Stochastic Differential Equation

Mid-term Presentation

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Introduction

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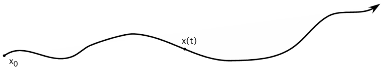

Figure: ODE Trajectory

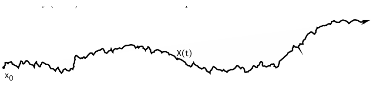

Figure: SDE Sample Path

Quick Overview

- Understanding Randomness
- Brownian Motion
- Ito Calculus
- Stochastic Differential Equations (SDEs)
- Numerical Approaches
- Future Directions

Understanding Randomness

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- Brownian Motion $(W(t))$: A continuous-time stochastic process

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- $W(0) = 0$ almost surely
- For $0 \leq s < t$, $W(t) - W(s) \sim N(0, t-s)$ (normally distributed increments)
- The process has independent increments: for any $0 \leq t_1 < t_2 < \cdots < t_n$, the increments $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ are independent

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- **Continuity:** Continuous sample paths
- **Non-differentiability:** Nowhere differentiable paths

Explanation of SDE and Why Ito Calculus is Needed

- **SDE Form:**

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t)$$

- $b(X(t))$: Drift term, describing deterministic evolution.
- $\sigma(X(t))$: Diffusion term, accounting for randomness.
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Why Ito Calculus is Needed

- Brownian motion is not differentiable, and traditional calculus doesn't apply.
- Ito calculus allows integration with respect to stochastic processes.

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- Evaluated as the limit of sums over random increments.

Properties of the Ito Integral

- **Linearity:**

$$\int_0^t (af(s) + bg(s))dW(s) = a\int_0^t f(s)dW(s) + b\int_0^t g(s)dW(s) \quad (7)$$

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- **Isometry:**

$$E\left[\left(\int_0^t f(s)dW(s)\right)^2\right] = \int_0^t f(s)^2 ds \quad (9)$$

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Using $(dX)^2 = \sigma^2(X(t), t)dt$, the full form of **Ito's Lemma** becomes:

$$du = \left(\frac{\partial u}{\partial t} + b(X(t), t)\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X(t), t)\frac{\partial^2 u}{\partial X^2} \right) dt + \sigma(X(t), t)\frac{\partial u}{\partial X} dW(t)$$

Stochastic Differential Equations (SDEs)

- SDEs describe systems with both deterministic trends and random noise.
- The general form of an SDE is:

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- $b(X(t), t)$: Drift term (deterministic part)
- $\sigma(X(t), t)$: Diffusion term (random part)
- $dW(t)$: Wiener process (Brownian motion)

Existence and Uniqueness Conditions

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Linear Growth Condition:

$$|b(x,t)| + |\sigma(x,t)| \leq C(1+|x|) \tag{14}$$

Numerical Methods: Euler-Maruyama and Milstein Method

Euler-Maruyama Method:

$$X_{n+1} = X_n + b(X_n, t_n)\Delta t + \sigma(X_n, t_n)\Delta W_n \tag{15}$$

where $\Delta W_n = W(t_{n+1}) - W(t_n)$ and $\Delta t = t_{n+1} - t_n$.

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Analytical Solution:

- Exact solutions are often available for linear SDEs (e.g., $dX(t) = \lambda X(t)dt + \mu X(t)dW(t)$).
- Example solution:

$$X(t) = X_0 \exp\left((\lambda - \frac{\mu^2}{2})t + \mu W(t)\right)$$

- The analytical solution provides a reference to measure the accuracy of numerical schemes.

Comparison and Future Directions

Comparison of Methods:

- **Euler-Maruyama:** Simple to implement, widely used for approximating SDEs. Suitable for many practical problems with acceptable precision.
- **Milstein:** More accurate due to the inclusion of additional terms accounting for the nonlinearity in the diffusion function, useful when higher precision is required.

Future Directions:

- Explore advanced stochastic control techniques, focusing on optimal control problems in random systems.
- Investigate the relationship between dynamic programming and the maximum principle for optimal control strategies in SDEs.

Thank You!