

Question. Give an example of an open set in \mathbb{R}^2 (with the product metric) that is not of the form $U \times V$, where $U, V \subset \mathbb{R}$ are open.

Example. Let

$$P = (\mathbb{R}, |\cdot|), \quad Q = (\mathbb{R}, |\cdot|)$$

. Consider the product space

$$P \times Q = (\mathbb{R}^2, d),$$

where d is the max (product) metric defined by

$$d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}.$$

Let

$$A = \overline{B}_{\|\cdot\|_2}((0, 0), 1) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\},$$

the closed unit ball in \mathbb{R}^2 with respect to the Euclidean norm.

Claim 1. A^c is open in (\mathbb{R}^2, d) .

Proof. Let $(a, b) \in A^c$. Since A^c is open in $(\mathbb{R}^2, \|\cdot\|_2)$, there exists $r > 0$ such that

$$B_{\|\cdot\|_2}((a, b), r) \subseteq A^c.$$

If $(x, y) \in B_{\|\cdot\|_2}((a, b), r)$, then

$$(x, y) \in A^c, \quad \text{so} \quad \sqrt{x^2 + y^2} > 1,$$

and

$$\sqrt{(x - a)^2 + (y - b)^2} < r. \tag{*}$$

Hence,

$$(x - a)^2 + (y - b)^2 < r^2.$$

Therefore, if we take

$$(x - a)^2 < \frac{r^2}{2} \quad \text{and} \quad (y - b)^2 < \frac{r^2}{2},$$

then (*) is satisfied and

$$(x, y) \in B_{\|\cdot\|_2}((a, b), r) \subseteq A^c.$$

This forces,

$$|x - a| < \frac{r}{\sqrt{2}} \quad \text{and} \quad |y - b| < \frac{r}{\sqrt{2}}.$$

Let $r' = \frac{r}{\sqrt{2}}$. Then

$$\max\{|x - a|, |y - b|\} < r',$$

and hence

$$d((a, b), (x, y)) < r'.$$

Therefore,

$$B_d((a, b), r') \subseteq A^c.$$

Thus, A^c is open in (\mathbb{R}^2, d) . □

Claim 2. A^c cannot be written in the form $U \times V$, where $U \subset \mathbb{R}$ and $V \subset \mathbb{R}$ are open.

Proof. Assume, for contradiction, that

$$A^c = U \times V$$

for some open sets $U, V \subset \mathbb{R}$.

Consider the horizontal line $\{(x, 2) : x \in \mathbb{R}\}$. Since $x^2 + 2^2 > 1$ for all $x \in \mathbb{R}$, this entire line lies in A^c . Thus, for every $x \in \mathbb{R}$,

$$(x, 2) \in U \times V,$$

which implies $\mathbb{R} \subset U$.

Similarly, consider the vertical line $\{(2, y) : y \in \mathbb{R}\}$. Again, $2^2 + y^2 > 1$ for all $y \in \mathbb{R}$, so this entire line lies in A^c , implying

$$\mathbb{R} \subset V.$$

Hence,

$$U = \mathbb{R}, \quad V = \mathbb{R},$$

and therefore

$$U \times V = \mathbb{R}^2,$$

which contradicts $A^c \neq \mathbb{R}^2$.

Thus A^c cannot be written as a product of open sets. □