Linear Algebra: The Geometry of Linear Equations

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My notes for the Linear Algebra: The Geometry of Linear Equations, by Professor Gilbert Strang. All scribing errors are mine.

1. Introduction

The fundemental problem of linear algebra is: given n linear equations, solve for n unknowns.

There are three ways of looking at the problem:

1. Matrix form
2. Row picture (1 equation at a time)
3. ⋆ **Column picture (the most important method of this course, also the best way)**

2. 2-Dimensions

Suppose the equations are:

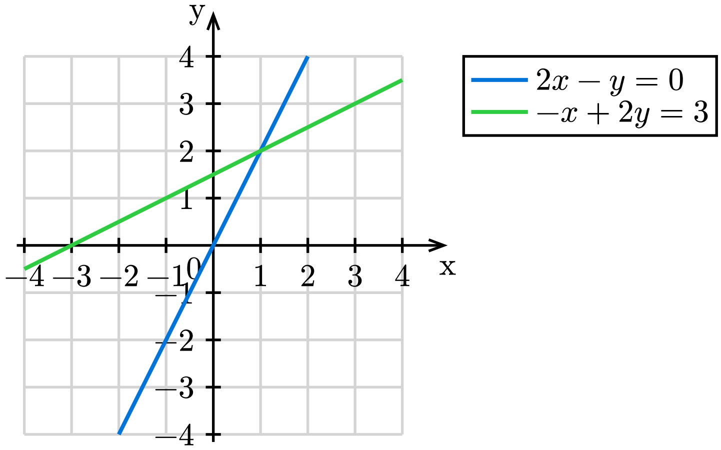
$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

2.1. Matrix form

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{\mathbf{b}}$$

2.2. Row picture

One line for each equation, can be graphed out in a 2d graph. Both equations will meet at a point $x = 1, y = 2$, which is the solution.



2.3. Column picture

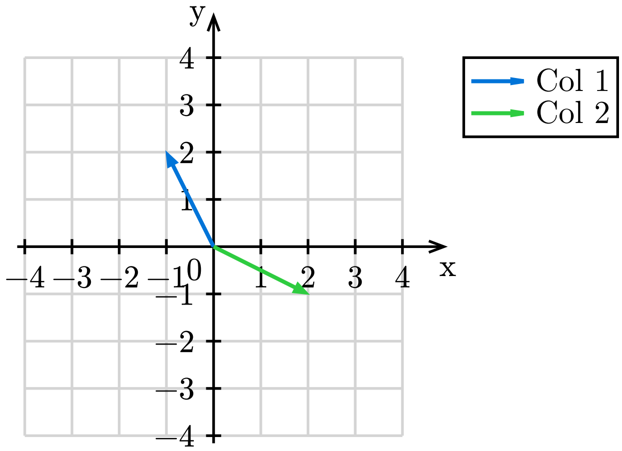
The algebra form of the column picture is:

$$x\begin{bmatrix}2\\-1\end{bmatrix} + y\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix} \tag{1}$$

where $\begin{bmatrix}2\\-1\end{bmatrix}$ (Col 1) and $\begin{bmatrix}-1\\2\end{bmatrix}$ (Col 2) are columns of the matrix. Hence, Equation 1 is asking us to find the right *linear combinations* of the columns to get $\begin{bmatrix}0\\3\end{bmatrix}$.

⋆ **Linear combinations is the entire meat of this course.**

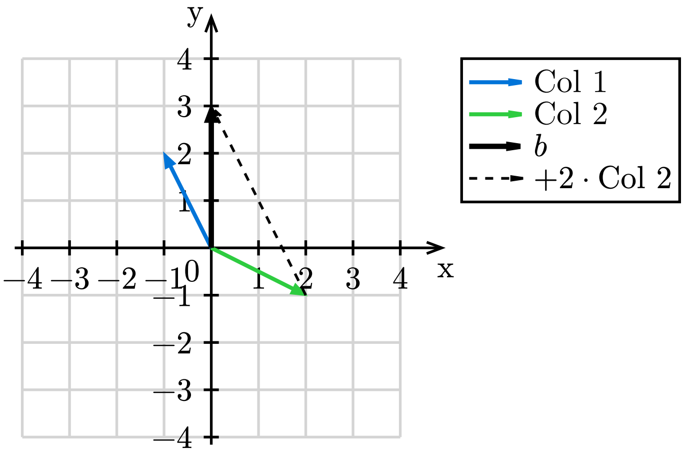
The geometry form of the column picture is:



Since we already know the solution previously, we know the right combination must be

$$1\begin{bmatrix}2\\-1\end{bmatrix} + 2\begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\3\end{bmatrix}$$

The new picture is:



I get $\vec{b}$ after adding $2 \cdot$ Col 2 to $1 \cdot$ Col 1.

Question: What are all the combinations? If I take all x, all y, what is the result?

Answer: I can get any vector as the result. It would fill the whole plane $\mathbb{R}^2$ (a vector space).

3. 3-Dimensions

Suppose the equations are:

$$\begin{aligned} 2x - y \quad &= 0 \\ -x + 2y - z &= -1 \\ -3y + 4z &= 4 \end{aligned}$$

How do we understand these equations? Again, we can use the three methods:

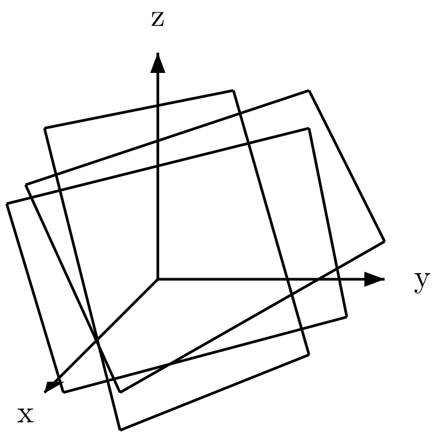
3.1. Matrix form

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -3 & 4 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$$

3.2. Row picture

This is getting a bit more complicated, so the drawing is not entirely accurate, but we can imagine that they all meet at a point somehow:



- We can get 3 points from equation 2: $(1,0,0)$, $(0,0,1)$, $\left(0,-\frac{1}{2},0\right)$.
- 3 points form a plane, so each equation is actually a plane.
- Two planes meet in a line.
- Three planes meet in a point.

We don't know what is the point that they will meet, but the main idea is that as the 3 planes are not parallel, they have a solution.

⋆ **The problem with the row picture: it is getting harder to visualize the problem as the amount of dimensions increase.** What about 4D, 5D, etc? Quite impossible.

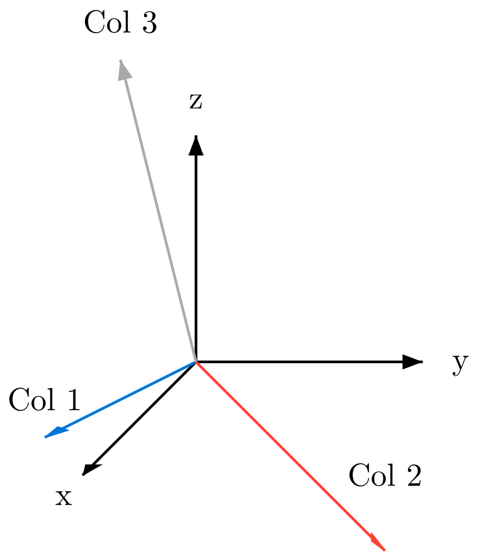
3.3. Column picture

The algebra form of the column picture is:

$$x\begin{bmatrix}2\\-1\\0\end{bmatrix}+y\begin{bmatrix}-1\\2\\-3\end{bmatrix}+z\begin{bmatrix}0\\-1\\4\end{bmatrix}=\begin{bmatrix}0\\-1\\4\end{bmatrix} \tag{2}$$

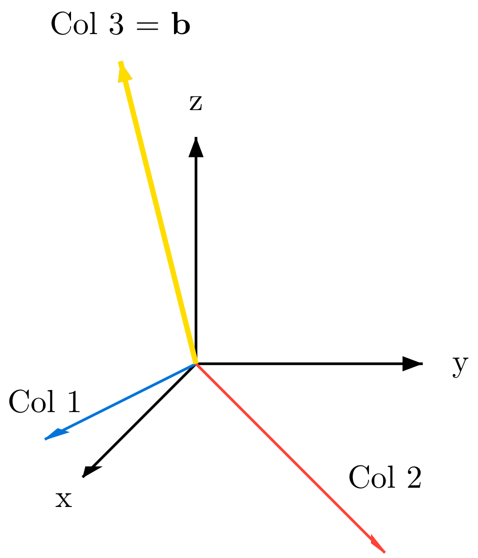
So it is the linear combination of 3 vectors, and the vectors are 3 dimensions.

The geometry form of the column picture is:



Again, what the set of equation is asking, is to find the right combination to produce $\mathbf{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}$.

This problem is specially chosen, so that the solution is deliberately $x = 0, y = 0, z = 1$.



Of course, not all problems will have such an obvious solution (next lecture will cover elimination, which will help us systematically solve the equations).

But let's come back to the big picture: what if right-hand side is different?

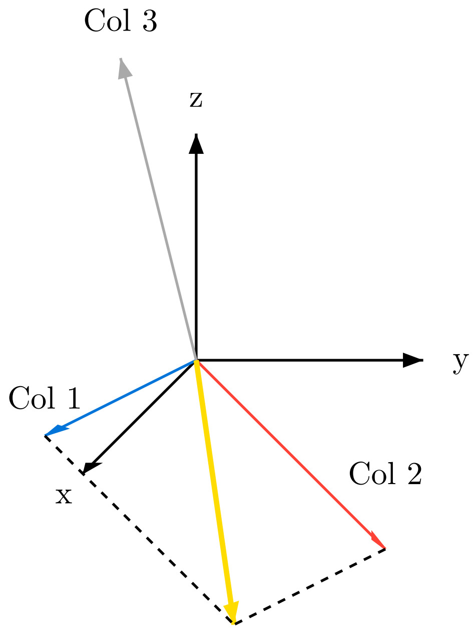
Change it, such that we add Col 1 to Col 2 to get our new $\mathbf{b}$:

$$x \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} + z \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \tag{3}$$

Solution is $x = 1, y = 1, z = 0$.

What happens to the row picture and col picture?

- In the row picture: 3 new planes, brand new picture. :(
- $\star$ **In the col picture: still the same arrows:**



$\mathbf{b}$ = Col 1 + Col 2

Now think: can we solve this for **all** right hand side? Is there a solution for all $\mathbf{b}$?

Formally:

1. Can I solve $A\mathbf{x} = \mathbf{b}$ for every $\mathbf{b}$?
2. Do the linear combinations of the columns fill up the entire 3D space?

It appears as if both questions are different from a different POV, but both are actually asking the same thing - I get a combination of the columns.

For this matrix A, the answer is YES. (Our example is a non-singular, invertible matrix).

When would the answer be NO? When will I not be able to produce $\mathbf{b}$?

$\star$ **If all 3 columns lie on the same plane, then all combinations will still lie on the same plane.**

Another example: Let Col 3 be the sum of Col 1 + Col 2. Col 3 is in the same plane as Col 1 + Col 2. All $\mathbf{b}$ in the same plane as Col 1 + Col 2 are solvable, but $\mathbf{b}$ outside the plane cannot be reached. $\rightarrow$ matrix is **not** invertible.

4. n-Dimensions

Pretend that we have 9 equations and 9 unknowns.

4.1. Matrix form

$$\underbrace{\begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix}}_{9\times 9} \underbrace{\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}}_{9\times 1} = \underbrace{\begin{bmatrix} \cdots \\ \cdots \end{bmatrix}}_{9\times 1}$$

4.2. Row picture

We cannot draw this. 9D plane in a 9D space? We give up at this stage.

4.3. Col picture

It is just 9 columns, so 9 vectors in 9D space, and we would still be finding their linear combination to hit $\mathbf{b}$.

We can also still ask: can we get all RHS $\mathbf{b}$?

If columns chosen are not independent: Col 9 is the same as Col 8, then Col 9 contributes nothing new, possible to have a $\mathbf{b}$ that I cannot get.

Think about 9 vectors in 9D, take combination - this is the central thought in linear algebra. Even though 9D space is still hard to visualize, we can still easily visualize the vectors.

⋆ In Col picture, everything is still just arrows, making visualization less of a problem.

5. Matrix Form of the Equation

Up till now, we have been using a very core concept that we haven't really explained.

How to multiple a matrix by a vector? For example:

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2 ways:

1. Rows way / dot product (which is what we usually learn first when doing matrix multiplication):

$$\begin{bmatrix} 2 * 1 + 5 * 2 \\ 1 * 1 + 3 * 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

2. ⋆ **Columns way** (what we have been doing the entire lecture):

$$1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

The columns way is basically saying that $A\mathbf{x}$ is a combination of the columns of A.

In linear algebra, columns way of thinking is more useful.