Solutions to
Understanding Analysis by Stephen Abbott

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Exercises

Exercise 1.2.1. (a) Prove that $\sqrt{3}$ is irrational. Does a similar argument work to show $\sqrt{6}$ is irrational?
(b) Where does the proof of Theorem 1.1.1 break down if we try to use it to prove $\sqrt{4}$ is irrational?
Solution. (a) Consider to the contrary that $\sqrt{3}$ is rational. Therefore, we can express $\sqrt{3}$ as some $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and p and q are co-prime.
Thus, $\sqrt{3} = \frac{p}{q}$.

$$\Rightarrow 3 = \left(\frac{p}{q}\right)^2 \Rightarrow p^2 = 3q^2.$$

Hence, p^2 is divisible by 3.

$$\Rightarrow p \text{ must also be divisible by 3.}$$

(The prime factorisation of p^2 must contain 3, and since 3 is a prime, 3 appears two times in the factorisation of p^2.)
Let $p = 3m$ for some $m \in \mathbb{Z}$. Then,

$$\Rightarrow 9m^2 = 3q^2 \Rightarrow q^2 = 3m^2 \Rightarrow q \text{ is also divisible by 3.}$$

But we assumed that p and q have no common factors.
Contradiction.
Yes. Assume $\sqrt{6}$ is rational. Then, we can express $\sqrt{6}$ as some $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and p and q are co-prime.
Thus, $\sqrt{6} = \frac{p}{q}$.

$$\Rightarrow 6 = \left(\frac{p}{q}\right)^2 \Rightarrow p^2 = 6q^2.$$

Hence, p^2 is divisible by 6. Thus, p^2 is divisible by 3. Thus, p is divisible by 3. Let $p = 3m$ for some $m \in \mathbb{Z}$. Then,

$$\Rightarrow 9m^2 = 6q^2 \Rightarrow 3m^2 = 2q^2 \Rightarrow q^2 = \frac{3m^2}{2}$$

Now, since q^2 is an integer, thus, q^2 is divisible by 2 and 3, and q is divisible by 2 and 3. But we assumed that p and q have no common factors.

Contradiction.
(b) Let us assume $\sqrt{4}$ is rational. Therefore, we can express $\sqrt{4}$ as some $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and p and q are co-prime.
Thus, $\sqrt{4} = \frac{p}{q}$.

$$\Rightarrow 4 = \left(\frac{p}{q}\right)^2 \Rightarrow p^2 = 4q^2.$$

Hence, p^2 is divisible by 4, thus, p^2 is even, thus, p is even. (Square of an odd number is odd.)
Now, Let $p = 2m$ for some $m \in \mathbb{Z}$. Then,

$$\Rightarrow 4m^2 = 4q^2 \Rightarrow q^2 = m^2 \Rightarrow q = \pm m$$

Thus,

$$\sqrt{4} = \frac{2m}{\pm m} = \pm 2$$

There is no contradiction here. We just showed that we can express $\sqrt{4}$ as $\frac{2}{1}$.
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Exercise 1.2.2. Show that there is no rational number r satisfying $2^r = 3$.
Solution. Since r is a rational number. Let r $= \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and p and q are co-prime.
Thus,

$$2^{\frac{p}{q}} = 3 \Rightarrow 2^p = 3^q$$

Case 1: p ¿ 0 and q ¡ 0
Then, $2^p \geq 2$ and $3^q \leq 1$

Case 2: p ¡ 0 and q ¿ 0
Similar to Case 1.

Case 3: p and q ¿ 0
$2^p \mid 3^q$ implies 2 divides 3 (since 2 and 3 both are primes) which is not the case. (If $a \mid b$ and $b \mid a$, where $a, b \in \mathbb{Z}$, then $a = b$ or $a = -b$.

Case 4: p and q ¡ 0 is case 3.

Hence, there is no rational number r satisfying $2^r = 3$.
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Exercise 1.2.3. Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \dots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

Solution. False. Consider $A_n = \{n, n+1, n+2, \dots\}$ where $n \in \mathbb{N}$. Then,

$$\bigcap_{i=1}^{\infty} A_i = \varnothing.$$

If the intersection were not empty, then there would exist a natural number $m \in \mathbb{N}$ such that

$$m \in \bigcap_{i=1}^{\infty} A_i.$$

But this implies $m \in A_{m+1}$, which is false because $A_{m+1} = \{m+1, m+2, m+3, \ldots\}$, and hence $m \notin A_{m+1}$.

Contradiction.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

Solution.

Proof. Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \supseteq \cdots$ be finite, nonempty sets. Assume, for the sake of contradiction, that

$$\bigcap_{n=1}^{\infty} A_n = \varnothing.$$

This implies that there exists some element a_n such that

$$a_n \in A_n \quad \text{but} \quad a_n \notin A_{n+1}, \qquad n \geq 1.$$

Since we have $\mathbb{N}$ (the set of natural numbers) indexing the sets, the sequence $\{a_n\}$ has $\aleph_0$ (countably infinitely) many elements. Moreover, since A_1 is the superset containing all of the subsequent sets, this means that

$$\{a_1, a_2, a_3, \ldots\} \subseteq A_1.$$

Thus, A_1 has infinitely many elements, which contradicts the fact that A_1 is finite. Therefore, our assumption was false, and

$$\bigcap_{n=1}^{\infty} A_n$$

must be non-empty.

It must be finite since A_1 is the superset containing all of the subsequent sets, and since $|A_1|$ (the cardinality of A_1) is finite. Moreover, since

$$\left|\bigcap_{n=1}^{\infty} A_n\right| \leq |A_n| \quad \text{for any } n,$$

it follows that the intersection is finite.

□

(c) $(A \cup B) \cup (A \cap B) = A \cup B$

Solution. True.

$$\begin{aligned}
& x \in A \cap B \\
\Rightarrow\ & x \in A \text{ and } x \in B \\
\Rightarrow\ & x \in A \text{ or } x \in B \\
\Rightarrow\ & x \in A \cup B \\
\Rightarrow\ & A \cap B \subseteq A \cup B \\
\Rightarrow\ & (A \cup B) \cup (A \cap B) = A \cup B
\end{aligned}$$

(d) $A \cap (B \cup C) = (A \cap B) \cup C$

Solution. True.

$$\begin{aligned} & (x \in A) \text{ and } (x \in B \cup C) \\ \Rightarrow\quad & (x \in A) \text{ and } (x \in B \text{ or } x \in C) \\ \Rightarrow\quad & (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ \Rightarrow\quad & x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution. Let $x \in A \cap (B \cup C)$.

$$\begin{aligned} \Rightarrow\quad & x \in A \text{ and } (x \in B \cup C) \\ \Rightarrow\quad & x \in A \text{ and } (x \in B \text{ or } x \in C) \\ \Rightarrow\quad & (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ \Rightarrow\quad & x \in A \cap B \text{ or } x \in A \cap C \\ \Rightarrow\quad & x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

So, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Now, let $x \in (A \cap B) \cup (A \cap C)$.

$$\begin{aligned} \Rightarrow\quad & x \in A \cap B \text{ or } x \in A \cap C \\ \Rightarrow\quad & (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ \Rightarrow\quad & x \in A \text{ and } (x \in B \text{ or } x \in C) \\ \Rightarrow\quad & x \in A \cap (B \cup C) \end{aligned}$$

So, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

Hence,

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

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Exercise 1.2.4. Produce an infinite collection of sets $A_1, A_2, A_3, \ldots$, with the property that every A_n has an infinite number of elements, and for all $i \neq j$, $A_i \cap A_j = \emptyset$.

Solution.

Let $A_1 = \{2k : k \in \mathbb{N}\}$.

Let $A_2 = \{3k : k \in \mathbb{N}\} \setminus A_1$.

Let $A_3 = \{5k : k \in \mathbb{N}\} \setminus (A_1 \cup A_2)$.

Proceeding in this manner, define

$$A_i = \{k \cdot p_i : k \in \mathbb{N}\} \setminus \bigcup_{j=1}^{i-1} A_j$$

where p_i denotes the i^{th} smallest prime number.

Each A_i is infinite because it contains all positive integral powers of a distinct prime p_i, i.e.,

$$p_i,\ p_i^2,\ p_i^3,\ \ldots$$

which are all unique to A_i.
Also, all the A_i are pairwise disjoint by construction.

We claim that $\bigcup_{i=1}^{\infty} A_i = \mathbb{N}$.
Given any $n \in \mathbb{N}$, let its prime factorisation be

$$n = a_1^{a'_1} \cdot a_2^{a'_2} \cdot \dots \cdot a_k^{a'_k},$$

where each $a_j \in \mathbb{N}$ is a prime number, and $a'_j \in \mathbb{N}$ are their respective powers.
Now, let $a = \min\{a_1, a_2, \dots, a_k\}$, i.e., the smallest prime dividing n.
Then, $a = p_i$ for some $i \in \mathbb{N}$, where p_i is the i^{th} prime.
Since n is divisible by p_i, and A_i contains all multiples of p_i not already included in earlier sets, we must have $n \in A_i$.
Hence, every natural number n belongs to some A_i, and so:

$$\bigcup_{i=1}^{\infty} A_i = \mathbb{N}.$$

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Exercise 1.2.5 (De Morgan's Laws). Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

Solution. If $x \in (A \cap B)^c$, then

$$\begin{gathered} x \notin A \cap B \\ \implies x \notin A \text{ or } x \notin B \text{ (or both)} \\ \implies x \in A^c \text{ or } x \in B^c \text{ (or both)} \\ \implies x \in A^c \cup B^c. \end{gathered}$$

Therefore,

$$(A \cap B)^c \subseteq A^c \cup B^c.$$

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

Solution.
The reverse implication can be proved by following the above implications in reverse order.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution. Let $x \in A^c \cap B^c$. Then

$$\begin{gathered} x \in A^c \text{ and } x \in B^c \\ \implies x \notin A \text{ and } x \notin B \\ \implies x \notin A \cup B \\ \implies x \in (A \cup B)^c. \end{gathered}$$

So,

$$A^c \cap B^c \subseteq (A \cup B)^c.$$

Reversing the implications gives

$$(A \cup B)^c \subseteq A^c \cap B^c,$$

and thus,

$$(A \cup B)^c = A^c \cap B^c.$$

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Exercise 1.2.6.

(a) Verify the triangle inequality in the special case where the two numbers are the same sign.

Solution. If a and b have the same sign, then

$$a + b \text{ has the same sign as } a \text{ and } b.$$

Case 1: a and b are both positive.

$$\begin{gathered} a > 0 \text{ and } b > 0 \\ \implies |a| = a \text{ and } |b| = b \\ \implies a + b = |a| + |b| \\ \implies |a + b| = a + b = |a| + |b| \\ \implies |a + b| \leq |a| + |b|. \end{gathered}$$

Case 2: a and b are both negative.

$$\begin{gathered} a < 0 \text{ and } b < 0 \\ \implies a + b < 0 \\ \implies |a + b| = -(a + b) \\ \implies |a| = -a \text{ and } |b| = -b \\ \implies |a| + |b| = -a - b = -(a + b) \\ \implies |a + b| = |a| + |b| \\ \implies |a + b| \leq |a| + |b|. \end{gathered}$$

In both cases, the triangle inequality holds.

(b) Find an efficient proof for all the cases at once by first demonstrating that $|a+b|^2 \leq (|a| + |b|)^2$.

Solution.

$$\begin{gathered} a^2 = |a|^2 \quad \text{and} \quad b^2 = |b|^2 \\ (a + b)^2 = a^2 + b^2 + 2ab \\ (|a| + |b|)^2 = |a|^2 + |b|^2 + 2|a||b| \\ \implies (a + b)^2 = a^2 + b^2 + 2ab \\ \implies (|a| + |b|)^2 = a^2 + b^2 + 2|ab| \\ ab \leq |ab| \\ \implies (a + b)^2 \leq (|a| + |b|)^2. \end{gathered}$$

Since $|a| + |b| \geq 0$,

$$\sqrt{(|a| + |b|)^2} = |a| + |b|.$$

Also,

$$\sqrt{(a+b)^2} = |a+b|.$$

Therefore,

$$|a+b| \leq |a| + |b|.$$

(c) Prove $|a-b| = |b-a| = |c-d| = |d-c|$ for all a, b, c, d.

Solution.

$$|a-b| \leq |a-c| + |c-d| + |d-b|$$

for all a, b, c, d.

Solution.

$$\begin{aligned} a-b &= (a-c)+(c-d)+(d-b) \\ \implies |a-b| &= |(a-c)+(c-d)+(d-b)| \\ \implies |a-b| &\leq |a-c| + |c-d| + |d-b|. \end{aligned}$$

(d) Prove $|a| + |b| \geq |a-b|$. (The unremarkable identity $a - b = a + (-b)$ may be useful.)

Solution.

$$\begin{aligned} \big|\, |a| - |b| \,\big| &= \Big|\, \big|(a-b)+b\big| - |b| \,\Big| \\ &\leq \big|\, |a-b| + |b| - |b| \,\big| \\ &= |a-b|. \end{aligned}$$

—

Exercise 1.2.7. Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A, that is,

$$f(A) = \{f(x) : x \in A\}$$

(a) Let $f(x) = x^2$. If $A = [0,2]$ and $B = [1,4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$? Does $f(A \cup B) = f(A) \cup f(B)$?

Solution.

$$f(A) = [0,4], \qquad f(B) = [1,16].$$

Yes,

$$f(A \cap B) = f(A) \cap f(B).$$

Yes,

$$f(A \cup B) = f(A) \cup f(B).$$

(b) If $f : \mathbb{R} \to \mathbb{R}$ and B is fixed, define $f^{-1}(B) = \{x \in \mathbb{R} : f(x) \in B\}$. Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$. Find an example to show that the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ is not arbitrary inclusion.

Solution. Let

$$A = [-2,1], \qquad B = [0,3].$$

(c) Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$,

$$g(A \cap B) \subseteq g(A) \cap g(B)$$

for all sets $A, B \subseteq \mathbb{R}$.

Solution. Let $x \in g(A \cap B)$. Then

$$\exists\, y \in A \cap B \text{ such that } g(y) = x.$$

$$\begin{aligned} y \in A &\implies g(y) \in g(A) \\ y \in B &\implies g(y) \in g(B) \\ &\implies g(y) = x \in g(A) \cap g(B). \end{aligned}$$

Thus,

$$g(A \cap B) \subseteq g(A) \cap g(B).$$

□

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$.

Conjecture.

$$g(A \cup B) = g(A) \cup g(B).$$

Proof. Let $y \in g(A \cup B)$. Then

$$\exists\, x \in A \cup B \text{ such that } g(x) = y$$

$$\begin{aligned} &\implies x \in A \text{ or } x \in B \\ &\implies g(x) \in g(A) \text{ or } g(x) \in g(B) \\ &\implies y \in g(A) \cup g(B). \end{aligned}$$

Therefore,

$$g(A) \cup g(B) \subseteq g(A \cup B).$$

Let $y \in g(A) \cup g(B)$. Then

$$y \in g(A) \text{ or } y \in g(B)$$

$$\begin{aligned} &\implies \exists\, x_1 \in A \text{ such that } g(x_1) = y \quad \text{or} \quad \exists\, x_2 \in B \text{ such that } g(x_2) = y \\ &\implies \exists\, x_1 \in A \cup B \text{ such that } g(x_1) = y \quad \text{or} \quad \exists\, x_2 \in A \cup B \text{ such that } g(x_2) = y \\ &\implies y \in g(A \cup B). \end{aligned}$$

Therefore,

$$g(A \cup B) = g(A) \cup g(B).$$

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Exercise 1.2.8. Let $f : A \to B$. State whether the following are possible or not. If possible, provide an example.

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

Solution. Impossible.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

Solution.

$$f(x) = x^2.$$

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

Solution. Impossible.

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Exercise 1.2.9. Given a function $f : D \to \mathbb{R}$ and a subset $B \subseteq \mathbb{R}$, let $f^{-1}(B)$ be the set of all points from the domain D that get mapped into B; that is,

$$f^{-1}(B) = \{x \in D : f(x) \in B\}.$$

This set is called the *preimage* of B.

(a) Let $f(x) = x^2$. If A is the closed interval $[0, 4]$ and B is the closed interval $[-1, 1]$, find $f^{-1}(A)$ and $f^{-1}(B)$. Does

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

in this case? Does

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)?$$

Solution.

$$f^{-1}(A) = [-2, 2], \qquad f^{-1}(B) = [-1, 1].$$
$$f^{-1}(A \cap B) = f^{-1}([0, 1]) = [-1, 1].$$
$$f^{-1}(A \cup B) = f^{-1}([-1, 4]) = [-2, 2].$$

Hence,

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B), \qquad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$

(b) The good behavior of preimages demonstrated in (a) is completely general. Show that for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that

$$g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B) \quad \text{and} \quad g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B)$$

for all sets $A, B \subseteq \mathbb{R}$.

Solution. Let $x \in g^{-1}(A \cap B)$. Then

$$\begin{aligned} &g(x) \in A \cap B \\ \implies &g(x) \in A \text{ and } g(x) \in B \\ \implies &x \in g^{-1}(A) \text{ and } x \in g^{-1}(B) \\ \implies &x \in g^{-1}(A) \cap g^{-1}(B). \end{aligned}$$

Thus,

$$g^{-1}(A \cap B) \subseteq g^{-1}(A) \cap g^{-1}(B).$$

The reverse inclusion follows by reversing the above implications, and hence

$$g^{-1}(A \cap B) = g^{-1}(A) \cap g^{-1}(B).$$

The proof for unions is analogous:

$$g^{-1}(A \cup B) = g^{-1}(A) \cup g^{-1}(B).$$

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Exercise 1.2.10. Decide which of the following statements are true. Provide a short justification for those that are valid and a counterexample for those that are not.

1. Two real numbers satisfy $a < b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

   Solution.

   ($\Rightarrow$) If $a < b$, then for any $\varepsilon > 0$,

   $$a < b < b + \varepsilon,$$

   so $a < b + \varepsilon$.

   ($\Leftarrow$) This implication is false. Counterexample: let $a = 1$ and $b = 1$. Then $a < b+\varepsilon$ for every $\varepsilon > 0$, but $a \not< b$.

2. Two real numbers satisfy $a < b$ if $a < b + \varepsilon$ for every $\varepsilon > 0$.

   Solution.

   This statement is false. Counterexample: let $a = 1$ and $b = 1$. Then $a < b + \varepsilon$ for every $\varepsilon > 0$, but $a \not< b$.

3. Two real numbers satisfy $a \leq b$ if and only if $a < b + \varepsilon$ for every $\varepsilon > 0$.

   Solution.

   ($\Rightarrow$) If $a \leq b$, then for any $\varepsilon > 0$,

   $$a \leq b < b + \varepsilon \ \Rightarrow \ a < b + \varepsilon.$$

   ($\Leftarrow$) Assume $a < b + \varepsilon$ for every $\varepsilon > 0$. Define

   $$A = \{\, b + \varepsilon : \varepsilon > 0 \,\}.$$

   Then $\inf A = b$. Since a is a lower bound for A, we conclude

   $$a \leq \inf A = b.$$

—

Exercise 1.2.11. This exercise deals with the formal logic of negation and claim. One main goal is to practice identifying the negation of a mathematical claim.
Let a statement be given. Then decide whether the statement is true, and justify your answer. If false, state its **negation** and determine whether it is true.

(a) There exists a real number $x > 0$ such that $x < 1/n$ for all $n \in \mathbb{N}$.

   Solution. Original statement:

   $$\forall a, b \in \mathbb{R},\ a < b \implies \exists n \in \mathbb{N} \text{ such that } a + \frac{1}{n} < b.$$

   Truth: True.

   Reason: Let $e = b - a > 0$. By the Archimedean property, choose $N \in \mathbb{N}$ such that $\frac{1}{N} < e$. Then

   $$a + \frac{1}{N} < b.$$

   Negation:

   $$\exists a, b \in \mathbb{R},\ a < b \text{ such that } \forall n \in \mathbb{N},\ a + \frac{1}{n} \geq b.$$

(b) Between every two distinct real numbers there is a rational number.

Solution. Original statement:

$$\exists x \in \mathbb{R},\ x > 0 \text{ such that } x < \frac{1}{n} \text{ for all } n \in \mathbb{N}.$$

Truth: False.

Negation:

$$\forall x \in \mathbb{R},\ x > 0 \implies \exists n \in \mathbb{N} \text{ such that } \frac{1}{n} < x.$$

This is true by the Archimedean property.

(c) Between every two distinct real numbers there is a rational number.

Solution. Original statement:

Between every two distinct real numbers there is a rational number.

Truth: True.

Negation:

$$\exists a, b \in \mathbb{R},\ a \neq b \text{ such that no rational number lies between } a \text{ and } b.$$

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Exercise 1.2.12. Let $y_1 = 6$, and for each $n \in \mathbb{N}$ define $y_{n+1} = \frac{2y_n}{3}$.

(a) Use induction to prove that the sequence satisfies $y_n > 6 - \epsilon$ for all $n \in \mathbb{N}$.

Solution. Base step:

$$y_1 = 6 > -6.$$

Induction hypothesis: Assume $y_n > -6$.

Inductive step: We must show that $y_{n+1} > -6$.

$$y_{n+1} = \frac{2y_n - 6}{3}.$$

Since $y_n > -6$, we have

$$2y_n - 6 > 2(-6) - 6 = -12 - 6 = -18.$$

Dividing by 3,

$$y_{n+1} = \frac{2y_n - 6}{3} > \frac{-18}{3} = -6.$$

Hence $y_{n+1} > -6$.

$\square$

(b) Use another induction argument to show the sequence $(y_1, y_2, y_3, \ldots)$ is decreasing.

Solution. Base step:

$$y_2 < y_1, \qquad 2 < 6.$$

Induction hypothesis: Assume $y_n < y_{n-1}$.

Inductive step: We must show that $y_{n+1} < y_n$.

$$y_n = \frac{2y_{n-1} - 6}{3} < y_{n-1}.$$

Multiplying by 3,

$$2y_{n-1} - 6 < 3y_{n-1}.$$

Rewriting,

$$3y_{n-1} + 6 > 2y_{n-1}.$$

Thus,

$$y_{n-1} > \frac{2y_{n-1} - 6}{3} = y_n.$$

Now,

$$y_{n+1} = \frac{2y_n - 6}{3}.$$

From the above inequality,

$$y_n > y_{n+1}.$$

Hence $y_{n+1} < y_n$.

□

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Exercise 1.2.13. For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

Solution. Base step: For $n = 2$,

$$(A_1 \cup A_2)^c = A_1^c \cap A_2^c.$$

Induction hypothesis: Assume

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c.$$

Inductive step:

$$(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1})^c = (A_1 \cup \cdots \cup A_n)^c \cap A_{n+1}^c$$
$$\implies A_1^c \cap A_2^c \cap \cdots \cap A_n^c \cap A_{n+1}^c.$$

□

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for **every finite** $n \in \mathbb{N}$, but this does not imply the validity of the infinite case.

Explain why **the infinite version is not a consequence of induction**, and illustrate with a counterexample.

Solution. Let

$$B_n = \left(0, \frac{1}{n}\right).$$

For every $n \in \mathbb{N}$,

$$\bigcap_{i=1}^{n} B_i = \left(0, \frac{1}{n}\right) \neq \varnothing.$$

However, if

$$x \in \bigcap_{i=1}^{\infty} B_i,$$

then

$$0 < x < \frac{1}{n} \quad \text{for all } n \in \mathbb{N},$$

which is impossible. Hence,

$$\bigcap_{i=1}^{\infty} B_i = \varnothing.$$

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does **not** use induction.

Solution. Let

$$x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c.$$

$$\Longrightarrow \; x \notin A_i \text{ for all } i \in \mathbb{N}$$

$$\Longrightarrow \; x \in A_1^c \text{ and } x \in A_2^c \text{ and } \cdots$$

$$\Longrightarrow \; x \in A_1^c \cap A_2^c \cap \cdots$$

Hence,

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c.$$

□