appendix A

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Here, we will analytically solve the Black-Scholes Equation for a European option.

We want to specify the initial conditions and boundary conditions to adapt the PDE to the specific security we want to price. We are looking for ways to solve the Black-Scholes analytically, so we will first transform and simplify our equation to the heat equation. Since we have already known the solution to the heat equation, we then can derive the solution of Black-Scholes Equation.

Before transforming the equation, by observation, we can know that the Black-Scholes equation is a backward equation, with the current stock price $S(t)$ and the terminal condition $V(T) = \max(S(T) - K, 0)$ given. All we need to do is determining the current price of the option, which requires us to go backward. By setting $\tau = T - t$ as the time until maturity, the time can be reversed. Then the remaining maturity at the terminal date will be zero. The price at maturity and the price of the option at time T can be rewritten as $S(0)$ and $V(0)$, the current price and option price at time t can be rewritten as $S(\tau)$ and $V(\tau)$.

Since the Black-Scholes Equation is derived based on the geometric Brownian motion and we know the price $S(T)$ of the underlying asset at the time of option expiration follows Brownian motion. We then have the solution of $S(T)$ below:

$$S(T) = e^{\ln S(t) + \left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(W_T - W_t)}$$

By taking Feyman-Kac theorem and the equivalence between stochastic and PDE representations into consideration, the European option price can be represented both as expectation of a stochastic term or via PDE, then the stochastic term $\sigma(W_T - W_t)$ of $S(T)$ can be ignored. A new variable x can then be defined by taking the log of $S(T)$ as below in order to cancel out the S^2 and S terms

$$x = \ln S(\tau) + \left(r - \frac{1}{2}\sigma^2\right)\tau$$

Also, in order to transform the value of current value of the option price to its forward value and represent it by $F(\tau)$, $F(\tau)$ is denoted as $V(\tau)e^{r\tau}$

First, we apply the chain rule on the first term of the Black-Scholes equation:

$$\frac{\partial V(t)}{\partial t} = \frac{\partial V(\tau)}{\partial \tau}\frac{\partial \tau}{\partial t} = -\frac{\partial V(\tau)}{\partial \tau}$$

Then, we apply the chain rule on the second and third terms:

$$\frac{\partial V(S,t)}{\partial S} = \frac{\partial V(S,\tau)}{\partial x}\frac{\partial x}{\partial S} = \frac{\partial V(S,\tau)}{\partial x}\frac{1}{S}\frac{\partial^2 V(S,t)}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V(S,\tau)}{\partial x}\frac{1}{S}\right) = -\frac{\partial V(S,\tau)}{\partial x}\frac{1}{S^2} + \frac{1}{S}\frac{\partial}{\partial S}\left(\frac{\partial V(S,\tau)}{\partial x}\right.$$

We then also transform the derivative with respect to τ because x is a function of τ. When τ changes, the option price changes and x also changes, also, V is a function of x. Then, an additional term will be included in $V(\tau)$ because of total derivative and get:

$$\frac{\partial V(\tau)}{\partial \tau} = \frac{\partial V(\tau)}{\partial \tau} + \frac{\partial V(\tau)}{\partial x}\frac{\partial x}{\partial \tau} = \frac{\partial V(\tau)}{\partial \tau} + \frac{\partial V(\tau)}{\partial x}\left(r - \frac{1}{2}\sigma^2\right)$$

The Black-Scholes equation then becomes:

$$0 = -\frac{\partial v}{\partial \tau} - \frac{\partial v}{\partial x}\left(r - \frac{1}{2}\sigma^2\right) + \frac{1}{2}\sigma^2\left(-\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) + r\frac{\partial v}{\partial x} - rv = -\frac{\partial v}{\partial \tau} + \frac{1}{2}\sigma^2\frac{\partial^2 v}{\partial x^2} - rv$$

Since we have already defined our $F(\tau) := V(\tau)e^{r\tau}$, then we have:

$$F_\tau = v_\tau e^{r\tau} v = Fe^{-r\tau}\frac{\partial v}{\partial x} = e^{-r\tau}\frac{\partial F}{\partial x}\frac{\partial^2 v}{\partial x^2} = e^{-r\tau}\frac{\partial^2 F}{\partial x^2}$$

We then can get $\frac{\partial V}{\partial \tau}$ by substituting the equations above:

$$\frac{\partial v}{\partial \tau} = e^{-r\tau}\frac{\partial F}{\partial \tau} + F\frac{\partial}{\partial \tau}e^{-r\tau} = e^{-r\tau}\frac{\partial F}{\partial \tau} - rFe^{-r\tau}$$

The Black-Scholes equation then can be transformed into a heat equation:

$$-e^{-r\tau}\frac{\partial F}{\partial \tau} + rFe^{-r\tau} + \frac{1}{2}\sigma^2 e^{-r\tau}\frac{\partial^2 F}{\partial x^2} - rFe^{-r\tau} = 0 - e^{-r\tau}\frac{\partial F}{\partial \tau} + \frac{1}{2}\sigma^2 e^{-r\tau}\frac{\partial^2 F}{\partial x^2} = 0\frac{\partial F}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}$$

For any heat equation $\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$, $f(x,0) = \psi(x)$ the solution is shown below:

$$f(x,t) = \int_{-\infty}^{\infty}\psi(z)\frac{1}{\sqrt{4\pi Dt}}e^{-\frac{(x-z)^2}{4Dt}}dz$$

The expression of $F(x,0) = \max(e^x - K, 0)$ will be non-negative only when $e^z - K \geq 0$, which means $z \geq \ln K$

First, substitute $D = \frac{1}{2}\sigma^2$ into the solution above and get:

$$F(x,\tau) = \int_{-\infty}^{\infty}(e^z - K)^+\frac{1}{\sqrt{4\pi\frac{1}{2}\sigma^2\tau}}e^{-\frac{(x-z)^2}{4\frac{1}{2}\sigma^2\tau}}dz = \int_{-\infty}^{\infty}(e^z - K)^+\frac{1}{\sigma\sqrt{2\pi\tau}}e^{-\frac{(x-z)^2}{2\sigma^2\tau}}dz = \int_{\ln K}^{\infty}(e^z - K)\frac{1}{\sigma\sqrt{2}}$$

We first transform the first term by expanding $(x-z)^2$ and put e^z inside:

$$\int_{\ln K}^{\infty} e^{z} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{x^2+z^2-2xz}{2\sigma^2\tau}} dz = \int_{\ln K}^{\infty} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{-2z\sigma^2\tau+x^2+z^2-2xz}{2\sigma^2\tau}} dz = \int_{\ln K}^{\infty} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{x^2}{2\sigma^2\tau}} e^{-\frac{z^2-2xz-2z\sigma^2\tau}{2\sigma^2\tau}} dz =$$

We can say z follows the normal distribution with mean= $x+\sigma^2\tau$, variance= $\sigma^2\tau$

$$\int_{\ln K}^{\infty} e^{z} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{x^2+z^2-2xz}{2\sigma^2\tau}} dz = \int_{\ln K}^{\infty} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{-2z\sigma^2\tau+x^2+z^2-2xz}{2\sigma^2\tau}} dz = e^{x+\frac{\sigma^2\tau}{2}} \Phi\left(\frac{-\ln K + x + \sigma^2\tau}{\sigma\sqrt{\tau}}\right)$$

By observation, we know the second term of the equation above is the probability density of a normal distribution with mean= x and variance $= \sigma^2\tau$. If we define $Z := \frac{z-x}{\sigma\sqrt{\tau}}$, we then know Z follows a normal distribution with mean= 0 and variance= 1 and $z = x + \sigma\sqrt{\tau}Z$

$$K \int_{\ln K}^{\infty} \frac{1}{\sigma\sqrt{2\pi\tau}} e^{-\frac{(x-2)^2}{2\sigma^2\tau}} dz = Prob(z \geq \ln K) = Prob(x+\sigma\sqrt{\tau}Z \geq \ln K) = Prob\left(Z \geq \frac{\ln K - x}{\sigma\sqrt{\tau}}\right) = Prob\left(\right.$$

Then our solution can be rewritten as below:

$$F(x,\tau) = e^{x+\frac{\sigma^2\tau}{2}} \Phi\left(\frac{-\ln K + x + \sigma^2\tau}{\sigma\sqrt{\tau}}\right) - K\Phi\left(\frac{-\ln K + x}{\sigma\sqrt{\tau}}\right) V(\tau)e^{r\tau} = e^{x+\frac{\sigma^2\tau}{2}} \Phi\left(\frac{-\ln K + x + \sigma^2\tau}{\sigma\sqrt{\tau}}\right) - K$$

We then get the solution of Black-Scholes equation as below:

$$V(\tau) = e^{\ln S(\tau)} \Phi\left(\frac{-\ln K + \ln S(\tau) + \left(r - \frac{1}{2}\sigma^2\right)\tau + \sigma^2\tau}{\sigma\sqrt{\tau}}\right) - Ke^{-r\tau} \Phi\left(\frac{-\ln K + \ln S(\tau) + \left(r - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)$$