Introductory Calculus

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My notes for the Introductory Calculus: Oxford Mathematics 1st Year Student Lecture, by Professor Dan Ciubotaru. All scribing errors are mine.

1. ODEs

ODEs = ordinary differential equations.

Definition 1.1 An ODE is a kind of equation that involves the independent variable x, a function of x (call it y, a dependent variable), and the derivatives of y w.r.t. x.

Example 1.2 Common ODE forms include:

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}^2 y}{\mathrm{d}x^2}, \ldots$$

The highest order of degree is the order of the differential equation. (Hence, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is of order 2).

The simplest kind of ODE is of the form: $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$, which can be solved by direct integration: $y = \int f(x)\,\mathrm{d}x$. ($y$ is the anti-derivative of $f(x)$).

1.1. Applications of ODEs

Example 1.3 From physical sciences, mechanics, the Newton's second law says that:

$$\vec{F} = m\vec{a} \tag{1}$$

where

- $\vec{a}$ = acceleration (a derivative of velocity wr.t. time).
- $\vec{a} = \frac{\mathrm{d}\vec{v}}{\mathrm{d}t}$ (where $\vec{v}$ = velocity)
- $\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t}$ (where $\vec{r}$ = displacement)
- $\therefore \vec{a} = \frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2}$ (which is a second order equation)

Example 1.4 From engineering, electrical circuits, we have a simple series circuit (RLC).

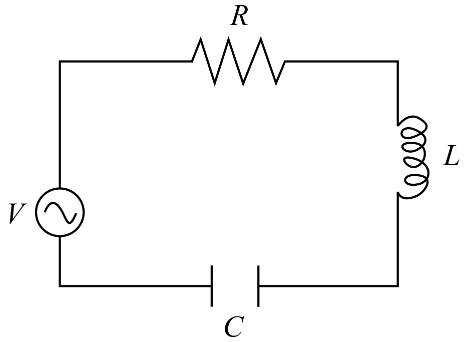

Figure 1: An example circuit

where from the symbols in Figure 1:

- capacitor C has capacitance
- resistor R has resistance
- inductor L has inductance

R, L, C are constants, independent of time.

We are interested in the current of the circuit $I(t)$ (dependent variable), which is the current across the circuit, w.r.t. function of time t (independent variable).

Let $Q(t)$ = charge on the capacitor.

The relationship is:

$$I = \frac{\mathrm{d}Q}{\mathrm{d}t} \tag{2}$$

The Kirchhoff's law states: total voltage is 0 around the circuit.

$$V(t) = V_R + V_L + V_C \tag{3}$$

where

- $V_R = R \cdot I(t)$ (ohm's law)
- $V_L = L \cdot \frac{\mathrm{d}I}{\mathrm{d}t}$ (faraday's law)
- $V_C = \frac{1}{C} \cdot Q(t)$

We can rewrite the equation of $V(t)$ using $Q(t)$ to make it become a differential equation:

$$L \cdot \frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + R \cdot \frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{1}{C} \cdot Q = V \tag{4}$$

This is a second order differential equation (the highest derivative is $\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2}$). It has constant coefficients (L, R and C), and is inhomogeneous (V does not have to be 0).

Exercise 1.5 The rate at which a radioactive substance decays is proportional to the remaining number of atoms. What is the ODE for the rate?

2. Review of simplest integration

Simplest is where direct integration is possible.

2.1. Integration by parts

It comes from the product rule for derivative (Leibniz rule).

Suppose two functions f and g.

$$
\begin{aligned}
(f \cdot g)' &= f' \cdot g + f \cdot g' \\
\Rightarrow f \cdot g' &= (f \cdot g)' - f' \cdot g \quad \text{(integrate both sides)} \\
\therefore \int f \cdot g' \, \mathrm{d}x &= f \cdot g - \int f' \cdot g \, \mathrm{d}x
\end{aligned}
\tag{5}
$$

or the definite integral version:

$$
\int_a^b f \cdot g' \, \mathrm{d}x = \Big[f \cdot g \Big]_a^b - \int_a^b f' \cdot g \, \mathrm{d}x \tag{6}
$$

Example 2.1

$$
I = \int \underbrace{x^2}_{f} \underbrace{\sin x}_{g'} \, \mathrm{d}x \tag{7}
$$

Eq. 7 gives the solution to $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x^2 \sin x$.

For Eq. 7, $g = -\cos x$.

Hence, using the integration by parts:

$$
\begin{aligned}
I &= x^2(-\cos x) - \int 2x(-\cos x) \, \mathrm{d}x \\
&= -x^2 \cos x + 2 \int \underbrace{x}_{f_2} \underbrace{\cos x}_{g_2'} \, \mathrm{d}x \\
&= -x^2 \cos x + 2(x \sin x) - 2 \int \sin x \, \mathrm{d}x \\
&= -x^2 \cos x + 2x \sin x + 2 \cos x + C
\end{aligned}
\tag{8}
$$

Example 2.2

$$
I = \int \underbrace{(2x-1)}_{g'} \underbrace{\ln(x^2+1)}_{f} \, \mathrm{d}x \tag{9}
$$

We want to get rid of ln, hence the selection of f and g', and $g = x^2 - x$.

$$
\begin{aligned}
I &= (x^2 - x) \ln(x^2+1) - \int (x^2 - x) \cdot \frac{2x}{x^2+1} \, \mathrm{d}x \\
&= (x^2 - x) \ln(x^2+1) - 2 \underbrace{\int \frac{x^3 - x^2}{x^2+1} \, \mathrm{d}x}_{J}
\end{aligned}
\tag{10}
$$

Do long division for the fraction in J.

$$
\begin{array}{rrrr|l}
x^3 & -x^2 & & & x-1 \\ \cline{5-5}
-x^3 & & -x & & x^2+1 \\ \cline{1-4}
& -x^2 & -x & & \\
& x^2 & & 1 & \\ \cline{2-4}
& & -x & 1 &
\end{array}
$$

$\Rightarrow \dfrac{x^3 - x^2}{x^2+1} = (x-1) + \dfrac{-x+1}{x^2+1}$

$$
\begin{aligned}
J &= \int \left[(x-1) + \frac{-x+1}{x^2+1} \right] \mathrm{d}x \\
&= \frac{1}{2}x^2 - x - \underbrace{\int \frac{x}{x^2+1}\,\mathrm{d}x}_{L} + \underbrace{\int \frac{\mathrm{d}x}{x^2+1}}_{\arctan x (= \tan^{-1} x)}
\end{aligned} \tag{11}
$$

Solve L by substitution rule: $u = x^2+1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$.

$$
\begin{aligned}
L &= \int \frac{x}{x^2+1}\,\mathrm{d}x \\
&= \frac{1}{2}\int \frac{\mathrm{d}u}{u} \\
&= \frac{1}{2}\ln|u| \\
&= \frac{1}{2}\ln(x^2+1)
\end{aligned} \tag{12}
$$

$$
\therefore J = \frac{1}{2}x^2 - x - \frac{1}{2}\ln(x^2+1) + \tan^{-1} x + C \tag{13}
$$

$$
\therefore I = (x^2 - x)\ln(x^2+1) - x^2 + 2x + \ln(x^2+1) - 2\tan^{-1} x + C \quad \square \tag{14}
$$

There are some cases where both f & g will not get simplified, but you do it twice, it will come back to *what you started with* (see the next example).

Example 2.3

$$
I = \int \underbrace{e^x}_{f} \underbrace{\sin x}_{g'}\,\mathrm{d}x \tag{15}
$$

If you integrate $\sin x \to -\cos x$, $-\cos x \to -\sin x$, then can equal each other to find actual answer:

$$
\begin{aligned}
I &= \ldots \\
&= \frac{1}{2}e^x(\sin x - \cos x) + C
\end{aligned} \tag{16}
$$

More difficult problems cannot solve in one go, need to find recursive formula.

Example 2.4 Reduction/recursive formula

(We will label it as I_n, because we can get I_{n-1}, I_{n-2}, etc)

$$
\begin{aligned}
I_n &= \int \cos^n x \,\mathrm{d}x \\
&= \int \underbrace{\cos^{n-1} x}_{f} \underbrace{\cos x}_{g'} \,\mathrm{d}x \\
&= \cos^{n-1} x \sin x - \int (n-1)\cos^{n-2} x(-\sin x)\sin x \,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \underbrace{\sin^2 x}_{=1-\cos^2 x} \,\mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1)\underbrace{\int \cos^{n-2} x \,\mathrm{d}x}_{I_{n-2}} - (n-1)\underbrace{\int \cos^n x \,\mathrm{d}x}_{I_n} \qquad (17)
\end{aligned}
$$

Solve for I_n:

$$nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2} \tag{18}$$

Recursive formula:

$$I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}I_{n-2} \tag{19}$$

where $n \geq 2$.

We also need to know I_0 & I_1, as it drops down by 2 each iteration, so these are base cases.

$$I_0 = \int \mathrm{d}x = x(+C) \tag{20}$$

$$I_1 = \int \cos x \,\mathrm{d}x = \sin x(+C) \tag{21}$$

With that, we can get any integral we want. For example:

$$
\begin{aligned}
I_6 &= \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}I_4 \\
&= \frac{1}{6}\cos^5 x \sin x + \frac{5}{6}\left(\frac{1}{4}\cos^3 x \sin x + \frac{3}{4}I_2\right) \qquad (22)
\end{aligned}
$$

$$I_2 = \frac{1}{2}\cos x \sin x + \frac{1}{2}\underbrace{x}_{I_0} \tag{23}$$

$$\therefore I_6 = \frac{1}{6}\cos^5 x \sin x + \frac{5}{6\cdot 4}\cos^3 x \sin x + \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cos x \sin x + \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}x + C \tag{24}$$

2.2. Separable ODEs

The next simplest ODEs, where:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a(x) \cdot b(y) \tag{25}$$

where $a(x)$ is **function of x only**, and $b(y)$ is **function of y only**.

Assume that $b(y) \neq 0$:

$$\begin{aligned} \frac{1}{b(y)} \frac{\mathrm{d}y}{\mathrm{d}x} &= a(x) \\ \Rightarrow \int \frac{1}{b(y)} \,\mathrm{d}y &= \int a(x) \,\mathrm{d}x \end{aligned} \tag{26}$$

The final form is two direct integrations that we can solve separately.

Example 2.5 Find the genearl solution to the separable differential equation ($0 < x < 1$ to avoid continuity issues): $x(y^2 - 1) - y(x^2 - 1)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

$$\begin{aligned} x(y^2 - 1) - y(x^2 - 1) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ y(x^2 - 1) \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= -x(y^2 - 1) \\ \frac{y}{y^2 - 1} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{x^2 - 1} \\ \underbrace{\int \frac{y}{y^2 - 1} \,\mathrm{d}y}_{\text{use absolute to reuse } D} &= \underbrace{\int \frac{x}{1 - x^2} \,\mathrm{d}x}_{\text{derivative of a log } = D} \end{aligned} \tag{27}$$

$$\boxed{D : [\ln(1 - x^2)]' = \frac{-2x}{1 - x^2}}$$

$$\begin{gathered} \dots \\ \frac{1}{2} \ln|y^2 - 1| = -\frac{1}{2} \ln(1 - x^2) + C \end{gathered} \tag{28}$$

Get rid of log by using the property of log ($\log a + \log b = \log a \cdot b$)

$$\begin{aligned} &\dots \\ \frac{1}{2} \ln|(y^2 - 1)|(1 - x^2) &= C \\ |(y^2 - 1)|(1 - x^2) &= e^{2C} \Rightarrow \mathcal{C}(> 0) \end{aligned} \tag{29}$$

Therefore, the answer is $|y^2 - 1| \underbrace{(1 - x^2)}_{>0} = \mathcal{C}$ where $\mathcal{C} > 0$.

We can drop the absolute to drop the assumption of $\mathcal{C} > 0$ and allow negatives, to get a better looking answer.

$$(1-y^2)(1-x^2)=\mathcal{C} \tag{30}$$

...except $\mathcal{C}$ still cannot be 0, because even after allowing negatives, there's still no scenario where $|e^x|=0$, so it still doesn't look nice. We lost that case in ($\leftsquigarrow$), which requires $y^2-1\neq 0$, but we need to allow that because y could be ± 1 (which is legal).

$y=\pm 1$ is included in the solution if we allow $\mathcal{C}=0$ in the answer. Hence $\mathcal{C}$ can now be any constants, and Eq. 30 is the final answer.