$$

where: $P(\mathrm{D/O})$ = probability that a target vessel is detected, given that a Coast Guard vessel is on patrol
$P(\mathrm{O/A})$ = probability that a Coast Guard vessel is on patrol, given that it is in the right area
$P(\mathrm{A})$ = probability that the Coast Guard vessel is in the right area.

This model helps test different methods of organising patrol vessels, and helps increase the efficiency of the service.

Source: Kimbrough S.O., Oliver J.R. and Pritchett C.W., On post-evaluation analysis, *Interfaces*, vol. 23(3), 1993.

CHAPTER REVIEW

This chapter introduced the idea of probability as a way of measuring uncertainty.

- Some problems are deterministic, where everything is known with certainty; but most problems are stochastic or probabilistic, which means that there is uncertainty.
- Probabilities give a way of measuring uncertainty. They define the likelihood of that an event will occur, or its relative frequency. Probabilities can either be calculated *a priori* or observed empirically (with less reliable estimates coming from subjective probabilities).
- Probability gives a measure of relative frequency, so is defined on a scale of 0 (meaning there is no chance of the event happening) to 1 (meaning that it is certain to happen).
- Two basic rules for probabilities are:
  - For independent events, $P(a \text{ AND } b) = P(a) \times P(b)$
  - For mutually exclusive events, $P(a \text{ OR } b) = P(a) + P(b)$
- Conditional probabilities occur when two events are dependent – so that:

  $$P(a) \neq P(a/b) \neq P(a/\bar{b})$$

  The most important result for conditional probabilities is Bayes' theorem:

  $$P(a/b) = \frac{P(b/a) \times P(a)}{P(b)} = \frac{P(a \text{ AND } b)}{P(b)}$$

- A probability tree shows the relationships between events as branches coming from nodes.

CASE STUDY The *Gamblers' Press*

The *Gamblers' Press* is a weekly paper that publishes large amounts of information that is used by gamblers. Its main contents are detailed sections on horse racing, greyhound racing, football, and other major sporting activities. It also runs regular features on card games, casino games and any other areas that gamblers find interesting.

The *Gamblers' Press* was founded in 1897 and now has a regular circulation of 50,000 copies. It is considered a highly respectable paper and has a strict policy of giving only factual information. It never gives tips or advice. Last year it decided to run a special feature on misleading or dishonest practices. This idea was suggested when four unconnected reports were passed to the editors.

The first report concerned an 'infallible' way of winning at roulette. Customers were charged $500 for the details of the scheme, which was based on a record of all the numbers that won on the roulette wheel during an evening. Then the customers were advised to bet on two sets of numbers:

- those that had come up more often, because the wheel might be biased in their favour
- those that had come up least often, because the laws of probability say that numbers which appear less frequently on one night must appear more frequently on another night.

The second report showed that a number of illegal chain letters were circulating in Germany. These letters contained a list of eight names. Individuals were asked to send €10 to the name at the top of the list. Then they should delete the name at the top, insert their own name at the bottom, and send a copy of the letter to eight of their friends. As each name moved to the top of the list they would receive payments from people who joined the chain later. The advertising accompanying these letters guaranteed to make respondents millionaires, claiming that 'you cannot lose!!!' It also said that people who did not respond would be letting down their friends and would inevitably be plagued by bad luck.

The third report was about 'a horse racing consultant', who sent people a letter saying which horse would win a race the following week. A week later he sent a second letter saying how the selected horse had won, and giving another tip for the following week. This was repeated for a third week. Then after three wins the 'consultant' said he would send the name of another horse which was guaranteed to win next week – but this time there would be a cost of $5,000. This seemed a reasonable price, as the horse was certain to win and gamblers could place bets of any size. Unfortunately, this scheme had a drawback. Investigators thought that the 'consultant' sent out about 10,000 of the original letters, and randomly tipped each horse in a five-horse race. The second letter was sent only to those people who had been given the winning horse. The next two letters followed the same pattern, with follow-up letters sent only to those who had been given the winning horse.

The fourth report concerned a European lottery, where people entered by paying €1 and choosing six numbers in the range 00 to 99. At the end of a week a computer generated a set of six random numbers, and anyone with the same six numbers would win the major prize (typically several million euros), and people with four or five matching numbers would win smaller prizes. A magazine reported a way of dramatically increasing the chances of winning. This suggested taking your eight favourite numbers and then betting on all combinations of six numbers from these eight. The advertisement explained the benefit of this by saying, 'Suppose there is a chance of one in a million of winning the first prize. If one of your lucky numbers is chosen by the computer, you will have this number in over a hundred entries, so your chances of winning are raised by 100 to only one in 10,000.'

Question

- **The *Gamblers' Press* finds schemes like these four almost every day, and often publishes articles on them. What do such schemes have in common? If you were asked to write an article about these four schemes, what would you say? You might start by explaining why the four schemes mentioned do not work, and then expand the study to include other examples of such schemes.**

PROBLEMS

14.1 An office has the following types of employees.

	Female	Male
Administrative	20	21
Managerial	12	10
Operational	42	38

If one person from the office is chosen at random, what is the probability that the person is: (a) a male administrator, (b) a female manager, (c) male, (d) an operator, (e) either a manager or an administrator, (f) either a female administrator or a female manager?

14.2 A quality control test has five equally likely outcomes, A, B, C, D and E.
(a) What is the probability of C occurring?
(b) What is the probability of A or B or C occurring?
(c) What is the probability that neither A nor B occur?

14.3 Four mutually exclusive events A, B, C and D have probabilities of 0.1, 0.2, 0.3 and 0.4 respectively. What are the probabilities of the following occurring: (a) A and B, (b) A or B, (c) neither A nor B, (d) A and B and C, (e) A or B or C, (f) none of A, B or C?

14.4 If you choose a card at random from a pack, what is the probability that it is: (a) an ace, (b) a heart, (c) an ace and a heart, (d) an ace or a heart, (e) neither an ace nor a heart?

14.5 Bert Klassen schedules three calls for a particular day, and each call has a probability of 0.5 of making a sale. What are his probabilities of making (a) three sales, (b) two or more sales, (c) no sales?

14.6 There are 20 people in a room. What is the probability that they all have different birthdays?

14.7 If $P(a) = 0.4$ and $P(b/a) = 0.3$, what is $P(a \text{ AND } b)$? If $P(b) = 0.6$, what is $P(a/b)$?

14.8 The probabilities of two events X and Y are 0.4 and 0.6 respectively. The conditional probabilities of three other events A, B and C occurring, given that X or Y has already occurred, are:

	A	B	C
X	0.2	0.5	0.3
Y	0.6	0.1	0.3

What are the conditional probabilities of X and Y occurring, given that A, B or C has already occurred?

14.9 Kenny Lam works in a purchasing department that buys materials from three main suppliers, with X supplying 35% of the department's needs, Y supplying 25% and Z the rest. The quality of the materials is described as good, acceptable or poor, with the following proportions from each supplier:

	Good	Acceptable	Poor
X	0.2	0.7	0.1
Y	0.3	0.65	0.05
X	0.1	0.8	0.1

What information can you find using Bayes' theorem on these figures? How would you show these results on a probability tree?

14.10 Data collected from Cape Town shows that 60% of drivers are above 30 years old. Five per cent of all the drivers over 30 will be prosecuted for a driving offence during a year, compared with 10% of drivers aged 30 or younger. If a driver has been prosecuted, what is the probability they are 30 or younger?

RESEARCH PROJECTS

14.1 Spreadsheets include a lot of statistical analyses, but these are often rather difficult to use and understand. Have a look at the statistics available on a typical spreadsheet and see what functions they include. If you want to do some large-scale statistical analysis, it is better to use specialised programs such as SPSS, SAS, Minitab or S+. Do a small survey of statistical packages and see what extra features they have.

14.2 The *Gamblers' Press* case study showed some misleading uses of statistics. Unfortunately this is fairly common. Find some other example where statistics are used to give the wrong impression – either intentionally or unintentionally.

14.3 Marius Gensumara found that during a typical period the following numbers of people did not turn up to work at his company in consecutive days:

13 16 24 21 15 23 15 26 25 11 10 24 27 30 15 31 25 19 15 27

He wanted to improve this, and introduced a new scheme of incentives and payments in general. He did not discuss this with the workforce or anyone else, and when he rechecked the numbers of absentees he found the following numbers:

31 29 27 30 26 28 38 34 40 25 29 34 33 30 28 26 41 45 30 28

Marius felt that there was some unhappiness with his new scheme and the way that it had been introduced. So he had various negotiations and agreed another incentive scheme. Now, rechecking the absentees, he found the following numbers each day:

09 12 16 08 24 09 15 16 20 21 09 11 10 10 25 17 16 18 09 08

What do these figures suggest? Find some examples of statistics that have been used in negotiations and say how they have been interpreted.

14.4 A lot of websites give tutorials on topics of statistics that are useful for managers. These sites are produced by universities, institutions, publishers, training companies, software providers, tutoring services, consultants, and so on. Do some searches on the Web to find sites that are useful for this course.

Sources of information

References

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2 CIS, promotional material, Co-operative Insurance Society, Manchester, 2006.

3 CIA, *World Factbook*, Brassey's, Inc., Dulles, VA, 2000.

Further reading

There are many books on statistics, ranging from the trivial through to the highly mathematical. You might look at the following as a starting point. These all include information that is also useful for the other chapters.

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CHAPTER 15

Probability distributions

Contents

Chapter outline

Probability distributions are relative frequency distributions, which show the probabilities of related events. You can draw an empirical distribution for any set of data – but standard distributions describe many real problems. This chapter describes three important distributions. The binomial distribution calculates the probable number of successes in a series of trials; the Poisson distribution describes random events; and the Normal distribution is the most widely used distribution, describing data in many different circumstances.

After finishing this chapter you should be able to:

- Understand the role of probability distributions
- Draw an empirical probability distribution
- Describe the difficulties of sequencing and scheduling
- Calculate numbers of combinations and permutations
- Know how to use a binomial distribution and calculate probabilities
- Know how to use a Poisson distribution and calculate probabilities
- Understand a Normal distribution and do related calculations.

Frequency distributions

In Chapter 5 we described a relative frequency distribution, which shows the proportion of observations in different classes. The last chapter described probabilities as measures of relative frequency. Now we are going to combine these two ideas into **probability distributions** – which describe the frequency of probabilities in different classes.

WORKED EXAMPLE 15.1

Every night the Luxor Hotel has a number of people who book rooms by telephone, but do not actually turn up. Managers recorded the numbers of no-shows over a typical period:

2 4 6 7 1 3 3 5 4 1 2 3 4 3 5 6 2 4 3 2 5 5

0 3 3 2 1 4 4 4 3 1 3 6 3 4 2 5 3 2 4 2 5 3 4

Draw a frequency table for the data. How can you use this to draw a probability distribution? What is the probability that there are more than four no-shows?

Solution

To draw a frequency table we add the number of nights with various numbers of no-shows, giving the following results.

No-shows	0	1	2	3	4	5	6	7
Frequency	1	4	8	12	10	6	3	1
Relative frequency or probability	0.02	0.09	0.18	0.27	0.22	0.13	0.07	0.02

Dividing each of the frequencies by the total number of observations (45) gives the relative frequency – or probability. For instance, there were seven no-shows on one night, so the probability of seven no-shows is 7/45 = 0.02. Repeating this gives figures for the probability distribution, which is drawn as a bar chart in Figure 15.1.

The probability of more than four no-shows is:

$$P(5) + P(6) + P(7) = 0.13 + 0.07 + 0.02 = 0.22$$

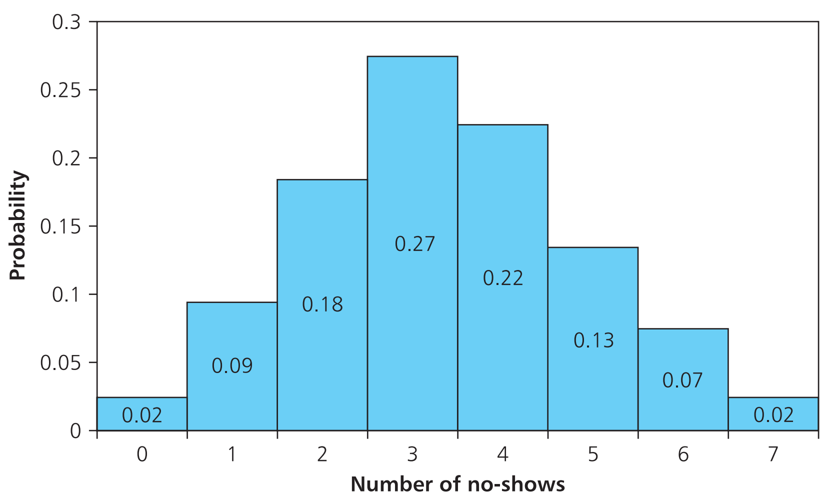


Figure 15.1 A probability distribution for worked example 15.1

The probability distribution in this example is empirical, which means that it came from actual observations. You can use this approach for any problem where you have enough observations. The result is a useful format for summarising data – but the weakness is that the resulting distribution is specific to a particular problem, and you cannot use the same distribution for other problems. The probability distribution in Figure 15.1 describes the number of no-shows in the Luxor Hotel, but you cannot assume that the same distribution works for any other hotel, or any other problem.

If you look at a lot of empirical distributions, you see that there are often standard patterns. If you look at the size of purchases from websites, the distribution looks very similar to the distribution of the annual production from a fruit farm. This suggests that some probability distributions do not just refer to a specific problem, but are more widely applicable. Three common distributions of this type are:

- binomial distribution
- Poisson distribution
- Normal (or Gaussian) distribution.

Review questions

15.1 What is the purpose of a probability distribution?

15.2 Is a probability distribution the same as a relative frequency distribution?

15.3 What are empirical probability distributions?

Combinations and permutations

The binomial distribution is the first standard probability distribution that we consider. However, before describing this we have to do some calculations for sequencing problems.

Sequencing problems occur whenever you have to do a number of activities, and can take these in different orders. The order in which you do the activities can affect your overall performance. For example, the time taken for a bus to travel between a suburb and a city centre depends on the order in which it visits stops; the time it takes you to cook a meal depends on the order in which you do all the different tasks; the efficiency of a computer system depends on the order in which it does a series of tasks; the time taken to build a house depends on the order in which the builder does the jobs.

When you have a sequencing problem, it seems fairly straightforward to consider all the possible orders in which you can do activities, and then choose the best. Unfortunately, the number of possible sequences you have to consider makes this notoriously difficult. Suppose you have to complete n activities, and want to find the best order in which to do them. You can choose the first activity as any one of the n. Then you can choose the second activity as any one of the remaining $(n - 1)$, so there are $n(n - 1)$ possible sequences for the first two activities. You can choose the third activity as any one of the remaining $(n - 2)$, the fourth as any of $(n - 3)$, and so on. And by the time you have chosen the last activity, the total number of possible sequences for n activities is:

$$\text{number of sequences} = n(n - 1)(n - 2)(n - 3) \ldots \times 3 \times 2 \times 1 = n!$$

(Remember that $n!$ – pronounced 'n factorial' – is an abbreviation for $n(n - 1)(n - 2) \ldots \times 3 \times 2 \times 1$. Notes also that $1! = 0! = 1$.)

If you have a trivial problem with five activities, there are already $5 \times 4 \times 3 \times 2 \times 1 = 120$ possible sequences. When you have 10 activities you have more than 3.6 million possible sequences, and with 15 activities this has risen to 1.3×10^{12}.

Instead of finding the best sequence of all n activities, suppose that you only want to find the best sequence of r of them. There are two important calculations for this, which are called permutations and combinations.

When you want to select r things from n, and you are not interested in the order in which the r things are selected but only whether they are selected or not, then you are concerned with the **combination** of r things from n.

> The **combination** of r things from n is the number of ways of selecting r things when the order of selection does not matter, and this is:
>
> $$^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

When you enter a lottery you are interested only in the numbers that are chosen, not the order in which they are chosen, so you are interested in combinations. If there is a pool of 10 cars and three customers arrive to use them, it does not matter in which order the customers arrive, so there are $^{10}C_3$ ways of allocating cars:

$$^{10}C_{3} = \frac{10!}{3! \times (10-3)!} = 120$$

If you have n things and want to select r of these, but this time you are concerned with the order of selection, then you are interested in the **permutation** of r things from n.

> The **permutation** of r things from n is the number of ways of selecting r things when the order of selection is important, and this is:
>
> $$^{n}P_{r} = \frac{n!}{(n-r)!}$$

Suppose there are 10 applicants for a social club committee consisting of a chairman, deputy chairman, secretary and treasurer. You want to select four from 10, and the order in which you select them is important as it corresponds to the different jobs. Then the number of ways of choosing the committee of four is:

$$^{n}P_{r} = \frac{n!}{(n-r)!} = \frac{10!}{(10-4)!} = 5{,}040$$

If you want to select four ordinary committee members – so they all do the same job and the order in which you choose them is not important – then you are interested in the combinations of four from 10:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{10!}{4!(10-4)!} = 210$$

Permutations depend on order of selection and combinations do not, so there are always a lot more permutations than combinations. The number of combinations of four letters of the alphabet is $26!/(4! \times 22!) = 14{,}950$.

One combination has the four letters A, B, C and D, and it does not matter in which order these are taken. The number of permutations of four letters from the alphabet is 26!/22! = 358,800, and this time it matters in which order the letters are taken, so now we start listing the permutations as ABCD, ABDC, ACBD, ACDB, ADBC, ADCB . . . and so on.

WORKED EXAMPLE 15.2

(a) In their annual recruiting, ALM Holdings has eight applicants to fill eight different jobs. In how many different ways can it assign applicants to jobs?

(b) There is a sudden reorganisation in ALM and the number of jobs falls to six. In how many different ways can the jobs be filled with the eight applicants?

(c) Suppose the reorganisation leads to a reclassification of jobs and the six jobs are now identical. In how many different ways can they be filled?

Solution

(a) This essentially asks, 'How many different sequences of eight candidates are there?'. The answer is 8!. The applicants can be assigned to jobs in 8! = 40,320 different ways.

(b) This asks the number of ways in which six candidates can be selected from eight. As the jobs are different, we are interested in the order of selection and hence the permutations. The number of permutations of six different jobs from eight applicants is:

$$^{n}P_{r} = \frac{n!}{(n-r)!} = \frac{8!}{(8-6)!} = 20{,}160$$

(c) This again asks the number of ways in which six candidates can be selected from eight, but now the jobs are identical so the order of selection is not important. The number of combinations of six identical jobs from eight applicants is:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{8!}{6!(8-6)!} = 28$$

You can look at this the other way around and ask the number of ways of rejecting two applicants from eight. Not surprisingly, with $r = 2$ we again have:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{8!}{2!(8-2)!} = 28$$

WORKED EXAMPLE 15.3

Twelve areas in Alaska become available for oil exploration and the US government's policy of encouraging competition limits the allocation of these to at most one area for any exploration company.

(a) If 12 exploration companies bid for the areas, in how many ways can the areas be allocated?

(b) Initial forecasts show that each area is equally likely to produce oil, so they are equally attractive. If 20 exploration companies put in bids, how many ways are there of selecting winning companies?

(c) Suppose that a report shows the probabilities of major oil discoveries in each area, and 16 companies now put in bids. How many different ways are there of choosing companies?

Solution

(a) Twelve companies each receive one area, so the companies can be sequenced in 12! possible ways, or 4.79×10^{8}.

(b) Now there are 20 companies, only 12 of which will be selected. As each area is equally attractive, the order of choosing companies does not matter, so we want the number of combinations:

$$^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{20!}{12!(20-12)!} = 125{,}970$$

(c) Now the areas are different, so the order of choosing 12 of the 16 companies is important, and therefore we want the number of permutations:

$$^{n}P_{r} = \frac{n!}{(n-r)!} = \frac{16!}{(16-12)!} = 8.72 \times 10^{11}$$

Review questions

15.4 In how many ways can n different activities be sequenced?

15.5 What is the difference between a permutation and a combination?

15.6 When selecting r things from n, are there more combinations or permutations?

Binomial distribution

We can use the ideas of sequencing in the first standard probability distribution, which is the **binomial distribution**. This is used whenever there is a series of **trials** with the following characteristics:

- Each trial has two possible outcomes, conventionally called success and failure.
- The two outcomes are mutually exclusive.
- There is a constant probability of success, p, and of failure, $q = 1 - p$.
- The outcomes of successive trials are independent.

Tossing a coin is an example of a binomial process – each toss is a trial; each head, say, is a success with a constant probability of 0.5; each tail is a failure with a constant probability of 0.5; and each trial is independent. Quality control inspections give another example of a binomial process. Each inspection of a unit is a trial, each fault is a success and each good unit is a failure.

The binomial distribution gives the probabilities of different numbers of successes in a series of trials. Specifically, it gives the probability of r successes in n trials. The important point is that we can calculate this probability. In each trial the probability of a success is constant at p. Then for independent trials the probability that the first r are successes is p^r. Similarly, the probability that the next $n - r$ trials are failures is q^{n-r}. So the probability that the first r trials are successes and then the next $n - r$ trials are failures is $p^r q^{n-r}$.

But the sequence of r successes followed by $n - r$ failures is only one way of getting r successes in n trials. We also need to look at the other possible sequences. In the last section we found that the number of sequences of r things chosen from n, when the order of selection does not matter, is ${}^nC_r = n!/r!(n - r)!$. So there are ${}^nC_r$ possible sequences of r successes and $n - r$ failures, each with probability $p^r q^{n-r}$. We find the overall probability of r successes by multiplying the number of sequences by the probability of each sequence, to give the binomial probability distribution:

$$P(r \text{ successes in } n \text{ trials}) = {}^nC_r p^r q^{n-r} = \frac{n!p^r q^{n-r}}{r!(n-r)!}$$

WORKED EXAMPLE 15.4

The probability of success in a binomial trial is 0.3. What is the probability of two successes in five trials? What is the probability of four successes?

Solution

Here $p = 0.3$, $q = 1 - p = 0.7$, $n = 5$ and we want $P(2)$. Substituting in the equation for the binomial distribution gives:

$$P(r) = \frac{n!\,p^r q^{n-r}}{r!(n-r)!}$$

so

$$P(2) = \frac{5! \times 0.3^2 \times 0.7^3}{2! \times 3!} = 0.309$$

Similarly:

$$P(4) = \frac{5! \times 0.3^4 \times 0.7^1}{4! \times 1!} = 0.028$$

WORKED EXAMPLE 15.5

Jenny Albright knows that in the long term she has a 50% chance of making a sale when calling on a customer. One morning she arranges six calls.

(a) What is her probability of making exactly three sales?
(b) What are her probabilities of making other numbers of sales?
(c) What is her probability of making fewer than three sales?

Solution

The problem is a binomial process in which the probability of success (that is, making a sale) is $p = 0.5$, the probability of failure (not making a sale) is $q = 1 - p = 0.5$, and the number of trials, n, is 6.

(a) Her probability of making exactly three sales (so $r = 3$) is:

$$P(r \text{ successes in } n \text{ trials}) = {}^{n}C_{r}p^{r}q^{n-r}$$

so

$$P(3 \text{ sales in 6 calls}) = {}^{6}C_{3} \times 0.5^{3} \times 0.5^{6-3}$$
$$= \frac{6!}{3!3!} \times 0.125 \times 0.125$$
$$= 0.3125$$

(b) We can substitute other values for r into the equation and get corresponding values for the probabilities. However, we can take a short cut, and in Figure 15.2 we have used Excel's standard 'BINOMDIST' function to automatically calculate the probabilities.

(c) The probability of making fewer than three sales is the sum of the probabilities of making no, one and two sales:

$$P(\text{fewer than 3 sales})$$
$$= P(0 \text{ sales}) + P(1 \text{ sale}) + P(2 \text{ sales})$$
$$= 0.0156 + 0.0938 + 0.2344 = 0.3438$$

	A	B	C	D	E	F	G	H
1	**Binomial distribution**							
2								
3	**Trials**	6	***r***	***P*(*r*)**				
4	***P*(success)**	0.5	0	0.0156				
5			1	0.0938				
6			2	0.2344				
7			3	0.3125				
8			4	0.2344				
9			5	0.0938				
10			6	0.0156				
11								

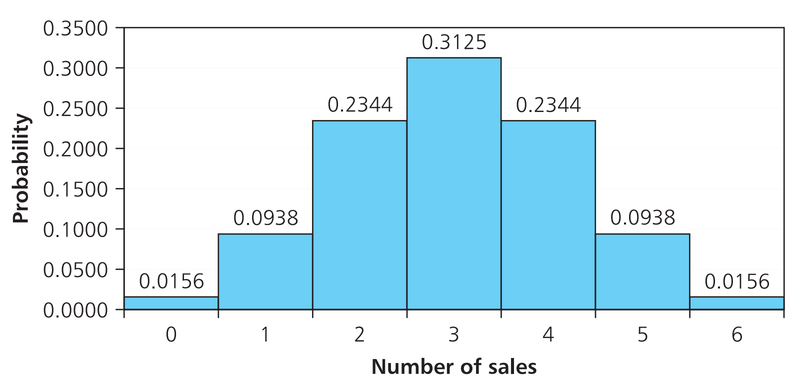


Figure 15.2 Binomial probability distribution for Jenny Albright

The shape of the binomial distribution varies with p and n. For small values of p the distribution is asymmetrical and the peak is to the left of centre. As p increases the peak moves to the centre of the distribution and with $p = 0.5$ the distribution is symmetrical. As p increases further the distribution again becomes asymmetrical but this time the peak is to the right of centre. For larger values of n the distribution is flatter and broader. Figure 15.3 shows these effects for a sample of values.

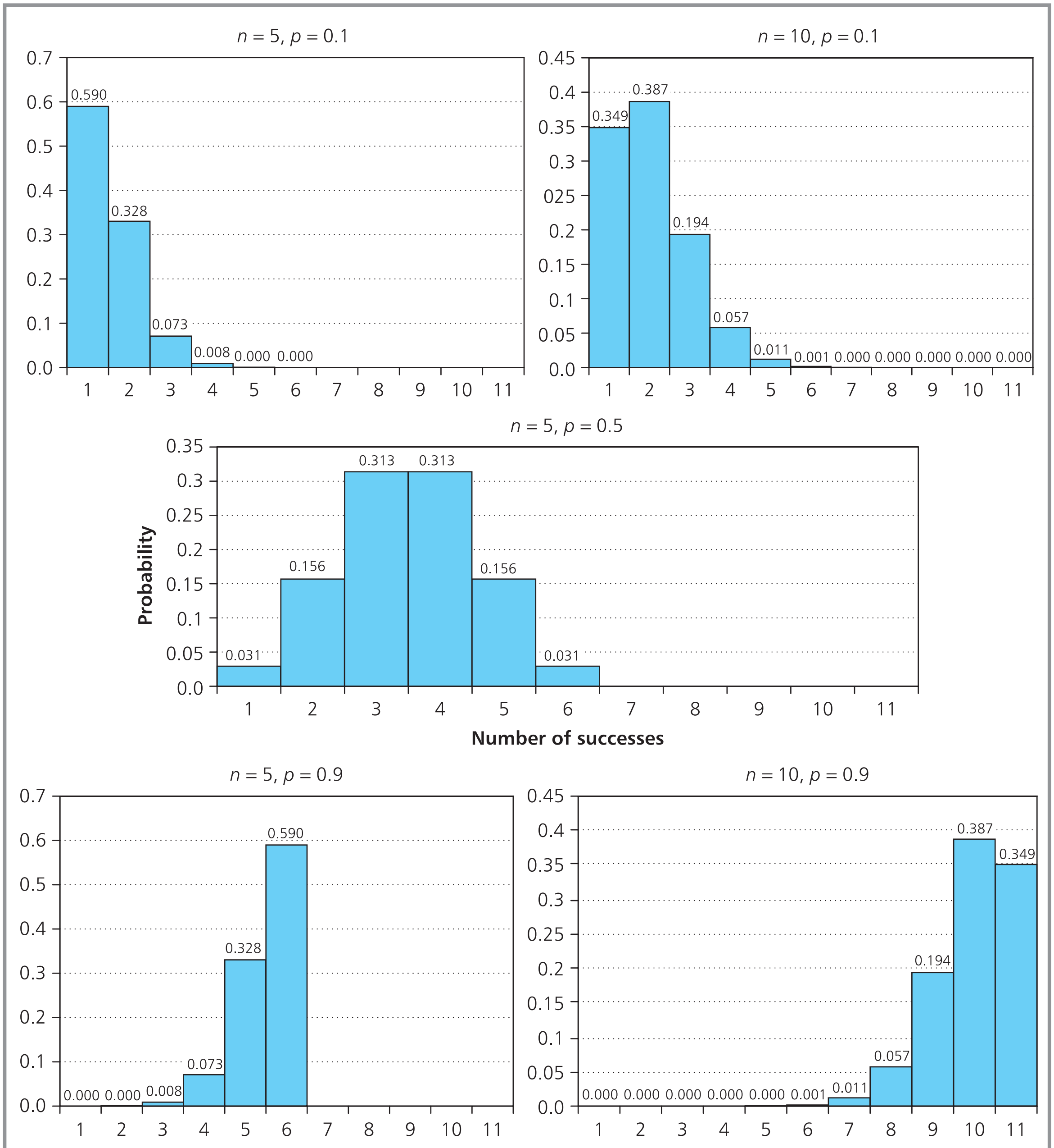


Figure 15.3 The shape of the binomial distribution changes with n and p

The mean, variance and standard deviation of a binomial distribution are calculated from the following formulae:

For a binomial distribution:

- mean = $\mu = np$
- variance = $\sigma^2 = npq$
- standard deviation = $\sigma = \sqrt{(npq)}$

Notice that in these definitions we have used the Greek letter μ (mu) for the mean rather than $\bar{x}$, and the Greek letter σ (sigma) for the standard deviation rather than s. This follows a standard notation, where:

- the mean and standard deviation of a sample are called $\bar{x}$ and s respectively
- the mean and standard deviation of a population are called μ and σ respectively.

WORKED EXAMPLE 15.6

A company makes a particularly sensitive electronic component, and 10% of the output is defective. If the components are sold in boxes of 12, describe the distribution of defects in each box. What is the probability that a box has no defects? What is the probability that a box has one defect?

Solution

This is a binomial process, with success being a faulty unit. In each box $n = 12$ and $p = 0.1$. So:

- mean number of faulty units in a box: $np = 12 \times 0.1 = 1.2$
- variance: $npq = 12 \times 0.1 \times 0.9 = 1.08$
- standard deviation: $\sqrt{npq} = \sqrt{1.08} = 1.039$

The probability of no defects in a box is:

$$P(0) = 0.9^{12} = 0.2824$$

The probability of one defect comes from the binomial distribution calculation:

$$P(r) = {}^nC_r p^r q^{n-r}$$

so

$$P(1) = {}^{12}C_1 \times 0.1^1 \times 0.9^{12-1} = \frac{12! \times 0.1 \times 0.3138}{11! \times 1!}$$

$$= 0.3766$$

We have already mentioned that Excel's 'BINOMDIST' function automatically does these calculations, but there is another option, which is to use standard tables. Appendix B contains tables for various combinations of n and p. If you look up the table for $n = 12$ and move across to the column where $p = 0.1$, you see a set of 12 numbers. These are the probabilities for each number of successes. The first number is 0.2824, which confirms our calculations for the probability of no defects in a box; the second number is 0.3766, which confirms our calculation for the probability of one defect in a box.

WORKED EXAMPLE 15.7

Boris Schwartz is a market researcher who has to visit 10 houses in a given area between 7.30 and 9.30 one evening. Previous calls suggest that there will be someone at home in 85% of houses. Describe the probability distribution of the number of houses with people at home. What is the probability there will be someone at home in at least eight houses?

Solution

This is a binomial process with visiting a house as a trial and finding someone at home a success. Then

Worked example 15.7 continued

we have $n = 10$, $p = 0.85$ and $q = 0.15$. Substituting in the standard equations gives:

- mean number of houses with someone at home $= np = 10 \times 0.85 = 8.5$
- variance $= npq = 10 \times 0.85 \times 0.15 = 1.275$
- standard deviation $= \sqrt{npq} = \sqrt{1.275} = 1.129$

You have three options to find these values. Firstly, you can do the calculation to find the probability that there is someone at home in, say, eight houses from:

$$P(8) = {}^{10}C_8 \times 0.85^8 \times 0.15^2$$
$$= 45 \times 0.2725 \times 0.0225 = 0.2759$$

Secondly, you can look up the figures in the standard tables in Appendix B. Values are given only for p up to 0.5, so to use the tables you must redefine 'success' as finding a house with no-one at home – and then $p = 0.15$. Looking up the entry for $n = 10$, $p = 0.15$ and $r = 2$ (finding eight houses with someone at home being the same as finding two houses with no-one at home) confirms the value as 0.2759.

Thirdly, you can use a spreadsheet – or other standard software – to do the calculations automatically. Figure 15.4 shows the probability distribution in a spreadsheet. In this you can also see that the probability that there is someone at home in at least eight houses is:

$$P(\geq 8) = P(8) + P(9) + P(10)$$
$$= 0.2759 + 0.3474 + 0.1969 = 0.8202$$

You could have found this from the cumulative probabilities, by saying:

$$P(\geq 8) = 1 - P(<7) = 1 - 0.1798 = 0.8202$$

	A	B	C	D	E	F	G	H
1	**Binomial distribution**							
2								
3	**Trials** / ***P*(success)**	10 / 0.85	***r***	***P*(*r*)**	**Cumulative probability**			
4			0	0.0000	0.0000			
5			1	0.0000	0.0000			
6			2	0.0000	0.0000			
7			3	0.0001	0.0001			
8			4	0.0012	0.0014			
9			5	0.0085	0.0099			
10			6	0.0401	0.0500			
11			7	0.1298	0.1798			
12			8	0.2759	0.4557			
13			9	0.3474	0.8031			
14			10	0.1969	1.0000			
15								

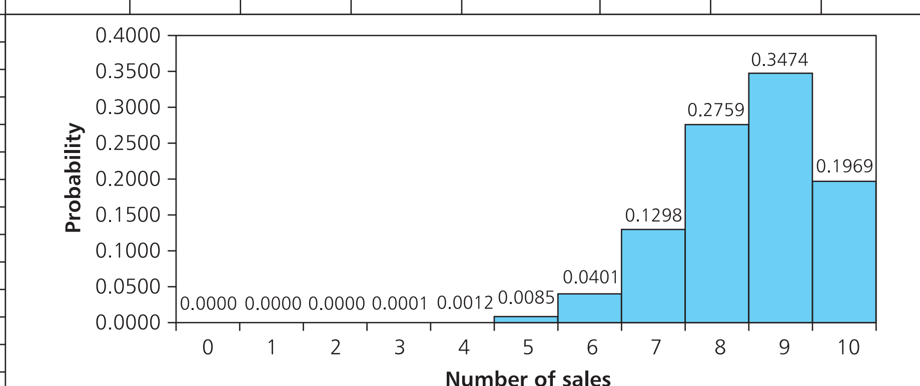


Figure 15.4 Probability distribution for Boris Schwartz

With the choice of calculation, standard tables and statistical software, you probably think that the arithmetic for binomial probabilities is straightforward. But sometimes it becomes difficult, and the following example illustrates the problem when n is very large and p is very small.

WORKED EXAMPLE 15.8

The accounts department of a company sends out 10,000 invoices a month, and has an average of five returned with an error. What is the probability that exactly four invoices are returned in a month?

Solution

This is a typical binomial process, where a trial is sending out an invoice, and a success is having an error. Unfortunately, when you start doing the arithmetic, you see the problem. The values we have are $n = 10{,}000$, $r = 4$, $p = 5/10{,}000 = 0.0005$, $q = 9{,}995/10{,}000 = 0.9995$. Substituting these into the binomial distribution:

$$P(r \text{ returns}) = \frac{n! \times p^r q^{n-r}}{r!(n-r)!} = \frac{10{,}000!}{4! \times 9{,}996!} \times (0.0005)^4 \times (0.9995)^{9{,}996}$$

Although we can do this calculation – and find the result that $P(4 \text{ returns}) = 0.1755$, it seems rather daunting to raise figures to the power of 9,996 or to contemplate 10,000 factorial. Fortunately, there is an alternative. When n, the number of trials, is large and p, the probability of success, is small, we can approximate the binomial distribution by a Poisson distribution.

Review questions

15.7 When would you use a binomial distribution?

15.8 Define all the terms in the equation $P(r) = {}^nC_r p^r q^{n-r}$

15.9 How can you calculate the mean and variance of a binomial distribution?

15.10 From the tables in Appendix B, find the probability of two successes from seven trials, when the probability of success is 0.2.

Poisson distribution

The **Poisson distribution** is a close relative of the binomial distribution and can be used to approximate it when:

- the number of trials, n, is large (say greater than 20)
- the probability of success, p, is small (so that np is less than 5).

As n gets larger and p gets smaller the approximation becomes better.

The Poisson distribution is also useful in its own right for solving problems where events occur at random. So you could use a Poisson distribution to describe the number of accidents each month in a company, the number of defects in a metre of cloth, the number of phone calls received each hour in a call centre, and the number of customers entering a shop each hour.

The binomial distribution uses the probabilities of both success and failure, but the Poisson uses only the probability of success. This is because it assumes that the probability of success is very small, so it is effectively looking for a few successes in a continuous background of failures. If you look at the number of spelling mistakes in a long report, the number of faults in a pipeline, or the number of accidents in a month, you are interested only in the small number of successes – and are not at all concerned with the large number of failures.

A Poisson distribution is described by the equation:

$$P(r \text{ successes}) = \frac{e^{-\mu}\mu^r}{r!}$$

where: e = exponential constant = 2.7183
μ = mean number of successes.

WORKED EXAMPLE 15.9

On a North Sea oil rig there have been 40 accidents that were serious enough to report in the past 50 weeks. In what proportion of weeks would you expect no, one, two, three and more than three accidents?

Solution

A small number of accidents occur, presumably at random, over time. We are not interested in the number of accidents that did *not* occur, so we have a Poisson process, with:

$$P(r \text{ successes}) = \frac{e^{-\mu}\mu^r}{r!}$$

The mean number of accidents in a week is 40/50 = 0.8, so substituting $\mu = 0.8$ and $r = 0$ gives:

$$P(0) = \frac{e^{-0.8} \times 0.8^0}{0!} = 0.4493$$

Similarly:

$$P(1) = \frac{e^{-0.8} \times 0.8^1}{1!} = 0.3595$$

$$P(2) = \frac{e^{-0.8} \times 0.8^2}{2!} = 0.1438$$

$$P(3) = \frac{e^{-0.8} \times 0.8^3}{3!} = 0.0383$$

and

$$\begin{aligned} P(>3) &= 1 - P(\leq 3) \\ &= 1 - P(0) - P(1) - P(2) - P(3) \\ &= 1 - 0.4493 - 0.3595 - 0.1438 - 0.0383 \\ &= 0.0091 \end{aligned}$$

WORKED EXAMPLE 15.10

How does the Poisson distribution deal with the situation in worked example 15.8, where an accounts department sends out 10,000 invoices a month and has an average of five returned with an error? What is the probability that exactly four invoices will be returned in a given month?

Solution

Here n is large and $np = 5$ (fairly high but the result should still be reasonable), so we can use a Poisson distribution to approximate the binomial distribution. The variables are $r = 4$ and $\mu = 5$.

$$P(r \text{ successes}) = \frac{e^{-\mu}\mu^r}{r!}$$

so

$$P(4 \text{ successes}) = \frac{e^{-5}\,5^4}{4!} = (0.0067 \times 625)/24$$

$$= 0.1755$$

When events occur at random you can usually use a Poisson distribution, but strictly speaking there are a number of other requirements. In particular, a Poisson process requires that:

- the events are independent
- the probability that an event happens in an interval is proportional to the length of the interval
- in theory, an infinite number of events should be possible in an interval.

Then:

For a Poisson distribution:

- mean, $\mu = np$
- variance, $\sigma^2 = np$
- standard deviation, $\sigma = \sqrt{(np)}$

WORKED EXAMPLE 15.11

A Poisson process has a mean of five events a day. Describe the probability distribution of events.

Solution

The mean number of events, μ, is 5. This is also the variance, and the standard deviation is $\sqrt{5} = 2.236$.

The probability of r events is:

$$P(r) = \frac{e^{-\mu}\mu^r}{r!}$$

so, for example,

$$P(2) = \frac{e^{-5} \times 5^2}{2!} = 0.0842$$

We could do the calculations for other numbers of successes, but as with the binomial distribution, there are alternatives. Again there are standard tables, as shown in Appendix C. If you look up the value for $\mu = 5$ and $r = 2$, you can confirm the probability as 0.0842. And you can use statistical packages or spreadsheets. Figure 15.5 shows the results from Excel's standard 'POISSON' function.

	A	B	C	D	E	F	G	H
1	**Poisson distribution**							
2								
3	**Mean**	5	***r***	***P*(*r*)**	**Cumulative probability**			
4			0	0.0067	0.0067			
5			1	0.0337	0.0404			
6			2	0.0842	0.1247			
7			3	0.1404	0.2650			
8			4	0.1755	0.4405			
9			5	0.1755	0.6160			
10			6	0.1462	0.7622			
11			7	0.1044	0.8666			
12			8	0.0653	0.9319			
13			9	0.0363	0.9682			
14			10	0.0181	0.9863			
15			11	0.0082	0.9945			
16			12	0.0034	0.9980			
17								
18								
19								
20								
21								
22								
23								
24								
25								
26								
27								
28								
29								
30								
31								

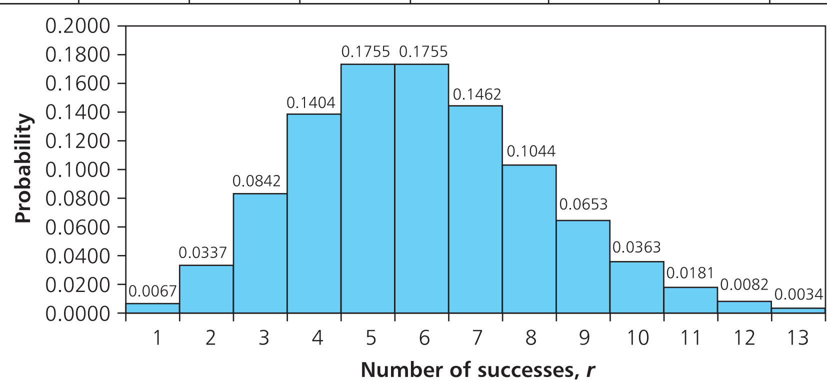


Figure 15.5 Spreadsheet of Poisson probabilities

Figure 15.5 also shows the shape of a Poisson distribution, which you can see is similar to that of a binomial distribution. The shape and position of the Poisson distribution are determined by the single parameter, μ. For small μ the distribution is asymmetrical with a peak to the left of centre; then as μ increases the distribution becomes more symmetrical, as you can see from Figure 15.6.

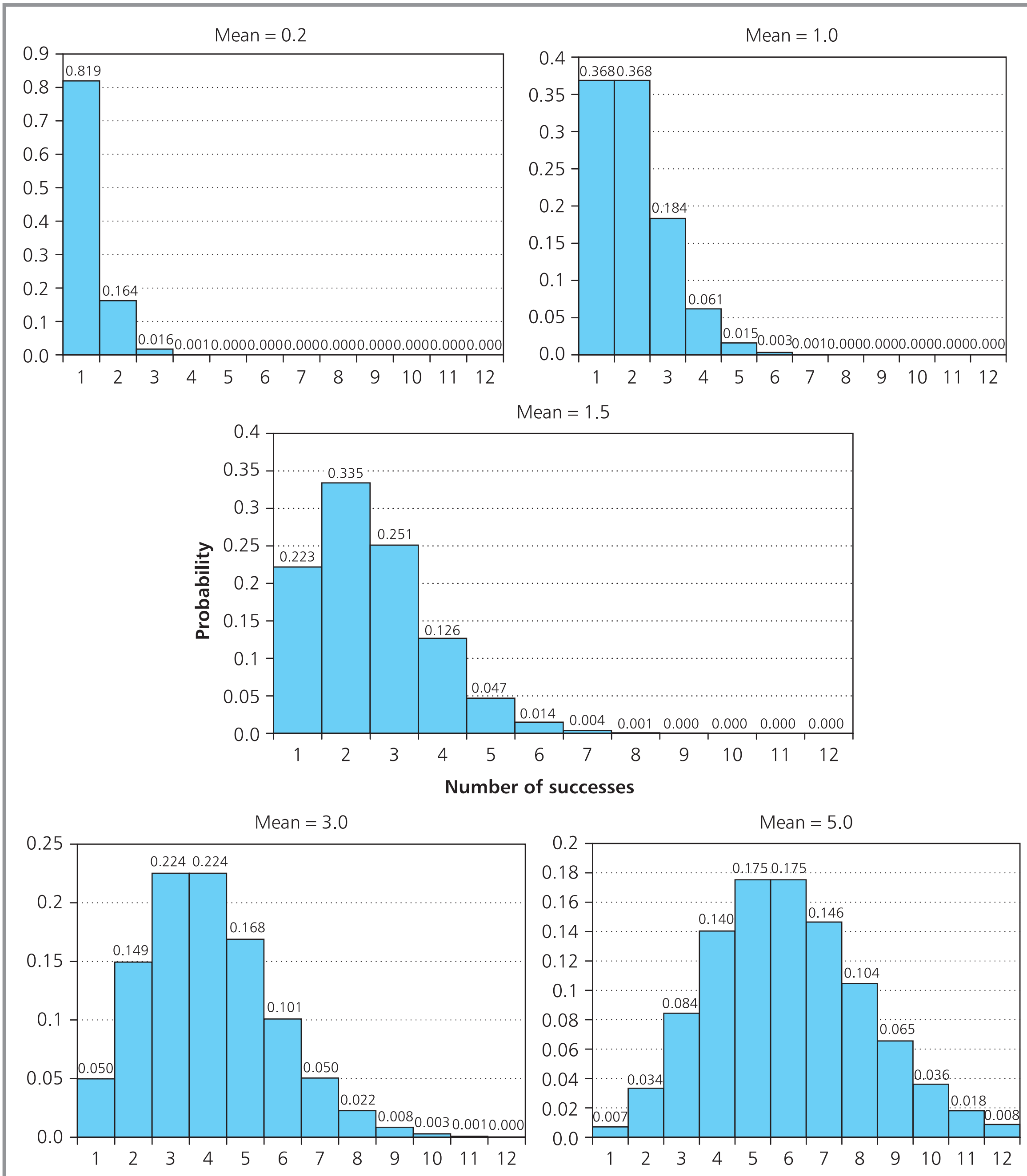


Figure 15.6 The shape of a Poisson distribution varies with μ

WORKED EXAMPLE 15.12

Hellier council ran a test to see whether a road junction should be improved. During this test they found that cars arrive randomly at the junction at an average rate of four cars every five minutes. What is the probability that more than eight cars will arrive in a five-minute period?

Solution

Random arrivals mean that we have a Poisson process. The mean number of successes (that is, cars arriving at the junction in five minutes) is $\mu = 4$. Then the probability of exactly, say, three cars arriving is:

$$P(r) = \frac{e^{-\mu}\mu^r}{r!}$$

so

$$P(3) = \frac{e^{-4} \times 4^3}{3!} = 0.1954$$

You can check this by looking at the tables in Appendix C, and Figure 15.7 confirms the result using Excel's 'POISSON' function. You can find the probability that more than eight cars arrive at the junction in a five-minute period, $P(\geq 8)$, from the cumulative probability in Figure 15.7.

$$P(\geq 8) = 1 - P(\leq 7) = 1 - 0.9489 = 0.0511$$

	A	B	C	D	E	F	G	H
1	**Poisson distribution**							
2								
3	**Mean**	4	***r***	***P*(*r*)**	**Cumulative probability**			
4			0	0.0183	0.0183			
5			1	0.0733	0.0916			
6			2	0.1465	0.2381			
7			3	0.1954	0.4335			
8			4	0.1954	0.6288			
9			5	0.1563	0.7851			
10			6	0.1042	0.8893			
11			7	0.0595	0.9489			
12			8	0.0298	0.9786			
13			9	0.0132	0.9919			
14			10	0.0053	0.9972			
15			11	0.0019	0.9991			
16			12	0.0006	0.9997			
17								

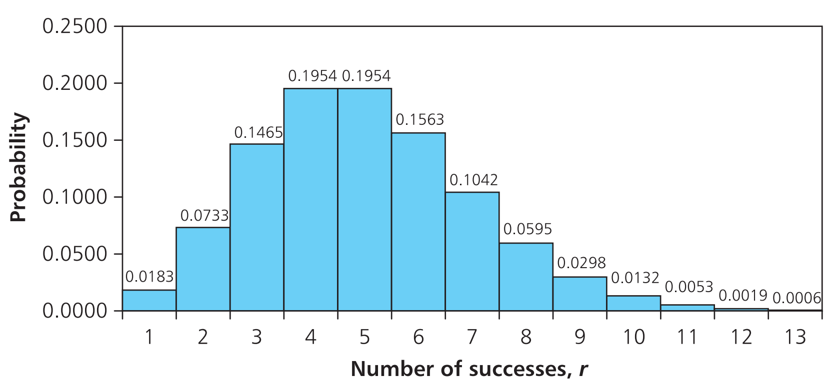


Figure 15.7 Spreadsheet of Poisson probabilities for Hellier council

IDEAS IN PRACTICE Shingatsu Industries

In 2006 Shingatsu Industries bought a network of oil pipelines in Eastern Russia. Shingatsu knew the network was not in good condition, and they found that 40% of the oil pumping stations had faults, and there was an average of one fault in the pipeline every 20 km. In a key 100 km stretch of pipeline there are 20 pumping stations, and Shingatsu wanted an idea of the likely number of problems.

They looked at the problem in two parts – faults in pumping stations, and faults along the pipeline. Faults in the pumping station follow a binomial distribution, as they are either faulty or not faulty. The number of trials is the number of stations (20), and the probability of success (a faulty station) $p = 0.4$. The mean number of faults in the pumping stations $= np = 20 \times 0.4 = 8$, and Figure 15.8 shows the probabilities of different numbers of faulty stations.

Assuming faults along the pipeline are random, they follow a Poisson distribution. The average number of faults in a kilometre of pipeline is 1/20 = 0.05, so in 100 km of pipeline the mean number of faults $\mu = 100 \times 0.05 = 5$. Figure 15.8 also shows the probabilities of different numbers of faults in the pipeline.

	A	B	C	D	E	F	G
1	**Pipeline faults**						
2							
3	**Pumping stations**				**Pipeline**		
4							
5	**Mean faults =**	**8**			**Mean faults =**	**5**	
6	**Standard deviation =**	**2.19**			**Standard deviation =**	**2.24**	
7							
8	**Faults**	**Probability**	**Cumulative**		**Faults**	**Probability**	**Cumulative**
9	0	0.0000	0.0000		0	0.0067	0.0067
10	1	0.0005	0.0005		1	0.0337	0.0404
11	2	0.0031	0.0036		2	0.0842	0.1246
12	3	0.0123	0.0159		3	0.1404	0.2650
13	4	0.0350	0.0509		4	0.1755	0.4405
14	5	0.0746	0.1256		5	0.1755	0.6159
15	6	0.1244	0.2500		6	0.1462	0.7621
16	7	0.1659	0.4159		7	0.1044	0.8666
17	8	0.1797	0.5956		8	0.0653	0.9319
18	9	0.1597	0.7553		9	0.0363	0.9681
19	10	0.1171	0.8724		10	0.0181	0.9863
20	11	0.0710	0.9434		11	0.0082	0.9945
21	12	0.0355	0.9789		12	0.0034	0.9979
22	13	0.0146	0.9935		13	0.0013	0.9993
23	14	0.0049	0.9984		14	0.0005	0.9997
24	15	0.0013	0.9996		15	0.0002	0.9999
25	16	0.0003	0.9999		16	0.0000	0.9999

Figure 15.8 Probability distributions for Shingatsu Industries oil pipeline

The arithmetic for calculating Poisson probabilities is straightforward, but it can become tedious for large numbers, as you can see in the following problem.

WORKED EXAMPLE 15.13

A motor insurance policy is available only to drivers with a low risk of accidents. One hundred drivers holding the policy in a certain area would expect an average of 0.2 accidents each a year. What is the probability that less than 15 drivers will have accidents in one year?

Solution

This is a binomial process with success as having an accident in the year. The mean number of accidents a year is $np = 100 \times 0.2 = 20$. So the probability that exactly r drivers have accidents in the year is:

$$^{20}C_r \times 0.2^r \times 0.8^{100-r}$$

To find the probability that fewer than 15 drivers have accidents in a year, we add this calculation for all values of r from 0 to 14:

$$P(<15) = \sum_{r=0}^{14} {}^{20}C_r \times 0.2^r \times 0.8^{100-r}$$

Although we can do this calculation (finding that the probability is 0.1056), it is rather messy, and we should look for a Poisson approximation. Unfortunately $np = 20$, which does not meet the requirement that np be less than 5. So we need to look for another approach, and this time we shall use the most common probability distribution of all. When n is large and np is greater than 5 we can approximate the binomial distribution by the Normal distribution.

Review questions

15.11 In what circumstances can you use a Poisson distribution?

15.12 What are the mean and variance of a Poisson distribution?

15.13 The average number of defects per square metre of material is 0.8. Use the tables in Appendix C to find the probability that a square metre has exactly two defects.

15.14 In what circumstances can you use a Poisson distribution to approximate a binomial distribution?

Normal distribution

Both the binomial and Poisson distributions describe discrete data – showing the number of successes. But we often want a probability distribution to describe continuous data – such as the weight of a product. Although these two are similar in principle, there is a key difference – with discrete probabilities you want the probability of, say, exactly three successes, but with continuous data you cannot find the probability that a person weighs exactly 80.456456456 kg. If you make the measurement precise enough, the probability of this happening is always very close to zero. It would be far more useful to know the probability that a person weighs between 80.4 kg and 80.5 kg. This is the approach of continuous probability distributions, which find the probability that a value is within a specified range.

Several common distributions describe continuous data, including the most widely used distribution of all – the Gaussian or **Normal distribution**. This distribution is a bell-shaped curve (illustrated in Figure 15.9) that describes many natural features such as the heights of trees, harvest from a hectare of land, weight of horses, flows in rivers, daily temperature, etc. It also describes many business activities, such as daily receipts, sales volumes, number of customers a week, production in a factory, and so on. The distribution is

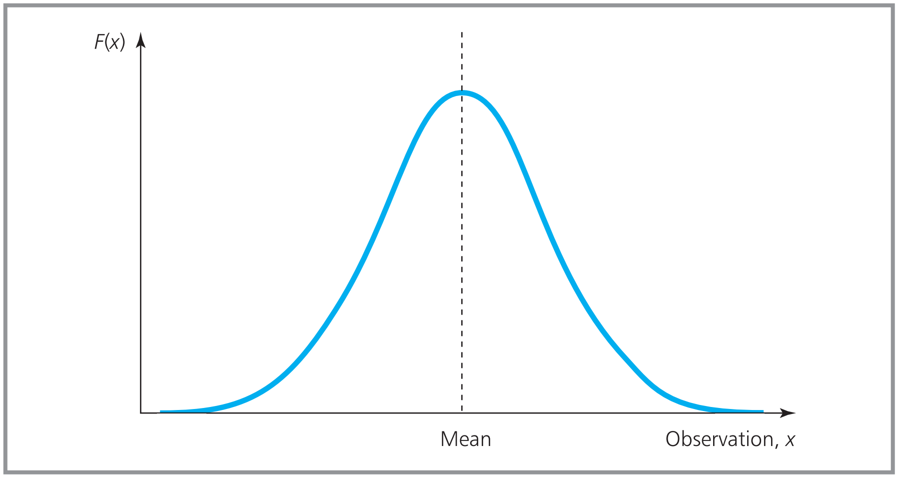


Figure 15.9 Shape of the Normal distribution

so common that a rule of thumb suggests that 'when you have a lot of observations, use the Normal distribution'.

> The Normal distribution has the properties of:
>
> - being continuous
> - being symmetrical about the mean, μ
> - having mean, median and mode all equal
> - having the total area under the curve equal to 1
> - in theory, extending to plus and minus infinity on the *x*-axis.

With continuous data it is the area under the curve that gives the probabilities. The height of the probability distribution does not have much meaning – which is fortunate, as its equation is rather messy.

Suppose a factory makes boxes of chocolates with a mean weight of 1,000 grams. There are small variations in the weight of each box, and if the factory makes a large number of boxes the weights will follow a Normal distribution. Managers in the factory are not interested in the number of boxes that weigh, say, exactly 1,005.0000 g, but they are interested in the number of boxes that weigh more than 1,005 g. This is represented by the area under the right-hand tail of the distribution, as shown in Figure 15.10.

It is difficult to calculate the area in the tail of the distribution, but there are two other ways of finding the probabilities – either using standard software (such as Excel's 'NORMDIST' and 'NORMINV' functions), or looking up the values in tables. We will start by showing how the calculations work with tables.

Normal distribution tables are based on a value, Z, which we find from the mean of a distribution and its standard deviation. To be precise, Z is the number of standard deviations a point is away from the mean. Then Normal tables show the probability of a value greater this. With the boxes of chocolates above, the mean weight is 1,000 g and the standard deviation might be 3 g. The point of interest is 1,005 g, so Z is given by:

$$Z = \text{number of standard deviations from the mean}$$
$$= \frac{\text{value} - \text{mean}}{\text{standard deviation}} = \frac{x - \mu}{\sigma} = \frac{1{,}005 - 1{,}000}{3} = 1.67$$

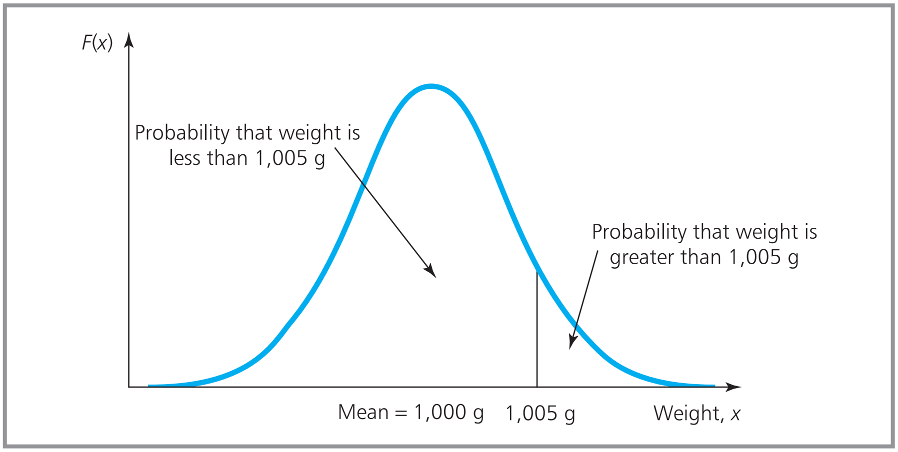


Figure 15.10 Continuous distribution shows the probability by the area under the curve

Appendix D gives a table of areas in the tail of a Normal curve, and if you look up 1.67 you see the value of 0.0475. This is the probability that a box weighs more than 1,005 g. In practice, tables for the Normal distribution have slight differences, so you must be careful when using them.

Because the Normal distribution is symmetrical about its mean, we can do some other calculations. For example, the probability that a box of chocolates weighs less than 995 g is the same as the probability that it weighs more than 1,005 g and we have calculated this as 0.0475 (shown in Figure 15.11). Furthermore,

$$\begin{aligned} P(\text{box is between 995 and 1,005 g}) &= 1 - P(<995\text{ g}) - P(>1{,}005\text{ g}) \\ &= 1 - 0.0475 - 0.0475 = 0.905 \end{aligned}$$

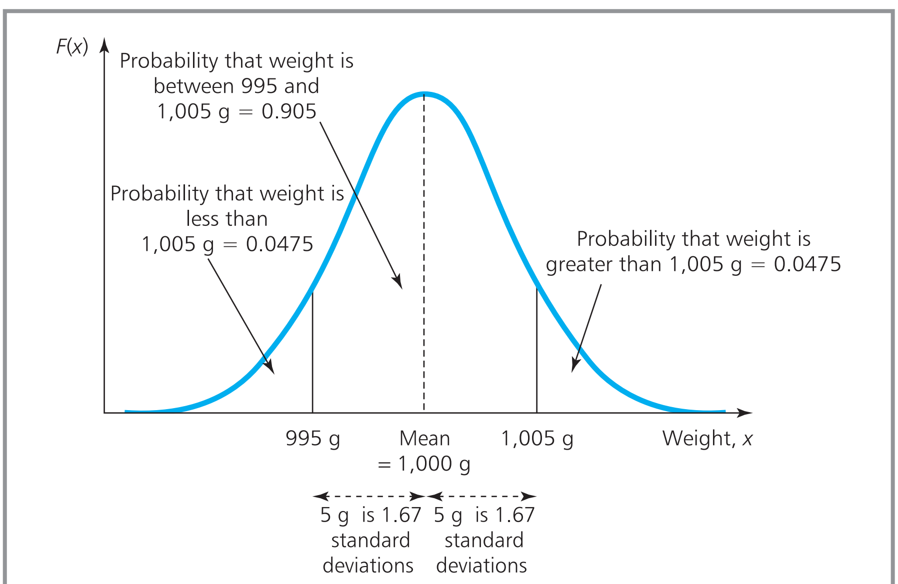


Figure 15.11 The Normal distribution is symmetrical about its mean

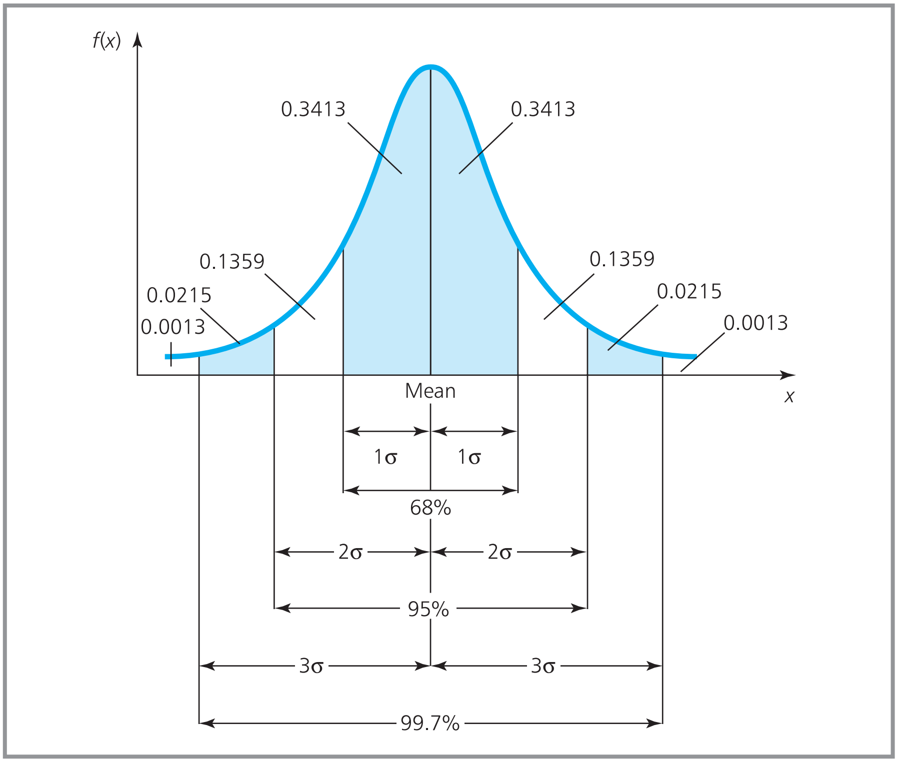
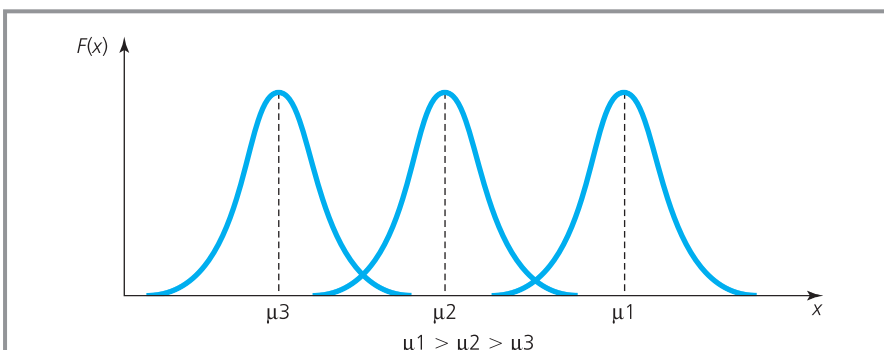


Figure 15.12 Normal probabilities and the standard deviation

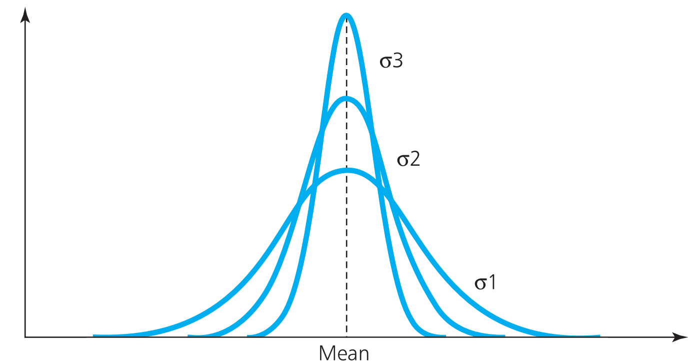
We can find these probabilities because the Normal distribution has the property that about 68% of observations are always within one standard deviation of the mean, 95% are within two standard deviations, and 99.7% are within three standard deviations (shown in Figure 15.12).

This is true regardless of the actual values of the mean and standard deviation. These determine the height and position of the curve (larger standard deviation means there is more spread, and the mean gives the position of the distribution on the x-axis, as shown Figure 15.13), but they do not affect its basic shape.



(a) Normal distributions with same standard deviation but different means

Figure 15.13 The mean and standard deviation affect the shape of the distribution



(b) Normal distributions with same means but different standard deviations

Figure 15.13 (*continued*)

WORKED EXAMPLE 15.14

Figures kept by McClure and Hanover Auctioneer for the past five years show that the weight of cattle brought to market has a mean of 950 kg and a standard deviation of 150 kg. What proportion of the cattle have weights:

(a) more than 1,250 kg
(b) less than 850 kg
(c) between 1,100 kg and 1,250 kg
(d) between 800 kg and 1,300 kg?

Solution

We can assume that a large number of cattle are brought to market, so their weights are Normally distribution with $\mu = 950$ and $\sigma = 150$. For each of the probabilities we have to find Z, the number of standard deviations the point of interest is away from the mean, and use this to look up the associated probability in standard tables.

(a) For a weight greater than 1,250 kg we have:

Z = number of standard deviations from the mean
$= (1{,}250 - 950)/150 = 2.0$

Looking this up in the Normal tables in Appendix D gives a value of 0.0228, which is the probability we want (shown in Figure 15.14).

(b) We can find the probability that the weight is less than 850 kg in the same way:

$Z = (850 - 950)/150 = -0.67$

The table shows only positive values, but as the distribution is symmetrical we can use the value for +0.67, which is 0.2514.

(c) Because the tables show only probabilities under the tail of the distribution, we often have to do some juggling to get the values we want. There is often more than one way of doing the calculations, but they should all give the same results. Here we want the probability that the weight is between 1,100 kg and 1,250 kg (as shown in Figure 15.15). For this we can say that:

P(between 1,100 kg and 1,250 kg)
$= P(>1{,}100 \text{ kg}) - P(>1{,}250 \text{ kg})$

For weight above 1,100 kg: $Z = (1{,}100 - 950)/150 = 1$, probability = 0.1587
For weight above 1,250 kg: $Z = (1{,}250 - 950)/150 = 2$, probability = 0.0228
So the probability that the weight is between these two is $0.1587 - 0.0228 = 0.1359$.

Worked example 15.14 continued

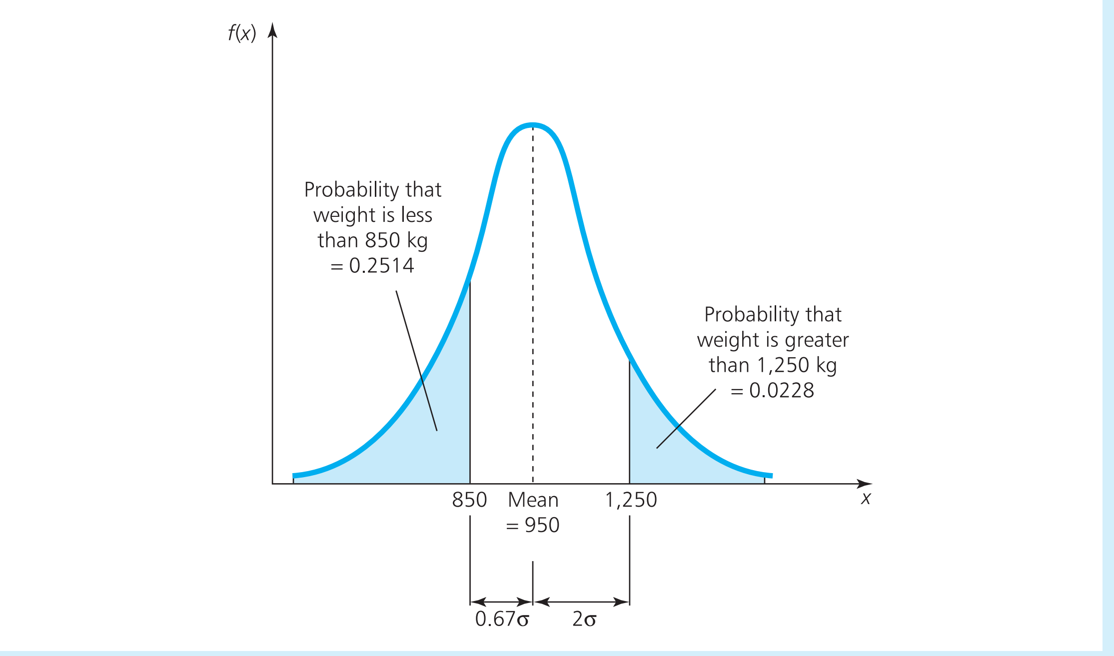


Figure 15.14 Normal probabilities for worked example 15.14

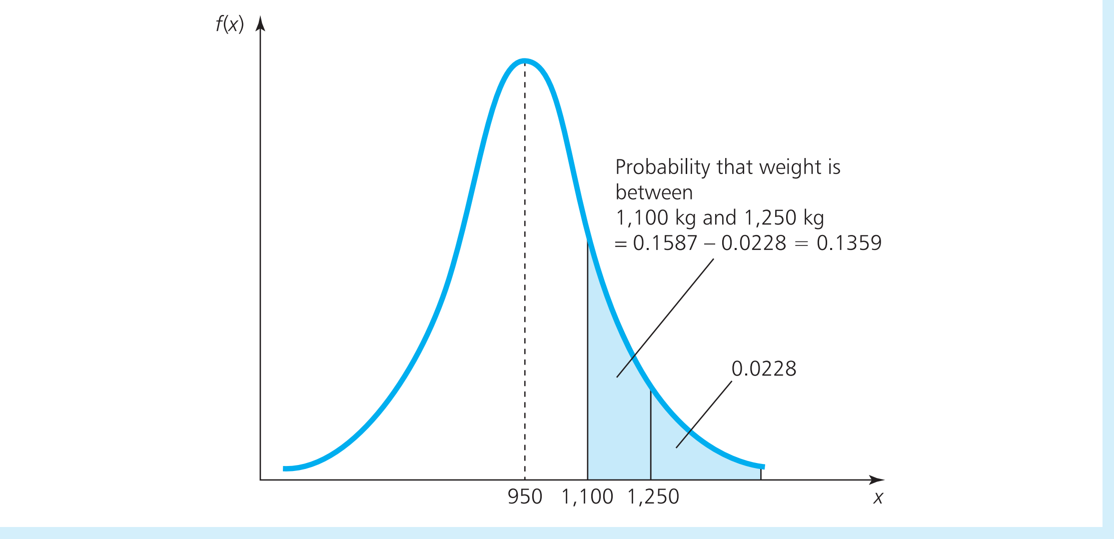


Figure 15.15 Probabilities for part (c) of worked example 15.14

Worked example 15.14 continued

(d) To find the probability that the weight is between 800 kg and 1,300 kg (as shown in Figure 15.16) we can say:

P(between 800 kg and 1,300 kg)
$= 1 - P(<800 \text{ kg}) - P(>1{,}300 \text{ kg})$

For weight below 800 kg: $Z = (800 - 950)/150 = -1$, probability = 0.1587
For weight above 1,300 kg: $Z = (1{,}300 - 950)/150 = 2.33$, probability = 0.0099
So the probability that the weight is between these two is 1 – 0.1587 – 0.0099 = 0.8314.

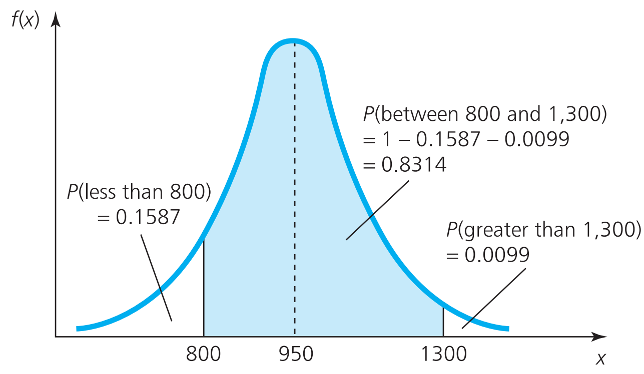


Figure 15.16 Probabilities for part (d) of worked example 15.14

WORKED EXAMPLE 15.15

In worked example 15.13, we tried unsuccessfully to use the Poisson distribution for motor insurance that is sold only to drivers with a low risk of accidents. One hundred drivers with the insurance policy in a certain area expect an average of 0.2 accidents each a year. Use a Normal distribution to find the probability that fewer than 15 drivers will have accidents in a year. How can you take into account the integer number of accidents?

Solution

This is a binomial process with $n = 100$ and $p = 0.2$, so the mean $= np = 100 \times 0.2 = 20$. The standard deviation of a binomial distribution is $\sqrt{npq} = \sqrt{100 \times 0.2 \times 0.8} = \sqrt{16} = 4$. We cannot use a Poisson approximation for the binomial (as np is not less than 5), but we can use a Normal approximation. Then to find the probability of fewer than 15 drivers having an accident:

$$Z = (15 - 20)/4 = -1.25$$

And we use tables to find that this gives a probability of 0.1056.

The only query here is that the number of accidents is discrete, while the Normal distribution assumes that data is continuous. Provided the numbers are reasonably large, the effect of this is small, but to be safe we can add a 'continuity correction'. Here we are looking for the probability of fewer than 15 accidents, but it is clearly impossible to have *between* 14 and 15 accidents. So we can interpret 'less than 15' as 'less than 14.5'. Then

$$Z = (14.5 - 20)/4 = -1.375, \text{ probability} = 0.0846$$

If the question had asked for '15 or less' accidents, we could have interpreted this as 'less than 15.5', with:

$$Z = (15.5 - 20)/4 = -1.125, \text{ probability} = 0.1303$$

WORKED EXAMPLE 15.16

The O'Hare Electric Cable Company finds an average of 30 faults in a week's production. What is the probability of more than 40 faults in a week?

Solution

As there are random faults in cable, we can assume a Poisson process, with:

$\mu = 30 = \text{variance}$

so

$\sigma = \sqrt{30} = 5.477$

Figure 15.17 shows the results from a spreadsheet using the standard function 'POISSON' to find the probability of more than 40 faults. This probability is 0.9677.

When there are many observations, you can reasonably use a Normal distribution to approximate the Poisson process. The easiest way to calculate Normal probabilities is by using the 'NORMSDIST' function in a spreadsheet – where you specify the value of Z and the function returns the cumulative probability. Figure 15.17 also shows the probability of more than 40 faults as:

$Z = (40 - 30)/5.477 = 1.826$, probability $= 0.9661$

As you can see, the two results are close, showing that the Normal distribution gives a good approximation to the Poisson.

	A	B	C	D	E	F	G	H
1	**O'Hare Electric Cable Company**							
2								
3	**Mean**	**30**	**Variance**	**30**	**Standard deviation**		**5.477**	
4								
5	**Poisson calculation**							
6	***r***	**Cumulative probability**						
7	40	0.9677			cell B7 contains POISSON(A8,B3,TRUE)			
8								
9	**Normal calculation**							
10	***r***	***Z***	**Probability**					
11	40	1.826	0.9661		cell C11 contains NORMSDIST(B13)			

Figure 15.17 Probabilities for O'Hare Electric Cable Company

WORKED EXAMPLE 15.17

On average a supermarket sells 500 litres of milk a day with a standard deviation of 50 litres.

(a) If the supermarket has 600 litres in stock at the beginning of a day, what is the probability that it will run out of milk?
(b) What is the probability that demand is between 450 and 600 litres in a day?
(c) How many litres should the supermarket stock if it wants the probability of running out to be 0.05?
(d) How many should it stock if it wants the probability of running out to be 0.01?

Solution

(a) We find the probability of running out of stock with 600 litres (shown in Figure 15.18) from:

$Z = (600 - 500)/50 = 2.0$

which gives probability $= 0.0228$.

(b) The probability of demand greater than 600 litres is 0.0228. The probability of demand less than 450 litres is:

$Z = (450 - 500)/50 = -1.0$

Worked example 15.17 continued

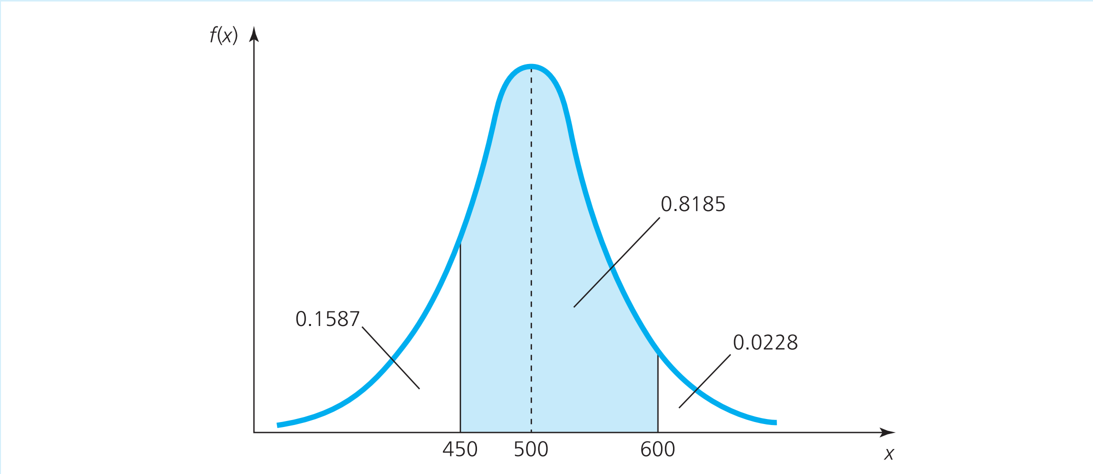


Figure 15.18 Probability that the supermarket runs out of milk

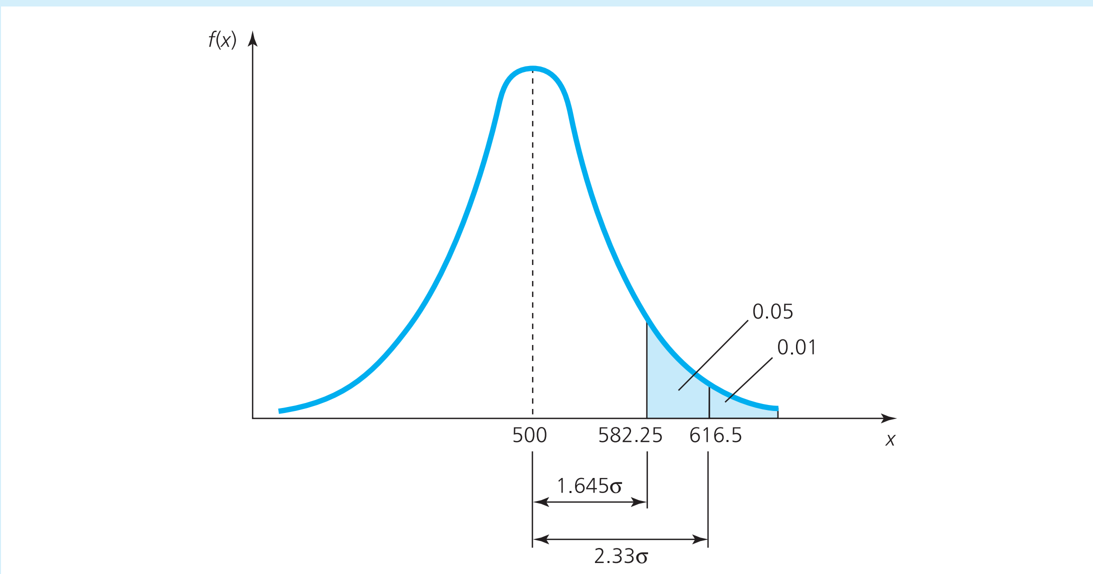


Figure 15.19 Setting the stock level to achieve target probabilities of running out of milk

which gives probability = 0.1587. Then the probability of demand between 450 and 600 litres is:

$$1 - 0.0228 - 0.1587 = 0.8185$$

(c) Here we know the probability and can use this to find the corresponding value of Z, and hence the value that we want. We have a required probability of 0.05, so you look this up in the body of the Normal tables in Appendix D. This is mid-way between values of $Z = 1.64$ and $Z = 1.65$, so we set $Z = 1.645$. Now 1.645 standard deviations is $1.645 \times 50 = 82.25$ litres from the mean. So Figure 15.19 shows that the supermarket needs $500 + 83 = 583$ litres at the beginning of the day (rounding up to make sure the maximum probability of a shortage is 0.05).

Worked example 15.17 continued

(d) A probability of 0.01 corresponds to a value of $Z = 2.33$. So the point of interest is 2.33 standard deviations, or $2.33 \times 50 = 116.5$ litres from the mean. This means that the supermarket needs $500 + 117 = 617$ litres at the beginning of the day.

Of course, we can do all of these calculations on a spreadsheet. Figure 15.20 shows results using Excel's two key functions:

- 'NORMSDIST', where you enter the value of Z and it returns the cumulative probability. Alternatively, you can use 'NORMDIST' where you enter the mean, standard deviation, and point of interest, and it returns the probability of values less than the point of interest.
- 'NORMSINV', where you enter the probability and it returns the corresponding value of Z. Alternatively, you can use 'NORMINV' where you also enter the mean and standard deviation, and it returns the point where there is the given probability of falling below.

	A	B	C	D	E	F	G	H	I
1	**Normal probabilities**								
2									
3	**Dairy example**								
4									
5	**Mean**	**500**	**Standard deviation**		**50**				
6									
7					**Calculation**				
8	(a) Probability demand >600 litres								
9		0.0228			1-NORMDIST(600,B5,E5,1)				
10									
11	(b) Probability demand between 450 and 600								
12		0.8186			NORMDIST(600,B5,E5,1)-NORMDIST(450,B5,E5,1)				
13									
14	(c) Number to give probability <0.05								
15		582.24			NORMINV(0.95,B5,E5)				
16									
17	(c) Number to give probability <0.01								
18		616.32			NORMINV(0.99,B5,E5)				

Figure 15.20 Using standard functions to calculate probabilities for the Normal distribution

WORKED EXAMPLE 15.18

Van Meerson are wholesalers who supply an average of 100,000 litres of white paint a day to retailers, with a standard deviation of 10,000 litres.

(a) If they have 110,000 litres in stock at the start of the day, what is the probability that they will run out?

(b) What is the probability that demand is between 110,000 and 90,000 litres?

(c) What amount are they 90% sure that the daily demand is below?

(d) Within what range are they 90% sure that the demand will fall?

Worked example 15.18 continued

	A	B	C	D	E	F	G	H
1	**Normal probabilities**							
2								
3	**van Meerson example**							
4								
5	**Mean**	**100,000**	**Standard deviation**		**10,000**			
6								
7					**Calculation**			
8	(a) Probability demand >110,000 litres							
9		0.1587			1-NORMDIST(110000,B5,E5,1)			
10								
11	(b) Probability demand between 110,000 and 90,000 litres				NORMDIST(110000,B5,E5,1)			
12		0.6827				-NORMDIST(90000,B5,E5,1)		
13								
14	(c) Number to give probability <0.1							
15		112815.52			NORMINV(0.9,B5,E5)			
16								
17	(d) Range that includes 90% of demands							
18	5% greater than	116448.54			NORMINV(0.95,B5,E5)			
19	5% less than	83551.46			NORMINV(0.05,B5,E5)			

Figure 15.21 Probabilities for demand at the paint wholesaler

Solution

Figure 15.21 shows these calculations in a spreadsheet, using the standard functions NORMDIST and NORMINV.

(a) NORMDIST finds the probability that demand is greater than 110,000 litres – and subtracting this from 1 gives the probability that demand is less than 110,000 litres, which is 0.1587.

(b) Here NORMDIST calculates the probability that demand is greater than 90,000 litres, and you subtract from this the probability that demand is greater than 110,000 litres.

(c) Now we know the probability and want to see where this occurs. NORMINV finds the point where demand has the given probability of falling below – and there is a probability of 0.9 that demand is less than 112,815.52 litres.

(d) Here NORMINV shows that there is a probability of 0.05 that demand is above 116,448.54 litres, and there is a corresponding probability that demand is below 83,551.46 litres. It follows that there is a 90% chance that demand is between these two limits.

Review questions

15.15 When would you use a Normal distribution?

15.16 What is the most obvious difference between a Normal distribution and a binomial or Poisson distribution?

15.17 What are the two most important measures of a Normal distribution?

15.18 When can you use a Normal distribution as an approximation to a binomial distribution?

15.19 If the mean of a set of observations is 100 and the standard deviation is 10, what proportion of observations will be between 90 and 110?

15.20 What is a ‘continuity correction’ for discrete data?

IDEAS IN PRACTICE Decisions with uncertainty

Most management problems involve uncertainty. However, probabilistic models are more complicated, so managers often use simpler deterministic models as approximations. They give results that are reasonably good, but with far less effort.

Consider the break-even analysis described in Chapter 8, where the break-even point is the number of sales at which revenue covers all costs and a product begins to make a profit:

break-even point = fixed costs / (price charged – variable cost)

In principle, managers should introduce a new product when forecast sales are above the break-even point, but they should not introduce a product when forecast demand is less than this. The problem is that sales forecasts are never entirely accurate. Typically a forecast will give an expected mean demand and a measure of possible spread about this. And to be accurate, managers also have to include uncertainty in future costs (both fixed and variable) and prices.

Combining values for possible demands, costs and revenues allows managers to get some idea of the expected returns and corresponding probabilities. Then they can decide whether the returns justify the risks.

Uncertainty makes managers' jobs very difficult. Sometimes actual events work in their favour, sometimes they do not – which is why Napoleon liked to be surrounded by lucky generals.

CHAPTER REVIEW

This chapter introduced the ideas of probability distributions and described three of the most useful.

- A probability distribution is a relative frequency distribution of events. You can draw empirical distributions to describe specific situations, but each problem has its own unique distribution.
- In practice, some standard probability distributions can be used for many different problems, rather than being restricted to a specific problem. The three most widely used distributions are the binomial, Poisson and Normal.
- The binomial distribution depends on some results for sequencing problems, particularly the concepts of combinations and permutations. Then it is used when a trial has two independent, mutually exclusive outcomes. Specifically, it calculates the probability of r successes in n trials as:

$$P(r \text{ successes in } n \text{ trials}) = {}^nC_r p^r q^{n-r}$$

- The Poisson distribution can be used as an approximation to the binomial distribution when the probability of success is small. It is more generally used to describe infrequent, random events.

$$P(r \text{ successes}) = \frac{e^{-\mu} \mu^r}{r!}$$

- Large numbers of observations usually follow a Normal probability distribution. This is a continuous distribution, where the area under the curve shows the probability of observations being within a certain range. These probabilities are always found from standard tables or different kinds of package.

CASE STUDY Machined components

The operations manager was speaking calmly to the marketing manager. 'I said it usually takes 70 days to make a batch of these components. We have to buy parts and materials, make sub-assemblies, set up machines, schedule operations, make sure everything is ready to start production – then actually make the components, check them and shift them to the finished goods stores. Actually making the components involves 187 distinct steps taking a total of 20 days. The whole process usually takes 70 days, but there is a lot of variability. This batch you are shouting about is going to take about 95 days because we were busy working on other jobs, and an important machine broke down so we had to wait for parts to be flown in from Tokyo and that took another five days. It is your fault that you heard my estimate and then assumed I was exaggerating so you promised the customer delivery in 65 days.'

The marketing manager looked worried. 'Why didn't you rush through this important job? Why is there such variation in time? Why did the breakdown of one machine disrupt production by so much? What am I going to say to our customer?'

The operations manager's reply was, 'To answer your questions in order. Because I was rushing through other important jobs. The variation isn't really that much; our estimates are usually within 10 days. It is a central machine that affects the capacity of the whole plant. I can only suggest you apologise and say you will listen to the operations manager more carefully in the future.'

Despite his apparent calmness, the operations manager was concerned about the variability in production times. He could see why there was some variability, but the total amount for the component they were considering did seem a lot. As an experiment he had once tried to match capacity exactly with expected throughput. Then he found that operations near the beginning of the process performed reasonably well, but at the end of the process the variability seemed to have been magnified and the throughput times went out of control. At one point he had eight machines in a line, each of which processed a part for 10 minutes before passing it to the next machine. Although this arrangement seemed perfectly balanced, he found that stocks of work in progress built up dramatically. Some people suggested that this was because the actual processing time could vary between 5 and 15 minutes. Whatever the reason, the experiment was stopped.

Question

- **Operations really need a study to see why there is variability, how much is acceptable, what its effects are, how it can be reduced, what benefits this will bring, and so on. Such a study needs funding – and your job is to write an initial proposal for this funding, including a detailed proposal for a larger study.**

PROBLEMS

15.1 Find the probability distribution of the following set of observations:

10 14 13 15 16 12 14 15 11 13 17 15 16 14 12 13 11 15 15 14 12 16 14 13 13 14 13 12 14 15 16 14 11 14 12 15 14 16 13 14

15.2 Paul La Sauvage forecasts likely profit next year with the following probabilities.

Profit	–€100,000	–€50,000	€0	€50,000	€100,000	€150,000
Probability	0.05	0.15	0.3	0.3	0.15	0.05

What is the probability that the company will make a profit next year? What is the probability that the profit will be at least €100,000?

15.3 Find the values of ^nC_r and ^nP_r when (a) $r = 5$ and $n = 15$, (b) $r = 2$ and $n = 10$, (c) $r = 8$ and $n = 10$.

15.4 An open-plan office has 10 desks. If 10 people work in the area, how many different seating arrangements are there? If two people leave, how many arrangements are there?

15.5 A salesman lists 12 customers to visit each day. In how many different ways can he visit the customers? One day he can visit only eight customers. In how many different ways can he select the eight? As the salesman has to travel between customers, the order in which his visits are scheduled is important. How many different schedules are there for eight customers?

15.6 A binomial process has a probability of success of 0.15. If eight trials are run, what are the mean number of successes and the standard deviation? What is the probability distribution for the number of successes?

15.7 In a town, 60% of families are known to drive European cars. In a sample of 10 families, what is the probability that at least eight drive European cars? In a sample of 1,000 families, what is the probability that at least 800 drive European cars?

15.8 Norfisk Oil is drilling some exploratory wells on the mainland of Norway. The results are described as either a dry well or a producer well. Past experience suggests that 10% of exploratory wells are producer wells. If the company drills 12 wells, what is the probability that all 12 wells are producer wells? What is the probability that all 12 wells are dry wells? What is the probability distribution for the number of dry wells?

15.9 One hundred trials are run for a Poisson process. If the probability of a success is 0.02, what are the mean number of successes and the standard deviation? What is the probability distribution for the number of successes? What is the probability of at least six successes?

15.10 During a typical hour an office receives 13 phone calls. What is the distribution of phone calls in a five-minute period?

15.11 During a busy period at an airport, planes arrive at an average rate of 10 an hour. What is the probability distribution for the number of planes arriving in an hour?

15.12 A machine makes a product, with 5% of units having faults. In a sample of 20 units, what is the probability that at least one is defective? In a sample of 200 units, what is the probability that at least 10 are defective?

15.13 A set of observations follow a Normal distribution with mean 40 and standard deviation 4. What proportions of observations have values: (a) greater than 46, (b) less than 34, (c) between 34 and 46, (d) between 30 and 44, (e) between 43 and 47?

15.14 A large number of observations have a mean of 120 and variance of 100. What proportion of observations is: (a) below 100, (b) above 130, (c) between 100 and 130, (d) between 130 and 140, (e) between 115 and 135?

15.15 The number of meals served in a week at Cath's Café is Normally distributed with a mean of 6,000 and a standard deviation of 600. What is the probability that in a given week the number of meals served is less than 5,000? What is the probability that more than 7,500 meals are served? What is the probability that between 5,500 and 6,500 are served? There is a 90% chance that the number of meals served in a week exceeds what value? There is a 90% chance that the number of meals served will fall within what range?

15.16 A service consists of two parts. The first part takes an average of 10 minutes with a standard deviation of 2 minutes; the second part takes an average of 5 minutes with a standard deviation of 1 minute. Describe how long it takes to complete the service. What is the probability that a customer can be served in less than 12 minutes? What is the probability that service to a customer will take more than 20 minutes?

RESEARCH PROJECTS

15.1 Why are scheduling problems so difficult? Plane, bus and train timetables show expected schedules – so how do you think these are designed? Choose a convenient service and collect data to show how actual arrival times compare with expected times. What can you say about these results?

15.2 The number of people visiting a shop each working hour for the past week has been recorded as follows:

12	23	45	09	16	74	58	21	31	07	26	22	14	24	50
23	30	35	68	47	17	08	54	11	24	33	55	16	57	27
02	97	54	23	61	82	15	34	46	44	37	26	28	21	07
64	38	71	79	18	24	16	10	60	50	55	34	44	42	47

What do these results show? Are they typical of the distribution of customer numbers at other shops? How would you set about collecting and analysing data to get more information about the distribution of customers?

15.3 Sometimes a binomial distribution seems close to a Normal distribution; sometimes a Poisson distribution seems close to a Normal distribution. Examine these three distributions and see how similar they really are.

15.4 The probability distributions described in this chapter are not the only ones available. What other distributions can you find, and when are they used?

Sources of information

Further reading

The statistics books mentioned in the last chapter contain descriptions of probability distributions. Two books specifically on distributions are:

Balakrishnan N. and Nevzorov V., *A Primer of Statistical Distributions*, John Wiley, Chichester, 2003.

Hastings N., Peacock B. and Evans M., *Statistical Distributions* (3rd edition), John Wiley, Chichester, 2000.

CHAPTER 16

Using samples

Contents

Chapter outline

The last two chapters have developed ideas of probabilities and probability distributions. This chapter applies these ideas to sampling. We have already met sampling in Chapter 4, where we looked at ways of collecting data. Essentially, sampling chooses a representative sample from a population, analyses the sample, and uses the results to estimate properties of the population. This is the basis of statistical inference. This chapter asks how reliable statistical inference is, and what sample size is needed.

After finishing this chapter you should be able to:

- Understand how and why to use sampling
- Appreciate the aims of statistical inference
- Use sampling distributions to find point estimates for population means
- Calculate confidence intervals for means and proportions
- Use one-sided distributions
- Use *t*-distributions for small samples.

Purpose of sampling

In Chapter 4 we described different ways of collecting data, which almost invariably involves samples. Here we are going to look again at sampling and see how big a sample we need, and how reliable the results are.

Statistical inference

The aim of sampling is to get reliable data about a population by looking at a few observations, rather than every possible observation. Suppose that an election is approaching, then many people are interested in the number of votes that each party can expect – not least the party campaigners. There are two ways of finding this:

- ask every person eligible to vote what their intentions are (giving a **census**), or
- take a **sample** of eligible people, ask their intentions, and use these results to estimate the voting intentions of the population as a whole.

Often it is impossible to test all of a population. For example, it makes no sense to test every can of beans to make sure that they all taste good. Even when a census is possible, it is unlikely to be completely accurate. In the political opinion poll, it is unlikely that everyone will answer the questions, tell the truth, not make a mistake, and not change their mind before polling day. So a census has the disadvantages of being difficult, time-consuming, expensive, and still not entirely reliable. Then it is difficult to justify the cost of a census, when a much smaller, well-organised sample is easier, faster and cheaper and gives equally reliable results.

So the aim of sampling is to estimate the properties of a population by looking at the properties of a smaller sample. Remember that a population is all the things that we could examine (rather than its more general use for populations of people) and a sample is the smaller number of things that we actually do examine.

Unfortunately, alongside the obvious advantages of a sample, we have to accept that there is an inherent risk. Despite our best endeavours, the results from a sample might not give an accurate representation of the population – and there is inevitably some variation between samples. When your breakfast cereal contains 20% fruit, you do not expect every spoonful to contain exactly this amount – and sometimes your sample contains no fruit at all.

Realistically, the more effort you put into sampling, the more accurate the results. So an obvious question concerns the size of the sample. It should be big enough to give a fair representation of the population, but small enough to be practical and cost-effective. We can do some calculations to find a reasonable sample size, but remember that these always rely on the assumption that the sample is chosen at random. The fact that every member of the population has the same probability of being chosen is important for statistical analyses, and if this basic condition is not met, the analyses are not valid.

In Chapter 4 we discussed the general area of data collection, and here we are focusing on the reliability of samples. For this we concentrate on the process that:

- collects data from a random sample of the population
- uses this data to estimate features of the whole population.

This process is called **statistical inference**.

Review questions

16.1 What is the purpose of sampling?

16.2 What is statistical inference?

IDEAS IN PRACTICE Renewable energy statistics

Most governments are encouraging the development of new and reusable sources of energy – illustrated by the UK government's target of generating 10% of electricity from renewable sources by 2010. But there are so many diverse initiatives, and such a broad range of organisations taking part, that it is difficult to see exactly what is happening. The government clearly needs to collect and analyse information to monitor progress, check the effects of different policies, and compare the UK's performance with European and world standards.

To collect this information the government contracts Future Energy Solutions, which is a part of AEA Technology Environment. They have collected statistics since 1989, and the database – called RESTATS – contains records of all known renewable energy projects in the UK. These include projects for solar energy (both active and passive), onshore and offshore wind power, wave power, large- and small-scale hydroelectricity, geothermal aquifers, and a range of biofuels.

Information for the database is collected from a number of sources.

- Large projects – annual surveys through questionnaires sent to project managers
- Small projects – estimates based on data collected from a sample of projects through:
  - renewable projects survey
  - waste-to-energy questionnaire
- Mail shots to interested organisations
- Telephone follow-up of non-respondents
- Online survey forms
- Estimation where data is not available
- Expert review to identify any gaps in the data and improve collection.

Results of the survey are published annually in the RESTATS database, with information passed to UK Energy Statistics, Eurostat, the International Energy Agency and the World Energy Council.[1–5]

Sources: websites at www.restats.org.uk and www.future-energy-solutions.com.

Sampling distribution of the mean

Probably the most common use of statistical inference is to estimate the mean of some variable in a population by looking at the values in a sample. For example, you might want to estimate the mean weight of boxes of apples delivered to a wholesaler by weighing a sample of boxes, or estimate the average time taken to solve customers' problems by timing a sample of calls to a call centre, or estimate the average cost of a service by looking at a sample of invoices.

The problem is that when we take a series of samples from a population, there is inevitably some variation between samples. Suppose that your boxes of apples have a nominal weight of 10 kg – if you take a sample of 10 boxes you would expect the mean weight to be about 10 kg, but would not be surprised by small variations about this. For instance, samples of 10 boxes taken over consecutive days might have mean weights of 10.2 kg, 9.8 kg, 10.3 kg, 10.1 kg, 9.6 kg, and so on. If you continue taking samples over some period, you can build a distribution of the sample means.

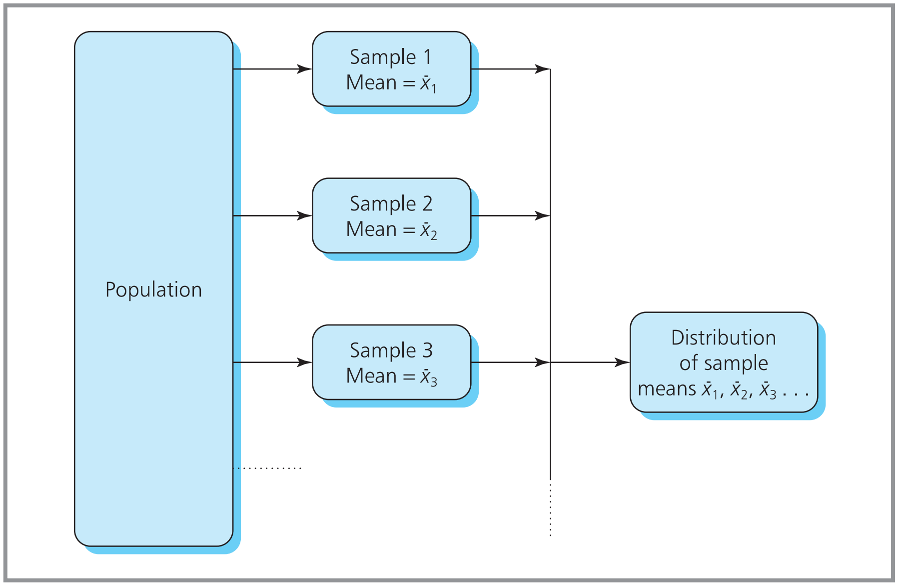


Figure 16.1 Creating the sampling distribution of the mean

Any distribution that is found from samples is called a sampling distribution. When we build a distribution of sample means it is a **sampling distribution of the mean** (illustrated in Figure 16.1).

We want to relate the properties of a sampling distribution of the mean back to the original population. To do this we rely on a result of the **central limit theorem**. This says that if we take large random samples from a population, the sample means are Normally distributed. This is true regardless of the distribution of the original population.

The central limit theorem gives us some other information, but for this we have to remember the standard notation that we mentioned in the last chapter. This has:

- a population of size N, mean μ (the Greek letter mu) and standard deviation σ (the Greek letter sigma);
- a sample of size n, mean $\bar{x}$ and standard deviation s.

Then the central limit theorem says:

- If a population is Normally distributed, the sampling distribution of the mean is also Normally distributed.
- If the sample size is large (say more than 30), the sampling distribution of the mean is Normally distributed regardless of the population distribution.
- The sampling distribution of the mean has a mean μ and standard deviation $\sigma/\sqrt{n}$.

So the sampling distribution of the mean is Normally distributed (provided the sample size is more than 30 or the population is Normally distributed), with the same mean as the population and with a smaller standard deviation (as shown in Figure 16.2). And as the sample size increases, the standard deviation gets smaller, confirming the intuitive result that larger samples have smaller variation and give more reliable results.

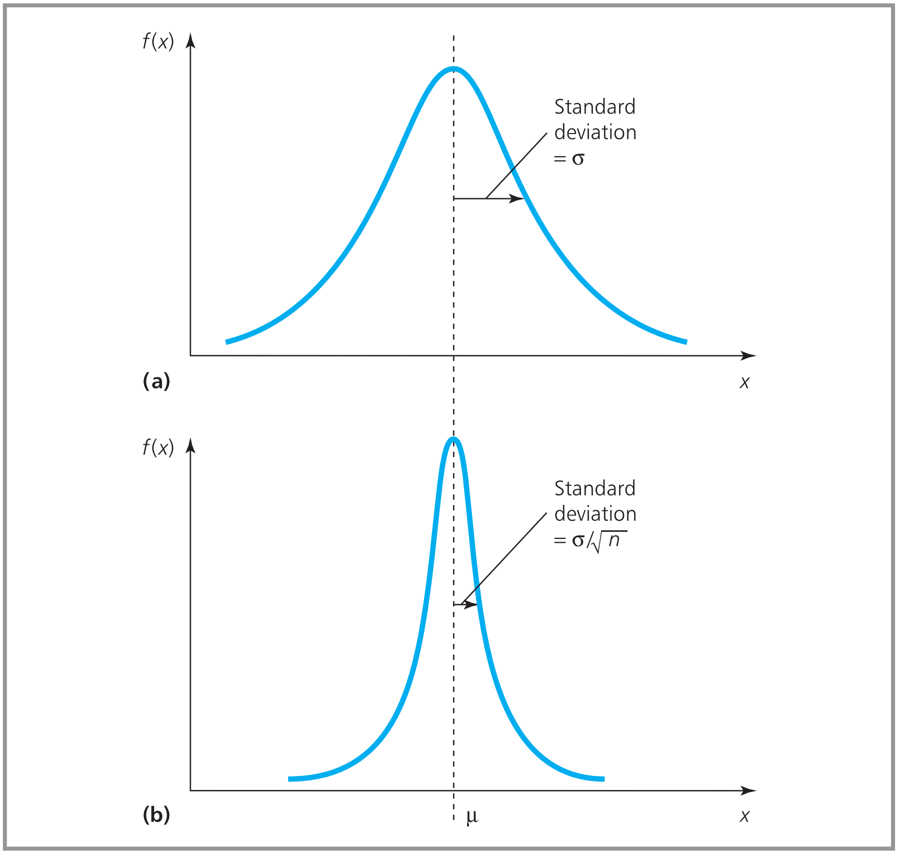


Figure 16.2 Comparisons of the distribution of (a) a population, and (b) a sampling distribution of the mean

WORKED EXAMPLE 16.1

A process makes units with a mean length of 60 cm and standard deviation of 1 cm. What is the probability that a sample of 36 units has a mean length of less than 59.7 cm?

Solution

Imagine what happens when you take a large number of samples of 36 units. You find the mean length of each sample, and the distribution of these means – the sampling distribution of the mean – is:

- Normally distributed
- with a mean length = μ = 60 cm
- and with a standard deviation = $\sigma/\sqrt{n}$ = $1/\sqrt{36}$ = 0.167 cm.

You can find the probability that one sample has a mean length less than 59.7 cm from the area in the tail of this sampling distribution of the mean. To find this area you need Z, the number of standard deviations the point of interest (59.7) is away from the mean (have another look at the last chapter if you are unsure about this):

$$Z = (59.7 - 60)/0.167 = -1.80$$

Looking this up in the Normal tables in Appendix D, or using a computer, shows that this corresponds to a probability of 0.0359. So we expect 3.59% of samples to have a mean length of less than 59.7 cm (as shown in Figure 16.3).

Worked example 16.1 continued

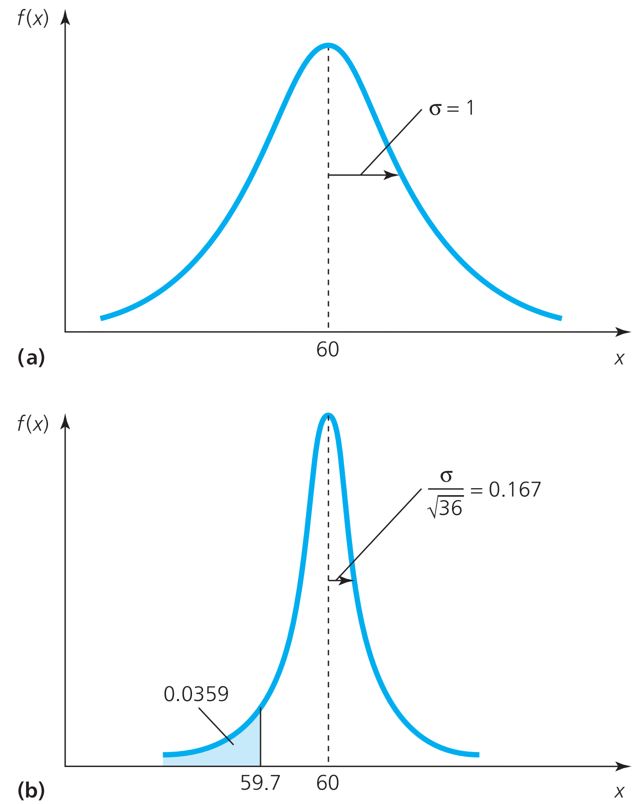


Figure 16.3 (a) Population and (b) sampling distribution of the mean for worked example 16.1

You have probably already noticed an obvious problem with statistical inference, which is the clumsy statements needed to describe, for example, 'the mean of the sampling distribution of the mean'. To make things a bit easier, the standard deviation of the sampling distribution of the mean is usually called the **standard error**. The ideas behind these phrases are fairly straightforward – but you have to be clear about what they describe. Remember that we have a population, with a certain mean. We take samples from this, and each sample has its own mean. The distribution of these sample means is the sampling distribution of the mean – and this in turn has its own mean and standard deviation.

WORKED EXAMPLE 16.2

Soft drinks are put into bottles that hold a nominal 200 ml, but the filling machine introduces a standard deviation of 10 ml. These bottles are packed into cartons of 25 and exported to a market which insists that the mean weight of a carton is at least the quantity specified by the

Worked example 16.2 continued

manufacturer. To make sure this happens, the bottler sets the machine to fill bottles to 205 ml. What is the probability that a carton chosen at random fails the quantity test?

Solution

The mean volume per bottle is 205 ml and has a standard deviation of 10 ml. Taking a random sample of 25 cans gives a sampling distribution of the mean with mean 205 ml and standard deviation of $10/\sqrt{25} = 2$ ml. A case fails the quantity test if the average quantity per can is less than 200 ml. That is:

$$Z = (200 - 205)/2 = -2.5$$

This corresponds to a probability of 0.0062, meaning that 62 cases in 10,000 will still fail the test (as shown in Figure 16.4).

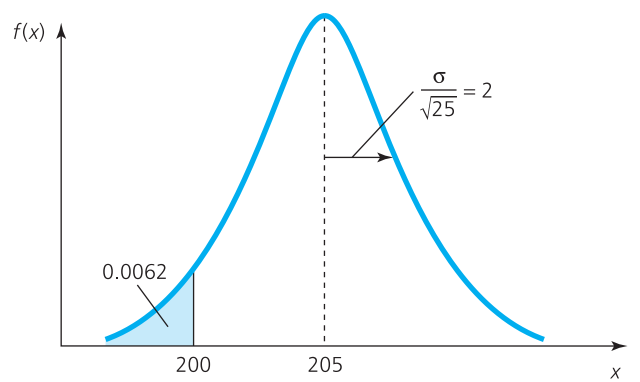


Figure 16.4 Sampling distribution of the mean for bottles (worked example 16.2)

Review questions

16.3 What is the sampling distribution of the mean?

16.4 Describe the shape of the sampling distribution of the mean.

Confidence intervals

The last two worked examples found the features of a sample from the known features of the population. Usually, we work the other way around and estimate the features of a population from a sample.

Suppose we take a sample of 100 units of a product and find that the mean weight is 30 g. How can we estimate the mean weight of the population? The obvious answer is to say that we have chosen the sample carefully and it fairly represents the population – so we estimate the population mean at 30 g. This single value is a **point estimate**.

A point estimate is our best estimate for the population mean, but we know that it comes from a sample and is unlikely to be exactly right. It should be close to the population mean – especially with a big sample – but there is still likely to be some error. A better approach is to define a range that the population mean is likely to be within. This gives an **interval estimate**, and for this we need two measures:

- the limits of the interval
- our confidence that the mean is within the interval.

If we set the interval very wide, we should be very confident that the population mean is within the range – but as the interval gets narrower, our confidence that the mean is still within its limits decreases. In our sample of 100 units with mean weight 30 g we might be 99% confident that the population mean is in the interval 20 to 40 g; we might be 95% confident that the mean is between 25 and 35 g; and we might be 90% confident that the mean is between 27 and 33 g. This kind of range is called a **confidence interval**.

> A 95% (for instance) **confidence interval** defines the range within which we are 95% confident that the population mean lies.

We can calculate the 95% confidence interval using the following argument. The sample mean, $\bar{x}$, is the best point estimate for the population mean, μ. But this point estimate is one observation from the sampling distribution of the mean. This sampling distribution of the mean is Normal, with mean μ and standard deviation $\sigma/\sqrt{n}$. As it is Normal, 95% of observations lie within 1.96 standard deviations of the mean, that is within the range:

$$\mu - 1.96\sigma/\sqrt{n} \quad \text{to} \quad \mu + 1.96\sigma/\sqrt{n}$$

We can phrase this as:

$$P(\mu - 1.96\sigma/\sqrt{n} \le \bar{x} \le \mu + 1.96\sigma/\sqrt{n}) = 0.95$$

And we can rearrange this to give the confidence interval for the population:

$$P(\bar{x} - 1.96\sigma/\sqrt{n} \le \mu \le \bar{x} + 1.96\sigma/\sqrt{n}) = 0.95$$

Repeating this calculation for different confidence intervals for the population mean gives:

> - 90% confidence interval: $\bar{x} - 1.645\sigma/\sqrt{n}$ to $\bar{x} + 1.645\sigma/\sqrt{n}$
> - 95% confidence interval: $\bar{x} - 1.96\sigma/\sqrt{n}$ to $\bar{x} + 1.96\sigma/\sqrt{n}$
> - 99% confidence interval: $\bar{x} - 2.58\sigma/\sqrt{n}$ to $\bar{x} + 2.58\sigma/\sqrt{n}$

WORKED EXAMPLE 16.3

A machine produces parts that have a standard deviation in length of 1.4 cm. A random sample of 100 parts has a mean length of 80 cm. What is the 95% confidence interval for the mean length of all parts?

Solution

The sample of 100 parts has a mean length of 80 cm, so the point estimate for the population mean is 80 cm.

The sampling distribution of the mean has a mean of 80 cm and standard deviation of $\sigma/\sqrt{n} = 1.4/\sqrt{100} = 0.14$ cm. Ninety-five per cent of observations are within 1.96 standard deviations of the mean, so the 95% confidence interval (as shown in Figure 16.5) is:

$$\bar{x} - 1.96\sigma/\sqrt{n} \quad \text{to} \quad \bar{x} + 1.96\sigma/\sqrt{n}$$
$$80 - 1.96 \times 0.14 \quad \text{to} \quad 80 + 1.96 \times 0.14$$
$$79.73 \text{ cm} \quad \text{to} \quad 80.27 \text{ cm}$$

Worked example 16.3 continued

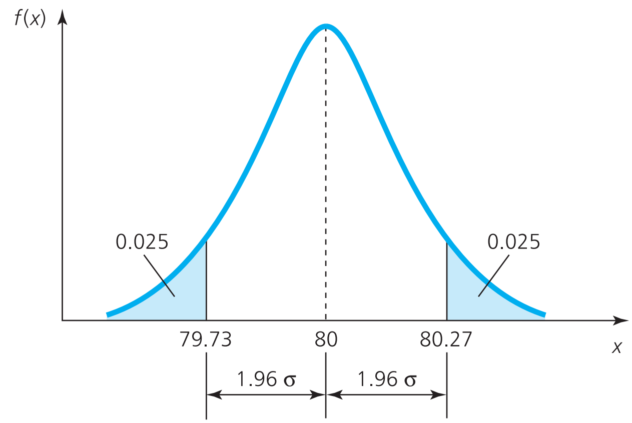


Figure 16.5 Confidence interval for worked example 16.3

In this last example we estimated the population mean from a sample mean – but assumed that we knew the standard deviation of the population. In reality, it is unlikely that we would know the standard deviation of a population, but not its mean. It is much more likely that we have information only from a sample and use this to estimate both the population mean and standard deviation.

The obvious estimate of the population standard deviation is the sample standard deviation, s. Then the 95% confidence interval becomes:

$$\bar{x} - 1.96s/\sqrt{n} \quad \text{to} \quad \bar{x} + 1.96s/\sqrt{n}$$

WORKED EXAMPLE 16.4

Homelock Security employs night watchmen to patrol warehouses and they want to find the average time needed to patrol warehouses of a certain size. On a typical night they recorded the time to patrol 40 similar warehouses. These showed a mean time of 76.4 minutes, with a standard deviation of 17.2 minutes. What are the 95% and 99% confidence intervals for the population mean?

Solution

The point estimate for the population mean is 76.4 minutes. The standard deviation of the sample is 17.2 minutes, and using this as an approximation for the standard deviation of the population gives:

- 95% confidence interval:

  $\bar{x} - 1.96s/\sqrt{n}$ to $\bar{x} + 1.96s/\sqrt{n}$

  $76.4 - 1.96 \times 17.2/\sqrt{40}$ to
  $76.4 + 1.96 \times 17.2/\sqrt{40}$

  71.07 to 81.73

  meaning that we are 95% confident that the population mean is between 71.07 minutes and 81.73 minutes.

- 99% confidence interval:

  $\bar{x} - 2.58s/\sqrt{n}$ to $\bar{x} + 2.58s/\sqrt{n}$

  $76.4 - 2.58 \times 17.2/\sqrt{40}$ to
  $76.4 + 2.58 \times 17.2/\sqrt{40}$

  69.38 to 83.42

  meaning that we are 99% confident that the population mean is between 69.38 minutes and 83.42 minutes.

WORKED EXAMPLE 16.5

A company wants to find the average value of its customer accounts. An initial sample suggests that the standard deviation of the value is £60. What sample size would give a 95% confidence interval for the population mean that is (a) £25 wide, (b) £20 wide, (c) £15 wide?

Solution

The standard deviation of the initial sample is £60, so we can use this as an approximation for the standard deviation of the population, giving a standard error of $60/\sqrt{n}$.

(a) A 95% confidence interval is:

$$\text{mean} - 1.96 \times 60/\sqrt{n} \quad \text{to}$$
$$\text{mean} + 1.96 \times 60/\sqrt{n}$$

giving a range of $2 \times 1.96 \times 60/\sqrt{n}$. We want this range to be £25 wide, so:

$$2 \times 1.96 \times 60/\sqrt{n} = 25 \quad \text{or} \quad \sqrt{n} = 9.41$$
$$\text{or} \quad n = 88.5$$

In other words, a sample size of 88.5 (rounded to 89) gives a confidence interval for the population mean that is £25 wide.

(b) Repeating this calculation with a confidence interval of £20 has:

$$2 \times 1.96 \times 60/\sqrt{n} = 20 \quad \text{or} \quad \sqrt{n} = 11.76$$
$$\text{or} \quad n = 138.3$$

(c) Again repeating the calculation with a confidence interval of £15 has:

$$2 \times 1.96 \times 60/\sqrt{n} = 15 \quad \text{or} \quad \sqrt{n} = 15.68$$
$$\text{or} \quad n = 245.9$$

As expected, larger samples give narrower confidence intervals. But notice that decreasing the range from £25 to £20 increased the sample size by 138.3 – 88.5 = 49.8, while decreasing the range from £20 to £15 increased the sample size by 245.9 – 138.3 = 107.6. There are clearly diminishing returns with increasing sample size. As the standard deviation of the sampling distribution is proportional to $1/\sqrt{n}$, reducing the range to a half would need a sample four times as large; reducing the range to a third would need a sample nine times as large, and so on.

Correcting the standard deviation

It is safe to use a sample standard deviation as an approximation to the population standard deviation when the sample size is large – say more than about 30. But with smaller samples this tends to underestimate the population standard deviation. We can compensate for this bias with a small adjustment, using $\sigma = s/\sqrt{(n-1)}$ rather than $\sigma = s/\sqrt{n}$. This is Bessel's correction, and although it seems rather arbitrary, there is a sound theoretical reason for using it.

WORKED EXAMPLE 16.6

MLP Mail-order collects a random sample of 40 customer orders, as shown in the following table. What is the 95% confidence interval for the population mean?

Size of order	Number of customers
€0–€100	4
€100–€200	8
€200–€300	14
€300–€400	8
€400–€500	4
€500–€600	2

Worked example 16.6 continued

Solution

Remember (from Chapter 6) that for grouped data the mean and standard deviation are:

$$\bar{x} = \frac{\sum fx}{\sum f} \quad s = \sqrt{\frac{\sum f(x - \bar{x})^2}{\sum f}}$$

where x is the midpoint of each range and f is the number of observations in each range. Doing these calculations, you find that:

$\bar{x} = 265$ and $s = 127.57$

These are the best point estimates for the population mean and standard deviation. Then the 95% confidence interval is 1.96 standard deviations from the mean, and using Bessel's correction we get the range:

$\bar{x} - 1.96s/\sqrt{(n-1)}$ to $\bar{x} + 1.96s/\sqrt{(n-1)}$

$265 - 1.96 \times 127.57/\sqrt{39}$ to
$265 + 1.96 \times 127/57/\sqrt{39}$

224.96 to 305.04

This range is wide because of the large variance of the data and the relatively small sample size.

You can see in this example that Bessel's correction increased the estimated population standard deviation by about 1%, making very little difference to the calculation. This is usually true, and the multiplier makes a difference only when the sample size is very small.

Estimating population proportions

Sometimes, instead of estimating the value of some variable in a population, we want to estimate the proportion of the population that share some characteristic. This is typical of a survey, which might show that that '25% of respondents believe this', or quality control that finds the proportion of output that is faulty, or financial analysts who find the proportion of invoices smaller than some amount, or personal records that show the proportion of people who work overtime. Then statistical inference takes a sample, finds the proportion of the sample with the required property, and then estimates the proportion of the population with that property.

Suppose the proportion of a population with a certain property is π (the Greek letter pi) and a sample is taken that contains a proportion p with the same property. Another result of the central limit theorem is that for large sample sizes (say over 30) the sample proportions are:

- Normally distributed
- with mean π
- and standard deviation $\sqrt{(\pi(1-\pi)/n)}$.

We have a sample with a proportion, p, and this is the best point estimate for the population proportion, π. But this point estimate is one observation from the sampling distribution. Using exactly the same reasoning as before, we can find a 95% confidence interval as:

$$p - 1.96 \times \sqrt{(\pi(1-\pi)/n)} \quad \text{to} \quad p + 1.96 \times \sqrt{(\pi(1-\pi)/n)}$$

Unfortunately, this range contains the term π, which is the proportion we are trying to find. As before, though, we can use the sample value, p, to estimate π. Then:

The 95% confidence interval for a population proportion is:

$$p - 1.96 \times \sqrt{(p(1-p)/n)} \quad \text{to} \quad p + 1.96 \times \sqrt{(p(1-p)/n)}$$

WORKED EXAMPLE 16.7

Queen Charlotte's Hospital gives a random sample of 50 patients a new treatment for an illness. Sixty per cent of these are cured. Find the 95% confidence interval for the proportion of all patients who will be cured by the treatment.

Solution

The proportion of patients in the sample who are cured, p, is 0.6. This is the point estimate for the proportion who will be cured in the population, π.

The 95% confidence interval for the proportion in the population is:

$$p - 1.96 \times \sqrt{(p(1-p)/n)} \quad \text{to} \quad p + 1.96 \times \sqrt{(p(1-p)/n)}$$

$$0.6 - 1.96 \times \sqrt{(0.6 \times 0.4/50)} \quad \text{to} \quad 0.6 + 1.96 \times \sqrt{(0.6 \times 0.4/50)}$$

$$0.6 - 0.136 \quad \text{to} \quad 0.6 + 0.136$$

$$0.464 \quad \text{to} \quad 0.736$$

We are 95% confident that between 46.4% and 73.6% of patients given the new treatment will be cured. This seems a wide range, but we are dealing with small samples – and real medical trials are conducted on thousands of patients.

WORKED EXAMPLE 16.8

Last month an opinion poll in Helmsburg suggested that 30% of people would vote for the Green Party. This month the poll is being rerun. How many people must be interviewed for the poll to be within 2% of actual voting intentions with a 95% level of confidence?

Solution

The best point estimate for the proportion of people who will vote for the Green Party is $p = 0.3$, found in last month's poll. With the next poll of size n, the 95% confidence interval for the proportion of people voting for the Green Party is:

$$p - 1.96 \times \sqrt{(p(1-p)/n)} \quad \text{to} \quad p + 1.96 \times \sqrt{(p(1-p)/n)}$$

$$0.3 - 1.96 \times \sqrt{(0.3 \times 0.7/n)} \quad \text{to} \quad 0.3 + 1.96 \times \sqrt{(0.3 \times 0.7/n)}$$

But we want the result to be within 2% of the mean, so that:

$$0.02 = 1.96 \times \sqrt{(0.3 \times 0.7/n)}$$

or

$$n = 2{,}017$$

The poll needs a sample of 2,017 people to get the desired accuracy.

IDEAS IN PRACTICE Opinion polls

Many organisations – such as MORI, ICM, NOP and Gallup – routinely collect large amounts of information in opinion polls. You often see their results just before an election – but then you probably see only the headline result, which is the point estimate for the population. Occasionally you will see a warning along the lines of 'this result is within 2% nineteen times out of twenty'.

An opinion poll of 2,127 people in 2006 suggested the following support for political parties. What does this mean for, say, the Liberal Democrat Party?

Labour	35%	Green Party	3%
Conservatives	31%	BNP	2%
Liberal Democrats	22%	UKIP	0%
Scottish Nationalists	6%	Others	1%

Ideas in practice continued

The point estimate for the proportion of people supporting the Liberal Democrats is 22%. Then $p = 0.22$, $(1 - p) = 0.78$ and $n = 2{,}127$, and we can substitute these values to find the 95% confidence interval of:

$p - 1.96 \times \sqrt{(p(1 - p)/n)}$ to
$p + 1.96 \times \sqrt{(p(1 - p)/n)}$

$0.22 - 1.96 \times \sqrt{(0.22 \times 0.78/2{,}179)}$ to
$0.22 + 1.96 \times \sqrt{(0.22 \times 0.78/2{,}179)}$

0.2026 to 0.2374

In other words, the Liberal Democrats can be 95% confident that their support is between 20.26% and 23.74% of the electorate.

Review questions

16.5 Why is a point estimate for the population mean unlikely to be exactly right?

16.6 What is the 95% confidence interval for a value?

16.7 Is a 95% confidence interval wider or narrower than a 90% interval?

16.8 If a sample of size n produces a confidence interval that is w wide, how big a sample would you need to give a confidence interval that is $w/5$ wide?

16.9 When would you use Bessel's correction?

One-sided confidence intervals

So far we have used a confidence interval that is symmetrical about the mean, and assumed that we use both sides of the sampling distribution. Then we are 95% confident that the true value is within a certain range, and there is a 2.5% chance that the true value is above the top of this range, and 2.5% chance that the true value is below the bottom of it. Often we are interested in only one side of the sampling distribution. For example, we might want to be 95% confident that the mean number of defects is below some maximum, or the weight of goods is above some minimum, or the cost is below some maximum. Then we are interested in only one tail of the distribution (shown in Figure 16.6).

To find a one-sided confidence interval we use the same general approach as for the two-sided interval. But while a two-sided 95% confidence interval is 1.96 standard deviations away from the mean (with 2.5% of the distribution in each tail), a one-sided 95% confidence interval is 1.645 standard deviations from the mean (with 5% of the distribution in one tail). Then we use the following rules for finding the one-sided 95% confidence interval:

- To find the value that we are 95% confident the population mean is above, use:

  $\bar{x} - 1.645$ standard errors

- To find the value that we are 95% confident the population mean is below, use:

  $\bar{x} + 1.645$ standard errors

Of course, we can use other levels of confidence, but 95% is convenient and generally the most common.

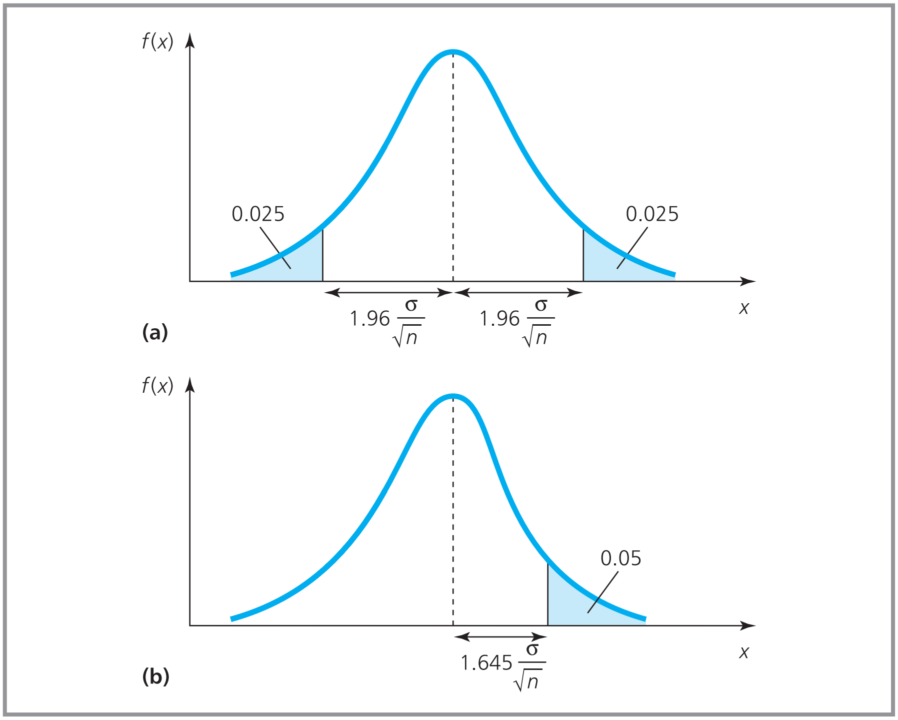


Figure 16.6 Comparison of (a) two-sided and (b) one-sided 95% confidence interval

WORKED EXAMPLE 16.9

Yamatsumo Electric use an automated process that introduces some variability into the weight of each unit. One day a sample of 60 units has a mean weight of 45 kg and standard deviation of 5 kg. What weight are Yamatsumo 95% confident the population mean is below? What weight are they 95% confident the population mean is above?

Solution

The best estimate of the population mean is 45 kg, and the best estimate of the standard error is $s/\sqrt{(n-1)} = 5/\sqrt{59} = 0.65$. So Yamatsumo are 95% confident that the population mean is less than:

$\bar{x} + 1.645 \times$ standard error or
$45 + 1.645 \times 0.65 = 46.07$ kg

And they are 95% confident that the population mean is more than:

$\bar{x} - 1.645 \times$ standard error or
$45 - 1.645 \times 0.65 = 43.93$ kg

WORKED EXAMPLE 16.10

A sample of 40 accounts at PrixMin shows that customers owe an average of €842 with a standard deviation of €137. PrixMin can be 95% confident that customers owe an average of less than what amount? What amount can they be 99% confident that customers owe less than?

Solution

The best estimate of the standard error is:

$s/\sqrt{(n-1)} = 137/\sqrt{39} = 21.94$

Worked example 16.10 continued

The point estimate for customer debt is €842. PrixMin can be 95% confident that the average debt is below:

$\bar{x} + 1.645 \times$ standard error or
$842 + 1.645 \times 21.94 = €878$

They can be 99% confident that the average debt is 2.33 standard errors from the mean (found from tables):

$\bar{x} + 2.33 \times$ standard error or
$842 + 2.33 \times 21.94 = €893$

WORKED EXAMPLE 16.11

A quality assurance programme takes a random sample of 40 invoices and finds that eight have mistakes. What proportion of mistakes is the company 95% sure that the population is below? What proportion of mistakes is the company 95% confident the population is above? What is the 95% two-sided confidence interval?

Solution

The proportion of defects in the sample, p, is 8/40 = 0.2. And we know that the best estimate for the standard error of a proportion is $\sqrt{(p(1 - p)/n)} = \sqrt{(0.2 \times 0.8/40)} = 0.063$. So the company is 95% confident that the proportion of mistakes in the population is less than:

$\bar{x} + 1.645 \times$ standard error or
$0.2 + 1.645 \times 0.063 = 0.304$

Similarly, the company is 95% confident that the population mean is more than:

$\bar{x} - 1.645 \times$ standard error or
$0.2 - 1.645 \times 0.063 = 0.096$

The two-sided 95% confidence limits are 1.96 standard errors from the mean, giving an interval of:

$\bar{x} - 1.96 \times 0.063$	to	$\bar{x} + 1.96 \times 0.063$
$0.2 - 1.96 \times 0.063$	to	$0.2 + 1.96 \times 0.063$
0.077	to	0.323

Review questions

16.10 When would you use a one-sided confidence interval?

16.11 Put the following in order of nearest the mean: a one-sided 95% confidence interval, a one-sided 99% confidence interval, and a two-sided 95% confidence interval.

Using small samples

Much of statistical inference is based on the central limit theorem – but this works only when a population is Normally distributed, or with a large sample. What happens when these conditions are not met? Suppose that you do not know the population distribution, and can take only a small sample (where 'small' is below about 30). Then you cannot assume the sampling distribution is Normal.

The problem is that small samples are always less representative of the population than large samples – and in particular, small samples include fewer outlying results and show less variation than the population. Once we recognise this pattern, we can allow for it by using a different probability distribution – the ***t*-distribution**, which is often called the **Student-*t* distribution**.

A t-distribution looks similar to the Normal, but its shape depends on the **degrees of freedom**. For our purpose, when we take a sample of size n, the degrees of freedom are simply defined as $n - 1$. This result comes from the definition, which says that the degrees of freedom are the number of independent pieces of information used. You might ask why a sample of size n has $n - 1$ pieces of information rather than n. The answer is that we fix a value for the mean, so only $n - 1$ values can vary. Suppose you have four numbers whose mean is 5; the first three numbers can take any value (3, 5 and 7 perhaps) but then the fourth number is fixed (at 5) to get the correct mean.

When the sample size is close to 30, the t-distribution is virtually the same as a Normal distribution. But as the degrees of freedom get smaller – meaning the sample size gets smaller – the distribution gets wider and lower, as shown in Figure 16.7.

You use t-distributions in the same way as Normal distributions, and can find values from either tables (shown in Appendix E) or statistical packages. With the Excel function 'TINV' you enter the probability in the tails of the distribution and the degrees of freedom, and it returns the number of standard deviations the point of interest is away from the mean (that is, the equivalent of Z). There are slight differences in the tables, so you have to be careful. In Appendix E the figures show the probability in each tail. A two-sided 95% confidence interval means that the probability of being in each tail is 0.025, so you look at this column and see that with one degree of freedom this is 12.706 standard errors away from the mean; with two degrees of freedom it is 4.303, with three degrees of freedom, 3.182, and so on. As the number of degrees of freedom gets higher, the t-distribution gets closer to the Normal, and the number of standard deviations gets closer to the Normal result of 1.96.

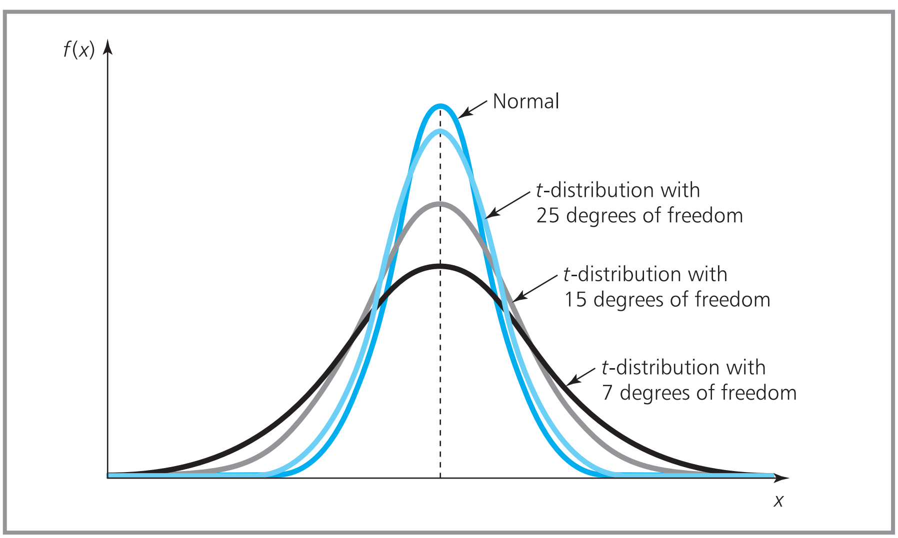


Figure 16.7 Comparison of Normal and t-distributions

WORKED EXAMPLE 16.12

A survey of 10 entries in a sales ledger has a mean value of £60 and a standard deviation of £8. What is the 95% confidence interval for the population of entries?

Solution

The point estimate for the population mean is £60. As the sample size is only 10, we cannot assume a Normal distribution, and must use a *t*-distribution with $10 - 1 = 9$ degrees of freedom.

A 95% confidence interval has a probability of 0.025 in each tail, and looking this up in Appendix E shows that with 9 degrees of freedom this corresponds to 2.262 standard errors. So the confidence limits are 2.262 standard errors from the mean at:

$\bar{x} - 2.262 \times s/\sqrt{(n-1)}$ to $\bar{x} + 2.262 \times s/\sqrt{(n-1)}$

$60 - 2.262 \times 8/\sqrt{9}$ to $60 + 2.262 \times 8/\sqrt{9}$

53.97 to 66.03

We are 95% confident that the population mean is within the range £54 to £66.

WORKED EXAMPLE 16.13

The time taken for eight people working in an office to travel to work has a mean of 37 minutes and a standard deviation of 12 minutes.

(a) What is the 90% confidence interval for the mean travel time of everyone in the office?
(b) What is the 95% confidence interval?
(c) What is the 95% confidence interval with a sample size of 20?
(d) What would the last result be using a Normal distribution?

Solution

(a) The sample size is 8, so there are 7 degrees of freedom. A 90% confidence interval has a probability of 0.05 in each tail, and looking up this value in Appendix E shows that it corresponds to 1.895 standard errors. So the 90% confidence limits are 1.895 standard errors from the mean at:

$\bar{x} - 1.895 \times s/\sqrt{(n-1)}$ to
$\bar{x} + 1.895 \times s/\sqrt{(n-1)}$

$37 - 1.895 \times 12/\sqrt{7}$ to
$37 + 1.895 \times 12/\sqrt{7}$

28.40 to 45.60

(b) For the 95% confidence interval we look up a probability of 0.025 with 7 degrees of freedom and get a value of 2.365. Then the 95% confidence interval is:

$37 - 2.365 \times 12/\sqrt{7}$ to $37 + 2.365 \times 12/\sqrt{7}$

26.27 to 47.73

(c) With a sample of 20 the standard error becomes $12/\sqrt{19}$ and there are 19 degrees of freedom. Then the 95% confidence interval is within 2.093 standard errors of the mean:

$37 - 2.093 \times 12/\sqrt{19}$ to
$37 + 2.093 \times 12/\sqrt{19}$

31.24 to 42.76

(d) 95% confidence limits with a Normal distribution are 1.96 standard errors from the mean, so the interval is:

$37 - 1.96 \times 12/\sqrt{19}$ to $37 + 1.96 \times 12/\sqrt{19}$

31.61 to 42.39

The small sample has not allowed for the full variability of the data, so the Normal distribution has assumed the data is less spread out than it actually is. The confidence interval tends to be too narrow, but you can see that the differences are often small. Figure 16.8 shows all of these calculations in a spreadsheet.

Worked example 16.13 continued

	A	B	C	D	E	F	G
1	**Student-t distribution**						
2							
3	**Sample size**			**8**			
4	**Mean**			**37**			
5	**Standard deviation**			**12**			
6							
7	**Part (a)**						
8	Degrees of freedom			7	D3 – 1		
9	Standard error			4.536	D5/SQRT(D3-1)		
10							
11	Confidence interval			90			
12	Number of standard deviations			1.895	TINV((100-D11/100,D8)		
13	Confidence interval		From	28.407	D4 – (D12*D9)		
14			To	45.593001	D4 + (D12*D9)		
15							
16	**Part (b)**						
17	Confidence interval			95			
18	Number of standard deviations			2.365	TINV((100-D17/100,D8)		
19	Confidence interval		From	26.275	D4 – (D18*D9)		
20			To	47.725	D4 + (D18*D9)		
21							
22	**Part (c)**						
23	Sample size			20			
24	Degrees of freedom			19			
25	Standard error			2.753	D5/SQRT(D23-1)		
26	Confidence interval			95			
27	Number of standard deviations			2.093	TINV((100-D26/100,D24)		
28	Confidence interval		From	31.238	D4 – (D27*D25)		
29			To	42.762	D4 + (D27*D25)		
30							
31	**Part (d)**						
32	Normal distribution						
33	Sample size			20			
34	Standard error			2.753	D5/SQRT(D33-1)		
35	Confidence interval			95			
36	Number of standard deviations			1.960	NORMSINV((100-D35/200)		
37	Confidence interval		From	31.604	D4 – (D36*D34)		
38			To	42.396	D4 + (D36*D34)		

Figure 16.8 Spreadsheet of calculations for worked example 16.13

Review questions

16.12 Why are sampling distributions not Normal when samples are small?

16.13 What are the 'degrees of freedom'?

CHAPTER REVIEW

This chapter discussed sampling and the reliability of samples

- Data collection usually needs sampling, which is easier, cheaper, faster and often as reliable as a census. Sampling collects data from a representative sample of the population, and uses this to estimate features for the population as a whole. This is the basis of statistical inference.
- When you take samples from a population, values of the mean (say) follow a sampling distribution of the mean. When the sample size is large, or the population is Normally distributed, the sampling distribution of the mean is Normally distributed, with mean μ and standard deviation $\sigma/\sqrt{n}$. For small samples you should use Bessel's correction to the standard error.
- A sample mean gives a point estimate for the population mean, and a sample standard deviation gives a point estimate for the population standard deviation.
- Confidence intervals can be more useful, as they define the range in which you have a specified level of confidence within which the population mean lies. The confidence interval is:

  $$\bar{x} - Zs/\sqrt{(n-1)} \quad \text{to} \quad \bar{x} + Zs/\sqrt{(n-1)}$$

- You can adjust this approach to find the confidence interval for the proportion of the population sharing some feature.
- Sometimes you are interested in only one tail of a distribution, and you can again adjust the standard approach to find a one-sided confidence interval.
- Small samples tend to underestimate the variability in a population, and you can allow for this by using a t-distribution. This is similar to the Normal distribution, but its shape is affected by the degrees of freedom, and hence the sample size.

CASE STUDY Kings Fruit Farm

In the 1920s Edward Filbert became the tenant of Kings Farm in Cambridgeshire. In 1978 his grandson James Filbert became the latest manager. But in the intervening years the farm has changed considerably. It has grown from 195 acres to over 3,000 acres and is owned by an agricultural company that owns several other farms in the area. Kings Farm grows a variety of vegetables, cereals and fruit, with Kings Fruit Farm as a subsidiary that focuses on their apple, pear, plum, damson and cherry orchards.

Recently James has been looking at the sales of plums. These are graded and sold as fruit to local shops and markets, for canning to a local cannery, or for jam to a more distant processor. The plums sold for canning earn about half as much income as those sold for fruit, but twice as much as those sold for jam.

James is trying to estimate the weight of plums sold each year. He does not know this, as the plums are sold by the basket rather than by weight, with each basket holding about 25 kg of plums. For a pilot study, James set up some scales to see whether he could weigh the amount of fruit in a sample of baskets. On the first day he weighed 10 baskets, six of which were sold as fruit, three for canning and one for jam. The weights of fruit, in kilograms, were as follows:

25.6 20.8 29.4 28.0 22.2 23.1 25.3 26.5 20.7 21.9

Case study continued

This trial seemed to work, so James then weighed a sample of 50 baskets on three consecutive days. The weights of fruit, in kg, were as follows:

- Day 1 24.6 23.8 25.1 26.7 22.9 23.6 26.6 25.0 24.6 25.2 25.7 28.1 23.0 25.9 24.2 21.7 24.9 27.7 24.0 25.6 26.1 26.0 22.9 21.6 28.2 20.5 25.8 22.6 30.3 28.0 23.6 25.7 27.1 26.9 24.5 23.9 27.0 26.8 24.3 19.5 31.2 22.6 29.4 25.3 26.7 25.8 23.5 20.5 18.6 21.5
- Day 2 26.5 27.4 23.8 24.8 30.2 28.9 23.6 27.5 19.5 23.6 25.0 24.3 25.3 23.3 24.0 25.1 22.2 20.1 23.6 25.8 24.9 23.7 25.0 24.9 27.2 28.3 29.1 22.1 25.0 23.8 18.8 19.9 27.3 25.6 26.4 28.4 20.8 24.9 25.4 25.6 24.9 25.0 24.1 25.5 25.2 26.8 27.7 20.6 31.3 29.5
- Day 3 27.2 21.9 30.1 26.9 23.5 20.7 26.4 25.1 25.7 26.3 18.0 21.0 21.9 25.7 28.0 26.3 25.9 24.7 24.9 24.3 23.9 23.0 24.1 23.6 21.0 24.6 25.7 24.7 23.3 22.7 22.9 24.8 22.5 26.8 27.4 28.3 31.0 29.4 25.5 23.9 29.5 23.3 18.6 20.6 25.0 25.3 26.0 22.2 23.9 25.7

He also recorded the number of each sample sent to each destination:

	Fruit	Cans	Jam
Day 1	29	14	7
Day 2	25	15	10
Day 3	19	15	16

Pickers are paid by the basket, and the payments book showed the number of baskets picked on the three days as 820, 750 and 700 respectively. During a good harvest, a total of around 6,000 baskets are picked.

Questions

- **What information can James find from these figures? How can he use this information?**
- **How should he set about a complete survey of the fruit crop?**

PROBLEMS

16.1 A production line makes units with a mean weight of 80 g and standard deviation of 5 g. What is the probability that a sample of 100 units has a mean weight of less than 79 g?

16.2 A machine makes parts with a variance of 14.5 cm in length. A random sample of 50 parts has a mean length of 106.5 cm. What are the 95% and 99% confidence intervals for the length of parts?

16.3 A frozen food packer specifies the mean weight of a product as 200 g. The output is Normally distributed with a standard deviation of 15 g. A random sample of 20 has a mean of 195 g. Does this suggest that the mean weight is too low?

16.4 Hamil Sopa took a random sample of 60 invoices from his year's records. The mean value of invoices in this sample was £125.50 and the standard deviation was £10.20. What are the 90% and 95% confidence intervals for the mean value of all invoices?

16.5 Sheila Brown times 60 people doing a job. The mean time is 6.4 minutes, with a standard deviation of 0.5 minutes. How long would it take the population to do this job?

16.6 Wade (Retail) looked at a random sample of 100 invoices from a large population. Eight of these contain an error. What are the 90% and 95% confidence intervals for the proportion of invoices with errors?

16.7 A company wants to find the average weight of its products. A large initial sample shows the standard deviation of the weight is 20 g. What sample size would give a 95% confidence interval for the population that is (a) 10 g wide, (b) 8 g wide, (c) 5 g wide?

16.8 Last year a trial survey found that 65% of houses in a Morrisey Township had a computer. A follow-up survey wants to find the actual number of houses with a computer to within 3% with a 95% confidence interval. How many houses should it survey?

16.9 Henry Lom feels that the quantity of chocolates in a particular type of packet has decreased. To test this feeling he takes a sample of 40 packets and finds the mean weight is 228 g with a standard deviation of 11 g. What is the weight Henry can be 95% confident the mean falls below? What are the two-sided confidence limits on this weight?

16.10 BC's quality assurance programme chooses a random sample of 50 units, and finds that 12 are defective. What is the number of defectives they can be 95% confident that the population mean is below? What is the number of defectives they can be 95% confident that the population mean is above? How do these compare with the two-sided 90% confidence interval?

16.11 A survey of 20 items in a sales ledger has a mean value of €100 and standard deviation of €20. What is the 95% confidence interval for the population of items? What is the 99% confidence interval?

16.12 The time taken for a sample of eight pieces of equipment to do a task has a mean of 52 minutes and a standard deviation of 18 minutes. What is the 90% confidence interval for the mean time of all equipment to do the task? What is the 95% confidence interval? If the same results had been found from a sample of 20 pieces of equipment, what would be the 95% confidence interval? What would be the last result if a Normal distribution had been used?

RESEARCH PROJECTS

16.1 The central limit theorem gives a fundamental result for sampling – that large samples, or any samples from a Normal distribution, are Normally distributed. Test this result to see that it really works. For this you can:

- use a spreadsheet to generate a population of random numbers
- draw a frequency distribution of the numbers and confirm that they follow a uniform distribution (where each number has the same probability)
- take large samples from this population of numbers, and calculate the mean of each sample
- draw a frequency distribution of these means (i.e. the sampling distribution of the mean)
- confirm that the result is Normally distributed.

Now repeat this process for different sample sizes and initial distributions of values to see what effect this has.

16.2 For small samples we have to use the *t*-distribution instead of the Normal. How would you describe – and measure – the differences between these two distributions?

Sources of information

References

1 DTI, *Renewable Energy Statistics Database*, Department of Trade and Industry, London, 2006.

2 DTI, *Digest of UK Energy Statistics*, Department of Trade and Industry, London, 2005.

3 Statistical Office of the European Communities at www.eurostat.ec.europa.eu.

4 International Energy Agency at www.iea.org.

5 World Energy Council at www.worldenergy.com.

Further reading

Most of the statistics books that we listed in Chapter 14 contain material on sampling. More material is covered in books on quality control sampling and market research sampling. The following list gives some more specialised books on sampling.

Chandra M.J., *Statistical Quality Control*, CRC Press, Boca Raton, FL, 2001.

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CHAPTER 17

Testing hypotheses

Contents

Chapter outline

Hypothesis testing starts with a statement describing some aspect of a population – giving the hypothesis to be tested. Then it examines a sample from the population, and sees if there is evidence to support the hypothesis. Either the evidence supports the hypothesis, or it does not support it and by implication supports some alternative hypothesis. You can use this general approach to hypothesis testing in many circumstances. One of the most common situations tests whether data follows a specified probability distribution.

After finishing this chapter you should be able to:

- Understand the purpose of hypothesis testing
- List the steps involved in hypothesis testing
- Understand the errors involved and the use of significance levels
- Test hypotheses about population means
- Use one and two-tail tests
- Extend these tests to deal with small samples
- Use the tests for a variety of problems
- Consider non-parametric tests, particularly the chi-squared test.

Aim of hypothesis testing

In the last chapter we saw how statistical inference used data from a sample to estimate values for a population. In this chapter we are going to extend this idea by testing whether a belief about a population is supported by the evidence from a sample. This is the basis of **hypothesis testing**.

Suppose you have some preconceived idea about the value taken by a population variable. For instance, you might believe that domestic telephone bills have fallen by 10% in the past year. This is a hypothesis that you want to test. Then you take a sample from the population and see whether or not the results support your hypothesis. This gives the formal procedure:

- Define a simple, precise statement about a population (the hypothesis).
- Take a sample from the population.
- Test this sample to see whether it supports the hypothesis, or makes the hypothesis highly improbable.
- If the hypothesis is highly improbable reject it, otherwise accept it.

This seems a reasonable approach – but it needs a small adjustment. Statisticians are more cautious than this, and they do not talk about 'accepting' a hypothesis. Instead they say that they 'can reject the hypothesis' if it is highly unlikely, or they 'cannot reject the hypothesis' if it is more likely.

WORKED EXAMPLE 17.1

Aceituna GmbH fills bottles with a nominal 400 ml of olive oil. There are small variations around this nominal amount and the actual contents are Normally distributed with a standard deviation of 20 ml. The company takes periodic samples to make sure that they are filling the bottles properly. If a sample bottle contains 446 ml, are the bottles being overfilled?

Solution

We start with an initial hypothesis that the bottles still contain 400 ml. We have a very limited amount of data from a single sample and can use this to test the hypothesis. The distribution of bottle contents should be Normal, with mean 400 ml and standard deviation 20 ml. Assuming this is correct, we can find the probability of finding a sample bottle containing 446 ml. The number of standard deviations from the mean is:

$$Z = (446 - 400)/20 = 2.3$$

which corresponds to a probability of 0.01. If our hypothesis that the bottles contain 400 ml is correct, finding a bottle with 446 ml is highly improbable, occurring on only 1% of occasions. So we can reasonably reject the initial hypothesis that the bottles still contain 400 ml.

The original statement is called the **null hypothesis**, which is usually labelled H_0. The name 'null' implies there has been no change in the value being tested since the hypothesis was formulated. If we reject the null hypothesis then we implicitly accept an alternative. In the worked example above we reject the null hypothesis that the bottles contain 400 ml, so we accept the **alternative hypothesis** that they do not contain 400 ml. For each null hypothesis there is always an alternative hypothesis, which is usually called H_1. If the null hypothesis, H_0, is that domestic telephone bills have fallen by 10% in the last year, the alternative hypothesis, H_1, is that they have not fallen by 10%.

The null hypothesis must be a simple, specific statement – while the alternative hypothesis is less precise and suggests only that some statement other than the null hypothesis is true. In practice, this means that the null hypothesis is phrased in terms of one thing equalling another, while the alternative hypothesis says that the equality is not true. So a null hypothesis, H_0, says that the average salary in an office is €50,000, and the alternative hypothesis, H_1, is that the average salary is not €50,000.

Errors in hypothesis testing

Sampling always contains uncertainty, and a sample may not give a completely accurate view of a population. So we can never be certain of the result when testing a hypothesis. In worked example 17.1, we said that the result was unlikely – occurring only one time in 100 – so we rejected the null hypothesis. But if the null hypothesis is actually true, we would still get this result in 1% of samples, and we would be rejecting a true hypothesis. Conversely, a sample might give evidence to support a null hypothesis, even when it is not true. So there are two ways of getting the wrong answer with hypothesis testing (shown in Figure 17.1):

- ***Type I error***, when we reject a null hypothesis that is true
- ***Type II error***, when we do not reject a null hypothesis that is false.

Ideally we would like to arrange things so that the probabilities of both Type I and Type II errors are close to zero – and the only way of doing this is to use a large sample. Otherwise, any adjustments to reduce the probability of Type I errors inevitably increase the probability of Type II errors, and vice versa. With a limited sample size, we have to accept a compromise between the two errors.

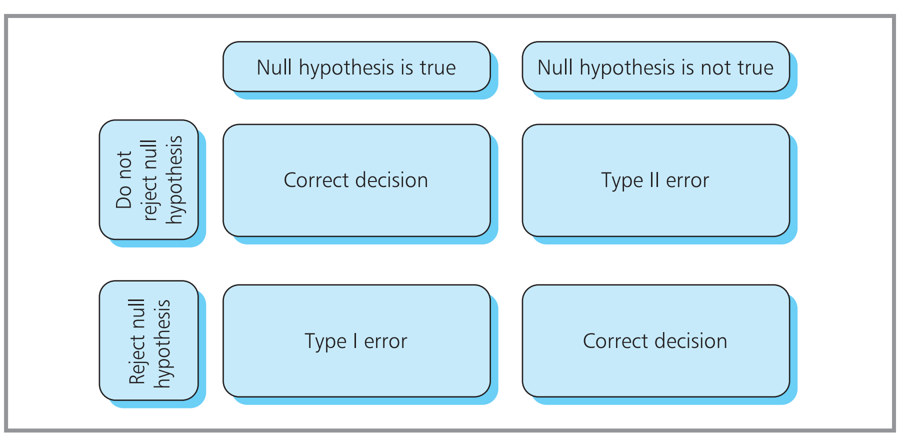


Figure 17.1 Errors in hypothesis testing

WORKED EXAMPLE 17.2

The City of Halifax, Nova Scotia, takes a survey of monthly costs of food and housing for a particular type of family. They think that the mean cost is \$1,600 with a standard deviation of \$489. A sample of 100 families has an average expenditure of \$1,712.50. Does this support their initial views?

Solution

We start by defining the null hypothesis, H_0, that the monthly cost of food and housing is \$1,600, while the alternative hypothesis, H_1, is that it does not equal \$1,600.

Hypothesis tests assume the null hypothesis is true for the test, so we assume the population has a mean of \$1,600 and a standard deviation of \$489. Then we find the probability that a sample with a mean of \$1,712.50 comes from this population. With a large sample of 100 the sampling distribution of the mean is Normal, with standard error $\sigma/\sqrt{n} = 489/\sqrt{100} = 48.9$. Then:

$$Z = (1{,}712.5 - 1{,}600)/48.9 = 2.3$$

which corresponds to a probability of 0.0107. If the null hypothesis is true, there is a probability of 0.0107 that the monthly cost of a sample is \$1,712.50. This is very unlikely, so the evidence does not support the null hypothesis. We can reject this and accept the alternative hypothesis. But remember that in about 1% of cases we are making a Type I error and rejecting a hypothesis that is actually true.

WORKED EXAMPLE 17.3

Marcia Lopez says that the mean wage of her employees is €300 a week with a standard deviation of €60. She checks a random sample of 36 wages and decides to revise her views if the mean of the sample is outside the range €270 to €330. What is the probability she makes a Type I error?

Solution

The null hypothesis, H_0, is that the mean wage is €300, and the alternative hypothesis, H_1, is that the mean wage is not €300.

Assuming that the population has a mean of €300 and standard deviation of €60, with a sample of 36 the sampling distribution of the mean is Normal with standard error $\sigma/\sqrt{n} = 60/\sqrt{36} = 10$. Then:

$$Z = (330 - 300)/10 = 3$$

which corresponds to a probability of 0.0013. By symmetry, the probability of a sample mean being less than €270 is also 0.0013. So when the null hypothesis is true, there is a probability of 0.0013 + 0.0013 = 0.0026 that the sample mean will be outside the acceptable range. This is the probability that Jenny rejects the null hypothesis when it is actually true.

Review questions

17.1 What is the purpose of hypothesis testing?

17.2 Which is a more precise statement, H_0 or H_1?

17.3 What are Type I and Type II errors?

IDEAS IN PRACTICE **Politics and hypotheses**

Most – but not all – legal systems work with a belief that someone is innocent until proven guilty. So they have a null hypothesis that someone is innocent, with the alternative hypothesis that they are guilty. No justice system is infallible, and a Type I error occurs when an innocent person is punished; a Type II error occurs when a guilty person is not punished. There is always a balance between these errors, with people having different opinions about where this balance should be. This effect can be broadened into other areas, such as public welfare payments – where a Type I error gives welfare payments to someone who does not deserve it, and a Type II error fails to give payments to someone who really needs it.

In the US, Hooke[1] has suggested that liberals and conservatives have different views of these errors. For example, with justice, liberals avoid Type I errors, while conservatives avoid Type II errors. With welfare payments it is the other way around: conservatives avoid Type I errors, while liberals avoid Type II errors. It is impossible to eliminate one type of error without increasing the other type of error, so the two political philosophies can never reach agreement.

Significance levels

So far we have rejected the null hypothesis if we consider the result from the sample to be unlikely. However, our judgement of what is 'unlikely' has been purely subjective. We can formalise this judgement in a **significance level**.

> A **significance level** is the minimum acceptable probability that a value actually comes from the hypothesised population.

With a 5% significance level, we do not reject a null hypothesis if there is a probability greater than 5% that a value comes from the specified population – and we reject the null hypothesis only when there is a probability of less than 5% that the value comes from the population. But this leaves 5% of observations that fall outside the acceptance range when the null hypothesis is actually true, and then we reject a null hypothesis that is true. As this is a Type I error, you can see that the significance level is the maximum acceptable probability of making a Type I error.

You can use any value for a significance level, but the most common is 5%, followed by 1% and occasionally 0.1%. With a large sample, the sampling distribution is Normal, and 95% of observations are within 1.96 standard deviations of the mean. So this defines the acceptance level (shown in Figure 17.2). With a 1% significance level, you cannot reject the null hypothesis if the observation is within 2.58 standard deviations of the mean. This is clearly a less stringent test – and it shows that lower significance levels need stronger evidence to reject the null hypothesis.

Of course, you may find that you reject a hypothesis at a 5% level, but cannot reject it at a 1% level – or even a 4% level. So this seems a rather arbitrary 'reject/cannot reject' decision, and it might be better to simply calculate the exact probability that a hypothesis is true. Then you could say, 'there is a probability of 0.002 that this sample comes from the hypothesised population'. In principle, this would be better – but the standard format has been used for many years and is unlikely to change.

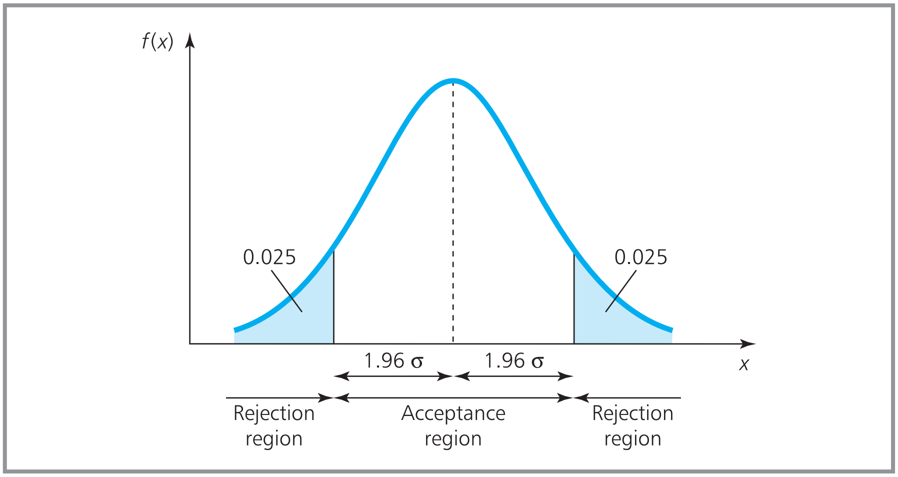


Figure 17.2 Acceptance and rejection regions for 5% significance level

WORKED EXAMPLE 17.4

John Lo thinks that the mean value of orders received by his firm is €260. He checks a sample of 36 accounts and finds a mean of €240 and standard deviation of €45. Does this evidence support his belief?

Solution

The null hypothesis is that the mean value of accounts is €260, and the alternative hypothesis is that the mean is not €260. Then:

H_0: $\mu = 260$ H_1: $\mu \neq 260$

We do not know the population standard deviation, but can estimate it from the sample standard deviation using $\mu = s/\sqrt{(n-1)}$. Then with a sample of 36, the sampling distribution of the mean is Normal with mean 260 and standard error $45/\sqrt{35} = 7.61$. With a significance level of 5% we do not reject values that are within 1.96 standard deviations of the mean. So the acceptance range is:

$260 - 1.96 \times 7.61$ to $260 + 1.96 \times 7.61$

or

245.1 to 274.9

The actual observation of €240 is outside this range, so we reject the null hypothesis, and accept the alternative hypothesis that the mean value of orders is not equal to €260.

Now we have seen all of the steps for hypothesis testing and can list them as follows.

1 State the null and alternative hypotheses.
2 Specify the significance level.
3 Calculate the acceptance range for the variable tested.
4 Find the actual value for the variable tested.
5 Decide whether or not to reject the null hypothesis.
6 State the conclusion.

WORKED EXAMPLE 17.5

The Central Tax Office says that the average income in Port Elizabeth is \$15,000. A sample of 45 people found their mean income to be \$14,300 with a standard deviation of \$2,000. Use a 5% significance level to check the claim. What is the effect of using a 1% significance level?

Solution

For this, you use the standard six-step procedure.

1 *State the null and alternative hypotheses*:

H_0: $\mu = 15{,}000$ H_1: $\mu \neq 15{,}000$

2 *Specify the level of significance*. This is given as 5%.

3 *Calculate the acceptance range for the variable tested*. With a sample of 45, the sampling distribution of the mean is Normal with mean 15,000 and standard error approximated by $s/\sqrt{(n-1)} = 2{,}000/\sqrt{44} = 301.51$. For a 5% significance level you cannot reject points that are within 1.96 standard deviations of the mean. So the acceptance range is:

$15{,}000 - 1.96 \times 301.51$ to
$15{,}000 + 1.96 \times 301.51$

or

14,409 to 15,591

4 *Find the actual value for the variable tested*. This is \$14,300.

5 *Decide whether or not to reject the null hypothesis*. The actual value is outside the acceptance range, so you reject the null hypothesis.

6 *State the conclusion*. At a 5% significance level, the evidence from the sample does not support the claim that the average income per capita in Port Elizabeth is \$15,000. Instead, it supports the alternative hypothesis that the average income is not \$15,000. With a 1% significance level, the acceptance range is within 2.58 standard deviations of the mean, or:

$15{,}000 - 2.58 \times 301.51$ to
$15{,}000 + 2.58 \times 301.51$

or

14,222 to 15,778

The actual observation of \$14,300 is within this range, so you cannot reject the null hypothesis (as shown in Figure 17.3).

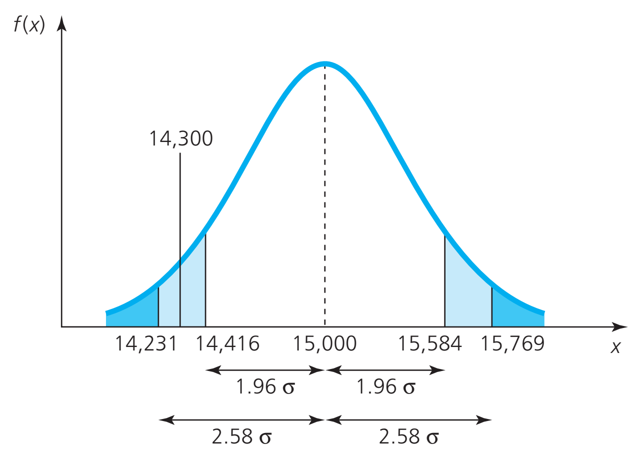


Figure 17.3 Acceptance range for incomes in Port Elizabeth (worked example 17.5)

One-sided tests

In the problems we have looked at so far, we have stated a null hypothesis of the form:

H_0: $\mu = 10$

and an alternative hypothesis in the form:

H_1: $\mu \neq 10$

In practice, we often want to test whether an actual value is above (or sometimes below) the claimed value. If we buy boxes of chocolates, we want to be sure that their weight is not below the specified weight; and if we are delivering parcels, we want to know that their weight is not above the claimed weight. We can tackle problems like this using the standard procedure, but with adjustments to the phrasing of the alternative hypothesis and calculation of the acceptance range.

If we are buying boxes of chocolates with a specified weight of 500 g, and want to be sure that the actual weight is not below this, we use:

Null hypothesis, H_0: $\mu = 500$ g
Alternative hypothesis, H_1: $\mu < 500$ g

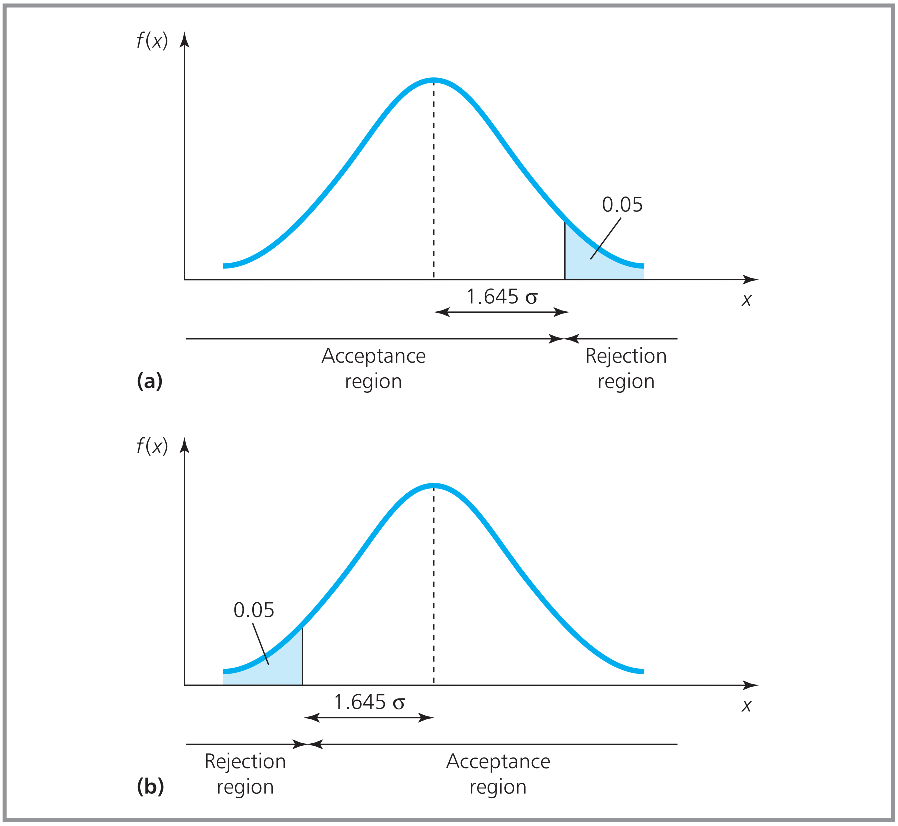


Figure 17.4 One-sided test for 5% significance level: (a) when concerned with a maximum value; (b) when concerned with a minimum value

If we are delivering parcels with a claimed weight of 25 kg, and want to be sure the actual weight is not above this, we use:

Null hypothesis, H_0: $\mu = 25$ kg
Alternative hypothesis, H_1: $\mu > 25$ kg

Now we are interested in only one tail of the sampling distribution, so the acceptance range is altered. In particular, a 5% significance level has the 5% area of rejection in one tail of the distribution. In a Normal distribution this point is 1.645 standard deviations from the mean, as shown in Figure 17.4.

WORKED EXAMPLE 17.6

BookCheck Mail-Order charge customers a flat rate for delivery based on a mean weight for packages of 1.75 kg with a standard deviation of 0.5 kg. Postage costs have risen and it seems likely that the mean weight is greater than 1.75 kg. The company checked a random sample of 100 packages and found a mean weight of 1.86 kg. Does this support the view that the mean weight is more than 1.75 kg?

Solution

We use the standard procedure.

1 *State the null and alternative hypotheses*. We want to test that the mean weight is not above 1.75 kg, so we have:

$$H_0: \mu = 1.75 \text{ kg} \quad H_1: \mu > 1.75 \text{ kg}$$

2 *Specify the level of significance*. This is not given, so we assume 5%.

3 *Calculate the acceptance range for the variable tested*. With a sample of 100, the sampling distribution of the mean is Normal with mean of 1.75 kg and standard deviation $\sigma/\sqrt{n} = 0.5/\sqrt{99} = 0.050$ kg. For a 5% significance level and a one-sided test, we look at points that are more than 1.645 standard deviations above the mean. The acceptance range is below $1.75 + 1.645 \times 0.05 = 1.83$ kg.

4 *Find the actual value for the variable tested*. The observed weight of parcels is 1.86 kg.

5 *Decide whether or not to reject the null hypothesis*. The actual value is outside the acceptance range, so we reject the null hypothesis.

6 *State the conclusion*. The evidence from the sample does not support the view that the mean weight of packages is 1.75 kg. The evidence supports the alternative hypothesis, that the mean weight is more 1.75 kg. This is illustrated in Figure 17.5.

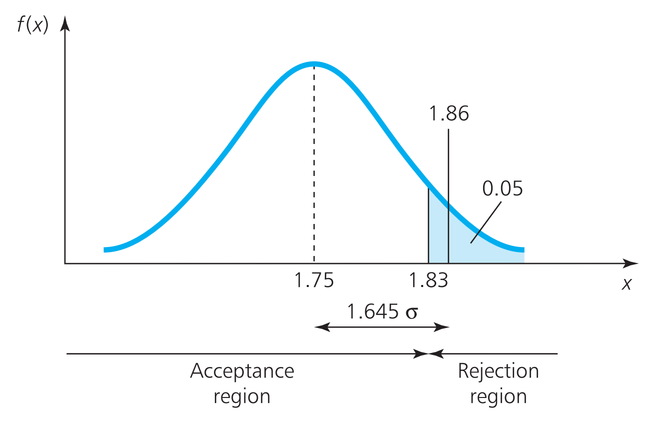


Figure 17.5 Acceptance region for BookCheck Mail-Order (worked example 17.6)

WORKED EXAMPLE 17.7

Elisabeta Horst is a management consultant who has recently introduced new procedures to a reception office. The receptionist should do at least 10 minutes of paperwork in each hour. Elisabeta made a check on 40 random hours of work and found the mean time spent on paperwork is 8.95 minutes with a standard deviation of 3 minutes. Can she reject the hypothesis that the new procedures meet specifications at a 1% level of significance?

Solution

1 *State the null and alternative hypotheses*. Elisabeta wants to check that the time spent on paperwork is at least 10 minutes in an hour. So:

H_0: $\mu = 10$ minutes H_1: $\mu < 10$ minutes

2 *Specify the level of significance*. This is given as 1%.

3 *Calculate the acceptance range for the variable tested*. With a sample of 40, the sampling distribution of the mean is Normal with a mean of 10 minutes and standard deviation $s/\sqrt{(n-1)}$ $= 3/\sqrt{39} = 0.480$ minutes. For a 1% significance level and a one-sided test, Elisabeta should accept points that are more than 2.33 standard deviations below the mean. Then the rejection range is below $10 - 2.33 \times 0.48 = 8.88$ minutes.

4 *Find the actual value for the variable tested*. The observed number of minutes spent on paperwork in each hour is 8.95.

5 *Decide whether or not to reject the null hypothesis*. The actual value is inside the acceptance range and Elisabeta cannot reject the null hypothesis.

6 *State the conclusion*. The evidence from the sample supports the view that the mean time spent on paperwork is at least 10 minutes an hour.

Review questions

17.4 What is a significance level?

17.5 Is the probability of a Type II error lower with a 5% significance level or a 1% significance level?

17.6 If a value is in the acceptance range, does this prove the null hypothesis is true?

17.7 When would you use a one-sided hypothesis test?

Tests with small samples

In the last chapter we noted that sampling distributions are Normal only when the population is Normal or the sample size is more than 30. When this is not true, the sampling distribution follows a *t*-distribution. Remember that the shape of a *t*-distribution depends on the degrees of freedom, which is the sample size minus one.

WORKED EXAMPLE 17.8

A coffee machine is set to fill cups with 200 ml of coffee. A sample of 10 cups contained 210 ml with a standard deviation of 10 ml. Is the machine working properly?

Solution

We can use the standard approach for hypothesis testing, with a two-tail test, but as the sample size is small we have to use a *t*-distribution.

1 *State the null and alternative hypotheses*. The null hypothesis is that the dispenser is filling cups with 200 ml, while the alternative hypothesis is that it is not filling cups with 200 ml.

H_0: $\mu = 200$ ml H_1: $\mu \neq 200$ ml

2 *Specify the level of significance*. We can use the standard 5%.

Worked example 17.8 continued

3 *Calculate the acceptance range for the variable tested*. With a sample of size 10, the sampling distribution of the mean follows a t-distribution with $n - 1 = 10 - 1 = 9$ degrees of freedom, a mean of 200 ml and standard deviation $s/\sqrt{(n - 1)} = 10/\sqrt{9} = 3.33$ ml. For a 5% significance level and a two-sided test, we look up (either in tables or using the 'TINV' function in a spreadsheet) a probability of 0.025 in each tail, and with 9 degrees of freedom the value is 2.262. Then the acceptance range is:

$$200 - 2.262 \times 3.33 \quad \text{to} \quad 200 + 2.262 \times 3.33$$
$$192.47 \quad \text{to} \quad 207.53$$

4 *Find the actual value for the variable tested*. The actual mean of the sample was 210 ml.

5 *Decide whether or not to reject the null hypothesis*. The actual value is outside the acceptance range, so we reject the null hypothesis.

6 *State the conclusion*. The evidence from the sample does not support the view that the machine is working properly.

WORKED EXAMPLE 17.9

A supermarket is getting complaints that its tins of strawberries contain a lot of juice, but few strawberries. A team from the supermarket make a surprise visit to the supplier who is about to deliver another batch. Each tin in this batch is claimed to have a minimum of 300 g of fruit, but a random sample of 15 tins found only 287 g with a standard deviation of 18 g. What conclusion can the supermarket make?

Solution

1 *State the null and alternative hypotheses*. The null hypothesis is that the mean weight of the fruit is 300 g, while the alternative hypothesis is that the weight is less than this.

$$H_0: \mu = 300 \text{ g} \quad H_1: \mu < 300 \text{ g}$$

2 *Specify the level of significance*. We can use the standard 5%.

3 *Calculate the acceptance range for the variable tested*. With a sample of size 15, the sampling distribution of the mean follows a t-distribution with $n - 1 = 15 - 1 = 14$ degrees of freedom and a mean of 300 g. The estimated standard error is $s/\sqrt{(n - 1)} = 18/\sqrt{14} = 4.81$ g. For a 5% significance level and a one-sided test, we look up (either in tables or using the 'TINV' function in a spreadsheet) a probability of 0.05. With 14 degrees of freedom the value is 1.761. Then the acceptance range is above $300 - 1.761 \times 4.81 = 291.53$ g.

4 *Find the actual value for the variable tested*. The actual mean of the sample was 287 g.

5 *Decide whether or not to reject the null hypothesis*. The actual value is outside the acceptance range, so we reject the null hypothesis.

6 *State the conclusion*. The evidence from the sample does not support the view that the mean weight is 300 g. It supports the alternative hypothesis that the mean weight of fruit is less than 300 g.

Review questions

17.8 Why do we not use the Normal distribution for small samples?

17.9 What shape is the t-distribution for large samples?

17.10 When would you use the approximation $s/\sqrt{(n - 1)}$ for $\sigma/\sqrt{n}$?

Testing other hypotheses

So far we have focused on hypothesis tests for population means, but we can use the same approach for a variety of other problems.

Population proportions

In the last chapter we mentioned that sampling could test the proportion of a population that shared some common feature. In particular, we used the standard result that when the proportion in the population is π, the sampling distribution of the proportion has mean of π and standard deviation of $\sqrt{(\pi(1-\pi)/n)}$. Now we can use this result to test hypotheses about proportions.

WORKED EXAMPLE 17.10

High street banks claim that they lend the money for 20% of all house purchases. To test this, a sample of 100 people with mortgages was interviewed, 15 of whom arranged their loan through a bank. Does this support the original claim?

Solution

Hypothesis tests always use the same procedure, and the only difference with this problem is that we are interested in a proportion, π, rather than a mean.

1. *State the null and alternative hypotheses*. The null hypothesis is that banks lend 20% of funds for mortgages, so using proportions we have:

   H_0: $\pi = 0.2$ $\quad H_1$: $\pi \neq 0.2$

2. *Specify the level of significance*. This is not given, so we assume 5%.
3. *Calculate the acceptance range for the variable tested*. With a sample of 100, the sampling distribution is Normal with mean 0.2 and standard deviation $\sqrt{(\pi(1-\pi)/n)} = \sqrt{(0.2 \times 0.8/100)} = 0.04$. For a 5% significance level we want points that are within 1.96 standard deviations of the mean. Then the acceptance range is:

   $0.2 - 1.96 \times 0.04$ to $0.2 + 1.96 \times 0.04$

   or

   0.122 to 0.278

4. *Find the actual value for the variable tested*. The sample had a proportion of $15/100 = 0.15$.
5. *Decide whether or not to reject the null hypothesis*. The actual value is within the acceptance range, so we cannot reject the null hypothesis.
6. *State the conclusion*. We cannot reject the claim that banks lend money for 20% of mortgages.

Testing for differences in means

Managers often want to compare two populations, to see whether there are significant differences. For example, they might have two shops and want to know whether each has the same profitability, or they might want to check sales before and after an advertising campaign.

We can use hypothesis testing to see whether the means of two populations are the same. For this we take a sample from each population, and if the sample means are fairly close we can assume the population means are the same, but if there is a large difference in the sample means we have to assume that the population means are different. So the procedure is to take the means of two samples, $\bar{x}_1$ and $\bar{x}_2$, and find the difference, $\bar{x}_1 - \bar{x}_2$. Then we use a standard result (which we are not bothering to prove) that for large samples the sampling distribution of $\bar{x}_1 - \bar{x}_2$ is Normal with:

$$\text{mean} = 0 \quad \text{and} \quad \text{standard error} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$

where: n_1 = sample size from population 1
n_2 = sample size from population 2
s_1 = standard deviation of sample 1
s_2 = standard deviation of sample 2.

WORKED EXAMPLE 17.11

Krinkle Kut Krisps uses two machines to fill packets of crisps. A sample of 30 packets from the first machine has a mean weight of 180 g and a standard deviation of 40 g. A sample of 40 packets from the second machine has a mean weight of 170 g and a standard deviation of 10 g. Are the two machines putting the same amount in packets?

Solution

1 *State the null and alternative hypotheses*. We want to check that the two machines are putting the same amounts in packets, so the null hypothesis is that the means from each machine are the same. The alternative hypothesis is that the means are not the same.

$$H_0\text{: } \mu_1 = \mu_2 \quad H_1\text{: } \mu_1 \neq \mu_2$$

2 *Specify the level of significance*. We can use the standard 5%.

3 *Calculate the acceptance range for the variable tested*. We are looking at the sampling distribution of $\bar{x}_1 - \bar{x}_2$, with sample sizes $n_1 = 30$ and $n_2 = 40$, and standard deviations $s_1 = 14$ and $s_2 = 10$. This sampling distribution is Normal with:

mean = 0 and

$$\text{standard error} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{14^2}{30} + \frac{10^2}{40}} = 3.01$$

For a 5% significance level and a two-sided test, the acceptance range is within 1.96 standard deviations of the mean. This defines the range:

$$0 - 1.96 \times 3.01 \quad \text{to} \quad 0 + 1.96 \times 3.01$$

or

$$-5.90 \quad \text{to} \quad +5.90$$

4 *Find the actual value for the variable tested*. The observed difference in samples is $\bar{x}_1 - \bar{x}_2 = 180 - 170 = 10$.

5 *Decide whether or not to reject the null hypothesis*. The actual value is outside the acceptance range, so we reject the null hypothesis.

6 *State the conclusion*. The evidence from the samples does not support the view that the mean weight put into packets is the same from each machine.

Paired tests

If you want to see whether a diet works, then you will weigh a set of people before the diet, and weigh them again after the diet. This gives a set of paired data – two weights for each person in your test – and you want to see whether there is a difference between the two. This is the kind of problem that managers meet when, for example, they interview people before and after an advertising campaign, or to see whether two people interviewing candidates for a job give different opinions.

To test for differences between paired observations, we find the difference between each pair. If the two sets of observations are similar, the mean difference should be around zero – but if there is a real distinction between the observations the mean difference becomes bigger. So we use hypothesis testing to see whether the differences between samples are small enough to suggest that the two samples are the same, or are big enough to suggest that they are different.

WORKED EXAMPLE 17.12

Amethyst Interviews counted the number of interviews that a sample of eight of their staff did in a day. Then they adjusted the way the questions were presented, and again counted the number of interviews the eight staff did. From the following results, can you say whether the adjustments had any effect?

Interviewer	1	2	3	4	5	6	7	8
Original interviews	10	11	9	6	8	10	7	8
Later interviews	10	9	11	10	9	12	9	11

Solution

Here we subtract the number of original interviews from the number of later interviews to get:

Interviewer	1	2	3	4	5	6	7	8
Difference	0	–2	2	4	1	2	2	3

Now we can use the standard approach on the sample differences.

1 *State the null and alternative hypotheses*. We want to test the null hypothesis that the mean difference is zero, and the alternative hypothesis that the mean difference is not zero.

$$H_0: \mu = 0 \quad H_1: \mu \neq 0$$

2 *Specify the level of significance*. We use the standard 5%.

3 *Calculate the acceptance range for the variable tested*. The mean of the differences is $(0 - 2 + 2 + 4 + 1 + 2 + 2 + 3)/8 = 1.5$, the variance is $(1.5^2 + 3.5^2 + 0.5^2 + 2.5^2 + 0.5^2 + 0.5^2 + 0.5^2 + 1.5^2)/8 = 3.0$, so the standard deviation is $\sqrt{3.0} = 1.732$. With a small sample of eight pairs of observations, the sampling distribution is a t-distribution with $8 - 1 = 7$ degrees of freedom and standard error $s/\sqrt{(n - 1)} = 1.732/\sqrt{7} = 0.655$. For a two-tail 5% significance level, the t-distribution with 7 degrees of freedom is 2.365. So the acceptance range is:

$$0 - 2.365 \times 0.655 \quad \text{to} \quad 0 + 2.365 \times 0.655$$

or

$$-1.548 \quad \text{to} \quad 1.548$$

4 *Find the actual value for the tested variable*. The mean of the differences is 1.5.

5 *Decide whether or not to reject the null hypothesis*. The actual value is within the acceptable range, so we cannot reject the null hypothesis.

6 *State the conclusion*. The evidence says that we cannot reject the view that there is no difference between the number of interviews before and after the adjustment. This is an interesting result, as it seems fairly clear that the adjustments have made a difference – but the explanation is that the sample size is very small.

We have done the calculations for hypothesis testing by hand, but we could clearly have used a computer. Figure 17.6 shows results from putting data from the last worked example into Excel's data analysis option which does calculations for paired samples. Here the data is on the left, and the analysis is on the right. You can see that the computer presents the results in a slightly different way. We have calculated the limits within which we accept the null hypothesis, and then we see whether the actual value lies within these limits. An alternative is to state the number of standard errors the acceptable range is from the mean, and then find how many standard errors the actual value is away from the mean. In this example, the 5% significance level sets the acceptable range for a two-tail test as within 2.3646 standard errors of the mean (called the critical t value), while the actual value is $1.5/0.655 = 2.29$ standard errors from the mean. As the actual value is within the acceptable range, we cannot reject the null hypothesis. However,

	A	B	C	D	E	F
1	**Paired tests**					
2						
3	**Amethyst Interviews**					
4						
5	**Data**			***t*-Test: Paired Two Sample for Means**		
6	**Original interviews**	**Later interviews**			**Original interviews**	**Later interviews**
7	10	10		Mean	8.625	10.125
8	11	9		Variance	2.8393	1.2679
9	9	11		Observations	8	8
10	6	10		Hypothesised Mean Difference	0	
11	8	9		Degrees of freedom	7	
12	10	12		*t* Statistic	2.291	
13	7	9		*P*(*T*<=*t*) one-tail	0.0279	
14	8	11		*t* Critical one-tail	1.8946	
15				*P*(*T*<=*t*) two-tail	0.0557	
16				*t* Critical two-tail	2.3646	

Figure 17.6 Spreadsheet of paired test for Amethyst Interviews (worked example 17.12)

you can see that the probability that the value is within the acceptable range is only 0.0557, so it only just passes at this significance level.

Review questions

17.11 In what circumstances would you use hypothesis tests?

17.12 Is the significance level the same as a confidence interval?

IDEAS IN PRACTICE Testing systems

Many automatic systems are programmed to make decisions under conditions of uncertainty. Then they check a response and follow prescribed rules to reach a decision. As there is uncertainty, there is always the chance that they make a Type I or Type II error.

You can imagine this with an airport security system that is designed to detect passengers carrying weapons. When you walk through an airport metal detector, the detector has a null hypothesis that you are carrying a weapon, and it takes electromagnetic measurements to test this hypothesis. When it does not have enough evidence to support the null hypothesis it lets you through; when the evidence supports the null hypothesis it stops you. But there is uncertainty in its readings, caused by other things that you might be carrying, and it makes two kinds of error. Either it stops people who are not carrying a weapon, or it does not stop people who are.

Another example is email spam filters, which block junk messages or those that are somehow objectionable. The filter has a null hypothesis that a message is spam, and then examines the contents to test this hypothesis. If it finds evidence to support the hypothesis that the message is spam (the presence of key words, patterns in senders' address, multiple copies transmitted, types of content, etc.), the filter blocks the message.

You can find many other examples of such systems, including Internet search engines, automatic recorders of earthquakes, burglar alarms, roadside speed cameras, and so on.

Chi-squared test for goodness of fit

Variables that take a specific value during an investigation are called parameters, and hypothesis tests that concern the value of a parameter are **parametric tests**. Often we want to test a hypothesis, but there is no appropriate variable – or parameter – to measure. You can imagine this with nominal data like the type of industry, value, quality, colour, and so on. For instance, you might suggest a hypothesis that one type of product offers better value than another – but there is no convincing parameter you can use to measure value. When you cannot use parametric hypothesis tests, you have to use the alternative **non-parametric** or **distribution-free tests**. These have the major benefit of making no assumptions about the distribution of the population.

The most important non-parametric test is the **chi-squared** or **χ^2 test**. χ is the Greek letter chi (pronounced 'kie', rhyming with 'lie') – and in this context χ is always squared, with the individual value χ having no meaning.

The chi-squared test is still a hypothesis test, so its general approach is the same as that of parametric tests. The difference is that it looks at the *frequencies* of observations and sees whether these match the expected frequencies.

Suppose we have a series of frequencies for some observations $O_1, O_2, O_3, \ldots, O_n$, and were expecting the frequencies $E_1, E_2, E_3, \ldots, E_n$. The difference between these tells us how closely the observations match expectations. We can square the difference between observed and expected values to remove negative values, and then dividing by the expected value gives a distribution with a standard shape. So we define χ^2 as:

$$\chi^2 = \frac{(O_1 - E_1)^2}{E_1} + \frac{(O_2 - E_2)^2}{E_2} + \frac{(O_3 - E_3)^2}{E_3} + \ldots + \frac{(O_n - E_n)^2}{E_n}$$

$$\text{chi-squared} = \chi^2 = \sum \frac{(O - E)^2}{E}$$

When observed frequencies are close to the expected frequencies, χ^2 has a value close to zero; but when the observed frequencies are not close to the expected ones, χ^2 has a larger value. So we define a **critical value** for χ^2: when the actual value is above this critical value we reject the hypothesis, and when the actual value is below the critical value we cannot reject the hypothesis. There are standard tables of critical values (shown in Appendix F) and they are calculated by standard software, such as Excel's CHIINV function.

Figure 17.7 shows how the shape of the χ^2 distribution depends on the degrees of freedom. We met degrees of freedom with the t-distribution in Chapter 16, where we said that they measure the number of pieces of information that are free to take any value. With the χ^2 distribution the number of degrees of freedom is defined as:

$$\text{degrees of freedom} = \text{number of classes} - \text{number of estimated variables} - 1$$

Now we can use this information in the standard procedure for hypothesis testing.

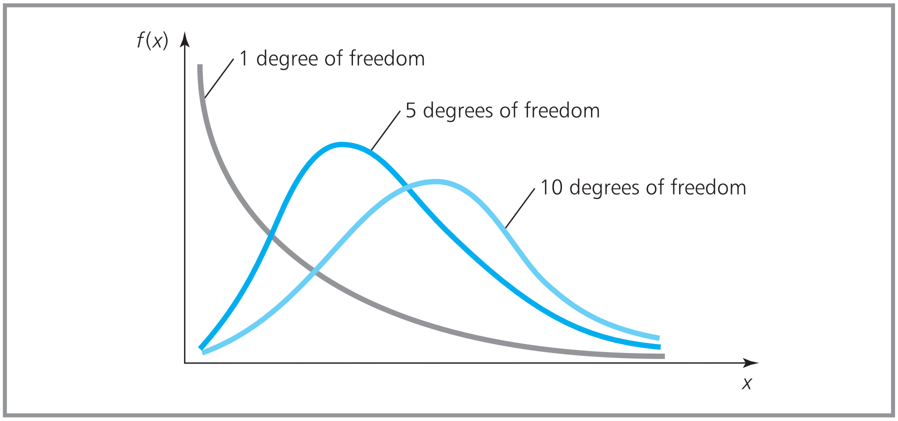


Figure 17.7 Chi-squared distribution with varying degrees of freedom

WORKED EXAMPLE 17.13

Since they began keeping records, five supermarkets have recorded the following numbers of minor accidents.

Supermarket	1	2	3	4	5
Number of accidents	31	42	29	35	38

Does this suggest that some supermarkets have more accidents than others?

Solution

We use the standard procedure for hypothesis testing.

1. *State the null and alternative hypotheses*. The null hypothesis, H_0, is that each supermarket has the same number of accidents. The alternative hypothesis, H_1, is that each supermarket does not have the same number of accidents.
2. *Specify the level of significance*. We can take this as 5%.
3. *Calculate the critical value of χ^2*. In this problem there are five classes, and no variables have been estimated, so the degrees of freedom are 5 – 0 – 1 = 4. With a 5% significance level we look up 0.05 in χ^2 tables (or use the CHIINV function) and with 4 degrees of freedom we get a critical value of 9.49.
4. *Find the actual value of χ^2, where*:

$$\chi^2 = \sum \frac{(O - E)^2}{E}$$

There are a total of 175 accidents. If each supermarket expects the same number of accidents, each expects 175/5 = 35. Then we do the following calculations, which show that the actual value of χ^2 is 3.143.

Supermarket	O	E	$O - E$	$(O - E)^2$	$(O - E)^2/E$
1	31	35	–4	16	0.457
2	42	35	7	49	1.400
3	29	35	–6	36	1.029
4	35	35	0	0	0.000
5	38	35	3	9	0.257
Total	175	175			3.143

5. *Decide whether or not to reject the null hypothesis*. The actual value (3.143) is less than the critical value (9.4877), so we cannot reject the null hypothesis.
6. *State the conclusion*. The evidence supports the view that each supermarket has the same number of accidents, with any variation due to chance.

The last worked example effectively tested whether there was a uniform distribution of accidents across supermarkets. In other words, we hypothesised that the accident rate was uniformly distributed and checked whether the data fitted this. We can use the same approach for other distributions, to see whether a set of data follows a specific distribution.

WORKED EXAMPLE 17.14

A survey asked 10 people whether they were prepared to pay higher charges for a service. This was repeated in each of 100 postcodes. Do the following results follow a binomial distribution?

Number in sample willing to pay higher charges	0	1	2	3	4
Number of postcodes	26	42	18	9	5

Solution

If we define success as being willing to pay higher charges and failure as being unwilling to pay, we have a binomial process. We would expect the number of successes in 10 trials to follow a binomial distribution, and can test this using the standard procedure.

1 *State the null and alternative hypotheses*. The null hypothesis, H_0, is that the distribution of successes is binomial. The alternative hypothesis, H_1, is that the distribution is not binomial.

2 *Specify the level of significance*. We take this as 5%.

3 *Calculate the critical value of χ^2*. There are five classes – but we also need to find one parameter for the binomial distribution, which is the probability of success. So the number of degrees of freedom is:

number of classes – number of estimated parameters – 1 = 5 – 1 – 1 = 3

Looking up the critical value for a significance level of 5% and 3 degrees of freedom gives a critical value of 7.81.

4 *Find the actual value of χ^2*. We have 10 results from each of 100 postcodes, giving 1,000 opinions, and of these the total number willing to pay higher charges is:

$$(26 \times 0) + (42 \times 1) + (18 \times 2) + (9 \times 3) + (5 \times 4) = 125$$

So the probability that someone is willing to pay higher charges is 125/1,000 = 0.125. Now we can find the probabilities of 0, 1, 2, 3, 4, etc. successes out of 10 when the probability of success is 0.125, using the binomial calculations described in Chapter 15 (or tables in Appendix B or statistical software). The probability of no successes is 0.263, the probability of one success is 0.376, the probability of two successes is 0.242, and so on. Multiplying these probabilities by the number of postcodes, 100, gives the expected number of postcodes with each number willing to pay higher charges. For convenience, we have taken the last class as '4 or more'.

Number willing to pay higher charges	0	1	2	3	4 or more
Probability	0.263	0.376	0.242	0.092	0.027
Expected number of postcodes	26.3	37.6	24.2	9.2	2.7

Now we have both the expected and observed distributions of postcodes, and can calculate the actual value of χ^2 in the following table.

Frequency	O	E	$O - E$	$(O - E)^2$	$(O - E)^2/E$
0	26	26.3	–0.3	0.09	0.003
1	42	37.6	4.4	19.36	0.515
2	18	24.2	–6.2	38.44	1.588
3	9	9.2	–0.2	0.04	0.004
4 or more	5	2.7	2.3	5.29	1.959
Total	100	100			4.069

5 *Decide whether or not to reject the null hypothesis*. The actual value of χ^2 (4.069) is less than the critical value (7.81), so we cannot reject the null hypothesis.

6 *State the conclusion*. The evidence supports the view that the observations follow a binomial distribution.

WORKED EXAMPLE 17.15

Performance Cables record the following numbers of faults per kilometre in a cable. Does this data follow a Poisson distribution?

Number of faults	0	1	2	3	4	5	6
Number of kilometres	37	51	23	7	4	2	1

Solution

1 *State the null and alternative hypotheses*. The null hypothesis, H_0, is that the distribution is Poisson. The alternative hypothesis, H_1, is that the distribution is not Poisson.

2 *Specify the level of significance*. We take this as 5%.

3 *Calculate the critical value of χ^2*. A small problem here is that the χ^2 distribution does not work well with expected frequencies of less than 5. We should really combine small adjacent classes so that the expected number of observations becomes greater than 5. Because of this adjustment, we will return to the critical value of χ^2 a little later.

4 *Find the actual value of χ^2*. We have 125 km of data, and the total number of defects is:

$$(37 \times 0) + (51 \times 1) + (23 \times 2) + (7 \times 3) + (4 \times 4) + (2 \times 5) + (1 \times 6) = 150$$

The mean number of defects is 150/125 = 1.2 per kilometre. Using this as the mean of a Poisson distribution, we can find the expected number of defects per kilometre in the usual ways. Then multiplying these probabilities by the number of observations, 125, gives the expected frequency distribution of defects. Figure 17.8 shows a spreadsheet with these results in columns A to C.

Now we should combine adjacent classes so that each class has more than five observations. Adding the last four classes gives the revised table of expected values in column D, with subsequent calculations in columns F and G. Now you can see why we delayed the calculation of the critical value; there are now only four classes, so the number of degrees of freedom is:

number of classes − number of estimated parameters − 1 = 4 − 1 − 1 = 2

Here the parameter estimated is the mean of the distribution. Looking up the critical value for a significance level of 5% and 2 degrees of freedom gives a value of 5.99.

5 *Decide whether or not to reject the null hypothesis*. The actual value (1.459) is less than the critical value (5.99), so we cannot reject the null hypothesis.

6 *State the conclusion*. The evidence supports the view that the observations follow a Poisson distribution.

	A	B	C	D	E	F	G
1	**Chi-squared test**						
2							
3	**Number of defects**	**Probability**	**Expected frequency**	**Revised frequency (*E*)**	**Actual frequency (*O*)**	**(*O* − *E*)**	**$(O - E)^2/E$**
4	0	0.301	37.649	37.649	37	−0.649	0.011
5	1	0.361	45.179	45.179	51	5.821	0.750
6	2	0.217	27.107	27.107	23	−4.107	0.622
7	3	0.087	10.843	15.064	14	−1.064	0.075
8	4	0.026	3.253				
9	5	0.006	0.781				
10	>=6	0.002	0.188				
11	**Totals**	**1.000**	**125.000**	**125.000**	**125.000**		**1.459**

Figure 17.8 Spreadsheet calculations for the chi-squared test in worked example 17.15

WORKED EXAMPLE 17.16

Humbolt Farm Products is about to analyse the weights of materials received from suppliers, but the analyses are valid only if the weights are Normally distributed. The population of materials is known to have a mean weight of 45 g and a standard deviation of 15 g. A sample of 500 units was taken, with weights given in the following distribution. Are these Normally distributed?

Weight (in grams)	Number of observations	Weight (in grams)	Number of observations
less than 10	9	40 to 49.99	115
10 to 19.99	31	50 to 59.99	94
20 to 29.99	65	60 to 69.99	49
30 to 39.99	97	70 to 79.99	24
		80 to 89.99	16

Solution

This is an important test, as many statistical analyses are valid only if there is a Normal distribution.

1. *State the null and alternative hypotheses*. The null hypothesis, H_0, is that the distribution is Normal; the alternative hypothesis, H_1, is that the distribution is not Normal.
2. *Specify the level of significance*. We take this as 5%.
3. *Calculate the critical value of χ^2*. The number of degrees of freedom is 9 – 0 – 1 = 8. Then Appendix F shows that the critical value for χ^2 with 8 degrees of freedom at a 5% significance level is 15.5.
4. *Find the actual value of χ^2*. The probability that an observation is in the range 10 to 19.99 is:

$$P(\text{between 10 and 19.99}) = P(\text{less than 20}) - P(\text{less than 10})$$

Now 20 is (20 – 45)/15 = –1.67 standard deviations from the mean, which corresponds to a probability of 0.048, and 10 is (10 – 45)/15 = –2.33 standard deviations from the mean, which corresponds to a probability of 0.010, so:

$$P(\text{between 10 and 19.99}) = 0.048 - 0.010 = 0.038$$

The expected number of observations in this range is 0.038 × 500 = 19. Figure 17.9 shows a spreadsheet of these calculations and the calculated value of χ^2 as 43.56.

5. *Decide whether or not to reject the null hypothesis*. The actual value (43.56) is greater than the critical value (15.5), so we reject the hypothesis that the sample is Normally distributed.
6. *State the conclusion*. The evidence does not support the view that observations follow a Normal distribution.

	A	B	C	D	E	F
1	**Chi-squared test – Humbolt Farm Products**					
2						
3	**Weight from**	**to**	**Probability**	**Expected frequency**	**Observed frequency**	**$(O - E)^2/E$**
4	0	10.00	0.010	4.9	9	3.41
5	10	20.00	0.038	19.0	31	7.61
6	20	30.00	0.111	55.4	65	1.65
7	30	40.00	0.211	105.4	97	0.67
8	40	50.00	0.261	130.5	115	1.85
9	50	60.00	0.211	105.4	94	1.23
10	60	70.00	0.111	55.4	49	0.75
11	70	80.00	0.038	19.0	24	1.32
12	80	90.00	0.010	4.9	16	25.07
13			1.000	500.0	500	**43.56**

Figure 17.9 Calculations for Humbolt Farm Products (worked example 17.16)

Review questions

17.13 What is the main difference between a parametric and a non-parametric test?

17.14 When would you use a non-parametric test?

17.15 'When you cannot use a parametric test, you can always use a non-parametric test instead.' Do you think this is true?

17.16 Why does a χ^2 test have only a critical value rather than an acceptance range?

17.17 What is χ (the square root of χ^2) used for?

Tests of association

We can also use a χ^2 test for checking association that is described in a **contingency table**. Suppose that you are looking at the results of a survey into the source of finance for buying a house, and find the following pattern of loan size and source of loan – which is a contingency table. You might ask whether there is any relationship or association between the size of a loan and the source. You can use a chi-squared test to see whether there is any statistical association between the two.

		Size of loan			Total
		Less than £80,000	£80,000 to £150,000	More than £150,000	
Source of mortgage	Building society	30	55	40	**125**
	Bank	23	29	3	**55**
	Elsewhere	12	6	2	**20**
Total		**65**	**90**	**45**	**200**

WORKED EXAMPLE 17.17

Is there any association between the size of loan and its source in the contingency table above?

Solution

This is a hypothesis test, so we can again use the standard procedure.

1 *State the null and alternative hypotheses*. When testing for association we normally use a null hypothesis that there is no association. Then the null hypothesis, H_0, is that there is no association between the size of mortgage and its source – in other words, the two are independent. The alternative hypothesis, H_1, is that there is an association.

2 *Specify the level of significance*. We take this as 5%.

3 *Calculate the critical value of χ^2*. For a contingency table we calculate the degrees of freedom from:

degrees of freedom = (number of rows – 1) × (number of columns – 1)

Here there are three rows and three columns (ignoring the totals), so there are

$(3 - 1) \times (3 - 1) = 4$ degrees of freedom

Looking up χ^2 tables for a 5% significance level and 4 degrees of freedom gives a critical value of 9.49.

4 *Find the actual value of χ^2*. For this we have to calculate the expected number of replies in each cell of the matrix. The top left-hand cell

Worked example 17.17 continued

shows the number of people who have a loan of less than £80,000 from a building society. A total of 125 loans come from a building society, so the probability that a particular loan comes from a building society is 125/200 = 0.625. A total of 65 loans are less than £80,000, so the probability that a particular loan is less than £80,000 is 65/200 = 0.325. Then the probability that a loan comes from a building society and is less than £80,000 is $0.625 \times 0.325 = 0.203$. Since there are 200 loans, the expected number of this type is $0.203 \times 200 = 40.625$. Repeating this calculation for every cell in the matrix gives the following results.

		Size of loan			Total
		Less than £80,000	£80,000 to £150,000	More than £150,000	
Source of mortgage	Building society	40.625	56.250	28.125	**125**
	Bank	17.875	24.750	12.375	**55**
	Elsewhere	6.500	9.000	4.500	**20**
Total		**65**	**90**	**45**	**200**

```
001 > read data columns c1–c3
002 > 30 55 40
003 > 23 29 3
004 > 12 6 2
005 > column titles <80, 80–150, >150
006 > row titles building society, bank, elsewhere
007 > end data
008 > chisquare calculate c1–c3

Expected counts are printed below observed counts

                   <80     80–150    >150     Total

building society   30      55        40       125
                   40.62   56.25     28.12

bank               23      29         3        55
                    7.88   24.75     12.37

elsewhere          12       6         2        20
                    6.50   9.00       4.50

Total              65      90        45       200

ChiSq = 2.779 + 0.028 + 5.014 +
        1.469 + 0.730 + 7.102 +
        4.654 + 1.000 + 1.389 = 24.165
df = 4

* WARNING *
1 cell with expected counts less than 5.0
  Merge rows or columns?
09 > no

010 > significance = 0.05
011 > chisquare test c1–c3

Critical value of ChiSq = 9.488
Calculated value of ChiSq = 24.165

Conclusion = reject null hypothesis
```

Figure 17.10 Printout from statistics package for chi-squared test

Worked example 17.17 continued

Now we have a set of nine observed frequencies, and a corresponding set of nine expected frequencies. When we calculated the expected frequencies, we assumed that there is no connection between the loan size and its sources. Any significant differences between expected and observed values are caused by an association between the loan size and its source. The closer the association, the larger is the difference, and the larger the calculated value of χ^2. So now we have to find the actual value of χ^2, and as you can see from the following table this is 24.165.

O	*E*	*O* – *E*	$(O - E)^2$	$(O - E)^2/E$
30	40.625	– 10.625	112.891	2.779
55	56.250	–1.25	1.563	0.028
40	28.125	11.875	141.016	5.014
23	17.875	5.125	26.266	1.469
29	24.750	4.25	18.063	0.730
3	12.375	–9.375	87.891	7.102
12	6.500	5.500	30.250	4.654
6	9.000	–3.000	9.000	1.000
2	4.500	–2.500	6.250	1.389
200	200			24.165

5 *Decide whether or not to reject the null hypothesis*. The actual value (24.165) is greater than the critical value (9.49), so we reject the null hypothesis and accept the alternative hypothesis.

6 *State the conclusion*. The evidence supports the view that there is an association, and the size of a mortgage is related to its source.

Figure 17.10 shows the printout from a statistics package for this problem.

You should always be careful when doing chi-squared tests, because they do not work well if the number of expected observations in any class falls below 5. Here one cell has an expected frequency of 4.5, so we should really combine this cell with others, perhaps combining the rows for banks and other sources.

Review questions

17.18 What is a test of association?

17.19 Why would you use a statistical package for χ^2 tests?

CHAPTER REVIEW

This chapter described the approach of hypothesis testing, which sees whether a statement about a population is supported by the evidence in a sample.

- Hypothesis testing starts with a null hypothesis, which is a precise statement about a population. Then it tests a sample from the population to see whether there is evidence to support the null hypothesis. If the evidence does not support the null hypothesis, it is rejected, otherwise it cannot be rejected.
- Samples always involve uncertainty, and in hypothesis testing there are two types of error: Type I errors reject a null hypothesis that is true; and Type II errors do not reject a null hypothesis that is false.
- A significance level is the minimum acceptable probability that a value is a random sample from the hypothesised population. It is equivalent to the probability of making a Type I error.

- A common use of hypothesis testing checks that the mean of a population has a specified value. Sometimes we are interested in testing whether a mean is above or below a specified value, where we use a one-sided analysis.
- We extended the standard analysis to deal with small samples using a t-distribution, proportions of a population sharing some feature, differences between means and paired observations.
- When there is no parameter to test, typically with nominal data, we have to use a distribution-free, or non-parametric, test. We illustrated this with chi-squared tests, which are used to see whether data follows a specified distribution. You can also use a chi-squared distribution to test the association between two parameters in a contingency table.

CASE STUDY Willingham Consumer Protection Department

Willingham Consumer Protection Department (WCPD) is responsible for administering all weights and measures laws in its area of North Carolina. A part of its service makes sure that packages of food and drink contain the quantities stated. One week WCPD decided to test containers of milk. Most of these tests were done at dairies, where procedures and historical data were also examined, with other random samples taken from local shops and milk delivery services.

On two consecutive days WCPD bought 50 containers with a nominal content of four pints or 2.27 litres. The actual contents of these, in litres, are as follows.

- Day 1: 2.274 2.275 2.276 2.270 2.269 2.271 2.265 2.275 2.263 2.278 2.260 2.278 2.280 2.275 2.261 2.280 2.279 2.270 2.275 2.263 2.275 2.781 2.266 2.277 2.271 2.273 2.283 2.260 2.259 2.276 2.286 2.275 2.271 2.273 2.291 2.271 2.269 2.265 2.258 2.283 2.274 2.278 2.276 2.281 2.269 2.259 2.291 2.289 2.276 2.283
- Day 2: 2.270 2.276 2.258 2.259 2.281 2.265 2.278 2.270 2.294 2.255 2.271 2.284 2.276 2.293 2.261 2.270 2.271 2.276 2.269 2.268 2.272 2.272 2.273 2.280 2.281 2.276 2.263 2.260 2.295 2.257 2.248 2.276 2.284 2.276 2.270 2.271 2.269 2.278 2.276 2.274 2.291 2.257 2.281 2.276 2.274 2.273 2.273 2.270 2.272 2.278

When they were collecting these figures, WCPD inspectors were convinced that there were no problems with the main dairies, but some small operations were not so reliable. This was because large dairies invariably used modern, well-designed equipment, and they employed special quality assurance staff. Smaller operators tended to use older, less reliable equipment, and could not afford to run a quality assurance department. Two companies, in particular, were identified as needing further checks. WCPD took random samples of 15 containers from each of these dairies, with the following results.

- Company 1: 2.261 2.273 2.250 2.268 2.268 2.262 2.272 2.269 2.268 2.257 2.260 2.270 2.254 2.249 2.267
- Company 2: 2.291 2.265 2.283 2.275 2.248 2.286 2.268 2.271 2.284 2.256 2.284 2.255 2.283 2.275 2.276

Question

- **What could the milk inspectors report about their findings? What follow-up action could they recommend? Are there any improvements they could make to their data collection and analysis?**

PROBLEMS

17.1 The mean wage of people living in Upper Hemmington is said to be £400 a week with a standard deviation of £100. A random sample of 36 people was examined. What is the acceptance range for a 5% significance level? What is the acceptance range for a 1% significance level?

17.2 The weight of packets of biscuits is claimed to be 500 g. A random sample of 50 packets has a mean weight of 495 g and a standard deviation of 10 g. Use a significance level of 5% to see whether the data from the sample supports the original claim.

17.3 Hamil Coaches Ltd say that their long-distance coaches take 5 hours for a particular journey. Last week a consumer group tested these figures by timing a sample of 30 journeys. These had a mean time of 5 hours 10 minutes with a standard deviation of 20 minutes. What report can the consumer group make?

17.4 A food processor specifies the mean weight of a product as 200 g. A random sample of 20 has a mean of 195 g and a standard deviation of 15 g. Does this evidence suggest that the mean weight is too low?

17.5 An emergency breakdown service suggests that 50% of all drivers are registered with their service. A random sample of 100 people had 45 who were registered. Does this sample support the original claim?

17.6 Quality Managers at CentralGen say that 12% of letters they post contain errors. A sample of 200 letters was checked and 31 of them contained errors. What do these results suggest?

17.7 Health service managers say that doctors should not spend more than 2 hours a day doing paperwork. A sample of 40 doctors spends an average of 2 hours 25 minutes a day doing paperwork, with a standard deviation of 55 minutes. What does this show?

17.8 A mobile phone has an advertised life of 30,000 hours. A sample of 50 phones had a life of 28,500 hours with a standard deviation of 1,000 hours. What can you say about the advertisements?

17.9 Dorphmund Industries have two similar factories. There is some disagreement, because people working in each factory think those in the other factory are getting higher wages. A sample of wages was taken from each factory with the following results:

- Sample 1: size = 45, mean = \$250, standard deviation = \$45
- Sample 2: size = 35, mean = \$230, standard deviation = \$40

What can you say about the wages?

17.10 A car manufacturer says that its cars cost €500 a year less to maintain than those of its competitors. To test this, a consumer group found the cost of maintaining 10 cars for a year, and the mean saving was €79 with a standard deviation of €20. What does this say about the manufacturer's claim?

17.11 Five factories reported the following numbers of minor accidents in a year:

Factory	1	2	3	4	5
Number of accidents	23	45	18	34	28

Does this suggest that some factories have more accidents than others?

17.12 The following figures show the number of defective components supplied each day by a factory. Does this data follow a binomial distribution?

Number of defects	0	1	2	3	4	5
Number of days	8	22	33	29	15	3

17.13 The number of road accident victims reporting to a hospital emergency ward is shown in the following table. Do these figures follow a Poisson distribution?

Number of accidents	0	1	2	3	4	5	6
Number of days	17	43	52	37	20	8	4

17.14 Do the following figures follow a Normal distribution?

Weight (in grams)	Number of observations	Weight (in grams)	Number of observations
less than 5	5	65 to 79.99	97
5 to 19.99	43	80 to 94.99	43
20 to 34.99	74	95 to 109.99	21
35 to 49.99	103	110 and more	8
50 to 64.99	121		

17.15 Figure 17.11 shows a spreadsheet doing the calculations for a *t*-test on the mean of two samples. Explain the results and check the calculations. How could you improve the format?

	A	B	C	D	E	F	G
1	**Two samples – *t*-test**						
2							
3	**Data**			***t*-Test: Two-Sample Assuming Equal Variances**			
4	**Variable 1**	**Variable 2**					
5	10	8			***Variable 1***		***Variable 2***
6	16	10		Mean	10.6		7.786
7	13	9		Variance	12.489		4.335
8	6	6		Observations	10		14
9	8	4		Pooled Variance	7.671		
10	9	10		Hypothesised mean difference	0		
11	16	9		df	22		
12	9	9		*t* Stat	2.454		
13	12	7		*P*(*T*<=*t*) one-tail	0.011		
14	7	4		*t* Critical one-tail	1.717		
15		10		*P*(*T*<=*t*) two-tail	0.023		
16		9		*t* Critical two-tail	2.074		
17		6					
18		8					

Figure 17.11 Calculations for Problem 17.15

RESEARCH PROJECTS

17.1 The chapter mentioned several examples of automatic systems that implicitly include hypothesis tests – including airport security systems, email spam filters, Internet search results, automatic recorders of earthquakes, burglar alarms and roadside speed cameras. What other example can you find? How do such systems actually incorporate hypothesis testing?

17.2 Hypothesis testing comes in many different forms, and it always seems to involve judgement. This makes it difficult to design a package that automatically takes data and does an appropriate hypothesis test. Do a small survey to see what facilities statistical packages have for hypothesis testing. How do they get around the practical problems?

17.3 Supermarkets and other retailers often claim that they offer the lowest prices in their area. How can you check their claims? Collect some data from competing stores and analyse the results. What conclusions can you reach?

17.4 Often a hypothesis may seem 'obvious' but on closer examination there is no evidence to support it. Find some real examples of this effect. What are the consequences?

Sources of information

Reference

1 Hooke R., *How to Tell the Liars from the Statisticians*, Marcel Dekker, New York, 1983.

Further reading

There are virtually no books specifically about hypothesis testing – and the odd ones are very technical. The best place to look is in general statistics books, like the ones listed in Chapter 14.

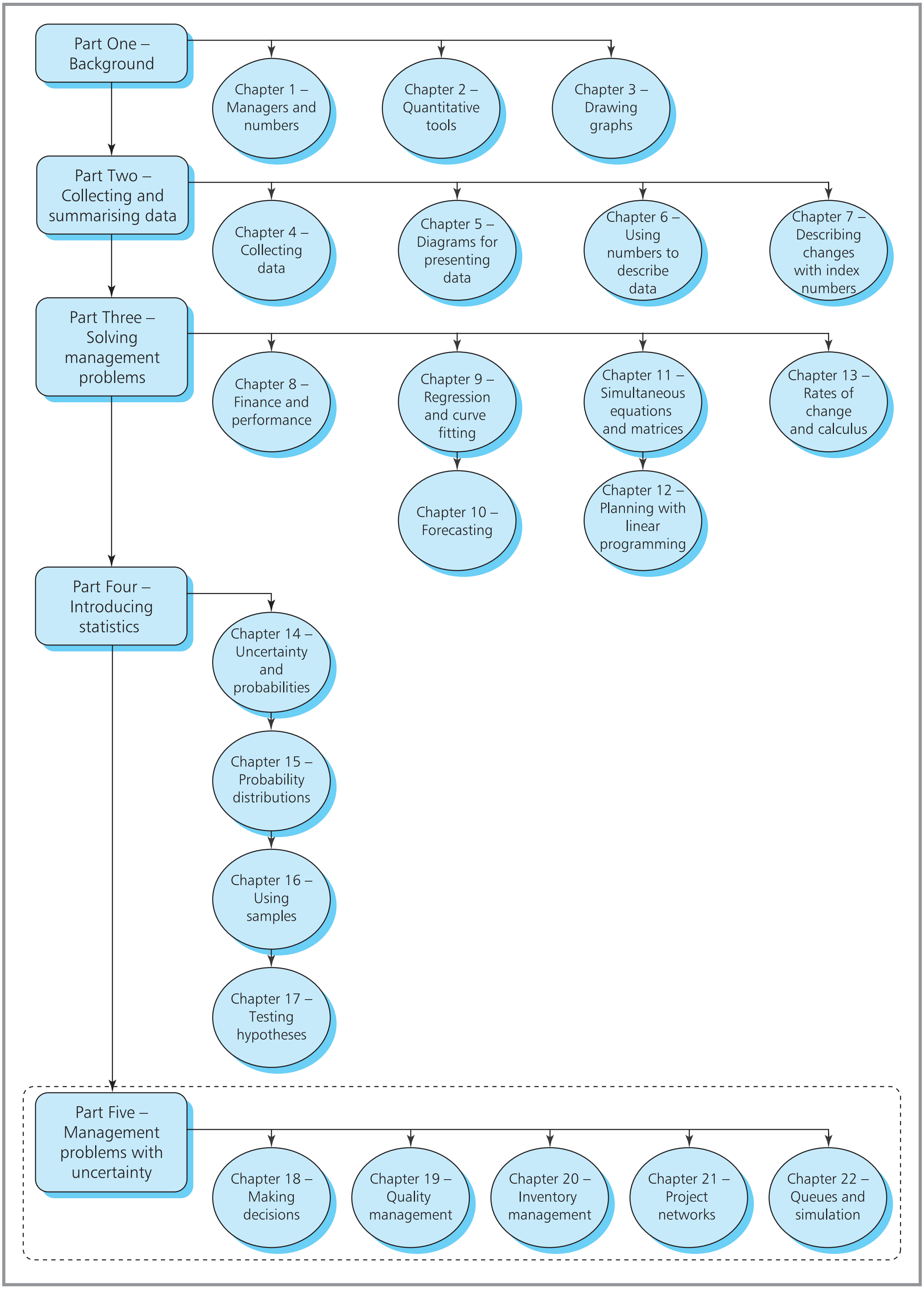
PART 5

Management problems with uncertainty

This book is divided into five parts, each of which covers a different aspect of quantitative methods. The first part gave the background and context for the rest of the book. The second part showed how to collect, summarise and present data. The third part used this data to solve deterministic problems, where we knew conditions with certainty. The fourth part showed how uncertainty can be measured and analysed using probabilities. This is the fifth part, which uses statistical ideas to tackle problems that contain uncertainty.

There are five chapters in this part. Chapter 18 describes decision analysis, which allows managers to give structure to problems and make decisions in conditions of uncertainty. Chapter 19 looks at the use of statistics in quality control and broader quality management. Chapter 20 describes some models for inventory management, while Chapter 21 shows how to use network analysis for planning and scheduling projects. Chapter 22 looks at the management of queues, and broader uses of simulation.

Map 5 shows how these chapters fit into the rest of the book.



Map 5 Map of chapters – Part Five

CHAPTER 18

Making decisions

Contents

Chapter outline

This chapter discusses the ways in which managers can approach decision-making. It starts with the idea that managers work in complex and uncertain conditions, and they need some structure to help with their decisions. We describe several ways of giving this structure, starting with simple maps and moving on to payoff matrices and decision trees. These structures can help analyse problems in different circumstances, including certainty, uncertainty and risk.

After finishing this chapter you should be able to:

- Appreciate the need to structure decisions and draw maps of problems
- List the main elements of a decision and construct a payoff matrix
- Make decisions under certainty
- Describe situations of uncertainty and use decision criteria to suggest decisions
- Describe situations of risk and use expected values to suggest decisions
- Use Bayes' theorem to update conditional probabilities
- Appreciate the use of utilities
- Use decision trees to solve problems with sequential decisions.

Giving structure to decisions

Everybody has to make decisions – choosing the best car to buy, whether to invest in a personal pension, where to eat, which software to use, where to go on holiday, which phone to buy, and whether to have tea or coffee. These decisions come in a steady stream. Most of them are fairly unimportant and we can make them using a combination of experience, intuition, judgement and common sense. But when decisions are more important, we have to use a more rigorous approach. For instance, suppose that you work for a company that is not making enough profit. Two obvious remedies are to reduce costs or to increase prices. But if the company increases prices, demand may fall – while reducing the costs might allow a lower price and demand may rise. If demand changes, the company may have to reschedule operations, change capacity and adjust marketing strategies. But changing the operations schedules can affect the flows of materials in supply chains, stocks, employment prospects, and so on. And then changing the flows of materials can affect relations with suppliers.

We could continue with these more or less random thoughts, showing how one adjustment triggers a whole series of related changes. But you already get the idea that interactions are complex; it is easy to get bogged down in the detail and lose track of the main arguments. Managers always work in complex conditions, where every decision has a series of consequences, not all of which are foreseeable. They try to balance many factors, but the job of juggling all possible consequences is very difficult. However, a useful starting point is to have a way of describing the interactions. A **problem map** – sometimes called a **relationship diagram** or **mind map** – gives a simple diagram for doing this. Figure 18.1 shows the start of a problem map for the discussion above.

As you can see, this map gives an informal way of presenting a stream of connected ideas, showing the interactions, and giving some structure to a problem. You can extend these basic diagrams to add features and symbols to give a more formal structure, but they do not really help in making a decision, as they present the circumstances but do not identify the best options or even give more information. If we want methods that actually help with decisions, we have to do some more analysis. We can start this by looking at the features of a decision. In any situation where a decision is needed there must be:

- a decision maker – the manager – who is responsible for making the decision
- a number of alternatives available to the decision maker
- an aim of choosing the best alternative
- after the decision has been made, events occurring over which the decision maker has no control
- each combination of an alternative chosen followed by an event happening leading to an outcome that has some measurable value.

To illustrate these elements, consider someone who owns a house valued at €200,000, and who has to decide whether to take out fire insurance at an annual cost of €600. The decision maker is the person who owns the house; they have an aim of minimising costs and must choose the best alternative from:

1 insure the house, or
2 do not insure the house.

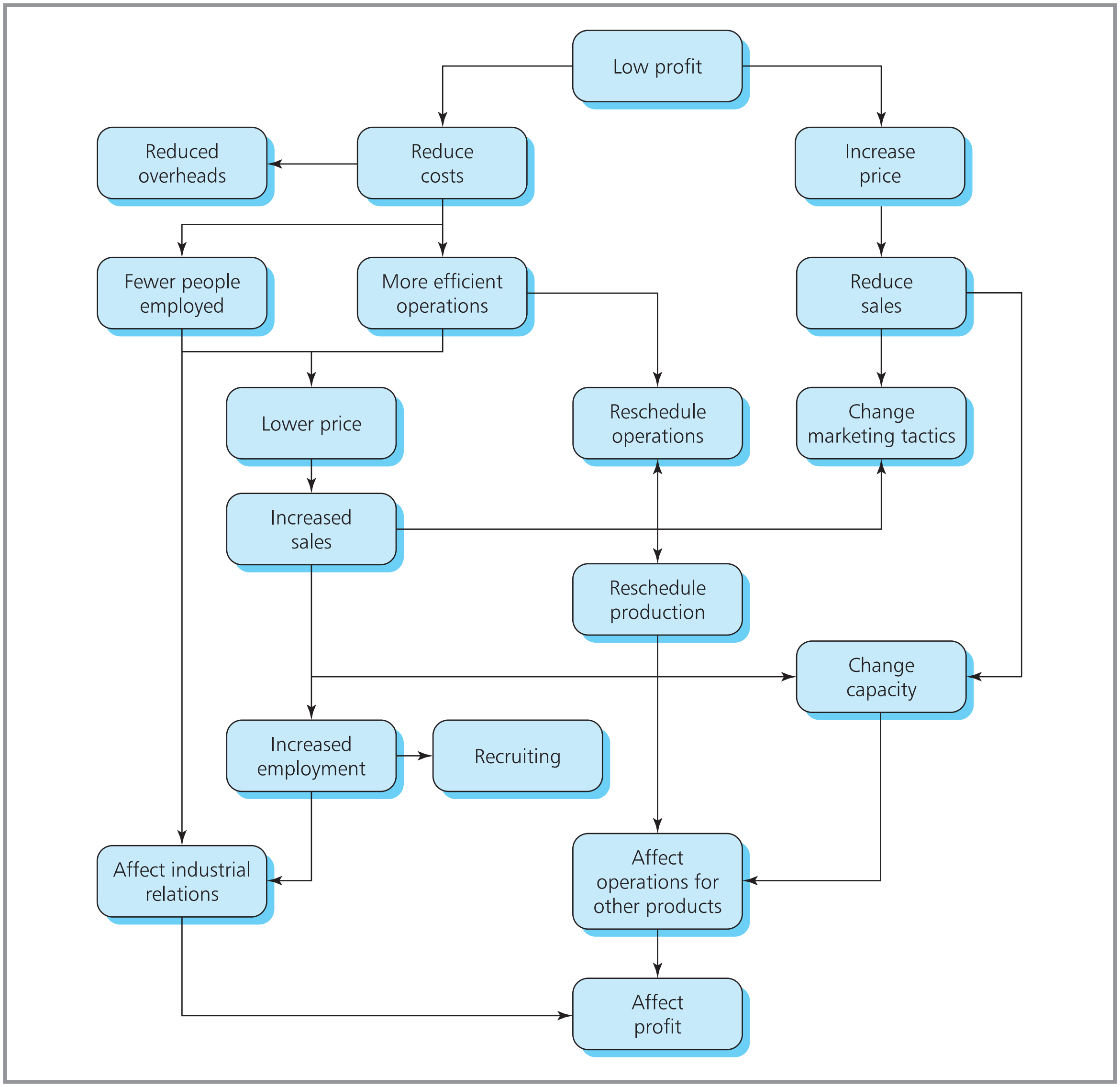


Figure 18.1 Start of a problem map for increasing profit

Then an event happens, but the decision maker cannot control whether it is:

1 the house burns down, or
2 the house does not burn down.

Each combination of choice (insure or not insure) and event (burns down or does not) has an outcome that we can show in the following table. This is a **payoff matrix** or **payoff table** and it shows the cost to the house owner of every combination of alternative and event.

		Event	
		House burns down	House does not burn down
Alternative	Insure house	€600	€600
	Do not insure house	€200,000	€0

Obviously, we have simplified the problem here, and in reality there is a choice of many insurance companies and policies, the house may be damaged but not destroyed by fire, there may be costs of inconvenience, and so on. But the payoff matrix shows the underlying structure of the problem.

Review questions

18.1 Why are maps and payoff matrices useful?

18.2 What are the five main elements in a decision?

Decision making with certainty

The characteristic of decision making under certainty is that we know, with certainty, which event will occur. So we have to consider only the one event we know will happen, and the payoff matrix has only one column. Then the method of solution is obvious – as we list the outcomes for each alternative and simply choose the alternative that gives the best outcome.

Suppose you have $1,000 to invest for a year. With certainty, you can list all the alternatives that you want to consider and get the following payoff matrix.

		Event
		Earns interest
Alternative	Bank	$1,065
	Building society	$1,075
	Government stock	$1,085
	Stock market	$1,100
	Others	$1,060

Notice that this time the outcomes are benefits rather than costs. And again, this is clearly a simplified view, as we have used forecast returns for investments, and included a wide range of alternatives in ‘others’. There is only one event – ‘earns interest’ – and by looking down the list of outcomes you can identify the best alternative. The highest value at the end of the year comes from investing in the stock market.

In reality, even decisions under certainty can be difficult. For instance, would it be better for a health service to invest money in providing more kidney dialysis machines, giving nurses higher wages, doing open-heart surgery, funding research into cancer, or providing more parking spaces at hospitals? Most decisions contain subjectivity, and even when managers know all the circumstances – which is almost impossible – they are still unlikely to agree on the best decision. Politicians can agree on an aim of economic growth, and they have as much information as is practical to collect – but they rarely agree on the means of achieving this.

WORKED EXAMPLE 18.1

The manager of La Pigalle Restaurant has a booking for a large wedding banquet. She has a number of ways of providing staff, each with different costs. Her most convenient alternatives are to pay full-time staff to work overtime (costing £600), hire current part-time staff for the day (£400), hire new temporary staff (£500) or use an agency (£750). What is her best alternative?

Solution

The payoff matrix for this decision under certainty is shown below. The entries are costs, so we want to identify the lowest. This is £400 for hiring current part-time staff for the day.

		Event
		Pay staff
Alternative	Pay full-time staff for overtime	£600
	Hire current part-time staff	£400
	Hire new temporary staff	£500
	Use an agency	£750

Review questions

18.3 What is meant by 'decision making with certainty'?

18.4 In reality, are you ever likely to meet decisions with certainty?

18.5 Are decisions with certainty always trivial?

Decision making with uncertainty

Most decisions do not have a single event that will definitely occur, but there are several possible events. This uncertainty means that we can list events that might occur, but we do not know in advance which one actually will happen. Often we cannot even give realistic probabilities to the events. For example, when you decide to accept a job offer, a number of events can happen: you may not like the new job and quickly start looking for another; you may get the sack; you may like the job and stay; you may be moved by the company. One event will occur, but they are largely outside your control and it is impossible even to give reliable probabilities.

When we cannot give probabilities to events, we are dealing with **uncertainty** or **strict uncertainty**. The most common way of solving problems with uncertainty is to use simple rules – called **decision criteria** – to recommend a solution. There are many different criteria, and we will show how they work by using three common ones.

Laplace decision criterion

As we cannot give probabilities to the events, the Laplace criterion says that we should treat them as equally likely. The procedure is as follows:

1 For each alternative find the mean value of the outcomes (that is, the average of each row in the payoff matrix).
2 Choose the alternative with the best average outcome (which is the lowest average cost or the highest average gain).

WORKED EXAMPLE 18.2

A restaurateur is going to set up a cream tea stall at a local gala. On the morning of the gala she visits the wholesale market and has to decide whether to buy a large, medium or small quantity of strawberries, scones, cream and other materials. Her profit depends on the number of people attending the gala, and this in turn depends on the weather. If her matrix of gains (in thousands of pounds) for different weather conditions is given below, what quantity of materials should she buy?

		Event – weather is		
		good	average	poor
Alternative – buy	large quantity	10	4	–2
	medium quantity	7	6	2
	small quantity	4	1	4

Solution

Following the procedure described:

1 Take the average value of outcomes for each alternative:
- Large quantity: (10 + 4 – 2)/3 = 4
- Medium quantity: (7 + 6 + 2)/3 = **5** (best)
- Small quantity: (4 + 1 + 4)/3 = 3

2 Choose the best average outcome. As these figures are profits, the best is the highest, which is to buy a medium quantity.

Wald decision criterion

Most organisations have limited resources and cannot afford to risk a big loss. This is the basis of the Wald decision criterion, which assumes that decision makers are cautious – or even pessimistic – and want to avoid big potential losses. The steps are:

1 For each alternative find the worst outcome.
2 Choose the alternative from the best of these worst outcomes.

With a payoff matrix showing costs, this is sometimes known as the 'minimax cost' criterion, as it looks for the maximum cost of each alternative and then chooses the alternative from the minimum of these – expressed as the minimum[maximum cost].

WORKED EXAMPLE 18.3

Use the Wald decision criterion on the example of the cream tea stall described in worked example 18.2.

Solution

Following the procedure described:

1 Find the worst outcome for each alternative, and as the entries are gains the worst is the lowest:
- Large quantity: minimum of [10, 4, –2] = –2
- Medium quantity: minimum of [7, 6, 2] = 2
- Small quantity: minimum of [4, 1, 4] = 1

		Event – weather is			
		good	average	poor	Worst
Alternative – buy	large quantity	10	4	–2	–2
	medium quantity	7	6	2	**2** (best)
	small quantity	4	1	4	1

2 Choose the best of these worst outcomes. As the figures are profits, the best is the highest (in this case, 2), which comes from buying a medium quantity.

Savage decision criterion

Sometimes we are judged not by how well we actually did, but by how well we could possibly have done. Students who get 70% in an exam might be judged by the fact that they did not get 100%; investment brokers who advised a client to invest in platinum may be judged not by the fact that platinum rose 15% in value, but by the fact that gold rose 25%. This happens particularly when performance is judged by someone other than the decision maker.

At such times there is a **regret**, which is the difference between actual outcome and best possible outcome. A student who gets 70% has a regret of 100 – 70 = 30%; an investor who gains 15% when they could have gained 25% has a regret of 25 – 15 = 10%. And if you choose the best option, there is clearly no regret. The Savage criterion is based on these regrets. It is essentially pessimistic and minimises the maximum regret, with the following steps:

1. For each event find the best possible outcome (that is, the best entry in each column of the payoff matrix).
2. Find the regret for every entry in the column, which is the difference between the entry itself and the best entry in the column.
3. Put the regrets found in Step 2 into a 'regret matrix'. There should be at least one zero in each column (for the best outcome) and regrets are always positive.
4. For each alternative find the highest regret (that is, the highest number in each row).
5. Choose the alternative with the best (that is, lowest) of these highest regrets.

As you can see, steps 1 to 3 build a regret matrix, and then steps 4 and 5 apply the Wald criterion to the regret matrix.

WORKED EXAMPLE 18.4

Use the Savage decision criterion on the example of the cream tea stall described in worked example 18.2.

Solution

1. The best outcome for each event is underlined (that is, with good weather a large quantity, with average weather a medium quantity, and with poor weather a small quantity).

		Event – weather is		
		good	average	poor
Alternative – buy	large quantity	<u>10</u>	4	–2
	medium quantity	7	<u>6</u>	2
	small quantity	4	1	<u>4</u>

2. The regret for every other entry in the column is the difference between this underlined value and the actual entry. So when the weather is good and the caterer bought a medium quantity, the regret is 10 – 7 = 3; when the weather is good and the caterer bought a small quantity the regret is 10 – 4 = 6; when the weather is good and the caterer bought a large quantity, the regret is zero. Repeat this calculation for every column.
3. Put the regrets into a matrix, replacing the original profit figures.

		Event – weather is			
		good	average	poor	Worst
Alternative – buy	large quantity	0	2	6	6
	medium quantity	3	0	2	**3** (best)
	small quantity	6	5	0	6

4. For each alternative find the highest regret:
   - Large quantity: maximum of [0, 2, 6] = 6
   - Medium quantity: maximum of [3, 0, 2] = **3** (best)
   - Small quantity: maximum of [6, 5, 0] = 6
5. Choose the alternative with the lowest of these maximum regrets. This is the medium quantity.

Choosing the criterion to use

Different criteria often suggest the same alternative (as you can see in the worked examples above), and this can reduce the importance of finding the 'right' one for a particular problem. But there is no guarantee of this – and when they recommend different alternatives you should choose the most relevant. For example, if you are working as a consultant and other people judge the quality of your decisions, you might use the Savage criterion; if the decision is made for a small company that cannot afford high losses, then Wald may be best; if there really is nothing to choose between different events, Laplace may be useful.

Although it is difficult to go beyond such general guidelines, you should notice one other factor. Both the Wald and Savage criteria effectively recommend their decision based on one outcome – the worst for Wald and the one that leads to the highest regret for Savage. So the choice might be dominated by a few atypical results. The Laplace criterion is the only one that uses all values to make its recommendation.

Of course, you might have a problem that does not suit any of the criteria we have described – but remember that we have only given some illustrations and there are many other options. For example, an ambitious organisation might aim for the highest profit and use a criterion that suggests the alternative with the highest return (a 'maximax profit' criterion). Or it may try to balance the best and worst outcomes for each event and use a criterion based on the value of:

$$\alpha \times \text{best outcome} + (1 - \alpha) \times \text{worst outcome}$$

where α is a parameter between 0 and 1.

Remember that decision criteria are useful tools – and their strength is not necessarily in identifying the best alternative, but to give structure to a problem, show relationships, and allow an informed debate of options.

WORKED EXAMPLE 18.5

Lawrence Pang has a problem with the following payoff matrix of costs. Use the Laplace, Wald and Savage decision criteria to show the best alternatives.

		Event		
		1	2	3
Alternative	A	14	22	6
	B	19	18	12
	C	12	17	15

Solution

As the entries are costs, Laplace recommends the alternative with the lowest average costs – which is alternative A.

	1	2	3	Mean
A	14	22	6	**14.0** (best)
B	19	18	12	16.3
C	12	17	15	14.7

Worked example 18.5 continued

Wald assumes that the highest cost will occur for each alternative, and then chooses the lowest of these – which is alternative C.

	1	2	3	Highest
A	14	22	6	22
B	19	18	12	19
C	12	17	15	**17** (best)

Savage forms the regret matrix, finds the highest regret for each alternative, and chooses the alternative with the lowest of these – which is alternative A.

Regret	1	2	3	Highest
A	2	5	0	**5** (best)
B	7	1	6	7
C	0	0	9	9

Review questions

18.6 What is meant by decision making under strict uncertainty?

18.7 List three useful decision criteria.

18.8 How many of these criteria take into account all outcomes for the alternatives?

18.9 Are the criteria described the only ones available? If not, can you suggest others that might be useful?

IDEAS IN PRACTICE Paco Menendes

Paco Menendes ran a plumbing wholesale business based in the Mexican city of Guadalajara. In the late 1990s he developed a simple valve mechanism for controlling the flow of water in domestic solar heating systems. He had to decide how to market his idea, and in the short term his options can be summarised as sell the valve locally, sell nationally through a website, enter a partnership with an existing company, or sell the patent. His returns depend on demand, which he described as high, medium or low. Using this simple model, he developed the matrix of potential annual gains shown in Figure 18.2.

He started with a decision criterion that balanced the best and worst outcomes comparing alternatives by:

$$\alpha \times \text{best outcome} + (1 - \alpha) \times \text{worst outcome}$$

This is the Hurwicz criterion, where α is chosen to show how optimistic the decision maker is. Paco Menendes used a value of 0.4, showing that he was slightly pessimistic. The criterion suggested that his best option was to sell through a website. However, when he explored his options more carefully, he decided to go into partnership with a national distributor.

	A	B	C	D	E	F	G
1	**Annual gains for Paco Menendes**						
2							
3				**Demand**		**Alpha**	
4			**High**	**Medium**	**Low**	**0.4**	
5	**Options**	**Market locally**	50	25	–20	8	
6		**Use website**	85	55	–10	28	Best
7		**Partnership**	40	25	10	22	
8		**Sell patent**	25	25	25	25	

Figure 18.2 Calculations for Paco Menendes

Decision making with risk

With uncertainty we know that there are a number of possible events, one of which will occur – but we have no idea of the likelihood of each event. With decision making under risk, we again have a number of events, but now we can give each of them a probability. We should include every relevant event, so these probabilities add to 1. A simple example of decision making under risk is spinning a coin. Possible events are the coin coming down heads or tails; the probability of each of these is 0.5, and these add to 1.

Expected values

The usual way of solving problems with risk is to calculate the **expected value** of each alternative and choose the alternative with the best expected value. The expected value of each alternative is defined as the sum of the probabilities multiplied by the value of the outcomes.

expected value = $\sum$ (probability of event $\times$ value of outcome)

If you spin a coin and win €20 if it comes down heads and lose €40 if it comes down tails:

$$\text{expected value} = 0.5 \times 20 - 0.5 \times 40 = -10$$

The expected value of an alternative is the average gain (or cost) when the event is repeated a large number of times. It is not the value of every event. With every spin of the coin you will either win €20 or lose €40, but over the long term you would expect to lose an average of €10 on every spin.

For decision making with risk there are two steps:

1. Calculate the expected value for each alternative.
2. Choose the alternative with the best expected value (that is, the highest value for gains, and the lowest value for costs).

WORKED EXAMPLE 18.6

What is the best alternative for the following matrix of gains?

		Event			
		1	2	3	4
		$P = 0.1$	$P = 0.2$	$P = 0.6$	$P = 0.1$
Alternative	A	10	7	5	9
	B	3	20	2	10
	C	3	4	11	1
	D	8	4	2	16

Solution

The expected value for each alternative is the sum of the probability times the value of the outcome:

- Alternative A:
  $0.1 \times 10 + 0.2 \times 7 + 0.6 \times 5 + 0.1 \times 9 = 6.3$
- Alternative B:
  $0.1 \times 3 + 0.2 \times 20 + 0.6 \times 2 + 0.1 \times 10 = 6.5$
- Alternative C:
  $0.1 \times 3 + 0.2 \times 4 + 0.6 \times 11 + 0.1 \times 1 = \mathbf{7.8}$ (best)
- Alternative D:
  $0.1 \times 8 + 0.2 \times 4 + 0.6 \times 2 + 0.1 \times 16 = 4.4$

As these are gains, the best alternative is C with an expected value of 7.8. If this decision is made repeatedly, the average return in the long run will be 7.8; if the decision is made only once, the gain could be any of the four values 3, 4, 11 or 1.

WORKED EXAMPLE 18.7

H.J. Symonds Haulage Contractors bids for a long-term contract to move newspapers from a printing works to wholesalers. It can submit one of three tenders: a low one that assumes newspaper sales will increase and unit transport costs will go down; a medium one that gives a reasonable return if newspaper sales stay the same; or a high one that assumes newspaper sales will decrease and unit transport costs will go up. The probabilities of newspaper sales and profits (in thousands of pounds) for the firm are shown in the following table. Based on this information, which tender should it submit?

		Newspaper sales		
		decrease	stay the same	increase
		$P = 0.4$	$P = 0.3$	$P = 0.3$
Alternative	low tender	10	15	16
	medium tender	5	20	10
	high tender	18	10	–5

Solution

Calculating the expected value for each alternative:

- Low tender:
  $0.4 \times 10 + 0.3 \times 15 + 0.3 \times 16 =$ **13.3** (best)
- Medium tender:
  $0.4 \times 5 + 0.3 \times 20 + 0.3 \times 10 = 11.0$
- High tender:
  $0.4 \times 18 + 0.3 \times 10 - 0.3 \times 5 = 8.7$

As these are profits, the best alternative is the one with highest expected value, which is the low tender.

Using Bayes' theorem to update probabilities

In Chapter 14 we showed how Bayes' theorem updates conditional probabilities:

$$\textbf{Bayes' theorem: } P(a/b) = \frac{P(b/a) \times P(a)}{P(b)}$$

where: $P(a/b)$ = probability of a happening given that b has already happened
$P(b/a)$ = probability of b happening given that a has already happened
$P(a)$, $P(b)$ = probabilities of a and b respectively.

The following example shows how Bayes' theorem can help with decisions under uncertainty.

WORKED EXAMPLE 18.8

The crowd for a sports event might be small (with a probability of 0.4) or large. The organisers can pay a consultant to collect and analyse advance ticket sales a week before the event takes place. Then advanced sales can be classified as high, average or low, with the probability of advanced sales conditional on crowd size given by the following table.

		Advance sales		
		high	average	low
Crowd size	large	0.7	0.3	0.0
	small	0.2	0.2	0.6

Worked example 18.8 continued

The organisers must choose one of two plans in running the event, and the table below gives the net profit in thousands of euros for each combination of plan and crowd size.

		Plan 1	Plan 2
Crowd size	large	20	28
	small	18	10

If the organisers use information on advance sales, what decisions would maximise their expected profits? How much should they pay for the information on advance sales?

Solution

We start by defining the abbreviations:

- CL and CS for crowd size large and crowd size small
- ASH, ASA and ASL for advance sales high, average and small.

If the organisers do not use the information on advance sales, the best they can do is use the probabilities of large and small crowds (0.6 and 0.4 respectively) to calculate expected values for the two plans.

- Plan 1: $0.6 \times 20 + 0.4 \times 18 = 19.2$
- Plan 2: $0.6 \times 28 + 0.4 \times 10 =$ **20.8** (better plan)

Then they should use plan 2 with an expected profit of €20,800.

The information on advance ticket sales gives the conditional probabilities *P*(ASH/CL), *P*(ASH/CS), etc. The organisers would like these the other way around, *P*(CL/ASH), *P*(CS/ASH), etc. – and for this they use Bayes' theorem. The calculations for Bayes' theorem are shown in the following table (if you have forgotten the details of these calculations, refer back to Chapter 14).

	ASH	ASA	ASL		ASH	ASA	ASL
CL	0.7	0.3	0.00	0.6	0.42	0.18	0.00
CS	0.2	0.2	0.6	0.4	0.08	0.08	0.24
					0.50	0.26	0.24
				CL	0.84	0.69	0.00
				CS	0.16	0.31	1.00

The probability that advance sales are high is 0.5. If this happens, the probability of a large crowd is 0.84 and the probability of a small crowd is 0.16. Then, if the organisers choose plan 1, the expected value is $0.84 \times 20 + 0.16 \times 18 = 19.68$; if the organisers choose plan 2, the expected value is $0.84 \times 28 + 0.16 \times 10 = 25.12$. So with high advance sales the organisers should choose plan 2 with an expected profit of €25,120.

Extending this reasoning to the other results gives the following expected values and best choices:

- ASH: Plan 1: $0.84 \times 20 + 0.16 \times 18 = 19.68$
  Plan 2: $0.84 \times 28 + 0.16 \times 10 =$ **25.12** (better plan)
- ASA: Plan 1: $0.69 \times 20 + 0.31 \times 18 = 19.38$
  Plan 2: $0.69 \times 28 + 0.31 \times 10 =$ **22.42** (better plan)
- ASL: Plan 1: $0.00 \times 20 + 1.00 \times 18 =$ **18.00** (better plan)
  Plan 2: $0.00 \times 28 + 1.00 \times 10 = 10.00$

The decisions that maximise the organiser's profit are: when advance sales are high or average, choose plan 2; when advance sales are low, choose plan 1.

We can go one step further with this analysis, as we know that the probability of high, average and low advance sales are respectively 0.5, 0.26 and 0.24. So we can calculate the overall expected value of following the recommended decisions as:

$$0.5 \times 25.12 + 0.26 \times 22.42 + 0.24 \times 18.00 = 22.71$$

So the expected profit of using the advance sales information is €22,710. This compares with €20,800 when the advance sales information is not used, and the benefit of using the additional information is $22{,}710 - 20{,}800 =$ €1,910, or over 9%.

WORKED EXAMPLE 18.9

Humbolt Oil drills an exploratory well in deep water off the Irish coast. The company is uncertain about the amount of recoverable oil it will find, but experience suggests that it might be minor (with a probability of 0.3), significant (with probability 0.5) or major. The company has to decide how to develop the find and has a choice of either moving quickly to minimise its long-term debt, or moving slowly to guarantee continuing income. The profits for every combination of size and development speed are given in the following table, where entries are in millions of dollars.

		Size of find		
		minor	significant	major
Development	quickly	100	130	180
	slowly	80	150	210

Further geological tests can give a more accurate picture of the size of the find, but these cost $2.5 million and are not entirely accurate. The tests give three results – A, B and C – with the following conditional probabilities of results given the size of find.

		Test result		
		A	B	C
Find size	minor	0.3	0.4	0.3
	significant	0.5	0.0	0.5
	major	0.25	0.25	0.5

What should Humbolt do to maximise its expected profits?

Solution

We define the abbreviations:

- MIN, SIG and MAJ for minor, significant and major finds
- QUICK and SLOW for the quick and slow development.

Without using further geological testing, the probabilities of minor, significant and major finds are 0.3, 0.5 and 0.2 respectively, so the expected profits for each speed of development are:

- QUICK: $0.3 \times 100 + 0.5 \times 130 + 0.2 \times 180 = 131$
- SLOW: $0.3 \times 80 + 0.5 \times 150 + 0.2 \times 210 = \mathbf{141}$ (better)

The company should develop the find slowly with an expected value of $141 million.

The information from further geological tests is in the form *P*(A/MIN), *P*(B/SIG), etc., but Humbolt want it in the form *P*(MIN/A), *P*(SIG/B), etc. It finds these using Bayes' theorem.

	A	B	C		A	B	C
MIN	0.3	0.4	0.3	0.3	0.09	0.12	0.09
SIG	0.5	0.0	0.5	0.5	0.25	0.00	0.25
MAJ	0.25	0.25	0.5	0.2	0.05	0.05	0.10
					0.39	0.17	0.44
				MIN	0.23	0.71	0.20
				SIG	0.64	0.00	0.57
				MAJ	0.13	0.29	0.23

If the test result is A, the probabilities of minor, significant and major finds are 0.23, 0.64 and 0.13 respectively. Developing the well quickly gives an expected profit of $0.23 \times 100 + 0.64 \times 130 + 0.13 \times 180 = 129.6$. Repeating this calculation for the other results gives the following expected values and best choices:

- Result A: QUICK: $0.23 \times 100 + 0.64 \times 130 + 0.13 \times 180 = 129.6$
  SLOW: $0.23 \times 80 + 0.64 \times 150 + 0.13 \times 210 = \mathbf{141.7}$ (better)
- Result B: QUICK: $0.71 \times 100 + 0.00 \times 130 + 0.29 \times 180 = \mathbf{123.2}$ (better)
  SLOW: $0.71 \times 80 + 0.00 \times 150 + 0.29 \times 210 = 117.7$
- Result C: QUICK: $0.20 \times 100 + 0.57 \times 130 + 0.23 \times 180 = 135.5$
  SLOW: $0.20 \times 80 + 0.57 \times 150 + 0.23 \times 210 = \mathbf{149.8}$ (better)

The best policy is to develop the field slowly with test result A or C, and develop it quickly with test result B. As the probabilities of test results A, B and C are 0.39, 0.17 and 0.44 respectively, the overall expected profit is:

$$0.39 \times 141.7 + 0.17 \times 123.2 + 0.44 \times 149.8 = 142.12$$

Worked example 18.9 continued

The profit without doing further tests is $141 million, while doing the test raises it to $142.12 minus the cost of $2.5 million. Clearly it is not worth doing the tests, and would not be worth doing them unless their cost is less than $1.12 million.

As you can see, analyses of this kind can become quite messy, but you can find standard software to help. Figure 18.3 shows the results for Humbolt Oil from an add-in to Excel.

	A	B	C	D	E	F	G	H
1	**Humbolt Oil**		**Expected profit from drilling**					
2								
3	**First analysis**							
4								
5			**Size of find**					
6		**Minor**	**Significant**	**Major**		**Expected value**		
7	**Probability**	0.3	0.5	0.2				
8	**Quick dev.**	100	130	180		131		
9	**Slow dev.**	80	150	210		141	****	
10								
11	**Bayes analysis**							
12								
13			**Conditional**		**Prior**		**Revised**	
14		**A**	**B**	**C**		**A**	**B**	**C**
15	**Minor**	0.3	0.4	0.3	0.3	0.09	0.12	0.09
16	**Significant**	0.5	0	0.5	0.5	0.25	0	0.25
17	**Major**	0.25	0.25	0.5	0.2	0.05	0.05	0.1
18						0.39	0.17	0.44
19				**Predictions**	**Minor**	0.23	0.71	0.20
20					**Significant**	0.64	0.00	0.57
21					**Major**	0.13	0.29	0.23
22								
23	**Calculations**							
24								
25					**Probabilities**		**Expected value**	
26	**A**	**Quick dev.**	129.6		0.39			
27		**Slow dev.**	141.7	****	0.39		55.26	
28	**B**	**Quick dev.**	123.2	****	0.17		20.94	
29		**Slow dev.**	117.7		0.17			
30	**C**	**Quick dev.**	135.5		0.44			
31		**Slow dev.**	149.8	****	0.44		65.91	
32						**Total**	**142.12**	
33								
34	**Conclusion**							
35	First analysis		141.00					
36	Using Bayes		142.12	****				
37	Extra profit		1.12					

Figure 18.3 Spreadsheet of calculations for Humbolt Oil

Utilities

Expected values are easy to use but they have some drawbacks. In particular, they do not always reflect real preferences. Have a look at the investment in the following payoff matrix.

		Event	
		gain	lose
		$P = 0.1$	$P = 0.9$
Alternative	invest	£500,000	–£50,000
	do not invest	£0	£0

The expected values are:

- Invest: $0.1 \times 500{,}000 - 0.9 \times 50{,}000 = £5{,}000$
- Do not invest: $0.1 \times 0 + 0.9 \times 0 = £0$

Expected values suggest investing – but you can see that there is a 90% chance of losing money. The reason is that the expected value shows the average return when a decision is repeated a large number of times, but it does not show the value for a single decision. When there is a single decision, the expected value can give misleading advice. In this example, if you repeat the decision many times, you will, on average, gain £5,000 – but if you make only one decision you are likely to lose £50,000.

Another weakness with expected values is that they assume a linear relationship between the amount of money and its value. So €1,000 has a value a thousand times as great as €1, and €1,000,000 has a value a thousand times as great as €1,000. In practice, this rigid linear relationship is not accurate. **Utilities** give a more accurate view of the value of money. Figure 18.4 shows a utility function with three distinct regions. At the top, near point A, the

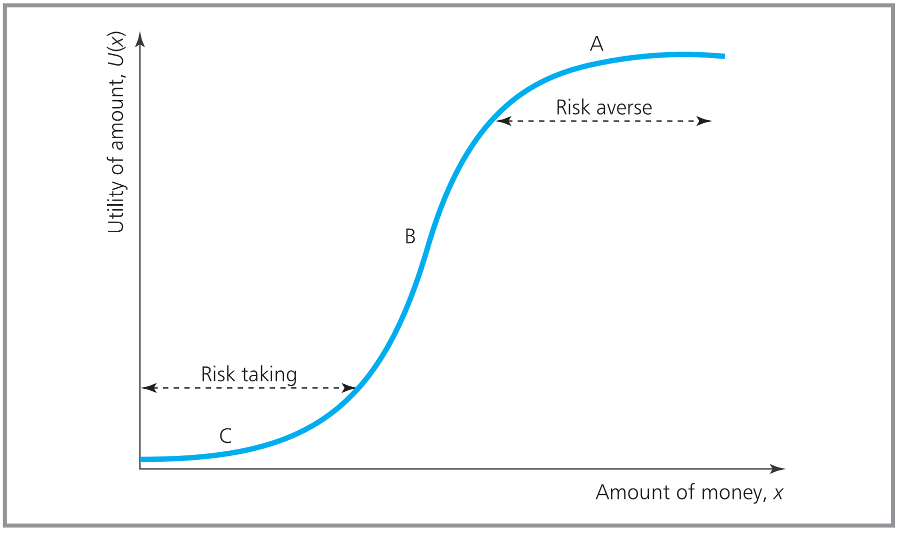


Figure 18.4 Utility curve showing the value of money

utility is rising slowly with the amount of money. A decision maker in this region already has a lot of money and would not put a high value on even more. However, the decision maker would certainly not like to lose money and move nearer to point B where the utility falls rapidly. Gaining more money is not very attractive, but losing it is very unattractive – so this suggests a conservative decision maker who does not take risks.

Region B on the graph has the utility of money almost linear, which is the assumption of expected values. A decision maker here is likely to look for a balance of risk and benefit. Finally, a decision maker at point C has little money, so losing some would not appreciably affect their utility. On the other hand, gaining money and moving nearer to B would have a high value. A decision maker here is keen to make a gain and does not unduly mind a loss – which suggests a risk taker.

Utilities are useful in principle, but it is very difficult to define a reasonable function. Each individual and organisation has a different view of the value of money and works with a different utility function. And to make things even more difficult, these curves change quickly over time – you might notice that you feel confident and risk-taking in the morning and conservative and risk-averse in the afternoon. In principle, though, when we can establish a reasonable utility function, the process of choosing the best alternative is the same as with expected values, but with expected utilities replacing expected values.

WORKED EXAMPLE 18.10

Mahendra Musingh's utility curve is a reasonable approximation to $\sqrt{x}$. What is his best decision with the following gains matrix?

		Event		
		X	Y	Z
		$P = 0.7$	$P = 0.2$	$P = 0.1$
Alternative	A	14	24	12
	B	6	40	90
	C	1	70	30
	D	12	12	6

Solution

The calculations are similar to those for expected values, except that the amount of money, x, is replaced by its utility, $U(x)$, which in this case is the square root, $\sqrt{x}$.

- Alternative A: $0.7 \times \sqrt{14} + 0.2 \times \sqrt{24} + 0.1 \times \sqrt{12} = \mathbf{3.95}$ (best)
- Alternative B: $0.7 \times \sqrt{6} + 0.2 \times \sqrt{40} + 0.1 \times \sqrt{90} = 3.93$
- Alternative C: $0.7 \times \sqrt{1} + 0.2 \times \sqrt{70} + 0.1 \times \sqrt{30} = 2.92$
- Alternative D: $0.7 \times \sqrt{12} + 0.2 \times \sqrt{12} + 0.1 \times \sqrt{6} = 3.36$

Although the difference is small, the best alternative is A. You can compare this with the expected values, which are 15.8, 21.2, 17.7 and 11.4 respectively, suggesting that alternative B is the best.

Review questions

18.10 What is 'decision making with risk'?

18.11 What is the expected value of a course of action?

18.12 Could you use subjective probabilities for events under risk?

18.13 When can you use Bayes' theorem to calculate expected values?

18.14 Why might expected utilities be better than expected values?

Sequential decisions

There are many circumstances where managers are not concerned with a single decision, but have to consider a series of related decisions. For example, when you buy a car your initial decision might be to choose a new or second-hand one. If you choose a new car, this opens the choice of Japanese, French, German, Italian, etc. If you choose a Japanese car, you then have a choice of Toyota, Nissan, Mitsubishi, Honda, and so on. If you choose a Toyota, you then have a choice of models and then a choice of options. At each stage, choosing one alternative opens up a series of other choices – or sometimes it opens up a series of events that might occur. We can describe such problems in a **decision tree**, where the branches of a horizontal tree represent alternatives and events.

WORKED EXAMPLE 18.11

Patrick O'Doyle asked his bank for a loan to expand his company. The bank managers have to decide whether or not to grant the loan. If they grant the loan, Patrick's expansion may be successful or it may be unsuccessful. If the bank managers do not grant the loan, Patrick may continue banking as before, or he may move his account to another bank. Draw a decision tree of this situation.

Solution

A decision tree shows the sequence of alternatives and events. There is a notional time scale going from left to right with early decisions or events on the left followed by later ones towards the right. There is only one decision in this example followed by events over which the bank managers have no control, so the sequence is:

- The managers make a decision either to grant Patrick's loan or not.
- If they grant the loan, the expansion may be successful or unsuccessful.
- If they do not grant the loan, Patrick may continue or he may move his account.

Figure 18.5 shows these in a basic decision tree.

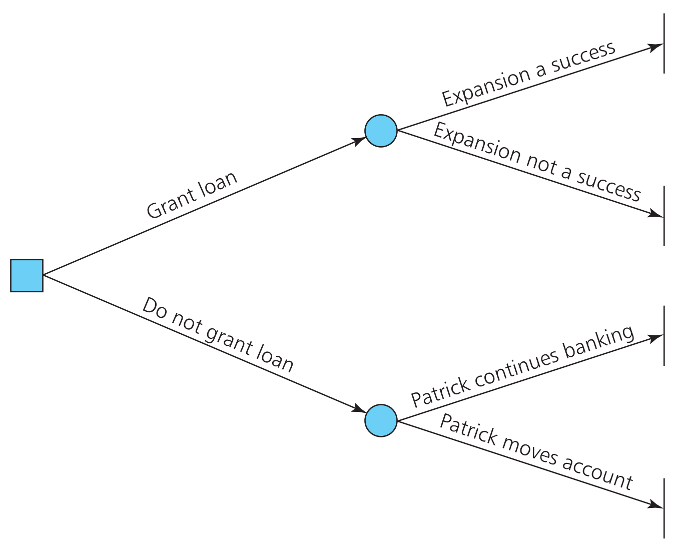


Figure 18.5 Decision tree for Patrick O'Doyle (worked example 18.11)

In the last example, you can see that we have drawn the alternatives and events as the branches of a tree, with each branch representing a different path (decision or event) that may be followed through the tree. There are three types of nodes, which are the points between or at the end of branches.

- | **Terminal nodes** are at the right-hand side of the tree, and are the ends of all sequences of decisions and events.
- ○ **Random nodes** represent points at which things happen, so that branches leaving a random node are events with known probabilities.
- □ **Decision nodes** represent points at which decisions are made, so that branches leaving a decision node are alternatives.

We now have the basic structure of a tree, and the next stage is to add the probabilities and outcomes.

WORKED EXAMPLE 18.12

Continuing the problem in worked example 18.11, suppose that Patrick O'Doyle's bank currently values his business at €20,000 a year. If the manager grants the loan and the expansion succeeds, the value to the bank of increased business and interest charges is €30,000 a year. If the expansion does not succeed, the value to the bank declines to €10,000, because of lower volumes and an allowance writing-off bad debt. There is a probability of 0.7 that the expansion plan will succeed. If the manager does not grant the loan, there is a probability of 0.6 that Patrick will transfer his account to another bank.

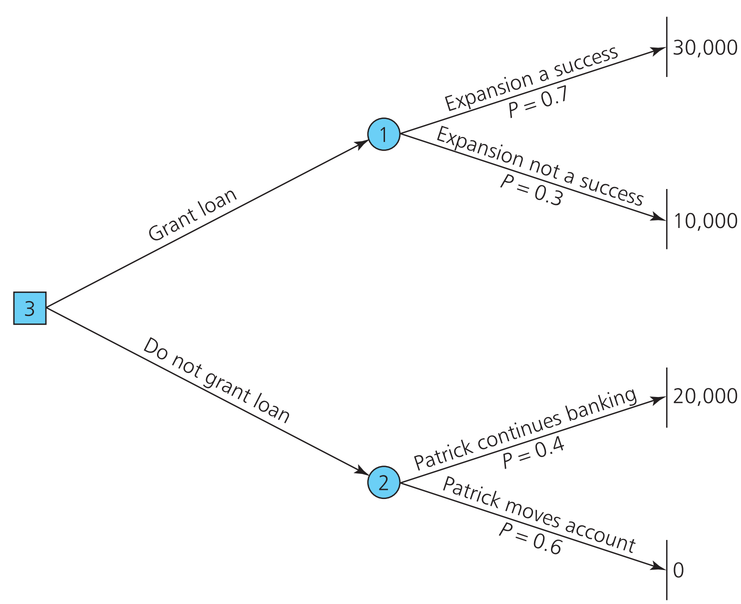


Figure 18.6 Decision tree with added probabilities and terminal values

Worked example 18.12 continued

Solution

We can add these figures to the decision tree. Figure 18.6 shows the probabilities added to event branches, making sure that all events are included and the sum of the probabilities from each random node is 1. It also has values on the terminal nodes, giving the total value (in this case the annual business expected by the bank) of moving through the tree and reaching the terminal node. This completes the drawing of the tree.

The next step for a decision tree is to analyse it, moving from right to left through the tree and putting a value on each node in turn. We do this by finding the best decision at each decision node and the expected value at each random node.

- At each decision node, alternative branches leaving are connected to following nodes. We compare the values on these following nodes, and identify the best – which shows the branch we would choose to move along. Then we transfer the value from the best following node to this decision node.
- At each random node, we find the expected value of following events. So for each branch we find the probability of leaving down the branch, and multiply this by the value of the following node. Adding these together for all branches from the random node gives its expected value.

Following this procedure from right to left, we eventually get back to the originating node, and the value on this is the overall expected value of making the best decisions.

WORKED EXAMPLE 18.13

Analyse the problem tree for Patrick O'Doyle's bank loan.

Solution

Figure 18.7 shows the node values added to the tree. The calculations for this are:

- At random node 1 calculate expected value:

$$0.7 \times 30{,}000 + 0.3 \times 10{,}000 = 24{,}000$$

- At random node 2 calculate expected value:

$$0.4 \times 20{,}000 + 0.6 \times 0 = 8{,}000$$

- At decision node 3 select the best alternative:

$$\max[24{,}000, 8{,}000] = 24{,}000$$

The best decision for the bank is to grant Patrick a loan, and this gives them an expected value of €24,000.

Worked example 18.13 continued

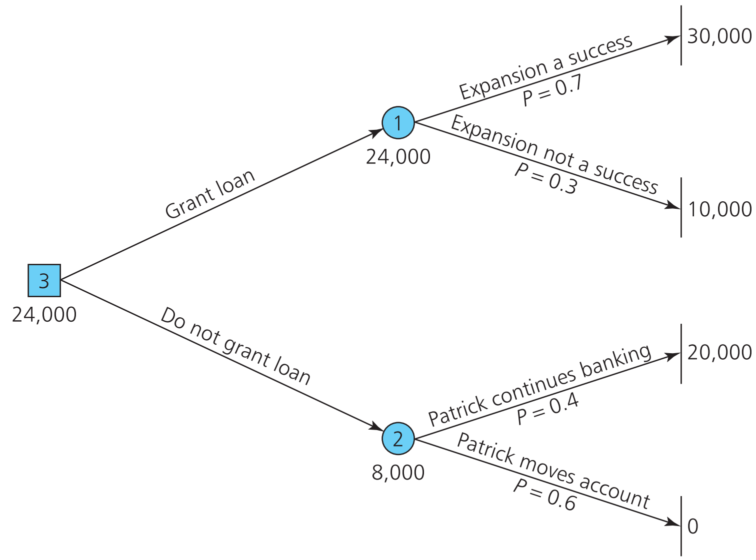


Figure 18.7 Analysing the decision tree

WORKED EXAMPLE 18.14

Lars Van Hoek is about to install a new machine for making parts for domestic appliances. Three suppliers have made bids to supply the machine. The first supplier offers the Basicor machine, which automatically produces parts of acceptable, but not outstanding, quality. The output from the machine varies (depending on the materials used and a variety of settings) and might be 1,000 a week (with probability 0.1), 2,000 a week (with probability 0.7) or 3,000 a week. The notional profit for this machine is €4 a unit. The second supplier offers a Superstamp machine, which makes higher quality parts. The output from this might be 700 a week (with probability 0.4) or 1,000 a week, with a notional profit of €10 a unit. The third supplier offers the Switchover machine, which managers can set to produce either 1,300 high-quality parts a week at a profit of €6 a unit, or 1,600 medium-quality parts a week with a profit of €5 a unit.

If the machine produces 2,000 or more units a week, Lars can export all production as a single bulk order. Then there is a 60% chance of selling this order for 50% more profit, and a 40% chance of selling for 50% less profit.

What should Lars do to maximise the expected profit?

Solution

Figure 18.8 shows the decision tree for this problem. Here the terminal node shows the weekly profit, found by multiplying the number of units produced by the profit per unit. If 1,000 are produced on the Basicor machine, the profit is €4 a unit, giving a node value of €4,000, and so on. When the output from Basicor is exported, profit may be increased by 50% (that is to €6 a unit) or reduced by 50% (to €2 a unit). Then the calculations at nodes are:

- Node 1: expected value at random node
  $= 0.6 \times 12{,}000 + 0.4 \times 4{,}000 = 8{,}800$
- Node 2: expected value at random node
  $= 0.6 \times 18{,}000 + 0.4 \times 6{,}000 = 13{,}200$
- Node 3: best alternative at decision node
  = maximum of 8,800 and 8,000 = 8,800
- Node 4: best alternative at decision node
  = maximum of 13,200 and 12,000
  = 13,200
- Node 5: expected value at random node
  $= 0.1 \times 4{,}000 + 0.7 \times 8{,}800 + 0.2 \times 13{,}200$
  $= 9{,}200$
- Node 6: expected value at random node
  $= 0.4 \times 7{,}000 + 0.6 \times 10{,}000 = 8{,}800$

Worked example 18.14 continued

- Node 7: best alternative at decision node
  = maximum of 7,800 and 8,000 = 8,000
- Node 8: best alternative at decision node
  = maximum of 9,200, 8,800 and 8,000
  = 9,200

The best decisions are to buy the Basicor machine and, if it produces more than 2,000 units, to export all production. The expected profit from this policy is €9,200 a week.

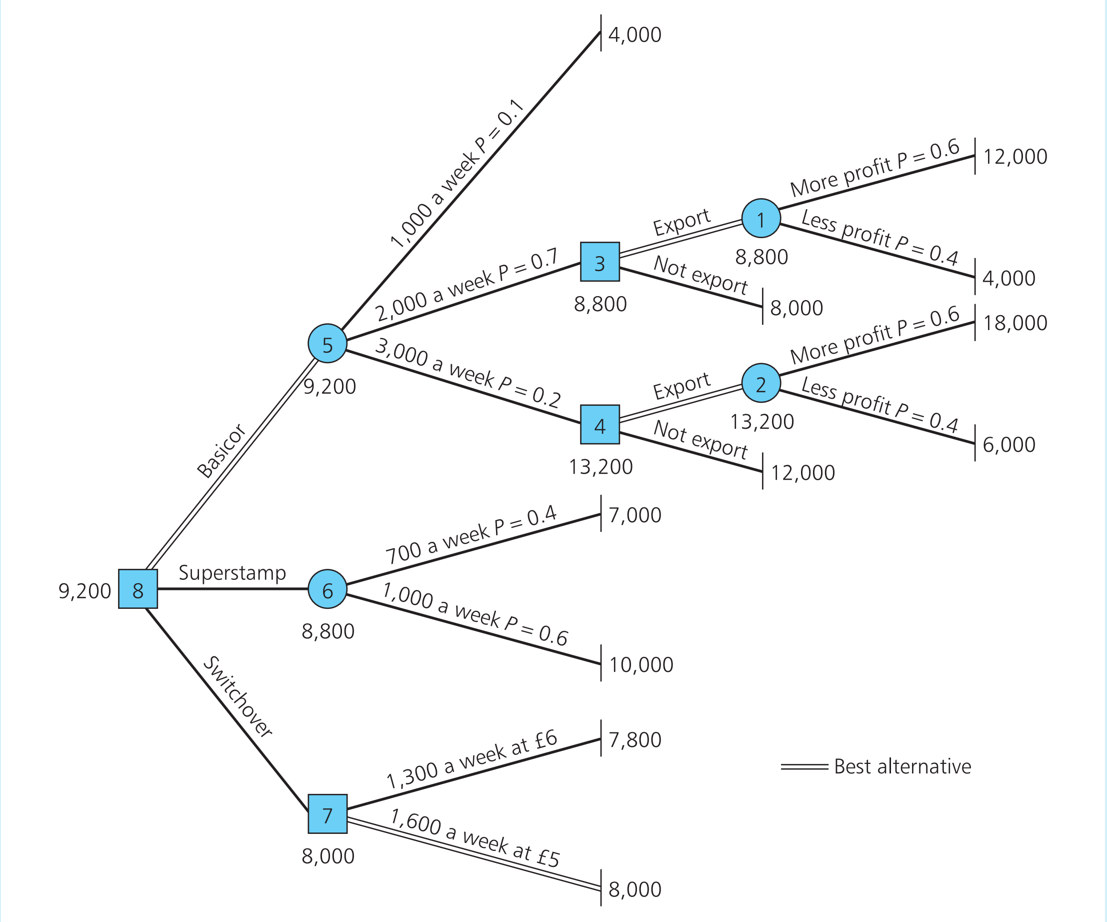


Figure 18.8 Decision tree for worked example 18.14

WORKED EXAMPLE 18.15

Draw a decision tree for the problem of planning a sports event described in worked example 18.8.

Solution

You can get specialised software for drawing decision trees that ranges from simple tools to do the calculations, through spreadsheet add-ins to sophisticated analysis packages. Figure 18.9 shows the results from one package as a sketch of the decision tree. Figure 18.10 shows the results from a package that simply organises the calculations in a spreadsheet. Both of these confirm our expected value of £22,710 advanced sales information.

Worked example 18.15 continued

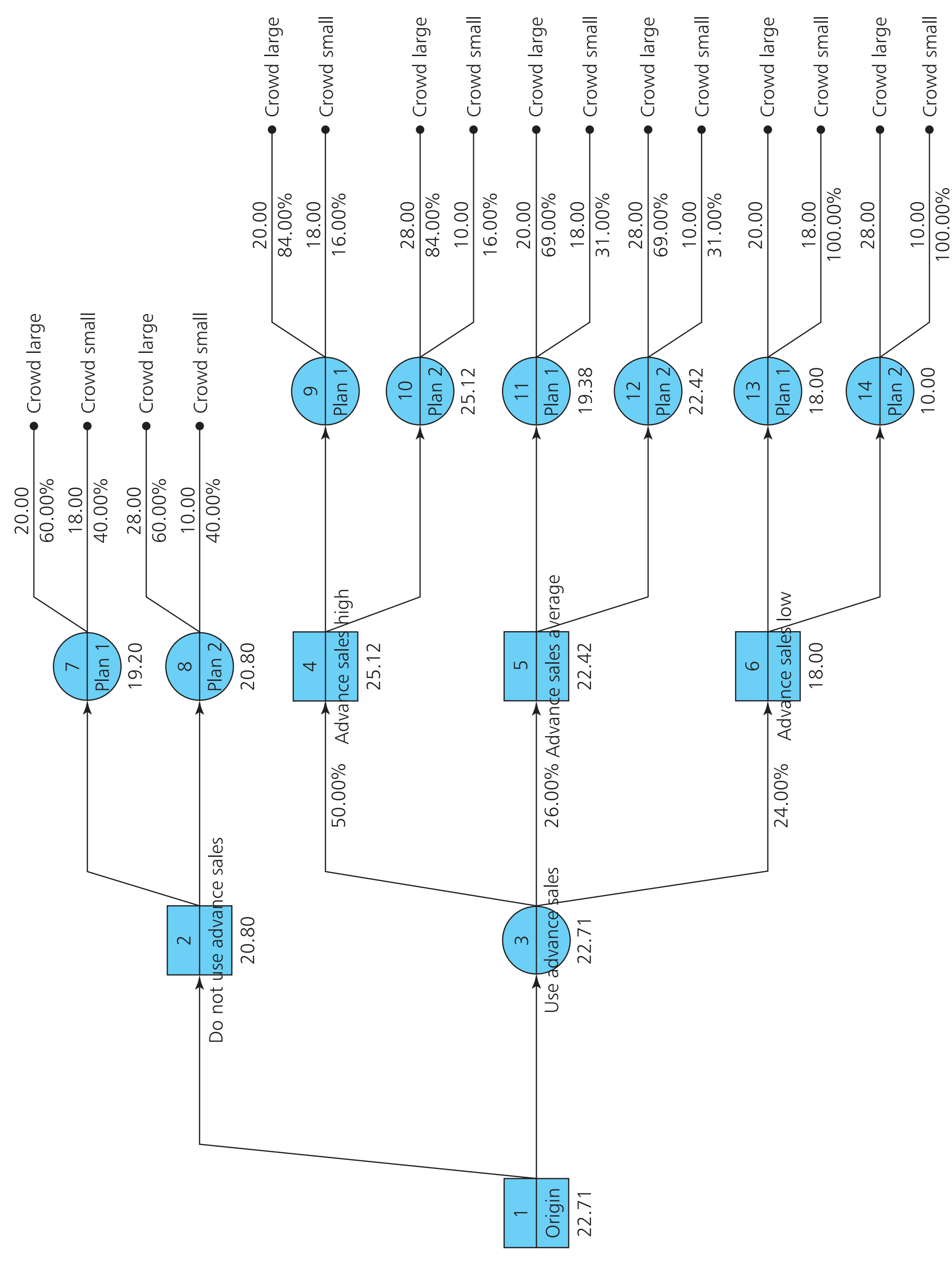


Figure 18.9 Computer sketch of decision tree for worked example 18.15

Worked example 18.15 continued

	A	B	C	D	E	F	G
1	**Decision tree for organising a sports event**						
2							
3	Node #	**Node name**	**Type**	**Following**	**Probability**	**Expected value**	**Decision**
4	1	Origin	Decision	2,3		22.71	Use advance sales
5	2	Do not use advance sales	Decision	7,8		20.80	Plan 2
6	3	Use advance sales	Chance	4,5,6		22.71	
7	4	Advance sales high	Decision	9,10	0.50	25.12	Plan 2
8	5	Advance sales average	Decision	11,12	0.26	22.42	Plan 2
9	6	Advance sales low	Decision	13,14	0.24	18.00	Plan 1
10	7	Plan 1	Chance	15,16		19.20	
11	8	Plan 2	Chance	17,18		20.80	
12	9	Plan 1	Chance	19,20		19.68	
13	10	Plan 2	Chance	21,22		25.12	
14	11	Plan 1	Chance	23,24		19.38	
15	12	Plan 2	Chance	25,26		22.42	
16	13	Plan 1	Chance	27,28		18.00	
17	14	Plan 2	Chance	29,30		10.00	
18	15	Crowd large			0.60	20.00	
19	16	Crowd small			0.40	18.00	
20	17	Crowd large			0.60	28.00	
21	18	Crowd small			0.40	10.00	
22	19	Crowd large			0.84	20.00	
23	20	Crowd small			0.16	18.00	
24	21	Crowd large			0.84	28.00	
25	22	Crowd small			0.16	10.00	
26	23	Crowd large			0.69	20.00	
27	24	Crowd small			0.31	18.00	
28	25	Crowd large			0.69	28.00	
29	26	Crowd small			0.31	10.00	
30	27	Crowd large			0.00	20.00	
31	28	Crowd small			1.00	18.00	
32	29	Crowd large			0.00	28.00	
33	30	Crowd small			1.00	10.00	
34	31	Overall				22.71	

Figure 18.10 Calculations for the decision tree

Review questions

18.15 How do you calculate the node value at a terminal node, a decision node and a random node?

18.16 How do you find the expected value of following the best options in a decision tree?

IDEAS IN PRACTICE Yield management in airlines

Yield management includes different types of analysis for allocating scarce resources to different types of customers. It is most common in airlines and hotels, where it is said to have brought significant benefits – for example, early work by American Airlines in the 1990s increased revenue by $500 million a year,[1] Delta airlines generated an additional $300 million[2] and Marriott Hotels generated an extra $100 million.

For airlines, there are two key issues in yield management. The first is overbooking, where airlines forecast the number of people who book seats and then cancel or do not turn up for the flight – and then they overbook this number of seats. In effect, they sell more seats than are available on a flight, in the belief that some passengers will not turn up. This increases revenue, but airlines have to juggle numbers carefully so that they do not have more people turn up than the aircraft will hold – or at least, they do not do this too often.

The second issue is the allocation of seats to different types of passengers. You can imagine this with the decision to sell a seat at a discount. Imagine a point several months before a flight, when an airline gets a request for a discount seat. The airline has to balance the certainty of getting some cash now, with the expected value of waiting and seeing whether they can sell the seat later at a higher price. Then the airline sells a seat at a discount only when the discounted price is greater than the expected value from selling the seat later. There are many types of passengers and fare offers, so this is not such a straightforward decision. And the probability of selling a seat changes right up to the point of filling the aircraft, so the calculations are continually updated. Sometimes when you leave buying a ticket until the last minute you get a bargain (suggesting that the airline is unlikely to sell the seat for a higher price) – and sometimes you have to pay a supplement (when many people are still wanting to buy tickets).

CHAPTER REVIEW

This chapter has shown how managers can approach decisions, describing the foundations of decision analysis.

- Managers make decisions in complex situations, and it is useful to give some structure to their decisions. Problem maps give a simple way of doing this, with another format describing payoff matrices.
- The main elements of a decision are a decision maker, their aims, alternatives they can choose, events that happen, and outcomes for each combination of chosen alternative and uncontrolled event.
- With decision making under certainty there is only one event. Then managers compare outcomes and choose the alternative that gives the best.
- With decision making under uncertainty there are several possible events, but we do not know which will occur or cannot even give them probabilities. The usual way of tackling these problems is to use decision criteria. We illustrated these with Laplace, Wald and Savage criteria, but there are many others for different circumstances.
- With decision making under risk there are several possible events, and we can give each a probability. The expected value for each alternative is:

  $$\sum (\text{probability of event occurring} \times \text{outcome})$$

  Expected values may not reflect real preferences, and in principle it is better to use expected utilities.

- One decision often leads to a series of others. You can draw sequential decisions on a decision tree, where branches show the events and alternatives that follow each node. To analyse the tree you choose the best alternative at a decision node and calculate the expected value at a random node.

CASE STUDY The Newisham Reservoir

Newisham has a population of about 30,000. It traditionally got its water supply from the nearby River Feltham, but increasing quantities of water were being extracted from the river by industry upstream. When the flow reaching the Newisham water treatment works became too small to supply the town's needs, the council decided to build a reservoir by damming the Feltham and diverting tributaries. This work was finished in 2002 and gave a guaranteed supply of water to the town.

Unfortunately, the dam reduced the amount of water available to farmers downstream, and two recently found the water supply to their cattle had effectively dried up. They now face the option of either connecting to the local mains water supply at a cost of £44,000, or drilling a new well. The drilling company cannot give an exact cost for the work, but suggest guidelines of £32,000 (with a probability of 0.3), £44,000 (with a probability of 0.3) or £56,000, depending on the underground rock structure and depth of water.

A local water survey company can do some more on-site tests. For a cost of £600 they will give either a favourable or an unfavourable report on the chances of easily finding water. The reliability of this report (phrased in terms of the probability of a favourable report, given that the drilling cost will be low, etc.) is given in the following table.

	Drilling well cost		
	£32,000	£44,000	£56,000
Favourable report	0.8	0.6	0.2
Unfavourable report	0.2	0.4	0.8

Questions

- **What would a decision tree of the farmers' problem look like?**
- **What are their best choices and expected costs?**

PROBLEMS

18.1 O'Brian's pub on the seafront at Blackpool notices that its profits are falling. The landlord has a number of alternatives for increasing his profits (attracting more customers, increasing prices, getting customers to spend more, etc.) but each of these leads to a string of other effects. Draw a map showing the interactions for this situation.

18.2 Choose the best alternative in the following matrix of gains.

	Event	
Alternative	A	100
	B	950
	C	–250
	D	0
	E	950
	F	500

18.3 Use the Laplace, Wald and Savage decision criteria to select alternatives in the following matrices. What results would you get for other decision criteria?

(a) Cost matrix

		Event				
		1	2	3	4	5
Alternative	A	100	70	115	95	60
	B	95	120	120	90	150
	C	180	130	60	160	120
	D	80	75	50	100	95
	E	60	140	100	170	160

(b) Gains matrix

		Event			
		1	2	3	4
Alternative	A	1	6	3	7
	B	2	5	1	4
	C	8	1	4	2
	D	5	2	7	8

18.4 Figure 18.11 shows a printout from a program which does the calculations for decision criteria. Describe the criteria that it uses and design your own spreadsheet to check the results. What results would other criteria give?

18.5 Which is the best alternative in the following gains matrix? Would this decision change using a utility function $U(x) = \sqrt{x}$?

		Event		
		1	2	3
		$P = 0.4$	$P = 0.3$	$P = 0.3$
Alternative	A	100	90	120
	B	80	102	110

18.6 GKR WebSpace can launch one of three versions of a new product, X, Y or Z. The profit depends on market reaction and there is a 30% chance that this will be good, a 40% chance it will be medium and a 30% chance it will be poor. Which version should the

-=*=- **INFORMATION ENTERED** -=*=-

NUMBER OF STATES:	5
NUMBER OF ALTERNATIVES:	5
NUMBER OF CRITERIA CHOSEN:	5
HURWICZ COEFFICIENT:	0.3

PAYOFF TABLE

	VALUE OF EACH ALTERNATIVE				
	1	2	3	4	5
1	1.00	5.00	9.00	2.00	6.00
2	3.00	7.00	3.00	5.00	1.00
3	6.00	4.00	4.00	6.00	8.00
4	8.00	2.00	7.00	5.00	6.00
5	6.00	9.00	4.00	1.00	2.00

-=*=- **RESULTS** -=*=-

CRITERION	ALTERNATIVE	PAYOFF
1. MAXIMAX	A2	9.00
2. MINIMIN	A3	3.00
3. LIKELIHOOD	A2	5.40
4. MINIMAX REGRET	A3	5.00
5. HURWICZ RULE	A3	4.80

-- END OF ANALYSIS --

Figure 18.11 Computer printout for decision criteria company launch with the profits given in the following table?

		Market reaction		
		Good	Medium	Poor
Version	X	100	110	80
	Y	70	90	120
	Z	130	100	70

The company can do another survey to give more information about market reaction. Experience suggests these surveys give results A, B or C with probabilities *P*(A/Good), *P*(A/Medium), etc., shown in the following table.

		Result		
		A	B	C
Market reaction	good	0.2	0.2	0.6
	medium	0.2	0.5	0.3
	poor	0.4	0.3	0.3

How much should the company pay for this survey?

18.7 Schwartz Transport owns a lorry with a one-year-old engine. It has to decide now, whether or not to replace the engine at a cost of €2,000. If it does not replace the engine, there is an increased chance that it will break down during the year and the cost of an emergency replacement is €3,200. Then at the end of next year, the company again has to decide whether to replace the engine. When an engine is replaced any time during the year, it is assumed to be one year old at the end of the year. The probabilities that an engine breaks down during the next year are as follows:

	Age of engine		
	0	1	2
Probability of breakdown	0.0	0.2	0.7

Draw a decision tree for this problem and find the decisions that minimise the total cost over the next two years. If a three-year-old engine is virtually certain to break down sometime in the next year, what is the minimum expected cost over three years?

18.8 Wilshaw Associates is considering launching an entirely new service. If the market reaction to this service is good (which has a probability of 0.2), they will make $30,000 a month; if market reaction is medium (with probability 0.5), they will make $10,000; and if reaction is poor (with probability 0.3), they will lose $15,000 a month. Wilshaw can run a survey to test market reaction with results A, B or C. Experience suggests the reliability of such surveys is described by the following matrix of *P*(A/good), *P*(A/medium), etc. Use a decision tree to find the most that Wilshaw should pay for this survey.

		Results		
		A	B	C
Market reaction	good	0.7	0.2	0.1
	medium	0.2	0.6	0.2
	poor	0.1	0.4	0.5

18.9 A television company has an option on a new six-part series. They could sell the rights to this series to the network for £100,000, or they could make the series themselves. If they make the series themselves, advertising profit from each episode is not known exactly but could be £15,000 (with a probability of 0.25), £24,000 (with a probability of 0.45) or £29,000, depending on the success of the series.

A local production company can make a pilot for the series. For a cost of £30,000 they will give either a favourable or an unfavourable report on the chances of the series being a success. The reliability of their report (phrased in terms of the probability of a favourable report, given the likely advertising profit, etc.) is given in the following table. What should the television company do?

	Advertising profit		
	£15,000	£24,000	£29,000
Unfavourable report	0.85	0.65	0.3
Favourable report	0.15	0.35	0.7

RESEARCH PROJECTS

18.1 We have described several formats for presenting decisions – problem maps, payoff matrices and decision trees. But these are not the only options. What other formats are available? Find some examples where different formats have been used in practice.

18.2 Some spreadsheet packages – or special add-ins – draw decision trees automatically. Do a small survey to see what features these contain.

You can also draw a decision tree on a standard spreadsheet, as illustrated in Figure 18.12. This uses the DRAW options for drawing the skeleton of the tree, with

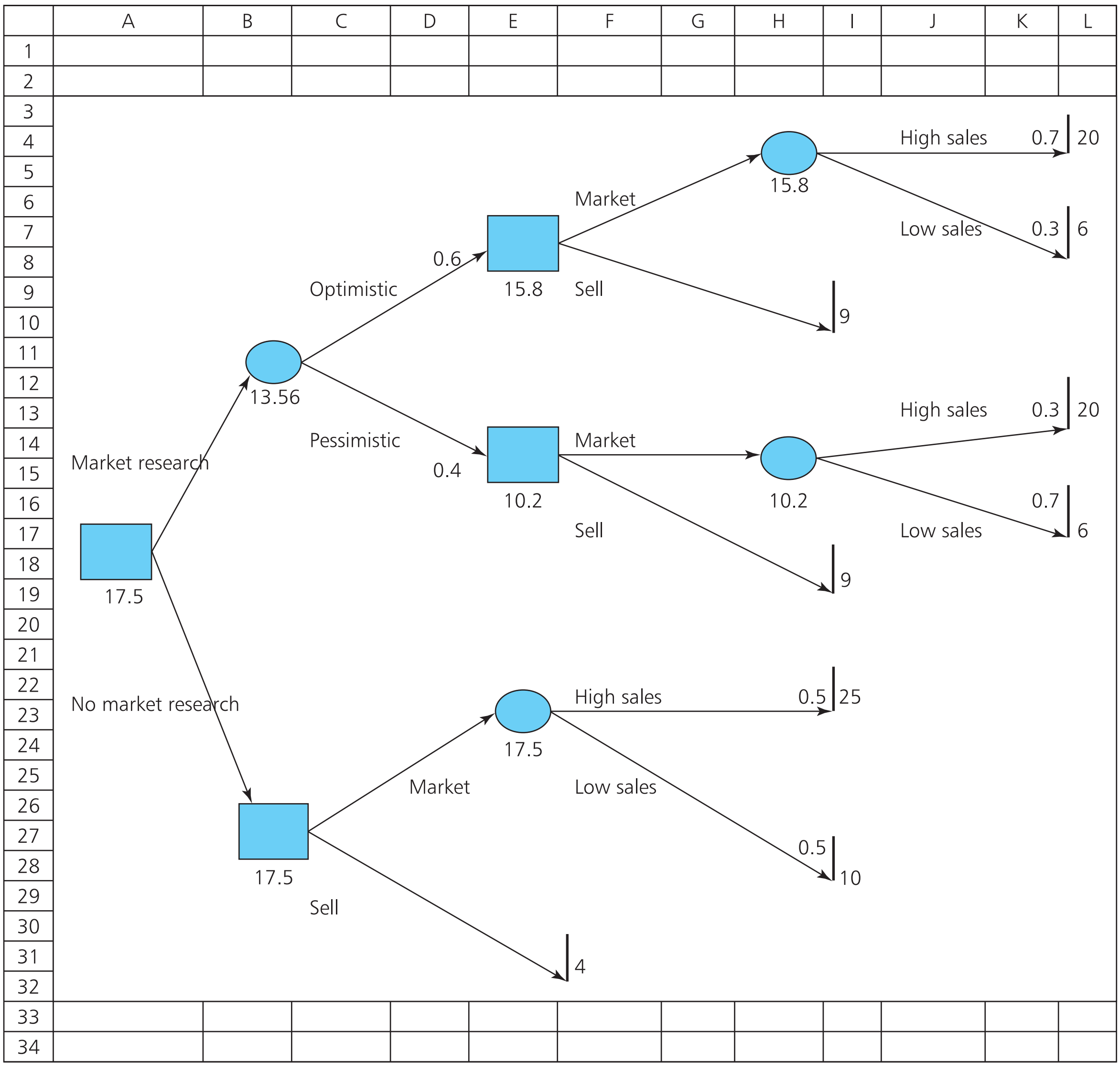


Figure 18.12 Example of a decision tree in a spreadsheet

calculations described in convenient cells using normal spreadsheet functions. In particular, the MAX function identifies the best alternatives at a decision node, and expected values are calculated as usual at random nodes. A useful point here is the GOAL SEEK function, which you can use to find the probabilities needed to achieve a specific return. See how this works and explore the possibilities it offers.

Sources of information

References

1 Smith B.C., Leimkuhler J.F. and Darrow R.M., Yield management at American Airlines, *Interfaces*, vol. 22(1), 1992.

2 Boyd A., Airline alliance revenue management, *OR/MS Today*, vol. 25, 1998.

Further reading

Material in this section is covered in books on management science, operational research and operations management. The following list includes more specialised books on decision analysis.

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CHAPTER 19

Quality management

Contents

Chapter outline

Quality is a measure of how good a product is. A high-quality product meets, and preferably exceeds, the requirements of both customers and producers. Quality management is the function responsible for all aspects of quality. The basic problem for quality management is that some variation is inevitable in a product. Quality management uses a range of tools to reduce this variation and make products that are of consistently high quality.

After finishing this chapter you should be able to:

- Discuss the meaning of quality and appreciate its importance
- Describe the costs of quality management
- Review the principles of Total Quality Management (TQM)
- See how quality control forms part of the broader quality management function
- Discuss the variation in a process and the need to control it
- Describe some key tools of quality control
- Design sampling plans for acceptance sampling
- Draw process control charts.

Measuring quality

The first problem with talking about quality management is defining exactly what we mean by 'quality'. You might start with your own experiences and say that you are happy with the quality of, say, a pen when it writes easily and clearly; you think an airline gives a high-quality service when you get to your destination on time and without too much hassle; an electricity supplier gives high quality if you never have to worry about supplies or costs. So we can suggest a fairly obvious statement that customers view products as having high quality if they do the jobs they were designed for.

> In its broadest sense, **quality** is the ability of a product to meet – and preferably exceed – customer expectations.

But this definition is still vague – especially as different customers have different expectations. And each judges quality by a number of different criteria, including innate excellence, fitness for intended use, performance, reliability, durability, features, level of technology, conformance to design specifications, uniformity, perception of high quality by customers, convenience of use, attractive appearance, value, after-sales service, on-time delivery, and so on. When you look at a television set you might judge its quality by how expensive it is, how attractive the cabinet is, how big it is, how easy it is to use, how clear the picture is, how accurate the colours are, how often it needs repairing, how long it will last, how many channels it can pick up, how good the sound is, what technology it uses, and the additional features it has.

Any reasonable view of quality must take into account all of these factors, and it would be foolish to judge a product by some factors and ignore others. However, to make things easier, we bundle all of these considerations together into the general concept of 'customer satisfaction'.

An important point is that when customers judge quality, they do not demand products with the highest technical quality. We want some balance of features that gives an acceptable overall picture. A Rolls-Royce car has the highest possible quality of engineering, but most people include price in their judgement and buy a cheaper make. We can use this distinction to describe two views of quality:

- **Designed quality** is the quality that a product is designed to have. This takes an external view and judges a product by how well it satisfies customers – so a bar of chocolate is high quality if it tastes good, satisfies hunger, etc.
- **Achieved quality** shows how closely the product actually made meets its designed quality. This takes an internal view, showing how closely production conforms to specifications. Then a bar of chocolate is high quality if it is close to the specified weight, contains the right amount of cocoa, and so on.

Quality management

All decisions about quality in an organisation are brought together under the general heading of **quality management**.

> **Quality management** is the function responsible for all aspects of a product's quality.

It is easy to see why organisations have to make high-quality products. If they make poor products, customers do not buy them and simply move to a competitor that is better at meeting their expectations. If you buy a pair of shoes that get a hole the first time you wear them, you will not buy another pair, no matter how cheap they are. So making high-quality products is the only way that an organisation can survive in the long term. And while high-quality products may not guarantee success, low-quality ones will certainly guarantee failure.

The question, of course, is how to achieve high quality. We can begin to answer this by looking at what happens when you go into a clothes shop to buy a suit. You will be satisfied only if the suit is well designed, if it is well made, if there are no faults in the material, if the price is reasonable, if the salesperson is helpful, if the shop is pleasant, and so on. This means that everyone concerned – from the person who designs the suit to the person who sells it, and from the person who owns the organisation to the person who keeps it clean – is directly involved in the quality of their product. If even a single person does something that the customers do not like, it is enough to make customers look for other suppliers. This is the view taken by **Total Quality Management** (TQM).

- **Total Quality Management** has the whole organisation working together to guarantee – and systematically improve – quality.
- The aim of TQM is to satisfy customers by making products with no defects.
- A defect is any aspect of the product that reduces customer satisfaction.

In recent years there have been so many developments in quality management that some people refer to a 'quality revolution'. This happened for four main reasons:

1 Improved operations can make products with guaranteed high quality.
2 Producers use high quality to get a competitive advantage.
3 Consumers have become used to high-quality products and will not accept anything less.
4 High quality reduces costs.

The first three of these are fairly obvious, but the view that high quality reduces costs seems to go against the commonsense view that you can only buy higher quality products at a higher price. But if you look at the costs more carefully, you see that some really do go down with higher quality.

Costs of quality

Imagine that you buy a washing machine that is faulty. You complain, and the manufacturer arranges for the machine to be repaired. The manufacturer could have saved money by finding the fault before the machine left the factory – and it could have saved even more by making a machine that did not have a fault in the first place. If, say, 5% of machines are faulty, the manufacturer has to increase production by 5% just to cover the defects, and it has to maintain systems for dealing with customer complaints, collecting defective machines, inspecting, repairing or replacing them, and returning them to customers. By eliminating the defects, the manufacturer increases productivity, reduces costs, eliminates customer complaints, and removes all the systems needed to correct faults.

Of course, some costs must rise with increasing quality, and to consider these we separate the total cost of quality into four components.

1 **Prevention costs** are the costs of preventing defects happening. These include direct costs spent on the product, such the use of better materials, adding extra features, and extra time to make the product. They also include indirect costs of employee training, pilot runs, testing prototypes, designing and maintaining control systems, improvement projects, etc. All things being equal, prevention costs rise with the quality of the product.
2 **Appraisal costs** are the costs of making sure the designed quality is actually achieved. These costs include sampling, inspecting, testing, checking and all the other elements of quality control. Generally, the more effort that is put into quality control, the higher is the final quality of the product – and the higher the costs of achieving this.
3 **Internal failure costs** are the costs of making defective products that are detected somewhere within the process. This includes allowances for units that are scrapped, returned to an earlier point in the process, or repaired. Part of the internal failure costs come directly from the loss of material, wasted labour, wasted machine time in making the defective item, extra testing, duplicated effort, and so on. Another part comes from the indirect costs of higher stock levels, longer lead times, extra capacity needed to allow for scrap and rejections, loss of confidence, etc.
4 **External failure costs** are the costs of having a unit go through the entire production process and being delivered to a customer, who then finds a fault. External failure faults are usually the highest costs of quality management and are the ones that you should avoid. Like internal failure costs, external failure costs generally decline with higher quality.

Adding together these four components gives the total cost of quality, and the result is often surprisingly high. Failure costs, in particular, are very high, and as they fall with increasing quality we get the pattern shown in Figure 19.1. From this you can see that the lowest total cost comes with products of perfect quality, where every unit is guaranteed to be fault-free.

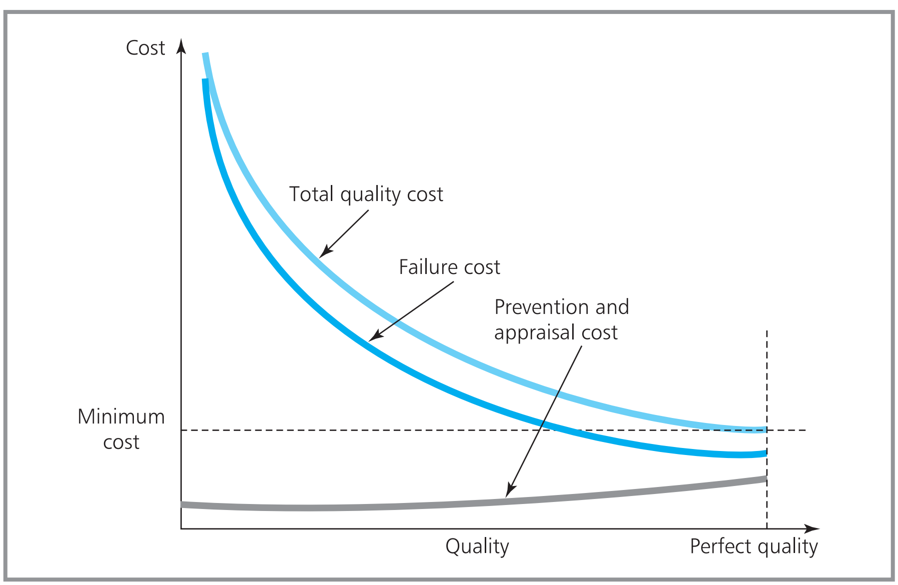


Figure 19.1 The lowest overall cost comes with perfect quality

WORKED EXAMPLE 19.1

Ying Shu Tang recorded her company's costs (in thousands of dollars a year) during a period when they introduced a new quality management programme. How effective do you think the new programme has been?

Year		−3	−2	−1	0	1	2	3
Sales value		1,225	1,247	1,186	1,150	1,456	1,775	1,865
Costs ($'000)	Prevention	7.3	8.1	9.1	26.8	30.6	32.9	35.2
	Appraisal	27.6	16.9	20.1	47.4	59.7	59.6	65.5
	Internal failure	72.8	71.9	75.0	40.3	24.0	20.0	19.4
	External failure	66.5	59.9	65.8	27.3	18.8	15.6	12.5

Solution

The best way of judging the quality management programme is to calculate the total cost of quality as a percentage of sales. The results for this are shown in the spreadsheet in Figure 19.2.

The quality management programme was introduced in year zero. This put more emphasis on prevention and appraisal, where costs have risen. Product quality has clearly risen, giving lower failure costs. Customers have apparently noticed the improvement, with sales no longer falling but rising sharply. Overall, quality costs have fallen and sales have risen, so we must judge the programme a success.

	A	B	C	D	E	F	G	H	I
1	**Costs of quality**								
2									
3	**Year**		**−3**	**−2**	**−1**	**0**	**1**	**2**	**3**
4	**Costs (% of sales)**								
5		Prevention	0.60	0.65	0.77	2.33	2.10	1.85	1.89
6		Appraisal	2.25	1.36	1.69	4.12	4.10	3.36	3.51
7		Internal failure	5.94	5.77	6.32	3.50	1.65	1.13	1.04
8		External failure	5.43	4.80	5.55	2.37	1.29	0.88	0.67
9	**Total (% of sales)**		**14.22**	**12.57**	**14.33**	**12.33**	**9.14**	**7.22**	**7.11**
10									

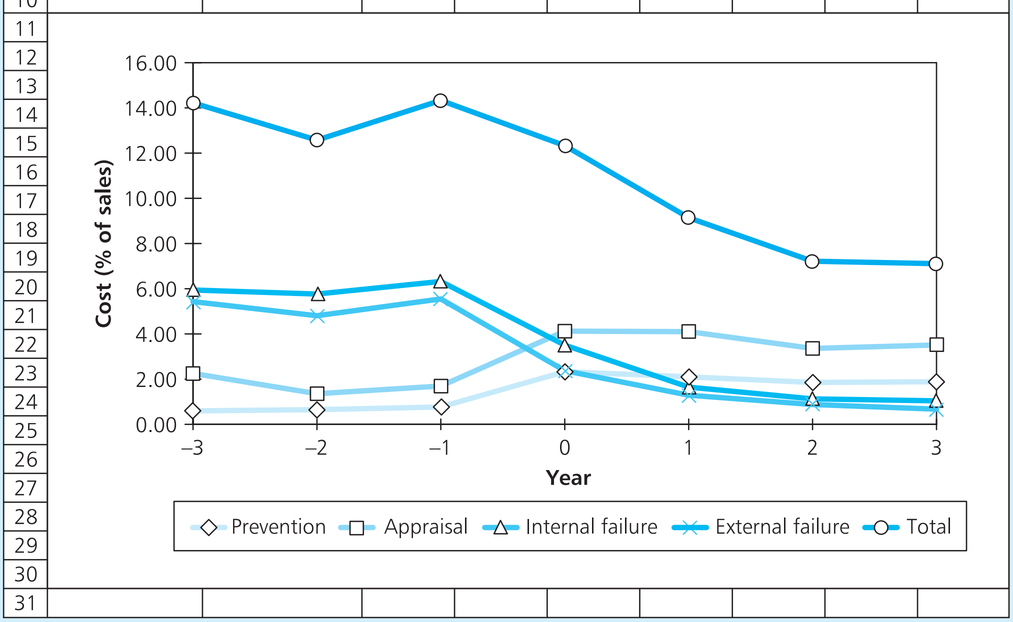


Figure 19.2 Changing costs with Ying Shu Tang's quality management programme

'Quality gurus'

Many people contributed to the growth of quality management, and a group of them have become known as the 'quality gurus'. Different people claim to be in this group, but the main members are:

- *Edwards Deming*,[1] who emphasised the role of management in setting quality and the importance of reducing variability in the process.
- *Armand Fiegenbaum*,[2] who looked at failure costs, and developed the idea of 'total quality' involving everyone in the organisation.
- *Joseph Juran*,[3] who emphasised the role of senior management and the definition of good quality as satisfying customer demand.
- *Philip Crosby*,[4] who analysed the total costs of quality and described straightforward methods for implementing quality management.
- *Genichi Taguchi*,[5] who showed the importance of product designs that allow high quality, with suitable control of the process.
- *Kaoru Ishikawa*,[6] who emphasised the contribution of 'workers' to quality and introduced the idea of quality circles.

Review questions

19.1 If the price is right, people will buy a product regardless of its quality. Do you think this is true?

19.2 Why is it so difficult to define 'quality'?

19.3 What is 'quality management'?

19.4 Why is quality management important to an organisation?

19.5 Higher quality inevitably comes at a higher cost. Is this true?

19.6 How would you find the best level of quality for a product?

IDEAS IN PRACTICE New York Containers

New York Containers make a range of scented products in spray cans. These include hair sprays, deodorants and room fresheners. In 2005 they appointed George Steinway as Director of Quality Assurance, with clear instructions to improve the quality of the company's products.

George spent the first few weeks talking to people and trying to find the real problems with quality. He quickly found one problem with the production department's ambition of meeting output quotas – almost regardless of price. So when a quality inspector rejected some aerosols as being over-filled and asked an operator to set them aside until she could find the cause of the problem, the production supervisor was concerned about his schedule and told the operator not to bother with the faults, but to release a little pressure from the cans and ship them out as usual. Later the quality inspector found that the pressure gauge on the filling machine was not working properly, the spray can nozzles delivered by a regular supplier were not up to standard, the production supervisor was judged by the number of cans produced with no concern for quality, and the machine operator was new and not fully trained.

Quality control

Traditionally, quality management developed as a separate function to check the output of production departments. But TQM says that everyone within an organisation is involved with quality – and in particular, quality management should move back to the people actually doing the work, so that it is no longer a separate function but is an integral part of the process. This move has brought changes to the timing and role of inspections. Traditionally, most effort was put into inspections in the later stages of the process, often just before finished products were delivered to customers. At first, this seems sensible, as all faults can be found in one big inspection. However, the longer a unit is in a process, the more time and money is spent on it. This suggests that faults should be found as early as possible, before any more money is wasted on a unit that is already defective. For instance, it is cheaper for a baker to find bad eggs when they arrive, rather than use the eggs in cakes and then scrap these when they fail a later inspection.

WORKED EXAMPLE 19.2

Svenson Electrics make light fittings on an assembly line. When the electric wiring is fitted, faults are introduced to 4% of units. An inspection at this point would find 95% of faults, with costs of €2 for the inspection and €3 to correct a fault. Any fault not found continues down the line and is detected and corrected later at a cost of €20.

Without the inspection after wiring, later tests cost an extra €1.20 a unit and each fault corrected costs €40. Is it worth inspecting light fittings after the wiring?

Solution

We can answer this by comparing the expected cost per unit of doing the inspection and not doing it.

- With an inspection after wiring, the expected costs per unit are:
  - cost of inspection = €2.00
  - cost of faults detected and corrected after wiring

    = proportion of faults detected × cost of repairing each
    = $0.04 \times 0.95 \times 3$ = €0.114
  - cost of faults not found until later

    = proportion not detected × cost of later repair
    = $0.04 \times (1 - 0.95) \times 20$ = €0.04

  This gives a total of $2.00 + 0.114 + 0.04$ = €2.154 a unit.
- Without an inspection after wiring, the expected costs per unit are:
  - additional cost of later inspection = €1.20
  - cost of faults detected and corrected after wiring

    = proportion with faults × cost of repair
    = 0.04×40 = €1.60

  This gives a total of $1.20 + 1.60$ = €2.80 a unit.

It is clearly cheaper to do an inspection when the wiring is fitted and to correct faults as soon as they are found.

Product variability

Of course, you may ask why inspections are needed if everyone is following TQM's advice and doing everything possible to make sure that products are always perfect. The answer is that no matter how good a process, there is always some variation in the products. Differences in materials, weather,

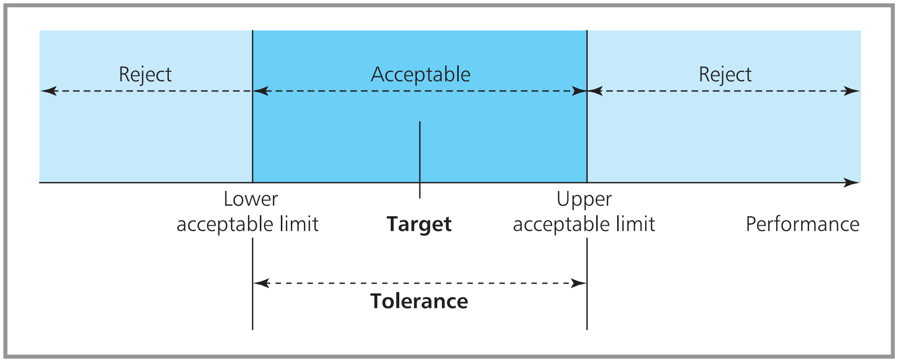


Figure 19.3 Traditional view of acceptable performance

tools, employees, moods, time, stress, and a whole range of other things combine to give these, apparently random, variations. The variations may be small, but they are always present. This is why marathon runners never finish a series of races in exactly the same times, and products never finish their process with exactly the same performance.

The design of products and operations must be robust enough to allow for these variations, and still give perfect quality. The traditional way of arranging this is to give a tolerance in the specifications. Provided a unit's performance is within a specified range, it is considered acceptable. A 250 g bar of chocolate might weigh between 249.9 g and 250.1 g and still be considered the right weight. Then a unit is considered faulty only if its performance is outside this tolerance, as shown in Figure 19.3.

Taguchi[5] pointed out that this approach has an inherent weakness. Suppose a bank sets the acceptable time to open a new account as between 20 and 30 minutes. If the time taken is 20, 25 or 30 minutes, the traditional view says that these are equally acceptable – the process is achieving its target, so there is no need for improvement. But customers would probably not agree that taking 30 minutes is as good as taking 20 minutes. On the other hand, there might be little real difference between taking 30 minutes (which is acceptable) and 31 minutes (which is unacceptable). The answer, of course, is that there are not such clear cut-offs. If you are aiming for a target, then the further you are away from the target, the worse your performance. This effect is described by a **loss function**, which gives a notional cost of missing the target (see Figure 19.4).

To minimise costs, managers have to get actual performance as close to the target as possible, and this means reducing the variability in a process. And to see whether this is actually happening, managers have to monitor performance over time. So they inspect units, test them, and make sure that everything is working properly and that the variation between units is small. This is the purpose of **quality control**.

Quality control uses a series of independent inspections and tests to make sure that designed quality is actually being achieved.

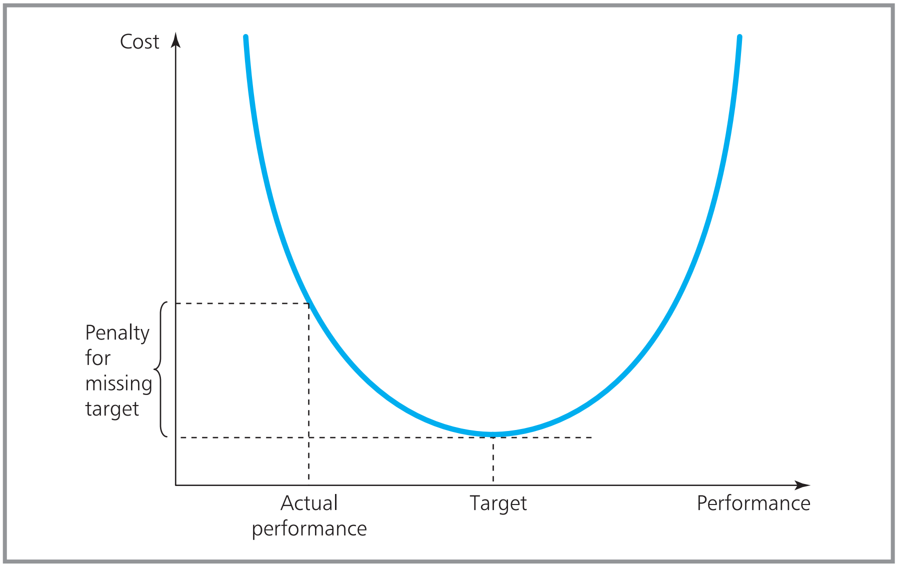


Figure 19.4 Loss function showing the cost of missing a specified target

TQM has everybody in the organisation working to make sure that no defects are made, so the purpose of quality control is not to find faults, but to give independent evidence that the process is working properly and that there really are no defects. If it finds a defective unit, it means that something has gone wrong with the process, and managers should find the cause of the problem and correct it before any more defects are made. Typical causes of faults are:

- human errors
- faults in equipment
- poor materials
- faults in operations, such as speed or temperature changes
- changes in the environment, such as humidity, dust or temperature
- errors in monitoring equipment, such as errors in measuring tools.

Review questions

19.7 What is the difference between quality control and quality management?

19.8 What is a loss function?

19.9 'With proper quality control, production departments can eliminate all variability.' Is this true?

Tools for quality control

When something goes wrong with a process, perhaps a sudden increase in customer complaints, it is often surprisingly difficult to find the cause. To help with this – and subsequent analyses – several tools have been developed for quality control. The simplest tool continues to ask a series of questions until the cause becomes clearer. You can imagine a session of this kind starting as follows.

Question: What is the problem?
Answer: A customer complained because we couldn't serve her.
Question: Why?
Answer: Because we had run out of stock.
Question: Why?
Answer: Because our suppliers were late in delivering.
Question: Why?
Answer: Because our order was sent in late.
Question: Why?
Answer: Because the purchasing department got behind with its orders.
Question: Why?
Answer: Because it used new staff who were not properly trained.

By this point it is clear that something has gone wrong in the purchasing department, and with more questions you could pinpoint the cause of the problem more accurately. For obvious reasons, this is called the **'5 whys' method**. Other tools that help with quality control include frequency counts, cause-and-effect diagrams and sampling.

Other tools

Another simple way of finding the cause of a problem is to record the number of times a specific problem is mentioned. For instance, when customers repeatedly mention the time taken to deliver a service, or the reliability of a product, this pinpoints an area where something has clearly gone wrong. The easiest way to record these is in a 'checksheet', which lists possible problems and records the number of times each is mentioned. A more formal version of this is a **Pareto chart**, which uses the 'rule of 80/20' to suggest that 80% of problems come from 20% of causes, while the remaining 20% of problems come from 80% of causes. So Wal-Mart might find that 80% of customer complaints come from 20% of their products. A Pareto chart lists the possible causes of problems, counts the number of faults that come from each, shows the results on a bar chart, and identifies the few areas that cause most problems. Then managers can concentrate on those areas that need special attention.

IDEAS IN PRACTICE Pengelly's Restaurant

Pengelly's Restaurant is a well-established business near the centre of Cape Town. It serves business lunches, and there is a healthy demand for its high-quality, expensive dinners. Jonas Subello is the owner of Pengelly's and looks after all the administration personally. There are few complaints from customers, but Jonas always keeps a record of them. Over the past three years he has collected the figures shown in Figure 19.5, where a bar chart highlights the areas for concern.

There were almost no complaints about the food, so customers were clearly pleased with what they were eating. Over half of the complaints came from faults in the bill. Jonas reduced these by installing new computerised cash registers. Sometimes the service was slow, particularly at busy times or when one of the staff was away. Jonas contacted an agency that could provide waiters at very short notice. These two measures alone dealt with almost three-quarters of complaints. When the restaurant needs refurbishing, Jonas can get some more comfortable chairs and increase the size of the non-smoking area. This would deal with another 19% of complaints. By these simple procedures, Jonas had dealt with 90% of complaints.

Ideas in practice continued

	A	B	C	D	E
1	**Complaints at Pengelly's Restaurant**				
2					
3	**Cause**	**Number of complaints**	**Percentage of complaints**		
4	Faults in the bill	80	51		
5	Slow service	31	20		
6	Smokers too near non-smokers	19	12		
7	Comfort of the chairs	11	7		
8	Wine	5	3		
9	Temperature of the restaurant	5	3		
10	Wait for a table	2	1		
11	Too limited menu	2	1		
12	Food: ingredients	2	1		
13	Food: cooking	1	1		
14					

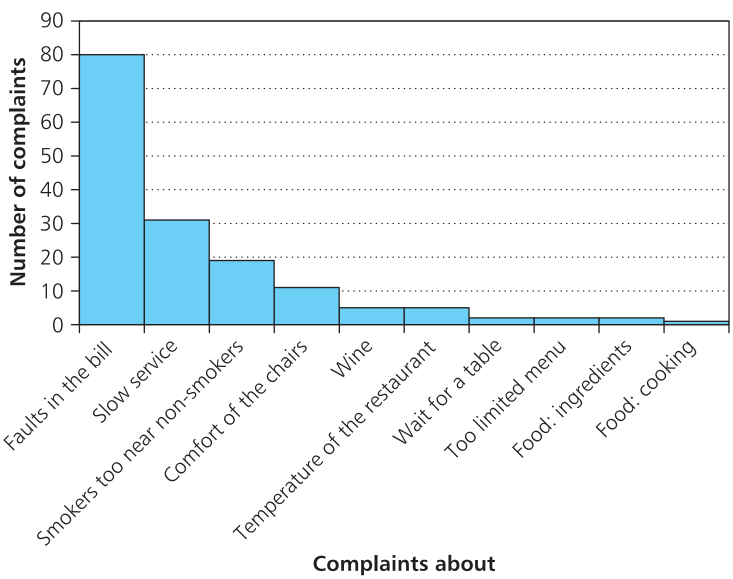


Figure 19.5 Pareto chart for complaints at Pengelly's Restaurant

Cause-and-effect diagrams – also called Ishikawa and fishbone diagrams – give a different view of the sources of problems. Suppose a customer complains at a hamburger restaurant. The problem may be caused by the raw materials, the cooking, the staff or the facilities. Problems with the raw materials may, in turn, be caused by suppliers, storage or costs. Then we could go into more details about the problems with, say, suppliers. A cause-and-effect diagram draws these relationships as coming from spines, like a fish bone, as shown in Figure 19.6.

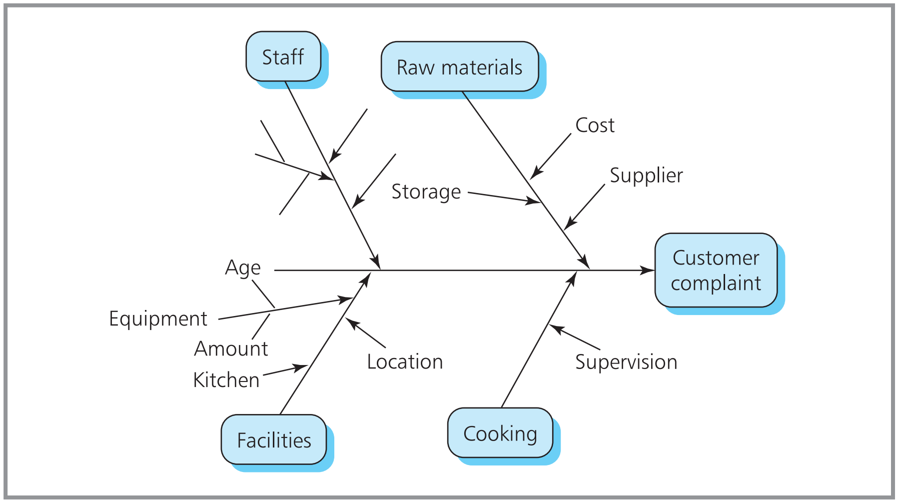


Figure 19.6 Cause-and-effect (or fishbone) diagram for complaints in a restaurant

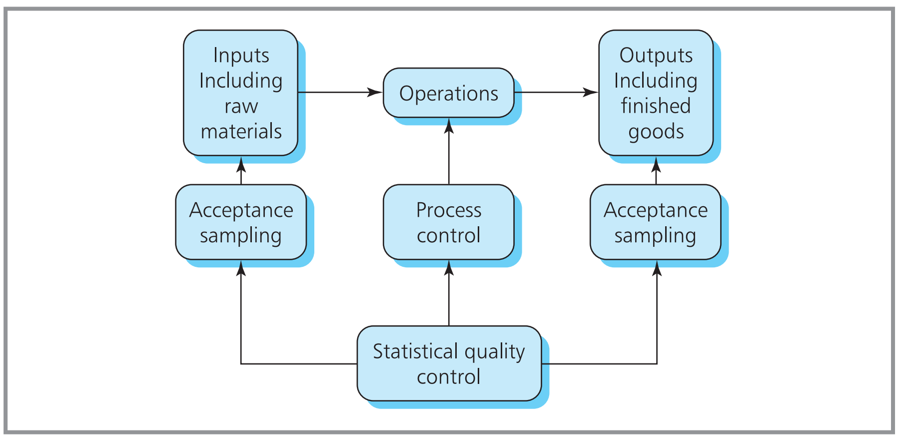


Figure 19.7 Aspects of statistical quality control

Probably the best known tool for quality control is routine sampling. There are really two types of sampling used for quality control:

- **Acceptance sampling** checks the quality of a batch of products. It takes a sample of units from a batch and checks whether the whole batch is likely to reach an acceptable level of quality, or whether it should be rejected. These checks focus on materials entering a process and on products leaving the process (as shown in Figure 19.7).
- **Process control** checks the performance of the process. It takes a sample to see whether the process is working within acceptable limits, or whether it needs adjusting.

Together these two types of sampling form the core of statistical quality control.

Review questions

19.10 'The best way to get high-quality products is to have a lot of inspections to find faults.' Is this true?

19.11 'Many diagrams can help identify the main causes of problems with quality.' Do you agree?

19.12 Who is responsible for the quality of a product?

Acceptance sampling

Acceptance sampling checks the quality of products – in particular, it considers a batch of products, takes a sample from this batch and uses this to test whether the whole batch reaches designed levels of quality and should be accepted, or whether it is defective and should be rejected. We can show the approach of acceptance sampling by considering some continuous property of a product, such as its weight, length, time or strength. This is called **sampling by variable**.

WORKED EXAMPLE 19.3

A batch of materials arrives at a service bay, where a sample of 40 units is found to have a mean weight of 25 kg and a standard deviation of 1 kg. Within what range is the bay 95% certain that the mean weight of units in the batch lies?

Solution

The best estimate for the mean weight of units in the batch is 20 kg, with a standard deviation of 1 kg. Then the sampling distribution of the mean is Normally distributed with mean 20 kg and unbiased standard deviation of $1/\sqrt{(n-1)} = 1/\sqrt{39} = 0.16$ kg. The 95% confidence interval for the mean weight in the batch is within 1.96 standard deviations of the mean, giving a range of:

$$25 + 1.96 \times 0.16 \quad \text{to} \quad 25 - 1.96 \times 0.16$$

or

$$25.31 \text{ kg} \quad \text{to} \quad 24.69 \text{ kg}$$

So the service bay is 95% certain that the mean weight of units in the batch is within this range.

The alternative to sampling by variables is **sampling by attribute**, which needs some criterion that describes a unit as either 'acceptable' or 'defective'. Sometimes this criterion is obvious. A light bulb either works or does not; boxes either contain at least 1 kg of soap powder or do not; a train either arrives on time or does not. Sometimes the criterion relies less on measurement and more on judgement. For instance, a piece of furniture may be rejected because its polished finish does not look good enough to an experienced inspector, or a service person might be considered rude.

Sampling by attribute needs a result that we saw in Chapter 16, which says that when the proportion of defective units in a population is π, the proportion of defects in samples of size n is Normally distributed with mean $= \pi$ and standard deviation $= \sqrt{\dfrac{\pi(1-\pi)}{n}}$.

WORKED EXAMPLE 19.4

SemiShan Communications use outside agents to check details of their contracts with customers. They insist that the agents make errors in fewer than 4% of contracts. One day they receive a large shipment of contracts from the agents. They take a sample of 200 contracts and check them. What criterion should SemiShan use to reject a batch if it wants to be 97.5% sure of not making a mistake?

Solution

If the proportion of errors is 4%, $\pi = 0.04$. Samples of size n have a Normally distributed proportion of defective units with:

- mean $= \pi = 0.04$
- standard deviation $= \sqrt{(\pi(1 - \pi)/n)} = \sqrt{(0.04 \times 0.96/200)} = 0.014$.

When SemiShan reject a batch of contracts, they want to be 97.5% sure that the mean is above 0.04. With a Normal distribution the point with 2.5% of the population in the tail is 1.96 standard deviations from the mean (as shown in Figure 19.8). So they should reject a batch when the number of errors is above:

$$0.04 + 1.96 \times 0.014 = 0.067$$

With a sample of 200 units this means $0.067 \times 200 = 13.4$ defects.

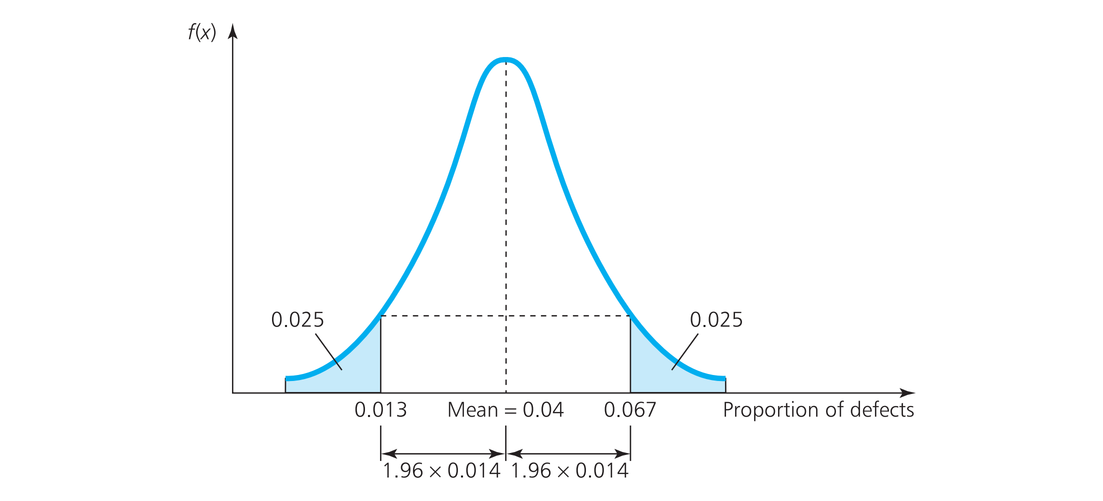


Figure 19.8 Sampling for SemiShan Communications

The last worked example illustrates the approach of acceptance sampling:

- Specify a sample size.
- Take a random sample of this size from a batch.
- Specify a maximum allowed number of defects in the sample.
- Test the sample to find the number that are actually defective.
- If the number of defects is greater than the allowed maximum number, reject the batch.
- If the number of defects is less than the allowed maximum number, accept the batch.

The maximum allowed number of defects in a sample is largely a matter of management policy, as it relies on their judgement about acceptable levels of quality. But remember that this idea of ‘defects’ can be misleading, as it suggests that products are not working properly. In practice, variability is small and a product can be described as defective even when it works properly and satisfies customers. Being defective means that a unit does not meet the supplier's internal targets – which might be considerably more demanding than those of the customers. You can imagine this with a call centre, where most customers are happy to wait three seconds before their call is answered, but operators describe a response as defective if it is not answered before the second ring.

Suppose that managers are prepared to accept a batch of products when fewer than 2% are defective. Ideally, they will take a sample and reject the batch if more than 2% of the sample is defective. But you know that this does not give perfect results, as even the best sample is not a perfect reflection of the population. So managers will reject some batches that are good (Type I errors) and accept some batches that are defective (Type II errors). The best we can do is to give batches with few defects a high probability of acceptance, and batches with more defects a high probability of rejection. The way to achieve this is with big samples – but more generally we have to consider four related measures:

- **Acceptable quality level** (AQL) is the poorest level of quality that we will accept – in other words, the maximum proportion of defects that still allows us to describe a batch as 'good'. We accept any batch with fewer defects than AQL, with typical figures around 1%.
- **Lot tolerance percent defective** (LTPD) is the quality that is unacceptable – or the highest proportion of defects that customers are willing to accept in a batch. We reject any batch with more defects than LTPD.

We want a low probability of rejecting a good batch, which we define as one with fewer defects than AQL. We can formalise this by defining:

- **producer's risk** (α) – the highest acceptable probability of rejecting a good batch, with fewer defects than the AQL. This is typically set around 5%.

We also want a low probability of accepting a bad batch, which we define as one with more defects than the LTPD. We can formalise this by defining:

- **consumer's risk** (β) – the highest acceptable probability of accepting a bad batch, with more defects than LTPD. This is typically set around 10%.

Using these four measures, we can use standard analyses to find values for n, the sample size, and c, the maximum number of allowed defects. A huge amount of work has been done on quality control statistics, and we do not have to duplicate this. The easiest way of finding values for n and c is to use a standard quality control package, many of which are available.

WORKED EXAMPLE 19.5

Juliet Ndalla buys components in batches from a supplier. The supplier uses an acceptance quality level of 2% defective, while Juliet accepts batches with a maximum of 6% defective. What are appropriate values of n and c?

Solution

We know that:

AQL = 0.02

LTPD = 0.06

Worked example 19.5 continued

We can find values for n and c from standard sampling plans. Traditionally, managers would calculate the ratio of LTPD/AQL and find the entry in sampling tables that is equal to, or just greater than, this value. Here LTPD/AQL = 0.06/0.02 = 3. Figure 19.9 shows an extract from sampling tables, and the value that is slightly greater than this is 3.21, which corresponds to $c = 6$. Then we use the third column of the table to find an implied sample size. The corresponding value of $n \times$ AQL is 3.29. We know that AQL = 0.02, so $n \times 0.02 = 3.29$, or $n = 164.5$. This gives the sampling plan:

- Take samples of 165 units.
- If six or fewer units are defective, accept the batch.
- If more than six units are defective, reject the batch.

Now managers do not have to use sampling tables, as all the work is done by standard programs. Figure 19.10 shows a simple printout, where four alternative plans are suggested based on different values of α and β.

LTPD/AQL	*c*	*n* × AQL
44.89	0	0.05
10.95	1	0.36
6.51	2	0.82
4.89	3	1.37
4.06	4	1.97
3.55	5	2.61
3.21	6	3.29
2.96	7	3.98
2.77	8	4.70
2.62	9	5.43
2.50	10	6.17

Figure 19.9 Extract from a table of sampling statistics

****** **QUALITY CONTROL STATISTICS** ******

Title: **Design of quality control sampling plan**
For: **Attribute sampling**

DATA ENTERED
- Acceptance quality level AQL = 0.02
- Lot tolerance percent defective LTPD = 0.06
- Producer's risk α = 0.05
- Consumer's risk β = 0.10

CRITICAL VALUES

LTPD/AQL = 3.00
Inferred *n* = 6 maximum number of defects
n × AQL = 3.29
Inferred *n* = 165 sample size

SUGGESTED PLANS

Take a sample of 165 units from a batch
If 6 or less units are defective accept the batch
If more than 6 units are defective reject the batch

SENSITIVITY AND ALTERNATIVE PLANS

Plan Number	Sample size (*n*)	Number of failures (*c*)	Actual alpha	Actual beta
1	165	6	0.051	0.137
2	176	6	0.067	0.099
3	200	7	0.051	0.090
4	197	7	0.048	0.098

Figure 19.10 Example of a printout giving alternative sampling plans

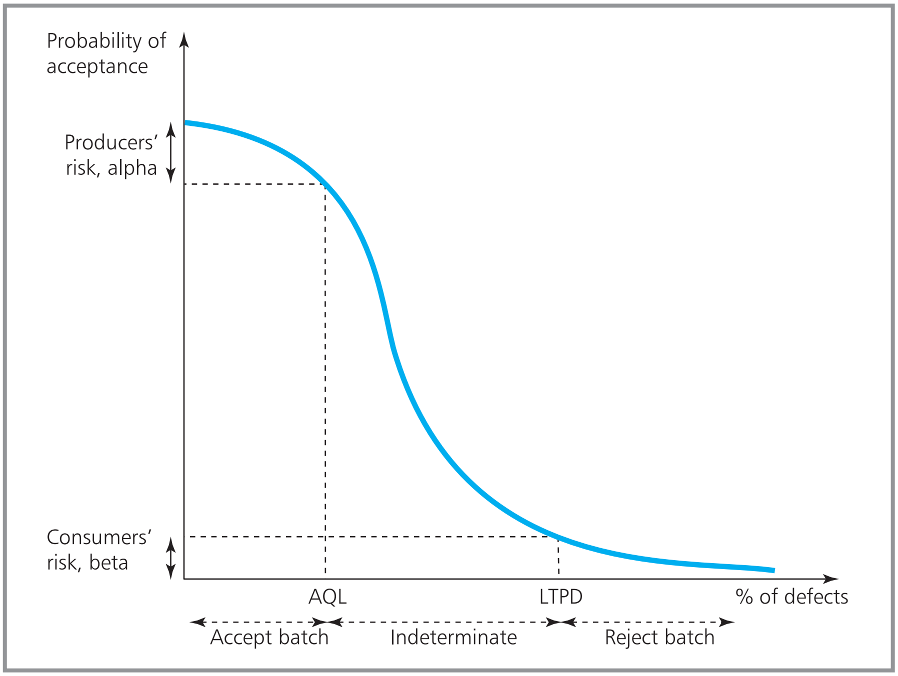


Figure 19.11 Operating characteristic curve for sampling plan

Each sampling plan has an **operating characteristic curve** (OC curve) which shows how well it actually separates good and bad batches. An OC curve shows the probability that a sampling plan accepts batches with different proportions of defects. Each combination of n and c has a distinct curve with the general shape shown in Figure 19.11. The shape of this curve is set by two points, one defined by AQL and α, and the second defined by LTPD and β.

We want to make a clear distinction between good and bad batches, so the OC curve should be as steep as possible. Ideally it would be vertical, differentiating perfectly between a good batch (with a probability of acceptance of 1) and a bad batch (with a probability of acceptance of 0). The only realistic way of getting close to this is to take large samples.

Review questions

19.13 What is the difference between acceptance sampling and process control?

19.14 What is the difference between sampling by variable and sampling by attribute?

19.15 Why is an ideal operating characteristic curve vertical?

Process control

Acceptance sampling checks the quality of products, while process control checks that the process making the products is working as planned. For this it makes sure that the random variation in products stays within acceptable limits. More specifically, it monitors samples over time to see whether there are any noticeable trends. For instance, if an increasing number of units are being rejected, we know that the process performance is deteriorating and it needs adjusting.

Process control charts give an easy format for monitoring performance. A basic chart takes a series of samples over time and plots a graph of the proportion of defectives. This gives a *p*-chart. The proportion of defects is usually around the proportion of defects expected in the population. Provided it does not vary far from this value, we can say that the process is working as planned. But we can define two limits to show when a process is out of control: an upper control limit (UCL) above the mean level, and a lower control limit (LCL) below the mean level. Provided the output stays between these two limits, we say that the process is in control – but if it moves outside the limits there is something wrong (as shown in Figure 19.12).

We can calculate control limits from the results we already know. If the proportion of defects in a population is π, the proportion of defects in a sample of size n is Normally distributed, with mean π and standard deviation $\sqrt{(\pi(1 - \pi)/n)}$. Then we can calculate the control limits from:

- upper control limit = UCL = $\mu + Z \times$ standard deviation
- lower control limit = LCL = $\mu - Z \times$ standard deviation

where Z is the number of standard deviations of the specified confidence limit.

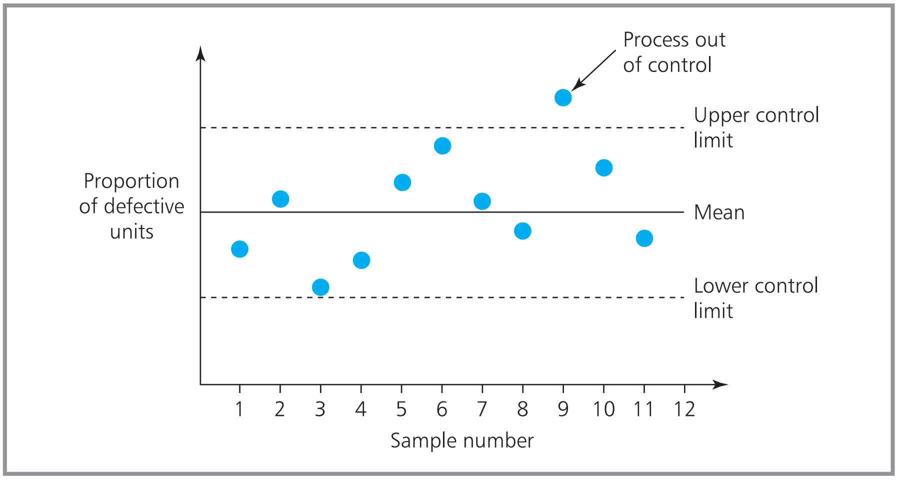


Figure 19.12 Typical process control chart

WORKED EXAMPLE 19.6

Joan McKenzie collected a sample of 500 units of the output from a process for each of 30 working days when it was known to be working normally. She tested these samples and recorded the number of defects as follows.

Day	Number of defects	Day	Number of defects	Day	Number of defects
1	70	11	45	21	61
2	48	12	40	22	57
3	66	13	53	23	65
4	55	14	51	24	48
5	50	15	60	25	42
6	42	16	57	26	40
7	64	17	55	27	67
8	47	18	62	28	70
9	51	19	45	29	63
10	68	20	48	30	60

What are the process control limits with 95% confidence?

Solution

The average proportion of defects is:

$$\pi = \frac{\text{total number of defects}}{\text{number of observations}} = \frac{1{,}650}{30 \times 500} = 0.11$$

$$\begin{aligned}\text{standard deviation} &= \sqrt{(\pi(1-\pi)/n)} \\ &= \sqrt{(0.11 \times 0.89/500)} = 0.014\end{aligned}$$

The 95% confidence limit shows the range within which 95% of samples lie when the process is working normally, and this has $Z = 1.96$, so:

- UCL $= p + Z \times$ standard deviation $= 0.11 + 1.96 \times 0.014 = 0.137$
- LCL $= p - Z \times$ standard deviation $= 0.11 - 1.96 \times 0.014 = 0.083$

Joan can assume that with samples of 500 the process is under control when the number of defects is between $0.083 \times 500 = 42$ and $0.137 \times 500 = 69$. If the proportion of defects moves outside this range, the process is out of control.

Notice that the data for drawing the control charts was collected when the process was known to be working normally. Obviously, if the process was already out of control when the data were collected, the results would be meaningless.

Some observations will be outside the control limits purely by chance – and with a 95% confidence interval, random variations leave 5% of samples outside. So managers should check every observation outside the limits to see whether the process is really out of control or whether it is actually working normally.

As well as checking the proportion of defects, we can use control charts to monitor the value of some variable, such as weight or cost. Then the usual approach is to plot two charts, one showing the mean values of the samples and a second showing the ranges (where the range is the difference between the largest and smallest observation in the sample). For example, suppose that a mobile telephone company takes samples to monitor the duration of calls. It can plot two control charts, one showing the mean length of calls in each sample and a second showing the range. Provided samples keep within control limits for both charts, the process is in control. If a sample moves outside the control limits on either chart, the process is out of control.

WORKED EXAMPLE 19.7

A company has taken samples of 10 units from a process in each of the past 20 days. Each unit in the sample was weighed, and the mean weight and range were recorded (shown in Figure 19.13). Draw process control charts for the sample means and ranges.

Solution

We could calculate the ranges, but these analyses are done so often that the easiest way of finding control limits is to use standard software. Figure 19.13 shows the results from one package. This has estimated the sample mean and standard deviation, and used these to set control limits that are three standard deviations away from the mean.

****** **QUALITY CONTROL STATISTICS** ******

Analysis by weight
Designing control charts

DATA ENTERED

Sample size = 10

Sample	Mean	Range	Sample	Mean	Range
1	12.2	4.2	11	12.5	3.3
2	13.1	4.6	12	12.3	4.0
3	12.5	3.0	13	12.5	2.9
4	13.3	5.1	14	12.6	2.7
5	12.7	2.9	15	12.8	3.9
6	12.6	3.1	16	12.1	4.2
7	12.5	3.2	17	13.2	4.8
8	13.0	4.6	18	13.0	4.6
9	12.2	4.3	19	13.2	5.0
10	12.0	5.0	20	12.6	3.8

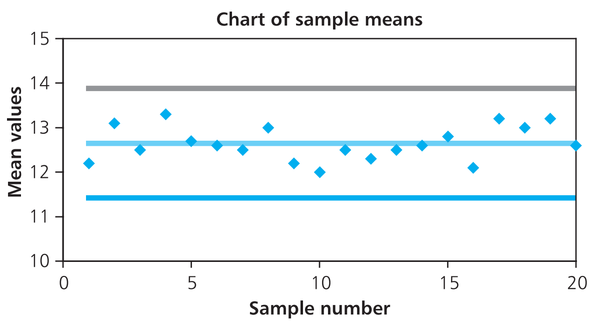
CONTROL LIMITS

Population mean = 12.65
Sample standard deviation = 0.41

Mean of sample ranges = 3.96
Sample standard deviation = 1.03

Control limits on sample means:
- Lower control limit = 11.42
- Centre line = 12.65
- Upper control limit = 13.88

Control limits on sample ranges:
- Lower control limit = 0.87
- Centre line = 3.96
- Upper control limit = 7.05



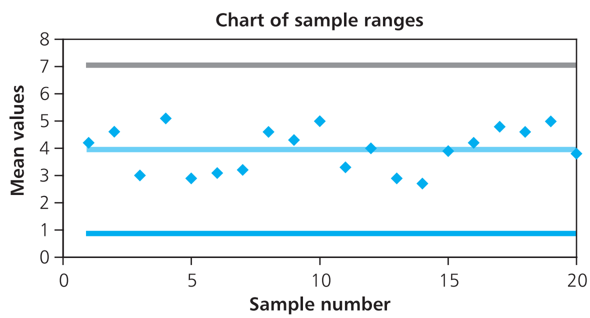


Figure 19.13 Sample of a printout for process control charts

Review questions

19.16 What does it mean if an observation is outside the limits in a process control chart?

19.17 What patterns should you investigate in a control chart?

IDEAS IN PRACTICE Stroh Brewery Company

Until 1999 when they sold their brands, the Stroh Brewery Company was the third-largest producer of beer in the USA. One of their plants is the Winston-Salem brewery, which occupied over 100,000 square metres and made 200 million gallons of beer a year.

Quality control of beer was rigorous, with the brewery checking everything from taste to the quantity in each can. For this they employed 38 people in three separate laboratories for microbiology, brewing and packaging. These people did 1,100 separate tests on each batch of beer. If they found problems, the quality control department stopped production and investigated.

A typical test in the brewing laboratory took a small sample during fermentation, diluted it, and counted the yeast cells. Beer must have a standard 16 million yeast cells (± 2 million) per millilitre of beer.

A typical test in the packaging laboratory checked the amount of air in a beer can. Because air can affect the taste, the company allowed a maximum of 1 cm^3 of air in a can. This was checked by testing three cans from the production line, five times a shift. If a sample was found with more than 1 cm^3 of air, the entire batch was put into 'quarantine' and systematically tested to find the point where the canning went wrong. As each line fills 1,600 cans a minute, this could mean a lot of testing.

CHAPTER REVIEW

This chapter introduced the broad area of quality management, which is the function responsible for all aspects of quality.

- It is difficult to give a general definition of quality. A common view says that it is the ability to meet – and preferably exceed – customer expectations. Then design quality means that product designs satisfy all requirements, particularly customer demand; achieved quality makes sure that products actually meet designed specifications.
- The four components of quality cost are prevention, appraisal, internal failure and external failure costs. Failure costs can be particularly high, but fall with increasing quality. This means that minimum overall costs come by making products with perfect quality. Total Quality Management focuses the effort of everyone in the organisation on making products of perfect quality.
- Even the best processes have some variation. High quality comes by reducing the amount of variation and keeping actual performance close to targets. This is monitored by quality control. Several tools help with quality control, including '5-whys', Pareto analyses, and cause-and-effect diagrams.
- Statistical sampling is at the core of quality control. This considers either acceptance sampling or process control.
- Acceptance sampling checks that a batch of products reaches the designed quality. It takes a random sample from a batch and checks that the number of defects is below a maximum permitted number.
- Sampling is also used for process control, to check that a process continues to work normally. The usual format has a chart with control limits to monitor performance over time.

CASE STUDY Bremen Engineering

Jurgen Hansmann is the Quality Control Manager of Bremen Engineering. On Tuesday morning he got to work at 7.30 and was immediately summoned by the General Manager. As Jurgen approached, the General Manager threw him a letter that had obviously come in the morning mail. Jurgen saw that the General Manager had circled two sections of the letter in red ink.

> 'We have looked at recent figures for the quality of one of the components you supply, AM74021-74222. As you will recall, we have an agreement that requires 99.5% of delivered units of this product to be within 5% of target output ratings. While your supplies have been achieving this, we are concerned that there has been no apparent improvement in performance over time.'
>
> 'We put considerable emphasis on the quality of our materials, and would like to discuss a joint initiative to raise the quality of your components. By working together we can share ideas and get mutual benefits.'

The General Manager waited for a few minutes and said:

> 'I find it incredible that we are sending poor quality goods to one of our biggest customers. We have a major complaint about our quality. Complete strangers think that we can't do our job properly, so they'll come and show us how to do it. This is your problem. I suggest you start solving it immediately.'

The General Manager's tone made Jurgen rather defensive and his reply was less constructive than normal.

> 'There is nothing wrong with our products. We agreed measures for quality and are consistently achieving these. We haven't improved quality because we didn't agree to improve it, and any improvement would increase our costs. We are making 995 units in 1,000 at higher quality than they requested, and the remaining 0.5% are only just below it. To me, this seems a level of quality that almost anyone would be proud of.'

The process for making AM74021-74222 is in five stages, each of which is followed by an inspection. The units then have a final inspection before being sent to customers. Jurgen now considered more 100% inspections, but each manual inspection costs about €0.60 and the selling price of the unit is only €24.75. There is also the problem that manual inspections are only 80% accurate. Automatic inspections cost €0.30 and are almost completely reliable, but they cannot cover all aspects of quality and at least three inspections have to remain manual.

Jurgen produced a weekly summary of figures to show that things were really going well.

	Inspection											
	A		B		C		D		E		F	
Week	Inspect	Reject	Inspect	Reject	Inspect	Reject	Inspect	Reject	Inspect	Reject	Inspect	Reject
1	4,125	125	350	56	287	0	101	53	3,910	46	286	0
2	4,086	136	361	0	309	0	180	0	3,854	26	258	0
3	4,833	92	459	60	320	0	194	0	4,651	33	264	0
4	3,297	43	208	0	186	0	201	0	3,243	59	246	0
5	4,501	83	378	0	359	64	224	65	4,321	56	291	0
6	4,772	157	455	124	401	0	250	72	4,410	42	289	0
7	4,309	152	420	87	422	0	266	123	3,998	27	287	64
8	4,654	101	461	0	432	0	278	45	4,505	57	310	0
9	4,901	92	486	0	457	0	287	0	4,822	73	294	0
10	5,122	80	512	0	488	0	301	0	5,019	85	332	0
11	5,143	167	524	132	465	48	290	61	4,659	65	287	0
12	5,119	191	518	0	435	0	256	54	4,879	54	329	0
13	4,990	203	522	83	450	0	264	112	4,610	55	297	0
14	5,231	164	535	63	475	0	276	0	5,002	32	267	0
15	3,900	90	425	56	288	0	198	0	3,820	37	290	58

Case study continued

Week	Inspection A Inspect	Reject	B Inspect	Reject	C Inspect	Reject	D Inspect	Reject	E Inspect	Reject	F Inspect	Reject
16	4,277	86	485	109	320	0	229	0	4,109	38	328	0
17	4,433	113	435	0	331	0	265	67	4,259	29	313	0
18	5,009	112	496	0	387	0	198	62	4,821	52	269	0
19	5,266	135	501	65	410	0	299	58	5,007	51	275	64
20	5,197	142	488	0	420	72	301	73	4,912	48	267	0
21	4,932	95	461	0	413	0	266	0	4,856	45	286	0
22	5,557	94	510	0	456	0	160	64	5,400	39	298	61
23	5,106	101	488	74	488	0	204	131	4,795	36	326	0
24	5,220	122	472	0	532	0	277	125	4,989	29	340	56
25	5,191	111	465	0	420	0	245	185	4,927	42	321	0
26	5,620	87	512	45	375	0	223	134	5,357	48	332	0

Notes on inspections

For sampling inspections, all production is considered in notional batches of 1 hour's output. Random samples are taken from each batch and if the quality is too low the whole batch is rejected, checked and reworked as necessary.

- A – automatic inspection of all units: rejects all defects
- B – manual inspection of 10% of output: rejects batch if more than 1% of batch is defective
- C – manual inspection of 10% of output: rejects batch if more than 1% of batch is defective
- D – manual inspection of 5% of output: rejects batch if more than 2% of batch is defective
- E – automatic inspection of all units: rejects all defects
- F – manual inspection of 5% of output: rejects batch if more than 1% of batch is defective

Questions

- **Do you think the General Manager's view is reasonable? What about Jurgen Hansmann's reaction?**
- **How effective is quality control at Bremen Engineering?**
- **Do you think the product quality needs to be improved? How would you set about this?**

PROBLEMS

19.1 Amwal Corporation had the following costs (in thousands of pounds) over the past six years. Describe what has been happening.

Year	1	2	3	4	5	6
Sales value	623	625	626	635	677	810
Costs:						
Design	6	8	18	24	37	43
Appraisal	15	17	22	37	45	64
Internal failure	91	77	32	36	17	10
External failure	105	101	83	51	27	16

19.2 Hung Gho Chan make a part on an assembly line. At one point they find that 2% of units are defective. It costs $1 to inspect each unit at this point, but the inspection would find only 70% of faults. If the faults are left, all parts will be found and corrected further down the line at a cost of $8. Is it worth inspecting all units at this point?

19.3 Sentinal Phoneback answers customer enquiries with telephone calls. When they timed a sample of 40 calls, they found a mean duration of 14.9 minutes and a standard deviation in duration of 2 minutes. What are the 95% and 99% confidence intervals for the true length of calls?

19.4 Eriksonn Catering says that its suppliers should send at most 2% of units that do not meet its 'outstanding' standard of quality. It receives a large shipment and takes a sample of 100 units. The company wants to be 95% sure that a rejected batch is really unsatisfactory. What criteria should it use to reject a batch?

19.5 Carn Bay Components make batches of a basic product in Toronto and transfer it to their main manufacturing plant in Chicago. When the product is made, an acceptance quality level of 1% defective is used, but transferred batches are allowed a maximum of 4% defective. The company accept a 5% risk of rejecting good batches, and a 10% risk of accepting bad batches. What would be a reasonable sampling plan for the component?

19.6 A service provider checks 24 samples of 200 clients to see whether they are giving an acceptable level of service. The numbers of unsatisfactory results were as follows.

Day	Number of defects	Day	Number of defects	Day	Number of defects
1	21	9	15	17	20
2	32	10	13	18	19
3	22	11	16	19	25
4	17	12	17	20	16
5	16	13	20	21	15
6	14	14	19	22	13
7	21	15	17	23	24
8	17	16	22	24	25

Draw control charts with 95% and 99% confidence limits on the process.

19.7 Gunta Hans took 30 samples of 15 units from a process. The average sample range for the 30 samples is 1.025 kg and the average mean is 19.872 kg. Draw control charts for the process.

19.8 Pioneer Remedial found that a particular repair takes a mean time of 75.42 minutes with a standard deviation of 2.01 minutes. If samples of eight are taken, find the control limits that include 99% of sample means if the process is working normally.

RESEARCH PROJECTS

19.1 Find a product that you have been particularly pleased with. Describe the aspects of its quality that you like. How many of these can you measure? How many people – from initial designers through to the person who delivered it – were involved in supplying this high-quality product? How could you make the product even better?

19.2 Quality management has undergone a revolution in recent years, with customers no longer willing to accept defective products. A key element of this has been the changing role of quality control. How would you describe the changes that have occurred in quality control?

19.3 A lot of software is available for quality control. Figure 19.14 shows a printout from a simple package that takes data, suggests a sampling plan, and shows the operating curve for this plan and the average outgoing quality. How does this compare with other software? What features do you think there should be?

19.4 Despite the attention paid to quality management, many products still do not meet acceptable standards. Give some examples of products that you think are unsatisfactory. Why is this? What can be done to improve these products?

Sampling Plan for Test Example: Single Sampling for Attributes

01-01-2010	Plan	Value
1	Given AQL	1%
2	Given RQL	10%
3	Desired producer's risk (alpha)	5%
4	Desired consumer's risk (beta)	10%
5	Lot size	10000
6	Recommended sample size (*n*)	37
7	Recommended acceptance number (*c*)	1
8	Resulted producer's risk (alpha)	5.2878%
9	Resulted consumer's risk (beta)	10.3631%
10	Resulted AQL	2.2914%

Test example

Randomly choose 37 items from a lot of 10,000 and inspect them.

If the number of defects is greater than 1 reject the lot.

Otherwise accept the lot.

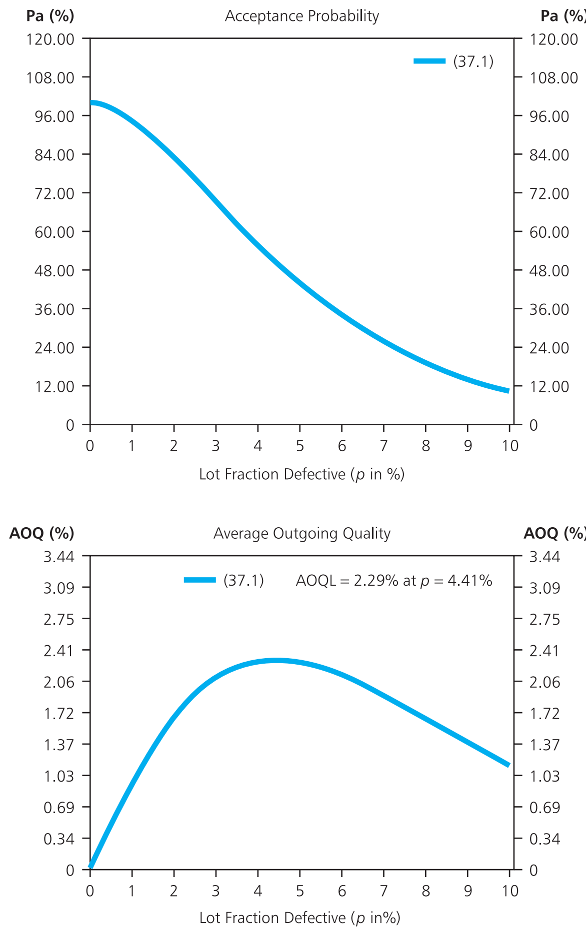


Figure 19.14 Printout from quality control package

Sources of information

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Further reading

Quality management has been such a popular topic in recent years that there is no shortage of books. The following list gives some ideas, but you can find many others.

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CHAPTER 20

Inventory management

Contents

Chapter outline

Stocks are the stores of materials that are held in every organisation. These stocks are surprisingly expensive, so managers look for ways of minimising their costs. This chapter describes some models for inventory management, starting with the classic economic order quantity. There are many extensions to this basic model. In practice, organisations increasingly attempt to minimise their stock by organising efficient flows of materials through supply chains.

After finishing this chapter you should be able to:

- Appreciate the need for stocks and the associated costs
- Discuss different approaches to inventory management
- Calculate an economic order quantity and reorder level
- Calculate the effects of fixed production rates
- Appreciate the need for safety stock and define a service level
- Calculate safety stock when lead time demand is Normally distributed
- Describe periodic review models and calculate target stock levels
- Do ABC analyses of inventories.

Background to stock control

If you look around any organisation you will find **stocks**. These are the stores of materials that an organisation holds until it needs them. The problem is that stocks always incur costs for tied-up capital, storage, warehousing, deterioration, loss, insurance, movement, and so on. So you might ask the obvious question, 'Why do organisations hold stock?' There are several answers to this, but the main one is that stocks give a buffer between supply and demand.

Imagine a supermarket that keeps a stock of goods on its shelves and in its stockroom. It holds the stock because lorries make large deliveries at relatively infrequent intervals, while customers make small demands that are almost continuous. So there is a mismatch between supply and demand, and the supermarket overcomes this by holding stock.

- Stocks are the stores of materials that an organisation holds until it needs them.
- The main purpose of stocks is to act as a buffer between supply and demand.

The short-term mismatch between supply and demand is only one reason for holding stock, and others include:

- to act as a buffer between different production operations – 'decoupling' consecutive operations
- to allow for demands that are larger than expected, or at unexpected times
- to allow for deliveries that are delayed or too small
- to take advantage of price discounts on large orders
- to buy items when the price is low and expected to rise
- to buy items that are going out of production or are difficult to find
- to make full loads and reduce transport costs
- to give cover for emergencies.

Types of stock

Just about everything is held as stock somewhere, whether it is raw materials in a factory, finished goods in a shop, or tins of baked beans in your pantry. We can classify these stocks as:

- **Raw materials** – the materials, parts and components that have been delivered to an organisation, but are not yet being used.
- **Work-in-progress** – materials that have started but not yet finished their journey through operations.
- **Finished goods** – finished goods that have finished their operations and are waiting to be delivered to customers.

This is a fairly arbitrary classification, as one company's finished goods are another company's raw materials. Some organisations (notably retailers and wholesalers) have stocks of finished goods only, while others (like manufacturers) have all three types in different proportions. Some items do not fall easily into these categories, and we can define two additional types (illustrated in Figure 20.1):

- **Spare parts** for machinery, equipment, etc.
- **Consumables** such as oil, fuel, paper, etc.

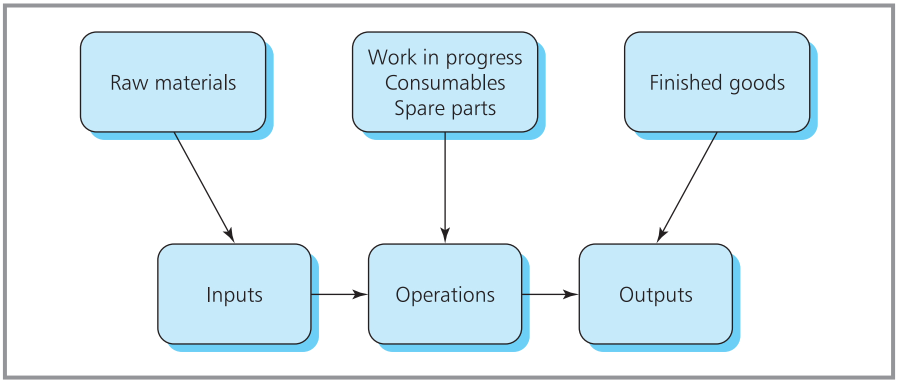

Figure 20.1 Classification of stocks

Approaches to inventory management

Organisations have dramatically changed their views of stock in recent years. Historically, they saw stock as a benefit, with high stocks ensuring continuing operations, giving cover for any problems – and even giving a measure of wealth. This thinking encouraged organisations to maximise their stock – and is still the reason why countries keep reserves of gold and why individuals keep food in their freezer. But in the twentieth century, it became clear that these stocks had costs that could be surprisingly high. Then organisations began to view stocks not as an unreserved benefit, but as a resource that needed careful control. In particular, they looked for ways of minimising the overall costs.

More recently, organisations have gone further in reducing stocks, and they try to work with very low levels. There has been a trend towards operations that move materials quickly and efficiently through supply chains, matching supply to demand so that stocks are not accumulated. When this works it gives considerable savings, but it is not a realistic option for all operations. Most organisations cannot work properly without stock, and then they have to consider its management. There are two ways of doing this – assuming independent or dependent demand.

The conventional approach to inventory management assumes that overall demand for a product is made up of individual demands from many separate customers. These demands are independent of each other, so that the demand from one customer is not related to the demand from another. If you are selling Nike shoes, the overall demand comes from hundreds of separate customers, all independently asking for a pair of shoes. This gives an **independent demand**.

There are many situations where demands are not independent. One demand for a product is not independent of a second demand for the product; demand for one product is not independent of demand for a second product. When a manufacturer uses a number of components to make a product, the demands for all components are clearly related, since they all depend on the production plan for the final product. This gives **dependent demand**.

These two patterns of demand need different means of inventory management. Independent demand uses forecasts of future demand, usually based on past patterns (which we discussed with projective forecasting in Chapter 10).

Dependent demand typically uses a production schedule to design timetables for the delivery of materials. You can see the differences between independent and dependent demand approaches in the way that a restaurant chef plans the ingredients for a week's meals. An independent demand system sees what ingredients were used in previous weeks, uses these past demands to forecast future demands, and then buys enough to make sure that there is enough in the pantry to cover these forecast demands. The alternative dependent demand approach looks at the meals the chef plans to cook each day, analyses these to see what ingredients are needed, and then orders the specific ingredients to arrive when they are needed.

Here we will describe some models for independent demand. These look at ways of minimising overall costs, so we should start by taking a closer look at the costs involved.

Costs of holding stock

The cost of holding stock is typically around 25% of its value a year. This is made up of four components, for unit, reorder, holding and shortage costs.

- **Unit cost** (U). This is the price of an item charged by the supplier, or the cost to an organisation of acquiring one unit of an item. It may be fairly easy to find this by looking at quotations or recent invoices from suppliers. But sometimes it is more difficult when several suppliers offer alternative products or give different purchase conditions. If a company makes an item itself, it can be difficult to set a production cost or to calculate a reasonable transfer price.
- **Reorder cost** (R). This is the cost of placing a repeat order for an item and includes all the cost of administration, correspondence, delivery, insurance, receiving, checking, follow-up, expediting, and so on. Sometimes, costs such as quality control, transport, finding suppliers, negotiation, and a whole range of other things are included in the reorder cost. These can be difficult to find, and in practice you often get a good estimate for the reorder cost by dividing the total annual budget of a purchasing department by the number of orders it sends out.
- **Holding cost** (H). This is the cost of holding one unit of an item in stock for a period of time (typically a year). The obvious cost is tied-up money which is either borrowed (with interest payable) or could be put to other use (in which case there are opportunity costs). Other holding costs are for storage space, damage, deterioration, obsolescence, handling, special packaging, administration and insurance.
- **Shortage cost** (S). If an item is needed but cannot be supplied from stock, there is usually a cost associated with this shortage. In the simplest case a retailer may lose profit from a sale, but the effects of shortages are usually much more widespread. There may also be some loss of customer goodwill and future sales, as well as lost reputation. When there is a shortage of raw materials for production, there can be severe disruption, rescheduled production, re-timing of maintenance, laying-off of employees, and so on. There can also be allowances for positive action to overcome the shortage, perhaps sending out emergency orders, paying for special deliveries, storing partly finished goods or using other more expensive suppliers.

Shortage costs are always difficult to find – but there is general agreement that they can be very high. This allows us to look at the purpose of stocks again and rephrase our earlier statement by saying, 'the cost of shortages can be very high and, to avoid these costs, organisations are willing to incur the relatively lower costs of carrying stock'.

Review questions

20.1 What is the main reason for holding stock?

20.2 What is independent demand?

20.3 List four types of cost associated with stock.

IDEAS IN PRACTICE **Stock holdings at Schultz-Heimleich**

Schultz-Heimleich make veterinary pharmaceuticals in their Swiss laboratories, and they use stocks of finished goods to give a buffer between production and sales. This gives two important benefits. Firstly, the company can smooth its operations so that production does not have to follow the seasonal pattern of demand. The result is more efficient operations, easier planning, regular schedules, routine workflow, fewer changes, and so on. Secondly, the company does not have to install enough capacity to match peak sales, when this would be under-utilised during quieter periods. It installs enough capacity to meet the average demand – when production is higher than sales, stock builds up; and when sales are higher than production, stock declines.

Managers at Schultz-Heimleich collected figures over eight quarters to illustrate this effect. The following table (where values are in millions of Swiss francs) shows that production remained stable, with two small adjustments during a period when sales varied by up to 57%. In quarter 3, the company met a record high demand without increasing production, and their costs fell as they reduced investment in stock. If the company had installed enough capacity to meet this peak demand, its utilisation would have been only 52% in quarter 1.

Quarter	1	2	3	4	5	6	7	8
Sales	14	22	27	21	14	16	15	22
Percentage change	–	+57	+23	−22	−33	+14	−6	+47
Production	20	21	21	19	19	19	19	19
Percentage change	–	+5	0	−10	0	0	0	0
Change in stock	+6	−1	−6	−2	+5	+3	+4	−3
Average stock level	10	9	3	1	6	9	13	10

The economic order quantity

There are two basic policies for dealing with independent demand.

- **Fixed order quantity**, where an order of fixed size is placed whenever stock falls to a certain level. For example, a central heating plant orders 25,000 litres of oil whenever the amount in its tank falls to 5,000 litres. Such systems need continuous monitoring of stock levels and are better suited to systems with low, irregular demand for relatively expensive items.
- **Periodic review**, where orders of varying size are placed at regular intervals. For example, displays of clothes in a shop might be refilled every evening to replace whatever was sold during the day. The operating cost of this system is generally lower, so it is better suited to high, regular demand of low-value items.

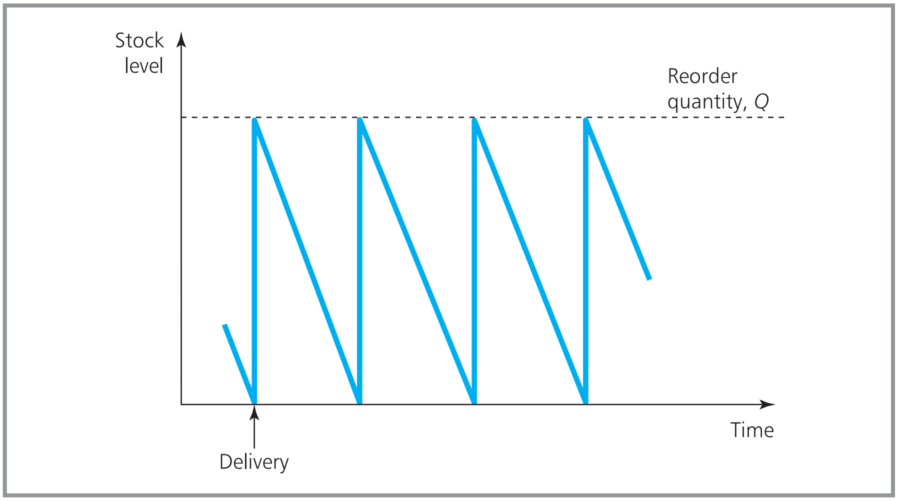

Figure 20.2 Stock level over time with fixed order quantities

We will start by looking at fixed order models, and calculate an **economic order quantity**, which finds the order size that minimises costs for a simple inventory system. The analysis considers a single item whose demand is known to be continuous and constant at D per unit time. It assumes that we know the unit cost (U), reorder cost (R) and holding cost (H), while the shortage cost (S) is so high that all demand must be met and no shortages are allowed.

With fixed order quantities the stock level alternately rises with deliveries and falls more slowly as units are removed to meet demand, giving the saw-tooth pattern shown in Figure 20.2.

Consider one cycle of this pattern, shown in Figure 20.3. At some point a delivery of an amount Q arrives and this is sold to customers at a constant

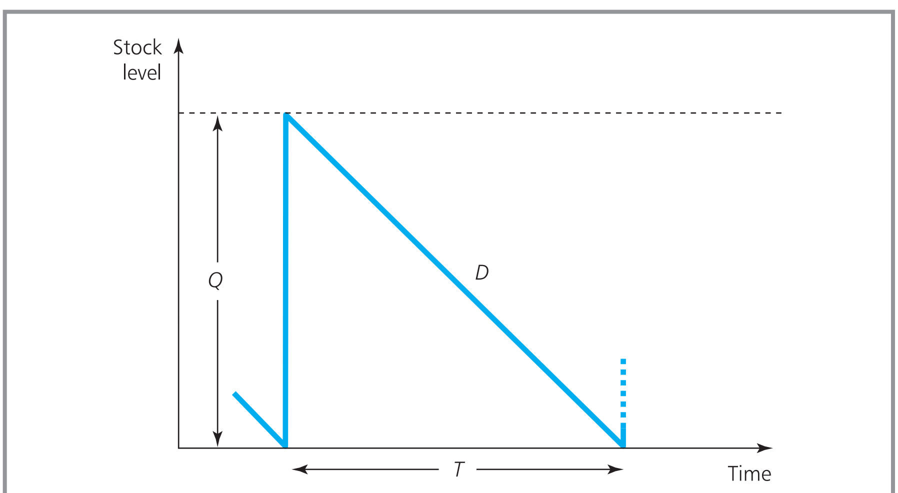

Figure 20.3 A single stock cycle

rate D until no stock remains, and we arrange for another delivery to arrive at this point. The stock cycle has length T and we know that:

$$\text{amount entering stock in the cycle} = \text{amount leaving stock in the cycle}$$
$$Q = D \times T$$

We also know that the stock level varies between Q and 0, so the average level is $(Q + 0)/2 = Q/2$.

We can find the total cost for the cycle by adding the four components of cost – unit, reorder, holding and shortage. As no shortages are allowed, we can ignore these costs, and the cost of buying materials is constant regardless of the ordering policy, so we can leave both of these out of the calculations and focus on the other two costs. The variable cost for the cycle is:

- total reorder cost = number of orders (1) × reorder cost (R) = R
- total holding cost = average stock level ($Q/2$) × time held (T) × holding cost (H) = $HQT/2$

Adding these two gives the variable cost for the cycle, and if we divide this by the cycle length, T, we get the variable cost per unit time, VC, as:

$$\text{VC} = (R + HQT/2)/T = R/T + HQ/2$$

But we know that $Q = DT$, or $T = Q/D$, and substituting this gives:

$$\text{VC} = RD/Q + HQ/2$$

We can plot the two parts on the right-hand side of this equation separately against Q, as shown in Figure 20.4. This curve has a distinct minimum which is the optimal order size, or economic order quantity. We can find a value for this by differentiating the equation for variable cost with respect to Q, and setting the result to equal zero (check in Chapter 13 if you are unsure about this):

$$0 = -RD/Q^2 + H/2$$

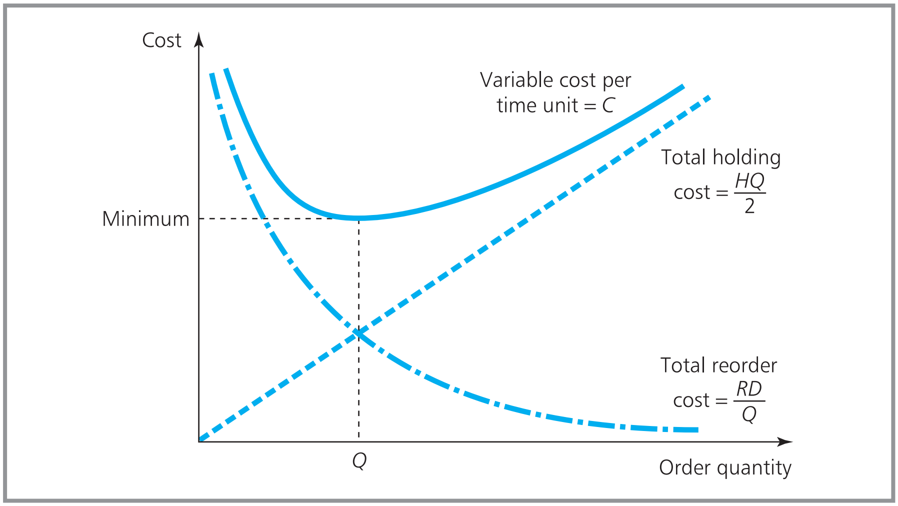

Figure 20.4 Finding the economic order quantity

or

economic order quantity $= Q_0 = \sqrt{\dfrac{2RD}{H}}$

If you order less than this, you have to place small, frequent orders, giving high reorder costs; if you order more than this, you place large, infrequent orders, giving high stocks and holding costs.

Now if we add the fixed cost of purchasing the materials, UD, to the variable cost, and do some arithmetic on the result, we get the optimal values as:

- total cost $= TC_0 = UD + VC_0$
- variable cost $= VC_0 = \sqrt{2RHD}$

WORKED EXAMPLE 20.1

The demand for an item is constant at 20 units a month. Unit cost is £50, cost of processing an order and arranging delivery is £60, and holding cost is £18 per unit per year. What are the economic order quantity, corresponding cycle length and costs?

Solution

Listing the values we know in consistent units:

$D = 20 \times 12 = 240$ units per year

$U =$ £50 per unit

$R =$ £60 per order

$H =$ £18 per unit per year.

Then substituting in the standard equations gives:

$Q_0 = \sqrt{(2RD/H)} = \sqrt{(2 \times 60 \times 240/18)}$
$= 40$ units

$VC_0 = \sqrt{(2RHD)} = \sqrt{(2 \times 60 \times 18 \times 240)}$
$=$ £720 a year

$TC_0 = U \times D + VC_0 = 50 \times 240 + 720$
$=$ £12,720 a year

We can find the cycle length, T_0, from $Q = DT$, so $40 = 240T$, or $T = 1/6$ years or 2 months.

The optimal policy (with total costs of £12,720 a year) is to order 40 units every 2 months.

Reorder level

The economic order quantity shows how much to order, but it does not say when to place an order. For this decision we have to know the lead time, L, between placing an order and having it arrive in stock. For simplicity, we will assume that this is fixed, so we get the pattern shown in Figure 20.5. To make sure that a delivery arrives just as stock is running out, we must place an order a time L before stock runs out. The easiest way of finding this point is to look at the current stock and place an order when there is just enough left to last the lead time. With constant demand of D, this means that we place an order when the stock level falls to lead time $\times$ demand, and this point is the reorder level.

reorder level $=$ ROL $=$ lead time $\times$ demand $= LD$

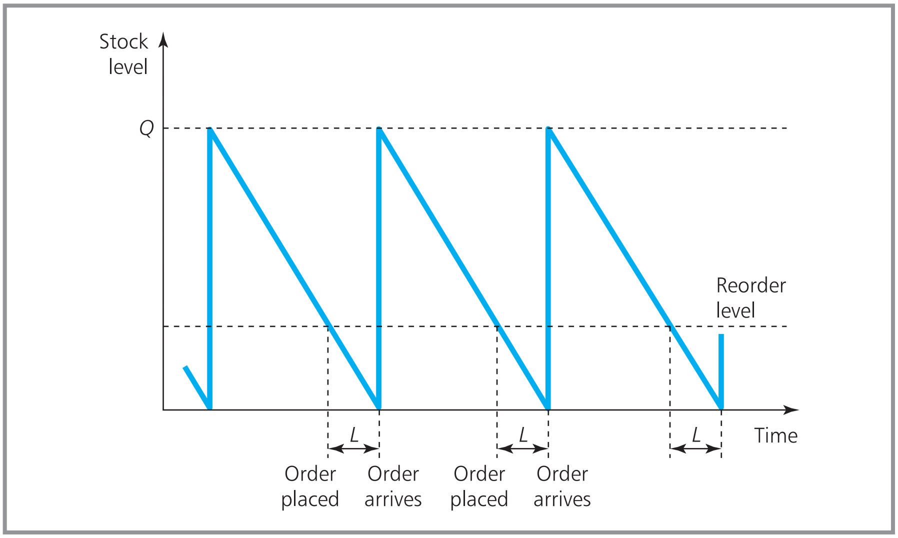

Figure 20.5 Stock level with a fixed lead time, *L*

WORKED EXAMPLE 20.2

Demand for an item is constant at 20 units a week, reorder cost is £125 an order and holding cost is £2 per unit per week. If suppliers guarantee delivery within 2 weeks, what is the best ordering policy?

Solution

Listing the variables in consistent units:

D = 20 units per week

R = £125 per order

H = £2 per unit per week

L = 2 weeks

Substituting these gives:

- economic order quantity: $Q_0 = \sqrt{(2RD/H)} = \sqrt{(2 \times 125 \times 20/2)} = 50$ units
- reorder level: ROL $= LD = 2 \times 20 = 40$ units

The best policy is to place an order for 50 units whenever stock falls to 40 units. We can find the cycle length from:

$$Q_0 = DT_0 \quad \text{so} \quad T_0 = 50/20 = 2.5 \text{ weeks}$$

The variable cost is:

$$\text{VC}_0 = \sqrt{(2RHD)} = \sqrt{(2 \times 125 \times 2 \times 20)} = £100 \text{ a week}$$

In practice, a ‘two-bin system’ gives a useful way of timing orders. This has stock kept in two bins, one of which holds the reorder level while the second holds all remaining stock. Demand is met from the second bin until this is empty. At this point you know that stock has declined to the reorder level and it is time to place an order.

Review questions

20.4 What is the economic order quantity?

20.5 If small orders are placed frequently (rather than placing large orders infrequently) does this:
(a) reduce total costs
(b) increase total costs
(c) either increase or decrease total costs?

20.6 What is the reorder level?

20.7 How would you calculate a reorder level?

IDEAS IN PRACTICE El Asiento Rojolo

El Asiento Rojolo makes parts for speedboats, most of which it sells to a major manufacturer whose factory is in the same area of San Diego. They made products in batches. A typical product had demand of 150 units a month and notional batch set-up and holding costs of \$2,000 and \$10 respectively, giving an optimal batch size of 245 units. El Asiento rounded this to 300 units, and made one delivery every two months to their biggest customer. But the company had to change its operations completely when this customer moved to just-in-time operations and said that they would accept deliveries in batches of no more than 20 units. El Asiento did not want to keep stocks themselves, as this would considerably increase their costs. Their only option was to dramatically reduce the batch set-up cost – notionally from \$2,000 to \$13. In practice, this meant that they completely redesigned their production, introducing continuous, flexible automation.

Stock control for production

The economic order quantity makes a series of assumptions, but we can remove these to give models that are useful in many different circumstances. Here we consider one extension, which assumes that replenishment is done at a fixed rate rather than having all units delivered at the same time.

If a company manufactures a product at a rate of 10 units an hour, the stock of finished goods will increase at this rate. In other words, there is not a single large delivery, but the stock level slowly rises over some time. We can allow for this by a simple adjustment to the economic order quantity. When the rate of production, P, is less than the rate of demand, D, there is no problem with stock holding: supply is not keeping up with demand and as soon as a unit is made it is passed straight out to customers. There are stock problems only when the rate of production is higher than the demand (which means that $P > D$). Then stock builds up at a rate $(P - D)$ for as long as production continues. At some point managers must stop production and transfer operations to make other items. So after some time T_P, production stops – demand from customers continues at a rate D, and this is met from the accumulated stock. After some further time, T_D, the stock is exhausted and production must restart. Figure 20.6 shows the resulting stock level.

We want to find an optimal value for the batch size. This is equivalent to finding the economic order quantity, so we use the same approach – finding the total cost for a single stock cycle, dividing this by the cycle length to give a cost per unit time, and then minimising this cost.

If we make batches of size Q, this would be the highest stock level with instantaneous replenishment. But as units are actually fed into stock at a fixed rate and are continuously removed to meet demand, the maximum stock level is lower than Q, and occurs at the point where production stops. We can find the value for A, the highest actual stock level, from the following argument. During the productive part of the cycle, T_P, we have:

$$A = (P - D) \times T_P$$

We also know that total production during the period is:

$$Q = P \times T_P \quad \text{or} \quad T_P = Q/P$$

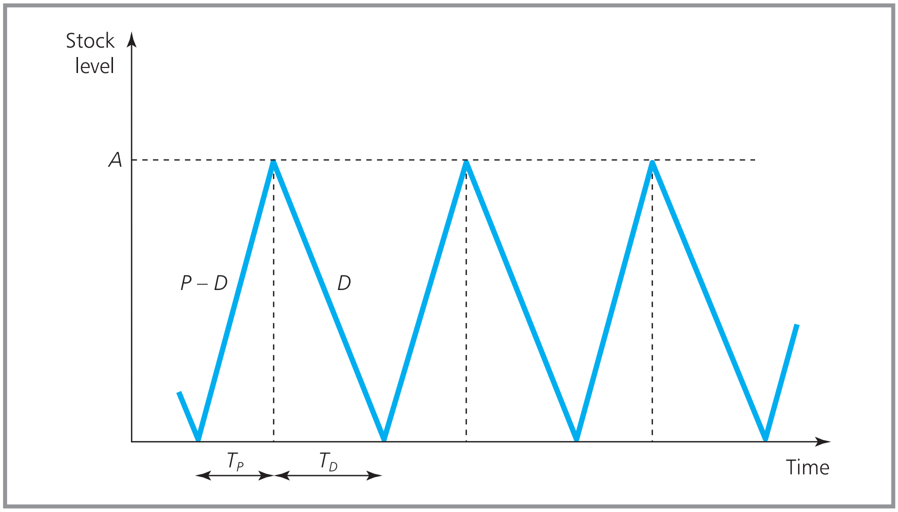

Figure 20.6 Variation in stock level with a fixed production rate, P

Substituting this value for T_P into the equation for A gives:

$$A = Q(P - D)/P$$

So the stock level is lower than it would be by the factor $(P - D)/P$.

We could continue the analysis, remembering that R is really a production setup cost, and again find that the results differ from the economic order quantity only by the factor $(P - D)/P$ (as shown on the Companion Website **www.pearsoned.co.uk/waters**).

- optimal order quantity $= Q_0 = \times \sqrt{\dfrac{2RD}{H}} \times \sqrt{\dfrac{P}{P - D}}$
- total cost $= TC_0 = UD + VC_0$
- variable cost $= VC_0 = \sqrt{2RHD} \times \sqrt{\dfrac{P - D}{P}}$

WORKED EXAMPLE 20.3

Joanna Lum notices that demand for an item is 600 units a month with relevant costs of:

- production setup \$640 per order
- administration \$500 per order
- scheduling \$110 per order
- insurance at 1% of unit cost per year
- obsolescence, deterioration and depreciation of 2% of unit cost per year
- capital at 20% of unit cost per year
- storage space at \$50 a unit per annum
- handling of \$60 a unit per annum
- shortage costs are so large that no shortages are allowed.

Each unit costs the company \$200 and the rate of production is 1,200 units per month. What are the optimal batch quantity and the minimum variable cost per year?

Solution

Every cost must be classified as unit, reorder or holding (with no shortage costs). Then:

$D = 600 \times 12 = 7{,}200$ units a year

$P = 1{,}200 \times 12 = 14{,}400$ units a year

$U = \$200$ a unit

Worked example 20.3 continued

Collecting together all costs for an order:

$R = 640 + 500 + 110 = \$1,250$ an order

There are two parts to the holding cost: a percentage (1%, 2% and 20%) of unit costs, and a fixed amount ($50 + $60) per unit per year. So:

$H = (50 + 60) + (0.01 + 0.02 + 0.2) \times 200$
$= \$156$ per unit per year

Substituting these values gives:

$Q_0 = \sqrt{(2RD/H)} \times \sqrt{(P/(P - D))}$
$= \sqrt{(2 \times 1,250 \times 7,200/156)} \times \sqrt{(14,400/(14,400 - 7,200))}$
$= 339.68 \times 1.414 = 480$ units

$VC_0 = \sqrt{(2RHD)} / \sqrt{(P/(P - D))}$
$= \sqrt{(2 \times 1,250 \times 156 \times 7,200)} / \sqrt{(14,400/(14,400 - 7,200))}$
$= 52,991/1.414 = \$37,476$ a year

Review questions

20.8 Are fixed production rates important for stock control when the production rate is greater than demand, or less than demand?

20.9 Does a fixed production rate give larger or smaller batches than instantaneous replenishment?

Variable demand

The economic order quantity assumes that demand is constant and known exactly. In practice this is rarely true and the demand for almost any item is uncertain and varies over time. Fortunately, these effects are generally small and the economic order quantity still gives useful guidelines. Sometimes, though, the variations are too large and we have to use another model. This starts with the view that shortage costs are usually much higher than holding costs, so organisations are willing to hold additional stocks – above their perceived needs – to add a margin of safety and avoid the risk of shortages. These **safety stocks** are available if the normal working stock runs out (as shown in Figure 20.7).

In principle, we should be able to find the cost of shortages and balance this with the cost of holding stock. Unfortunately, this is rarely possible, as shortage costs are notoriously difficult to find and are generally no more than informed guesses. An alternative approach relies more directly on managers to define a **service level**. This is the probability that a demand can be met from stock, and it needs a positive decision by managers to specify the desired level – or, conversely, the maximum acceptable probability that a demand cannot be met from stock. Service level varies widely, but is commonly around 95%, implying a probability of 0.05 that a demand cannot be met.

There are really several ways of defining the service level, including percentage of orders fully met from stock, percentage of units met from stock, percentage of periods without shortages, percentage of stock cycles without shortages, percentage of time there is stock available, and so on. Here we use the probability of not running out of stock in a stock cycle, which is the **cycle service level**.

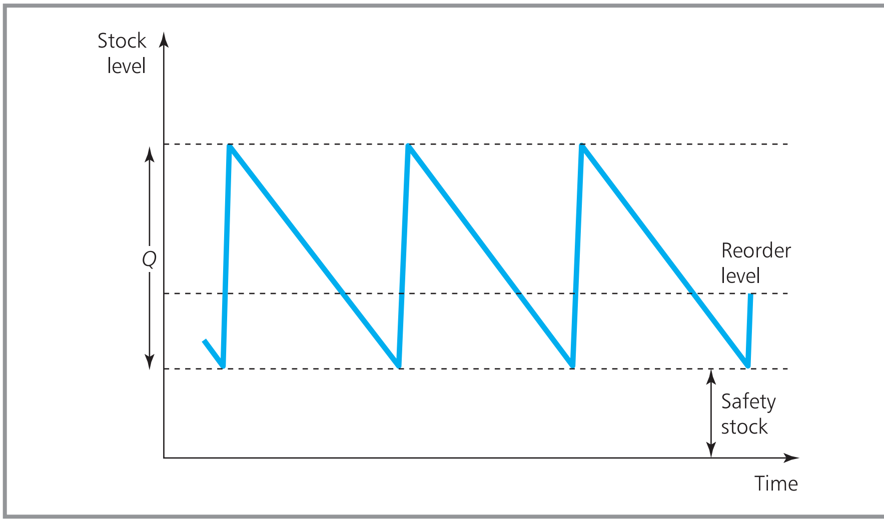

Figure 20.7 Stock level with an added safety stock

Normally distributed demand

Consider an item whose demand is Normally distributed with a mean of D per unit time and standard deviation of σ. We can add variances in demand, but not standard deviations. So:

- demand in a single period has mean D and variance σ^2,
- demand in two periods has mean $2D$ and variance $2\sigma^2$,
- demand in three periods has mean $3D$ and variance $3\sigma^2$, etc.

and when the lead time is constant at L:

- demand in L periods has mean LD and variance $L\sigma^2$.

When we assumed that demand is constant, we used the lead time demand, LD, as a reorder level. Now the lead time demand is Normally distributed with mean of LD, variance of $\sigma^2 L$ and standard deviation of $\sigma\sqrt{L}$. This means that it is greater than LD in half of stock cycles, and if we simply used LD as the reorder level there would be shortages in 50% of cycles. On the other hand, the lead time demand is less than the mean in 50% of stock cycles, and this gives spare stock (as shown in Figure 20.8). To give a cycle service level above 50% we have to add a safety stock, and then the reorder level becomes:

$$\text{reorder level} = \text{mean lead time demand} + \text{safety stock}$$

The size of the safety stock depends on the service level specified by managers. If they specify a high service level, the safety stock must also be high. In particular, when the lead time demand is Normally distributed the safety stock is:

$$\text{safety stock} = Z \times \text{standard deviation of lead time demand} = Z\sigma\sqrt{L}$$

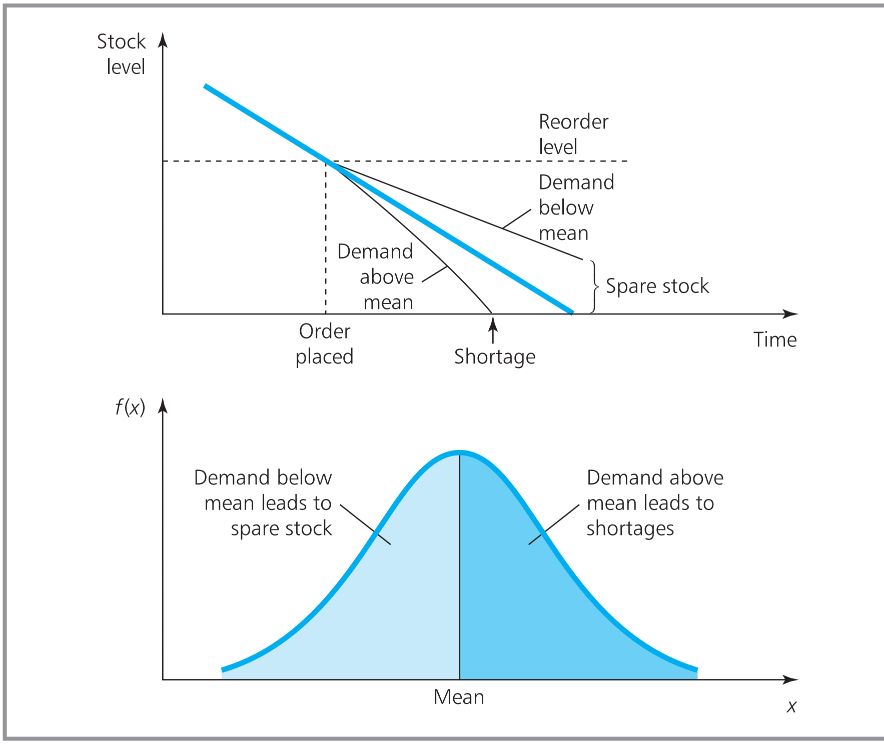

Figure 20.8 Service level with Normally distributed demand

Here Z is the number of standard deviations the safety stock is away from the mean, corresponding to the probability specified in the service level. With:

- a service level of 90%, $Z = 1.28$
- a service level of 95%, $Z = 1.645$
- a service level of 97%, $Z = 1.88$
- a service level of 99%, $Z = 2.33$.

When demand varies widely, the standard deviation of lead time demand is high, and we would need very high safety stocks to give a service level anywhere close to 100%. This may be too expensive, so organisations usually set a lower level, typically around 95%. Sometimes it is convenient to give items different service levels depending on their importance. Very important items may have levels close to 100%, while less important ones are set around 85%.

WORKED EXAMPLE 20.4

Rawcliffe Commercial send out service packs to meet demand that is Normally distributed with a mean of 200 units a week and a standard deviation of 40 units. Reorder cost for the packs, including delivery, is €200, holding cost is €6 per unit per year, and lead time is fixed at three weeks. What ordering pattern will give a 95% cycle service level? How much would costs rise with a 97% service level?

Worked example 20.4 continued

Solution

Listing the values we know:

$D = 200$ units per week

$\sigma = 40$ units

$R = €200$ per order

$H = €6$ per unit per year

$L = 3$ weeks

Substituting these values gives:

- order quantity, $Q_0 = \sqrt{(2RD/H)}$
  $= \sqrt{(2 \times 200 \times 200 \times 52/6)}$
  $= 832.67$
- reorder level, ROL $= LD +$ safety stock
  $= 3 \times 200 +$ safety stock
  $= 600 +$ safety stock

For a 95% service level, $Z = 1.645$ standard deviations from the mean. Then:

safety stock $= Z\sigma\sqrt{L} = 1.645 \times 40 \times \sqrt{3} = 113.97$

The best policy is to order 833 packs whenever stock falls to 600 + 114 = 714 units. On average, orders should arrive when there are 114 units left. And on average, none of the safety stock is used, so its cost is:

$Z\sigma\sqrt{L} \times H = 114 \times 6 = €684$ a year

Raising the service level to 97% gives $Z = 1.88$ and:

safety stock $= Z\sigma\sqrt{L} = 1.88 \times 40 \times \sqrt{3} = 130$

The cost of holding this is $130 \times 6 = €780$ a year, giving a rise of €96 or 14%.

Review questions

20.10 What is a service level and why is it used?

20.11 What is the purpose of safety stock?

20.12 How can you increase the service level?

Periodic review

Earlier we said that there are two different ordering policies:

- *fixed order quantity*, which we have been discussing, where an order of fixed size is placed whenever stock falls to a certain level
- *periodic review*, where orders of varying size are placed at regular intervals.

When demand is constant these two systems are identical, so differences appear only when the demand varies (shown in Figure 20.9).

Suppose that we want to design a periodic review system when the demand is Normally distributed. The usual format has a **target stock level**, and we order enough to raise the stock to this level. For this, we have to answer two basic questions:

- How long should the interval between orders be?
- What should the target stock level be?

The order interval can really be any convenient period. For example, it might be easiest to place an order at the end of every week, or every morning, or at the end of a month. If there is no obvious cycle we might aim for a certain number of orders a year or some average order size. We might calculate an economic order quantity, and then find the period that gives orders of about this size. The final decision is largely a matter for management judgement.

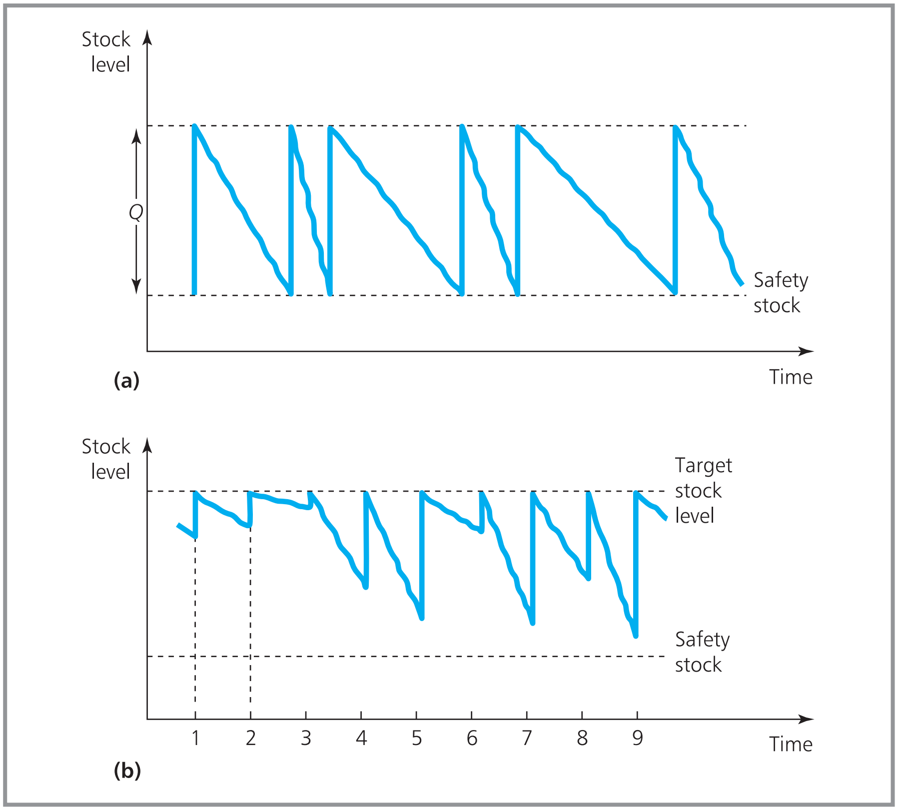

Figure 20.9 Two ways of dealing with varying demand: (a) fixed order quantity; (b) periodic review

Whatever interval is chosen, we need to find a suitable target stock level. Then the system works by finding the amount of stock on hand when it is time to place an order, and ordering the amount that brings this up to the target stock level.

order quantity = target stock level − stock on hand

Suppose the lead time is constant at L and orders are placed every period, T. When an order is placed, the stock on hand plus this order must be enough to last until the next order arrives, which is $T + L$ away (as shown in Figure 20.10).

The target stock level, TSL, should be high enough to cover mean demand over this period, so it must be at least $(T + L)D$. Using the same reasoning that we used for periodic review, when the demand is Normally distributed there must be some safety stock to allow for the 50% of cycles when demand is above average. Assuming both the cycle length and lead time are constant, the demand over $T + L$ is Normally distributed with mean of $D(T + L)$, variance of $\sigma^2(T + L)$ and standard deviation of $\sigma\sqrt{(T + L)}$. Then we can define a safety stock as:

$$\text{safety stock} = Z \times \text{standard deviation of demand over } T + L = Z\sigma\sqrt{(T + L)}$$

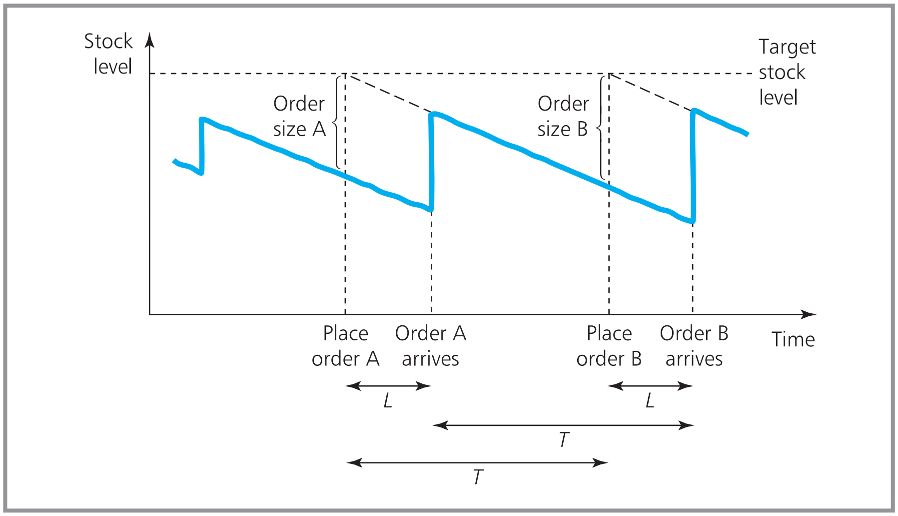

Figure 20.10 Calculating the target stock level

and:

target stock level = mean demand over $(T + L)$ + safety stock
$= D(T + L) + Z\sigma\sqrt{(T + L)}$

WORKED EXAMPLE 20.5

Paola Ricardo manages an item with mean demand of 200 units a week and standard deviation of 40 units. She places an order every four weeks, with a constant lead time of two weeks. How can she get a 95% service level? If the holding cost is €20 per unit per week, how much would a 98% service level cost?

Solution

The variables are:

$D = 200$ units

$\sigma = 40$ units

$H = €20$ per unit per week

$T = 4$ weeks

$L = 2$ weeks

For a 95% safety stock, Z is 1.645. Then:

- safety stock $= Z\sigma\sqrt{(T + L)} = 1.645 \times 40 \times \sqrt{6} = 161$
- target stock level $= D(T + L)$ + safety stock $= 200 \times (4 + 2) + 161 = 1{,}361$

When it is time to place an order, Paola's policy is to count the stock on hand, and place an order for:

order size = 1,361 – stock on hand

For instance, if she found 200 units in stock, she would order 1,361 – 200 = 1,161 units.

The cost of the 'working' stock is always the same, so the holding cost varies with the safety stock. On average, the safety stock is always held, so each unit costs H per period. Here the safety stock for a 95% service level costs $161 \times 20 = €3{,}220$ per week. If the service level is increased to 98%, $Z = 2.05$, and:

safety stock $= 2.05 \times 40 \times \sqrt{6} = 201$

The target stock level is then 1,200 + 201 = 1,401 units and the cost of the safety stock is $201 \times 20 = €4{,}020$ per week.

Review questions

20.13 How is the order size calculated for a periodic review system?

20.14 Is the safety stock higher for a fixed order quantity system or a periodic review system?

IDEAS IN PRACTICE Vancouver Electrical Factors

In 2005 Edwin Choi worked for Vancouver Electrical Factors, which makes and supplies around 10,000 products for their range of electric motors. He demonstrated the use of the economic order quantity for one of their core products. The annual demand for this product was constant at around 1,000 units, each unit cost the company \$400 to make, and each batch had setup costs of \$1,000. The annual holding costs for the product were 18% of unit cost for interest charges, 1% for insurance, 2% allowance for obsolescence, \$20 for building overheads, \$15 for damage and loss, and \$40 miscellaneous costs.

Edwin calculated the economic order quantity from the following figures:

Demand, $D = 1{,}000$ units a year

Unit cost, $\text{UC} = \$400$ a unit

Reorder cost, $\text{RC} = \$1{,}000$ an order

$$\text{Holding cost, HC} = (0.18 + 0.01 + 0.02) \times 400 + 20 + 15 + 40$$
$$= \$159 \text{ a unit a year}$$

Then he suggested ordering 120 units, with an order about every six weeks and annual costs of \$18,000. But operations managers felt that this schedule made too many assumptions, and they suggested an adjustment of ordering 85 units at the end of each month, with annual costs around \$18,500. They argued that their batch size was 35% below Edwin Choi's optimal, but variable costs increased by only 5% – and they thought that this was a reasonable cost for the improved schedules. It is common to find that costs rise slowly, so provided order quantities are somewhere close to the economic order quantity, the inventory costs should be close to a minimum.

ABC analysis of stock

Stock control systems are usually computerised, often with automated purchasing from suppliers. But some items are very expensive and need special care above the routine calculations. An **ABC analysis** is a way of putting items into categories that reflect the amount of effort worth spending on stock control. This is really a Pareto analysis – or 'rule of 80/20' – which we saw with quality defects in the last chapter. Here, it suggests that 80% of stock items need 20% of the attention, while the remaining 20% of items need 80% of the attention. To be specific, ABC analyses define:

- A items as expensive and needing special care
- B items as ordinary ones needing standard care
- C items as cheap and needing little care.

Typically an organisation might use automated procedures for all C items – or it might leave them out of the system, and leave any control to *ad hoc* procedures. B items are automated with regular monitoring to make sure that everything is working normally. Most effort is given to A items, with managers making final decisions.

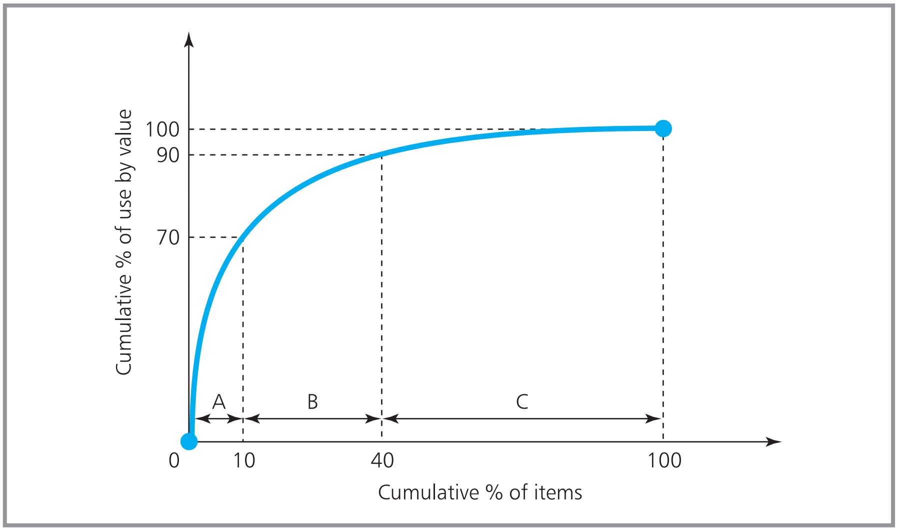

Figure 20.11 ABC analysis of stocks

An ABC analysis starts by calculating the total annual use of each item in terms of value, by multiplying the number of units used in a year by the unit cost. Usually, a few expensive items account for a lot of use, while many cheap ones account for little use. If we list the items in order of decreasing annual use by value, A items are at the top of the list, B items are in the middle, and C items are at the bottom. We might typically find:

Category	Percentage of items	Cumulative percentage of items	Percentage of use by value	Cumulative percentage of use by value
A	10	10	70	70
B	30	40	20	90
C	60	100	10	100

Plotting the cumulative percentage of annual use against the cumulative percentage of items gives the graph shown in Figure 20.11.

WORKED EXAMPLE 20.6

A small store has 10 categories of product with the following costs and annual demands:

Product	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
Unit cost (£)	20	10	20	50	10	50	5	20	100	1
Annual demand	250	5,000	2,000	6,600	1,500	600	1,000	500	100	5,000

Do an ABC analysis of these items. If resources for stock control are limited, which items should be given least attention?

Solution

The annual use of P1 in terms of value is $20 \times 250 = £5{,}000$. Repeating this calculation for the other items, and sorting them into order of decreasing annual use, gives the results shown in Figure 20.12.

Worked example 20.6 continued

The boundaries between categories of items are sometimes unclear, but in this case P4 is clearly an A item, P2, P3 and P6 are B items, and the rest are C items. The C items account for only 10% of annual use by value, and these should be given least attention when resources are limited.

	A	B	C	D	E	F	G	H	I
1	**ABC analysis of stock**								
2									
3	**Category**	**Product**	**Percentage of items**	**Cumulative percentage of items**	**Unit cost**	**Annual demand**	**Annual use (£s)**	**Cumulative annual use (£s)**	**Cumulative percentage of annual use**
4	A	P4	10	10	50	6,600	330,000	330,000	66
5	B	P2	10	20	10	5,000	50,000	380,000	76
6	B	P3	10	30	20	2,000	40,000	420,000	84
7	B	P6	10	40	50	600	30,000	450,000	90
8	C	P5	10	50	10	1,500	15,000	465,000	93
9	C	P8	10	60	20	500	10,000	475,000	95
10	C	P9	10	70	100	100	10,000	485,000	97
11	C	P1	10	80	20	250	5,000	490,000	98
12	C	P7	10	90	5	1,000	5,000	495,000	99
13	C	P10	10	100	1	5,000	5,000	500,000	100

Figure 20.12 ABC analysis for worked example 20.6

Review questions

20.15 What is the purpose of an ABC analysis of inventories?

20.16 Which items can best be dealt with by routine, automated control procedures?

CHAPTER REVIEW

This chapter described some quantitative models for stock control, where stocks are the stores of materials that organisations hold until they are needed.

- There are several reasons for holding stock, but the main one is to give a buffer between supply and demand. Then, depending on circumstances, there are several ways of controlling the stocks. The two main options consider either dependent or independent demand.
- Independent demand models look for a balance of unit, reorder, holding and shortage costs that minimises the overall inventory costs. The basic model calculates an optimal order size – the economic order quantity – with related calculations giving the costs and optimal cycle length.
- The reorder level shows when it is time to place an order. With constant lead time and demand, an order is placed when stock on hand falls to the lead time demand.

- The economic order quantity analysis can be extended in many ways. We illustrated this by adding a fixed production rate.
- With large variations in demand it is better to work with service level models. A cycle service level is the probability of meeting all demand in a cycle. When the lead time demand is Normally distributed, we can achieve a specified service level by adding extra safety stock.
- An alternative approach uses periodic reviews to place orders of variable size at regular intervals. This uses an order quantity that raises stock to a target level.
- ABC analyses categorise items according to their importance. Typically, 20% of items account for 80% of use by value (A items) while the bulk of items account for very little use by value (C items).

CASE STUDY Templar Manufacturing

James Templar founded his own manufacturing company when he was 21 years old. He has continued to run it for the past 35 years and through steady expansion it now employs over 200 people.

A management consultant recently suggested improving the stock control system, but Mr Templar is not sure that this is necessary. He was talking to a meeting of managers and said, 'I don't know how much the present stock control system costs, if it works as well as it could, or if the proposals would save money. I know that we have the things we need in stock, and if we have a shortage, enough people complain to make sure things get sorted out and we don't have problems again. What I want is someone to show me if the proposals are worth looking at.'

The management consultant asked what kind of demonstration Mr Templar would like, and was told, 'I know you wanted to run a pilot scheme before starting work on a revised stock control system. I still need convincing that it is even worth going ahead with the pilot scheme. I don't want anything fancy. Let me give you an example of one of the components we make and see what you can do.

'This component is basically a flanged orbital hub which costs us about £15 to make. We use about 2,000 a year. At the moment we can make them at a rate of 70 a week, but plan only one batch every quarter. Each time we set up the production it costs £345 to change the production line and £85 for preparation and scheduling costs. Other stock holding costs are related to the unit costs including insurance (1% a year), deterioration and obsolescence (2%) and capital (13%). I think that we could make them a bit faster, say up to 90 a week, and the unit cost could even fall a few per cent. Of course, we could make them a bit slower, but this would raise the cost by a few per cent.'

Questions

- **If you were the management consultant, how would you demonstrate the benefit of a new stock control system to Mr Templar?**
- **What information would you need for your demonstration?**

PROBLEMS

20.1 The demand for an item is constant at 100 units a year. Unit cost is £50, the cost of processing an order is £20 and holding cost is £10 a unit a year. What are the economic order quantity, cycle length and costs?

20.2 Beograd, Inc. works 50 weeks a year and has demand for a part that is constant at 100 units a week. The cost of each unit is $200 and the company aims for a return of 20% on capital invested. Annual warehouse costs are 5% of the value of goods stored. The purchasing department costs $450,000 a year and sends out an average of 2,000 orders. Find the optimal order quantity for the part, the time between orders and the minimum cost of stocking the part.

20.3 Demand for an item is steady at 20 units a week and the economic order quantity has been calculated at 50 units. What is the reorder level when the lead time is (a) 1 week, (b) 2 weeks?

20.4 How would the results for Problem 20.1 change if the part could be supplied only at a fixed rate of 10 units a week? Would there be any benefit in reducing production to five units a week?

20.5 H.R. Prewett Limited forecasts the demand for one component to average 18 a day over a 200-day working year. If there are any shortages, production will be disrupted with very high costs. The total cost of placing an order is £800 and holding costs are £400 a unit a year. What is the best inventory policy for the component? How does this compare with the option of making the component internally at a rate of 80 units a day?

20.6 A company advertises a 95% cycle-service level for all stock items. Stock is replenished from a single supplier who guarantees a lead time of four weeks. What reorder level should the company use for an item that has a Normally distributed demand with mean of 1,000 units a week and standard deviation of 100 units? What is the reorder level for a 98% cycle-service level?

20.7 An item of stock has a unit cost of £40, a reorder cost of £50 and a holding cost of £1 a unit a week. Demand for the item has a mean of 100 a week with a standard deviation of 10. Lead time is constant at three weeks. Design a stock policy that gives the item a service level of 95%. How would the costs change with a 90% service level?

20.8 Describe a periodic review system with an interval of two weeks for the company described in Problem 20.6.

20.9 A small store holds 10 categories of product with the following costs and annual demands:

Product	X1	X2	X3	Y1	Y2	Y3	Z1	Z2	Z3	Z4
Unit cost (€)	20	25	30	1	4	6	10	15	20	22
Annual demand	300	200	200	1,000	800	700	3,000	2,000	600	400

Do an ABC analysis of these items.

20.10 Annual demand for an item is 2,000 units, each order costs £10 to place and the annual holding cost is 40% of the unit cost. The unit cost depends on the quantity ordered as follows:

- for quantities less than 500, unit cost is £1
- for quantities between 500 and 1,000, unit cost is £0.80
- for quantities of 1,000 or more, unit cost is £0.60.

What is the best ordering policy for the item?

RESEARCH PROJECTS

20.1 Virtually all inventory control systems are automated. Not surprisingly, there is a lot of software for the routine calculations. Figure 20.13 shows the printout from a computer program that has done some basic calculations. How can managers use such information? What features would you expect to see in commercial inventory management software? Do a survey to compare available packages.

20.2 A small company wants to control the stocks of 20 items. It seems extravagant to buy an inventory control system for this number of items, and there is no-one in the company to write their own software. It has been suggested that a spreadsheet can record weekly sales and do related calculations. Do you think this is a reasonable approach? If so, show how you would start designing a spreadsheet for the company to use.

20.3 Current thinking has managers trying to eliminate their stocks. This has been assisted by just-in-time operations, e-business, efficient customer response, point-of-sales equipment, electronic fund transfer, efficient transport, and so on. Is this a sensible direction to move in? How do you think technology has affected views of inventory management? Give real examples to illustrate your points.

+++ === ECONOMIC ORDER QUANTITY === +++

EOQ Input Data:

Demand per year (D)	=	400
Order or setup cost per order (Co)	=	£650
Holding cost per unit per year (Ch)	=	£20
Shortage cost per unit per year (Cs)	=	£1,000
Shortage cost per unit, independent of time (π)	=	£100
Replenishment or production rate per year (P)	=	500
Lead time for a new order in year (LT)	=	0.25
Unit cost (C)	=	120

EOQ Output:

EOQ	=	360.56
Maximum inventory	=	72.11
Maximum backorder	=	0.00
Order interval	=	0.90 year
Reorder point	=	100.00
Ordering cost	=	721.11
Holding cost	=	721.11
Shortage cost	=	0.00

Subtotal of inventory cost per year	=	£ 1,442.22
Material cost per year	=	£48,000.00
Total cost per year	=	£49,442.22

Figure 20.13 Printout from an inventory control package

Sources of information

Further reading

Material about inventory management is generally included in books on operations management, management science and operational research. There are some books specifically on inventory management, and the following list gives a useful starting point.

Arnold J.R. and Chapman S.N., *Introduction to Materials Management* (5th edition), Prentice Hall, London, 2003.

Lewis C.D., *Demand Forecasting and Inventory Control*, John Wiley, Chichester, 1998.

Mecimore C. and Weeks J., *Techniques of Inventory Management and Control*, Institute of Management Accountants, London, 1987.

Muller M., *Essentials of Inventory Management*, Amacom, New York, 2003.

Tersine R.J., *Principles of Inventory and Materials Management* (4th edition), Prentice Hall, London, 1994.

Toomey J.W., *Inventory Management*, Kluwer Acedemic Publishers, Boston, MA, 2000.

Viale J.D., *Inventory Management*, Crisp Publications, Menlo Park, CA, 1996.

Waters D., *Inventory Control and Management* (2nd edition), John Wiley, Chichester, 2003.

Wild T., *Best Practices in Inventory Management* (2nd edition), John Wiley, New York, 1998.

Zipkin P.H., *Foundations of Inventory Management*, McGraw-Hill, Boston, MA, 2000.

CHAPTER 21

Project networks

Contents

Chapter outline

A project consists of a set of related activities, with a clear start and finish, and an aim of making a distinct product. Each project is largely unique and needs careful planning to keep it on time and within budget. Network analysis is the most widely used tool for planning projects. This identifies the activities that make up a project, shows the relationships between them, analyses the timing, and schedules the resources.

After finishing this chapter you should be able to:

- Appreciate the need to plan complex projects
- Divide a project into distinct activities and show the relationships between them
- Draw a project as a network of connected activities
- Calculate the timing of activities
- Identify critical paths and the overall project duration
- Reduce the duration of a project
- Draw Gantt charts
- Consider the resources needed during a project
- Use PERT when there is uncertainty in activity durations.

Network analysis

Organisations run many repetitive operations – for instance, a train operator runs a particular service at the same time every day, a manufacturer makes a stream of cars, a surgeon does a series of identical operations, and a shop serves a series of customers. But not all operations are repetitive, and some are unique pieces of work – which we describe as **projects**.

- A **project** is a unique job that makes a one-off product.
- It has a distinct start and finish, and all operations must be co-ordinated within this timeframe.

With this broad definition you can see that we all do small projects every day, such as preparing a meal, writing an essay, building a fence, or organising a party. Each of these projects needs planning, and in particular we have to identify:

- the activities that form the project
- the order in which these activities are done
- the timing of each activity
- the resources needed at each stage.

This is fairly easy for small projects, and a little thought is enough to make sure that they run smoothly. But business projects can be very large and expensive – such as the installation of a new information system, building a power station, organising the Olympic Games, writing a major consultancy report, launching a new product, or moving to new offices. You would expect such large projects to run smoothly only if they were careful planned, with **project network analysis** as the most widely used method of organising complex projects.

Review questions

21.1 What is a project?

21.2 What is project management?

21.3 Project management is concerned only with major capital projects. Is this true?

IDEAS IN PRACTICE The Channel Tunnel

In December 1990 Transmanche Link, a consortium of 10 British and French companies, finished the first continuous tunnel under the English Channel. The main tunnels were opened in 1994, and handed over to Eurotunnel to start operations. This was a significant step in a huge project.[1]

The Channel Tunnel was the world's biggest privately funded construction project, needing the largest banking syndicate ever assembled, with 200 participating banks and finance institutions. By 1994 the estimated cost of the tunnel was £9 billion, and by 1998 this had risen to £15 billion, with rail companies investing another £3 billion in rolling stock and infrastructure. At its peak, the project employed 14,500 people.

The idea of a tunnel under the Channel is not new. In 1802 Albert Mathieu, one of Napoleon's engineers, drew a crude plan, and at various times several trial tunnels were dug. This project had clearly been developing for a very long time, and it was executed by very successful and experienced companies. They dug a total of 150 km of tunnels, with two main rail tunnels and a third service tunnel. By all accounts, the tunnel was a triumph of construction. Nonetheless, its costs were several times the original estimates of £4.5 billion, the consortium was continually looking for additional funding, the opening date was delayed so much that extra interest charges, bankers' and lawyers' fees amounted to £1 billion, and participants were

Ideas in practice continued

plagued by legal disputes. And reports suggest that the final cost of the project is £10 billion more than the benefits.[2]

It is common for major projects to overrun their budgets and schedules. In 1994 the British Library was half-built after 12 years, the cost had tripled to £450 million, and a House of Commons Committee reported that 'no one – ministers, library staff, building contractors, anyone at all – has more than the faintest idea when the building will be completed, when it will be open for use, or how much it will cost.'[3] In a study of 1,449 projects by the Association of Project Managers, only 12 came in on time and under budget.

Drawing project networks

A project network shows the relationships between the activities that make up the project. It consists of a series of alternating circles – or nodes – connected by arrows. There are two formats for this:

1. *Activity on arrow*, where each arrow represents an activity and nodes represent the points when activities start and finish.
2. *Activity on node*, where each node represents an activity and arrows show the relationships between them.

Suppose you have a project with three activities, A, B and C, which have to be done in that order. B has to wait until A finishes before it can start, and it must then finish before C can start. We can represent this using the two formats in Figure 21.1.

The choice between these is largely a matter of personal preference. Activity on arrow networks are better at showing some relationships and the calculations are easier; activity on node networks are easier to draw and put into project planning software. In practice, activity on node networks have probably become more common, so we will stick to this format.

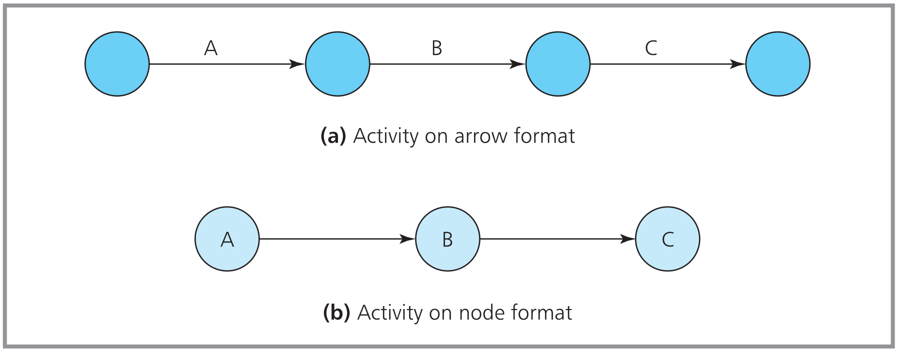

Figure 21.1 Alternative formats for project networks

WORKED EXAMPLE 21.1

A gardener is building a greenhouse from a kit. The instructions show that this is a project with four main activities:

- A, levelling the ground, which takes 2 days
- B, building the base, which takes 3 days
- C, building the frame, which takes 2 days
- D, fixing the glass, which takes 1 day.

Draw a network for the project.

Solution

The four activities must be done in a fixed order; levelling the ground must be done first, followed by building the base, building the frame and finally fixing the glass. We can describe this order by a **dependence table**. Here each activity is listed along with those activities that immediately precede it.

Activity	Duration (days)	Description	Immediate predecessor
A	2	level ground	–
B	3	build base	A
C	2	build frame	B
D	1	fix glass	C

Labelling the activities A, B, C and D is a convenient shorthand and allows us to say that activity B has activity A as its immediate predecessor – which is normally stated as 'B depends on A'. In this table we list only the immediate predecessors, so we do not need to say that C depends on A as well as B, since this follows from the other dependencies. Activity A has no immediate predecessors and can start whenever convenient.

Now we can draw a network from the dependence table, shown in Figure 21.2.

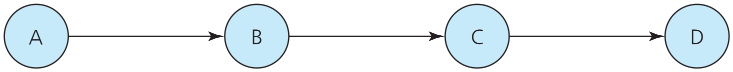

Figure 21.2 Network for building a greenhouse

The directions of the arrows show precedence – each preceding activity must finish before the following one starts – and following activities can start as soon as preceding ones finish. In the example above, levelling the ground must be done first, and as soon as this is finished the base can be built. The frame can be built as soon as the base is finished, and the glass can be fixed as soon as the frame is built.

WORKED EXAMPLE 21.2

Find the schedule for activities in the last example. What happens if the base takes more than 3 days, or the glass is delayed, or the frame takes less than 2 days?

Solution

The schedule gives the time when each activity is done. If we take a notional starting time of zero, we can finish levelling the ground by the end of day 2. Then we can start building the base, and as this takes 3 days we finish by the end of day 5. Then we can start building the frame, and as this takes 2 days we finish by the end of day 7. Finally, we can start fixing the glass, which takes 1 day, so we finish by the end of day 8.

If the base takes more than 3 days to build, or the glass is not delivered by day 7, the project is delayed. If building the frame takes less than 2 days, the project finishes early.

Now we have a timetable for the project showing when each activity starts and finishes – and we can use this timetable to schedule resources. We know exactly when we need a concrete mixer, when equipment is needed to clear the ground, when woodworkers should be hired, and so on.

This defines the major steps in project planning as:

- define the separate activities and their durations
- determine the dependence of activities
- draw a network
- analyse the timing of the project
- schedule resources.

Drawing larger networks

In principle, you can draw networks of any size, simply by starting at the left-hand side with activities that do not depend on any others. Then you add activities that depend only on these first activities; then add activities that depend only on the latest activities, and so on. The network expands systematically, working from left to right, until you have added all the activities and the network is complete. The two main rules are as follows:

- Before an activity can begin, all preceding activities must be finished.
- The arrows only show precedence and neither their length nor their direction has any significance.

There are several other rules to make sure the network is sensible:

- To make things clear, we add one 'start' and one 'finish' activity to a network to define the whole project.
- Every arrow must have a head and a tail connecting two different activities.
- Every activity (apart from the start and finish) must have at least one predecessor and at least one successor activity.
- There must be no loops in the network.

WORKED EXAMPLE 21.3

When Prahalad Commercial opened a new office, they defined the work as a project with the following activities and dependencies:

Activity	Description	Depends on
A	find office location	–
B	recruit new staff	–
C	make office alterations	A
D	order equipment needed	A
E	install new equipment	D
F	train staff	B
G	start operations	C, E, F

Draw a network of this project.

Worked example 21.3 continued

Solution

Activities A and B have no predecessors and can start as soon as convenient. As soon as activity A is finished, both C and D can start; E can start as soon as D is finished, and F can start as soon as B is finished. G can start only when C, E and F have all finished. Figure 21.3 shows the resulting network.

The network shows that the project starts with activities A and B, but this does not mean that these must start at the same time – only that they can both start as soon as convenient and must be finished before any following activity can start. On the other hand, activity G must wait until C, E and F are all finished. This does not mean that C, E and F must finish at the same time – only that they must all finish before G can start.

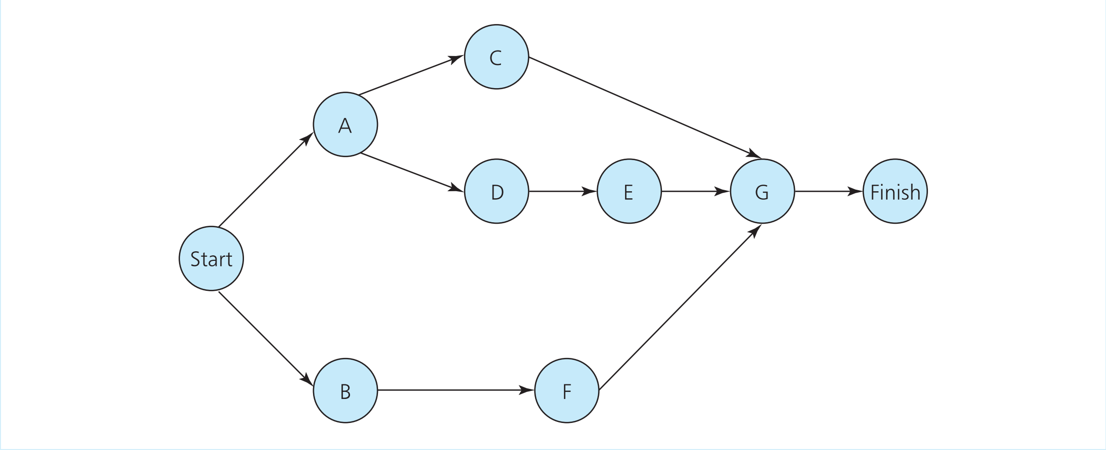

Figure 21.3 Network for Prahalad Commercial

WORKED EXAMPLE 21.4

The following dependence table describes a software development project. Draw a network of the project.

Activity	Depends on	Activity	Depends on
A	J	I	J
B	C, G	J	–
C	A	K	B
D	F, K, N	L	I
E	J	M	I
F	B, H, L	N	M
G	A, E, I	O	M
H	G	P	O

Solution

This seems a difficult network, but the steps are fairly straightforward. Activity J is the only one that does not depend on anything else, so this starts the network. Then we can add activities A, E and I, which depend only on J. Then we can add activities that depend on A and E. Continuing this systematic addition of activities leads to the network shown in Figure 21.4.

Worked example 21.4 continued

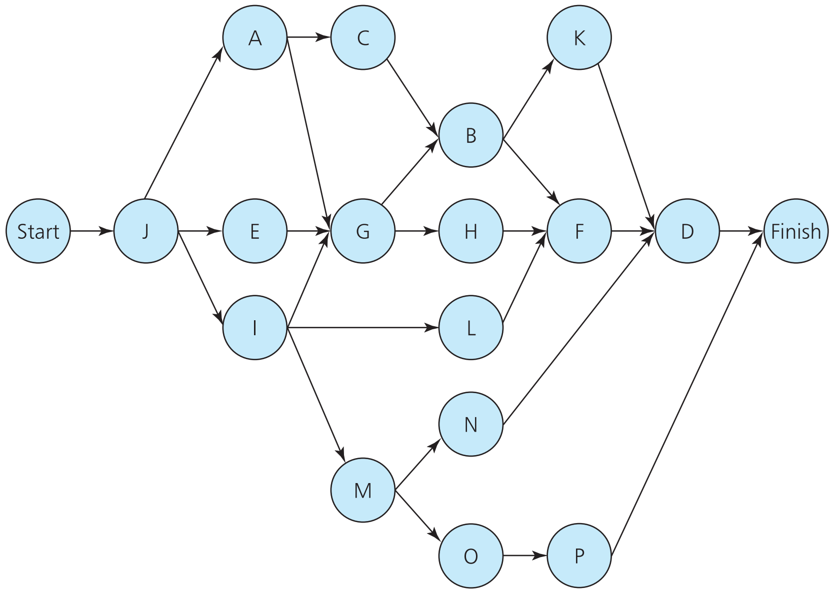

Figure 21.4 Network for software project

Bigger networks obviously take more effort to draw and analyse – even when this is done by computer. We divided our initial example of building a greenhouse into four activities. We could have used a lot more, perhaps clearing vegetation, laying hardcore, digging the foundations, and so on. As the complexity of the network increases, the significance of each activity declines. So we have to choose the best number of activities, balancing the usefulness of a network for planning with its complexity.

Agreeing a reasonable set of activities is usually the most difficult part of project planning, particularly with large projects. There can be a lot of disagreement between managers about what activities should be included, what order they should be done in, and how long they will take. When this difficult area has been agreed, the later stages of analysing the project often reduce to mechanical procedures.

Review questions

21.4 In the networks we have drawn, what do the nodes and arrows represent?

21.5 What basic information do you need to draw a project network?

21.6 What are the main rules of drawing a project network?

Timing of projects

After drawing the project network, we know what order the activities have to be done in, and can consider the detailed schedule. This means that we want the time when each activity must be done, and in particular the four key times:

- earliest start time of an activity, which is the earliest time at which all preceding activities are finished
- earliest finish time, which is the earliest start time plus the activity duration
- latest finish time of an activity, which is the time by which it must be finished
- latest start time, which is the latest finish time minus the activity duration.

These four times define the time slot that is available for the activity (illustrated in Figure 21.5).

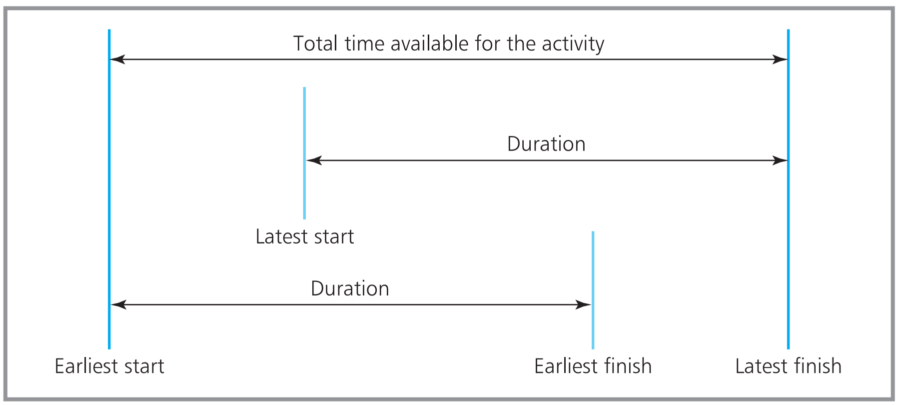

Figure 21.5 Timing of an activity

We can show the calculations for timing in an example. Suppose a project has the following dependence table, which includes the duration of each activity in weeks. Figure 21.6 shows the network for this project.

Activity	Duration	Depends on
A	3	–
B	2	–
C	2	A
D	4	A
E	1	C
F	3	D
G	3	B
H	4	G
I	5	E, F

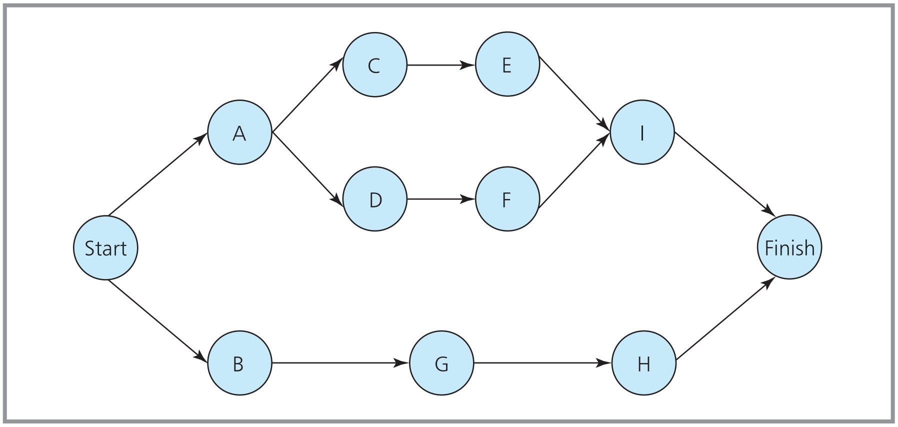

Figure 21.6 Network for timing example

Earliest times

The first part of the timing analysis finds the earliest possible time for starting and finishing each activity. For simplicity we will assume a notional start time of zero for the project. The earliest start of activity B is clearly 0, and as it takes 2 weeks the earliest finish is 0 + 2 = 2. When B finishes, G can start, so its earliest start is 2, and adding the duration of 3 gives an earliest finish of 2 + 3 = 5. When G finishes, H can start, so its earliest start is 5, and adding the duration of 4 gives an earliest finish of 5 + 4 = 9.

Similarly, the earliest start of A is clearly 0, and as it takes 3 weeks, the earliest finish is 3. When A finishes, both C and D can start, so the earliest start time for both of these is 3. Adding the durations gives earliest finish times of 5 and 7 respectively. Then E follows C, with an earliest start of 5 and earliest finish of 6; F follows D, with an earliest start of 7 and earliest finish of 10.

Activity I must wait until both E and F finish. The earliest finishes for these are 6 and 10 respectively, so activity I cannot start until week 10. Then we add the duration of 5 to get the earliest finish of 15. The finish of the project comes when both H and I are finished. These have earliest finish times of 9 and 15 respectively, so the earliest finish of the whole project is week 15. We can show these times in the following table. Alternatively we can add them directly to the network, with each node drawn as a box containing the times. Here we will use the notation in Figure 21.7, where the earliest start, duration and earliest finish are across the top of the box. Then we get the results shown in Figure 21.8.

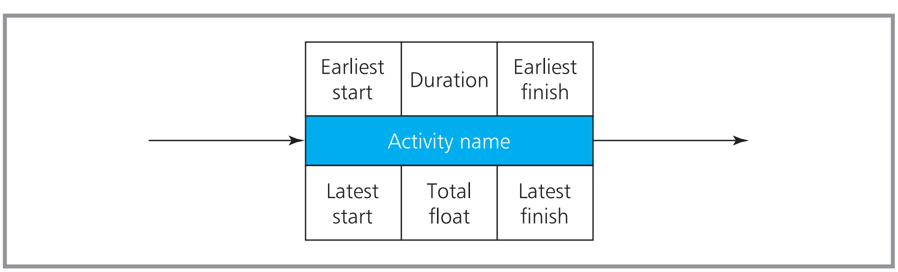

Figure 21.7 Format for the times added to activities

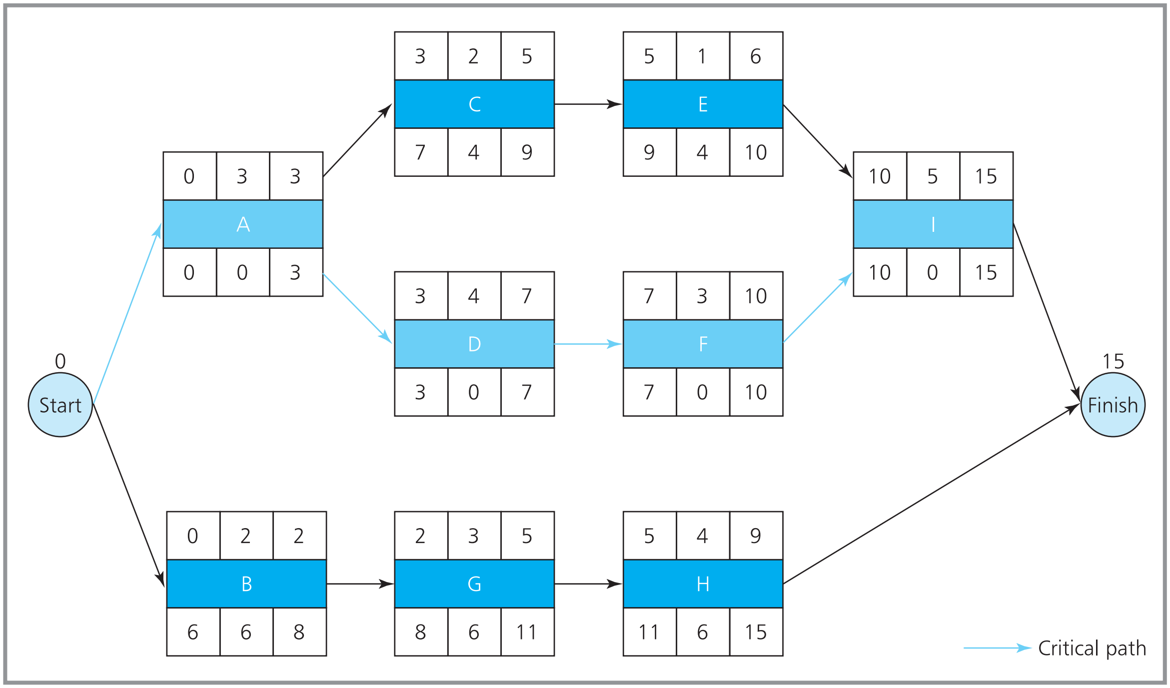

Figure 21.8 Network for example showing times

Activity	Duration	Earliest start	Earliest finish
A	3	0	3
B	2	0	2
C	2	3	5
D	4	3	7
E	1	5	6
F	3	7	10
G	3	2	5
H	4	5	9
I	5	10	15

Latest times

The next part of the timing analysis finds the latest time for starting and finishing each activity. The procedure for this is almost the reverse of the procedure for finding the earliest times.

The earliest finish time for the whole project is week 15. If we want the project to finish then, we set this as the latest finish time. The latest finish time of activity H is clearly 15, so its latest start time is the duration 4 earlier than this at 15 – 4 = 11. Activity G must finish before this, so its latest finish is 11, so its latest start is the duration of 3 earlier at 11 – 3 = 8. Activity B must be finished before this, so its latest finish is 8, and the latest start is the duration 2 earlier at 8 – 2 = 6.

Now activity I must also finish by week 15, so its latest start is its duration of 5 earlier than this, at 10. But for I to start at 10, both E and F must finish by 10, so this gives both of their latest finish times. They must start their durations earlier at time 10 – 1 = 9 for E and 10 – 3 = 7 for F. Activity C must finish by time 9, so its latest start is 9 – 2 = 7, and activity D must finish by time 7, so its latest start is 7 – 4 = 3.

Activity A must finish in time for both C and D to start. Activity C has a latest start of 7, and activity D has a latest start of 3, so A must be finished for both of these, giving a latest finish of 3 and a latest start of 0. Similarly the latest time to start the project must allow both A to start at 0 and B to start at 6, so it must start at time 0. This gives the timings in the following table, and you can also see these in the bottom row of figures for each activity in Figure 21.8.

Activity	Duration	Earliest start	Earliest finish	Latest start	Latest finish
A	3	0	3	0	3
B	2	0	2	6	8
C	2	3	5	7	9
D	4	3	7	3	7
E	1	5	6	9	10
F	3	7	10	7	10
G	3	2	5	8	11
H	4	5	9	11	15
I	5	10	15	10	15

Critical activities

You can see from these results that some activities have flexibility in their timing: activity G can start as early as week 2 or as late as week 8, while activity C can start as early as week 3 or as late as week 7. On the other hand, some activities have no flexibility at all: activities A, D, F and I have no flexibility, as their latest start time is the same as their earliest start time. The activities that have to be done at fixed times are the critical activities.

- Each of the **critical activities** has to be done at a fixed time.
- They form a continuous path through the network, called the **critical path**.

The length of the critical path sets the overall project duration. If one of the critical activities is extended by a certain amount, the overall project duration is extended by this amount; if one of the critical activities is delayed by some time, the overall project duration is extended by this delay. On the other hand, if one of the critical activities is made shorter, the overall project duration may be reduced.

The activities that have some flexibility in timing are the **non-critical activities** and these may be delayed or extended without necessarily affecting the overall project duration. However, there is a limit to the amount by which a non-critical activity can be extended without affecting the project duration, and this is measured by the **float**. The **total float** – sometimes called **slack** – is the difference between the amount of time available for an activity and the time it actually needs. If an activity takes three weeks, and there is a five-week slot during which it can be done, the total float is 5 – 3 = 2 weeks. It is the difference between the earliest and latest start times – which is clearly the same as the difference between the earliest and latest finish times.

total float = latest start time – earliest start time

or

total float = latest finish time – earliest finish time

The total float is zero for critical activities and has some positive value for non-critical activities. In the example above, the earliest and latest start of activity D are both 3, so the total float is 3 – 3 = 0, showing that this is one of the critical activities. The earliest and latest start of activity G are 2 and 8, so the total float is 8 – 2 = 6, showing that this is a non-critical activity. To be specific, it shows that the duration of G can expand by up to 6 weeks without affecting the duration of the project, but if it takes more than this the project is delayed. A negative total float means that an activity is already late, and the project cannot be finished within the proposed time. The following table shows the complete calculations for the example.

		Earliest time		Latest time		
Activity	Duration	Start	Finish	Start	Finish	Total float
A	3	0	3	0	3	0
B	2	0	2	6	8	6
C	2	3	5	7	9	4
D	4	3	7	3	7	0
E	1	5	6	9	10	4
F	3	7	10	7	10	0
G	3	2	5	8	11	6
H	4	5	9	11	15	6
I	5	10	15	10	15	0

WORKED EXAMPLE 21.5

ArcticCom build communication satellite receiving stations for isolated communities on Northern Europe. The following table shows the activities for building a small station, the expected durations (in days) and dependences. Draw the network for this project, find its duration and calculate the total float of each activity.

Activity	Description	Duration	Depends on
A	design internal equipment	10	–
B	design building	5	A
C	order parts for equipment	3	A
D	order material for building	2	B
E	wait for equipment parts	15	C
F	wait for building material	10	D
G	employ equipment assemblers	5	A
H	employ building workers	4	B
I	install equipment	20	E, G, J
J	complete building	30	F, H

Worked example 21.5 continued

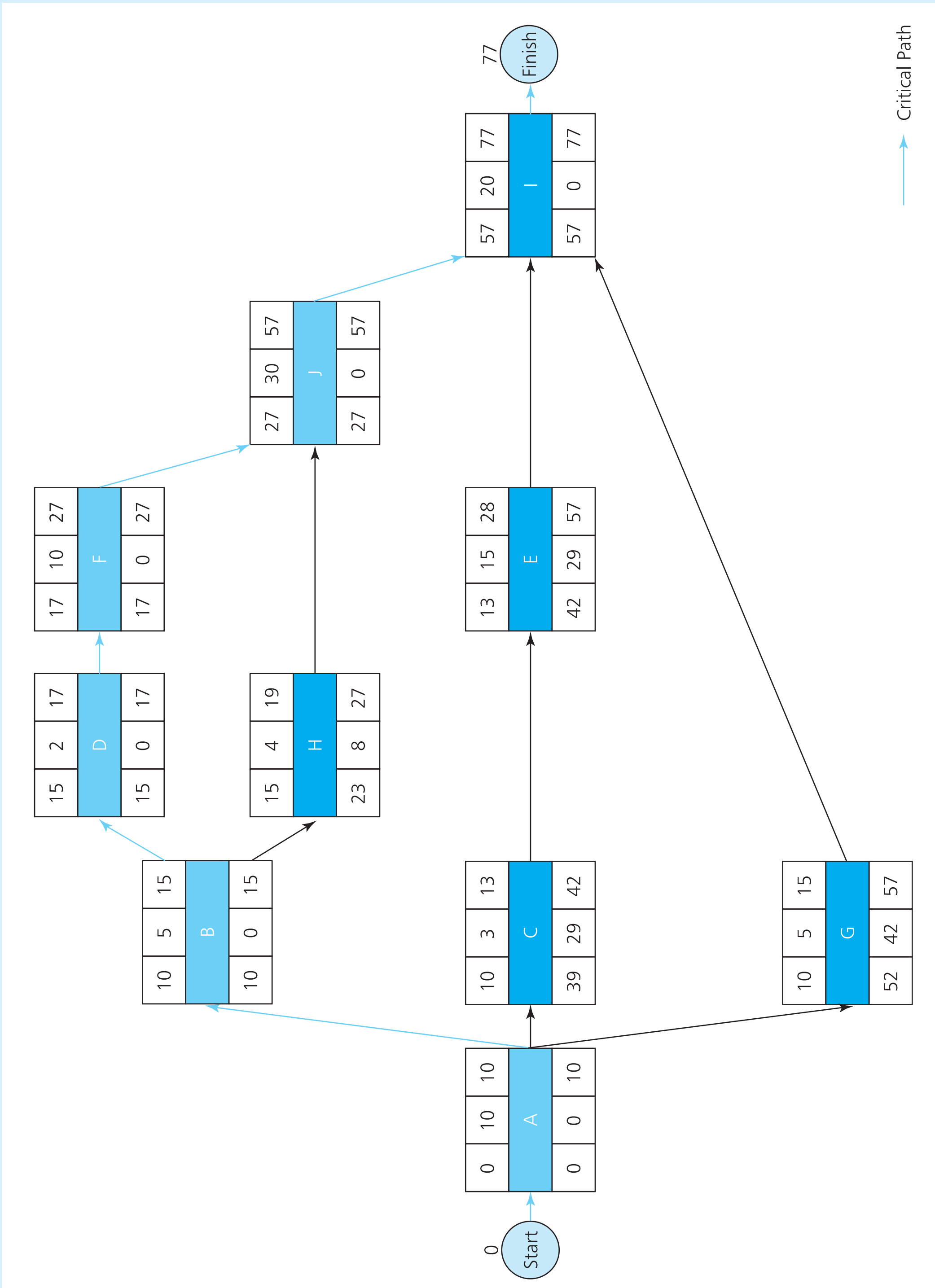

Figure 21.9 Network for building a communications station

Worked example 21.5 continued

Solution

Figure 21.9 shows the network for this problem. Repeating the calculations described above gives the following table of results.

Activity	Duration	Earliest time		Latest time		Total float
		Start	Finish	Start	Finish	
A	10	0	10	0	10	0 *
B	5	10	15	10	15	0 *
C	3	10	13	39	42	29
D	2	15	17	15	17	0 *
E	15	13	28	42	57	29
F	10	17	27	17	27	0 *
G	5	10	15	52	57	42
H	4	15	19	23	27	8
I	20	57	77	57	77	0 *
J	30	27	57	27	57	0 *

The duration of the project is 77 days, defined by the critical path A, B, D, F, I and J.

Reducing a project duration

Suppose that you have drawn a network and analysed the timing, only to find that the project takes too long. How can you reduce its length? For this you have to remember that the overall duration is set by the critical path, so you can reduce the overall duration only by reducing the durations of critical activities. Reducing the duration of non-critical activities has no effect on the project duration. But you have to be careful here. Small reductions are generally all right, but if you keep reducing the length of the critical path, there must come a point when some other path through the network becomes critical. You can find this point from the total float on paths parallel to the critical path. Each activity on a parallel path has the same total float, and when you reduce the critical path by more than this, the parallel path itself becomes critical.

WORKED EXAMPLE 21.6

The project network in Figure 21.10 has a duration of 14 with A, B and C as the critical path. If each activity can be reduced by up to 50% of the original duration, how would you reduce the overall duration to: (a) 13 weeks, (b) 11 weeks, (c) 9 weeks? If reductions cost an average of $1,000 per week, what is the cost of finishing the project by week 9?

Solution

The analysis of activity times is as follows.

Activity	Duration	Earliest		Latest		Total float
		Start	Finish	Start	Finish	
A	8	0	8	0	8	0 *
B	4	8	12	8	12	0 *
C	2	12	14	12	14	0 *
D	3	0	3	2	5	2
E	6	3	9	5	11	2
F	3	9	12	11	14	2
G	2	0	2	4	6	4
H	4	2	6	6	10	4
I	4	6	10	10	14	4

Worked example 21.6 continued

There are three parallel paths, A–B–C, D–E–F and G–H–I. The critical path is A–B–C and these critical activities have zero total float. The total float of activities on the other two paths are 2 and 4 respectively. This means that we can reduce the critical path A–B–C by up to 2, but if we reduce it any more the path D–E–F becomes critical. If we reduce the critical path by more than 4, the path G–H–I also becomes critical.

(a) To finish in 13 weeks we need a reduction of one week in the critical path. It is usually easiest to find savings in longer activities, so we reduce the duration of A to seven weeks.

(b) To finish in 11 weeks needs a further reduction of two weeks in the critical path. We can also remove this from A, but the path D–E–F has now become critical with a duration of 12 weeks. We can remove one week from E – again chosen as the longest activity in the critical path.

(c) To finish in nine weeks needs five weeks removed from the path A–B–C (say four from A and one from B), three weeks removed from the path D–E–F (say from E), and one week removed from the path G–H–I (say from H).

To get a five-week reduction in the project duration, we have reduced the durations of individual activities by a total of 5 + 3 + 1 = 9 weeks. This gives a total cost of 9 × 1,000 = $9,000.

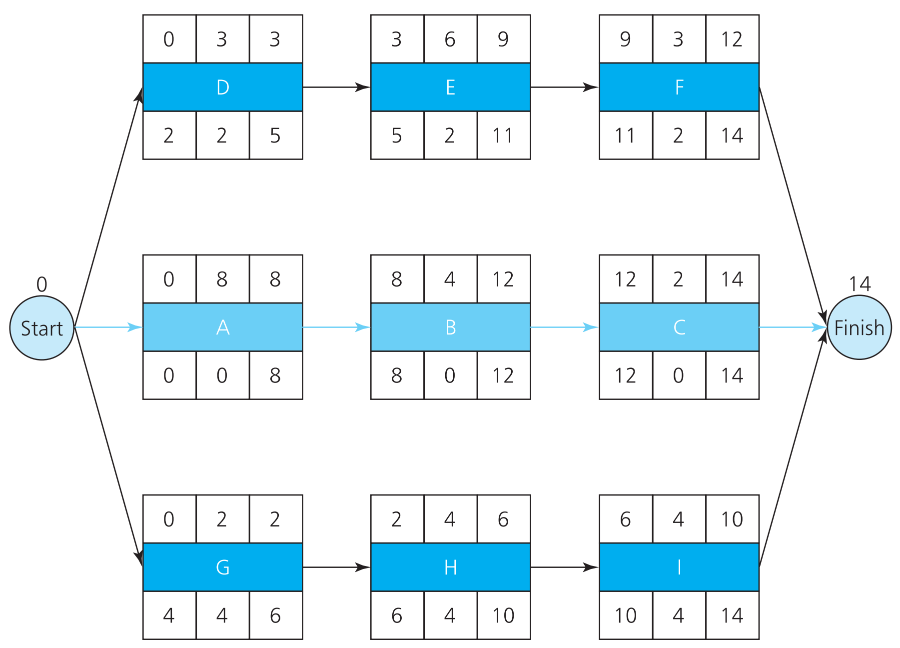

Figure 21.10 Network for worked example 21.6

Resource levelling

When a project is actually being executed, managers have to monitor progress to make sure that everything is done at the scheduled times. But these times are not really clear from a network. It is much easier to see the times in a **Gantt chart**, which is a form of bar chart. The chart consists of a time scale across the bottom, with activities listed down the left-hand side, and times when activities should be done blocked off in the body of the chart.

WORKED EXAMPLE 21.7

Draw a Gantt chart for the original data in the last worked example, assuming that each activity starts as early as possible.

Solution

We have already done the timing analysis for this project. If each activity starts as early as possible, we can show the times needed by the blocked-off areas in Figure 21.11. The total float of each activity is added afterwards as a broken line. The total float is the maximum expansion that still allows the project to finish on time, so provided an activity is completed before the end of the broken line, there should be no problem in keeping to the planned duration.

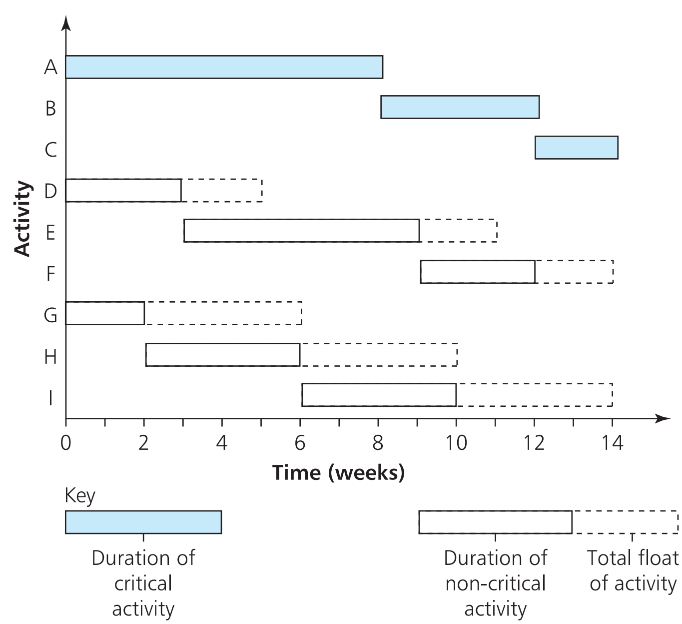

Figure 21.11 Gantt chart for worked example 21.7

The main benefit of Gantt charts is that they show clearly the state of each activity at any point in the project. They show which activities should be in hand, as well as those that should be finished, and those about to start. Gantt charts are also useful for planning the allocation of resources. For simplicity, suppose that each activity in the Gantt chart in Figure 21.11

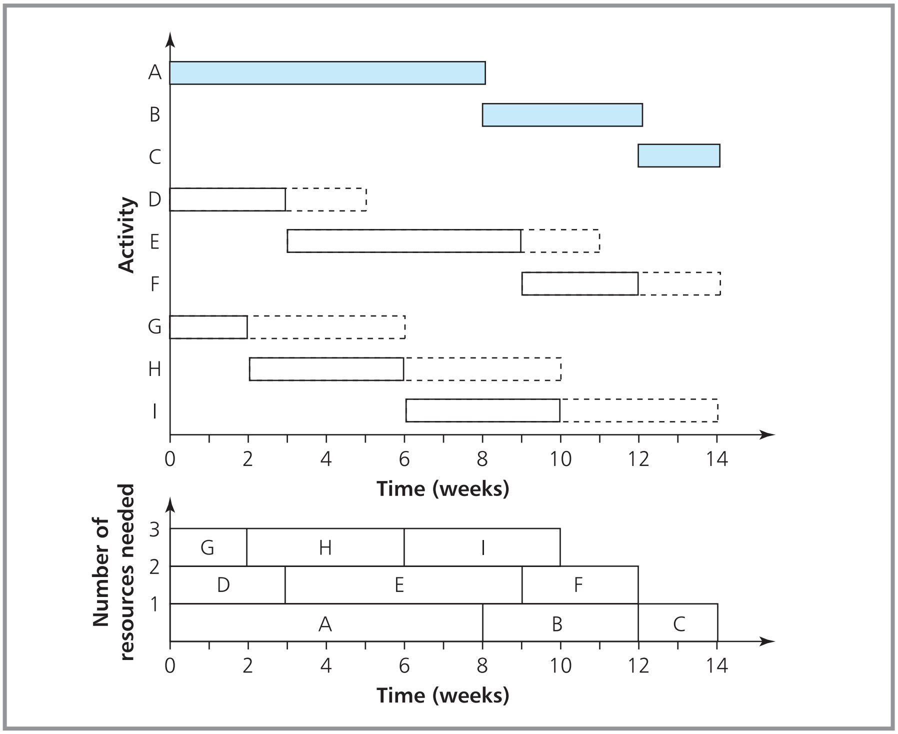

Figure 21.12 Gantt chart and resources used during a project (worked example 21.7)

uses one unit of a particular resource – perhaps one team of workers. If all activities start as soon as possible, we can draw a vertical bar chart to show the resources in use at any time. The project starts with activities A, D and G, so these need three teams. At the end of week 2 one team can move from G to H, but three teams will still be needed. Continuing these allocations gives the graph of resources shown in Figure 21.12.

In this example, the use of resources is steady for most of the project and begins to fall only near the end. It is rare to get such a smooth pattern of resource use, and usually there are a series of peaks and troughs. Smooth operations are always more efficient that widely varying ones, so managers try to smooth out the variations in workload. As critical activities are at fixed times, they must do this levelling by rescheduling non-critical activities, and in particular by delaying activities with relatively large total floats. Adjusting and monitoring schedules, workloads, times and costs needs a lot of arithmetic and this is best done using standard software. A lot of software is available – as usual ranging from the basic to the very sophisticated. A few widely used packages are ConceptDraw Project, Microsoft Project, Oracle Projects, PowerProject, Primavera Project Planner, SuperProject and TurboProject.

Review questions

21.7 How do you calculate the earliest and latest start times for an activity?

21.8 What is the total float of an activity?

21.9 How big is the total float of a critical activity?

21.10 What is the significance of the critical path?

21.11 Which activities must be shortened to reduce the overall duration of a project?

21.12 By how much can a critical path usefully be shortened?

21.13 What are the main benefits of Gantt charts?

21.14 How can the resource use be smoothed during a project?

Project evaluation and review technique

The approach we have described so far is the **critical path method** (CPM) where each activity is given a single, fixed duration. But as you know from experience, the time needed for any job can vary quite widely. **Project evaluation and review technique** (PERT) is a useful extension to CPM that allows for uncertainty in duration. In particular, it uses the observation that activity durations often follow a beta distribution. This looks like a skewed Normal distribution, and has the useful property that the mean and variance can be found from three estimates of duration:

- An *optimistic duration* (O) is the shortest time an activity takes if everything goes smoothly and without any difficulties.
- A *most likely duration* (M) is the duration of the activity under normal conditions.
- A *pessimistic duration* (P) is the time needed if there are significant problems and delays.

Then we can find the expected activity duration and variance from the **rule of sixths**:

$$\text{expected duration} = \frac{O + 4M + P}{6}$$

$$\text{variance} = \frac{(P - O)^2}{36}$$

If an activity has an optimistic duration of 4 days, a most likely duration of 5 days and a pessimistic duration of 12 days, then:

$$\text{expected duration} = (O + 4M + P)/6 = (4 + 4 \times 5 + 12)/6 = 6$$
$$\text{variance} = (P - O)^2/36 = (12 - 4)^2/36 = 1.78$$

We can use the expected values in the same way as the single estimate of CPM – but the variance allows us to do some more calculations with the timings.

WORKED EXAMPLE 21.8

The following table shows the dependences and estimated durations of nine activities in a project. What is the project's expected duration?

Activity	Depends on	Duration		
		Optimistic	Most likely	Pessimistic
A	–	2	3	10
B	–	4	5	12
C	–	8	10	12
D	A, G	4	4	4
E	B	3	6	15
F	B	2	5	8
G	B	6	6	6
H	C, F	5	7	15
I	D, E	6	8	10

Solution

We can find the expected duration and variance of each activity from the rule of sixths. For activity A:

$$\text{expected duration} = (O + 4M + P)/6 = (2 + 4 \times 3 + 10)/6 = 4$$

$$\text{variance} = (P - O)^2/36 = (10 - 2)^2/36 = 1.78$$

Repeating these calculations for other activities gives the results shown in Figure 21.13. We can use these to draw the network in Figure 21.14. From this you can see that the critical path for the project is B, G, D and I. The expected duration of the project is 24. The network shows the earliest and latest times for each activity, along with the floats, and we have added these timings to the spreadsheet in Figure 21.13.

	A	B	C	D	E	F	G	H	I	J	K
1	**PERT Analysis**										
2											
3		**Duration**					**Earliest**		**Latest**		
4	**Activity**	**Optimistic**	**Most likely**	**Pessimistic**	**Expected**	**Variance**	**Start**	**Finish**	**Start**	**Finish**	**Total float**
5	A	2	3	10	4	1.78	0	4	8	12	8
6	B	4	5	12	6	1.78	0	6	0	6	0
7	C	8	10	12	10	0.44	0	10	6	16	6
8	D	4	4	4	4	0.00	12	16	12	16	0
9	E	3	6	15	7	4.00	6	13	9	16	3
10	F	2	5	8	5	1.00	6	11	11	16	5
11	G	6	6	6	6	0.00	6	12	6	12	0
12	H	5	7	15	8	2.78	11	19	16	24	5
13	I	6	8	10	8	0.44	16	24	16	24	0

Figure 21.13 Activity timing analysis for worked example 21.8

The duration of the critical path is the sum of the durations of activities making up that path – and when the duration of each activity is variable, the overall duration of the project is also variable. The central limit theorem tells us that when there is a large number of activities on the critical path, and assuming that the duration of each activity is independent of the others, the overall duration of the project is Normally distributed. This distribution has:

- a mean equal to the sum of the expected durations of activities on the critical path
- a variance equal to the sum of the variances of activities on the critical path.

We can use these values to do some calculations for the project duration.

Worked example 21.8 continued

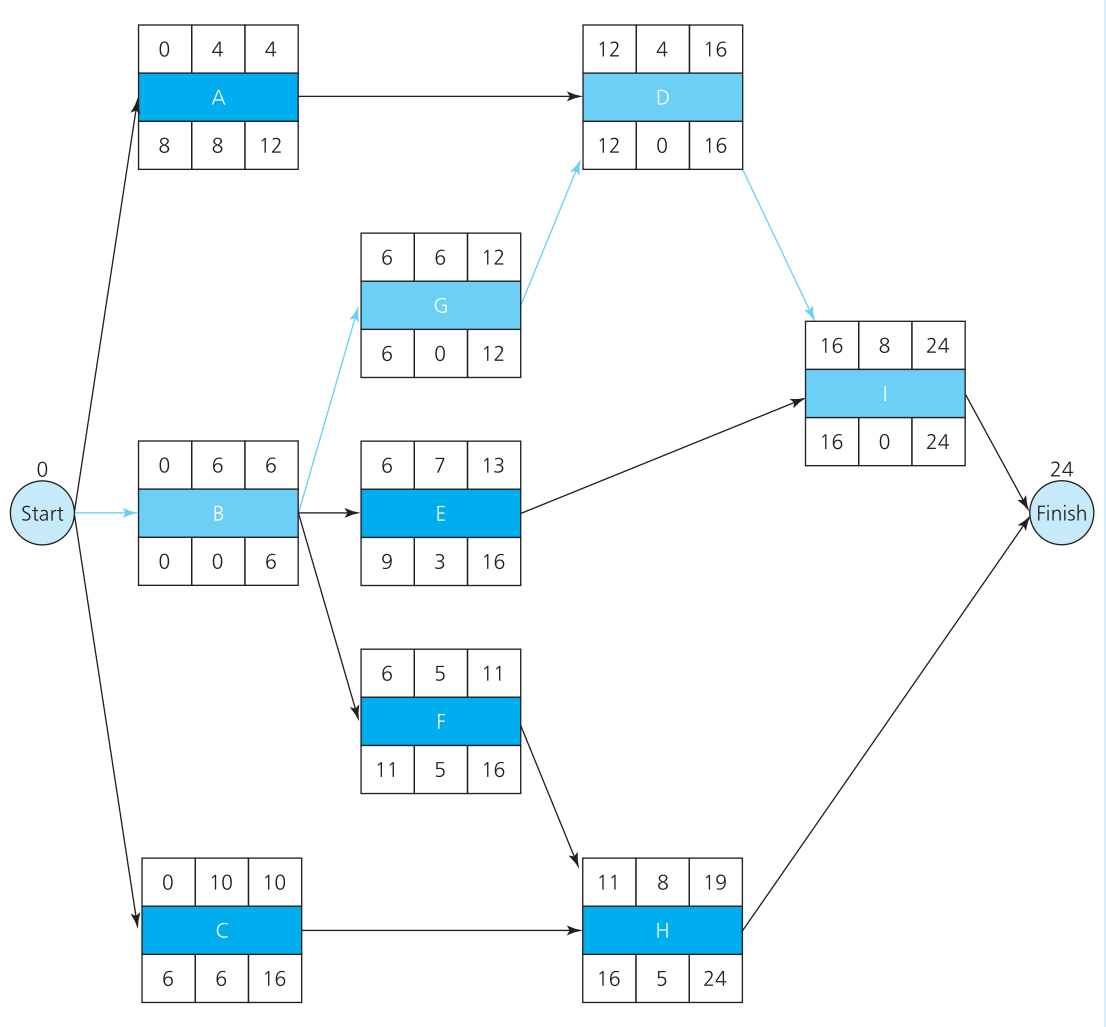

Figure 21.14 Network for PERT worked example 21.8

WORKED EXAMPLE 21.9

What are the probabilities that the project in the last worked example is finished before (a) day 26, (b) day 20?

Solution

The critical path is activities B, G, D and I with expected durations of 6, 6, 4 and 8 respectively and variances of 1.78, 0, 0 and 0.44 respectively. Although the number of activities on the critical path is small, we can reasonably assume the overall duration of the project is Normally distributed (at least to illustrate the calculation). The expected duration then has a mean of $6 + 6 + 4 + 8 = 24$. The variance in project duration is $1.78 + 0 + 0 + 0.44 = 2.22$, so the standard deviation is $\sqrt{2.22} = 1.49$.

Worked example 21.9 continued

(a) We can find the probability that the project is not finished before day 26 from the Normal distribution shown in Figure 21.15. Here Z is the number of standard deviations the point of interest is away from the mean:

$$Z = (26 - 24)/1.49 = 1.34 \text{ standard deviations}$$

This corresponds to a probability of 0.0901 (found from tables or a statistical package). So the probability that the project is finished by day 26 is 1 – 0.0901 = 0.9099.

(b) Similarly, the probability that the project is finished by day 20 has:

$$Z = (24 - 20)/1.49 = 2.68 \text{ standard deviations}$$

corresponding to a probability of 0.0037. So there is a probability of only 0.0037 that the project is completed before day 20.

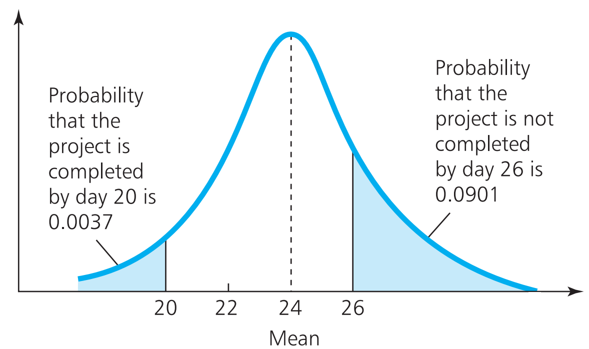

Figure 21.15 Normal distribution for project duration

Review questions

21.15 What is the difference between CPM and PERT?

21.16 What is the 'rule of sixths' and when is it used?

21.17 How can you calculate the expected duration of a project and its variance?

IDEAS IN PRACTICE Survey into use of quantitative methods

In 2002 Peder Kristensen sent a questionnaire to 187 managers asking them how much they used standard quantitative analyses. Some of these results were quite disappointing. Peder said, 'Some methods were widely used – such as break-even analyses, basic statistics and inventory control. On the other hand, some common methods – such as linear programming and regression analysis – were used surprisingly little. My survey contained small companies that are less likely to use sophisticated methods, but the results were still disappointing.'

Peder soon recognised a fault in his data collection. He explained, 'It was the most basic and embarrassing mistake. I assumed that most managers would be familiar with a range of quantitative methods. I simply asked questions like 'Do you use linear programming?' Actually, relatively few of the managers had any formal training in quantitative methods, and were unlikely to use them.'

In 2006 Peder repeated the survey, sponsored by a software company which was convinced that managers' needs were not being met. This time he asked questions like 'Do you know about linear

Ideas in practice continued

programming? If the answer is 'yes', do you use it in your work?' The following table shows some of his results.

Topic	Percent aware of	Percent of these using
Descriptive statistics	93	98
Discounted cash flow	78	87
Forecasting	74	83
Inventory control	69	65
Regression	67	79
Project planning – CPM	58	76
Project planning – PERT	51	61
Linear programming	47	53
Queuing models	41	38
Integer programming	25	27

Many people believe that managers do not use quantitative methods because they do not trust them, or they believe the analyses are too difficult or inappropriate. Peder showed that an important factor is that managers do not use quantitative methods because they do not know about them.

CHAPTER REVIEW

This chapter introduced the topic of project management, where a project is defined as a self-contained piece of work that has a clear start and finish. It consists of the activities needed to make a distinct product.

- Projects are often large and need detailed planning. Project network analysis is the most widely used tool for planning projects. This starts by dividing the project into separate activities, with a dependence table showing the relationships between activities. You can use this to draw a project network.
- After drawing a network, you can analyse the timing. In particular, you can find the earliest and latest start and finish times for each activity. Then the total float measures the amount of flexibility in timing. Some activities are at fixed times – which means that they have no float – and these form the critical path that sets the project duration. Other non-critical activities have flexibility in timing.
- When managers want to reduce the duration of a project, they have to reduce the length of the critical path. But they can reduce the critical path only by a certain amount before another parallel path becomes critical. This limit is set by the total float of activities on parallel paths.
- Gantt charts give another view of projects, emphasising the timing. They give useful formats for monitoring progress during project execution and for planning resources.
- PERT assumes that there is uncertainty in activity durations, and the rule of sixths gives an expected duration and variance. The overall project duration is Normally distributed with mean and variance given by adding values for activities on the critical path.

CASE STUDY Westin Contractors

William Purvis looked across his desk at the company's latest recruit and said: 'Welcome to Westin Contractors. This is a good company to work for, and I hope you settle in and will be very happy here. Everyone working for Westin has to be familiar with our basic tools, so you should start by looking at network analysis. Here is a small project we have just costed, and I have to give the customer some details about schedules, workloads and budgets by the end of the week. I would like a couple of alternative views, with your recommendation of the best. Everything you need is available in the office, so don't be afraid to ask for help and advice.'

William Purvis supplied the following data, and said that there is a penalty cost of £3,500 for every week the project finished after week 28. (Note: The duration of an activity can often be reduced by using more resources. This is called 'crashing'. Then an activity has a normal duration and a normal cost – and a shorter crashed duration and a higher crashed cost.)

Activity	Depends on	Normal		Crashed		Number of teams
		Time	Cost	Time	Cost	
A	–	3	13	2	15	3
B	A	7	25	4	28	4
C	B, E	5	16	4	19	4
D	C	5	12	3	24	2
E	–	8	32	5	38	6
F	E	6	20	4	30	1
G	F	8	30	6	35	5
H	–	12	41	7	45	6
I	H	6	25	3	30	4
J	E	4	18	3	26	6
K	I, J	12	52	10	60	4
L	I, J	6	20	3	30	1
M	D, G, I	2	7	1	14	1
N	B, E	6	18	5	24	5

Question

- **If you were the recruit, how would you set about this job, and what would you say in your report?**

PROBLEMS

21.1 A project has the activities shown in the following dependence table. Draw the network for this project.

Activity	Depends on	Activity	Depends on
A	–	G	B
B	–	H	G
C	A	I	E, F
D	A	J	H, I
E	C	K	E, F
F	B, D	L	K

21.2 (a) BiilsMoore Amateur Dramatic Society is planning its annual production and wants to use a network to co-ordinate the various activities. What activities do you think should be included in the network?

(b) If discussions lead to the following activities, what would the network look like?

- assess resources and select play
- prepare scripts
- select actors and cast parts
- rehearse
- design and organise advertisements
- prepare stage, lights and sound
- build scenery
- sell tickets
- final arrangements for opening

21.3 Draw a network for the following dependence table.

Activity	Depends on	Activity	Depends on
A	H	I	F
B	H	J	I
C	K	K	L
D	I, M, N	L	F
E	F	M	O
F	–	N	H
G	E, L	O	A, B
H	E	P	N

21.4 If each activity in Problem 21.3 has a duration of one week, find the earliest and latest start and finish times for each activity and the corresponding total floats.

21.5 Sven Sengler has divided a project into the following activities. What does the network for the project look like? If each activity can be reduced by up to two weeks, what is the shortest duration of the project and which activities should he reduce?

Activity	Duration (weeks)	Depends on
A	5	–
B	3	–
C	3	B
D	7	A
E	10	B
F	14	A, C
G	7	D, E
H	4	E
I	5	D

21.6 A project consists of ten activities with estimated durations (in weeks) and dependences shown in the following table. What are the estimated duration of the project and the earliest and latest times for activities?

Activity	Depends on	Duration	Activity	Depends on	Duration
A	–	8	F	C, D	10
B	A	6	G	B, E, F	5
C	–	10	H	F	8
D	–	6	I	G, H, J	6
E	C	2	J	A	4

If activity B needs special equipment, when should this be hired? A check on the project at week 12 shows that activity F is running two weeks late, that activity J will now take six weeks, and that the equipment for B will not arrive until week 18. How does this affect the overall project duration?

21.7 Draw a Gantt chart for the project described in problem 21.5. If each activity uses one team of people, draw a graph of the manpower needed. How can the manpower be smoothed?

21.8 Analyse the timing and resource use of the project described by the following values.

Activity	Depends on	Duration	Resources
A	–	4	1
B	A	4	2
C	A	3	4
D	B	5	4
E	C	2	2
F	D, E	6	3
G	–	3	3
H	G	7	1
I	G	6	5
J	H	2	3
K	I	4	4
L	J, K	8	2

21.9 In the project described in Problem 21.8, it costs €1,000 to reduce the duration of an activity by 1. If there is €12,000 available to reduce the duration, what is the shortest time the project can be completed within? What are the minimum resources needed by the revised schedule?

21.10 A project is shown in the following dependence table.

(a) What is the probability that the project will be completed before week 17?

(b) By what time is there a probability of 0.95 that the project will be finished?

Activity	Depends on	Duration (weeks)		
		Optimistic	Most likely	Pessimistic
A	–	1	2	3
B	A	1	3	6
C	B	4	6	10
D	A	1	1	1
E	D	1	2	2
F	E	3	4	8
G	F	2	3	5
H	D	7	9	11
I	A	0	1	4
J	I	2	3	4
K	H, J	3	4	7
L	C, G, K	1	2	7

RESEARCH PROJECTS

21.1 Most projects seem to be finished late and over budget. But why should this happen, when there are many tools to help and a lot of experience and knowledge in the area? Find some examples of particularly bad projects and say what went wrong.

21.2 Find a project with which you are familiar and break it into about 50 activities. Draw the network for the project and do the relevant analyses. What software did you use and how useful was it?

21.3 How can this kind of information be used in project management? Figure 21.16 shows a printout from a program that automatically crashes projects until it finds a minimum cost. How do you think this works?

Input Data for – CPM Demonstration

Activity number	Activity name	Start event	End event	Normal duration	Crash duration	Normal cost	Crash cost
1	Start 1	1	2	15.00	12.00	4500	5500
2	Start 2	1	3	10.00	8.00	3000	4500
3	Check	2	3	7.00	5.00	1500	1800
4	Build	2	4	8.00	6.00	800	1200
5	Employ	3	4	15.00	10.00	4000	5000
6	Purchase	3	5	12.00	10.00	3500	4000
7	Install	4	6	16.00	12.00	6000	8000
8	Operate	5	6	12.00	8.00	6000	8000

Figure 21.16 Printout for analysis of crashed costs (project 21.3)

CPM Analysis for – CPM Demonstration

Activity number	Activity name	Earliest Start	Latest Start	Earliest Finish	Latest Finish	Slack LS–ES
1	Start 1	0	0	15.00	15.00	Critical
2	Start 2	0	12.00	10.00	22.00	12.000
3	Check	15.000	15.00	22.00	22.00	Critical
4	Build	15.000	29.00	23.00	37.00	14.000
5	Employ	22.000	22.00	37.00	37.00	Critical
6	Purchase	22.000	29.00	34.00	41.00	7.000
7	Install	37.000	37.00	53.00	53.00	Critical
8	Operate	34.000	41.00	46.00	53.00	7.000

Completion time = 53 Total cost = 29300

Critical paths for – CPM Demonstration

Critical Path Number 1:
Activities Start 1 Check Employ Install
Events 1 ==========> 2 ==========> 3 ==========> 4 ==========> 6

Crash Analysis for – CPM Demonstration

Target Crashed Duration is 40

Activity number	Activity name	Earliest Start	Latest Start	Earliest Finish	Latest Finish	Slack LS–ES
1	Start 1	0	0	12.00	12.00	Critical
2	Start 2	0	7.00	10.00	17.00	7.000
3	Check	12.00	12.00	17.00	17.00	Critical
4	Build	12.00	19.00	20.00	27.00	7.000
5	Employ	17.00	17.00	27.00	27.00	Critical
6	Purchase	17.00	17.00	28.00	28.00	Critical
7	Install	27.00	27.00	40.00	40.00	Critical
8	Operate	28.00	28.00	40.00	40.00	Critical

Completion time = 40 Total cost = 33350

Critical paths for – CPM Demonstration

Critical Path Number 1:
Activities Start 1 Check Employ Install
Events 1 ==========> 2 ==========> 3 ==========> 4 ==========> 6

Critical Path Number 2:
Activities Start 1 Check Purchase Operate
Events 1 ==========> 2 ==========> 3 ==========> 5 ==========> 6

Analysis of Crashed Activities for – CPM Demonstration

Crash activity Start 1
by 3 time units: new duration = 12: incremental cost = 1000
Crash activity Check
by 2 time units: new duration = 5: incremental cost = 300
Crash activity Employ
by 5 time units: new duration = 10: incremental cost = 1000
Crash activity Purchase
by 1 time units: new duration = 11: incremental cost = 250
Crash activity Install
by 3 time units: new duration = 13: incremental cost = 1500

Crashed duration = 40: Additional cost = 4050: Crashed cost = 33350

Figure 21.16 (*continued*)

Sources of information

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Further reading

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CHAPTER 22

Queues and simulation

Contents

Chapter outline

We are all familiar with queues – but few of us enjoy them. But managers can shorten queues only by using more servers, and this means higher costs. Models of queues look for ways to reduce the time for which customers wait and still give acceptable costs. In practice, not all queues involve people, and there are many queues of inanimate – and even intangible – objects. Queuing problems are notoriously difficult, so analytical solutions are available only for relatively small problems. Simulation gives a more robust method of tackling bigger and more complex problems, by imitating the operations of a system over a typical period.

After finishing this chapter you should be able to:

- Appreciate the scope of queuing problems and describe the features of queues
- Calculate the characteristics of a single-server queue
- Describe the characteristic approach of simulation
- Do manual simulations of queuing systems
- Use computers for bigger simulations.

Features of queues

Queues form when customers want a service, but arrive to find the server busy – then they wait in a queue to be served. As you know, you are likely to meet a queue whenever you buy a train ticket, get money from a bank, go to a supermarket, wait for traffic lights to change – and in many other circumstances.

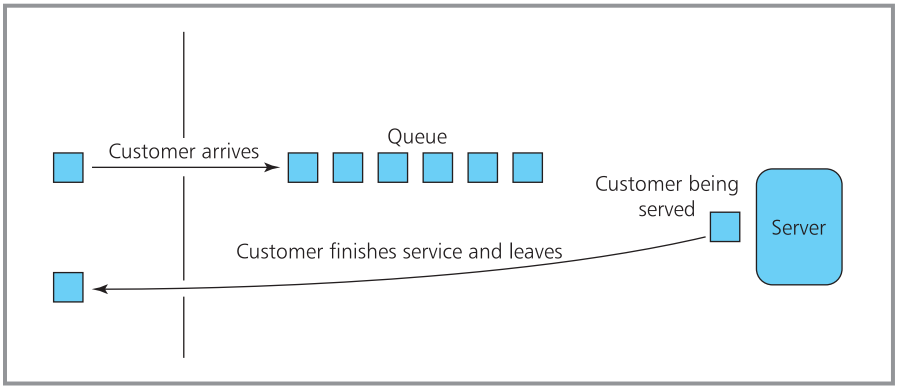


Figure 22.1 A single-server queuing system

But not all queues involve people, and there are also queues of programs waiting to be processed on a computer, telephone calls waiting to use a satellite, items moving along an assembly line, faulty equipment waiting to be repaired, ships queuing for a berth, aeroplanes queuing to land, and so on.

All queues have features in common, and by convention a customer is anyone or anything wanting a service, and a server is the person or thing providing that service. Queues form when a customer wants a service but arrives to find the server is already busy. The customer may decide not to use the service, particularly if other customers are already waiting, but more usually they decide to wait in the queue. Then a queuing system (illustrated in Figure 22.1) contains all the features associated with a queue.

There are many different configurations of queues, with variations including the following:

- Customers may form a single queue, or separate queues for each server.
- Customers may arrive singly or in batches (for example, when a bus-load of people arrive at a restaurant).
- Arrivals may be at random or organised through an appointment system.
- Customers may be served individually or in batches (for example, at an airport customers are served a plane-load at a time).
- Servers may be in parallel (where each does the same job) or in series (where each gives part of the service and then passes the customer on to the next stage).
- Service time may be constant or variable.
- Customers may be served in order of arrival or in some other order (for example, hospitals admit patients in order of urgency).

As you know from experience, you judge the quality of the service – at least in part – by the time you have to wait. And the length of the queue depends on three things:

- the rate at which customers arrive
- the time taken to serve each customer
- the number of servers available.

In a given situation, having a lot of servers gives a short queue, but it can also have high costs. Having few servers reduces the cost, but customers might see the length of the queue and go somewhere else. Managers look for a balance that seems to satisfy all parties – combining reasonable queue length with acceptable costs. The point of balance differs according to circumstances. When you visit a doctor's surgery you often have a long wait. This is because the doctor's time is considered expensive while patients' time is cheap; to make sure that doctors do not waste their valuable time waiting for patients, they make appointments close together, and patients are expected to wait. On the other hand, in petrol stations the cost of servers (petrol pumps) is low and customers can drive to a competitor when there is a queue. Then it is better to have a large number of servers with low utilisation, ensuring that customers only have a short wait in any queue.

Review questions

22.1 What causes a queue?

22.2 'Customers do not like to wait, so there should always be enough servers to eliminate queues.' Do you think this is true?

Single-server queues

The simplest type of queue has:

- a **single server** dealing with a queue of customers
- random arrival of customers to join the queue
- all customers waiting to be served in first-come-first-served order
- random service time.

Chapter 15 showed that a Poisson distribution describes random occurrences, so we can use this to describe customer arrivals. When the average number of customers arriving in a unit time is λ, the probability of r arrivals in unit time is given by the Poisson distribution:

$$P(r) = \frac{e^{-\lambda} \times \lambda^r}{r!}$$

where: r = number of arrivals
λ = mean number of arrivals
e = exponential constant (2.71828 . . .).

Service time is also random – but now the data is continuous. To describe this we can use a negative exponential distribution, which is related to the Poisson distribution but describes continuous data. You need not worry about its exact form (which is illustrated in Figure 22.2), except that it has the useful feature that the probability of service being completed within some specified time, T, is:

$$P(t \leq T) = 1 - e^{-\mu T}$$

where: μ = mean service rate
= the average number of customers served in a unit of time.

So the probability that service is not completed by time T is:

$$P(t > T) = 1 - P(t \leq T) = e^{-\mu T}$$

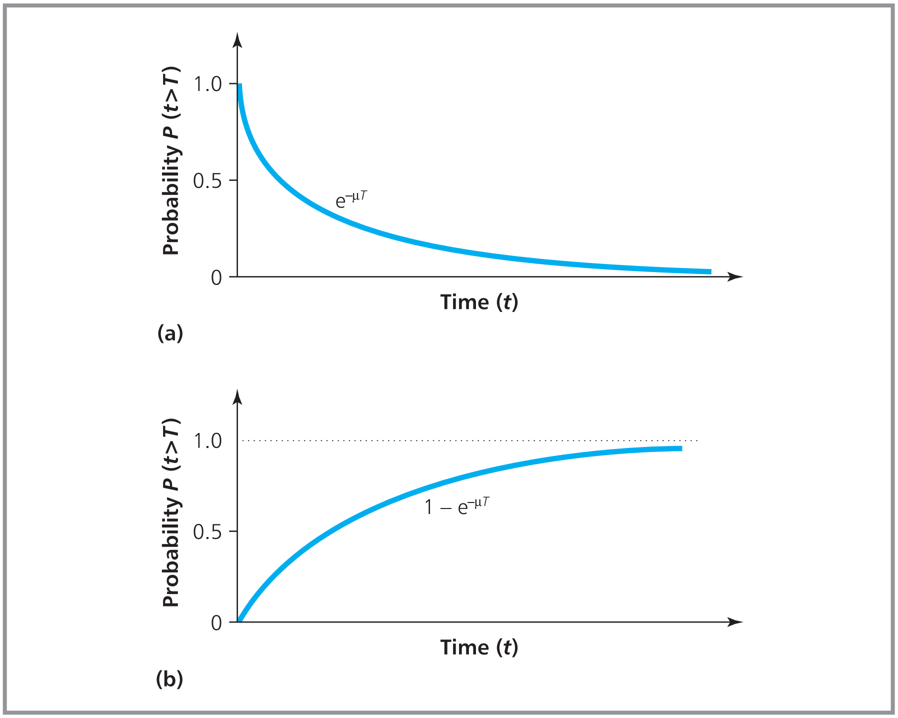


Figure 22.2 Random service times follow a negative exponential distribution: (a) probability that service is not completed within a time T; (b) probability that service is completed within a time T

Now we have descriptions of the random arrival of customers in terms of λ (the mean arrival rate) and of random service times in terms of μ (the mean service rate). If the mean arrival rate is greater than the mean service rate, the queue will never settle down to a steady state but will continue to grow indefinitely. So any analysis of queues must assume a steady state where μ is greater than λ.

Now we can derive some standard results for a single-server queue – which are called the operating characteristics. Unfortunately, the formal derivations are rather messy, so we will develop the results intuitively. Some of these are not obvious, so we will simply state them as standard results.

Imagine a queue where the mean arrival rate is two customers an hour and the mean service rate is four customers an hour – on average the system is busy for half the time. Here we define 'busy' as having at least one customer either being served or in the queue. From this you can imagine that in general a system is busy for a proportion of time λ/μ, and this is also the average number of customers being served at any time.

The probability that there is no customer in the system is:

$$P_0 = 1 - \lambda/\mu$$

This is the probability that a new customer is served without any wait.

The probability that there are n customers in the system is:

$$P_n = P_0(\lambda/\mu)^n$$

We can use this result – which is not intuitively obvious – to calculate some other characteristics of the queue. To start with, the average number of customers in the system is:

$$L = \sum_{n=0}^{\infty} nP_n = \frac{\lambda}{\mu - \lambda}$$

where the symbol ∞ ('infinity') indicates that the summation continues indefinitely.

The average number of customers in the queue is the average number in the system minus the average number being served:

$$L_q = L - \frac{\lambda}{\mu} = \frac{\lambda}{\mu - \lambda} - \frac{\lambda}{\mu} = \frac{\lambda^2}{\mu(\mu - \lambda)}$$

The average time a customer has to spend in the system is:

$$W = \frac{L}{\lambda} = \frac{1}{\mu - \lambda}$$

The average time spent in the queue is the average time in the system minus the average service time:

$$W_q = W - \frac{1}{\mu} = \frac{\lambda}{\mu(\mu - \lambda)}$$

WORKED EXAMPLE 22.1

People arrive randomly at a bank teller at an average rate of 30 an hour. If the teller takes an average of 0.5 minutes to serve each customer, what are the average number of customers in the queue and the time they wait to be served? What happens if average service time increases to one or two minutes?

Solution

The average arrival rate is $\lambda = 30$. If the teller takes an average of 0.5 minutes to serve each customer, this is equivalent to a service rate of 120 an hour. Then the average number of customers in the queue (excluding anyone being served) is:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{30^2}{120 \times (120 - 30)} = 0.083$$

The average time in the queue is:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{120 \times (120 - 30)} = 0.003 \text{ hours}$$
$$= 0.167 \text{ minutes}$$

We find the average number of people in the system from:

$$L = L_q + \lambda/\mu = 0.083 + 30/120 = 0.333$$

and the average time in the system is:

$$W = W_q + 1/\mu = 0.003 + 1/120 = 0.011 \text{ hours}$$
$$= 0.667 \text{ minutes}$$

With an average service time of one minute, $\mu = 60$ and:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{30^2}{60 \times (60 - 30)} = 0.5$$
$$L = L_q + \lambda/\mu = 0.5 + 30/60 = 1.0$$
$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{30}{60 \times (60 - 30)} = 0.017 \text{ hours}$$
$$= 1.0 \text{ minute}$$
$$W = W_q + 1/\mu = 1.0 + 1/60 = 1.017 \text{ minutes}$$

If the average service time increases to 2 minutes, the service rate is $\mu = 30$. This does not satisfy the condition that $\mu > \lambda$, so the system will not settle down to a steady state and the queue will continue to grow indefinitely.

WORKED EXAMPLE 22.2

Customers arrive randomly at a railway information desk at a mean rate of 20 an hour. There is one person at the desk who takes an average of two minutes with each customer. Find the characteristics of the queuing system.

Solution

The mean arrival rate, λ, is 20 an hour and the mean service rate, μ, is 30 an hour. The probability that there is no-one in the system is:

$P_0 = 1 - \lambda/\mu = 1 - 20/30 = 0.33$

So there is a probability of $1 - 0.33 = 0.67$ that a customer has to wait to be served.

The probability of n customers in the system is:

$P_n = P_0 \times (\lambda/\mu)^n = 0.33 \times (0.67)^n$

Therefore

$P_1 = 0.22$, $P_2 = 0.15$, $P_3 = 0.10$, $P_4 = 0.07$, etc.

The average number of customers in the system is:

$L = \lambda/(\mu - \lambda) = 20/(30 - 20) = 2$

The average number of customers in the queue is:

$$L_q = \frac{\lambda^2}{\mu(\mu - \lambda)} = \frac{20^2}{30 \times 10} = 1.33$$

The average time a customer spends in the system is:

$W = 1/(\mu - \lambda) = 1/(30 - 20) = 0.1$ hours
$= 6$ minutes

The average time a customer spends in the queue is:

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{20}{30 \times 10} = 0.0667 \text{ hours}$$
$= 4$ minutes

We have described the operating characteristics of a single-server queue, and could now go on to describe multi-server and other types of queue. As you can guess from the calculations that we have already described, the arithmetic becomes rather messy. Of course, we could use a computer for the arithmetic, and Figure 22.3 shows the printout from a package analysing a queue with four servers. Another option is to look for another method of solving these problems, and this is where we move on to simulation.

Review questions

22.3 What are the variables λ and μ in a queuing system?

22.4 What happens in a queue if $\lambda \geq \mu$?

22.5 What are the assumptions of the single-server queue model?

Simulation models

Simulation gives an alternative way of dealing with larger queuing problems – or any other complex problem. It does not solve a set of equations, but simulates the operations to see how they behave over a typical period. Simulation effectively imitates the working of a real situation, giving a dynamic view of a system over an extended time. An ordinary model looks at the system, collects data for some fixed point in time and draws conclusions; simulation follows the operations of the system and sees exactly what happens over time. A simple analogy has an ordinary model giving a snapshot of the system at some fixed point, while a simulation model takes a movie of the system.

QUEUE ANALYSIS

PROBLEM NAME: Example

MODEL: Multiple Channels

Arrival Rate (lambda)	=	100
Service Rate (mu)	=	30
Number of Channels	=	4

Average Number of Units in Waiting Line	=	3.2886
Average Number of Units in System	=	6.6219
Average Waiting Time in Line	=	0.0329
Average Time in System	=	0.0662
Probability of Idle System	=	0.0213
Probability of 1 units in the system	=	0.0710
Probability of 2 units in the system	=	0.1184
Probability of 3 units in the system	=	0.1315
Probability of 4 units in the system	=	0.1096
Probability of 5 units in the system	=	0.0914
Probability of 6 units in the system	=	0.0761
Probability of 7 units in the system	=	0.0634
Probability of 8 units in the system	=	0.0529
Probability of 9 units in the system	=	0.0441
Probability of 10 units in the system	=	0.0367
Probability of 11 units in the system	=	0.0306
Probability of 12 units in the system	=	0.0255
Probability of 13 units in the system	=	0.0212
Probability of 14 units in the system	=	0.0177
Probability of 15 units in the system	=	0.0148
Probability of 16 units in the system	=	0.0123
Probability of 17 units in the system	=	0.0102

COSTS

Average cost of units in the system	=	£20 per time period
Average cost per server	=	£35 per time period
Total cost per time period	=	£272.44

Figure 22.3 Example of a printout for a multi-server queue

We can show the general approach of simulation with a simple example, illustrated in Figure 22.4. Here an item is made on a production line at a rate of one unit every two minutes. At some point there is an inspection, which takes virtually no time. At this inspection 50% of units are rejected and the remaining 50% continue along the line to the next operations, which take three minutes a unit. Managers want to answer a series of questions about this system:

- How much space should they leave for the queue between the inspection and the next operations?
- How long will each unit stay in the system?
- What is the utilisation of equipment?
- Are there any bottlenecks?

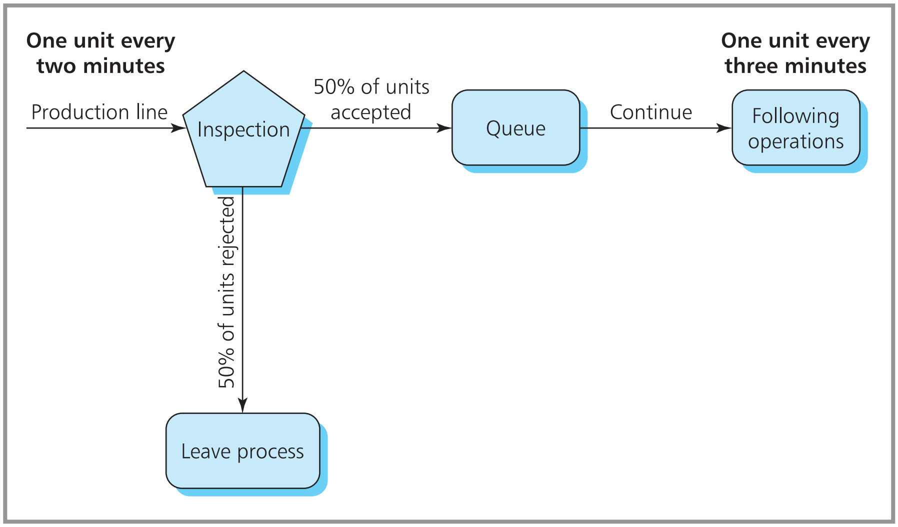


Figure 22.4 Example for simulation

This system is essentially a single-server queue where we know all the information, so we could use a queuing model. An alternative is to stand and watch the system working over a typical period and see what happens. We could follow a few units through the system and record information, perhaps using a table like that shown in Figure 22.5.

Here the first unit arrived for inspection at some time which we arbitrarily set to 0. The unit was accepted and moved straight through to operations that took three minutes. The unit was in the system – consisting of inspection, queue and operations – for three minutes.

The second unit arrived at time 2 from the arbitrary start time, was accepted and joined the queue (column E shows the number in the queue after this unit joins it). Operations could not start on unit 2 until unit 1 was finished at time 3, and then unit 2 left the system three minutes later at time 6.

	A	B	C	D	E	F	G	H	I
1	**Collecting data by observation**								
2									
3	**Unit number**	**Arrival time**	**Accept or reject**	**Time joins queue**	**Number in queue**	**Time operations start**	**Time operations finish**	**Time in queue**	**Time in system**
4	1	0	A	0	0	0	3	0	3
5	2	2	A	2	1	3	6	1	4
6	3	4	A	4	1	6	9	2	5
7	4	6	R		0				
8	5	8	R		0				
9	6	10	A	10	0	10	13	0	3
10	7	12	A	12	1	13	16	1	4
11	8	14	R		0				

Figure 22.5 Information collected by watching a process

We could stand and watch the operation for as long as we needed to get a reliable view of its operation. Then we could analyse the figures to get all the information we want. But this approach clearly has a number of disadvantages – it is time consuming and difficult to organise, needs many reliable observations, considers only one way of working (so it cannot compare different methods), and needs someone observing operations (which is unpopular with the people doing them). But there is an alternative, which is to collect enough data to show how the system works, and then make up other sets of typical figures. In other words, we generate observations with the same characteristics as those in Figure 22.5, without actually standing and watching the operations.

In this example, there is only one element of uncertainty, which is whether a unit is accepted or rejected. We need some method of randomly assigning these decisions to a unit, giving a 50% chance of acceptance and a 50% chance of rejection. An obvious way of doing this is to spin a coin – when it comes down heads we reject the unit, and when it comes down tails we accept it (or vice versa). A more formal method uses random numbers, which we described in Chapter 4. If we have the following string of random digits:

52847780169413567564547930177149431790465825

we could use even digits (including 0) for acceptance, and odd digits for rejection. Then we reject the first unit (based on 5), accept the second (based on 2), accept the third (based on 8), and so on. Now we can develop a typical set of results for the process without actually watching it, like those in Figure 22.6.

We know that one unit arrives for inspection every two minutes, so we can complete column B. Column C shows the sequence of random numbers, with the corresponding decision in column D. Units that are rejected leave the system, while those that are accepted join the queue at their arrival time, shown in column E. Column F shows the number in the queue after a unit arrives. Column G shows the time when operations start, which is the later

	A	B	C	D	E	F	G	H	I	J
1	**Simulation of process**									
2										
3	**Unit number**	**Arrival time**	**Random number**	**Accept or reject**	**Time joins queue**	**Number in queue**	**Time operations start**	**Time operations finish**	**Time in queue**	**Time in system**
4	1	0	5	R		0				
5	2	2	2	A	2	0	2	5	0	3
6	3	4	8	A	4	1	5	8	1	4
7	4	6	4	A	6	1	8	11	2	5
8	5	8	7	R		0				
9	6	10	7	R		0				
10	7	12	8	A	12	0	12	15	0	3
11	8	14	0	A	14	1	15	18	1	4
12	9	16	1	R		0				
13	10	18	6	A	18	0	18	21	0	3
14	**Mean**					0.30			0.67	3.67
15	**Maximum**					1.00			2.00	5.00

Figure 22.6 Simulating the process

of the arrival time and the time the previous unit finishes (from column H). Column H shows that operations finish three minutes after they start, column I shows the time in the queue (the difference between the arrival time in column B and the time operations start in column G), and column J shows the total time in the system (the difference between the arrival time in column B and the time operations finish in column H).

So the rules for generating entries in each column are as follows:

- Column A: the number increases by 1 for each unit entering.
- Column B: arrival time increases by 2 for each unit entering.
- Column C: from a string of random numbers, using the RAND function.
- Column D: a unit is accepted if the corresponding random number is even and rejected if it is odd.
- Column E: accepted units join the queue straight away at their arrival time; rejected ones leave the system.
- Column F: the number already in the queue is one more than it was for the last arrival, minus the number that left since the last arrival.
- Column G: operations start at the arrival time when they are idle, or when they finish work on the previous unit (the previous entry in column H).
- Column H: finishing time for operations, which is column G plus 3.
- Column I: time in the queue is the difference between the arrival time in the queue and the time operations start (column G minus column E).
- Column J: time in the system is the difference between the arrival time and the finish of operations (column H minus column B).

The simulation has been run for 10 units arriving and we can use the figures to give a number of results. For example, we can note that there was at most one unit in the queue for the processor. We can also find:

- number accepted = 6 (in the long run this would be 50% of units)
- number rejected = 4 (again this would be 50% in the long run)
- maximum time in queue = 2 minutes
- average time in queue = 4/6 minutes = 40 seconds
- maximum time in system = 5 minutes
- average time in system = 22/6 = 3.67 minutes
- average time in system including rejects = 22/10 = 2.2 minutes
- operations were busy for 18 minutes
- utilisation of operations = 18/21 = 86%.

It is important to ask how reliable these figures are. The simulation certainly gives a picture of the system working through a typical period of 21 minutes, but this is a very small number of observations and the results are not likely to be very accurate. So the next step is to extend the simulation for a much larger number of observations. Once we have built the simulation model and defined all the logic, extending it to give more repetitions is easy. And when we have information for several hundred arrivals we can be more confident that the results are reliable. As the model includes a random element, we can never be certain that the results are absolutely accurate, but with large numbers of repetitions we can be confident that they give a reasonable picture. This amount of repetition clearly needs a lot of arithmetic, so simulations are always done by computer.

Review questions

22.6 What does it mean when people describe simulation as a 'dynamic' representation?

22.7 Simulation can be used to model complex situations. Do you think this is true?

IDEAS IN PRACTICE Taco Bell

Taco Bell – a part of Yum! Brands, Inc., – is a fast-food restaurant chain that specialises in Mexican cuisine. Its restaurants aim at serving customers quickly – normally within three to five minutes – but they have to balance this level of customer service with the cost of providing it. A major part of their costs is employing people to work in their restaurants. Taco Bell wants these people to be fully utilised, but – in common with all fast-food restaurants – they have a problem with variable demand. Typically, there are considerable peaks in demand at lunchtime and in the early evening, with very quiet periods in the afternoon and early morning.

To tackle this problem Taco Bell developed its SMART (Scheduling Management And Restaurant Tool) system. This has three modules:

- A forecasting module, which predicts the number of customers arriving at a store in 30-minute – or even 15-minute – time slots. This gives a detailed picture of the number of customers and what they will buy throughout every day. Managers can add special events, holidays and other features to adjust the underlying forecasts.
- A simulation module, which takes the forecast demands, adds the features of the restaurant (such as size, opening hours, drive-through service, menu options, etc.) and shows how many employees of different types are needed throughout each day.
- An employee scheduling module, which takes the pattern of employees needed, the staff available in the store, and uses linear programming to produce schedules for staff members.

The resulting schedules are available four weeks in advance, and they allow for specific staff requirements, non-critical tasks scheduled during slack periods, performance monitoring, a broad history of data that gives forecasts for new products and stores, and identification of special risks.

Sources: www.tacobell.com; Bistritz M., Taco Bell finds recipe for success, *OR / MS Today*, October 1997.

Monte Carlo simulation

Simulation models that include a lot of uncertainty are described as **Monte Carlo simulation**. In the last example the only uncertainty was whether a unit was accepted or rejected, and we used random numbers for this decision. Most real problems have much more variability, and for this they use more complex procedures with random numbers. For example, suppose we want the probability of acceptance at an inspection to be 0.6. One way of arranging this is to use random digits 0 to 5 to represent acceptance and 6 to 9 to represent rejection. Then the string:

52847780169413567564547930177149431790465825

represents accept, accept, reject, accept, reject, and so on. We can extend this approach to sampling from more complex patterns. If 50% of units are accepted, 15% sent for reworking, 20% for reinspection and 15% rejected, we can split the stream of random digits into pairs:

52 84 77 80 16 94 13 56 75 64 54 79 30 17 71 etc.

Then:

- 00 to 49 (that is 50% of pairs) represent acceptance
- 50 to 64 (that is 15% of pairs) represent reworking
- 65 to 84 (that is 20% of pairs) represent reinspection, and
- 85 to 99 (that is 15% of pairs) represent rejection.

The stream of random digits then represents rework, reinspect, reinspect, reinspect, accept, and so on. In the long term the proportion of outcomes will match the requirements, but in the short term there will obviously be some variation. Here three of the first four units need reinspecting and while you might be tempted to 'adjust' such a figure, you should resist this temptation. Simulation needs many repetitions to give typical figures, and these include fairly unlikely occurrences.

WORKED EXAMPLE 22.3

Conal Fitzgerald checks the stock of an item at the beginning of each month and places an order so that:

order size = 100 – opening stock

The order is equally likely to arrive at the end of the month in which it is placed or one month later. Demand follows the pattern:

Monthly demand	10	20	30	40	50	60	70
Probability	0.1	0.15	0.25	0.25	0.15	0.05	0.05

There are currently 40 units in stock, and Conal want to simulate the system for the next 10 months. What information can he get from the results?

Solution

There is uncertainty in delivery time and demand, and Conal can take samples for these using the following schemes for random numbers.

- For *delivery time*, using single-digit random numbers:
  - an even random number means the delivery arrives in the current month
  - an odd random number means the delivery arrives in the next month.
- For *demand*, using a two-digit random number:

Demand	10	20	30	40	50	60	70
Probability	0.1	0.15	0.25	0.25	0.15	0.05	0.05
Random number	00–09	10–24	25–49	50–74	75–89	90–94	95–99

These schemes need two streams of random digits, which Conal can generate using a spreadsheet's RANDBETWEEN function. Then Figure 22.7 shows a set of results from following the system through 10 months. In this the stock at the end of a month is the initial stock plus arrivals minus demand. In month 1 the initial stock is 40, so Conal orders 60 units. The arrival random number determines that this arrives in the same month. The demand random number determines a demand of 50 in the month, so the closing stock is:

$$\begin{aligned}\text{closing stock} &= \text{opening stock} + \text{arrivals} - \text{demand}\\ &= 40 + 60 - 50 = 50\end{aligned}$$

There are no shortages and the closing stock is transferred to the opening stock for month 2. These calculations are repeated for the following 10 months.

The conclusions from this very limited simulation are not at all reliable. But if Conal continued the simulation for a much longer period – hundreds or thousands of months – he could find reliable figures for the distribution of opening and closing stocks, distribution of orders, mean demand, shortages, and mean lead time. Adding costs to the model would allow a range of other calculations.

Worked example 22.3 continued

	A	B	C	D	E	F	G	H	I	J	K
1	**Simulation of stocks**										
2											
3	**Month**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
4											
5	**Opening stock**	40	50	10	20	80	150	130	80	60	20
6	**Order**	60	50	90	80	20	–50	–30	20	40	80
7	**Arrival RN**	2	5	9	1	0	7	3	8	7	6
8	**Arrival month**	1	3	4	5	5	7	8	8	10	10
9											
10	**Demand RN**	83	50	56	49	37	15	84	2	66	41
11	**Demand size**	50	40	40	30	30	20	50	40	40	30
12	**Arrival**	60	0	50	90	100	0	0	20	0	120
13	**Closing stock**	50	10	20	80	150	130	80	60	20	110
14	**Shortages**	0	0	0	0	0	0	0	0	0	0

Figure 22.7 Simulation of stock for worked example 22.3 (RN = random number)

WORKED EXAMPLE 22.4

An office organises appointments for its customers so that one should arrive at a reception desk every eight minutes. After answering some standard questions, which takes an average of two minutes, customers are passed on to one of two offices. Thirty percent of customers (chosen randomly) go to office A, where they are served for an average of five minutes; the remaining customers go to office B where they are served for an average of seven minutes. Then all customers go to office D where they fill in forms for an average of six minutes before leaving. How would you set about simulating this system?

Solution

Figure 22.8 shows this system. As you can see, it is fairly straightforward, but for even the simplest simulation you need a computer. Spreadsheets can do many of the calculations, but they soon become complicated and it is far easier to use specialised software. Figure 22.9 shows the basic information given by a simple simulation package. In reality, most simulation is done with specialised simulation languages that are designed to build and run models efficiently. Some of these include GPSS, PARSEC,[1] ProModel, SIMSCRIPT, SIMULA, SLAM and SLX.

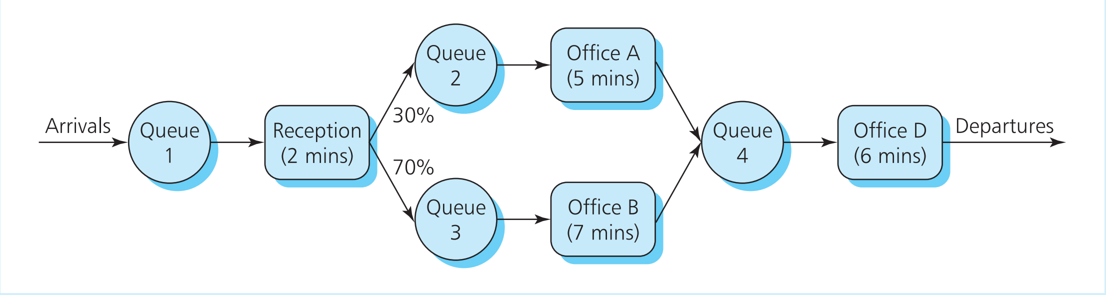


Figure 22.8 System for worked example 22.4

Worked example 22.4 continued

Customer Analysis for Queuing System

Data entered

Arrivals	Poisson	mean = 8	
Queue 1			
Reception	Normal	mean = 2	standard deviation = 0.5
Queue 2	Probability = 0.3		
Office A	Normal	mean = 5	standard deviation = 1.0
Queue 3	Probability = 0.7		
Office B	Normal	mean = 7	standard deviation = 2.0
Queue 4			
Office D	Normal	mean = 6	standard deviation = 1.0

Results

1	Total Number of Arrival	128
2	Total Number of Balking	0
3	Average Number in the System (L)	1.97
4	Maximum Number in the System	5
5	Current Number in the System	3
6	Number Finished	125
7	Average Process Time	13.74
8	Std Dev. of Process Time	2.10
9	Average Waiting Time (Wq)	1.69
10	Std Dev. of Waiting Time	2.85
11	Average Transfer Time	0
12	Std Dev. of Transfer Time	0
13	Average Flow Time (W)	15.61
14	Std Dev. of Flow Time	4.06
15	Maximum Flow Time	31.13

Data Collection: 0 to 1000 minutes
CPU Seconds = 0.1420

Figure 22.9 Sample printout for simulating queues in the office

Review questions

22.8 Why are random numbers used in simulation?

22.9 How many repetitions would you need to guarantee accurate results from simulation?

IDEAS IN PRACTICE SenGen Instrumental

SenGen Instrumental has a warehouse outside Geneva, from which it supplies materials to Central and Southern Europe. They use an inventory management system that was installed by a local IT specialist, and it has worked well for several years.

Before adjusting the system, the company wanted to check the effects of their proposed changes, and ran a series of simulations. These trials are too big to describe in detail, but we can illustrate their general approach.

Ideas in practice continued

The first requirement was a description of the sequence of activities in the current system, and for this SenGen used a comprehensive flow diagram. You can imagine this starting with the monthly demand for an item and using this to calculate an economic order quantity. Figure 22.10 shows the sequence of activities, starting by setting the known data, including costs and demand pattern, and uses this to calculate the features of the system, including economic order quantity and timing of deliveries. Then it follows the operations through a typical series of months. Starting with the first month, it checks the demand and deliveries due, sets the opening stock (which is last month's closing stock), finds the closing stock (which is opening stock plus deliveries minus demand) and calculate all the costs. Then it goes to the second month and repeats the analysis. The model repeats the calculations for as many months as are needed to get a reliable view of the operations, and then prepares summaries and analyses for all the figures. (In reality, the model looked at demand every hour and used a combination of ordering policies.)

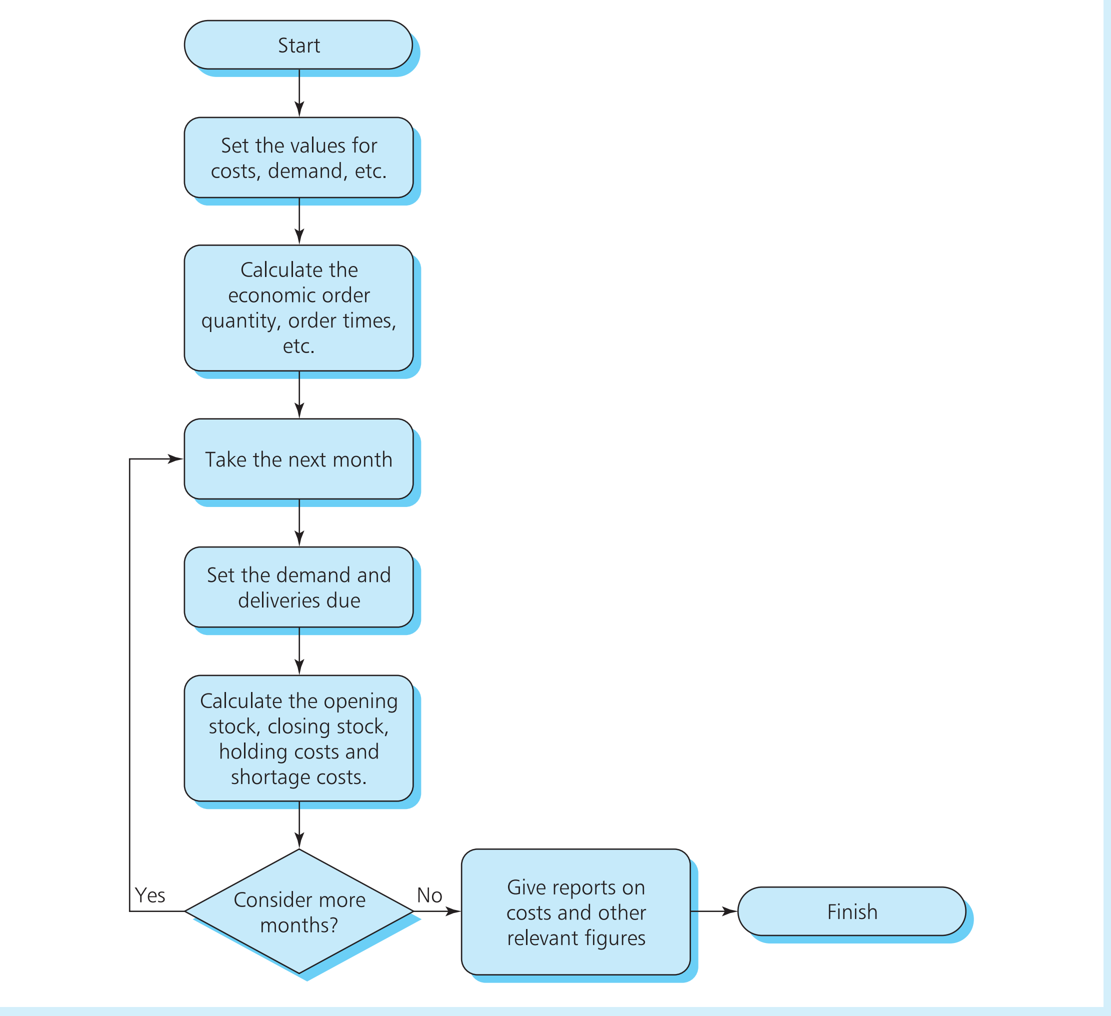


Figure 22.10 Flow diagram for basic inventory operations in SenGen

Ideas in practice continued

Having described the logic in a flow diagram, SenGen moved to the detailed simulation. They designed the basic calculations on a spreadsheet, illustrated in Figure 22.11. Here the economic order quantity is calculated as 200, corresponding to one delivery a month. The simulation follows the operations for 12 months, and you can see that there is a build-up of stock in the first half of the year followed by shortages in the second half. This gives high costs and suggests that the company might look for a different order pattern.

	A	B	C	D	E	F	G
1	**Simulation of stocks at SenGen**						
2							
3	**Notes:**	Effects of variable demand on costs					
4							
5	**Inputs**	Annual demand		2400			
6		Unit cost		SFr 10.00			
7		Holding cost amount		SFr 6.00			
8		Holding cost percentage		0%			
9		Reorder cost		SFr 50.00			
10		Shortage cost		SFr 100.00			
11							
12	**Results**	Annual holding cost		SFr 6.00			
13		Order size		200.00			
14		Time between orders		1.00			
15		Number of orders a year		12.00			
16		Fixed unit costs		SFr 24,000.00			
17		Variable cost		SFr 1,200.00			
18		Total annual cost		SFr 25,200.00			
19							
20	**Month**	**Opening stock**	**Demand**	**Delivery**	**Closing stock**	**Holding cost**	**Shortage cost**
21	1	0	140	200	60	SFr 30.00	SFr –
22	2	60	120	220	160	SFr 80.00	SFr –
23	3	160	145	190	205	SFr 102.50	SFr –
24	4	205	165	180	220	SFr 110.00	SFr –
25	5	220	295	225	150	SFr 75.00	SFr –
26	6	150	245	215	120	SFr 60.00	SFr –
27	7	120	315	200	5	SFr 2.50	SFr –
28	8	5	225	200	–20	SFr –	SFr 166.67
29	9	–20	245	220	–45	SFr –	SFr 375.00
30	10	–45	250	225	–70	SFr –	SFr 583.33
31	11	–70	205	210	–65	SFr –	SFr 541.67
32	12	–65	110	200	25	SFr 12.50	SFr –
33							
34					**Summary costs**		
35						Holding cost	SFr 472.50
36						Reorder cost	SFr 600.00
37						Shortage cost	SFr 1,666.67
38						Unit cost	SFr 24,000.00
39						**Total cost**	SFr 26,739.17

Figure 22.11 Basic simulation of stock at SenGen

Ideas in practice continued

Now SenGen had the basic steps of a simulation, and they added details to give a better picture of the real system. For example, they added more variability to all aspects of demand, lead time, amount delivered, costs, and so on. Then they transferred the ideas to specialised software for the actual runs (with a single run illustrated in Figure 22.12).

SIMULATION SYSTEMS

SYSTEM: Inventory Management **Originator: SenGen 43/df**
Created on 01.01.11 **Modified on 01.01.11** **This run at 0000 on 01.01.11**

DATA REVIEW

Order quantity	Automatic
Reorder level	Automatic
Forecasting method	Exponential smoothing
Forecasting parameters	Optimised
Number of stock items	1

Stock item 1			
Name	Component		
Unit cost	SFr 20	Demand history	Defined
Reorder cost	SFr 75	Demand distribution	Normal
Holding cost	SFr 1	Lowest demand	100
Fixed holding cost	SFr 0	Highest demand	500
Shortage cost	SFr 50	Lead time distribution	Uniform
Fixed shortage cost	SFr 0	Shortest lead time	3
Backorders	0	Longest lead time	5
Urgent orders	No	Service level	93%
Emergency orders	No		

Period	Week		
Number of periods	13	Number of runs	100
Random number seed	Random	Analysis	None
Reports	Transaction (#1,1),	Summary (1)	

INITIAL VALUES

Opening stock	1500	Mean lead time	4
Mean demand	300	Lead time demand	1200
Reorder quantity	212	Safety stock	83
Outstanding orders	Defined	Reorder level	1283

SIMULATION RESULTS

TRANSACTION REPORT(1) – first run

Week	**Opening**	**Demand**	**Closing**	**Shortage**	**Order**	**Arrival**
1	1500	331	1169			
2	1169	372	797		210 #1	
3	797	229	568		210 #2	
4	568	205	363		210 #3	
5	363	397	0	34	250 #4	
6	0	227	0	227	260 #5	
7	215	0	0	215	270 #6	210 #1
8	210	326	0	116	280 #7	210 #2
9	210	329	0	119	290 #8	210 #3
10	210	336	0	126	300 #9	250 #4
11	250	295	45		310 #10	260 #5
12	305	280	25		310 #11	270 #6
13	295	263	32		310 #12	280 #7

SUMMARY RESULTS(1)

Number of periods	13			
Number of runs	100			
Demand	296.7			
Lead time	5.13 weeks			
Stock level	396.2 units	Holding cost	SFr	396.20
Shortages	812.3 units	Shortage cost	SFr	40615.00
		Reorder cost	SFr	787.50
		Variable cost	SFr	41798.70
Deliveries	1681 units	Fixed cost	SFr	33620.00
		Total cost	**SFr**	**75418.70**

Figure 22.12 Example of more detailed simulation for SenGen

CHAPTER REVIEW

This chapter discussed the concept of queuing theory, and used this to introduce the idea of simulation as a means of tackling complex problems.

- A queue forms whenever a customer arrives at a server and finds that the server is already busy. In practice, queues occur in many situations, not all of which involve people.
- For any given queue, the length depends on the number of servers. More servers increase customer service but give higher costs. Managers look for a balance between service and costs.
- The simplest queue has a single server with random arrivals and service times. The operating characteristics show the key features of such systems.
- We could move on to more complex queuing systems, but the calculations become very difficult.
- Simulation gives an alternative way of tackling complex problems. Rather than solve a series of equations, it duplicates the operations over some period, producing a typical – but artificial – set of results. Monte Carlo simulation includes a lot of variability and uncertainty, which is achieved by using random numbers.

CASE STUDY The Palmer Centre for Alternative Therapy

Jane and Andrew Palmer run a business in Florida that they describe as an 'alternative therapy centre'. Their treatments range from fairly traditional ones such as osteopathy, to more experimental ones that work with the body's 'energy pathways'. Their clients have a range of problems of different severity and symptoms, but in practice their main concern is people who suffer from long-term back problems.

The Palmer Centre employs a receptionist who deals with around 20 customers an hour. Seventy per cent of these only ask for information and the receptionist can deal with them in a couple of minutes. The remainder are actual patients.

Each patient takes about three minutes to give the receptionist information before moving on to consultations with either Jane or Andrew. Roughly two-thirds of patients see Andrew, who spends around 15 minutes with each; the rest see Jane, who spends around 20 minutes with each. After they have finished treatment, the patients return to the receptionist, who takes two minutes to collect information, prepare bills and arrange follow-up appointments.

This arrangement seems to work well, but some patients complain of having to wait. This is particularly important, as the Palmers are thinking of expanding and employing another therapist. They will have to do some marketing to attract new clients, but realise that alternative treatments are becoming more popular and finding new business should not be difficult.

For the expansion, they will have to make some changes to the centre, and build another office and waiting area for the new therapist. If the expansion is a success, they might be tempted to expand even more in the future. This raises the question of how many offices the centre will need for varying levels of demand, and how big the waiting areas should be.

Questions

- **How can you judge the current performance of the centre?**
- **Can you find the resources that the centre will need to maintain – or improve – the current level of service with increasing numbers of patients?**

PROBLEMS

22.1 Describe the operating characteristics of a single-server queue with random arrivals at an average rate of 100 an hour and random service times at an average rate of 120 an hour.

22.2 A single-server queue has random arrivals at a rate of 30 an hour and random service times at a rate of 40 an hour. If it costs £20 for each hour a customer spends in the system and £40 for each hour of service time, how much does the queue cost?

22.3 Piotr Banasiewicz is a self-employed plumber who offers a 24-hour emergency service. For most of the year calls arrive randomly at a rate of six a day. The time he takes to travel to a call and do the repair is randomly distributed with a mean of 90 minutes. Forecasts for February suggest cold weather, and last time this happened Piotr received emergency calls at a rate of 18 a day. Because of repeat business, Piotr is anxious not to lose a customer and wants average waiting time to be no longer in February than during a normal month. How many assistants should he employ to achieve this?

22.4 Figure 22.13 shows the printout from a program which analyses queues. The title 'M/M/3' is an abbreviation to show that the queue has random arrivals, random service time and three servers. What do the results show? How can you check them?

Input Data of Queuing Example

M/M/3

Customer arrival rate (lambda)	=	20.000
Distribution	:	Poisson
Number of servers	=	3
Service rate per server	=	8.000
Distribution	:	Poisson
Mean service time	=	0.125 hour
Standard deviation	=	0.125 hour
Queue limit	=	Infinity
Customer population	=	Infinity

Solution for Queuing Example

M/M/3

With lambda = 20 customers per hour and μ = 8 customers per hour

Utilisation factor (p)	= 0.833
Average number of customers in the system (L)	= 6.011
Average number of customers in the queue (Lq)	= 3.511
Average time a customer in the system (W)	= 0.301
Average time a customer in the queue (Wq)	= 0.176
The probability that all servers are idle (Po)	= 0.004
The probability an arriving customer waits (Pw)	= 0.702

P(1) = 0.11236	P(2) = 0.14045	P(3) = 0.11704	P(4) = 0.09753
P(5) = 0.08128	P(6) = 0.06773	P(7) = 0.05644	P(8) = 0.04704
P(9) = 0.03920	P(10) = 0.03266		

$$\sum_{i=1}^{10} P(i) = 0.791736$$

Figure 22.13 Printout from a queuing analysis program

22.5 Customers arrive at a server at a rate of 100 an hour and each server can serve at a rate of 15 an hour. If customer time is valued at $20 an hour and server time costs $50 an hour, how can you identify the best number of servers?

22.6 A supermarket has an average of 240 customers an hour arriving at peak times. During these periods all 10 checkouts could be used, with each serving an average of 35 customers an hour. Simulate this process to show the effects of actually opening different numbers of checkouts.

RESEARCH PROJECTS

22.1 There are many specialised programs for simulation. Some of these are sophisticated simulation languages that include animation and many other features. If you were looking for a simulation program, what features would you expect to see in a major package? Illustrate your answer with references to real packages.

22.2 Spreadsheets can be used for some types of simulation. Look at a specific queuing problem and build a simple model on a spreadsheet. What results can you find? How would you expand your model to deal with more complex situations?

22.3 Find a real queuing problem and show how you would begin to analyse it. How can simulation help with your problem? How does your problem compare with the queuing problems that managers have solved in other circumstances?

22.4 Simulations of various kinds are widely used in practice. Find some examples where they have been particularly useful.

Sources of information

Reference

1 UCLA Parallel Computing Laboratory at www.cs.ucla/edu/project/parsec.

Further reading

Books on queuing – like the subject – soon become very mathematical. The following books give a useful starting point.

Banks J., Carson J.S., Nelson B. and Nicol D., *Discrete-event Simulation*, Prentice-Hall, Englewood Cliffs, NJ, 2005.

Gross D. and Harris C., *Fundamentals of Queuing Theory* (3rd edition), John Wiley, Chichester, 1998.

Hall R.W., *Queuing Methods*, Prentice Hall, Upper Saddle River, NJ, 1997.

Law A., *Simulation Modelling and Analysis* (4th edition), McGraw-Hill, New York, 2006.

Moore J.H. and Weatherford L.R., *Decision Modelling with Microsoft Excel* (6th edition), Prentice Hall, Upper Saddle River, NJ, 2001.

Oakshott L., *Business Modelling and Simulation*, FT Prentice Hall, London, 1997.

Pidd M., *Computer Simulation in Management Science* (4th edition), John Wiley, Chichester, 1997.

Prabhu N.U., *Foundations of Queuing Theory*, Kluwer Academic, Boston, MA, 1997.

Ragsdate C., *Spreadsheet Modelling and Decision Analysis* (4th edition), South-Western College Publishing, Cincinnati, OH, 2003.

GLOSSARY

[Figures in brackets show the chapter where the topic is discussed]

5-whys method [19] – repeatedly asking questions to find the cause of faults

ABC analysis [20] – Pareto analysis for inventory items

acceptable quality level (AQL) [19] – the poorest level of quality, or the most defects, that is acceptable

acceptance sampling [19] – tests a sample from a batch to see whether the whole batch reaches an acceptable level of quality

achieved quality [19] – shows how closely a product conforms to its designed specifications

aggregate index [7] – monitors the way that several variables change over time

algebra [2] – use of symbols to represent variables and describe relationships between them

alternative hypothesis [17] – hypothesis that is true when we reject the null hypothesis

annual equivalent rate or **annual percentage rate** [8] – true interest rate for borrowing or lending money

annuity [8] – amount invested to give a fixed income over some period

arithmetic [2] – calculations with numbers

arithmetic mean [6] – the 'average' of a set of numbers

autocorrelation [9] – a relationship between the errors in multiple regression

axes [3] – rectangular scales for drawing graphs

bar chart [5] – diagram that represents the frequency of observations in a class by the length of a bar

base [2] – the value of b when a number is represented in the logarithmic format of $n = b^p$

base period [7] – the fixed point of reference for an index

base-period weighted index or **base-weighted index** [7] – an index which assumes that quantities purchased do not change from the base period

base value [7] – value of a variable in the base period

Bayes' theorem [14] – theorem that calculates conditional probabilities

bias [4] – a systematic error in a sample

binomial distribution [15] – distribution that shows the probabilities of different numbers of successes in a number of trials

break-even point [8] – sales volume at which an organisation covers its costs and begins to make a profit

calculations [1] – arithmetic manipulation of numbers

capacity [8] – the maximum output that can be achieved in a specified time

cardinal data [4] – data that can be measured

causal forecasting [9] – using a relationship to forecast the value of a dependent variable that corresponds to a known value of an independent variable

causal methods [10] – quantitative methods of forecasting that analyse the effects of outside influences and use these to produce forecasts

cause-and-effect diagram, **Ishikawa diagram** or **fishbone diagram** [19] – diagram that shows the causes of problems with quality

census [4] – a sample of the entire population

central limit theorem [16] – theorem that describes the distribution of observations about the mean

chi-squared (or χ^2) **test** [17] – a non-parametric hypothesis test

class [5] – range or entry in a frequency distribution

cluster sample [4] – result of choosing a sample in clusters rather than individually

coefficients [3] – in an equation, the numbers that multiply the variables (for example, 3 in $y = 3x$)

coefficient of correlation or **Pearson's coefficient** [9] – a measure of the strength of a linear relationship

coefficient of determination [9] – proportion of the total sum of squared errors from the mean that is explained by a regression

coefficient of skewness [6] – a measure of the symmetry or skewness of data

coefficient of variation [6] – the ratio of standard deviation over mean

column vector [11] – matrix with only one column

combination [15] – number of ways of selecting r things from n, when the order of selection does not matter

common fraction or **fraction** [2] – part of a whole expressed as the ratio of a numerator over a denominator

common logarithm [2] – logarithm to the base 10

compound interest [8] – interest paid on both the principal and the interest previously earned

conditional probabilities [14] – probabilities for dependent events

confidence interval [16] – the interval that we are, for instance, 95% confident that a value lies within

constant [2] – a number or quantity that always has the same, fixed value, such as π, e or 2

constrained optimisation [12] – applied to problems with an aim of optimising some objective, subject to constraints

consumer's risk [19] – the highest acceptable probability of accepting a bad batch, with more defects than LTPD

contingency table [17] – table showing the relationship between two parameters

continuous data [4] – data that can take any value (rather than just discrete values)

co-ordinates [3] – values of x and y that define a point on Cartesian axes

critical activities [21] – activities at fixed times in a project

critical path [21] – a series of critical activities in a project

critical path method (CPM) [21] – a method of project planning that assumes each activity has a fixed duration

critical value [17] – the test value for a chi-squared test

cumulative frequency distribution [5] – a diagram showing the sum of frequencies in lower classes

cumulative percentage frequency distribution [5] – a diagram showing the sum of percentage frequencies in lower classes

current-period weighted index or **current-weighted index** [7] – index which assumes that the current amounts bought were also bought in the base period

curve fitting [9] – finding the function that best fits a set of data

cycle service level [20] – the probability that all demand can be met in a stock cycle

data [4] – raw facts that are processed to give information

data collection [4] – the gathering of facts that are needed for decisions

data presentation [5] – format for showing the characteristics of data and emphasising the underlying patterns

data reduction [5] – reducing the amount of detail in data to emphasise the underlying patterns

decimal fraction [2] – part of a whole described by a number following a decimal point, such as 0.5

decimal places [2] – the digits following a decimal point

decision criteria [18] – simple rules that recommend an alternative for decisions with uncertainty

decision nodes [18] – points in a decision tree where decisions are made

decision tree [18] – diagram that represents a series of alternatives and events by the branches of a tree

decision variables [12] – variables whose value we can choose

definite integral [13] – evaluation of the indefinite integral at two points to find the difference

degrees of freedom [16] – a measure of the number of independent pieces of information used in probability distributions

denominator [2] – bottom line of a common fraction

dependence table [21] – table showing the relationships between activities in a project

dependent demand [21] – situation in which demands for materials are somehow related to each other

dependent events [14] – events in which the occurrence of one event directly affects the probability of another

dependent variable [3] – a variable whose value is set by the value of the independent variable

depreciation [8] – amount by which an organisation reduces the value of its assets

derivative or **first derivative** [13] – result of differentiating a function (see differentiation)

designed quality [19] – the quality that a product is designed to have

deterministic [14] – describing a situation of certainty

deviation [6] – distance an observation is away from the mean

differentiation [13] – algebraic process to calculate the instantaneous rate of change of one variable, e.g. y, with respect to another, e.g. x; the derivative is written dy/dx

discount factor [8] – value of $(1 + i)^{-n}$ when discounting to present value

discount rate [8] – value of i when discounting to present value

discounting to present value [8] – calculating the present value of an amount available in the future

discrete data [4] – data that is limited to integer values

diseconomies of scale [8] – effect where the average cost per unit rises as the number of units produced increases

distribution-free tests [17] – see non-parametric tests

e or **exponential constant** [2] – a constant calculated from $(1 + 1/n)^n$, where n is an indefinitely large number; it equals approximately 2.7182818

economic order quantity [20] – order size that minimises costs for a simple inventory system

economies of scale [8] – effect where the average cost per unit declines as the number of units produced increases

elasticity of demand [13] – the proportional change in demand divided by the proportional change in price

element [11] – each entry in a matrix

equation [2] – algebraic formula that shows the relationship between variables, saying that the value of one expression equals the value of a second expression

expected value [18] – the sum of the probability multiplied by the value of the outcome

exponential constant or **e** [2] – a constant calculated from $(1 + 1/n)^n$, where n is an indefinitely large number; it equals approximately 2.7182818

exponential smoothing – weighting based on the idea that older data is less relevant and therefore should be given less weight

extrapolation [9] – causal forecasting with values outside the range used to define the regression

extreme point [12] – corner of the feasible region in linear programming

feasible region [12] – area of a graph in which all feasible solutions lie for a linear programme

feedback [1] – return of information to managers so that they can compare actual performance with plans

first derivative [13] – result of differentiating a function; see derivative

float, **total float** (or sometimes **slack**) [21] – difference between the amount of time available for an activity in a project and the time actually used

formulation [12] – getting a problem in the right form, particularly with linear programming

fraction [2] – usual name for a common fraction

frequency distribution [5] – diagram showing the number of observations in each class

frequency table [5] – table showing the number of observations in each class

Gantt chart [21] – diagram for showing the schedule of a project

global optima [13] – the overall maximum or minimum points of a graph

gradient [3] – a measure of how steeply a function is rising, expressed as dy/dx (see differentiation)

graph [3] – a pictorial view of the relationship between (usually) two variables

grouped data [6] – raw data already divided into classes

histogram [5] – frequency distribution for continuous data

hypothesis testing [17] – seeing whether a belief about a population is supported by the evidence from a sample

identity matrix [11] – matrix with 1's down the diagonal and all other entries zero

indefinite integral [13] – the reverse of differentiation, expressed as a function

independent demand [20] – demand where there is no link between demands for items

independent equations [11] – equations that are not related, and are not different versions of the same equation

independent events [14] – events for which the occurrence of one event does not affect the probability of a second

independent variable [3] – a variable that can take any value, and sets the value of a dependent variable

index or **index number** [7] – a number that compares the value of a variable at any point in time with its value in a base period

inequality [2] – a relationship that is less precise than an equation, typically with a form like $a \leq b$

information [4] – data that has been processed into a useful form

instantaneous gradient [13] – gradient of a curve at a single point

integer [2] – whole number without any fractional parts

integration [13] – the reverse of differentiation

intercept [3] – the point where the graph of a line crosses the y-axis

interest [8] – amount paid to lenders as reward for using their money

internal rate of return [8] – discount rate that gives a net present value of zero

interquartile range [6] – distance between the first and third quartiles

interval estimate [16] – estimated range within which the value for a population is likely to lie

inverse matrix [11] – matrix that multiplies the original matrix to give an identity matrix and is used instead of matrix division

judgemental forecasts [10] – forecasts that rely on subjective assessments and opinions

Laspeyre index [7] – base-weighted index

line graph [3] – graph that shows the relationship between two variables, usually on Cartesian axes

line of best fit [9] – line that minimises some measure of the error in a set of data

linear programming (LP) [12] – a method of solving some problems of constrained optimisation

linear regression [9] – a process that finds the straight line that best fits a set of data

linear relationship [3] – a relationship between two variables of the form $y = ax + b$, giving a straight line graph

local optima [13] – maximum or minimum points within a restricted range of a graph, rather than global optima

logarithm [2] – the value of p when a number is represented in the logarithmic format of $n = b^p$

loss function [19] – function that shows the notional cost of missing a performance target

lot tolerance percent defective (LTPD) [19] – the level of quality that is unacceptable, or the highest number of defects that customers are willing to accept

marginal benefit [4] – benefit from the last unit made, collected, etc.

marginal cost [4] – cost of making one extra unit of a product

marginal revenue [8] – revenue generated by selling one more unit of a product

matrix [11] – format for describing a table of figures and doing related calculations

matrix multiplication [11] – multiplying two matrices together

mean [6] – the ‘average’ of a set of numbers

mean absolute deviation [6] – average distance of observations from the mean

mean absolute error [9] – average error, typically in a forecast

mean price relative index [7] – mean value of indices for separate items

mean squared deviation or **variance** [6] – average of the squared distance from the mean

mean squared error [9] – average squared error, typically in a forecast

measure [1] – a numerical description of some attribute

measure of location [6] – showing the 'centre' or typical value for a set of data

measure of spread [6] – showing how widely data is dispersed about its centre

median [6] – the middle value of a set of numbers

mind map [18] – see problem map

mode [6] – the most frequent value in a set of numbers

model [1] – a simplified representation of reality

Monte Carlo simulation [22] – type of simulation model that includes a lot of uncertainty

mortgage [8] – amount borrowed for buying a house, or other capital facilities

moving average [10] – an average of the most recent periods of data

multicollinearity [9] – a relationship between the independent variables in multiple regression

multiple (linear) regression [9] – process that finds the line of best fit through a set of dependent variables

multi-stage sample [4] – sample that successively breaks a population into smaller parts, confining it to a small geographical area

mutually exclusive events [14] – events where only one can happen, but not both

natural logarithm [2] – logarithm to the base e

negative number [2] – number below zero

net present value [8] – result of subtracting the present value of all costs from the present value of all revenues

noise [9] – the random errors in observations

nominal data [4] – data for which there is no useful quantitative measure

non-critical activities [21] – activities in a project that have some flexibility in their timing

non-linear regression [9] – process that finds the function that best fits a set of data

non-linear relationship [3] – any relationship between variables that is not linear

non-negativity constraint [12] – constraint that sets all variables to be positive, especially in linear programmes

non-parametric tests or **distribution-free tests** [17] – hypothesis tests that make no assumptions about the distribution of the population

Normal distribution [15] – the most widely used probability distribution for continuous data

null hypothesis [17] – the original hypothesis that is being tested

numerator [2] – top line of a common fraction

objective function [12] – function to be optimised, especially in linear programming

ogive [5] – graph of the cumulative frequency against class for continuous data

operating characteristics [22] – features of a queuing system

operating characteristic curve [19] – curve that shows how well a sampling plan separates good batches from bad ones

operations [1] – all the activities that make an organisation's products

ordinal data [4] – data that cannot be precisely measured, but that can be ordered or ranked

origin [3] – the point where x and y Cartesian axes cross

Paasche index [7] – current-weighted index

parametric test [17] – hypothesis test that concerns the value of a parameter

Pareto chart [19] – the 'rule of 80/20' method of applying to identify the small number of causes that cause most problems

partial productivity [8] – the output achieved for each unit of a specified resource

payoff matrix or **payoff table** [18] – table that shows the outcomes for each combination of alternatives and events in a decision

Pearson's coefficient [9] – a measure of the strength of a linear relationship (same as coefficient of correlation)

percentage [2] – fraction expressed as a part of 100

percentage frequency distribution [5] – diagram showing the percentage of observations in each class

percentage point change [7] – change in an index between two periods

performance ratio [8] – actual performance divided by some standard reference value

permutation [15] – number of ways of selecting r things from n, when the order of selection is important

pictogram [5] – bar chart where the plain bar is replaced by some kind of picture

pie chart [5] – diagram that represents the frequency of observations in a class by the area of a sector of a circle

point estimate [16] – single estimate of a population value from a sample

Poisson distribution [15] – probability distribution largely used for describing random events

polynomial [3] – equation containing a variable raised to some power

population [4] – every source of data for a particular application

positive number [2] – number above zero

positive quadrant [3] – top right-hand quarter of a graph, where both x and y are positive

power [2] – value of b when a number is represented as a^b (i.e. the number of times a is multiplied by itself)

present value [8] – discounted value of a future amount

price elasticity of demand [13] – proportional change in demand divided by proportional change in price

price relative [7] – price of an item at any time divided by base price

primary data [4] – new data that is collected for a particular purpose

principal [8] – amount originally borrowed for a loan

probabilistic or **stochastic** [14] – containing uncertainty that is measured by probabilities

probability [14] – likelihood or relative frequency of an event

probability distribution [15] – a description of the relative frequency of observations

probability tree [14] – diagram to show related probabilities

problem map, **relationship diagram** or **mind map** [18] – a diagram that shows interactions and relationships in a problem

process control [19] – taking a sample of products to check that a process is working within acceptable limits

process control chart [19] – diagram for monitoring a process over time

producer's risk [19] – highest acceptable probability of rejecting a good batch, with fewer defects than the AQL

productivity [8] – amount of output for each unit of resource used

project [21] – a unique job that makes a one-off product

project (or **programme**) **evaluation and review technique (PERT)** [21] – method of project planning that assumes each activity has an uncertain duration

project network analysis [21] – the most widely used method of organising complex projects

projective methods [10] – quantitative methods of forecasting that extend the pattern of past demand into the future

quadratic equation [3] – equation with the general form $y = ax^2 + bx + c$

qualitative [1] – not using numbers, but based on opinions and judgement

quality [19] – ability of a product to meet, and preferably exceed, customer expectations

quality control [19] – using a series of independent inspections and tests to make sure that designed quality is actually being achieved

quality management [19] – all aspects of management related to product quality

quantitative [1] – using numbers

quantitative methods [1] – a broad range of numerical approaches to solving problems

quartile deviation [6] – half the interquartile range

quartiles [6] – points that are a quarter of the way through data when it is sorted by size

questionnaire [4] – set of questions used to collect data

queue [22] – line of customers waiting to be served

quota sample [4] – sample structured in such a way that it has the same characteristics as the population

random nodes [18] – points in a decision tree where events happen

random numbers [4] – a string of digits that follow no patterns

random sample [4] – sample in which every member of the population has the same chance of being selected

range [6] – difference between the largest and smallest values in a set of data

regression [9] – process to find the best equation to describe the relationship between variables

regret [18] – difference between the best possible outcome and the actual outcome in a decision

relationship diagram [18] – see problem map

reorder level [20] – stock level when it is time to place an order

roots of a quadratic equation [3] – points where the curve crosses the x-axis

round [2] – to state a number to a specified number of decimal places or significant figures

row vector [11] – matrix with only one row

rule of sixths [21] – rule to find the expected duration of an activity for PERT

safety stock [20] – additional stock that is used to cover unexpectedly high demand

sample [4] – members of the population chosen as sources of data

sampling by attribute [19] – taking a quality control sample where units are either acceptable or defective

sampling by variable [19] – taking a quality control sample where units have a measurable feature

sampling distribution of the mean [16] – distribution of the mean of samples from the population

sampling frame [4] – a list of every member of the population

scalar multiplication [11] – multiplying a matrix by a single number

scatter diagram [5] – graph of the unconnected set of points (x, y)

scientific notation [2] – representation of a number in the form $a \times 10^b$

seasonal index [10] – amount by which a deseasonalised value is multiplied to get a seasonal value

second derivative [13] – the result of differentiating the first derivative, called d^2y/dx^2 if the first derivative is dy/dx

secondary data [4] – data that already exists and can be used for a problem

semi-interquartile range [6] – half the interquartile range

sensitivity [10] – speed at which a forecast responds to changing conditions

sensitivity analysis [12] – seeing what happens when a problem (particularly a linear programme) is changed slightly

service level [20] – probability that demand can be met from stock

shadow price [12] – marginal value of resources in a linear programme

significance level [17] – minimum acceptable probability that a value actually comes from the hypothesised population

significant figures [2] – the main digits to the left of a number

simple aggregate index or **simple composite index** [7] – index that adds all prices (say) together and calculates an index based on the total price

simple interest [8] – interest paid on only the initial deposit, but not on interest already earned

simulation [22] – process that analyses problems by imitating real operations, giving a set of typical, but artificial, results

simultaneous equations [11] – independent equations that show the relationship between a set of variables

single-server queue [22] – queuing system with only one server

sinking fund [8] – a fund that receives regular payments so that a specified sum is available at some point in the future

slack [21] – see float

smoothing constant [10] – parameter used to adjust the sensitivity of exponential smoothing forecasts

solution [12] – finding an optimal solution to a problem (particularly a linear programme)

solving an equation [2] – using the known constants and variables in an equation to find the value of a previously unknown constant or variable

Spearman's coefficient (of rank correlation) [9] – a measure of the correlation of ranked data

spreadsheet [1] – a general program that stores values in the cells of a grid, and does calculations based on defined relationships between cells

square root [2] – the square root of n, $\sqrt{n}$, is the number that is multiplied by itself to give n

standard deviation [6] – a measure of the data spread, equal to the square root of the variance

standard error [16] – standard deviation of the sampling distribution of the mean

statistical inference [16] – process of collecting data from a random sample of a population and using it to estimate features of the whole population

stochastic or **probabilistic** [14] – containing uncertainty that is measured by probabilities

stocks [20] – stores of materials that organisations keep until needed

stratified sample [4] – a sample taken from each distinct group in the population

strict uncertainty [18] – see uncertainty

Student-*t* distribution [16] – see *t*-distribution

symbolic model [1] – model where real properties are represented by symbols, usually algebraic variables

systematic sample [4] – sample in which data is collected at regular intervals

***t*-distribution** or **Student-*t* distribution** [16] – a distribution used instead of the Normal distribution for small samples

target stock level [20] – stock level that determines the order size for stocks with periodic review

terminal nodes [18] – points in a decision tree at the end of each path

time series [10] – a series of observations taken at regular intervals of time

total float [21] – see float

total productivity [8] – total output per unit of resources used

Total Quality Management (TQM) [19] – the system of having the whole organisation working together to guarantee, and systematically improve, quality

tracking signal [10] – a measure to monitor the performance of a forecast

transpose [11] – to change the rows of a matrix to columns and vice versa, and the name given to the matrix thus changed

turning points [13] – maxima and minima on a graph

uncertainty or **strict uncertainty** [18] – situation in which we can list possible events for a decision, but cannot given them probabilities

utilisation [8] – proportion of available capacity that is actually used

utility [18] – a measure that shows the real value of money to a decision maker

variable [2] – a number or quantity that can take different values, such as x, a or P

variance [6] – a measure of the spread of data by the mean squared deviation

Venn diagram [14] – a diagram that represents probabilities as circles that may or may not overlap

weighted index [7] – a price index (say) that takes into account both prices and the importance of items

weighted mean [6] – a mean that gives a different weight to each observation

zero matrix [11] – a matrix where every element is zero

APPENDIX A

Solutions to review questions

Chapter 1 – Managers and numbers

1.1 They allow clear, precise and objective measures of features, calculations using these, and rational analysis of problems.

1.2 No – managers make decisions.

1.3 No – but they must be aware of the types of analysis available, understand the underlying principles, recognise the assumptions and limitations, do some analyses themselves, have intelligent discussions with experts, and interpret the results.

1.4 There are several reasons for this, including availability of computers, improving software, fiercer competition forcing better decisions, new quantitative methods, good experiences with earlier analyses, better education of managers, and so on.

1.5 To develop solutions for problems, allow experimentation without risk to actual operations, allow experiments that would not be possible in reality, check the consequences of decisions, see how sensitive operations are to change, and so on.

1.6 We describe four stages: identifying a problem, analysing it, making decisions and implementing the results.

1.7 Generally in the analysis stage.

1.8 No – we can use any approach that efficiently gets a good answer.

1.9 No – you can get a feel for the numbers without doing the detailed calculations.

1.10 Because they are widely available, they use standard formats that are familiar and easy to use, and you do not have to learn how to use a new program for each problem.

Chapter 2 – Quantitative tools

2.1 You might want to check the figures, do some easy calculations by hand, do initial calculations to get rough estimates, get a feel for the numbers involved, or a host of other reasons.

2.2 (a) 4, (b) 56/5 or 11.2, (c) 3.

2.3 There is no difference. The choice of best depends on circumstances.

2.4 1,745,800.362 and 1,750,000.

2.5 Because it gives a precise way of describing and solving quantitative problems.

2.6 Yes.

2.7 No – to find two unknowns you need two equations.

2.8 A constant always has the same fixed value; a variable can take any one of a range of values.

2.9 There is no best format for all occasions – you should use the format that best suits your needs.

2.10 By rearranging them to (a) $x = 11 - 8pr/3q$ and (b) $x = q/4 - 7pq/2r$.

2.11 A relationship that is not a precise equation but takes some other form, such as $a \leq b$.

2.12 In ascending order, $(\frac{1}{2})^4 = 0.0625$, $4^{-1} = 0.25$, $1^4 = 1$, $4^{1/2} = 2$, $4^1 = 4$, $(\frac{1}{2})^{-4} = 16$.

2.13 $9^{1.5}/4^{2.5} = (\sqrt{9})^3/(\sqrt{4})^5 = 3^3/2^5 = 27/32 = 0.84$.

2.14 1 – as anything raised to the power zero equals 1.

2.15 1.23×10^9 and 2.53×10^{-7}.

2.16 Logarithms are defined by the relationship that $n = b^p$ meaning that $p = \log_b n$. They are used mainly to solve equations that contain powers.

Chapter 3 – Drawing graphs

3.1 A variable whose value is set by the value taken by the independent variable.

3.2 No. Graphs show relationships, but do not suggest cause and effect.

3.3 (0, 0).

3.4 Yes – the distance between the two points is 10.

3.5 It is a straight line with a gradient of –2 that crosses the y-axis at –4.

3.6 (a) 0, (b) 1, (c) –6.

3.7 They correspond to the points where $y > 3x + 5$.

3.8 No – they are the same general shape, but differ in detail.

3.9 The points where the curve crosses the x-axis.

3.10 They are imaginary.

3.11 Because graphs are difficult to draw exactly and calculations give more accurate results.

3.12 A function containing x raised to some power.

3.13 A point where the gradient changes from positive to negative (or vice versa), corresponding to a peak or trough.

3.14 Using the usual procedure to draw a graph of the general form $y = ne^{mx}$, where n and m are positive constants.

3.15 You cannot draw this on a two-dimensional graph. You can draw it on a three-dimensional graph, but the results are generally difficult to interpret.

Chapter 4 – Collecting data

4.1 Data are the raw numbers, measurements, opinions, etc. that are processed to give useful information.

4.2 Because managers need reliable information to make their decisions, and the first stage of this is data collection.

4.3 No – there is an almost limitless amount of data that can be collected, but only some of it is useful and cost-effective.

4.4 Because data of different types is collected, analysed and presented in different ways.

4.5 There are several ways, including quantitative/qualitative, nominal/cardinal/ordinal, discrete/continuous, and primary/secondary.

4.6 Discrete data can take only integer values, while continuous data can take any values.

4.7 There are many possible examples.

4.8 In principle it is better, but in practice we have to balance its benefits with the cost and effort of collection.

4.9 Because it is too expensive, time-consuming or impractical to collect data from the whole population.

4.10 Yes (at least almost always).

4.11 Because using the wrong population would make the whole data collection and analysis pointless.

4.12 One classification has census, random, systematic, stratified, quota, multi-stage and cluster samples.

4.13 Every member of the population has the same chance of being selected.

4.14 Suggestions are: (a) telephone survey, (b) personal interview, (c) longitudinal survey, (d) observation.

4.15 (a) leading question, (b) too vague, (c) several questions in one, (d) speculative.

4.16 You can try contacting non-respondents and encourage them to reply. Realistically, this will have little success, so you should search for common features to ensure no bias is introduced.

4.17 Because interviewers keep asking people until they fill the required quotas.

4.18 No. All data collection must be carefully planned before it is started.

Chapter 5 – Diagrams for presenting data

5.1 Data are the raw numbers, measurements, opinions, etc. that are processed to give useful information.

5.2 To simplify raw data, remove the detail, and show underlying patterns.

5.3 Unfortunately not.

5.4 Using diagrams or numbers.

5.5 They can display lots of information, show varying attributes and highlight patterns.

5.6 A description of the number of observations in a set of data falling into each class.

5.7 This depends on the nature of the data and the purpose of the table. A guideline suggests between 4 and 10 classes.

5.8 Because they are a very efficient way of presenting a lot of detail. No other format can fit so much information into a small space.

5.9 To a large extent yes – but the choice often depends on personal preference, and a diagram of any kind may not be appropriate.

5.10 To show that they are scaled properly, accurately drawn, and give a true picture.

5.11 No – there are many possible variations and the best is often a matter of opinion.

5.12 Probably some kind of pictogram.

5.13 They are not very accurate, show a small amount of data, and can be misleading.

5.14 Unfortunately, you can find many of these.

5.15 No – it is true for bar charts, but in histograms the area shows the number of observations.

5.16 The average height of the two separate bars.

5.17 Often there is no benefit from histograms and it makes sense to stick with bar charts. Sometimes histograms help with further statistical analyses.

5.18 To show the cumulative frequency against class for continuous data.

5.19 Perhaps – the diagonal line shows equally distributed wealth, but we do not know whether this is fair.

Chapter 6 – Using numbers to describe data

6.1 They give an overall impression, but do not give objective measures.

6.2 A measure of the centre of the data – some kind of typical or average value.

6.3 No. These only partially describe a set of data.

6.4 A measure for the centre of the data or a typical value.

6.5 No.

6.6 The most common measures are: arithmetic mean = $\sum x/n$; median = middle observation; mode = most frequent observation.

6.7 $(10 \times 34 + 5 \times 37)/15 = 35$.

6.8 In Excel useful functions are AVERAGE, MEDIAN and MODE.

6.9 We concentrated on range, mean absolute deviation, variance, and standard deviation. Yes.

6.10 Because positive and negative deviations cancel and the mean deviation is always zero.

6.11 Metres^2 and metres respectively.

6.12 Because it gives standard results that we can interpret and use in other analyses.

6.13 210 is nine standard deviations away from 120, and the chances of this are very small. It is more likely that there has been a mistake – perhaps writing 210 instead of 120.

6.14 Useful ones include MAX, MIN, VARP, STDEVP and QUARTILE.

6.15 To give a relative view of spread that can be used to compare different sets of data.

6.16 The general shape of a distribution.

6.17 The coefficients of variation are 0.203 and 0.128 respectively (remembering to take the square root of the variance), which shows that the first set of data is more widely dispersed than the second set.

Chapter 7 – Describing changes with index numbers

7.1 To measure the changes in a variable over time.

7.2 The base index is not always 100. It is often used, but only for convenience.

7.3 A percentage rise of 10% increases the value by 10% of the value; a percentage point rise of 10 increases the value by 10% of the base value.

7.4 When circumstances change significantly or when the old index gets too high.

7.5 132 × 345 / 125 = 364.32.

7.6 The mean price relative index is the average of the separate price indices; the simple aggregate index is based on sum of prices.

7.7 They are sensitive to the units used and do not take into account the relative importance of variables.

7.8 Base-period weighting assumes that the basket of items used in the base period is always used; current-period weighting considers price changes based on the current basket of items.

7.9 Because the basket of items bought is affected by prices, with items whose prices rise rapidly replaced by ones with lower price rises.

7.10 Yes.

7.11 Not really. The RPI monitors the changing prices paid for some items by a 'typical' family – but some people question its reliability.

Chapter 8 – Finance and performance

8.1 Because they give some context for the measures.

8.2 Total productivity is the ratio of total output over total inputs; partial productivity is the ratio of the main products over (usually) one resource.

8.3 Yes.

8.4 No.

8.5 The number of units processed (made, sold, served, etc.).

8.6 The number of units that must be sold before covering all costs and making a profit.

8.7 No – there may also be diseconomies of scale.

8.8 The optimal production quantity is the point where the marginal cost equals the marginal revenue.

8.9 £1,000 now.

8.10 Because you also earn interest on the interest that has been paid earlier, so your returns increase over time.

8.11 No.

8.12 By reducing all costs and revenues to present values, and calculating either the net present value or the internal rate of return for each project.

8.13 An estimate of the proportional increase or decrease in the value of money in each time period.

8.14 NPV uses a fixed discount rate to get different present values; IRR uses different discount rates to get a fixed present value.

8.15 Straight-line depreciation reduces the value by a fixed amount each period; the reducing-balance method reduces the value by a fixed percentage each period.

8.16 A fund that receives regular payments so that a specified sum is available at some point in the future.

8.17 By using the equation $A_n = A_0 \times (1 + i)^n + [F \times (1 + i)^n - F]/i$, to find A_0 when $A_n = 0$ and F is the regular payment received.

8.18 No – other factors should be included.

Chapter 9 – Regression and curve fitting

9.1 The errors, or deviations from expected values.

9.2 Real relationships are almost never perfect, and errors are introduced by noise, incorrectly identifying the underlying pattern, changes in the system being modelled, etc.

9.3 The mean error is $(1/n) \times \sum E_i$. Positive and negative errors cancel each other, so the mean error should be around zero unless there is bias.

9.4 Mean absolute error and mean squared error.

9.5 By calculating the errors – mean errors, mean absolute deviations, and mean squared errors – for each equation. All things being equal, the stronger relationship is the one with smaller error.

9.6 To find the line of best fit relating a dependent variable to an independent one.

9.7 x_i and y_i are the ith values of independent and dependent variables respectively; a is the intercept of the line of best fit, and b is its gradient; E_i is the error from random noise.

9.8 The time period.

9.9 No – there is no implied cause and effect.

9.10 Interpolation considers values within the range used to define the regression; extrapolation considers values outside this range.

9.11 The proportion of the total sum of squared error that is explained by the regression.

9.12 Values from −1 to +1. The coefficient of correlation is the square root of the coefficient of determination.

9.13 They are essentially the same, but Pearson's coefficient is used for cardinal data, while Spearman's is used for ordinal data.

9.14 No. It shows that 90% of the variation in the dependent variable is explained by, but not necessarily caused by, the relationship with the independent variable.

9.15 Multiple (linear) regression and non-linear regression.

9.16 Yes.

9.17 By comparing the coefficients of determination.

9.18 There is no difference.

Chapter 10 – Forecasting

10.1 All decisions become effective at some point in the future, so they must take into account future circumstances – and these must be forecast.

10.2 No.

10.3 Judgemental, projective and causal forecasting.

10.4 Relevant factors include: what is to be forecast, why this is being forecast, availability of quantitative data, how the forecast affects other parts of the organisation, how far into the future forecasts are needed, reliability of available data, what external factors are relevant, how much the forecast will cost, how much errors will cost, how much detail is required, how much time is available, and so on.

10.5 Forecasts based on subjective views, opinions and intuition rather than quantitative analysis.

10.6 Personal insight, panel consensus, market surveys, historical analogy and Delphi method.

10.7 The drawbacks are that the methods can be unreliable, experts may give conflicting views, cost of data collection is high, there may be no available expertise, and so on. On the other hand, they can be the only methods available, are flexible, can give good results, include subjective data, etc.

10.8 Because observations contain random noise which cannot be forecast.

10.9 By using both forecasts over a typical period and comparing the errors.

10.10 Because older data tends to swamp more recent (and more relevant) data.

10.11 By using a lower value of n.

10.12 It can be influenced by random fluctuations.

10.13 By using a moving average with n equal to the length of the season.

10.14 Because the weight given to the data declines exponentially with age, and the method smoothes the effects of noise.

10.15 By using a higher value for the smoothing constant, α.

10.16 Deseasonalise the data, find seasonal adjustments, project the underlying trend, use the seasonal adjustments to get seasonal forecasts.

10.17 An additive model adds a seasonal adjustment; a multiplicative model multiplies by a seasonal index.

10.18 Regression is generally preferred.

Chapter 11 – Simultaneous equations and matrices

11.1 Independent equations that show the relationships between a set of variables.

11.2 Six.

11.3 The equations for both graphs are true, so this point identifies the solution to the simultaneous equations.

11.4 Because they are more accurate and you can use more than two variables.

11.5 Matrices provide a convenient and efficient way of describing some problems.

11.6 **F** is a (3×3) matrix with $f_{1,3} = 4$ and $f_{3,1} = 6$.

11.7 A matrix with only one row or column.

11.8 Matrices can be added only if they are of the same size, so this calculation cannot be done.

11.9 Scalar multiplication multiplies a matrix by a number; matrix multiplication multiplies two matrices together.

11.10 (4×6).

11.11 Multiply by a column vector containing 1's.

11.12 No – most matrices do not have inverses.

11.13 Solving sets of simultaneous equations.

11.14 No.

Chapter 12 – Planning with linear programming

12.1 A problem where managers want an optimal solution, but there are constraints that limit the options available.

12.2 A method of tackling some problems of constrained optimisation.

12.3 The problem is constrained optimisation, both constraints and objective function are linear with respect to decision variables, proportionality and additivity assumptions are valid, problem variables are non-negative, and reliable data is available.

12.4 You put a problem into a standard form.

12.5 Decision variables, an objective function, problem constraints, and a non-negativity constraint.

12.6 The area representing feasible solutions which satisfy all constraints, including the non-negativity conditions.

12.7 To give the measure by which solutions are judged and hence allow an optimal solution to be identified.

12.8 The corners of the feasible region (which is always a polygon); optimal solutions are always at extreme points.

12.9 By moving the objective function line as far away from the origin as possible (for a maximum), or moving it as close to the origin as possible (for a minimum), and identifying the last point it passes through in the feasible region.

12.10 This looks at the changes in the optimal solution with small changes to the constraints and objective function.

12.11 Its marginal value, or the amount the objective function changes with one more – or less – unit of the resource.

12.12 Until so many resources become available that the constraint is no longer limiting (or resources are reduced until a new constraint becomes limiting).

12.13 Because the solution needs a lot of simple arithmetic on matrices.

12.14 The usual information includes a copy of the problem solved, details of the optimal solution, limiting constraints and unused resources, shadow prices and ranges over which these are valid, and variations in the objective function that will not change the position of the optimal solution.

12.15 Not usually.

Chapter 13 – Rates of change and calculus

13.1 To find the instantaneous rate of change (or gradient) of a function at any point.

13.2 This is the notation used to describe the derivative of y with respect to x. It is the formula for the gradient at any point.

13.3 With a continuous function, a minimum point has dy/dx equal to zero and d^2y/dx^2 greater than zero.

13.4 The variable names have no significance, so $dp/dc = dq/dc + dr/dc$.

13.5 The instantaneous gradient of a function at any point is dy/dx. This gradient itself changes with x, and d^2y/dx^2 describes the rate of change.

13.6 The additional cost of producing one more unit of a product.

13.7 The average revenue is the total revenue divided by the number produced; the marginal revenue is found by differentiating the total revenue function.

13.8 The ratio of change in demand for a product over change in price.

13.9 Demand rises with increasing price.

13.10 It is easiest to view integration as the reverse of differentiation.

13.11 The integral of y with respect to x is a function of x – which means that y is the instantaneous gradient of $f(x)$.

13.12 By knowing some other information that allows c to be calculated.

13.13 To find the sum of a function between two limits.

Chapter 14 – Uncertainty and probabilities

14.1 To a large extent yes.

14.2 A measure of its likelihood or its relative frequency.

14.3 No – you can know a lot about a situation without knowing everything with certainty.

14.4 Yes.

14.5 $10{,}000 \times 0.01 = 100$

14.6 Events where the probability of one occurring is not affected by whether or not the other occurs.

14.7 Events that cannot both occur.

14.8 By adding the separate probabilities of each event.

14.9 By multiplying the separate probabilities of each event.

14.10 You can only say that $P(\mathrm{A}) = 1 - P(\mathrm{B}) - P(\mathrm{C})$

14.11 Two (or more) events are dependent if they are not independent – meaning that $P(a) \neq P(a/b) \neq P(a/\cancel{b})$.

14.12 Probabilities of the form $P(a/b)$, giving the probability of event a occurring, given that event b has already occurred.

14.13 Bayes' theorem states that $P(a/b) = P(b/a) \times P(a) \,/\, P(b)$ and it is used for calculating conditional probabilities.

14.14 It shows a diagrammatic view of a problem and a way of organising calculations.

Chapter 15 – Probability distributions

15.1 To describe the probabilities or relative frequencies of events or classes of observations.

15.2 Yes.

15.3 Probability distributions found from observation of events that actually occurred.

15.4 $n!$

15.5 The order of selection is not important for a combination, but it is important for a permutation.

15.6 Permutations.

15.7 When there is a series of trials; each trial has two possible outcomes; the two outcomes are mutually exclusive; there is a constant probability of success, p, and failure, $q = 1 - p$; the outcomes of successive trials are independent.

15.8 $P(r)$ is the probability of r successes, n is the number of trials, p is the probability of success in each trial, q is the probability of failure in each trial, ${}^nC_r$ is the number of ways of combining r items from n.

15.9 mean = np; variance = npq

15.10 0.2753.

15.11 When independent events occur infrequently and at random, the probability of an event in an interval is proportional to the length of the interval, and an infinite number of events should be possible in an interval.

15.12 Mean = variance = np.

15.13 0.1438

15.14 When the number of events, n, in the binomial process is large and the probability of success is small, so np is less than 5.

15.15 In many – arguably most – situations where there is a large number of observations.

15.16 Binomial and Poisson distributions describe discrete data, while the Normal distribution describes continuous data.

15.17 The mean and standard deviation.

15.18 When the number of events, n, is large and the probability of success is relatively large (with np greater than 5).

15.19 About 68% of observations are within one standard deviation of the mean.

15.20 The Normal distribution describes continuous data, so a small continuity correction should be used for discrete data (perhaps replacing 'between 3 and 6 people' by 'between 2.5 and 6.5 people').

Chapter 16 – Using samples

16.1 To take a sample of observations that fairly represents the whole population.

16.2 A process where some property (quality, weight, length, etc.) in a representative sample is used to estimate the property in the population.

16.3 When a series of samples are taken from a population and a mean value of some variable is found for each sample, these means form the sampling distribution of the mean.

16.4 If the sample size is greater than about 30, or the population is Normally distributed, the sampling distribution of the mean is Normally distributed with mean μ and standard deviation $\sigma/\sqrt{n}$.

16.5 Because it comes from a sample which is unlikely to be perfectly representative of the population.

16.6 The range within which we are 95% confident the actual value lies.

16.7 Wider.

16.8 $25n$

16.9 With small samples.

16.10 When you want to be confident that a value is either above or below a certain point.

16.11 One-sided 95%, two-sided 95%, one-sided 99%.

16.12 Because the samples are not representative of the population, and tend to underestimate variability.

16.13 The number of independent pieces of data.

Chapter 17 – Testing hypotheses

17.1 To test whether a statement about a population is supported by the evidence in a sample.

17.2 The null hypothesis, H_0.

17.3 Type I error rejects a null hypothesis that is true; type II error does not reject a null hypothesis that is false.

17.4 The minimum acceptable probability that an observation is a random sample from the hypothesised population.

17.5 5% significance.

17.6 We cannot be this confident – but the evidence does support the null hypothesis and means it cannot be rejected.

17.7 When you want to make sure that a variable is above or below a specified value.

17.8 Because a small sample underestimates the variability in the population.

17.9 Very close to a Normal distribution.

17.10 When you want an unbiased estimator (but it has little effect with large samples).

17.11 There are many circumstances – whenever you want to check a hypothesis about a population.

17.12 No – the confidence interval is a range that we are confident a value is within; a significance level is the maximum acceptable probability of making a Type I error.

17.13 A parametric test makes assumptions about the distribution of variables, and works only with cardinal data.

17.14 When the conditions needed for a parametric test are not met.

17.15 No – there may be no appropriate test.

17.16 Because the distribution takes only positive values, so the acceptance range is from 0 to the critical value.

17.17 Nothing.

17.18 A test to see whether there is a relationship between two parameters, or whether they are independent.

17.19 It is easier and more reliable than doing the calculations by hand – or using a spreadsheet.

Chapter 18 – Making decisions

18.1 Because they give structure to the situation, define relationships – and clearly show alternatives, events and consequences.

18.2 A decision maker, a number of alternatives, a number of events, a set of measurable outcomes, and an objective of selecting the best alternative.

18.3 There is only one event, so we can list the outcomes and identify the alternative that gives the best.

18.4 Probably not – but remember that models are simplifications to help with decisions, and they do not replicate reality.

18.5 No – as decision makers interpret aims, alternatives, events ad outcomes differently, and not agree about the best decision.

18.6 One of several events may occur, but there is no way of telling which events are more likely.

18.7 The three criteria described are due to Laplace, Wald and Savage.

18.8 Only the Laplace criterion.

18.9 No. Many criteria could be devised to fit particular circumstances.

18.10 There are several possible events and we can give probabilities to each of them.

18.11 The sum of the probabilities multiplied by the values of the outcomes: expected value = $\sum(P \times V)$.

18.12 Yes – but the results may be unreliable.

18.13 When the conditional probabilities are available in situations of risk.

18.14 Expected values do not reflect real preferences, and a utility function describes a more realistic relationship.

18.15 The value of a terminal node is the total cost or gain of reaching that node; the value of a decision node is the best value of nodes reached by leaving alternative branches; the value of a random node is the expected value of the leaving branches.

18.16 By doing the analysis back to the left-hand, originating node: the value at this node is the overall expected value of following the best policy.

Chapter 19 – Quality management

19.1 No.

19.2 Because there are so many opinions, viewpoints and possible measures.

19.3 The function that is responsible for all aspects of quality.

19.4 Because it has implications for survival, reputation, marketing effort needed, market share, prices charged, profits, costs, liability for defects, and almost every other aspect of an organisation's operations.

19.5 No.

19.6 By minimising the total quality cost – and this usually means perfect quality.

19.7 Quality control inspects products to make sure they conform to designed quality; quality management is a wider function that is responsible for all aspects of quality.

19.8 A measure of the cost of missing target performance.

19.9 Unfortunately not – there are always random variations.

19.10 No.

19.11 Yes.

19.12 Everyone in the organisation.

19.13 Acceptance sampling checks that products are conforming to design quality; process control checks that the process is working properly.

19.14 Sampling by attribute classifies units as either acceptable or defective; sampling by variable measures some continuous value.

19.15 Because it gives perfect differentiation between good batches (where the probability of acceptance is 1) and bad batches (where the probability of acceptance is 0).

19.16 The process needs adjusting – but check for random fluctuations before doing this.

19.17 A single reading outside the control limits – or an unexpected pattern such as a clear trend, several consecutive readings near to a control limit, several consecutive readings on the same side of the mean, very erratic observations, etc.

Chapter 20 – Inventory management

20.1 To act as a buffer between supply and demand.

20.2 Demands which are distinct from each other, so that there is no kind of relationship between separate demands.

20.3 Unit cost, reorder cost, holding cost and shortage cost.

20.4 The fixed order quantity that minimises costs (when a number of assumptions are made).

20.5 (c) – either increase or decrease total costs, depending on the economic order quantity.

20.6 The stock level when it is time to place an order.

20.7 From the lead time demand.

20.8 Greater than demand.

20.9 Larger batches (all things being equal).

20.10 The probability that a demand can be satisfied (we used cycle service level, which is the probability that an item remains in stock during a cycle). It is used because alternative analyses need shortage costs which are very difficult to find.

20.11 It reduces the probability of shortages and increases service levels.

20.12 By increasing the amount of safety stock.

20.13 The difference between current stock and target stock level. Target stock level equals expected demand over $T + L$ plus safety stock.

20.14 A periodic review system (all things being equal).

20.15 To see where most (and least) effort should be allocated in inventory management.

20.16 B or C items, depending on circumstances.

Chapter 21 – Project networks

21.1 A coherent piece of work with a clear start and finish, consisting of the set of activities that make a distinct product.

21.2 The function responsible for the planning, scheduling and controlling of activities in a project and hence the management of resources.

21.3 No.

21.4 Nodes represent activities; arrows show the relationships between activities.

21.5 A list of all activities in the project and the immediate predecessors of each one. Durations, resources needed and other factors can be added, but these are not essential for drawing the network.

21.6 The two main rules are as follows:
- Before an activity can begin, all preceding activities must be finished.
- The arrows representing activities imply precedence only and neither their length nor their orientation is significant.

21.7 The earliest time an activity can start is the latest time by which all preceding activities can finish. The latest time an activity can finish is the earliest time that allows all following activities to be started on time.

21.8 The difference between the maximum amount of time available for an activity and the time it actually needs.

21.9 Zero.

21.10 It is the chain of activities that determines the project duration. If any critical activity is extended or delayed, the whole project is delayed.

21.11 The critical activities – usually starting with the longest.

21.12 By the amount of total float of activities on a parallel path. Reductions beyond this make the parallel path critical.

21.13 They give a clear picture of what stage each activity in a project should have reached at any time.

21.14 By delaying non-critical activities to times when less resources are needed.

21.15 CPM assumes a fixed activity duration, while PERT assumes that activity durations follow a known distribution.

21.16 It assumes that the duration of an activity follows a beta distribution, in which expected duration = $(O + 4M + P)/6$ and variance = $(P - O)^2/36$.

21.17 The project duration is Normally distributed with mean equal to the sum of the expected durations of activities on the critical path, and variance equal to the sum of the variances of activities on the critical path.

Chapter 22 – Queues and simulation

22.1 Customers want a service but find the server is busy, so they have to wait.

22.2 No. A balance is needed between the costs of providing a large number of servers and losing potential customers.

22.3 λ is the average arrival rate and μ is the average service rate.

22.4 Customers arrive faster than they are served and the queue continues to grow.

22.5 Assumptions include: a single server, random arrivals, random service time, first-come-first-served service discipline, the system has reached its steady state, there is no limit to the number of customers allowed in the queue, there is no limit on the number of customers who use the service, and all customers wait until they are served.

22.6 Ordinary quantitative analyses describe a problem at a fixed point of time, while simulation models follow the operation of a process over some extended time.

22.7 Yes.

22.8 To give typical (i.e. random) values to variables.

22.9 This depends on circumstances, but a usual guideline suggests several hundred.

APPENDIX B

Probabilities for the binomial distribution

						p					
n	*r*	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
1	0	.9500	.9000	.8500	.8000	.7500	.7000	.6500	.6000	.5500	.5000
	1	.0500	.1000	.1500	.2000	.2500	.3000	.3500	.4000	.4500	.5000
2	0	.9025	.8100	.7225	.6400	.5625	.4900	.4225	.3600	.3025	.2500
	1	.0950	.1800	.2550	.3200	.3750	.4200	.4550	.4800	.4950	.5000
	2	.0025	.0100	.0225	.0400	.0625	.0900	.1225	.1600	.2025	.2500
3	0	.8574	.7290	.6141	.5120	.4219	.3430	.2746	.2160	.1664	.1250
	1	.1354	.2430	.3251	.3840	.4219	.4410	.4436	.4320	.4084	.3750
	2	.0071	.0270	.0574	.0960	.1406	.1890	.2389	.2880	.3341	.3750
	3	.0001	.0010	.0034	.0080	.0156	.0270	.0429	.0640	.0911	.1250
4	0	.8145	.6561	.5220	.4096	.3164	.2401	.1785	.1296	.0915	.0625
	1	.1715	.2916	.3685	.4096	.4219	.4116	.3845	.3456	.2995	.2500
	2	.0135	.0486	.0975	.1536	.2109	.2646	.3105	.3456	.3675	.3750
	3	.0005	.0036	.0115	.0256	.0469	.0756	.1115	.1536	.2005	.2500
	4	.0000	.0001	.0005	.0016	.0039	.0081	.0150	.0256	.0410	.0625
5	0	.7738	.5905	.4437	.3277	.2373	.1681	.1160	.0778	.0503	.0312
	1	.2036	.3280	.3915	.4096	.3955	.3602	.3124	.2592	.2059	.1562
	2	.0214	.0729	.1382	.2048	.2637	.3087	.3364	.3456	.3369	.3125
	3	.0011	.0081	.0244	.0512	.0879	.1323	.1811	.2304	.2757	.3125
	4	.0000	.0004	.0022	.0064	.0146	.0284	.0488	.0768	.1128	.1562
	5	.0000	.0000	.0001	.0003	.0010	.0024	.0053	.0102	.0185	.0312
6	0	.7351	.5314	.3771	.2621	.1780	.1176	.0754	.0467	.0277	.0156
	1	.2321	.3543	.3993	.3932	.3560	.3025	.2437	.1866	.1359	.0938
	2	.0305	.0984	.1762	.2458	.2966	.3241	.3280	.3110	.2780	.2344
	3	.0021	.0146	.0415	.0819	.1318	.1852	.2355	.2765	.3032	.3125
	4	.0001	.0012	.0055	.0154	.0330	.0595	.0951	.1382	.1861	.2344
	5	.0000	.0001	.0004	.0015	.0044	.0102	.0205	.0369	.0609	.0938
	6	.0000	.0000	.0000	.0001	.0002	.0007	.0018	.0041	.0083	.0516

		p									
n	r	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
7	0	.6983	.4783	.3206	.2097	.1335	.0824	.0490	.0280	.0152	.0078
	1	.2573	.3720	.3960	.3670	.3115	.2471	.1848	.1306	.0872	.0547
	2	.0406	.1240	.2097	.2753	.3115	.3177	.2985	.2613	.2140	.1641
	3	.0036	.0230	.0617	.1147	.1730	.2269	.2679	.2903	.2918	.2734
	4	.0002	.0026	.0109	.0287	.0577	.0972	.1442	.1935	.2388	.2734
	5	.0009	.0002	.0012	.0043	.0115	.0250	.0466	.0774	.1172	.1641
	6	.0000	.0000	.0001	.0004	.0013	.0036	.0084	.0172	.0320	.0547
	7	.0000	.0000	.0000	.0000	.0001	.0002	.0006	.0016	.0037	.0078
8	0	.6634	.4305	.2725	.1678	.1001	.0576	.0319	.0168	.0084	.0039
	1	.2793	.3826	.3847	.3355	.2670	.1977	.1373	.0896	.0548	.0312
	2	.0515	.1488	.2376	.2936	.3115	.2965	.2587	.2090	.1569	.1094
	3	.0054	.0331	.0839	.1468	.2076	.2541	.2786	.2787	.2568	.2188
	4	.0004	.0046	.0185	.0459	.0865	.1361	.1875	.2322	.2627	.2734
	5	.0000	.0004	.0026	.0092	.0231	.0467	.0808	.1239	.1719	.2188
	6	.0000	.0000	.0002	.0011	.0038	.0100	.0217	.0413	.0703	.1094
	7	.0000	.0000	.0000	.0001	.0004	.0012	.0033	.0079	.0164	.0312
	8	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0007	.0017	.0039
9	0	.6302	.3874	.2316	.1342	.0751	.0404	.0207	.0101	.0046	.0020
	1	.2985	.3874	.3679	.3020	.2253	.1556	.1004	.0605	.0339	.0176
	2	.0629	.1722	.2597	.3020	.3003	.2668	.2162	.1612	.1110	.0703
	3	.0077	.0446	.1069	.1762	.2336	.2668	.2716	.2508	.2119	.1641
	4	.0006	.0074	.0283	.0661	.1168	.1715	.2194	.2508	.2600	.2461
	5	.0000	.0008	.0050	.0165	.0389	.0735	.1181	.1672	.2128	.2461
	6	.0000	.0001	.0006	.0028	.0087	.0210	.0424	.0743	.1160	.1641
	7	.0000	.0000	.0000	.0003	.0012	.0039	.0098	.0212	.0407	.0703
	8	.0000	.0000	.0000	.0000	.0001	.0004	.0013	.0035	.0083	.0716
	9	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0008	.0020
10	0	.5987	.3487	.1969	.1074	.0563	.0282	.0135	.0060	.0025	.0010
	1	.3151	.3874	.3474	.2684	.1877	.1211	.0725	.0403	.0207	.0098
	2	.0746	.1937	.2759	.3020	.2816	.2335	.1757	.1209	.0763	.0439
	3	.0105	.0574	.1298	.2013	.2503	.2668	.2522	.2150	.1665	.1172
	4	.0010	.0112	.0401	.0881	.1460	.2001	.2377	.2508	.2384	.2051
	5	.0001	.0015	.0085	.0264	.0584	.1029	.1563	.2007	.2340	.2461
	6	.0000	.0001	.0012	.0055	.0162	.0368	.0689	.1115	.1596	.2051
	7	.0000	.0000	.0001	.0008	.0031	.0090	.0212	.0425	.0746	.1172
	8	.0000	.0000	.0000	.0001	.0004	.0014	.0043	.0106	.0229	.0439
	9	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0016	.0042	.0098
	10	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0010

		p									
n	*r*	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
11	0	.5688	.3138	.1673	.0859	.0422	.0198	.0088	.0036	.0014	.0005
	1	.3293	.3835	.3248	.2362	.1549	.0932	.0518	.0266	.0125	.0054
	2	.0867	.2131	.2866	.2953	.2581	.1998	.1395	.0887	.0513	.0269
	3	.0137	.0710	.1517	.2215	.2581	.2568	.2254	.1774	.1259	.0806
	4	.0014	.0158	.0536	.1107	.1721	.2201	.2428	.2365	.2060	.1611
	5	.0001	.0025	.0132	.0388	.0803	.1321	.1830	.2207	.2360	.2256
	6	.0000	.0003	.0023	.0097	.0268	.0566	.0985	.1471	.1931	.2256
	7	.0000	.0000	.0003	.0017	.0064	.0173	.0379	.0701	.1128	.1611
	8	.0000	.0000	.0000	.0002	.0011	.0037	.0102	.0234	.0462	.0806
	9	.0000	.0000	.0000	.0000	.0001	.0005	.0018	.0052	.0126	.0269
	10	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0007	.0021	.0054
	11	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0005
12	0	.5404	.2824	.1422	.0687	.0317	.0138	.0057	.0022	.0008	.0002
	1	.3413	.3766	.3012	.2062	.1267	.0712	.0368	.0174	.0075	.0029
	2	.0988	.2301	.2924	.2835	.2323	.1678	.1088	.0639	.0339	.0161
	3	.0173	.0852	.1720	.2362	.2581	.2397	.1954	.1419	.0923	.0537
	4	.0021	.0213	.0683	.1329	.1936	.2311	.2367	.2128	.1700	.1208
	5	.0002	.0038	.0193	.0532	.1032	.1585	.2039	.2270	.2225	.1934
	6	.0000	.0005	.0040	.0155	.0401	.0792	.1281	.1766	.2124	.2256
	7	.0000	.0000	.0006	.0033	.0115	.0291	.0591	.1009	.1489	.1934
	8	.0000	.0000	.0001	.0005	.0024	.0078	.0199	.0420	.0762	.1208
	9	.0000	.0000	.0000	.0001	.0004	.0015	.0048	.0125	.0277	.0537
	10	.0000	.0000	.0000	.0000	.0000	.0002	.0008	.0025	.0068	.0161
	11	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0010	.0029
	12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002
13	0	.5133	.2542	.1209	.0550	.0238	.0097	.0037	.0013	.0004	.0001
	1	.3512	.3672	.2774	.1787	.1029	.0540	.0259	.0113	.0045	.0016
	2	.1109	.2448	.2937	.2680	.2059	.1388	.0836	.0453	.0220	.0095
	3	.0214	.0997	.1900	.2457	.2517	.2181	.1651	.1107	.0660	.0349
	4	.0028	.0277	.0838	.1535	.2097	.2337	.2222	.1845	.1350	.0873
	5	.0003	.0055	.0266	.0691	.1258	.1803	.2154	.2214	.1989	.1571
	6	.0000	.0008	.0063	.0230	.0559	.1030	.1546	.1968	.2169	.2095
	7	.0000	.0001	.0011	.0058	.0186	.0442	.0833	.1312	.1775	.2095
	8	.0000	.0000	.0001	.0011	.0047	.0142	.0336	.0656	.1089	.1571
	9	.0000	.0000	.0000	.0001	.0009	.0034	.0101	.0243	.0495	.0873
	10	.0000	.0000	.0000	.0000	.0001	.0006	.0022	.0065	.0162	.0349
	11	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0012	.0036	.0095
	12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0016
	13	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

		p									
n	r	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
14	0	.4877	.2288	.1028	.0440	.0178	.0068	.0024	.0008	.0002	.0001
	1	.3593	.3559	.2539	.1539	.0832	.0407	.0181	.0073	.0027	.0009
	2	.1229	.2570	.2912	.2501	.1802	.1134	.0634	.0317	.0141	.0056
	3	.0259	.1142	.2056	.2501	.2402	.1943	.1366	.0845	.0462	.0222
	4	.0037	.0348	.0998	.1720	.2202	.2290	.2022	.1549	.1040	.0611
	5	.0004	.0078	.0352	.0860	.1468	.1963	.2178	.2066	.1701	.1222
	6	.0000	.0013	.0093	.0322	.0734	.1262	.1759	.2066	.2088	.1833
	7	.0000	.0002	.0019	.0092	.0280	.0618	.1082	.1574	.1952	.2095
	8	.0000	.0000	.0003	.0020	.0082	.0232	.0510	.0918	.1398	.1833
	9	.0000	.0000	.0000	.0003	.0018	.0066	.0183	.0408	.0762	.1222
	10	.0000	.0000	.0000	.0000	.0003	.0014	.0049	.0136	.0312	.0611
	11	.0000	.0000	.0000	.0000	.0000	.0002	.0010	.0033	.0093	.0222
	12	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0019	.0056
	13	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0009
	14	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001
15	0	.4633	.2059	.0874	.0352	.0134	.0047	.0016	.0005	.0001	.0000
	1	.3658	.3432	.2312	.1319	.0668	.0305	.0126	.0047	.0016	.0005
	2	.1348	.2669	.2856	.2309	.1559	.0916	.0476	.0219	.0090	.0032
	3	.0307	.1285	.2184	.2501	.2252	.1700	.1110	.0634	.0318	.0139
	4	.0049	.0428	.1156	.1876	.2252	.2186	.1792	.1268	.0780	.0417
	5	.0006	.0105	.0449	.1032	.1651	.2061	.2123	.1859	.1404	.0916
	6	.0000	.0019	.0132	.0430	.0917	.1472	.1906	.2066	.1914	.1527
	7	.0000	.0003	.0030	.0138	.0393	.0811	.1319	.1771	.2013	.1964
	8	.0000	.0000	.0005	.0035	.0131	.0348	.0710	.1181	.1647	.1964
	9	.0000	.0000	.0001	.0007	.0034	.0116	.0298	.0612	.1048	.1527
	10	.0000	.0000	.0000	.0001	.0007	.0030	.0096	.0245	.0515	.0916
	11	.0000	.0000	.0000	.0000	.0001	.0006	.0024	.0074	.0191	.0417
	12	.0000	.0000	.0000	.0000	.0000	.0001	.0004	.0016	.0052	.0139
	13	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0010	.0032
	14	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0005
	15	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

		p									
n	*r*	.05	.10	.15	.20	.25	.30	.35	.40	.45	.50
16	0	.4401	.1853	.0743	.0281	.0100	.0033	.0010	.0003	.0001	.0000
	1	.3706	.3294	.2097	.1126	.0535	.0228	.0087	.0030	.0009	.0002
	2	.1463	.2745	.2775	.2111	.1336	.0732	.0353	.0150	.0056	.0018
	3	.0359	.1423	.2285	.2463	.2079	.1465	.0888	.0468	.0215	.0085
	4	.0061	.0514	.1311	.2001	.2252	.2040	.1553	.1014	.0572	.0278
	5	.0008	.0137	.0555	.1201	.1802	.2099	.2008	.1623	.1123	.0667
	6	.0001	.0028	.0180	.0550	.1101	.1649	.1982	.1983	.1684	.1222
	7	.0000	.0004	.0045	.0197	.0524	.1010	.1524	.1889	.1969	.1746
	8	.0000	.0001	.0009	.0055	.0197	.0487	.0923	.1417	.1812	.1964
	9	.0000	.0000	.0001	.0012	.0058	.0185	.0442	.0840	.1318	.1746
	10	.0000	.0000	.0000	.0002	.0014	.0056	.0167	.0392	.0755	.1222
	11	.0000	.0000	.0000	.0000	.0002	.0013	.0049	.0142	.0337	.0667
	12	.0000	.0000	.0000	.0000	.0000	.0002	.0011	.0040	.0115	.0278
	13	.0000	.0000	.0000	.0000	.0000	.0000	.0002	.0008	.0029	.0085
	14	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0018
	15	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002
	16	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000
17	0	.4181	.1668	.0631	.0225	.0075	.0023	.0007	.0002	.0000	.0000
	1	.3741	.3150	.1893	.0957	.0426	.0169	.0060	.0019	.0005	.0001
	2	.1575	.2800	.2673	.1914	.1136	.0581	.0260	.0102	.0035	.0010
	3	.0415	.1556	.2359	.2393	.1893	.1245	.0701	.0341	.0144	.0052
	4	.0076	.0605	.1457	.2093	.2209	.1868	.1320	.0796	.0411	.0182
	5	.0010	.0175	.0668	.1361	.1914	.2081	.1849	.1379	.0875	.0472
	6	.0001	.0039	.0236	.0680	.1276	.1784	.1991	.1839	.1432	.0944
	7	.0000	.0007	.0065	.0267	.0668	.1201	.1685	.1927	.1841	.1484
	8	.0000	.0001	.0014	.0084	.0279	.0644	.1134	.1606	.1883	.1855
	9	.0000	.0000	.0003	.0021	.0093	.0276	.0611	.1070	.1540	.1855
	10	.0000	.0000	.0000	.0004	.0025	.0095	.0263	.0571	.1008	.1484
	11	.0000	.0000	.0000	.0001	.0005	.0026	.0090	.0242	.0525	.0944
	12	.0000	.0000	.0000	.0000	.0001	.0006	.0024	.0081	.0215	.0472
	13	.0000	.0000	.0000	.0000	.0000	.0001	.0005	.0021	.0068	.0182
	14	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0004	.0016	.0052
	15	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0003	.0010
	16	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001
	17	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Probabilities for the Poisson distribution

r	μ = .005	.01	.02	.03	.04	.05	.06	.07	.08	.09
0	.9950	.9900	.9802	.9704	.9608	.9512	.9418	.9324	.9231	.9139
1	.0050	.0099	.0192	.0291	.0384	.0476	.0565	.0653	.0738	.0823
2	.0000	.0000	.0002	.0004	.0008	.0012	.0017	.0023	.0030	.0037
3	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001

r	μ = 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	.9048	.8187	.7408	.6703	.6065	.5488	.4966	.4493	.4066	.3679
1	.0905	.1637	.2222	.2681	.3033	.3293	.3476	.3595	.3659	.3679
2	.0045	.0164	.0333	.0536	.0758	.0988	.1217	.1438	.1647	.1839
3	.0002	.0011	.0033	.0072	.0126	.0198	.0284	.0383	.0494	.0613
4	.0000	.0001	.0002	.0007	.0016	.0030	.0050	.0077	.0111	.0153
5	.0000	.0000	.0000	.0001	.0002	.0004	.0007	.0012	.0020	.0031
6	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0002	.0003	.0005
7	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

r	μ = 1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0	.3329	.3012	.2725	.2466	.2231	.2019	.1827	.1653	.1496	.1353
1	.3662	.3614	.3543	.3452	.3347	.3230	.3106	.2975	.2842	.2707
2	.2014	.2169	.2303	.2417	.2510	.2584	.2640	.2678	.2700	.2707
3	.0738	.0867	.0998	.1128	.1255	.1378	.1496	.1607	.1710	.1804
4	.0203	.0260	.0324	.0395	.0471	.0551	.0636	.0723	.0812	.0902
5	.0045	.0062	.0084	.0111	.0141	.0176	.0216	.0260	.0309	.0361
6	.0008	.0012	.0018	.0026	.0035	.0047	.0061	.0078	.0098	.0120
7	.0001	.0002	.0003	.0005	.0008	.0011	.0015	.0020	.0027	.0034
8	.0000	.0000	.0001	.0001	.0001	.0002	.0003	.0005	.0006	.0009
9	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002

	μ									
r	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0	.1225	.1108	.1003	.0907	.0821	.0743	.0672	.0608	.0550	.0498
1	.2527	.2438	.2306	.2177	.2052	.1931	.1815	.1703	.1596	.1494
2	.2700	.2681	.2652	.2613	.2565	.2510	.2450	.2384	.2314	.2240
3	.1890	.1966	.2033	.2090	.2138	.2176	.2205	.2225	.2237	.2240
4	.0992	.1082	.1196	.1254	.1336	.1414	.1488	.1557	.1662	.1680
5	.0417	.0476	.0538	.0602	.0668	.0735	.0804	.0872	.0940	.1008
6	.0146	.0174	.0206	.0241	.0278	.0319	.0362	.0407	.0455	.0504
7	.0044	.0055	.0068	.0083	.0099	.0118	.0139	.0163	.0188	.0216
8	.0011	.0015	.0019	.0025	.0031	.0038	.0047	.0057	.0068	.0081
9	.0003	.0004	.0005	.0007	.0009	.0011	.0014	.0018	.0022	.0027
10	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0005	.0006	.0008
11	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0002	.0002
12	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

	μ									
r	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0
0	.0450	.0408	.0369	.0334	.0302	.0273	.0247	.0224	.0202	.0183
1	.1397	.1304	.1217	.1135	.1057	.0984	.0915	.0850	.0789	.0733
2	.2165	.2087	.2008	.1929	.1850	.1771	.1692	.1615	.1539	.1465
3	.2237	.2226	.2209	.2186	.2158	.2125	.2087	.2046	.2001	.1954
4	.1734	.1781	.1823	.1858	.1888	.1912	.1931	.1944	.1951	.1954
5	.1075	.1140	.1203	.1264	.1322	.1377	.1429	.1477	.1522	.1563
6	.0555	.0608	.0662	.0716	.0771	.0826	.0881	.0936	.0989	.1042
7	.0246	.0278	.0312	.0348	.0385	.0425	.0466	.0508	.0551	.0595
8	.0095	.0111	.0129	.0148	.0169	.0191	.0215	.0241	.0269	.0298
9	.0033	.0040	.0047	.0056	.0066	.0076	.0089	.0102	.0116	.0132
10	.0010	.0013	.0016	.0019	.0023	.0028	.0033	.0039	.0045	.0053
11	.0003	.0004	.0005	.0006	.0007	.0009	.0011	.0013	.0016	.0019
12	.0001	.0001	.0001	.0002	.0002	.0003	.0003	.0004	.0005	.0006
13	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0002	.0002
14	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001

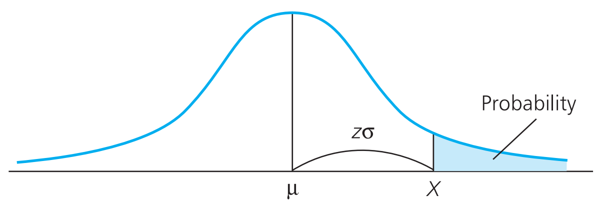
	μ									
r	4.1	4.2	4.3	4.4	4.5	4.6	4.7	4.8	4.9	5.0
0	.0166	.0150	.0136	.0123	.0111	.0101	.0091	.0082	.0074	.0067
1	.0679	.0630	.0583	.0540	.0500	.0462	.0427	.0395	.0365	.0337
2	.1393	.1323	.1254	.1188	.1125	.1063	.1005	.0948	.0894	.0842
3	.1904	.1852	.1798	.1743	.1687	.1631	.1574	.1517	.1460	.1404
4	.1951	.1944	.1933	.1917	.1898	.1875	.1849	.1820	.1789	.1755
5	.1600	.1633	.1662	.1687	.1708	.1725	.1738	.1747	.1753	.1755
6	.1093	.1143	.1191	.1237	.1281	.1323	.1362	.1398	.1432	.1462
7	.0640	.0686	.0732	.0778	.0824	.0869	.0914	.0959	.1002	.1044
8	.0328	.0360	.0393	.0428	.0463	.0500	.0537	.0575	.0614	.0653
9	.0150	.0168	.0188	.0209	.0232	.0255	.0280	.0307	.0334	.0363
10	.0061	.0071	.0081	.0092	.0104	.0118	.0132	.0147	.0164	.0181
11	.0023	.0027	.0032	.0037	.0043	.0049	.0056	.0064	.0073	.0082
12	.0008	.0009	.0011	.0014	.0016	.0019	.0022	.0026	.0030	.0034
13	.0002	.0003	.0004	.0005	.0006	.0007	.0008	.0009	.0011	.0013
14	.0001	.0001	.0001	.0001	.0002	.0002	.0003	.0004	.0004	.0005
15	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0001	.0002

	μ									
r	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
0	.0061	.0055	.0050	.0045	.0041	.0037	.0033	.0030	.0027	.0025
1	.0311	.0287	.0265	.0244	.0225	.0207	.0191	.0176	.0162	.0149
2	.0793	.0746	.0701	.0659	.0618	.0580	.0544	.0509	.0477	.0446
3	.1348	.1293	.1239	.1185	.1133	.1082	.1033	.0985	.0938	.0892
4	.1719	.1681	.1641	.1600	.1558	.1515	.1472	.1428	.1383	.1339
5	.1753	.1748	.1740	.1728	.1714	.1697	.1678	.1656	.1632	.1606
6	.1490	.1515	.1537	.1555	.1571	.1584	.1594	.1601	.1605	.1606
7	.1086	.1125	.1163	.1200	.1234	.1267	.1298	.1326	.1353	.1377
8	.0692	.0731	.0771	.0810	.0849	.0887	.0925	.0962	.0998	.1033
9	.0392	.0423	.0454	.0486	.0519	.0552	.0586	.0620	.0654	.0688
10	.0200	.0220	.0241	.0262	.0285	.0309	.0334	.0359	.0386	.0413
11	.0093	.0104	.0116	.0129	.0143	.0157	.0173	.0190	.0207	.0225
12	.0039	.0045	.0051	.0058	.0065	.0073	.0082	.0092	.0102	.0113
13	.0015	.0018	.0021	.0024	.0028	.0032	.0036	.0041	.0046	.0052
14	.0006	.0007	.0008	.0009	.0011	.0013	.0015	.0017	.0019	.0022
15	.0002	.0002	.0003	.0003	.0004	.0005	.0006	.0007	.0008	.0009
16	.0001	.0001	.0001	.0001	.0001	.0002	.0002	.0002	.0003	.0003
17	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001	.0001	.0001

r	μ 6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8	6.9	7.0
0	.0022	.0020	.0018	.0017	.0015	.0014	.0012	.0011	.0010	.0009
1	.0137	.0126	.0116	.0106	.0098	.0090	.0082	.0076	.0070	.0064
2	.0417	.0390	.0364	.0340	.0318	.0296	.0276	.0258	.0240	.0223
3	.0848	.0806	.0765	.0726	.0688	.0652	.0617	.0584	.0552	.0521
4	.1294	.1249	.1205	.1162	.1118	.1076	.1034	.0992	.0952	.0912
5	.1579	.1549	.1519	.1487	.1454	.1420	.1385	.1349	.1314	.1277
6	.1605	.1601	.1595	.1586	.1575	.1562	.1546	.1529	.1511	.1490
7	.1399	.1418	.1435	.1450	.1462	.1472	.1480	.1486	.1489	.1490
8	.1066	.1099	.1130	.1160	.1188	.1215	.1240	.1263	.1284	.1304
9	.0723	.0757	.0791	.0825	.0858	.0891	.0923	.0954	.0985	.1014
10	.0441	.0469	.0498	.0528	.0558	.0588	.0618	.0649	.0679	.0710
11	.0245	.0265	.0285	.0307	.0330	.0353	.0377	.0401	.0426	.0452
12	.0124	.0137	.0150	.0164	.0179	.0194	.0210	.0227	.0245	.0264
13	.0058	.0065	.0073	.0081	.0089	.0098	.0108	.0119	.0130	.0142
14	.0025	.0029	.0033	.0037	.0041	.0046	.0052	.0058	.0064	.0071
15	.0010	.0012	.0014	.0016	.0018	.0020	.0023	.0026	.0029	.0033
16	.0004	.0005	.0005	.0006	.0007	.0008	.0010	.0011	.0013	.0014
17	.0001	.0002	.0002	.0002	.0003	.0003	.0004	.0004	.0005	.0006
18	.0000	.0001	.0001	.0001	.0001	.0001	.0001	.0002	.0002	.0002
19	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0001	.0001	.0001

APPENDIX D

Probabilities for the Normal distribution



Example $Z = \frac{X - \mu}{\sigma}$

$P[Z > 2] = 0.0228$
$P[Z > 2] = 0.1587$

Normal deviate *z*	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
2.4	.0082	.0080	.0078	.0075	.0073	.0072	.0069	.0068	.0066	.0064
2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010

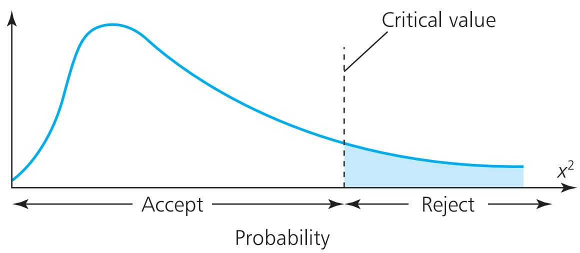
APPENDIX E

Probabilities for the t-distribution

Degrees of freedom	0.25	0.20	0.15	0.10	0.05	0.025	0.01	0.005
1	1.000	1.376	1.963	3.078	6.314	12.706	31.821	63.657
2	.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925
3	.765	.978	1.250	1.638	2.353	3.182	4.541	5.841
4	.741	.941	1.190	1.533	2.132	2.776	3.747	4.604
5	.727	.920	1.156	1.476	2.015	2.571	3.365	4.032
6	.718	.906	1.134	1.440	1.943	2.447	3.143	3.707
7	.711	.896	1.119	1.415	1.895	2.365	2.998	3.499
8	.706	.889	1.108	1.397	1.860	2.306	2.896	3.355
9	.703	.883	1.100	1.383	1.833	2.262	2.821	3.250
10	.700	.879	1.093	1.372	1.812	2.228	2.764	3.169
11	.697	.876	1.088	1.363	1.796	2.201	2.718	3.106
12	.695	.873	1.083	1.356	1.782	2.179	2.681	3.055
13	.694	.870	1.079	1.350	1.771	2.160	2.650	3.012
14	.692	.868	1.076	1.345	1.761	2.145	2.624	2.977
15	.691	.866	1.074	1.341	1.753	2.131	2.602	2.947
16	.690	.865	1.071	1.337	1.746	2.120	2.583	2.921
17	.689	.863	1.069	1.333	1.740	2.110	2.567	2.898
18	.688	.862	1.067	1.330	1.734	2.101	2.552	2.878
19	.688	.861	1.066	1.328	1.729	2.093	2.539	2.861
20	.687	.860	1.064	1.325	1.725	2.086	2.528	2.845
21	.686	.859	1.063	1.323	1.721	2.080	2.518	2.831
22	.686	.858	1.061	1.321	1.717	2.074	2.508	2.819
23	.685	.858	1.060	1.319	1.714	2.069	2.500	2.807
24	.685	.857	1.059	1.318	1.711	2.064	2.492	2.797
25	.684	.856	1.058	1.316	1.708	2.060	2.485	2.787
26	.684	.856	1.058	1.315	1.706	2.056	2.479	2.779
27	.684	.855	1.057	1.314	1.703	2.052	2.473	2.771
28	.683	.855	1.056	1.313	1.701	2.048	2.467	2.763
29	.683	.854	1.055	1.311	1.699	2.045	2.462	2.756
30	.683	.854	1.055	1.310	1.697	2.042	2.457	2.750
40	.681	.851	1.050	1.303	1.684	2.021	2.423	2.704
60	.679	.848	1.046	1.296	1.671	2.000	2.390	2.660
120	.677	.845	1.041	1.289	1.658	1.980	2.358	2.617
∞	.674	.842	1.036	1.282	1.645	1.960	2.326	2.576

APPENDIX F

Critical values for the χ^2 distribution



Degrees of freedom	0.250	0.100	0.050	0.025	0.010	0.005	0.001
1	1.32	2.71	3.84	5.02	6.63	7.88	10.8
2	2.77	4.61	5.99	7.38	9.21	10.6	13.8
3	4.11	6.25	7.81	9.35	11.3	12.8	16.3
4	5.39	7.78	9.49	11.1	13.3	14.9	18.5
5	6.63	9.24	11.1	12.8	15.1	16.7	20.5
6	7.84	10.6	12.6	14.4	16.8	18.5	22.5
7	9.04	12.0	14.1	16.0	18.5	20.3	24.3
8	10.2	13.4	15.5	17.5	20.3	22.0	26.1
9	11.4	14.7	16.9	19.0	21.7	23.6	27.9
10	12.5	16.0	18.3	20.5	23.2	25.2	29.6
11	13.7	17.3	19.7	21.9	24.7	26.8	31.3
12	14.8	18.5	21.0	23.3	26.2	28.3	32.9
13	16.0	19.8	22.4	24.7	27.7	29.8	34.5
14	17.1	21.1	23.7	26.1	29.1	31.3	36.1
15	18.2	22.3	25.0	27.5	30.6	32.8	37.7
16	19.4	23.5	26.3	28.8	32.0	34.3	39.3
17	20.5	24.8	27.6	30.2	33.4	35.7	40.8
18	21.6	26.0	28.9	31.5	34.8	37.2	42.3
19	22.7	27.2	30.1	32.9	36.2	38.6	43.8
20	23.8	28.4	31.4	34.2	37.6	40.0	45.3
21	24.9	29.6	32.7	35.5	38.9	41.4	46.8
22	26.0	30.8	33.9	36.8	40.3	42.8	48.3
23	27.1	32.0	35.2	38.1	41.6	44.2	49.7
24	28.2	33.2	36.4	39.4	43.0	45.6	51.2

Degrees of freedom	0.250	0.100	0.050	0.025	0.010	0.005	0.001
25	29.3	34.4	37.7	40.6	44.3	46.9	52.6
26	30.4	35.6	38.9	41.9	45.6	48.3	54.1
27	31.5	36.7	40.1	43.2	47.0	49.6	55.5
28	32.6	37.9	41.3	44.5	48.3	51.0	56.9
29	33.7	39.1	42.6	45.7	49.6	52.3	58.3
30	34.8	40.3	43.8	47.0	50.9	53.7	59.7
40	45.6	51.8	55.8	59.3	63.7	66.8	73.4
50	56.3	63.2	67.5	71.4	76.2	79.5	86.7
60	67.0	74.4	79.1	83.3	88.4	92.0	99.6
70	77.6	85.5	90.5	95.0	100	104	112
80	88.1	96.6	102	107	112	116	125
90	98.6	108	113	118	123	128	137
100	109	118	124	130	136	140	149

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