Solutions to *Understanding Analysis* by Stephen Abbott

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Exercises

Exercise 1.4.1. Recall that I stands for the set of irrational numbers.

1. Show that if $a, b \in \mathbb{Q}$, then ab and $a+b$ are elements of $\mathbb{Q}$ as well.

   Solution.

   Proof.

   Since $a, b \in \mathbb{Q}$,
   $$a = \frac{p}{q}, \quad b = \frac{m}{n},$$
   for some $p, m \in \mathbb{Z}$ and $q, n \in \mathbb{N}$ with $q \neq 0$ and $n \neq 0$.

   Then
   $$ab = \frac{pm}{qn}.$$
   Since $pm \in \mathbb{Z}$ and $qn \in \mathbb{N}$ with $qn \neq 0$,
   $$\Rightarrow ab \in \mathbb{Q}.$$

   Also,
   $$a + b = \frac{p}{q} + \frac{m}{n} = \frac{pn + qm}{qn}.$$
   Since $pn + qm \in \mathbb{Z}$ and $qn \in \mathbb{N}$ with $qn \neq 0$,
   $$\Rightarrow a + b \in \mathbb{Q}.$$

   Thus, $\mathbb{Q}$ is closed under addition and multiplication. □

2. Show that if $a \in \mathbb{Q}$ and $t \in I$, then $a + t \in I$ and $at \in I$ as long as $a \neq 0$.

3. Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is I closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st?

Exercise 1.4.2. Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $s \in \mathbb{R}$ have the property that for all $n \in \mathbb{N}$, $s + \frac{1}{n}$ is an upper bound for A and $s - \frac{1}{n}$ is not an upper bound for A. Show $s = \sup A$.

Solution.

Step 1: s is an upper bound for A.

Since

$$s + \frac{1}{n} \text{ is an upper bound for } A \quad \text{for all } n \in \mathbb{N},$$

we have

$$x \leq s + \frac{1}{n} \quad \text{for all } x \in A \text{ and all } n \in \mathbb{N}.$$

Letting $n \to \infty$, we obtain

$$x \leq s \quad \text{for all } x \in A.$$

$$\Rightarrow s \text{ is an upper bound for } A.$$

Step 2: s is the least upper bound of A.

Since

$$s - \frac{1}{n} \text{ is not an upper bound for } A \quad \text{for all } n \in \mathbb{N},$$

it follows that for each $n \in \mathbb{N}$, there exists an element $x_n \in A$ such that

$$s - \frac{1}{n} < x_n.$$

Now let $\varepsilon > 0$ be given.

By the Archimedean Property, there exists $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \varepsilon.$$

Since $s - \frac{1}{m}$ is not an upper bound for A, there exists $x \in A$ such that

$$s - \frac{1}{m} < x.$$

Thus,

$$s - \varepsilon < s - \frac{1}{m} < x \leq s.$$

$$\Rightarrow \text{for every } \varepsilon > 0, \text{ there exists } x \in A \text{ such that } s - \varepsilon < x \leq s.$$

Hence, no number smaller than s can be an upper bound for A.

$$\Rightarrow s \text{ is the least upper bound of } A.$$

Therefore,

$$\boxed{s = \sup A.}$$

Exercise 1.4.3. Prove that $\bigcap_{n=1}^{\infty}(0, 1/n) = \varnothing$. Notice that this demonstrates that the intervals in the Nested Interval Property must be closed for the conclusion of the theorem to hold.

Proof. Suppose, for contradiction, that

$$\bigcap_{n=1}^{\infty}(0,1/n) \neq \varnothing.$$

Then there exists some $x \in \mathbb{R}$ such that

$$x \in (0,1/n) \quad \text{for all } n \in \mathbb{N}.$$

Define

$$I_n = (0,1/n), \quad n \in \mathbb{N}.$$

Then

$$x \in I_n \ \forall n \in \mathbb{N} \ \Rightarrow\ 0 < x < \frac{1}{n} \ \forall n \in \mathbb{N}.$$

Thus,

$$x \leq \frac{1}{n} \quad \forall n \in \mathbb{N},$$

which implies that x is a lower bound of the set

$$\left\{\frac{1}{n} : n \in \mathbb{N}\right\}.$$

But

$$\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0,$$

since for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \varepsilon$$

(by the Archimedean Property).
Hence,

$$x \leq 0.$$

This contradicts the fact that $x > 0$.
Therefore, no such x exists, and we conclude

$$\bigcap_{n=1}^{\infty}(0,1/n) = \varnothing.$$

□

Exercise 1.4.4. Let $a < b$ be real numbers and consider the set $T = \mathbb{Q} \cap [a,b]$. Show $\sup T = b$.
Solution.
Step 1: b is an upper bound for T.
For all $x \in [a,b]$, we have

$$x \leq b.$$

Since $T \subseteq [a,b]$, it follows that

$$x \leq b \quad \text{for all } x \in T.$$

$$\Rightarrow b \text{ is an upper bound for } T.$$

Step 2: b is the least upper bound of T.
Let $\varepsilon > 0$ be given, with $\varepsilon \le b - a$.
Since the rational numbers are dense in $\mathbb{R}$, there exists a rational number $q \in \mathbb{Q}$ such that

$$b - \varepsilon < q < b.$$

Because $a \le b - \varepsilon < q < b \le b$, we have

$$q \in [a, b].$$

Hence,

$$q \in \mathbb{Q} \cap [a, b] = T.$$

Thus, for every $\varepsilon > 0$, there exists $q \in T$ such that

$$b - \varepsilon < q \le b.$$

$$\Rightarrow \text{no number smaller than } b \text{ can be an upper bound for } T.$$

Therefore,

$$\boxed{\sup T = b.}$$

Exercise 1.4.5. Using Exercise 1.4.1, supply a proof for Corollary 1.4.4 by considering the real numbers $a - \sqrt{2}$ and $b - \sqrt{2}$.
Proof. Let $a < b$ be real numbers. Consider the real numbers

$$a - \sqrt{2} \quad \text{and} \quad b - \sqrt{2}.$$

Since $a < b$, we have

$$a - \sqrt{2} < b - \sqrt{2}.$$

Because $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists a rational number

$$r \in \mathbb{Q}$$

such that

$$a - \sqrt{2} < r < b - \sqrt{2}.$$

Adding $\sqrt{2}$ throughout gives

$$a < r + \sqrt{2} < b.$$

Now, $r \in \mathbb{Q}$ and $\sqrt{2} \in I$, so by Exercise 1.4.1(b),

$$r + \sqrt{2} \in I.$$

Thus, there exists an irrational number $t = r + \sqrt{2}$ such that

$$a < t < b.$$

□

Exercise 1.4.6. Recall that a set B is *dense* in $\mathbb{R}$ if an element of B can be found between any two real numbers $a < b$. Which of the following sets are dense in $\mathbb{R}$? Take $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ in every case.

1. The set of all rational numbers p/q with $q \leq 10$.
2. The set of all rational numbers p/q with q a power of 2.
3. The set of all rational numbers p/q with $10|p| \geq q$.

Exercise 1.4.7. Finish the proof of Theorem 1.4.5 by showing that the assumption $\alpha^2 > 2$ leads to a contradiction of the fact that $\alpha = \sup T$.

Exercise 1.4.8. Give an example of each or state that the request is impossible. When a request is impossible, provide a compelling argument for why this is the case.

1. Two sets A and B with $A \cap B = \varnothing$, $\sup A = \sup B$, $A \not\subseteq A^c$ and $\sup B \notin B$.

   Example exists.

   Let
   $$A = \mathbb{Q} \cap (0,2), \qquad B = I \cap (0,2).$$
   Then
   $$A \cap B = \emptyset,$$
   because no number is both rational and irrational. Moreover,
   $$\sup A = \sup B = 2,$$
   and since $2 \notin (0,2)$,
   $$2 \notin A \quad \text{and} \quad 2 \notin B.$$

2. A sequence of nested open intervals $J_1 \supseteq J_2 \supseteq J_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} J_n$ nonempty but containing only a finite number of elements.

   Impossible.

3. A sequence of nested unbounded closed intervals $L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots$ with $\bigcap_{n=1}^{\infty} L_n = \varnothing$. (An unbounded closed interval has the form $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.)

   Example exists.

   Define
   $$L_n = [n, \infty).$$
   Then
   $$L_1 \supseteq L_2 \supseteq L_3 \supseteq \cdots,$$
   each L_n is closed and unbounded, and
   $$\bigcap_{n=1}^{\infty} L_n = \emptyset,$$
   since no real number is greater than or equal to every natural number.

4. A sequence of closed bounded (not necessarily nested) intervals $I_1, I_2, I_3, \ldots$ with the property that $\bigcap_{n=1}^{N} I_n \neq \varnothing$ for all $N \in \mathbb{N}$, but $\bigcap_{n=1}^{\infty} I_n = \varnothing$.

   Impossible.