Problems and Solutions of Atomic, Nuclear, and Particle Physics

Compiled by
The Physics Coaching Class
University of Science and Technology of China

Edited by
Yung-Kuo Lim

World Scientific

Problems and Solutions on Atomic, Nuclear and Particle Physics

This Page Intentionally Left Blank

Major American Universities Ph.D. Qualifying Questions and Solutions

Problems and Solutions on Atomic, Nuclear and Particle Physics

Compiled by

The Physics Coaching Class
University of Science and Technology of China

Edited by

Yung-Kuo Lim

National University of Singapore

World Scientific
Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P 0 Box 128, Farrer Road, Singapore 912805

USA office: Suite lB, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

Major American Universities Ph.D. Qualifying Questions and Solutions
PROBLEMS AND SOLUTIONS ON ATOMIC, NUCLEAR AND PARTICLE PHYSICS

ISBN 981-02-3917-3
981-02-3918-l (pbk)

This book is printed on acid-free paper.

Printed in Singapore by Uto-Print

PREFACE

This series of physics problems and solutions, which consists of seven volumes — Mechanics, Electromagnetism, Optics, Atomic, Nuclear and Particle Physics, Thermodynamics and Statistical Physics, Quantum Mechanics, Solid State Physics and Relativity, contains a selection of 2550 problems from the graduate-school entrance and qualifying examination papers of seven U.S. universities — California University Berkeley Campus, Columbia University, Chicago University, Massachusetts Institute of Technology, New York State University Buffalo Campus, Princeton University, Wisconsin University — as well as the CUSPEA and C.C. Ting's papers for selection of Chinese students for further studies in U.S.A., and their solutions which represent the effort of more than 70 Chinese physicists, plus some 20 more who checked the solutions.

The series is remarkable for its comprehensive coverage. In each area the problems span a wide spectrum of topics, while many problems overlap several areas. The problems themselves are remarkable for their versatility in applying the physical laws and principles, their uptodate realistic situations, and their scanty demand on mathematical skills. Many of the problems involve order-of-magnitude calculations which one often requires in an experimental situation for estimating a quantity from a simple model. In short, the exercises blend together the objectives of enhancement of one's understanding of physical principles and ability of practical application.

The solutions as presented generally just provide a guidance to solving the problems, rather than step-by-step manipulation, and leave much to the students to work out for themselves, of whom much is demanded of the basic knowledge in physics. Thus the series would provide an invaluable complement to the textbooks.

The present volume consists of 483 problems. It covers practically the whole of the usual undergraduate syllabus in atomic, nuclear and particle physics, but in substance and sophistication goes much beyond. Some problems on experimental methodology have also been included.

In editing, no attempt has been made to unify the physical terms, units and symbols. Rather, they are left to the setters' and solvers' own preference so as to reflect the realistic situation of the usage today. Great pains has been taken to trace the logical steps from the first principles to the final solution, frequently even to the extent of rewriting the entire solution.

In addition, a subject index to problems has been included to facilitate the location of topics. These editorial efforts hopefully will enhance the value of the volume to the students and teachers alike.

Yung-Kuo Lim
Editor

INTRODUCTION

Solving problems in course work is an exercise of the mental facilities, and examination problems are usually chosen, or set similar to such problems. Working out problems is thus an essential and important aspect of the study of physics.

The series *Major American University Ph.D. Qualifying Questions and Solutions* comprises seven volumes and is the result of months of work of a number of Chinese physicists. The subjects of the volumes and the respective coordinators are as follows:

1. Mechanics (Qiang Yan-qi, Gu En-pu, Cheng Jia-fu, Li Ze-hua, Yang De-tian)
2. Electromagnetism (Zhao Shu-ping, You Jun-han, Zhu Jun-jie)
3. Optics (Bai Gui-ru, Guo Guang-can)
4. Atomic, Nuclear and Particle Physics (Jin Huai-cheng, Yang Bao-zhong, Fan Yang-mei)
5. Thermodynamics and Statistical Physics (Zheng Jiu-ren)
6. Quantum Mechanics (Zhang Yong-de, Zhu Dong-pei, Fan Hong-yi)
7. Solid State Physics and Miscellaneous Topics (Zhang Jia-lu, Zhou You-yuan, Zhang Shi-ling).

These volumes, which cover almost all aspects of university physics, contain 2550 problems, mostly solved in detail.

The problems have been carefully chosen from a total of 3100 problems, collected from the China-U.S.A. Physics Examination and Application Program, the Ph.D. Qualifying Examination on Experimental High Energy Physics sponsored by Chao-Chong Ting, and the graduate qualifying examinations of seven world-renowned American universities: Columbia University, the University of California at Berkeley, Massachusetts Institute of Technology, the University of Wisconsin, the University of Chicago, Princeton University, and the State University of New York at Buffalo.

Generally speaking, examination problems in physics in American universities do not require too much mathematics. They can be characterized to a large extent as follows. Many problems are concerned with the various frontier subjects and overlapping domains of topics, having been selected from the setters own research encounters. These problems show a "modern" flavor. Some problems involve a wide field and require a sharp mind for their analysis, while others require simple and practical methods demanding a fine "touch of physics". Indeed, we believe that these problems, as a whole, reflect to some extent the characteristics of American science and culture, as well as give a glimpse of the philosophy underlying American education.

That being so, we considered it worthwhile to collect and solve these problems, and introduce them to students and teachers everywhere, even though the work was both tedious and strenuous. About a hundred teachers and graduate students took part in this time-consuming task.

This volume on Atomic, Nuclear and Particle Physics which contains 483 problems is divided into four parts: Atomic and Molecular Physics (142), Nuclear Physics (120), Particle Physics (90), Experimental Methods and Miscellaneous topics (131).

In scope and depth, most of the problems conform to the usual undergraduate syllabi for atomic, nuclear and particle physics in most universities. Some of them, however, are rather profound, sophisticated, and broad-based. In particular they demonstrate the use of fundamental principles in the latest research activities. It is hoped that the problems would help the reader not only in enhancing understanding of the basic principles, but also in cultivating the ability to solve practical problems in a realistic environment.

This volume was the result of the collective efforts of forty physicists involved in working out and checking of the solutions, notably Ren Yong, Qian Jian-ming, Chen Tao, Cui Ning-zhuo, Mo Hai-ding, Gong Zhu-fang and Yang Bao-zhong.

CONTENTS

PART I

ATOMIC AND MOLECULAR PHYSICS

This Page Intentionally Left Blank

1. ATOMIC PHYSICS (1001–1122)

1001

Assume that there is an announcement of a fantastic process capable of putting the contents of physics library on a very smooth postcard. Will it be readable with an electron microscope? Explain.

(*Columbia*)

Solution:

Suppose there are 10^6 books in the library, 500 pages in each book, and each page is as large as two postcards. For the postcard to be readable, the planar magnification should be $2 \times 500 \times 10^6 \approx 10^9$, corresponding to a linear magnification of $10^{4.5}$. As the linear magnification of an electron microscope is of the order of 800,000, its planar magnification is as large as 10^{11}, which is sufficient to make the postcard readable.

1002

At 10^{10} K the black body radiation weighs (1 ton, 1 g, 10^{-6} g, 10^{-16} g) per cm^3.

(*Columbia*)

Solution:

The answer is nearest to 1 ton per cm^3.

The radiant energy density is given by $u = 4\sigma T^4/c$, where $\sigma = 5.67 \times 10^{-8}$ Wm^{-2} K^{-4} is the Stefan–Boltzmann constant. From Einstein's mass-energy relation, we get the mass of black body radiation per unit volume as $u = 4\sigma T^4/c^3 = 4\times5.67\times10^{-8}\times10^{40}/(3\times10^8)^3 \approx 10^8$ kg/m^3 $= 0.1$ ton/cm^3.

1003

Compared to the electron Compton wavelength, the Bohr radius of the hydrogen atom is approximately

(a) 100 times larger.

(b) 1000 times larger.

(c) about the same.

(*CCT*)

Solution:

The Bohr radius of the hydrogen atom and the Compton wavelength of electron are given by $a = \frac{\hbar^2}{me^2}$ and $\lambda_c = \frac{h}{mc}$ respectively. Hence $\frac{a}{\lambda_c} = \frac{1}{2\pi}(\frac{e^2}{\hbar c})^{-1} = \frac{137}{2\pi} = 22$, where $e^2/\hbar c$ is the fine-structure constant. Hence the answer is (a).

1004

Estimate the electric field needed to pull an electron out of an atom in a time comparable to that for the electron to go around the nucleus.

(*Columbia*)

Solution:

Consider a hydrogen-like atom of nuclear charge Ze. The ionization energy (or the energy needed to eject the electron) is 13.6Z^2 eV. The orbiting electron has an average distance from the nucleus of $a = a_0/Z$, where $a_0 = 0.53 \times 10^{-8}$ cm is the Bohr radius. The electron in going around the nucleus in electric field E can in half a cycle acquire an energy eEa. Thus to eject the electron we require

$$eEa \gtrsim 13.6 \text{ Z}^2 \text{ eV},$$

or

$$E \gtrsim \frac{13.6 \text{ Z}^3}{0.53 \times 10^{-8}} \approx 2 \times 10^9 \text{ Z}^3 \text{ V/cm}.$$

1005

As one goes away from the center of an atom, the electron density

(a) decreases like a Gaussian.

(b) decreases exponentially.

(c) oscillates with slowly decreasing amplitude.

(*CCT*)

Solution:

The answer is (c).

1006

An electronic transition in ions of ^{12}C leads to photon emission near $\lambda = 500$ nm ($h\nu = 2.5$ eV). The ions are in thermal equilibrium at an ion temperature $kT = 20$ eV, a density $n = 10^{24}$ m^{-3}, and a non-uniform magnetic field which ranges up to $B = 1$ Tesla.

(a) Briefly discuss broadening mechanisms which might cause the transition to have an observed width $\Delta\lambda$ greater than that obtained for very small values of T, n and B.

(b) For one of these mechanisms calculate the broadened width $\Delta\lambda$ using order-of-magnitude estimates of needed parameters.

(*Wisconsin*)

Solution:

(a) A spectral line always has an inherent width produced by uncertainty in atomic energy levels, which arises from the finite length of time involved in the radiation process, through Heisenberg's uncertainty principle. The observed broadening may also be caused by instrumental limitations such as those due to lens aberration, diffraction, etc. In addition the main causes of broadening are the following.

Doppler effect: Atoms or molecules are in constant thermal motion at $T > 0$ K. The observed frequency of a spectral line may be slightly changed if the motion of the radiating atom has a component in the line of sight, due to Doppler effect. As the atoms or molecules have a distribution of velocity a line that is emitted by the atoms will comprise a range of frequencies symmetrically distributed about the natural frequency, contributing to the observed width.

Collisions: An atomic system may be disturbed by external influences such as electric and magnetic fields due to outside sources or neighboring atoms. But these usually cause a shift in the energy levels rather than broadening them. Broadening, however, can result from atomic collisions which cause phase changes in the emitted radiation and consequently a spread in the energy.

(b) *Doppler broadening:* The first order Doppler frequency shift is given by $\Delta\nu = \frac{\nu_0 v_x}{c}$, taking the x-axis along the line of sight. Maxwell's velocity distribution law then gives

$$dn \propto \exp\left(-\frac{Mv_x^2}{2kT}\right) dv_x = \exp\left[-\frac{Mc^2}{2kT}\left(\frac{\Delta\nu}{\nu_0}\right)^2\right] dv_x\,,$$

where M is the mass of the radiating atom. The frequency-distribution of the radiation intensity follows the same relationship. At half the maximum intensity, we have

$$\Delta\nu = \nu_0\sqrt{\frac{(\ln 2)2kT}{Mc^2}}\,.$$

Hence the line width at half the maximum intensity is

$$2\Delta\nu = \frac{1.67c}{\lambda_0}\sqrt{\frac{2kT}{Mc^2}}\,.$$

In terms of wave number $\tilde{\nu} = \frac{1}{\lambda} = \frac{\nu}{c}$ we have

$$\Gamma_D = 2\Delta\tilde{\nu} = \frac{1.67}{\lambda_0}\sqrt{\frac{2kT}{Mc^2}}\,.$$

With $kT = 20$ eV, $Mc^2 = 12 \times 938$ MeV, $\lambda_0 = 5 \times 10^{-7}$ m,

$$\Gamma_D = \frac{1.67}{5 \times 10^{-7}}\sqrt{\frac{2 \times 20}{12 \times 938 \times 10^6}} = 199 \text{ m}^{-1} \approx 2 \text{ cm}^{-1}\,.$$

Collision broadening: The mean free path for collision l is defined by $nl\pi d^2 = 1$, where d is the effective atomic diameter for a collision close enough to affect the radiation process. The mean velocity $\bar{v}$ of an atom can be approximated by its root-mean-square velocity given by $\frac{1}{2}M\overline{v^2} = \frac{3}{2}\,kT$. Hence

$$\bar{v} \approx \sqrt{\frac{3kT}{M}}\,.$$

Then the mean time between successive collisions is

$$t = \frac{l}{\bar{v}} = \frac{1}{n\pi d^2}\sqrt{\frac{M}{3kT}}\,.$$

The uncertainty in energy because of collisions, ΔE, can be estimated from the uncertainty principle $\Delta E \cdot t \approx \hbar$, which gives

$$\Delta \nu_c \approx \frac{1}{2\pi t},$$

or, in terms of wave number,

$$\Gamma_c = \frac{1}{2} n d^2 \sqrt{\frac{3kT}{Mc^2}} \sim \frac{3 \times 10^{-3}}{\lambda_0} \sqrt{\frac{2kT}{Mc^2}},$$

if we take $d \approx 2a_0 \sim 10^{-10}$ m, a_0 being the Bohr radius. This is much smaller than Doppler broadening at the given ion density.

1007

(I) The ionization energy E_I of the first three elements are

Z	Element	E_I
1	H	13.6 eV
2	He	24.6 eV
3	Li	5.4 eV

(a) Explain qualitatively the change in E_I from H to He to Li.

(b) What is the second ionization energy of He, that is the energy required to remove the second electron after the first one is removed?

(c) The energy levels of the $n = 3$ states of the valence electron of sodium (neglecting intrinsic spin) are shown in Fig. 1.1.

Why do the energies depend on the quantum number l?

(*SUNY, Buffalo*)

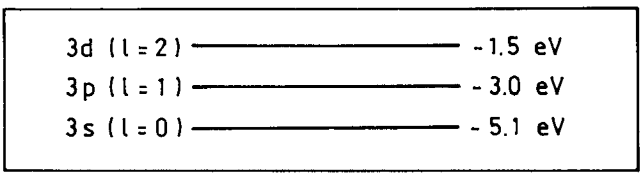

Fig. 1.1

Solution:

(a) The table shows that the ionization energy of He is much larger than that of H. The main reason is that the nuclear charge of He is twice than that of H while all their electrons are in the first shell, which means that the potential energy of the electrons are much lower in the case of He. The very low ionization energy of Li is due to the screening of the nuclear charge by the electrons in the inner shell. Thus for the electron in the outer shell, the effective nuclear charge becomes small and accordingly its potential energy becomes higher, which means that the energy required for its removal is smaller.

(b) The energy levels of a hydrogen-like atom are given by

$$E_n = -\frac{Z^2}{n^2} \times 13.6 \text{ eV}\,.$$

For $Z = 2$, $n = 1$ we have

$$E_I = 4 \times 13.6 = 54.4 \text{ eV}\,.$$

(c) For the $n = 3$ states the smaller l the valence electron has, the larger is the eccentricity of its orbit, which tends to make the atomic nucleus more polarized. Furthermore, the smaller l is, the larger is the effect of orbital penetration. These effects make the potential energy of the electron decrease with decreasing l.

1008

Describe briefly each of the following effects or, in the case of rules, state the rule:

(a) Auger effect
(b) Anomalous Zeeman effect
(c) Lamb shift
(d) Landé interval rule
(e) Hund's rules for atomic levels

(*Wisconsin*)

Solution:

(a) Auger effect: When an electron in the inner shell (say K shell) of an atom is ejected, a less energetically bound electron (say an L electron) may jump into the hole left by the ejected electron, emitting a photon. If the process takes place without radiating a photon but, instead, a higher-energy shell (say L shell) is ionized by ejecting an electron, the process is called Auger effect and the electron so ejected is called Auger electron. The atom becomes doubly ionized and the process is known as a nonradiative transition.

(b) Anomalous Zeeman effect: It was observed by Zeeman in 1896 that, when an excited atom is placed in an external magnetic field, the spectral line emitted in the de-excitation process splits into three lines with equal spacings. This is called normal Zeeman effect as such a splitting could be understood on the basis of a classical theory developed by Lorentz. However it was soon found that more commonly the number of splitting of a spectral line is quite different, usually greater than three. Such a splitting could not be explained until the introduction of electron spin, thus the name 'anomalous Zeeman effect'.

In the modern quantum theory, both effects can be readily understood: When an atom is placed in a weak magnetic field, on account of the interaction between the total magnetic dipole moment of the atom and the external magnetic field, both the initial and final energy levels are split into several components. The optical transitions between the two multiplets then give rise to several lines. The normal Zeeman effect is actually only a special case where the transitions are between singlet states in an atom with an even number of optically active electrons.

(c) Lamb shift: In the absence of hyperfine structure, the $2^2S_{1/2}$ and $2^2P_{1/2}$ states of hydrogen atom would be degenerate for orbital quantum number l as they correspond to the same total angular momentum $j = 1/2$. However, Lamb observed experimentally that the energy of $2^2S_{1/2}$ is 0.035 cm^{-1} higher than that of $2^2P_{1/2}$. This phenomenon is called Lamb shift. It is caused by the interaction between the electron and an electromagnetic radiation field.

(d) Landé interval rule: For LS coupling, the energy difference between two adjacent J levels is proportional, in a given LS term, to the larger of the two values of J.

(e) Hund's rules for atomic levels are as follows:

(1) If an electronic configuration has more than one spectroscopic notation, the one with the maximum total spin S has the lowest energy.

(2) If the maximum total spin S corresponds to several spectroscopic notations, the one with the maximum L has the lowest energy.

(3) If the outer shell of the atom is less than half full, the spectroscopic notation with the minimum total angular momentum J has the lowest energy. However, if the shell is more than half full the spectroscopic notation with the maximum J has the lowest energy. This rule only holds for LS coupling.

1009

Give expressions for the following quantities in terms of $e, \hbar, c, k, m_e$ and m_p.

(a) The energy needed to ionize a hydrogen atom.

(b) The difference in frequency of the Lyman alpha line in hydrogen and deuterium atoms.

(c) The magnetic moment of the electron.

(d) The spread in measurement of the π^0 mass, given that the π^0 lifetime is τ.

(e) The magnetic field B at which there is a 10^{-4} excess of free protons in one spin direction at a temperature T.

(f) Fine structure splitting in the $n = 2$ state of hydrogen.

(*Columbia*)

Solution:

(a)

$$E_I = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{m_e}{2\hbar^2} ,$$

ε_0 being the permittivity of free space.

(b) The difference of frequency is caused by the Rydberg constant changing with the mass of the nucleus. The wave number of the α line of hydrogen atom is

$$\tilde{\nu}_H = R_H \left(1 - \frac{1}{4}\right) = \frac{3}{4} R_H ,$$

and that of the α line of deuterium atom is

$$\tilde{\nu}_D = \frac{3}{4} R_D .$$

The Rydberg constant is given by

$$R = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{m_r}{m_e} = \frac{m_r}{m_e} R_\infty \,,$$

where m_r is the reduced mass of the orbiting electron in the atomic system, and

$$R_\infty = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{m_e}{4\pi\hbar^3 c} \,.$$

As for H atom, $m_r = \frac{m_p m_e}{m_p + m_e}$, and for D atom,

$$m_r = \frac{2m_p m_e}{2m_p + m_e} \,,$$

m_p being the nucleon mass, we have

$$\Delta\nu = c\Delta\tilde{\nu} = \frac{3}{4}c(R_D - R_H) = \frac{3}{4}cR_\infty \left(\frac{1}{1+\dfrac{m_e}{2m_p}} - \frac{1}{1+\dfrac{m_e}{m_p}}\right)$$

$$\approx \frac{3}{4}cR_\infty \frac{m_e}{2m_p} = \frac{3}{4}\left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{\pi^2}{h^3}\frac{m_e^2}{m_p} \,.$$

(c) The magnetic moment associated with the electron spin is

$$\mu_e = \frac{he}{4\pi m_e} = \mu_B \,,$$

μ_B being the Bohr magneton.

(d) The spread in the measured mass (in energy units) is related to the lifetime τ through the uncertainty principle

$$\Delta E \cdot \tau \gtrsim \hbar \,,$$

which gives

$$\Delta E \gtrsim \frac{\hbar}{\tau} \,.$$

(e) Consider the free protons as an ideal gas in which the proton spins have two quantized directions: parallel to B with energy $E_p = -\mu_p B$ and antiparallel to B with energy $E_p = \mu_p B$, where $\mu_p = \frac{\hbar e}{2m_p}$ is the magnetic moment of proton. As the number density $n \propto \exp(\frac{-E_p}{kT})$, we have

$$\frac{\exp\left(\frac{\mu_p B}{kT}\right) - \exp\left(\frac{-\mu_p B}{kT}\right)}{\exp\left(\frac{\mu_p B}{kT}\right) + \exp\left(\frac{-\mu_p B}{kT}\right)} = 10^{-4},$$

or

$$\exp\left(\frac{2\mu_p B}{kT}\right) = \frac{1+10^{-4}}{1-10^{-4}},$$

giving

$$\frac{2\mu_p B}{kT} \approx 2 \times 10^{-4},$$

i.e.

$$B = \frac{kT}{\mu_p} \times 10^{-4}.$$

(f) The quantum numbers of $n = 2$ states are: $n = 2$, $l = 1$, $j_1 = 3/2$, $j_2 = 1/2$ (the $l = 0$ state does not split and so need not be considered here). From the expression for the fine-structure energy levels of hydrogen, we get

$$\Delta E = -\frac{2\pi Rhc\alpha^2}{n^3}\left(\frac{1}{j_1+\frac{1}{2}} - \frac{1}{j_2+\frac{1}{2}}\right) = \frac{\pi Rhc\alpha^2}{8},$$

where

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$$

is the fine structure constant,

$$R = \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \frac{m_e}{4\pi\hbar^3 c}$$

is the Rydberg constant.

1010

As shown in Fig. 1.2, light shines on sodium atoms. Estimate the cross-section on resonance for excitation of the atoms from the ground to the

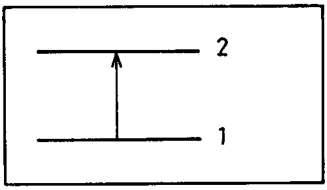

Fig. 1.2

first excited state (corresponding to the familiar yellow line). Estimate the width of the resonance. You need not derive these results from first principles if you remember the appropriate heuristic arguments.

(*Princeton*)

Solution:

The cross-section is defined by $\sigma_A = P_\omega / I_\omega$, where $P_\omega d\omega$ is the energy in the frequency range ω to $\omega+d\omega$ absorbed by the atoms in unit time, $I_\omega d\omega$ is the incident energy per unit area per unit time in the same frequency range. By definition,

$$\int P_\omega d\omega = B_{12} \hbar\omega N_\omega \,,$$

where B_{12} is Einstein's B-coefficient giving the probability of an atom in state 1 absorbing a quantum $\hbar\omega$ per unit time and $N_\omega d\omega$ is the energy density in the frequency range ω to $\omega + d\omega$. Einstein's relation

$$B_{12} = \frac{\pi^2 c^3}{\hbar\omega^3} \cdot \frac{g_1}{g_2} A_{21}$$

gives

$$B_{12} = \frac{\pi^2 c^3}{\hbar\omega^3} \cdot \frac{g_1}{g_2} \cdot \frac{1}{\tau} = \frac{\pi^2 c^3}{\hbar^2\omega^3} \cdot \frac{g_1}{g_2} \Gamma \,,$$

where τ is the lifetime of excited state 2, whose natural line width is $\Gamma \approx \frac{\hbar}{\tau}$, g_1, g_2 are respectively the degeneracies of states 1 and 2, use having been made of the relation $A_{12} = 1/\tau$ and the uncertainty principle $\Gamma\tau \approx \hbar$. Then as $N_\omega = I_\omega / c$, c being the velocity of light in free space, we have

$$P_\omega = \frac{\pi^2 c^2}{\hbar\omega^2} \cdot \frac{g_1}{g_2} \Gamma I_\omega \,.$$

Introducing the form factor $g(\omega)$ and considering ω and I_ω as average values in the band of $g(\omega)$, we can write the above as

$$P_\omega = \frac{\pi^2 c^2}{\hbar\omega^2} \cdot \frac{g_1}{g_2} \Gamma I_\omega g(\omega) .$$

Take for $g(\omega)$ the Lorentz profile

$$g(\omega) = \frac{\hbar}{2\pi} \frac{\Gamma}{(E_2 - E_1 - \hbar\omega)^2 + \dfrac{\Gamma^2}{4}} .$$

At resonance,

$$E_2 - E_1 = \hbar\omega ,$$

and so

$$g\left(\omega = \frac{E_2 - E_1}{\hbar}\right) = \frac{2\hbar}{\pi\Gamma} .$$

Hence

$$\sigma_A = \frac{\pi^2 c^2}{\hbar\omega^2} \cdot \frac{g_1}{g_2} \cdot \frac{2\hbar}{\pi} = \frac{2\pi c^2}{\omega^2} \cdot \frac{g_1}{g_2} .$$

For the yellow light of Na (D line), $g_1 = 2$, $g_2 = 6$, $\lambda = 5890$ Å, and

$$\sigma_A = \frac{1}{3} \cdot \frac{\lambda^2}{2\pi} = 1.84 \times 10^{-10}\ \text{cm}^2 .$$

For the D line of sodium, $\tau \approx 10^{-8}$ s and the line width at half intensity is

$$\Gamma \approx \frac{\hbar}{\tau} = 6.6 \times 10^{-8}\ \text{eV} .$$

As

$$\Gamma = \Delta E = \hbar\Delta\omega = \hbar\Delta\left(\frac{2\pi c}{\lambda}\right) = 2\pi\hbar c\Delta\tilde{\nu} ,$$

the line width in wave numbers is

$$\Delta\tilde{\nu} = \frac{\Gamma}{2\pi\hbar c} \approx \frac{1}{2\pi c\tau} = 5.3 \times 10^{-4}\ \text{cm}^{-1} .$$

1011

The cross section for electron impact excitation of a certain atomic level A is $\sigma_A = 1.4 \times 10^{-20}$ cm^2. The level has a lifetime $\tau = 2 \times 10^{-8}$ sec, and decays 10 per cent of the time to level B and 90 per cent of the time to level C (Fig. 1.3).

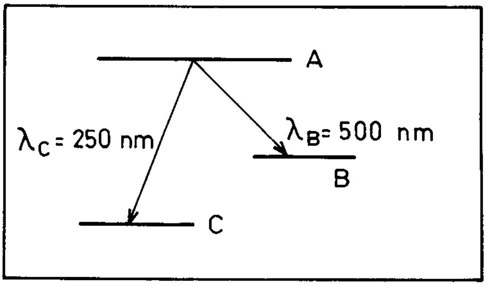

Fig. 1.3

(a) Calculate the equilibrium population per cm^3 in level A when an electron beam of 5 mA/cm^2 is passed through a vapor of these atoms at a pressure of 0.05 torr.

(b) Calculate the light intensity emitted per cm^3 in the transition $A \to B$, expressed in watts/steradian.

(*Wisconsin*)

Solution:

(a) According to Einstein's relation, the number of transitions B, $C \to A$ per unit time (rate of production of A) is

$$\frac{dN_{BC\to A}}{dt} = n_0 \sigma_A N_{BC} \, ,$$

and the number of decays $A \to B$, C per unit time is

$$\frac{dN_{A\to BC}}{dt} = \left(\frac{1}{\tau} + n_0 \sigma_A\right) N_A \, ,$$

where N_{BC} and N_A are the numbers of atoms in the energy levels B, C and A respectively, n_0 is the number of electrons crossing unit area per unit time. At equilibrium,

$$\frac{dN_{BC\to A}}{dt} = \frac{dN_{A\to BC}}{dt} \, ,$$

giving

$$N_A = \frac{n_0 \sigma_A N}{\frac{1}{\tau} + 2 n_0 \sigma_A} \approx n_0 \sigma_A N \tau \, , \qquad (N = N_A + N_{BC})$$

as $n_0 = 5 \times 10^{-3}/1.6 \times 10^{-19} = 3.1 \times 10^{16}$ cm^{-2} s^{-1} and so $\frac{1}{\tau} \gg 2 n_0 \sigma_A$.

Hence the number of atoms per unit volume in energy level A at equilibrium is

$$\begin{aligned} n &= \frac{N_A}{V} = \frac{\tau n_0 \sigma_A N}{V} = \frac{\tau n_0 \sigma_A p}{kT} \\ &= 2 \times 10^{-8} \times 3.1 \times 10^{16} \times 1.4 \times 10^{-20} \times \frac{0.05 \times 1.333 \times 10^3}{1.38 \times 10^{-16} \times 300} \\ &= 1.4 \times 10^4 \text{ cm}^{-3} , \end{aligned}$$

where we have taken the room temperature to be $T = 300$ K.

(b) The probability of atomic decay $A \to B$ is

$$\lambda_1 = \frac{0.1}{\tau} .$$

The wavelength of the radiation emitted in the transition $A \to B$ is given as $\lambda_B = 500$ nm. The corresponding light intensity I per unit volume per unit solid angle is then given by

$$4\pi I = n\lambda_1 hc/\lambda_B ,$$

i.e.,

$$\begin{aligned} I &= \frac{nhc}{40\pi\tau\lambda_B} = \frac{1.4 \times 10^4 \times 6.63 \times 10^{-27} \times 3 \times 10^{10}}{40\pi \times 2 \times 10^{-8} \times 500 \times 10^{-7}} \\ &= 2.2 \times 10^{-2} \text{ erg} \cdot \text{s}^{-1} \text{ sr}^{-1} = 2.2 \times 10^{-9} \text{ W sr}^{-1} . \end{aligned}$$

1012

The electric field that an atom experiences from its surroundings within a molecule or crystal can noticeably affect properties of the atomic ground state. An interesting example has to do with the phenomenon of angular momentum quenching in the iron atom of the hem group in the hemoglobin of your blood. Iron and hemoglobin are too complicated, however. So consider an atom containing one valence electron moving in a central atomic potential. It is in an $l = 1$ level. Ignore spin. We ask what happens to this level when the electron is acted on by the external potential arising from the atom's surroundings. Take this external potential to be

$$V_{\text{pert}} = Ax^2 + By^2 - (A+B)z^2$$

(the atomic nucleus is at the origin of coordinates) and treat it to lowest order.

(a) The $l = 1$ level now splits into three distinct levels. As you can confirm (and as we hint at) each has a wave function of the form

$$\Psi = (\alpha x + \beta y + \gamma z) f(r)\,,$$

where $f(r)$ is a common central function and where each level has its own set of constants (α, β, γ), which you will need to determine. Sketch the energy level diagram, specifying the *relative* shifts ΔE in terms of the parameters A and B (i.e., compute the three shifts up to a common factor).

(b) More interesting: Compute the expectation value of L_z, the z component of angular momentum, for each of the three levels.

(*Princeton*)

Solution:

(a) The external potential field V can be written in the form

$$V = \frac{1}{2}(A+B)r^2 - \frac{3}{2}(A+B)z^2 + \frac{1}{2}(A-B)(x^2-y^2)\,.$$

The degeneracy of the state $n = 2$, $l = 1$ is 3 in the absence of perturbation, with wave functions

$$\Psi_{210} = \left(\frac{1}{32\pi a^3}\right)^{\frac{1}{2}} \frac{r}{a} \exp\left(-\frac{r}{2a}\right) \cos\theta\,,$$

$$\Psi_{21\pm 1} = \mp \left(\frac{1}{64\pi a^3}\right)^{\frac{1}{2}} \frac{r}{a} \exp\left(-\frac{r}{2a}\right) \exp(\pm i\varphi) \sin\theta\,,$$

where $a = \hbar^2/\mu e^2$, μ being the reduced mass of the valence electron.

After interacting with the external potential field V, the wave functions change to

$$\Psi = a_1\Psi_{211} + a_2\Psi_{21-1} + a_3\Psi_{210}\,.$$

Perturbation theory for degenerate systems gives for the perturbation energy E' the following matrix equation:

$$\begin{pmatrix} C+A'-E' & B' & 0 \\ B' & C+A'-E' & 0 \\ 0 & 0 & C+3A'-E' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_2 \end{pmatrix} = 0\,,$$

where

$$\begin{aligned} C &= \langle \Psi_{211}|\frac{1}{2}(A+B)r^2|\Psi_{211}\rangle \\ &= \langle \Psi_{21-1}|\frac{1}{2}(A+B)r^2|\Psi_{21-1} \\ &= \langle \Psi_{210}|\frac{1}{2}(A+B)r^2|\Psi_{210}\rangle \\ &= 15a^2(A+B)\,, \\ A' &= -\langle \Psi_{211}|\frac{3}{2}(A+B)z^2|\Psi_{211}\rangle \\ &= -\langle \Psi_{21-1}|\frac{3}{2}(A+B)z^2|\Psi_{21-1}\rangle \\ &= -\frac{1}{3}\langle \Psi_{210}|\frac{3}{2}(A+B)z^2|\Psi_{210}\rangle \\ &= -9a^2(A+B)\,, \\ B' &= \langle \Psi_{211}|\frac{1}{2}(A-B)(x^2-y^2)|\Psi_{21-1}\rangle \\ &= \langle \Psi_{21-1}|\frac{1}{2}(A-B)(x^2-y^2)|\Psi_{211}\rangle \\ &= -\frac{3}{2}a^2(A-B)\,. \end{aligned}$$

Setting the determinant of the coefficients to zero, we find the energy corrections

$$E' = C+3A'\,, C+A' \pm B'\,.$$

For $E' = C+3A' = -12(A+B)a^2$, the wave function is

$$\Psi_1 = \Psi_{210} = \left(\frac{1}{32\pi a^3}\right)^{\frac{1}{2}} \frac{r}{a}\exp\left(-\frac{r}{2a}\right)\cos\theta = f(r)z\,,$$

where

$$f(r) = \left(\frac{1}{32\pi a^3}\right)^{\frac{1}{2}} \frac{1}{a} \cdot \exp\left(-\frac{r}{2a}\right) ,$$

corresponding to $\alpha = \beta = 0$, $\gamma = 1$.

For $E' = C + A' + B' = \frac{3}{2}(5A + 3B)a^2$, the wave function is

$$\Psi_2 = \frac{1}{\sqrt{2}}(\Psi_{211} + \Psi_{21-1}) = -i\left(\frac{1}{32\pi a^3}\right)^{\frac{1}{2}} \frac{r}{a} \exp\left(-\frac{r}{2a}\right) \sin\theta \sin\varphi$$
$$= -if(r)y ,$$

corresponding to $\alpha = \gamma = 0$, $\beta = -i$.

For $E' = C + A' - B' = \frac{3}{2}(3A + 5B)a^2$, the wave function is

$$\Psi_3 = \frac{1}{\sqrt{2}}(\Psi_{211} - \Psi_{21-1}) = -f(r)x ,$$

corresponding to $\alpha = -1$, $\beta = \gamma = 0$.

Thus the unperturbed energy level E_2 is, on the application of the perturbation V, split into three levels:

$$E_2 - 12(A + B)a^2 , \quad E_2 + \frac{3}{2}(3A + 5B)a^2 , \quad E_2 + \frac{3}{2}(5A + 3B)a^2 ,$$

as shown in Fig. 1.4.

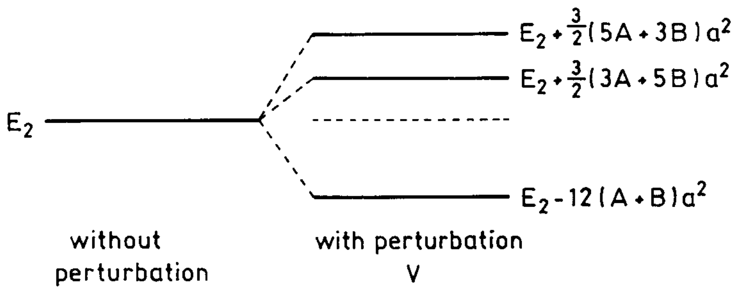

Fig. 1.4

(b) The corrected wave functions give

$$\langle\Psi_1|l_z|\Psi_1\rangle = \langle\Psi_2|l_z|\Psi_2\rangle = \langle\Psi_3|l_z|\Psi_3\rangle = 0 .$$

Hence the expectation value of the z component of angular momentum is zero for all the three energy levels.

1013

The Thomas-Fermi model of atoms describes the electron cloud in an atom as a continuous distribution $\rho(x)$ of charge. An individual electron is assumed to move in the potential determined by the nucleus of charge Ze and of this cloud. Derive the equation for the electrostatic potential in the following stages.

(a) By assuming the charge cloud adjusts itself locally until the electrons at Fermi sphere have zero energy, find a relation between the potential ϕ and the Fermi momentum p_F.

(b) Use the relation derived in (a) to obtain an algebraic relation between the charge density $\rho(x)$ and the potential $\phi(x)$.

(c) Insert the result of (b) in Poisson's equation to obtain a nonlinear partial differential equation for ϕ.

(Princeton)

Solution:

(a) For a bound electron, its energy $E = \frac{p^2}{2m} - e\phi(\mathbf{x})$ must be lower than that of an electron on the Fermi surface. Thus

$$\frac{p_{\max}^2}{2m} - e\phi(\mathbf{x}) = 0 \,,$$

where $p_{\max} = p_f$, the Fermi momentum.

Hence

$$p_f^2 = 2me\phi(\mathbf{x}) \,.$$

(b) Consider the electrons as a Fermi gas. The number of electrons filling all states with momenta 0 to p_f is

$$N = \frac{Vp_f^3}{3\pi^2\hbar^3} \,.$$

The charge density is then

$$\rho(\mathbf{x}) = \frac{eN}{V} = \frac{ep_f^3}{3\pi^2\hbar^3} = \frac{e}{3\pi^2\hbar^3}[2me\phi(\mathbf{x})]^{\frac{3}{2}} \,.$$

(c) Substituting $\rho(\mathbf{x})$ in Poisson's equation

$$\nabla^2\phi = 4\pi\rho(\mathbf{x})$$

gives

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi(\mathbf{x}) = \frac{4e}{3\pi\hbar^3}[2me\phi(\mathbf{x})]^{\frac{3}{2}} .$$

On the assumption that ϕ is spherically symmetric, the equation reduces to

$$\frac{1}{r}\frac{d^2}{dr^2}[r\phi(r)] = \frac{4e}{3\pi\hbar^3}[2me\phi(r)]^{\frac{3}{2}} .$$

1014

In a crude picture, a metal is viewed as a system of free electrons enclosed in a well of potential difference V_0. Due to thermal agitation, electrons with sufficiently high energies will escape from the well. Find and discuss the emission current density for this model.

(*SUNY, Buffalo*)

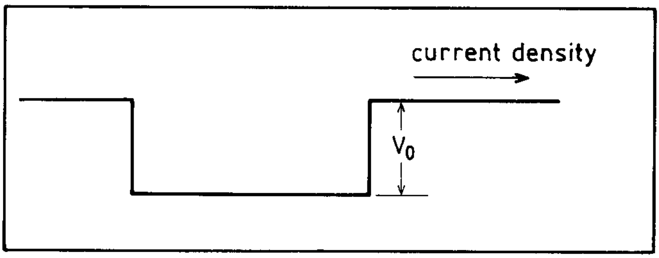

Fig. 1.5

Solution:

The number of states in volume element $dp_x dp_y dp_z$ in the momentum space is $dN = \frac{2}{h^3}dp_x dp_y dp_z$. Each state ε has degeneracy $\exp(-\frac{\varepsilon-\mu}{kT})$, where ε is the energy of the electron and μ is the Fermi energy.

Only electrons with momentum component $p_z > (2mV_0)^{1/2}$ can escape from the potential well, the z-axis being selected parallel to the outward normal to the surface of the metal. Hence the number of electrons escaping from the volume element in time interval dt is

$$dN' = Av_z dt \frac{2}{h^3} dp_x dp_y dp_z \exp\left(-\frac{\varepsilon - \mu}{kT}\right) ,$$

where v_z is the velocity component of the electrons in the z direction which satisfies the condition $mv_z > (2mV_0)^{1/2}$, A is the area of the surface of the metal. Thus the number of electrons escaping from the metal surface per unit area per unit time is

$$\begin{aligned}
R &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{(2mV_0)^{1/2}}^{+\infty} \frac{2v_z}{h^3}\exp\left(-\frac{\varepsilon-\mu}{kT}\right) dp_x dp_y dp_z \\
&= \frac{2}{mh^3}\exp\left(\frac{\mu}{kT}\right)\int_{-\infty}^{+\infty}\exp\left(-\frac{p_x^2}{2mkT}\right) dp_x \int_{-\infty}^{+\infty}\exp\left(-\frac{p_y^2}{2mkT}\right) dp_y \\
&\quad \times \int_{(2mV_0)^{1/2}}^{+\infty} p_z \exp\left(-\frac{p_z^2}{2mkT}\right) dp_z \\
&= \frac{4\pi mk^2T^2}{h^3}\exp\left(\frac{\mu - V_0}{kT}\right) ,
\end{aligned}$$

and the emission current density is

$$J = -eR = -\frac{4\pi mek^2T^2}{h^3}\exp\left(\frac{\mu - V_0}{kT}\right) ,$$

which is the Richardson–Dushman equation.

1015

A narrow beam of neutral particles with spin 1/2 and magnetic moment μ is directed along the x-axis through a "Stern-Gerlach" apparatus, which splits the beam according to the values of μ_z in the beam. (The apparatus consists essentially of magnets which produce an inhomogeneous field $B_z(z)$ whose force on the particle moments gives rise to displacements Δz proportional to $\mu_z B_z$.)

(a) Describe the pattern of splitting for the cases:

(i) Beam polarized along $+z$ direction.

(ii) Beam polarized along $+x$ direction.

(iii) Beam polarized along $+y$ direction.

(iv) Beam unpolarized.

(b) For those cases, if any, with indistinguishable results, describe how one might distinguish among these cases by further experiments which use the above Stern-Gerlach apparatus and possibly some additional equipment.

(*Columbia*)

Solution:

(a) (i) The beam polarized along $+z$ direction is not split, but its direction is changed.

(ii) The beam polarized along $+x$ direction splits into two beams, one deflected to $+z$ direction, the other to $-z$ direction.

(iii) Same as for (ii).

(iv) The unpolarized beam is split into two beams, one deflected to $+z$ direction, the other to $-z$ direction.

(b) The beams of (ii) (iii) (iv) are indistinguishable. They can be distinguished by the following procedure.

(1) Turn the magnetic field to $+y$ direction. This distinguishes (iii) from (ii) and (iv), as the beam in (iii) is not split but deflected, while the beams of (ii) and (iv) each splits into two.

(2) Put a reflector in front of the apparatus, which changes the relative positions of the source and apparatus (Fig. 1.6). Then the beam of (ii) does not split, though deflected, while that of (iv) splits into two.

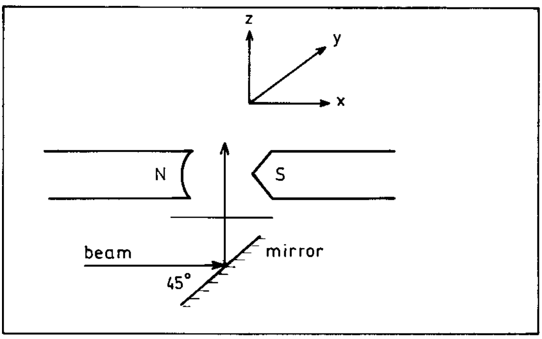

Fig. 1.6

1016

The range of the potential between two hydrogen atoms is approximately 4 Å. For a gas in thermal equilibrium, obtain a numerical estimate of the temperature below which the atom-atom scattering is essentially s-wave.

(MIT)

Solution:

The scattered wave is mainly s-wave when $ka \leq 1$, where a is the interaction length between hydrogen atoms, k the de Broglie wave number

$$k = \frac{p}{\hbar} = \frac{\sqrt{2mE_k}}{\hbar} = \frac{\sqrt{2m \cdot \frac{3}{2} k_B T}}{\hbar} = \frac{\sqrt{3mk_B T}}{\hbar},$$

where p is the momentum, E_k the kinetic energy, and m the mass of the hydrogen atom, and k_B is the Boltzmann constant. The condition

$$ka = \sqrt{3mk_B T} \cdot \frac{a}{\hbar} \leq 1$$

gives

$$T \leq \frac{\hbar^2}{3mk_B a^2} = \frac{(1.06 \times 10^{-34})^2}{3 \times 1.67 \times 10^{-27} \times 1.38 \times 10^{-23} \times (4 \times 10^{-10})^2}$$
$$\approx 1 \text{ K}$$

1017

(a) If you remember it, write down the differential cross section for Rutherford scattering in cm^2/sr. If you do not remember it, say so, and write down your best guess. Make sure that the Z dependence, energy dependence, angular dependence and dimensions are "reasonable". Use the expression you have just given, whether correct or your best guess, to evaluate parts (b–e) below.

An accelerator supplies a proton beam of 10^{12} particles per second and 200 MeV/c momentum. This beam passes through a 0.01-cm aluminum window. (Al density $\rho = 2.7$ gm/cm^3, Al radiation length $x_0 = 24$ gm/cm^2, $Z = 13$, $A = 27$).

(b) Compute the differential Rutherford scattering cross section in cm^2/sr at 30° for the above beam in Al.

(c) How many protons per second will enter a 1-cm radius circular counter at a distance of 2 meters and at an angle of 30° with the beam direction?

(d) Compute the integrated Rutherford scattering cross section for angles greater than 5°. (Hint: $\sin\theta d\theta = 4\sin\frac{\theta}{2}\cos\frac{\theta}{2}d\frac{\theta}{2}$)

(e) How many protons per second are scattered out of the beam into angles $> 5°$?

(f) Compute the projected rms multiple Coulomb scattering angle for the proton beam through the above window. Take the constant in the expression for multiple Coulomb scattering as 15 MeV/c.

(*UC, Berkeley*)

Solution:

(a) The differential cross section for Rutherford scattering is

$$\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2}{2mv^2}\right)^2 \left(\sin\frac{\theta}{2}\right)^{-4} .$$

This can be obtained, to a dimensionless constant, if we remember

$$\frac{d\sigma}{d\Omega} \sim \left(\sin\frac{\theta}{2}\right)^{-4} ,$$

and assume that it depends also on ze, Ze and $E = \frac{1}{2}mv^2$.

Let

$$\frac{d\sigma}{d\Omega} = K(zZe^2)^x E^y \left(\sin\frac{\theta}{2}\right)^{-4} ,$$

where K is a dimensionless constant. Dimensional analysis then gives

$$[L]^2 = (e^2)^x E^y .$$

As

$$\left[\frac{e^2}{r}\right] = [E] ,$$

the above gives

$$x = 2, y = -x = -2\,.$$

(b) For the protons,

$$\beta \equiv \frac{v}{c} = \frac{pc}{\sqrt{m^2c^4 + p^2c^2}} = \frac{200}{\sqrt{938^2 + 200^2}} = 0.2085\,.$$

We also have

$$\frac{e^2}{mv^2} = r_0 \left(\frac{m_e}{m}\right) \left(\frac{v}{c}\right)^{-2}\,,$$

where $r_0 = \frac{e^2}{m_e c^2} = 2.82 \times 10^{-13}$ cm is the classical radius of electron. Hence at $\theta = 30°$,

$$\begin{aligned}
\frac{d\sigma}{d\Omega} &= \left(\frac{13}{2}\right)^2 r_0^2 \left(\frac{m_e}{m}\right)^2 \left(\frac{v}{c}\right)^{-4} \left(\sin\frac{\theta}{2}\right)^{-4} \\
&= \left(\frac{6.5 \times 2.82 \times 10^{-13}}{1836 \times 0.2085^2}\right)^2 \times (\sin 15°)^{-4} \\
&= 5.27 \times 10^{-28} \times (\sin 15°)^{-4} = 1.18 \times 10^{-25}\ \text{cm}^2/\text{sr}\,.
\end{aligned}$$

(c) The counter subtends a solid angle

$$d\Omega = \frac{\pi(0.01)^2}{2^2} = 7.85 \times 10^{-5}\ \text{sr}\,.$$

The number of protons scattered into it in unit time is

$$\begin{aligned}
\delta n &= n \left(\frac{\rho t}{27}\right) A_v \left(\frac{d\sigma}{d\Omega}\right) \delta\Omega \\
&= 10^{12} \times \left(\frac{2.7 \times 0.01}{27}\right) \times 6.02 \times 10^{23} \times 1.18 \times 10^{-25} \times 7.85 \times 10^{-5} \\
&= 5.58 \times 10^3\ \text{s}^{-1}\,.
\end{aligned}$$

(d)

$$\sigma_I = \int \frac{d\sigma}{d\Omega} d\Omega = 2\pi \int_{5^\circ}^{180^\circ} \left(\frac{Ze^2}{2mv^2}\right)^2 \frac{\sin\theta}{\sin^4\dfrac{\theta}{2}} d\theta$$

$$= 8\pi \left(\frac{Ze^2}{2mv^2}\right)^2 \int_{5^\circ}^{180^\circ} \left(\sin\frac{\theta}{2}\right)^{-3} d\sin\frac{\theta}{2}$$

$$= 4\pi \left(\frac{Ze^2}{2mv^2}\right)^2 \left[\frac{1}{(\sin 2.5^\circ)^2} - 1\right]$$

$$= 4\pi \times 5.27 \times 10^{-28} \times \left[\frac{1}{(\sin 2.5^\circ)^2} - 1\right]$$

$$= 3.47 \times 10^{-24}\ \text{cm}^2 .$$

(e) The number of protons scattered into $\theta \geq 5^\circ$ is

$$\delta n = n \left(\frac{\rho t}{27}\right) A_v \sigma_I = 2.09 \times 10^9\ \text{s}^{-1} ,$$

where $A_v = 6.02 \times 10^{23}$ is Avogadro's number.

(f) The projected rms multiple Coulomb scattering angle for the proton beam through the Al window is given by

$$\theta_{\text{rms}} = \frac{kZ}{\sqrt{2}\beta p} \sqrt{\frac{t}{x_0}} \left[1 + \frac{1}{9} \ln\left(\frac{t}{x_0}\right)\right] ,$$

where k is a constant equal to 15 MeV/c. As $Z = 13$, $p = 200$ MeV/c, $\beta = 0.2085$, $t = 0.01 \times 2.7$ g cm^{-2}, $x_0 = 24$ g cm^{-2}, $t/x_0 = 1.125 \times 10^{-3}$, we have

$$\theta_{\text{rms}} = \frac{15 \times 13}{\sqrt{2} \times 0.2085 \times 200} \times \sqrt{1.125 \times 10^{-3}} \left[1 + \frac{1}{9} \ln(1.125 \times 10^{-3})\right]$$

$$= 2.72 \times 10^{-2}\ \text{rad} .$$

1018

Typical lifetime for an excited atom is $10^{-1}, 10^{-8}, 10^{-13}, 10^{-23}$ sec.

(*Columbia*)

Solution:

The answer is 10^{-8} s.

1019

An atom is capable of existing in two states: a ground state of mass M and an excited state of mass $M + \Delta$. If the transition from ground to excited state proceeds by the absorption of a photon, what must be the photon frequency in the laboratory where the atom is initially at rest?

(Wisconsin)

Solution:

Let the frequency of the photon be ν and the momentum of the atom in the excited state be p. The conservation laws of energy and momentum give

$$Mc^2 + h\nu = [(M+\Delta)^2c^4 + p^2c^2]^{1/2}\,,$$

$$\frac{h\nu}{c} = p\,,$$

and hence

$$\nu = \frac{\Delta c^2}{h}\left(1 + \frac{\Delta}{2M}\right)\,.$$

1020

If one interchanges the spatial coordinates of two electrons in a state of total spin 0:

(a) the wave function changes sign,
(b) the wave function is unchanged,
(c) the wave function changes to a completely different function.

(CCT)

Solution:

The state of total spin zero has even parity, i.e., spatial symmetry. Hence the wave function does not change when the space coordinates of the electrons are interchanged.

So the answer is (b).

1021

The Doppler width of an optical line from an atom in a flame is 10^6, 10^9, 10^{13}, 10^{16} Hz.

(*Columbia*)

Solution:

Recalling the principle of equipartition of energy $m\overline{v^2}/2 = 3kT/2$ we have for hydrogen at room temperature $mc^2 \approx 10^9$ eV, $T = 300$ K, and so

$$\beta = \frac{v}{c} \approx \frac{\sqrt{\overline{v^2}}}{c} = \sqrt{\frac{3kT}{mc^2}} \sim 10^{-5}\,,$$

where $k = 8.6 \times 10^{-5}$ eV/K is Boltzmann's constant.

The Doppler width is of the order

$$\Delta\nu \approx \nu_0\beta\,.$$

For visible light, $\nu_0 \sim 10^{14}$ Hz. Hence $\Delta\nu \sim 10^9$ Hz.

1022

Estimate (order of magnitude) the Doppler width of an emission line of wavelength $\lambda = 5000$ Å emitted by argon $A = 40$, $Z = 18$, at $T = 300$ K.

(*Columbia*)

Solution:

The principle of equipartition of energy $\frac{1}{2}m\bar{v}^2 = \frac{3}{2}kT$ gives

$$v \approx \sqrt{\overline{v^2}} = c\sqrt{\frac{3kT}{mc^2}}$$

with $mc^2 = 40 \times 938$ MeV, $kT = 8.6 \times 10^{-5} \times 300 = 2.58 \times 10^{-2}$ eV. Thus

$$\beta = \frac{v}{c} = 1.44 \times 10^{-6}$$

and the (full) Doppler width is

$$\Delta\lambda \approx 2\beta\lambda = 1.44 \times 10^{-2}\ \text{Å}\,.$$

1023

Typical cross section for low-energy electron-atom scattering is 10^{-16}, 10^{-24}, 10^{-32}, 10^{-40} cm^2.

(*Columbia*)

Solution:

The linear dimension of an atom is of the order 10^{-8} cm, so the cross section is of the order $(10^{-8})^2 = 10^{-16}$ cm^2.

1024

An electron is confined to the interior of a hollow spherical cavity of radius R with impenetrable walls. Find an expression for the pressure exerted on the walls of the cavity by the electron in its ground state.

(*MIT*)

Solution:

Suppose the radius of the cavity is to increase by dR. The work done by the electron in the process is $4\pi R^2 PdR$, causing a decrease of its energy by dE. Hence the pressure exerted by the electron on the walls is

$$P = -\frac{1}{4\pi R^2}\frac{dE}{dR}\,.$$

For the electron in ground state, the angular momentum is 0 and the wave function has the form

$$\Psi = \frac{1}{\sqrt{4\pi}}\frac{\chi(r)}{r}\,,$$

where $\chi(r)$ is the solution of the radial part of Schrödinger's equation,

$$\chi''(r) + k^2\chi(r) = 0\,,$$

with $k^2 = 2mE/\hbar^2$ and $\chi(r) = 0$ at $r = 0$. Thus

$$\chi(r) = A\sin kr\,.$$

As the walls cannot be penetrated, $\chi(r) = 0$ at $r = R$, giving $k = \pi/R$. Hence the energy of the electron in ground state is

$$E = \frac{\pi^2\hbar^2}{2mR^2}\,,$$

and the pressure is

$$P = -\frac{1}{4\pi R^2}\frac{dE}{dR} = \frac{\pi\hbar^2}{4mR^5}\,.$$

1025

A particle with magnetic moment $\boldsymbol{\mu} = \mu_0 \mathbf{s}$ and spin $\mathbf{s}$ of magnitude 1/2 is placed in a constant magnetic field $\mathbf{B}$ pointing along the x-axis. At $t = 0$, the particle is found to have $s_z = +1/2$. Find the probabilities at any later time of finding the particle with $s_y = \pm 1/2$.

(*Columbia*)

Solution:

In the representation $(\mathbf{s}^2, s_x)$, the spin matrices are

$$\sigma_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

with eigenfunctions $\binom{1}{0}$, $\binom{1}{1}$, $\binom{1}{i}$ respectively. Thus the Hamiltonian of interaction between the magnetic moment of the particle and the magnetic field is

$$H = -\boldsymbol{\mu} \cdot \mathbf{B} = -\frac{\mu_0 B}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the Schrödinger equation is

$$i\hbar \frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = -\frac{\mu_0 B}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix},$$

where $\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$ is the wave function of the particle at time t. Initially we have $\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$, and so the solution is

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp\left(i\dfrac{\mu_0 B t}{2\hbar}\right) \\ i\exp\left(-i\dfrac{\mu_0 B t}{2\hbar}\right) \end{pmatrix}.$$

Hence the probability of the particle being in the state $s_y = +1/2$ at time t is

$$\left|\frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}a(t)\\b(t)\end{pmatrix}\right|^2 = \frac{1}{4}\left|\exp\left(i\frac{\mu_0 Bt}{2\hbar}\right) + i\exp\left(-i\frac{\mu_0 Bt}{2\hbar}\right)\right|^2$$

$$= \frac{1}{2}\left(1 + \sin\frac{\mu_0 Bt}{\hbar}\right).$$

Similarly, the probability of the particle being in the state $s_y = -1/2$ at time t is $\frac{1}{2}(1 - \sin\frac{\mu_0 Bt}{\hbar})$.

1026

The ground state of the realistic helium atom is of course nondegenerate. However, consider a hypothetical helium atom in which the two electrons are replaced by two identical spin-one particles of negative charge. Neglect spin-dependent forces. For this hypothetical atom, what is the degeneracy of the ground state? Give your reasoning.

(*CUSPEA*)

Solution:

Spin-one particles are bosons. As such, the wave function must be symmetric with respect to interchange of particles. Since for the ground state the spatial wave function is symmetric, the spin part must also be symmetric. For two spin-1 particles the total spin S can be 2, 1 or 0. The spin wave functions for $S = 2$ and $S = 0$ are symmetric, while that for $S = 1$ is antisymmetric. Hence for ground state we have $S = 2$ or $S = 0$, the total degeneracy being

$$(2 \times 2 + 1) + (2 \times 0 + 1) = 6\,.$$

1027

A beam of neutrons (mass m) traveling with nonrelativistic speed v impinges on the system shown in Fig. 1.7, with beam-splitting mirrors at corners B and D, mirrors at A and C, and a neutron detector at E. The corners all make right angles, and neither the mirrors nor the beam-splitters affect the neutron spin. The beams separated at B rejoin coherently at D, and the detector E reports the neutron intensity I.

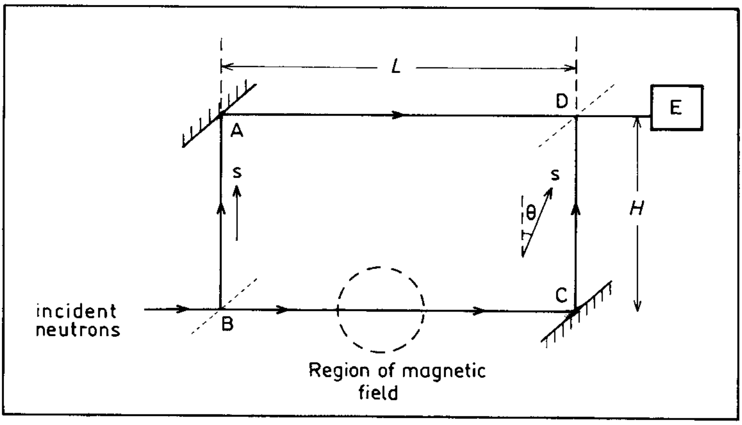

Fig. 1.7

(a) In this part of the problem, assume the system to be in a vertical plane (so gravity points down parallel to AB and DC). Given that detector intensity was I_0 with the system in a horizontal plane, derive an expression for the intensity I_g for the vertical configuration.

(b) For this part of the problem, suppose the system lies in a horizontal plane. A uniform magnetic field, pointing out of the plane, acts in the dotted region indicated which encompasses a portion of the leg BC. The incident neutrons are polarized with spin pointing along BA as shown. The neutrons which pass through the magnetic field region will have their spins pressed by an amount depending on the field strength. Suppose the spin expectation value presses through an angle θ as shown. Let $I(\theta)$ be the intensity at the detector E. Derive $I(\theta)$ as a function of θ, given that $I(\theta = 0) = I_0$.

(*Princeton*)

Solution:

(a) Assume that when the system is in a horizontal plane the two split beams of neutrons have the same intensity when they reach D, and so the wave functions will each have amplitude $\sqrt{I_0}/2$. Now consider the system in a vertical plane. As BA and CD are equivalent dynamically, they need not be considered. The velocities of neutrons v in BC and v_1 in AD are related through the energy equation

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_1^2 + mgH ,$$

giving

$$v_1 = \sqrt{v^2 - 2gH} .$$

When the two beams recombine at D, the wave function is

$$\Psi = \left[\frac{\sqrt{I_0}}{2}\exp\left(i\frac{mv_1}{\hbar}L\right) + \frac{\sqrt{I_0}}{2}\exp\left(i\frac{mv}{\hbar}L\right)\right]\exp\left(-i\frac{Et}{\hbar}\right)\exp(i\delta) ,$$

and the intensity is

$$I_g = |\Psi|^2 = \frac{I_0}{2} + \frac{I_0}{2}\cos\left[\frac{mL(v-v_1)}{\hbar}\right] = I_0\cos^2\left[\frac{mL(v-v_1)}{2\hbar}\right] .$$

If we can take $\frac{1}{2}mv^2 \gg mgH$, then $v_1 \approx v - \frac{gH}{v}$ and

$$I_g \approx I_0 \cos^2\left(\frac{mgHL}{2\hbar v}\right) .$$

(b) Take z-axis in the direction of BA and proceed in the representation of $(\mathbf{s}^2, s_z)$. At D the spin state is $\binom{1}{0}$ for neutrons proceeding along BAD and is $\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}$ for those proceeding along BCD. Recombination gives

$$\begin{aligned}\Psi &= \frac{\sqrt{I_0}}{2}\exp\left(-i\frac{Et}{\hbar}\right)\exp(i\delta)\left[\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix}\right] \\ &= \frac{\sqrt{I_0}}{2}\exp\left(-i\frac{Et}{\hbar}\right)\exp(i\delta)\begin{pmatrix}1+\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{pmatrix} ,\end{aligned}$$

and hence

$$I(\theta) = |\Psi|^2 = \frac{I_0}{4}\left[\left(1+\cos\frac{\theta}{2}\right)^2 + \sin^2\frac{\theta}{2}\right] = I_0\cos^2\frac{\theta}{4} .$$

1028

The fine structure of atomic spectral lines arises from

(a) electron spin-orbit coupling.
(b) interaction between electron and nucleus.
(c) nuclear spin.

(*CCT*)

Solution:

The answer is (a).

1029

Hyperfine splitting in hydrogen ground state is 10^{-7}, 10^{-5}, 10^{-3}, 10^{-1} eV.

(*Columbia*)

Solution:

For atomic hydrogen the experimental hyperfine line spacing is $\Delta\nu_{hf} = 1.42\times10^9\ \mathrm{s}^{-1}$. Thus $\Delta E = h\nu_{hf} = 4.14\times10^{-15}\times1.42\times10^9 = 5.9\times10^{-6}$ eV. So the answer is 10^{-5} eV.

1030

The hyperfine structure of hydrogen

(a) is too small to be detected.
(b) arises from nuclear spin.
(c) arises from finite nuclear size.

(*CCT*)

Solution:

The answer is (b).

1031

Spin-orbit splitting of the hydrogen $2p$ state is 10^{-6}, 10^{-4}, 10^{-2}, 10^0 eV.

(*Columbia*)

Solution:

For the $2p$ state of hydrogen atom, $n = 2$, $l = 1$, $s = 1/2$, $j_1 = 3/2$, $j_2 = 1/2$. The energy splitting caused by spin-orbit coupling is given by

$$\Delta E_{ls} = \frac{hcR\alpha^2}{n^3 l\left(l+\frac{1}{2}\right)(l+1)}\left[\frac{j_1(j_1+1)-j_2(j_2+1)}{2}\right],$$

where R is Rydberg's constant and $hcR = 13.6$ eV is the ionization potential of hydrogen atom, $\alpha = \frac{1}{137}$ is the fine-structure constant. Thus

$$\Delta E_{ls} = \frac{13.6 \times (137)^{-2}}{2^3 \times \frac{3}{2} \times 2} \times \frac{1}{2}\left(\frac{15}{4} - \frac{3}{4}\right) = 4.5 \times 10^{-5} \text{ eV}.$$

So the answer is 10^{-4} eV.

1032

The Lamb shift is

(a) a splitting between the $1s$ and $2s$ energy levels in hydrogen.
(b) caused by vacuum fluctuations of the electromagnetic field.
(c) caused by Thomas precession.

(CCT)

Solution:

The answer is (b)

1033

The average speed of an electron in the first Bohr orbit of an atom of atomic number Z is, in units of the velocity of light,

(a) $Z^{1/2}$.
(b) Z.
(c) Z/137.

(CCT)

Solution:

Let the average speed of the electron be v, its mass be m, and the radius of the first Bohr orbit be a. As

$$\frac{mv^2}{a} = \frac{Ze^2}{a^2}, \qquad a = \frac{\hbar^2}{mZe^2},$$

We have

$$v = \frac{Ze^2}{\hbar} = Zc\alpha,$$

where $\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}$ is the fine-structure constant. Hence the answer is (c).

1034

The following experiments were significant in the development of quantum theory. Choose TWO. In each case, briefly describe the experiment and state what it contributed to the development of the theory. Give an approximate date for the experiment.

(a) Stern-Gerlach experiment
(b) Compton Effect
(c) Franck-Hertz Experiment
(d) Lamb-Rutherford Experiment

(*Wisconsin*)

Solution:

(a) *Stern-Gerlach experiment.* The experiment was carried out in 1921 by Stern and Gerlach using apparatus as shown in Fig. 1.8. A highly collimated beam ($v \approx 500$ m/s) of silver atoms from an oven passes through the poles of a magnet which are so shaped as to produce an extremely non-uniform field (gradient of field $\sim 10^3$ T/m, longitudinal range ~ 4 cm) normal to the beam. The forces due to the interaction between the component μ_z of the magnetic moment in the field direction and the field gradient cause a deflection of the beam, whose magnitude depends on μ_z. Stern and Gerlach found that the beam split into two, rather than merely broadened, after crossing the field. This provided evidence for the space quantization of the angular momentum of an atom.

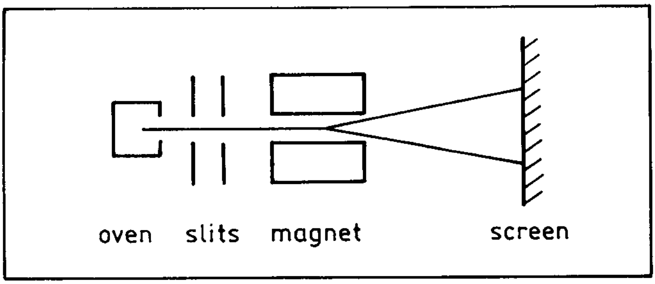

Fig. 1.8

(b) *Compton Effect.* A. H. Compton discovered that when monochromatic X-rays are scattered by a suitable target (Fig. 1.9), the scattered radiation consists of two components, one spectrally unchanged the other with increased wavelength. He further found that the change in wavelength of the latter is a function only of the scattering angle but is independent of the wavelength of the incident radiation and the scattering material. In 1923, using Einstein's hypothesis of light quanta and the conservation of momentum and energy, Compton found a relation between the change of wavelength and the scattering angle, $\Delta\lambda = \frac{h}{m_e c}(1-\cos\theta)$, which is in excellent agreement with the experimental results. Compton effect gives direct support to Einstein's theory of light quanta.

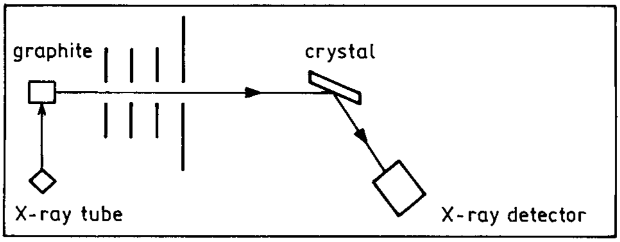

Fig. 1.9

(c) *Franck-Hertz experiment.* Carried out by Franck and Hertz in 1914, this experiment proved Bohr's theory of quantization of atomic energy states as well as provided a method to measure the energy spacing of quantum states. The experimental setup was as shown in Fig. 1.10. A glass

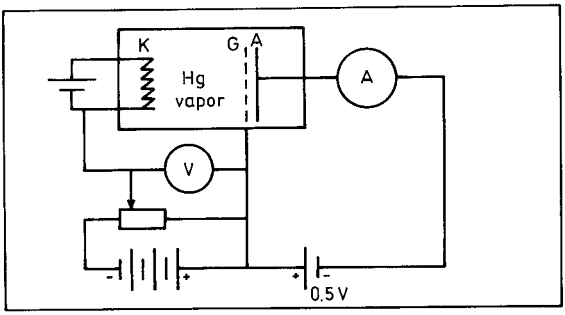

Fig. 1.10

vessel, filled with Hg vapor, contained cathode K, grid G and anode A. Thermoelectrons emitted from K were accelerated by an electric field to G, where a small retarding field prevented low energy electrons from reaching A. It was observed that the electric current detected by the ammeter A first increased with the accelerating voltage until it reached 4.1 V. Then the current dropped suddenly, only to increase again. At the voltages 9.0 V and 13.9 V, similar phenomena occurred. This indicated that the electron current dropped when the voltage increased by 4.9 V (the first drop at 4.1 V was due to the contact voltage of the instrument), showing that 4.9 eV was the first excited state of Hg above ground. With further improvements in the instrumentation Franck and Hertz were able to observe the higher excited states of the atom.

(d) *Lamb-Rutherford Experiment.* In 1947, when Lamb and Rutherford measured the spectrum of H atom accurately using an RF method, they found it different from the predictions of Dirac's theory, which required states with the same (n, j) but different l to be degenerate. Instead, they found a small splitting. The result, known as the Lamb shift, is satisfactorily explained by the interaction between the electron with its radiation field. The experiment has been interpreted as providing strong evidence in support of quantum electrodynamics.

The experimental setup was shown in Fig. 1.11. Of the hydrogen gas contained in a stove, heated to temperature 2500 K, about 64% was ionized (average velocity 8×10^3 m/s). The emitted atomic beam collided at B with a transverse electron beam of energy slightly higher than 10.2 eV and were excited to $2^2S_{1/2}$, $2^2P_{1/2}$, $2^2P_{3/2}$ states. The atoms in the P

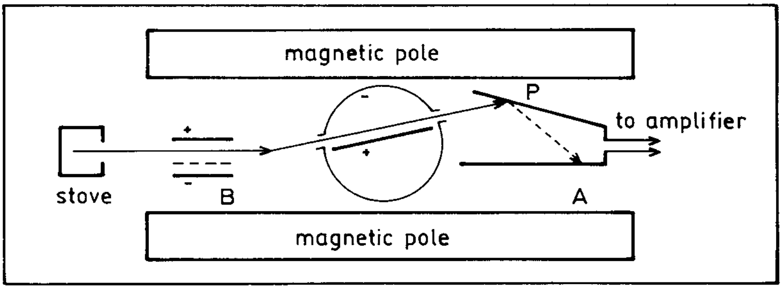

Fig. 1.11

states spontaneously underwent transition to the ground state $1^2S_{1/2}$ almost immediately whereas the $2^2S_{1/2}$ state, which is metastable, remained. Thus the atomic beam consisted of only $2^2S_{1/2}$ and $1^2S_{1/2}$ states when it impinged on the tungsten plate P. The work function of tungsten is less than 10.2 eV, so that the atoms in $2^2S_{1/2}$ state were able to eject electrons from the tungsten plate, which then flowed to A, resulting in an electric current between P and A, which was measured after amplification. The current intensity gave a measure of the numbers of atoms in the $2^2S_{1/2}$ state. A microwave radiation was then applied between the excitation and detection regions, causing transition of the $2^2S_{1/2}$ state to a P state, which almost immediately decayed to the ground state, resulting in a drop of the electric current. The microwave energy corresponding to the smallest electric current is the energy difference between the $2^2S_{1/2}$ and $2^2P_{1/2}$ states. Experimentally the frequency of Lamb shift was found to be 1057 MHz.

1035

(a) Derive from Coulomb's law and the simple quantization of angular momentum, the energy levels of the hydrogen atom.

(b) What gives rise to the doublet structure of the optical spectra from sodium?

(Wisconsin)

Solution:

(a) The Coulomb force between the electron and the hydrogen nucleus is

$$F = \frac{e^2}{4\pi\varepsilon_0 r^2} \, .$$

In a simplest model, the electron moves around the nucleus in a circular orbit of radius r with speed v, and its orbital angular momentum $p_\phi = mvr$ is quantized according to the condition

$$p_\phi = n\hbar \, ,$$

where $n = 1, 2, 3, \ldots$ and $\hbar = h/2\pi$, h being Planck's constant. For the electron circulating the nucleus, we have

$$m\frac{v^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2} \, ,$$

and so

$$v = \frac{e^2}{4\pi\varepsilon_0 n\hbar} \, .$$

Hence the quantized energies are

$$\begin{aligned} E_n = T + V &= \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r} = -\frac{1}{2}mv^2 \\ &= -\frac{1}{2}\frac{me^4}{(4\pi\varepsilon_0)^2\hbar^2 n^2} \, , \end{aligned}$$

with $n = 1, 2, 3, \ldots$.

(b) The doublet structure of the optical spectra from sodium is caused by the coupling between the orbital and spin angular momenta of the valence electron.

1036

We may generalize the semiclassical Bohr-Sommerfeld relation

$$\oint \mathbf{p} \cdot d\mathbf{r} = \left(n + \frac{1}{2}\right) 2\pi\hbar$$

(where the integral is along a closed orbit) to apply to the case where an electromagnetic field is present by replacing $\mathbf{p} \to \mathbf{p} - \frac{e\mathbf{A}}{c}$. Use this and the equation of motion for the linear momentum **p** to derive a quantized condition on the magnetic flux of a semiclassical electron which is in a magnetic field **B** in an arbitrary orbit. For electrons in a solid this condition can be restated in terms of the size S of the orbit in k-space. Obtain the quantization condition on S in terms of B. (Ignore spin effects)

(*Chicago*)

Solution:

Denote the closed orbit by C. Assume **B** is constant, then Newton's second law

$$\frac{d\mathbf{p}}{dt} = -\frac{e}{c}\frac{d\mathbf{r}}{dt} \times \mathbf{B}$$

gives

$$\oint_C \mathbf{p} \cdot d\mathbf{r} = -\frac{e}{c}\oint_C (\mathbf{r} \times \mathbf{B}) \cdot d\mathbf{r} = \frac{e}{c}\oint_C \mathbf{B} \cdot \mathbf{r} \times d\mathbf{r} = \frac{2e}{c}\int_S \mathbf{B} \cdot d\mathbf{S} = \frac{2e}{c}\Phi\,,$$

where Φ is the magnetic flux crossing a surface S bounded by the closed orbit. We also have, using Stokes' theorem,

$$-\frac{e}{c}\oint_C \mathbf{A} \cdot d\mathbf{r} = -\frac{e}{c}\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = -\frac{e}{c}\int_S \mathbf{B} \cdot d\mathbf{S} = -\frac{e}{c}\Phi\,.$$

Hence

$$\oint \left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right) \cdot d\mathbf{r} = \oint_C \mathbf{p} \cdot d\mathbf{r} - \frac{e}{c}\oint_C \mathbf{A} \cdot d\mathbf{r} = \frac{2e}{c}\Phi - \frac{e}{c}\Phi = \frac{e}{c}\Phi\,.$$

The generalized Bohr-Sommerfeld relation then gives

$$\Phi = \left(n + \frac{1}{2}\right)\frac{2\pi\hbar c}{e}\,,$$

which is the quantization condition on the magnetic flux.

On a plane perpendicular to **B**,

$$\Delta p \equiv \hbar\Delta k = \frac{e}{c}B\Delta r\,,$$

i.e.,

$$\Delta r = \frac{\hbar c}{eB}\Delta k\,.$$

Hence the orbital area S in k-space and A in r-space are related by

$$A = \left(\frac{\hbar c}{eB}\right)^2 S\,.$$

Using the quantization condition on magnetic flux, we have

$$A = \frac{\Phi}{B} = \left(n + \frac{1}{2}\right)\frac{2\pi\hbar c}{eB}\,,$$

or

$$\left(\frac{\hbar c}{eB}\right)^2 S = \left(n + \frac{1}{2}\right)\frac{2\pi\hbar c}{eB}\,.$$

Therefore the quantization condition on the orbital area S in k-space is

$$S = \left(n + \frac{1}{2}\right)\frac{2\pi e}{\hbar c}B\,.$$

1037

If a very small uniform-density sphere of charge is in an electrostatic potential $V(\mathbf{r})$, its potential energy is

$$U(\mathbf{r}) = V(\mathbf{r}) + \frac{r_0^2}{6}\nabla^2 V(\mathbf{r}) + \cdots$$

where $\mathbf{r}$ is the position of the center of the charge and r_0 is its very small radius. The "Lamb shift" can be thought of as the small correction to the energy levels of the hydrogen atom because the physical electron does have this property.

If the r_0^2 term of U is treated as a very small perturbation compared to the Coulomb interaction $V(\mathbf{r}) = -e^2/r$, what are the Lamb shifts for the $1s$ and $2p$ levels of the hydrogen atom? Express your result in terms of r_0 and fundamental constants. The unperturbed wave functions are

$$\psi_{1s}(\mathbf{r}) = 2a_B^{-3/2}\exp(-r/a_B)Y_0^0\,,$$

$$\psi_{2pm}(\mathbf{r}) = a_B^{-5/2} r \exp(-r/2a_B)Y_1^m/\sqrt{24}\,,$$

where $a_B = \hbar^2/m_e e^2$.

(*CUSPEA*)

Solution:

As

$$\nabla^2 V(\mathbf{r}) = -e^2 \nabla^2 \frac{1}{r} = 4\pi e^2 \delta(\mathbf{r}) \, ,$$

where $\delta(\mathbf{r})$ is Dirac's delta function defined by

$$\nabla^2 \frac{1}{r} = -4\pi \delta(\mathbf{r}) \, ,$$

we have

$$\int \psi^* \nabla^2 V(\mathbf{r}) \psi d^3\mathbf{r} = 4\pi e^2 \int \psi^*(\mathbf{r}) \psi(\mathbf{r}) \delta(\mathbf{r}) d^3\mathbf{r} = 4\pi e^2 \psi^*(0)\psi(0) \, .$$

Hence

$$\begin{aligned} \Delta E_{1s} &= \frac{r_0^2}{6} \cdot 4\pi e^2 \psi_{1s}^*(0)\psi_{1s}(0) \\ &= \frac{r_0^2}{6} \cdot 4\pi e^2 \cdot 4a_B^{-3} = \frac{8\pi e^2 r_0^2}{3} a_B^{-3} \, , \\ \Delta E_{2p} &= \frac{r_0^2}{6} \cdot 4\pi e^2 \psi_{2p}^*(0)\psi_{2p}(0) = 0 \, . \end{aligned}$$

1038

(a) Specify the dominant multipole (such as E1 (electric dipole), E2, E3 . . . , M1, M2, M3. . .) for spontaneous photon emission by an excited atomic electron in each of the following transitions,

$$2p_{1/2} \to 1s_{1/2} \, ,$$

$$2s_{1/2} \to 1s_{1/2} \, ,$$

$$3d_{3/2} \to 2s_{1/2} \, ,$$

$$2p_{3/2} \to 2p_{1/2} \, ,$$

$$3d_{3/2} \to 2p_{1/2} \, .$$

(b) Estimate the transition rate for the first of the above transitions in terms of the photon frequency ω, the atomic radius a, and any other necessary physical constants. Give a rough numerical estimate of this rate for a typical atomic transition.

(c) Estimate the ratios of the other transition rates (for the other transitions in (a)) relative to the first one in terms of the same parameters as in (b).

(*UC, Berkeley*)

Solution:

(a) In multipole transitions for spontaneous photon emission, angular momentum conservation requires

$$|j_i - j_f| \leq L \leq j_i + j_f ,$$

L being the order of transition, parity conservation requires

$$\Delta P = (-1)^L \text{ for electric multipole radiation},$$

$$\Delta P = (-1)^{L+1} \text{ for magnetic multipole radiation}.$$

Transition with the smallest order L is the most probable. Hence for

$$2p_{1/2} \to 1s_{1/2} : L = 1, \Delta P = -, \text{ transition is E1},$$

$$2s_{1/2} \to 1s_{1/2} : L = 0, \Delta P = +,$$

$$\text{transition is a double-photon dipole transition},$$

$$3d_{3/2} \to 2s_{1/2} : L = 1, 2, \Delta P = +, \text{ transition is M1 or E2},$$

$$2p_{3/2} \to 2p_{1/2} : L = 1, 2, \Delta P = +, \text{ transition is M1 or E2},$$

$$3d_{3/2} \to 2p_{1/2} : L = 1, 2, \Delta P = -, \text{ transition is E1}.$$

(b) The probability of spontaneous transition from $2p_{1/2}$ to $1s_{1/2}$ per unit time is

$$A_{E1} = \frac{e^2\omega^3}{3\pi\varepsilon_0\hbar c^3}|\mathbf{r}_{12}|^2 = \frac{4}{3}\alpha\omega^3\left(\frac{|\mathbf{r}_{12}|}{c}\right)^2 ,$$

where $\alpha = e^2/(4\pi\varepsilon_0\hbar c) = 1/137$ is the fine-structure constant. As $|\mathbf{r}_{12}| \approx a$,

$$A_{E1} \approx \frac{4}{3}\alpha\omega^3\left(\frac{a}{c}\right)^2 .$$

With $a \sim 10^{-10}$ m, $\omega \sim 10^{16}$ s^{-1}, we have $A_{E1} \sim 10^9$ s^{-1}.

(c)

$$\frac{A(2^2s_{\frac{1}{2}} \to 1^2s_{\frac{1}{2}})}{A_{E1}} \approx 10\left(\frac{\hbar}{mc\alpha}\right),$$

$$\frac{A(3d_{\frac{3}{2}} \to 2s_{\frac{1}{2}})}{A_{E1}} \approx (ka)^2,$$

$$\frac{A(2p_{\frac{3}{2}} - 2p_{\frac{1}{2}})}{A_{E1}} \approx (ka)^2,$$

where $k = \omega/c$ is the wave number of the photon,

$$\frac{A(3d_{3/2} \to 2p_{1/2})}{A_{E1}} \approx \left[\frac{\omega(3d_{3/2} \to 2p_{1/2})}{\omega(2p_{1/2} \to 1s_{1/2})}\right]^3.$$

1039

(a) What is the energy of the neutrino in a typical beta decay?

(b) What is the dependence on atomic number Z of the lifetime for spontaneous decay of the $2p$ state in the hydrogen-like atoms H, He^+, Li^{++}, etc.?

(c) What is the electron configuration, total spin S, total orbital angular momentum L, and total angular momentum J of the ground state of atomic oxygen?

(*UC, Berkeley*)

Solution:

(a) The energy of the neutrino emitted in a typical β-decay is $E_\nu \approx$ 1 MeV.

(b) The probability of spontaneous transition $2p \to 1s$ per unit time is (**Problem 1038(b)**) $A \propto |\mathbf{r}_{12}|^2\omega^3$, where

$$|\mathbf{r}_{12}|^2 = |\langle 1s(Zr)|\mathbf{r}|2p(Zr)\rangle|^2,$$

$|1s(Zr)\rangle$ and $|2p(Zr)\rangle$ being the radial wave functions of a hydrogen-like atom of nuclear charge Z, and

$$\omega = \frac{1}{\hbar}(E_2 - E_1).$$

As

$$1s(Zr)\rangle = \left(\frac{Z}{a_0}\right)^{\frac{3}{2}} 2e^{\frac{-Zr}{a_0}},$$

$$2p(Zr)\rangle = \left(\frac{Z}{2a_0}\right)^{\frac{3}{2}} \frac{Zr}{a_0\sqrt{3}} e^{-\frac{Zr}{2a_0}},$$

a_0 being a constant, we have for $Z > 1$,

$$|\mathbf{r}_{12}|^2 \propto Z^{-2}, \qquad \omega^3 \propto Z^6,$$

and so $A \propto Z^4$. Hence the lifetime τ is

$$\tau \propto \frac{1}{A} \propto Z^{-4}.$$

(c) The electron configuration of ground state atomic oxygen is $1s^2 2s^2$ $2p^4$. As the state has $S = 1$, $L = 1$, $J = 2$, it is designated $^3P_2$.

1040

Suppose that, because of small parity-violating forces, the $2^2S_{1/2}$ level of the hydrogen atom has a small p-wave admixture:

$$\begin{aligned}\Psi(n = 2, j = 1/2) = \Psi_s(n = 2, j = 1/2, l = 0) \\ + \varepsilon\Psi_p(n = 2, j = 1/2, l = 1).\end{aligned}$$

What first-order radiation decay will de-excite this state? What is the form of the decay matrix element? What dose it become if $\varepsilon \to 0$ and why?

(*Wisconsin*)

Solution:

Electric dipole radiation will de-excite the p-wave part of this mixed state: $\Psi_p(n = 2, j = 1/2, l = 1) \to \Psi_s(n = 1, j = 1/2, l = 0)$. The $\Psi_s(n = 2, j = 1/2, l = 0)$ state will not decay as it is a metastable state. The decay matrix, i.e. the T matrix, is

$$\langle\Psi_f|T|\Psi_i\rangle = \varepsilon \int \Psi_f^* V(\mathbf{r})\Psi_i d^3r,$$

where, for electric dipole radiation, we have

$$V(\mathbf{r}) = -(-e\mathbf{r}) \cdot \mathbf{E} = erE\cos\theta\,,$$

taking the z-axis along the electric field. Thus

$$\begin{aligned}
\langle \Psi_f|T|\Psi_i\rangle &= \varepsilon eE \int R_{10} r R_{21} r^2 dr \int Y_{00} Y_{10}\cos\theta d\Omega \\
&= \frac{\varepsilon eE}{\sqrt{2}a^3}\int_0^\infty r^3 \exp\left(-\frac{3r}{2a}\right) dr \\
&= \frac{32}{27\sqrt{6}}\varepsilon eaE \int_\Omega Y_{00} Y_{10}\cos\theta d\Omega\,.
\end{aligned}$$

As

$$\cos\theta Y_{10} = \sqrt{\frac{4}{15}} Y_{20} + \sqrt{\frac{1}{3}} Y_{00}\,,$$

the last integral equals $\sqrt{\frac{1}{3}}$ and

$$\langle \Psi_f|T|\Psi_i\rangle = \left(\frac{2}{3}\right)^4 \sqrt{2}\varepsilon eaE\,.$$

If $\varepsilon \to 0$, the matrix element of the dipole transition $\langle \Psi_f|T|\Psi_i\rangle \to 0$ and no such de-excitation takes place. The excited state $\Psi_s(n = 2, j = 1/2, l = 0)$ is metastable. It cannot decay to the ground state via electric dipole transition (because $\Delta l \neq 1$). Nor can it do so via magnetic dipole or electric quadruple transition. It can only decay to the ground state by the double-photons transition $2^2S_{1/2} \to 1^2S_{1/2}$, which however has a very small probability.

1041

(a) The ground state of the hydrogen atom is split by the hyperfine interaction. Indicate the level diagram and show from first principles which state lies higher in energy.

(b) The ground state of the hydrogen molecule is split into total nuclear spin triplet and singlet states. Show from first principles which state lies higher in energy.

(Chicago)

Solution:

(a) The hyperfine interaction in hydrogen arises from the magnetic interaction between the intrinsic magnetic moments of the proton and the electron, the Hamiltonian being

$$H_{\text{int}} = -\boldsymbol{\mu}_p \cdot \mathbf{B},$$

where $\mathbf{B}$ is the magnetic field produced by the magnetic moment of the electron and $\boldsymbol{\mu}_p$ is the intrinsic magnetic moment of the proton.

In the ground state, the electron charge density is spherically symmetric so that $\mathbf{B}$ has the same direction as the electron intrinsic magnetic moment $\boldsymbol{\mu}_e$. However as the electron is negatively charged, $\boldsymbol{\mu}_e$ is antiparallel to the electron spin angular momentum $\mathbf{s}_e$. For the lowest energy state of H_{int}, $\langle \boldsymbol{\mu}_p \cdot \boldsymbol{\mu}_e \rangle > 0$, and so $\langle \mathbf{s}_p \cdot \mathbf{s}_e \rangle < 0$. Thus the singlet state $F = 0$ is the ground state, while the triplet $F = 1$ is an excited state (see Fig. 1.12).

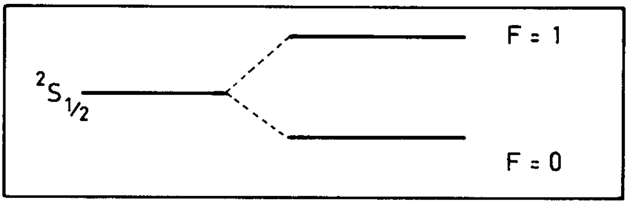

Fig. 1.12

(b) As hydrogen molecule consists of two like atoms, each having a proton (spin $\frac{1}{2}$) as nucleus, the nuclear system must have an antisymmetric state function. Then the nuclear spin singlet state ($S = 0$, antisymmetric) must be associated with a symmetric nuclear rotational state; thus $J = 0, 2, 4, \ldots$, with the ground state having $J = 0$. For the spin triplet state ($S = 1$, symmetric) the rotational state must have $J = 1, 3, \ldots$, with the ground state having $J = 1$. As the rotational energy is proportional to $J(J+1)$, the spin triplet ground state lies higher in energy.

1042

(a) In Bohr's original theory of the hydrogen atom (circular orbits) what postulate led to the choice of the allowed energy levels?

(b) Later de Broglie pointed out a most interesting relationship between the Bohr postulate and the de Broglie wavelength of the electron. State and derive this relationship.

(*Wisconsin*)

Solution:

(a) Bohr proposed the quantization condition

$$mvr = n\hbar\,,$$

where m and v are respectively the mass and velocity of the orbiting electron, r is the radius of the circular orbit, $n = 1, 2, 3, \ldots$. This condition gives descrete values of the electron momentum $p = mv$, which in turn leads to descrete energy levels.

(b) Later de Broglie found that Bohr's circular orbits could exactly hold integral numbers of de Broglie wavelength of the electron. As

$$pr = n\hbar = \frac{nh}{2\pi}\,,$$

$$2\pi r = n\frac{h}{p} = n\lambda\,,$$

where λ is the de Broglie wavelength, which is associated with the group velocity of matter wave.

1043

In radio astronomy, hydrogen atoms are observed in which, for example, radiative transitions from $n = 109$ to $n = 108$ occur.

(a) What are the frequency and wavelength of the radiation emitted in this transition?

(b) The same transition has also been observed in excited helium atoms. What is the ratio of the wavelengths of the He and H radiation?

(c) Why is it difficult to observe this transition in laboratory experiment?

(*Wisconsin*)

Solution:

(a) The energy levels of hydrogen, in eV, are

$$E_n = -\frac{13.6}{n^2}\,.$$

For transitions between excited states $n = 109$ and $n = 108$ we have

$$h\nu = \frac{13.6}{108^2} - \frac{13.6}{109^2}\,,$$

giving

$$\nu = 5.15 \times 10^9 \text{ Hz}\,,$$

or

$$\lambda = c/\nu = 5.83 \text{ cm}\,.$$

(b) For such highly excited states the effective nuclear charge of the helium atom experienced by an orbital electron is approximately equal to that of a proton. Hence for such transitions the wavelength from He approximately equals that from H.

(c) In such highly excited states, atoms are easily ionized by colliding with other atoms. At the same time, the probability of a transition between these highly excited states is very small. It is very difficult to produce such environment in laboratory in which the probability of a collision is very small and yet there are sufficiently many such highly excited atoms available. (However the availability of strong lasers may make it possible to stimulate an atom to such highly excited states by multiphoton excitation.)

1044

Sketch the energy levels of atomic Li for the states with $n = 2, 3, 4$. Indicate on the energy diagram several lines that might be seen in emission and several lines that might be seen in absorption. Show on the same diagram the energy levels of atomic hydrogen for $n = 2, 3, 4$.

(*Wisconsin*)

Solution:

As most atoms remain in the ground state, the absorption spectrum arises from transitions from $2s$ to np states ($n = 2, 3, 4$). In Fig. 1.13,

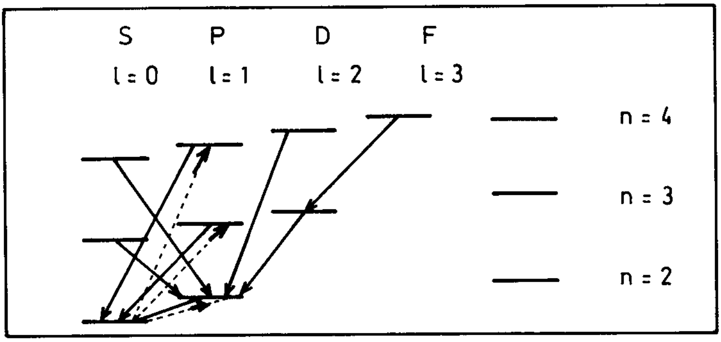

Fig. 1.13

the dashed lines represent absorption transitions, the solid lines, emission transitions.

1045

The "plum pudding" model of the atom proposed by J. J. Thomson in the early days of atomic theory consisted of a sphere of radius a of positive charge of total value Ze. Z is an integer and e is the fundamental unit of charge. The electrons, of charge $-e$, were considered to be point charges embedded in the positive charge.

(a) Find the force acting on an electron as a function of its distance r from the center of the sphere for the element hydrogen.

(b) What type of motion does the electron execute?

(c) Find an expression for the frequency for this motion.

(*Wisconsin*)

Solution:

(a) For the hydrogen atom having $Z = 1$, radius a, the density of positive charge is

$$\rho = \frac{e}{\frac{4}{3}\pi a^3} = \frac{3e}{4\pi a^3} \,.$$

When an electron is at a distance r from the center of the sphere, only the positive charge inside the sphere of radius r can affect the electron and so the electrostatic force acting on the electron is

$$F(r) = -\frac{e}{4\pi\varepsilon_0 r^2} \cdot \frac{4}{3}\pi r^3 \rho = -\frac{e^2 r}{4\pi\varepsilon_0 a^3}\,,$$

pointing toward the center of the sphere.

(b) The form of $F(r)$ indicates the motion of the electron is simple harmonic.

(c) $F(r)$ can be written in the form

$$F(r) = -kr\,,$$

where $k = \frac{e^2}{4\pi\varepsilon_0 a^3}$. The angular frequency of the harmonic motion is thus

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{e^2}{4\pi\varepsilon_0 a^3 m}}\,,$$

where m is the mass of electron.

1046

Lyman alpha, the $n = 1$ to $n = 2$ transition in atomic hydrogen, is at 1215 Å.

(a) Define the wavelength region capable of photoionizing a H atom in the ground level ($n = 1$).

(b) Define the wavelength region capable of photoionizing a H atom in the first excited level ($n = 2$).

(c) Define the wavelength region capable of photoionizing a He^+ ion in the ground level ($n = 1$).

(d) Define the wavelength region capable of photoionizing a He^+ ion in the first excited level ($n = 2$).

(*Wisconsin*)

Solution:

(a) A spectral series of a hydrogen-like atom has wave numbers

$$\tilde{\nu} = Z^2 R\left(\frac{1}{n^2} - \frac{1}{m^2}\right),$$

where Z is the nuclear charge, R is the Rydberg constant, and n, m are positive integers with $m > n$. The ionization energy of the ground state of H atom is the limit of the Lyman series ($n = 1$), the wave number being

$$\tilde{\nu}_0 = \frac{1}{\lambda_0} = R\,.$$

For the alpha line of the Lyman series,

$$\tilde{\nu}_\alpha = \frac{1}{\lambda_\alpha} = R\left(1 - \frac{1}{2^2}\right) = \frac{3}{4}R = \frac{3}{4\lambda_0}\,.$$

As $\lambda_\alpha = 1215$ Å, $\lambda_0 = 3\lambda_\alpha/4 = 911$ Å. Hence the wavelength of light that can photoionize H atom in the ground state must be shorter than 911 Å.

(b) The wavelength should be shorter than the limit of the Balmer series $(n = 2)$, whose wave number is

$$\tilde{\nu} = \frac{1}{\lambda} = \frac{R}{2^2} = \frac{1}{4\lambda_0}\,.$$

Hence the wavelength should be shorter than $4\lambda_0 = 3645$ Å.

(c) The limiting wave number of the Lyman series of He^+ $(Z = 2)$ is

$$\tilde{\nu} = \frac{1}{\lambda} = \frac{Z^2 R}{1^2} = 4R = \frac{4}{\lambda_0}\,.$$

The wavelength that can photoionize the He^+ in the ground state must be shorter than $\lambda_0/4 = 228$ Å.

(d) The wavelength should be shorter than $1/R = \lambda_0 = 1215$ Å.

1047

A tritium atom in its ground state beta-decays to He^+.

(a) Immediately after the decay, what is the probability that the helium ion is in its ground state?

(b) In the $2s$ state?

(c) In the $2p$ state?

(Ignore spin in this problem.)

(UC, Berkeley)

Solution:

At the instant of β-decay continuity of the wave function requires

$$|1s\rangle_H = a_1|1s\rangle_{He^+} + a_2|2s\rangle_{He^+} + a_3|2p\rangle_{He^+} + \cdots\,,$$

where

$$|1s\rangle = R_{10}(r)Y_{00}\,, \qquad |2s\rangle = R_{20}(r)Y_{00}\,, \qquad |2p\rangle = R_{21}(r)Y_{10}\,,$$

with

$$R_{10} = \left(\frac{Z}{a}\right)^{\frac{3}{2}} 2\exp\left(-\frac{Zr}{a}\right)\,, \qquad R_{20} = \left(\frac{Z}{2a}\right)^{\frac{3}{2}}\left(2-\frac{Zr}{a}\right)\exp\left(-\frac{Zr}{2a}\right)\,,$$

$$R_{21} = \left(\frac{Z}{2a}\right)^{\frac{3}{2}}\frac{Zr}{a\sqrt{3}}\exp\left(-\frac{Zr}{2a}\right)\,, \qquad a = \frac{\hbar^2}{me^2}\,.$$

(a)

$$\begin{aligned} a_1 = {}_{He^+}\langle 1s|1s\rangle_H &= \int_0^\infty \frac{2}{a^{3/2}}\exp\left(-\frac{r}{a}\right)\cdot 2\left(\frac{2}{a}\right)^{3/2} \\ &\times \exp\left(-\frac{2r}{a}\right)\cdot r^2 dr\int Y_{00}^2 d\Omega = \frac{16\sqrt{2}}{27}\,. \end{aligned}$$

Accordingly the probability of finding the He$^+$ in the ground state is

$$W\langle 1s\rangle = |a_1|^2 = \frac{512}{729}\,.$$

(b)

$$\begin{aligned} a_2 = {}_{He^+}\langle 2s|1s\rangle_H &= \int_0^\infty \frac{2}{a^{3/2}}\exp\left(-\frac{r}{a}\right)\cdot\frac{1}{\sqrt{2}}\left(\frac{2}{a}\right)^{3/2}\left(1-\frac{r}{a}\right) \\ &\times \exp\left(-\frac{r}{a}\right)\cdot r^2 dr\int Y_{00}^2 d\Omega = -\frac{1}{2}\,. \end{aligned}$$

Hence the probability of finding the He$^+$ in the $2s$ state is

$$W\langle 2s\rangle = |a_2|^2 = \frac{1}{4}\,.$$

(c)

$$a_3 = {}_{He^+}\langle 2p|1s\rangle_H = \int_0^\infty \frac{2}{a^{3/2}} \exp\left(-\frac{r}{a}\right) \cdot \frac{1}{2\sqrt{6}} \left(\frac{2}{a}\right)^{3/2} \cdot \frac{2r}{a}$$

$$\times \exp\left(-\frac{r}{a}\right) \cdot r^2 dr \int Y_{10}^* Y_{00} d\Omega = 0\,.$$

Hence the probability of finding the He^+ in the $2p$ state is

$$W\langle 2p\rangle = |a_3|^2 = 0\,.$$

1048

Consider the ground state and $n = 2$ states of hydrogen atom.

Indicate in the diagram (Fig. 1.14) the complete spectroscopic notation for all four states. There are four corrections to the indicated level structure that must be considered to explain the various observed splitting of the levels. These corrections are:

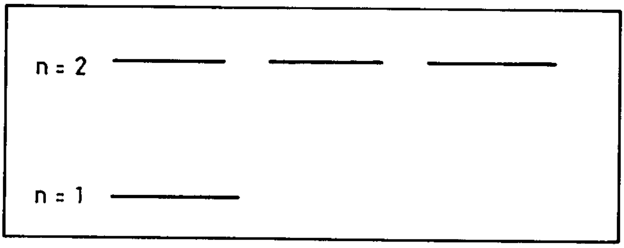

Fig. 1.14

(a) Lamb shift,
(b) fine structure,
(c) hyperfine structure,
(d) relativistic effects.

(1) Which of the above apply to the $n = 1$ state?

(2) Which of the above apply to the $n = 2$, $l = 0$ state? The $n = 2$, $l = 1$ state?

(3) List in order of decreasing importance these four corrections. (i.e. biggest one first, smallest last). Indicate if some of the corrections are of the same order of magnitude.

(4) Discuss briefly the physical origins of the hyperfine structure. Your discussion should include an appropriate mention of the Fermi contact potential.

(*Wisconsin*)

Solution:

The spectroscopic notation for the ground and first excited states of hydrogen atom is shown in Fig. 1.15.

Three corrections give rise to the fine structure for hydrogen atom:

$$E_f = E_m + E_D + E_{so}\,,$$

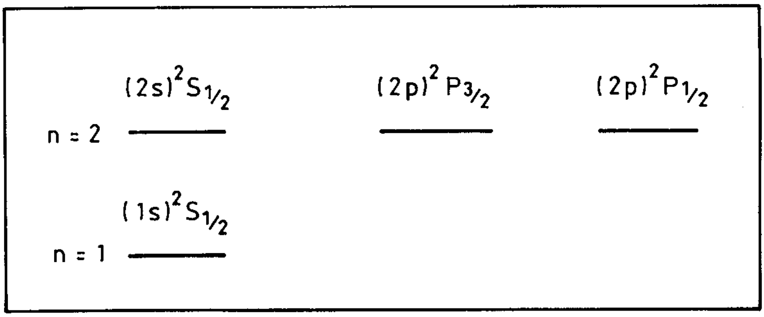

Fig. 1.15

where E_m is caused by the relativistic effect of mass changing with velocity, E_D, the Darwin term, arises from the relativistic non-locality of the electron, E_{so} is due to the spin-orbit coupling of the electron. They are given by

$$E_m = -\frac{\alpha^2 Z^4}{4n^4}\left(\frac{4n}{l+\frac{1}{2}} - 3\right) \times 13.6 \text{ eV}\,,$$

$$E_D = \frac{\alpha^2 Z^4}{n^3}\delta_{l0} \times 13.6 \text{ eV},$$

$$E_{so} = \begin{cases} (1-\delta_{l0})\dfrac{\alpha^2 Z^4 l}{n^3 l(l+1)(2l+1)} \times 13.6 \text{ eV}, & \left(j = l + \dfrac{1}{2}\right) \\ -(1-\delta_{l0})\dfrac{\alpha^2 Z^4 (l+1)}{n^3 l(l+1)(2l+1)} \times 13.6 \text{ eV}. & \left(j = l - \dfrac{1}{2}\right) \end{cases}$$

where α is the fine-structure constant, and δ_{l0} is the usual Kronecker delta.

Lamb shift arises from the interaction between the electron and its radiation field, giving rise to a correction which, when expanded with respect to $Z\alpha$, has the first term

$$E_L = k(l)\frac{\alpha(Z\alpha)^4 mc^2}{2\pi n^3}$$

$$= k(l)\frac{\alpha^3 Z^4}{\pi n^3} \times 13.6 \text{ eV},$$

where $k(l)$ is a parameter related to l.

Hyperfine structure arises from the coupling of the total angular momentum of the electron with the nuclear spin.

(1) For the $n = 1$ state ($l = 0$), E_m, E_D, E_L can only cause the energy level to shift as a whole. As $E_{so} = 0$ also, the fine-structure correction does not split the energy level. On the other hand, the hyperfine structure correction can cause a splitting as shown in Fig. 1.16.

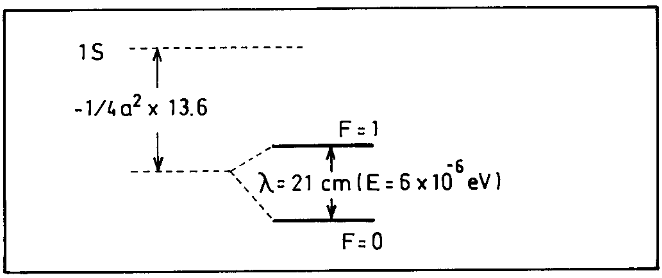

Fig. 1.16

(2) For the $n = 2$ state ($l = 0$ and $l = 1$), the fine-structure correction causes the most splitting in the $l = 1$ level, to which the hyperfine structure correction also contributes (see Fig. 1.17).

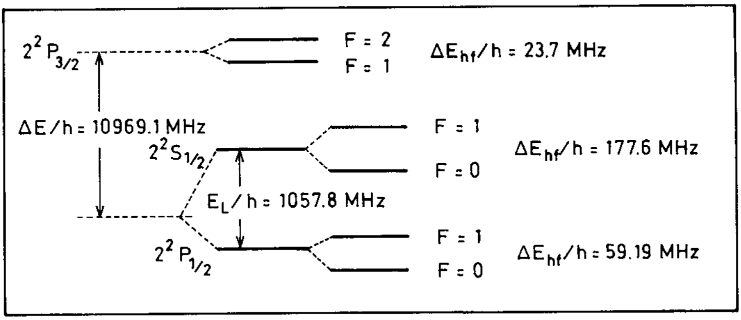

Fig. 1.17

(3) E_m, E_D, E_{so} are of the same order of magnitude > Lamb shift $\gtrsim$ hyperfine structure.

(4) The hyperfine structure can be separated into three terms:

(a) Interaction between the nuclear magnetic moment and the magnetic field at the proton due to the electron's orbital motion,

(b) dipole-dipole interaction between the electron and the nuclear magnetic moment,

(c) the Fermi contact potential due to the interaction between the spin magnetic moment of the electron and the internal magnetic field of the proton.

1049

Using the Bohr model of the atom,

(a) derive an expression for the energy levels of the He^+ ion.

(b) calculate the energies of the $l = 1$ state in a magnetic field, neglecting the electron spin.

(Wisconsin)

Solution:

(a) Let the radius of the orbit of the electron be r, and its velocity be v. Bohr assumed that the angular momentum L_ϕ is quantized:

$$L_\phi = mvr = n\hbar . \qquad (n = 1, 2, 3 \ldots)$$

The centripetal force is provided by the Coulomb attraction and so

$$m\frac{v^2}{r} = \frac{2e^2}{4\pi\varepsilon_0 r^2} .$$

Hence the energy of He^+ is

$$E_n = \frac{1}{2}mv^2 - \frac{2e^2}{4\pi\varepsilon_0 r} = -\frac{1}{2}mv^2 = -\frac{2me^4}{(4\pi\varepsilon_0)^2 n^2\hbar^2} .$$

(b) The area of the electron orbit is

$$A = \int_0^{2\pi} \frac{r}{2} \cdot r d\phi = \frac{1}{2}\int_0^T r^2 \omega dt = \frac{L_\phi}{2m} T ,$$

where $\omega = \frac{d\phi}{dt}$, the angular velocity, is given by $L_\phi = mr^2\omega$, and T is the period of circular motion. For $l = 1$, $L_\phi = \hbar$ and the magnetic moment of the electron due to its orbital motion is

$$\mu = IA = -\frac{e}{T}A = -\frac{e\hbar}{2m} ,$$

where I is the electric current due to the orbital motion of the electron. The energy arising from interaction between the $l = 1$ state and a magnetic field $\mathbf{B}$ is

$$\Delta E = -\boldsymbol{\mu} \cdot \mathbf{B} = \begin{cases} \dfrac{e\hbar}{2m}B, & (\boldsymbol{\mu} // \mathbf{B}) \\ 0 , & (\boldsymbol{\mu} \perp \mathbf{B}) \\ -\dfrac{e\hbar}{2m}B . & (\boldsymbol{\mu} // -\mathbf{B}) \end{cases}$$

1050

An atom has a nucleus of charge Z and one electron. The nucleus has a radius R, inside which the charge (protons) is uniformly distributed. We want to study the effect of the finite size of the nucleus on the electron levels:

(a) Calculate the potential taking into account the finite size of the nucleus.

(b) Calculate the level shift due to the finite size of the nucleus for the $1s$ state of ^{208}Pb using perturbation theory, assuming that R is much smaller than the Bohr radius and approximating the wave function accordingly.

(c) Give a numerical answer to (b) in cm^{-1} assuming $R = r_0 A^{1/3}$, $r_0 = 1.2$ fermi.

(*Wisconsin*)

Solution:

(a) For $r \geq R$.

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}\,.$$

For $r < R$,

$$V(r) = -\frac{Ze^2}{4\pi\varepsilon_0 r}\cdot\left(\frac{r}{R}\right)^3 - \int_r^R \frac{e\rho 4\pi r'^2}{r'}dr' = -\frac{Ze^2}{8\pi\varepsilon_0 R^3}(3R^2 - r^2)\,,$$

where

$$\rho = \frac{Ze}{\frac{4}{3}\pi r^3}\,.$$

(b) Taking the departure of the Hamiltonian from that of a point nucleus as perturbation, we have

$$H' = \begin{cases} \dfrac{Ze^2}{4\pi\varepsilon_0 r} - \dfrac{Ze^2}{4\pi\varepsilon_0 R}\left(\dfrac{3}{2} - \dfrac{r^2}{2R^2}\right) & \text{for } r < R\,, \\ 0 & \text{for } r \geq R\,. \end{cases}$$

The $1s$ wave function of ^{208}Pb is

$$|1s\rangle = 2\left(\frac{Z}{a_0}\right)^{3/2}\exp\left(-\frac{2r}{a_0}\right)\cdot\frac{1}{\sqrt{4\pi}}\,,$$

where $Z = 82$, a_0 is the Bohr radius. Taking the approximation $r \ll a_0$, i.e., $\exp(-\frac{2r}{a_0}) \approx 1$, the energy shift is

$$\begin{aligned}\Delta E &= \langle 1s|H'|1s\rangle \\ &= -\frac{4Z^4e^2}{4\pi\varepsilon_0 a_0^3}\int_0^R \left(\frac{3}{2R} - \frac{r^2}{2R^3} - \frac{1}{r}\right) r^2 dr \\ &= \frac{4}{5}Z^2|E_0|\left(\frac{R}{a_0}\right)^2 ,\end{aligned}$$

where $E_0 = -\frac{Z^2e^2}{(4\pi\varepsilon_0)2a_0}$ is the ground state energy of a hydrogen-like atom.

(c)

$$\Delta E = \frac{4}{5} \times 82^2 \times (82^2 \times 13.6) \times \left(\frac{1.2 \times 10^{-19} \times 208^{\frac{1}{3}}}{5.29 \times 10^{-9}}\right)^2 = 8.89 \text{ eV} ,$$

$$\Delta\tilde{\nu} = \frac{\Delta E}{hc} \approx 7.2 \times 10^4 \text{ cm}^{-1} .$$

1051

If the proton is approximated as a uniform charge distribution in a sphere of radius R, show that the shift of an s-wave atomic energy level in the hydrogen atom, from the value it would have for a point proton, is approximately

$$\Delta E_{ns} \approx \frac{2\pi}{5}e^2|\Psi_{ns}(0)|^2R^2 ,$$

using the fact that the proton radius is much smaller than the Bohr radius. Why is the shift much smaller for non-s states?

The $2s$ hydrogenic wave function is

$$(2a_0)^{-3/2}\pi^{-1/2}\left(1 - \frac{r}{2a_0}\right)\exp\left(-\frac{r}{2a_0}\right) .$$

What is the approximate splitting (in eV) between the $2s$ and $2p$ levels induced by this effect? [$a_0 \approx 5 \times 10^{-9}$ cm for H, $R \approx 10^{-13}$ cm.]

(*Wisconsin*)

Solution:

The perturbation caused by the finite volume of proton is (**Problem 1050**)

$$H' = \begin{cases} 0, & (r \geq R) \\ \dfrac{e^2}{r} - \dfrac{e^2}{R}\left(\dfrac{3}{2} - \dfrac{r^2}{2R^2}\right) . & (r < R) \end{cases}$$

The unperturbed wave function is

$$\Psi_{ns} = N_{n0} \exp\left(-\frac{r}{na_0}\right) F\left(-n+1, 2, \frac{2r}{na_0}\right) Y_{00} ,$$

where

$$N_{n0} = \frac{2}{(na_0)^{3/2}} \sqrt{\frac{n!}{(n-1)!}} \approx \frac{2}{(na_0)^{3/2}} ,$$

$$F\left(-n+1, 2, \frac{2r}{na_0}\right) = 1 - \frac{n-1}{2} \cdot \frac{2r}{na_0} + \frac{(n-1)(n-2)}{2 \cdot 3} \times \frac{1}{2!}\left(\frac{2r}{na_0}\right)^2 + \cdots .$$

Taking the approximation $r \ll a_0$, we have

$$F\left(-n+1, 2, \frac{2r}{na_0}\right) \approx 1 , \qquad \exp\left(-\frac{r}{na_0}\right) \approx 1 ,$$

and so

$$\Psi_{ns} = N_{n0} Y_{00} = \frac{2}{(na_0)^{3/2}} Y_{00} ,$$

$$\Delta E_{ns} = \langle \Psi^*_{ns} | H' | \Psi_{ns} \rangle = \int_0^R \left[\frac{e^2}{r} - \frac{e^2}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)\right] \Psi^*_{ns} \Psi_{ns} r^2 dr d\Omega$$

$$= \frac{2\pi}{5} \frac{e^2 R^2}{\pi (na_0)^3} .$$

Using

$$\Psi_{ns}(0) = \frac{2}{(na_0)^{3/2}} \cdot \frac{1}{\sqrt{4\pi}} = \frac{1}{\sqrt{\pi}(na_0)^{3/2}} ,$$

we have

$$\Delta E_{ns} = \frac{2\pi}{5} e^2 |\Psi_{ns}(0)|^2 R^2 \,.$$

As the non-s wave functions have a much smaller fraction inside the nucleus and so cause smaller perturbation, the energy shift is much smaller.

For hydrogen atom, since $\Delta E_{2p} \ll \Delta E_{2s}$,

$$\begin{aligned} \Delta E_{ps} &= \Delta E_{2s} - \Delta E_{2p} \approx \Delta E_{2s} \\ &= \frac{2\pi}{5} e^2 |\Psi_{2s}(0)|^2 R \,, \end{aligned}$$

where

$$\Psi_{2s}(0) = (2a_0)^{-3/2} \pi^{-1/2} \,.$$

Hence

$$\begin{aligned} \Delta E_{ps} &\approx \frac{2\pi}{5} e^2 [(2a_0)^{-3/2} \pi^{-1/2}]^2 R^2 \\ &= \frac{e^2 R^2}{20 a_0^3} = \left(\frac{e^2}{\hbar c} \right)^2 \cdot \frac{R^2 mc^2}{20 a_0^2} \\ &= \left(\frac{1}{137} \right)^2 \times \frac{10^{-26} \times 0.511 \times 10^6}{20 \times (5 \times 10^{-9})^2} \approx 5.4 \times 10^{-10} \text{ eV} \,. \end{aligned}$$

1052

The ground state of hydrogen atom is $1s$. When examined very closely, it is found that the level is split into two levels.

(a) Explain why this splitting takes place.

(b) Estimate numerically the energy difference between these two levels.

(*Columbia*)

Solution:

(a) In the fine-structure spectrum of hydrogen atom, the ground state $1s$ is not split. The splitting is caused by the coupling between the magnetic moments of the nuclear spin and the electron spin: $\hat{\mathbf{F}} = \hat{\mathbf{I}} + \hat{\mathbf{J}}$. As $I = 1/2$, $J = 1/2$, the total angular momentum is $F = 1$ or $F = 0$, corresponding to the two split energy levels.

(b) The magnetic moment of the nucleus (proton) is $\boldsymbol{\mu} = \mu_N \boldsymbol{\sigma}_N$, where $\boldsymbol{\sigma}_N$ is the Pauli matrix operating on the nuclear wave function, inducing a magnetic field $\mathbf{H}_m = \nabla \times \nabla \times (\frac{\mu_N \boldsymbol{\sigma}_N}{r})$. The Hamiltonian of the interaction between $\mathbf{H}_m$ and the electron magnetic moment $\boldsymbol{\mu} = -\mu_e \boldsymbol{\sigma}_e$ is

$$\hat{H} = -\boldsymbol{\mu} \cdot \hat{\mathbf{H}}_m = \mu_e \mu_N \boldsymbol{\sigma}_e \cdot \nabla \times \nabla \times \left(\frac{\boldsymbol{\sigma}_N}{r} \right) .$$

Calculation gives the hyperfine structure splitting as (**Problem 1053**)

$$\Delta E = A' \mathbf{I} \cdot \mathbf{J} ,$$

where

$$\begin{aligned} A' \sim \frac{\mu_e \mu_N}{e^2 a_0^3} &\approx \left(\frac{m_e}{m_N} \right) \frac{m_e c^2}{4} \cdot \left(\frac{e^2}{\hbar c} \right)^4 \\ &\approx \frac{1}{2000} \cdot \frac{0.51 \times 10^6}{4} \times \left(\frac{1}{137} \right)^4 \\ &\approx 2 \times 10^{-7} \text{ eV} , \end{aligned}$$

m_e, m_N, c, a_0 being the electron mass, nucleon mass, velocity of light, Bohr radius respectively.

1053

Derive an expression for the splitting in energy of an atomic energy level produced by the hyperfine interaction. Express your result in terms of the relevant angular momentum quantum numbers.

(*SUNY, Buffalo*)

Solution:

The hyperfine structure is caused by the interaction between the magnetic field produced by the orbital motion and spin of the electron and the nuclear magnetic moment $\mathbf{m}_N$. Taking the site of the nucleus as origin, the magnetic field caused by the orbital motion of the electron at the origin is

$$\mathbf{B}_e(0) = \frac{\mu_0 e}{4\pi} \frac{\mathbf{v} \times \mathbf{r}}{r^3} = -\frac{2\mu_0 \mu_B}{4\pi\hbar} \frac{\mathbf{l}}{r^3} ,$$

where $\mathbf{v}$ is the velocity of the electron in its orbit, $\mathbf{l} = m\mathbf{r} \times \mathbf{v}$ is its orbital angular momentum, $\mu_B = \frac{e\hbar}{2m}$, m being the electron mass, is the Bohr magneton.

The Hamiltonian of the interaction between the nuclear magnetic moment $\mathbf{m}_N$ and $\mathbf{B}_e(0)$ is

$$H_{lI} = -\mathbf{m}_N \cdot \mathbf{B}_e(0) = \frac{2\mu_0 g_N \mu_N \mu_B}{4\pi\hbar^2 r^3} \mathbf{l} \cdot \mathbf{I},$$

where $\mathbf{I}$ is the nuclear spin, μ_N the nuclear magneton, g_N the Landé g-factor of the nucleon.

At $\mathbf{r} + \mathbf{r}'$, the vector potential caused by the electron magnetic moment $\mathbf{m}_s = -\frac{2\mu_B \mathbf{s}}{\hbar}$ is $\mathbf{A} = \frac{\mu_0}{4\pi}\mathbf{m}_s \times \frac{\mathbf{r}'}{r'^3}$, $\mathbf{r}'$ being the radius vector from $\mathbf{r}$ to the field point. So the magnetic field is

$$\begin{aligned}
\mathbf{B}_s &= \nabla \times \mathbf{A} = \frac{\mu_0}{4\pi}\nabla \times \left(\mathbf{m}_s \times \frac{\mathbf{r}'}{r'^3}\right) \\
&= \frac{2\mu_0\mu_B}{4\pi\hbar}\nabla' \times \left(\mathbf{s} \times \nabla' \frac{1}{r'}\right) = \frac{2\mu_0\mu_B}{4\pi\hbar}\left[\mathbf{s}\nabla'^2 \frac{1}{r'} - (\mathbf{s} \cdot \nabla')\nabla' \frac{1}{r'}\right] \\
&= -\frac{2\mu_0\mu_B}{4\pi\hbar}\left[4\pi\mathbf{s}\delta(\mathbf{r}') + (\mathbf{s} \cdot \nabla')\nabla' \frac{1}{r'}\right] .
\end{aligned}$$

Letting $\mathbf{r}' = -\mathbf{r}$, we get the magnetic field caused by $\mathbf{m}_s$ at the origin:

$$\mathbf{B}_s(0) = -\frac{2\mu_0\mu_B}{4\pi\hbar}\left[4\pi\mathbf{s}\delta(\mathbf{r}) + (\mathbf{s} \cdot \nabla)\nabla \frac{1}{r}\right] .$$

Hence the Hamiltonian of the interaction between $\mathbf{m}_N = \frac{g_N \mu_N \mathbf{I}}{\hbar}$ and $\mathbf{B}_s(0)$ is

$$\begin{aligned}
H_{sI} &= -\mathbf{m}_N \cdot \mathbf{B}_s(0) \\
&= \frac{2\mu_0 g_N \mu_N \mu_B}{4\pi\hbar^2}\left[4\pi\mathbf{I} \cdot \mathbf{s}\delta(\mathbf{r}) + (\mathbf{s} \cdot \nabla)\left(\mathbf{I} \cdot \nabla \frac{1}{r}\right)\right] .
\end{aligned}$$

The total Hamiltonian is then

$$\begin{aligned}
H_{hf} &= H_{lI} + H_{sI} \\
&= \frac{2\mu_0 g_N \mu_N \mu_B}{4\pi\hbar^2}\left[\frac{\mathbf{l} \cdot \mathbf{I}}{r^3} + 4\pi\mathbf{s} \cdot \mathbf{I}\delta(\mathbf{r}) + (\mathbf{s} \cdot \nabla)\left(\mathbf{I} \cdot \nabla \frac{1}{r}\right)\right] .
\end{aligned}$$

In zeroth order approximation, the wave function is $|lsjIFM_F\rangle$, where l, s and j are respectively the quantum numbers of orbital angular momentum, spin and total angular momentum of the electron, I is the quantum number of the nuclear spin, F is the quantum number of the total angular momentum of the atom and M_F is of its z-component quantum number. Hence in first order perturbation the energy correction due to H_{hf} is

$$\Delta E = \langle lsjIFM_F|H_{hf}|lsjIFM_F\rangle \,.$$

If $l \neq 0$, the wave function is zero at the origin and we only need to consider H_{hf} for $\mathbf{r} \neq 0$. Thus

$$\begin{aligned} H_{hf} &= \frac{2\mu_0 g_N \mu_N \mu_B}{4\pi\hbar^2}\left[\frac{\mathbf{I}\cdot\mathbf{l}}{r^3} + (\mathbf{s}\cdot\nabla)\left(\mathbf{I}\cdot\nabla\frac{1}{r}\right)\right] \\ &= \frac{2\mu_0 g_N \mu_N \mu_B}{4\pi\hbar^2 r^3}\mathbf{G}\cdot\mathbf{I}\,, \end{aligned}$$

where

$$\mathbf{G} = \mathbf{l} + 3\frac{(\mathbf{s}\cdot\mathbf{r})\mathbf{r}}{r^2}\,.$$

Hence

$$\begin{aligned} \Delta E &= \frac{2\mu_0 g_N \mu_N \mu_B}{4\pi\hbar^2}\left\langle\frac{1}{r^3}\mathbf{G}\cdot\mathbf{I}\right\rangle \\ &= \frac{\mu_0 g_N \mu_N \mu_B}{4\pi}\cdot\frac{l(l+1)}{j(j+1)}\cdot[F(F+1) - I(I+1) - j(j+1)]\left\langle\frac{1}{r^3}\right\rangle \\ &= \frac{\mu_0 g_N \mu_N \mu_B}{4\pi}\cdot\frac{Z^3}{a_0^3 n^3\left(l+\frac{1}{2}\right)j(j+1)}\cdot[F(F+1) \\ &\quad - I(I+1) - j(j+1)]\,, \end{aligned}$$

where a_0 is the Bohr radius and Z is the atomic number of the atom.

For $l = 0$, the wave function is spherically symmetric and

$$\Delta E = \frac{2\mu_0 g_N \mu_N \mu_B}{4\pi\hbar^2}\left[4\pi\langle\mathbf{s}\cdot\mathbf{I}\delta(\mathbf{r})\rangle + \left\langle(\mathbf{s}\cdot\nabla)\left(\mathbf{I}\cdot\nabla\frac{1}{r}\right)\right\rangle\right]\,.$$

As

$$\left\langle (\mathbf{s}\cdot\nabla)\left(\mathbf{I}\cdot\nabla\frac{1}{r}\right)\right\rangle = \left\langle \sum_{i,j=1}^{3} s_i I_j \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right)\right\rangle$$

$$= \left\langle \sum_{i,j=1}^{3} s_i I_j \frac{\partial^2}{\partial x_i^2}\left(\frac{1}{r}\right)\right\rangle + \left\langle \sum_{\substack{i,j=1\\ i\neq j}}^{3} s_i I_j \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{r}\right)\right\rangle$$

$$= \frac{1}{3}\left\langle \mathbf{s}\cdot\mathbf{I}\nabla^2\left(\frac{1}{r}\right)\right\rangle = -\frac{4\pi}{3}\langle \mathbf{s}\cdot\mathbf{I}\delta(\mathbf{r})\rangle\,,$$

we have

$$\Delta E = \frac{2\mu_0 g_N \mu_N \mu_B}{4\pi\hbar^2}\cdot\frac{8\pi}{3}\langle \mathbf{s}\cdot\mathbf{I}\delta(\mathbf{r})\rangle$$

$$= \frac{\mu_0 g_N \mu_N \mu_B}{4\pi}[F(F+1)-I(I+1)-s(s+1)]\cdot\frac{8\pi}{3}\langle\delta(\mathbf{r})\rangle$$

$$= \frac{2\mu_0 g_N \mu_N \mu_B}{3\pi}\cdot\frac{Z^3}{a_0^3 n^3}\cdot[F(F+1)-I(I+1)-s(s+1)]\,.$$

1054

What is meant by the fine structure and hyperfine structure of spectral lines? Discuss their physical origins. Give an example of each, including an estimate of the magnitude of the effect. Sketch the theory of one of the effects.

(*Princeton*)

Solution:

(a) *Fine structure:* The spectral terms as determined by the principal quantum number n and the orbital angular momentum quantum number l are split due to a coupling between the electron spin $\mathbf{s}$ and orbital angular momentum $\mathbf{l}$. Consequently the spectral lines arising from transitions between the energy levels are each split into several lines. For example, the spectral line arising from the transition $3p \to 3s$ of Na atom shows a doublet structure, the two yellow lines D_1 (5896 Å), D_2 (5890 Å) which are close to each other.

As an example of numerical estimation, consider the fine structure in hydrogen.

The magnetic field caused by the orbital motion of the electron is $B = \frac{\mu_0 ev}{4\pi r^2}$. The dynamic equation $\frac{mv^2}{r} = \frac{e^2}{4\pi\varepsilon_0 r^2}$ and the quantization condition $mvr = n\hbar$ give $v = \alpha c/n$, where $\alpha = \frac{e^2}{\hbar c}$ is the fine-structure constant, $n = 1, 2, 3, \ldots$. For the ground state $n = 1$. Then the interaction energy between the spin magnetic moment μ_s of the electron and the magnetic field B is

$$\Delta E \approx -\mu_s B \approx \frac{\mu_0 \mu_B \alpha ec}{4\pi r^2},$$

where $\mu_s = -\frac{e\hbar}{2m} = -\mu_B$, the Bohr magnetron. Take $r \approx 10^{-10}$ m, we find

$$\Delta E \approx 10^{-7} \times 10^{-23} \times 10^{-2} \times 10^{-19} \times 10^8/10^{-20} \approx 10^{-23}\ \text{J} \approx 10^{-4}\ \text{eV}.$$

Considering an electron moving in a central potential $V(\mathbf{r}) = -\frac{Ze^2}{4\pi\varepsilon_0 r}$, the interaction Hamiltonian between its orbital angular momentum about the center, $\mathbf{l}$, and spin $\mathbf{s}$ can be obtained quantum mechanically following the same procedure as

$$H' = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}(\mathbf{s}\cdot\mathbf{l}).$$

Taking H' as perturbation we then obtain the first order energy correction

$$\Delta E_{nlj} = \langle H' \rangle = \frac{Rhc\alpha^2 Z^4 \left[j(j+1) - l(l+1) - \frac{3}{4}\right]}{2n^3 l\left(l + \frac{1}{2}\right)(l+1)},$$

where R is the Rydberg constant, $\mathbf{j}$ is the total angular momentum of the electron.

As states with different j have different ΔE_{nlj}, an energy level (n, l) is split into two levels with $j = l + 1/2$ and $j = l - 1/2$.

(b) *Hyperfine structure:* Taking into account the coupling between the nuclear spin I and the total angular momentum j of the orbiting electron, an energy level determined by j will be split further, forming a hyperfine structure. Using an instrument of high resolution, we can see that the D_1 spectral line of Na atom is actually composed of two lines with a separation of 0.023 Å, and the D_2 line is composed of two lines separated by 0.021 Å.

For ground state hydrogen atom, the magnetic field caused by the electron at the nucleus is $B = \frac{\mu_0 ev}{4\pi a^2}$, where a is the Bohr radius. The hyperfine structure splitting is

$$\begin{aligned}\Delta E &\approx \mu_N B \approx \frac{\mu_0}{4\pi}\frac{\mu_N e\alpha c}{a^2}\\ &\approx 10^{-7} \times \frac{5\times 10^{-27}\times 1.6\times 10^{-19}\times 3\times 10^{8}}{137\times(0.53\times 10^{-10})^2}\ \mathrm{J}\\ &\approx 10^{-7}\ \mathrm{eV}\,.\end{aligned}$$

A theory of hyperfine structure is outlined in **Problem 1053**.

1055

Calculate, to an order of magnitude, the following properties of the $2p$-$1s$ electromagnetic transition in an atom formed by a muon and a strontium nucleus $(Z = 38)$:

(a) the fine-structure splitting,

(b) the natural line width. (Hint: the lifetime of the $2p$ state of hydrogen is 10^{-9} sec)

(*Princeton*)

Solution:

Taking into account the hyperfine structure corrections, the energy levels of a hydrogen-like atom are given by

$$\begin{aligned}E &= E_0 + \Delta E_r + \Delta E_{ls}\\ &= \begin{cases} -\dfrac{RhcZ^2}{n^2} - \dfrac{Rhc\alpha^2 Z^4}{n^3}\left(\dfrac{1}{l} - \dfrac{3}{4n}\right), & \left(j = l - \dfrac{1}{2}\right)\\ -\dfrac{RhcZ^2}{n^2} - \dfrac{Rhc\alpha^2 Z^4}{n^3}\left(\dfrac{1}{l+1} - \dfrac{3}{4n}\right). & \left(j = l + \dfrac{1}{2}\right)\end{cases}\end{aligned}$$

The $1s$ state is not split, but the $2p$ state is split into two substates corresponding to $j = 1/2$ and $j = 3/2$. The energy difference between the two lines of $2p \to 1s$ is

$$\Delta E = \frac{Rhc\alpha^2 Z^4}{n^3}\left(\frac{1}{l} - \frac{1}{l+1}\right),$$

where $Z = 38$, $n = 2$, $l = 1$, $R = m_\mu R_H/m_e \approx 200R_H = 2.2 \times 10^9\ \text{m}^{-1}$, $\alpha = \frac{1}{137}$. Hence

$$\Delta E = \frac{2.2 \times 10^9 \times 4.14 \times 10^{-15} \times 3 \times 10^8 \times 38^4}{2^3 \times 137^2 \times 2} = 1.9 \times 10^4\ \text{eV}\,.$$

(b) The lifetime of the $2p$ state of μ-mesic atom is

$$\tau_\mu = \frac{1}{Z^4} \cdot \frac{m_e}{m_\mu} \tau_H = 2.4 \times 10^{-18}\ \text{s}\,.$$

The uncertainty principle gives the natural width of the level as

$$\Gamma \approx \hbar/\tau_\mu = 2.7 \times 10^2\ \text{eV}\,.$$

1056

The lowest-energy optical absorption of neutral alkali atoms corresponds to a transition $ns \to (n+1)p$ and gives rise to a characteristic doublet structure. The intensity ratio of these two lines for light alkalis is 2; but as Z increases, so does the ratio, becoming 3.85 for Cs ($6s \to 7p$).

(a) Write an expression for the spin-orbit operator $N(r)$.

(b) In a hydrogenic atom, is this operator diagonal in the principal quantum number n? Is it diagonal in J?

(c) Using the following data, evaluate approximately the lowest order correction to the intensity ratio for the Cs doublet:

E_n = energy of the np state in cm^{-1},

I_n = transition intensity for the unperturbed states from the $6s$ state to the np state,

$$I_6/I_7 = 1.25\,, \qquad I_8/I_7 = 0.5\,,$$

Δn = spin-orbit splitting of the np state in cm^{-1},

$$\Delta_6 = 554 \qquad E_6 = -19950\,,$$

$$\Delta_7 = 181 \qquad E_7 = -9550\,,$$

$$\Delta_8 = 80 \qquad E8 = -5660\,.$$

In evaluating the terms in the correction, you may assume that the states can be treated as hydrogenic.

HINT: For small r, the different hydrogenic radial wave functions are proportional: $f_m(r) = k_{mn} f_n(r)$, so that, to a good approximation, $\langle 6p|N(r)|6p\rangle \approx k_{67}\ \langle 7p|N(r)|6p\rangle \approx k_{67}^2 \langle 7p|N(r)|7p\rangle$.

(*Princeton*)

Solution:

(a) The spin-orbit interaction Hamiltonian is

$$\begin{aligned} N(r) &= \frac{1}{2\mu^2 c^2 r}\frac{dV}{dr}\hat{\mathbf{s}}\cdot\hat{\mathbf{l}} \\ &= \frac{1}{4\mu^2 c^2 r}\frac{dV}{dr}(\hat{\mathbf{j}}^2 - \hat{\mathbf{l}}^2 - \hat{\mathbf{s}}^2)\,, \end{aligned}$$

where μ is the reduced mass, and $V = -\frac{Ze^2}{4\pi\varepsilon_0 r}$.

(b) The Hamiltonian is $H = H_0 + N(r)$. For hydrogen atom, $[H_0, N(r)] \neq 0$, so in the principal quantum number n, $N(r)$ is not diagonal. Generally,

$$\langle nlm|N(r)|klm\rangle \neq 0\,.$$

In the total angular momentum j (with fixed n), since $[N(r), \hat{\mathbf{j}}^2] = 0$, $N(r)$ is diagonal.

(c) The rate of induced transition is

$$W_{k'k} = \frac{4\pi^2 e^2}{3\hbar^2}|\mathbf{r}_{k'k}|^2 \rho(\omega_{k'k})$$

and the intensity of the spectral line is $I(\omega_{k'k}) \propto \hbar\omega_{k'k} W_{k'k}$.

With coupling between spin and orbital angular momentum, each np energy level of alkali atom is split into two sub-levels, corresponding to $j = 3/2$ and $j = 1/2$. However as the s state is not split, the transition $ns \to (n+1)p$ will give rise to a doublet. As the splitting of the energy level is very small, the frequencies of the $ns \to (n+1)p$ double lines can be taken to be approximately equal and so $I \propto |\mathbf{r}_{k'k}|^2$.

The degeneracy of the $j = 3/2$ state is 4, with $j_z = 3/2, 1/2, -1/2, -3/2$; the degeneracy of the $j = 1/2$ state is 2, with $j_z = 1/2, -1/2$. In the zeroth order approximation, the intensity ratio of these two lines is

$$\frac{I\left(j=\frac{3}{2}\right)}{I\left(j=\frac{1}{2}\right)}=\frac{\sum\limits_{j_z}\left|\left\langle (n+1)p\frac{3}{2}\right|\mathbf{r}\left|ns\right\rangle\right|^2}{\sum\limits_{j_z}\left|\left\langle (n+1)p\frac{1}{2}\right|\mathbf{r}\left|ns\right\rangle\right|^2}\approx 2\,,$$

as given. In the above $|(n+1)p, 1/2\rangle$, $|(n+1)p, 3/2\rangle$ are respectively the zeroth order approximate wave functions of the $j=1/2$ and $j=3/2$ states of the energy level $(n+1)p$.

To find the intensity ratio of the two lines of $6s\to 7p$ transition of Cs atom, take $N(r)$ as perturbation. First calculate the approximate wave functions:

$$\Psi_{3/2}=\left|7p\frac{3}{2}\right\rangle+\sum_{n=6}^{\infty}{}'\frac{\left\langle np\frac{3}{2}\right|N(r)\left|7p\frac{3}{2}\right\rangle}{E_7-E_n}\left|np\frac{3}{2}\right\rangle\,,$$

$$\Psi_{1/2}=\left|7p\frac{1}{2}\right\rangle+\sum_{n=6}^{\infty}{}'\frac{\left\langle np\frac{1}{2}\right|N(r)\left|7p\frac{1}{2}\right\rangle}{E_7-E_n}\left|np\frac{1}{2}\right\rangle\,,$$

and then the matrix elements:

$$|\langle\Psi_{3/2}|\mathbf{r}|6s\rangle|^2=\left|\left\langle 7p\frac{3}{2}\right|\mathbf{r}\left|6s\right\rangle+\sum_{n=6}^{\infty}{}'\frac{\left\langle np\frac{3}{2}\right|N(\mathbf{r})\left|7p\frac{3}{2}\right\rangle}{E_7-E_n}\left\langle np\frac{3}{2}\right|\mathbf{r}\left|6s\right\rangle\right|^2$$

$$\approx\left|\left\langle 7p\frac{3}{2}\right|\mathbf{r}\left|6s\right\rangle\right|^2\left|1+\sum_{n=6}^{\infty}{}'\frac{\left\langle np\frac{3}{2}\right|N(\mathbf{r})\left|7p\frac{3}{2}\right\rangle}{E_7-E_n}\sqrt{\frac{I_n}{I_7}}\right|^2\,,$$

$$|\langle\Psi_{1/2}|\mathbf{r}|6s\rangle|^2=\left|\left\langle 7p\frac{1}{2}\right|\mathbf{r}\left|6s\right\rangle+\sum_{n=6}^{\infty}{}'\frac{\left\langle np\frac{1}{2}\right|N(\mathbf{r})\left|7p\frac{1}{2}\right\rangle}{E_7-E_n}\left\langle np\frac{1}{2}\right|\mathbf{r}\left|6s\right\rangle\right|^2$$

$$\approx\left|\left\langle 7p\frac{1}{2}\right|\mathbf{r}\left|6s\right\rangle\right|^2\left|1+\sum_{n=6}^{\infty}{}'\frac{\left\langle np\frac{1}{2}\right|N(\mathbf{r})\left|7p\frac{1}{2}\right\rangle}{E_7-E_n}\sqrt{\frac{I_n}{I_7}}\right|^2\,,$$

where

$$\frac{\left\langle np\frac{3}{2}\middle|\mathbf{r}\middle|6s\right\rangle}{\left\langle 7p\frac{3}{2}\middle|\mathbf{r}\middle|6s\right\rangle} \approx \frac{\left\langle np\frac{1}{2}\middle|\mathbf{r}\middle|6s\right\rangle}{\left\langle 7p\frac{1}{2}\middle|\mathbf{r}\middle|6s\right\rangle} \approx \sqrt{\frac{I_n}{I_7}}\,.$$

As

$$\begin{aligned} N(r) &= \frac{1}{4\mu^2c^2r}\frac{dV}{dr}(\hat{\mathbf{j}}^2 - \hat{\mathbf{l}}^2 - \hat{\mathbf{s}}^2) \\ &= F(r)(\hat{\mathbf{j}}^2 - \hat{\mathbf{l}}^2 - \hat{\mathbf{s}}^2)\,, \end{aligned}$$

where

$$F(r) \equiv \frac{1}{4\mu^2c^2r}\frac{dV}{dr}\,,$$

we have

$$\begin{aligned} \left\langle np\frac{3}{2}\middle|N(\mathbf{r})\middle|7p\frac{3}{2}\right\rangle &= \left\langle np\frac{3}{2}\middle| F(r)(\hat{\mathbf{j}}^2 - \hat{\mathbf{l}}^2 - \hat{\mathbf{s}}^2) \middle|7p\frac{3}{2}\right\rangle \\ &= \left[\frac{3}{2}\times\left(\frac{3}{2}+1\right) - 1\times(1+1) - \frac{1}{2}\times\left(\frac{1}{2}+1\right)\right]\hbar^2 \\ &\quad \times \langle np|F(r)|7p\rangle = \hbar^2\langle np|F(r)|7p\rangle\,, \end{aligned}$$

$$\left\langle np\frac{1}{2}\middle| N(\mathbf{r}) \middle|7p\frac{1}{2}\right\rangle = -\,2\hbar^2\langle np|F(r)|7p\rangle\,.$$

For $n = 7$, as

$$\Delta_7 = \left\langle 7p\frac{3}{2}\middle| N(r) \middle|7p\frac{3}{2}\right\rangle - \left\langle 7p\frac{1}{2}\middle| N(r) \middle|7p\frac{1}{2}\right\rangle = 3\hbar^2\langle 7p|F(r)|7p\rangle\,,$$

we have

$$\langle 7p|F(r)|7p\rangle = \frac{\Delta_7}{3\hbar^2}\,.$$

For $n = 6$, we have

$$\left\langle 6p\frac{3}{2}\middle| N(r) \middle|7p\frac{3}{2}\right\rangle = \hbar^2\langle 6p|F(r)|7p\rangle = \hbar^2 k_{67}\langle 7p|F(r)|7p\rangle = \frac{k_{67}}{3}\Delta_7\,,$$

$$\begin{aligned} \left\langle 6p\frac{1}{2}\middle| N(r) \middle|7p\frac{1}{2}\right\rangle &= -2\hbar^2\langle 6p|F(r)|7p\rangle \\ &= -2\hbar^2 k_{67}\langle 7p|F(r)|7p\rangle = -\frac{2k_{67}}{3}\Delta_7\,. \end{aligned}$$

For $n = 8$, we have

$$\left\langle 8p\frac{3}{2}\right| N(r) \left|7p\frac{3}{2}\right\rangle = \frac{k_{87}}{3}\Delta_7 \,,$$

$$\left\langle 8p\frac{1}{2}\right| N(r) \left|7p\frac{1}{2}\right\rangle = -\frac{2k_{87}}{3}\Delta_7 \,.$$

In the above

$$k_{67} = \frac{\langle 6p|F(r)|7p\rangle}{\langle 7p|F(r)|7p\rangle} \,,$$

$$k_{87} = \frac{\langle 8p|F(r)|7p\rangle}{\langle 7p|F(r)|7p\rangle} \,.$$

Hence

$$|\langle \Psi_{3/2}|\mathbf{r}|6s\rangle|^2 = \left|\left\langle 7p\frac{3}{2}\right| \mathbf{r} \left|6s\right\rangle\right|^2$$
$$\times \left|1 + \frac{k_{67}\Delta_7}{3(E_7 - E_6)}\sqrt{\frac{I_6}{I_7}} + \frac{k_{87}\Delta_7}{3(E_7 - E_8)}\sqrt{\frac{I_8}{I_7}}\right|^2 \,,$$

$$|\langle \Psi_{1/2}|\mathbf{r}|6s\rangle|^2 = \left|\left\langle 7p\frac{1}{2}\right| \mathbf{r} \left|6s\right\rangle\right|^2$$
$$\times \left|1 - \frac{2k_{67}\Delta_7}{3(E_7 - E_6)}\sqrt{\frac{I_6}{I_7}} - \frac{2k_{87}\Delta_7}{3(E_7 - E_8)}\sqrt{\frac{I_8}{I_7}}\right|^2 \,.$$

As

$$\Delta_6 = \left\langle 6p\frac{3}{2}\right| N(r) \left|6p\frac{3}{2}\right\rangle - \left\langle 6p\frac{1}{2}\right| N(r) \left|6p\frac{1}{2}\right\rangle = 3\hbar^2\langle 6p|F(r)|6p\rangle$$
$$= 3\hbar^2 k_{67}^2 \langle 7p|F(r)|7p\rangle = k_{67}^2\Delta_7 \,,$$

we have

$$k_{67} = \sqrt{\frac{\Delta_6}{\Delta_7}} \,,$$

and similarly

$$k_{87} = \sqrt{\frac{\Delta_8}{\Delta_7}} \,.$$

Thus

$$\frac{I\left(j=\frac{3}{2}\right)}{I\left(j=\frac{1}{2}\right)}=\frac{\sum\limits_{j_z}|\langle\Psi_{3/2}|\mathbf{r}|6s\rangle|^2}{\sum\limits_{j_z}|\langle\Psi_{1/2}|\mathbf{r}|6s\rangle|^2}$$

$$\approx 2\left|\frac{1+\frac{k_{67}\Delta_7}{3(E_7-E_6)}\sqrt{\frac{I_6}{I_7}}+\frac{k_{87}\Delta_7}{3(E_7-E_8)}\sqrt{\frac{I_8}{I_7}}}{1-\frac{2k_{67}\Delta_7}{3(E_7-E_6)}\sqrt{\frac{I_6}{I_7}}-\frac{2k_{87}\Delta_7}{3(E_7-E_8)}\sqrt{\frac{I_8}{I_7}}}\right|^2$$

$$=2\left|\frac{1+\frac{\sqrt{\Delta_6\Delta_7}}{3(E_7-E_6)}\sqrt{\frac{I_6}{I_7}}+\frac{\sqrt{\Delta_8\Delta_7}}{3(E_7-E_8)}\sqrt{\frac{I_8}{I_7}}}{1-\frac{2\sqrt{\Delta_6\Delta_7}}{3(E_7-E_6)}\sqrt{\frac{I_6}{I_7}}-\frac{2\sqrt{\Delta_8\Delta_7}}{3(E_7-E_8)}\sqrt{\frac{I_8}{I_7}}}\right|^2=3.94\,,$$

using the data supplied.

1057

An atomic clock can be based on the (21-cm) ground-state hyperfine transition in atomic hydrogen. Atomic hydrogen at low pressure is confined to a small spherical bottle ($r \ll \lambda = 21$ cm) with walls coated by Teflon. The magnetically neutral character of the wall coating and the very short "dwell-times" of the hydrogen on Teflon enable the hydrogen atom to collide with the wall with little disturbance of the spin state. The bottle is shielded from external magnetic fields and subjected to a controlled weak and uniform field of prescribed orientation. The resonant frequency of the gas can be detected in the absorption of 21-cm radiation, or alternatively by subjecting the gas cell to a short radiation pulse and observing the coherently radiated energy.

(a) The Zeeman effect of these hyperfine states is important. Draw an energy level diagram and give quantum numbers for the hyperfine substates of the ground state as functions of field strength. Include both the weak and strong field regions of the Zeeman pattern.

(b) How can the energy level splitting of the strong field region be used to obtain a measure of the g-factor for the proton?

(c) In the weak field case one energy-level transition is affected little by the magnetic field. Which one is this? Make a rough estimate of the maximum magnetic field strength which can be tolerated with the resonance frequency shifted by $\Delta\nu < 10^{-10}\ \nu$.

(d) There is no Doppler broadening of the resonance line. Why is this?

(*Princeton*)

Solution:

(a) Taking account of the hyperfine structure and the Zeeman effect, two terms are to be added to the Hamiltonian of hydrogen atom:

$$H_{hf} = A\mathbf{I}\cdot\mathbf{J}\,, \qquad (A > 0)$$

$$H_B = -\boldsymbol{\mu}\cdot\mathbf{B}\,.$$

For the ground state of hydrogen,

$$I = \frac{1}{2}, \qquad J = \frac{1}{2}$$

$$\boldsymbol{\mu} = -g_e\frac{e\hbar}{2m_ec}\frac{\mathbf{J}}{\hbar} + g_p\frac{e\hbar}{2m_pc}\frac{\mathbf{I}}{\hbar}\,.$$

Letting

$$\frac{e\hbar}{2m_ec} = \mu_B\,, \qquad \frac{e\hbar}{2m_pc} = \mu_N$$

and using units in which $\hbar = 1$ we have

$$\boldsymbol{\mu} = -g_e\mu_B\mathbf{J} + g_p\mu_N\mathbf{I}\,.$$

(1) *Weak magnetic field case.* $\langle H_{nf}\rangle \gg \langle H_B\rangle$, we couple $\mathbf{I}$, $\mathbf{J}$ as $\mathbf{F} = \mathbf{I} + \mathbf{J}$. Then taking H_{hf} as the main Hamiltonian and H_B as perturbation we solve the problem in the representation of $\{\hat{\mathbf{F}}^2, \hat{\mathbf{I}}^2, \hat{\mathbf{J}}^2, \hat{\mathbf{F}}_z\}$. As

$$H_{hf} = \frac{A}{2}(\hat{\mathbf{F}}^2 - \hat{\mathbf{I}}^2 - \hat{\mathbf{J}}^2) = \frac{A}{2}\left(\hat{\mathbf{F}}^2 - \frac{1}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot\frac{3}{2}\right) = \frac{A}{2}\left(\hat{\mathbf{F}}^2 - \frac{3}{2}\right),$$

we have

$$\Delta E_{hf} = \begin{cases} -\dfrac{3}{4}A & \text{for } F = 0 \\ \dfrac{1}{4}A & \text{for } F = 1\,. \end{cases}$$

In the subspace of $\{\hat{\mathbf{F}}^2, \hat{\mathbf{F}}_z\}$, the Wigner-Ecart theory gives

$$\langle \boldsymbol{\mu} \rangle = \frac{(-g_e \mu_B \mathbf{J} + g_p \mu_N \mathbf{I}) \cdot \mathbf{F}}{F^2} \mathbf{F} .$$

As for $I = J = \frac{1}{2}$,

$$\mathbf{J} \cdot \mathbf{F} = \frac{1}{2}(\hat{\mathbf{F}}^2 + \hat{\mathbf{J}}^2 - \hat{\mathbf{I}}^2) = \frac{1}{2}\hat{\mathbf{F}}^2 ,$$

$$\mathbf{I} \cdot \mathbf{F} = \frac{1}{2}(\hat{\mathbf{F}}^2 + \hat{\mathbf{I}}^2 - \hat{\mathbf{J}}^2) = \frac{1}{2}\hat{\mathbf{F}}^2 ,$$

we have

$$\langle \boldsymbol{\mu} \rangle = -\frac{g_e \mu_B - g_p \mu_N}{2} \hat{\mathbf{F}} .$$

Then as

$$H_B = -\boldsymbol{\mu} \cdot \mathbf{B} = \frac{g_e \mu_B - g_p \mu_N}{2} B \hat{F}_z ,$$

we have

$$\Delta E_B = \begin{cases} E_1, & (F_z = 1) \\ 0, & (F_z = 0) \\ -E_1, & (F_z = -1) \end{cases}$$

where

$$E_1 = \frac{g_e \mu_B - g_p \mu_N}{2} B .$$

(2) *Strong magnetic field case.* As $\langle H_B \rangle \gg \langle H_{hf} \rangle$, we can treat H_B as the main Hamiltonian and H_{hf} as perturbation. With $\{\hat{\mathbf{J}}^2, \hat{\mathbf{I}}^2, \hat{\mathbf{J}}_z, \hat{\mathbf{I}}_z\}$ as a complete set of mechanical quantities, the base of the subspace is $|++\rangle$, $|+-\rangle$, $|-+\rangle$, $|--\rangle$ (where $|++\rangle$ means $J_z = +1/2$, $I_z = +1/2$, etc.). The energy correction is

$$\Delta E = \langle H_{hf} + H_B \rangle = \langle A I_z J_z \rangle + g_e \mu_B B \langle J_z \rangle - g_p \mu_N B \langle I_z \rangle$$

$$= \begin{cases} E_1 + \dfrac{A}{4} & \text{for } |++\rangle, \\ E_2 - \dfrac{A}{4} & \text{for } |+-\rangle, \\ -E_2 - \dfrac{A}{4} & \text{for } |-+\rangle, \\ -E_1 + \dfrac{A}{4} & \text{for } |--\rangle, \end{cases}$$

where

$$E_1 = \frac{g_e \mu_B - g_p \mu_N}{2} B ,$$

$$E_2 = \frac{g_e \mu_B + g_p \mu_N}{2} B .$$

The quantum numbers of the energy sublevels are given below and the energy level scheme is shown in Fig. 1.18.

quantum numbers	(F, J, I, F_z),	(J, I, J_z, I_z)
sublevel	$(1, 1/2, 1/2, 1)$	$(1/2, 1/2, 1/2, -1/2)$
	$(1, 1/2, 1/2, 0)$	$(1/2, 1/2, 1/2, 1/2)$
	$(1, 1/2, 1/2, -1)$	$(1/2, 1/2, -1/2, -1/2)$
	$(0, 1/2, 1/2, 0)$	$(1/2, 1/2, -1/2, 1/2)$

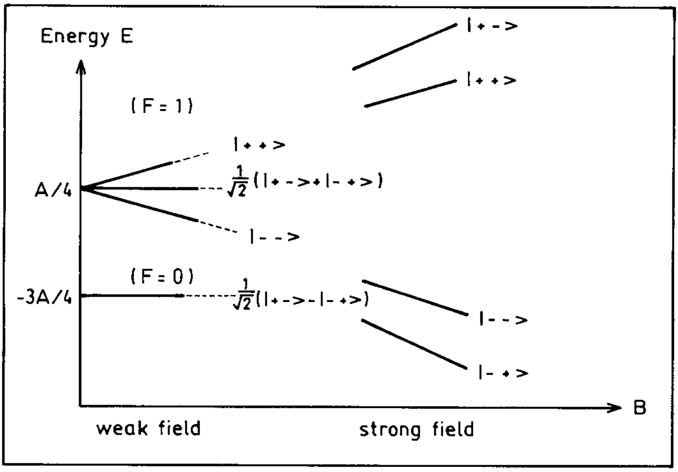

Fig. 1.18

(b) In a strong magnetic field, the gradients of the energy levels with respect to B satisfy the relation

$$\frac{\dfrac{\Delta E_{|+-\rangle}}{\Delta B}+\dfrac{\Delta E_{|--\rangle}}{\Delta B}}{\dfrac{\Delta E_{|+-\rangle}}{\Delta B}+\dfrac{\Delta E_{|++\rangle}}{\Delta B}}=\frac{g_p\mu_N}{g_e\mu_B}\,,$$

which may be used to determine g_p if the other quantities are known.

(c) In a weak magnetic field, the states $|F=1, F_z=0\rangle$, $|F=0, F_z=0\rangle$ are not appreciably affected by the magnetic field, so is the transition energy between these two states. This conclusion has been reached for the case of weak magnetic field ($A \gg E_1$) considering only the first order effect. It may be expected that the effect of magnetic field on these two states would appear at most as second order of E_1/A. Thus the dependence on B of the energy of the two states is

$$\left(\frac{E_1}{A}\right)^2\cdot A=\frac{E_1^2}{A}\,,$$

and so

$$\frac{\Delta\nu}{\nu}=\frac{\Delta E}{E}\approx\frac{\dfrac{E_1^2}{A}}{\dfrac{A}{4}-\left(-\dfrac{3A}{4}\right)}=\frac{E_1^2}{A^2}\approx\left(\frac{g_e\mu_B B}{2A}\right)^2\,,$$

neglecting $g_p\mu_N$. For $\Delta\nu/\nu<10^{-10}$ and the 21-cm line we have

$$A=\frac{1}{4}A-\left(-\frac{3}{4}A\right)=h\nu=\frac{2\pi\hbar c}{\lambda}\approx\frac{2\pi\times 2\times 10^{-5}}{21}=6\times 10^{-6}\ \mathrm{eV}\,,$$

and so

$$B\le\left(\frac{2A}{g_e\mu_B}\right)^2\times 10^{-5}=\left(\frac{2\times 6\times 10^{-6}}{2\times 6\times 10^{-9}}\right)\times 10^{-5}=10^{-2}\ \mathrm{Gs}\,.$$

(d) The resonance energy is very small. When photon is emitted, the ratio of the recoil energy of the nucleon to that of the photon E, $\Delta E/E\ll 1$. Hence the Doppler broadening caused by recoiling can be neglected.

1058

Consider an atom formed by the binding of an Ω^- particle to a bare Pb nucleus ($Z=82$).

(a) Calculate the energy splitting of the $n=10$, $l=9$ level of this atom due to the spin-orbit interaction. The spin of the Ω^- particle is 3/2. Assume a magnetic moment of $\boldsymbol{\mu}=\frac{e\hbar}{2mc}g\mathbf{p}_s$ with $g=2$ and $m=1672$ MeV/c^2.

Note:

$$\left\langle \frac{1}{r^3} \right\rangle = \left(\frac{mc^2}{\hbar c} \right)^3 (\alpha Z)^3 \frac{1}{n^3 l \left(l + \frac{1}{2} \right) (l+1)}$$

for a particle of mass m bound to a charge Z in a hydrogen-like state of quantum numbers (n, l).

(b) If the Ω^- has an electric quadrupole moment $Q \sim 10^{-26}$ cm^2 there will be an additional energy shift due to the interaction of this moment with the Coulomb field gradient $\partial E_z / \partial z$. Estimate the magnitude of this shift; compare it with the results found in (a) and also with the total transition energy of the $n = 11$ to $n = 10$ transition in this atom.

(*Columbia*)

Solution:

(a) The energy of interaction between the spin and orbital magnetic moments of the Ω^- particle is

$$\Delta E_{ls} = Z \boldsymbol{\mu}_l \cdot \boldsymbol{\mu}_s \left\langle \frac{1}{r^3} \right\rangle ,$$

where

$$\boldsymbol{\mu}_l = \frac{e}{2mc} \mathbf{p}_l , = \frac{e\hbar}{2mc} \mathbf{l} ,$$

$$\boldsymbol{\mu}_s = \frac{e}{mc} \mathbf{p}_s , = \frac{e\hbar}{mc} \mathbf{s} ,$$

$\mathbf{p}_l$, $\mathbf{p}_s$ being the orbital and spin angular momenta. Thus

$$\Delta E_{ls} = \frac{Ze^2\hbar^2}{2m^2c^2} \left\langle \frac{1}{r^3} \right\rangle \mathbf{l} \cdot \mathbf{s} .$$

As

$$\mathbf{l} \cdot \mathbf{s} = \frac{1}{2} [(\mathbf{l} + \mathbf{s})^2 - l^2 - s^2] ,$$

we have

$$\begin{aligned} \Delta E_{ls} &= \frac{Ze^2\hbar^2}{2m^2c^2} \left\langle \frac{1}{r^3} \right\rangle \frac{(\mathbf{j}^2 - \mathbf{l}^2 - \mathbf{s}^2)}{2} \\ &= \frac{(Z\alpha)^4 mc^2}{4} \left[\frac{j(j+1) - l(l+1) - s(s+1)}{n^3 l \left(l + \frac{1}{2} \right) (l+1)} \right] . \end{aligned}$$

With $Z = 82$, $m = 1672$ MeV/c^2, $s = 3/2$, $n = 10$, $l = 9$, $\alpha = \frac{1}{137}$, and $\langle 1/r^3 \rangle$ as given, we find $\Delta E_{ls} = 62.75 \times [j(j+1) - 93.75]$ eV. The results are given in the table below.

j	ΔE_{ls} (eV)	Level splitting (eV)
19/2	377	1193
17/2	−816	1067
15/2	−1883	941
13/2	−2824	

(b) The energy shift due to the interaction between the electric quadrupole moment Q and the Coulomb field gradient $\frac{\partial E_z}{\partial z}$ is

$$\Delta E_Q \approx Q \left\langle \frac{\partial E_z}{\partial z} \right\rangle ,$$

where $\frac{\partial E_z}{\partial z}$ is the average value of the gradient of the nuclear Coulomb field at the site of Ω^-. As

$$\left\langle \frac{\partial E_z}{\partial z} \right\rangle \approx - \left\langle \frac{1}{r^3} \right\rangle ,$$

we have

$$\Delta E_Q \approx -Q \left\langle \frac{1}{r^3} \right\rangle$$

in the atomic units of the hyperon atom which have units of length and energy, respectively,

$$a = \frac{\hbar^2}{me^2} = \frac{\hbar c}{mc^2}\left(\frac{\hbar c}{e^2}\right) = \frac{1.97 \times 10^{11}}{1672} \times 137 = 1.61 \times 10^{-12} \text{ cm} ,$$

$$\varepsilon = \frac{me^4}{\hbar^2} = mc^2\left(\frac{e^2}{\hbar c}\right)^2 = \frac{1672 \times 10^6}{137^2} = 8.91 \times 10^4 \text{ eV} .$$

For $n = 10$, $l = 9$, $\langle \frac{1}{r^3} \rangle = 1.53 \times 10^{35}$ cm^{-3} ≈ 0.6 a.u. With $Q \approx 10^{-26}$ cm^2 $\approx 4 \times 10^{-3}$ a.u., we have

$$\Delta E_Q \approx 2.4 \times 10^{-3} \text{ a.u.} \approx 2 \times 10^2 \text{ eV} .$$

The total energy resulting from a transition from $n = 11$ to $n = 10$ is

$$\Delta E = \frac{Z^2 mc^2}{2}\left(\frac{e^2}{\hbar c}\right)^2\left(\frac{1}{10^2} - \frac{1}{11^2}\right)$$
$$= \frac{82^2 \times 1672 \times 10^6}{2 \times 137^2}\left(\frac{1}{10^2} - \frac{1}{11^2}\right)$$
$$\approx 5 \times 10^5 \text{ eV}.$$

1059

What is the energy of the photon emitted in the transition from the $n = 3$ to $n = 2$ level of the μ^- mesic atom of carbon? Express it in terms of the γ energy for the electronic transition from $n = 2$ to $n = 1$ of hydrogen, given that $m_\mu/m_e = 210$.

(*Wisconsin*)

Solution:

The energy of the μ^- atom of carbon is

$$E_n(\mu) = \frac{Z^2 m_\mu}{m_e} E_n(H),$$

where $E_n(H)$ is the energy of the corresponding hydrogen atom, and $Z = 6$.

The energy of the photon emitted in the transition from $n = 3$ to $n = 2$ level of the mesic atom is

$$\Delta E = \frac{Z^2 m_\mu}{m_e}[E_3(H) - E_2(H)].$$

As

$$-E_n(H) \propto \frac{1}{n^2},$$

we have

$$\frac{36}{5}[E_3(H) - E_2(H)] = \frac{4}{3}[E_2(H) - E_1(H)],$$

and hence

$$\Delta E = \frac{5Z^2 m_\mu}{27 m_e}[E_2(H) - E_1(H)]$$
$$= 1400[E_2(H) - E_1(H)],$$

where $E_2(H) - E_1(H)$ is the energy of the photon emitted in the transition from $n = 2$ to $n = 1$ level of hydrogen atom.

1060

The muon is a relatively long-lived elementary particle with mass 207 times the mass of electron. The electric charge and all known interactions of the muon are identical to those of the electron. A "muonic atom" consists of a neutral atom in which one electron is replaced by a muon.

(a) What is the binding energy of the ground state of muonic hydrogen?

(b) What ordinary chemical element does muonic lithium ($Z = 3$) resemble most? Explain your answer.

(MIT)

Solution:

(a) By analogy with the hydrogen atom, the binding energy of the ground state of the muonic atom is

$$E_\mu = \frac{m_\mu e^4}{2\hbar^2} = 207E_H = 2.82 \times 10^3 \text{ eV}\,.$$

(b) A muonic lithium atom behaves chemically most like a He atom. As μ and electron are different fermions, they fill their own orbits. The two electrons stay in the ground state, just like those in the He atom, while the μ stays in its own ground state, whose orbital radius is $1/207$ of that of the electrons. The chemical properties of an atom is determined by the number of its outer most shell electrons. Hence the mesic atom behaves like He, rather than like Li.

1061

The Hamiltonian for a $(\mu^+ e^-)$ atom in the $n = 1$, $l = 0$ state in an external magnetic field is

$$H = a\mathbf{S}_\mu \cdot \mathbf{S}_e + \frac{|e|}{m_e c}\mathbf{S}_e \cdot \mathbf{B} - \frac{|e|}{m_\mu c}\mathbf{S}_\mu \cdot \mathbf{B}\,.$$

(a) What is the physical significance of each term? Which term dominates in the interaction with the external field?

(b) Choosing the z-axis along $\mathbf{B}$ and using the notation (F, M_F), where $\mathbf{F} = \mathbf{S}_\mu + \mathbf{S}_e$, show that $(1, +1)$ is an eigenstate of H and give its eigenvalue.

(c) An RF field can be applied to cause transition to the state (0,0). Describe quantitatively how an observation of the decay $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$ could be used to detect the occurrence of this transition.

(*Wisconsin*)

Solution:

(a) In the Hamiltonian, the first term, $a\mathbf{S}_\mu \cdot \mathbf{S}_e$, describes the electromagnetic interaction between μ^+ and e^-, the second and third terms respectively describe the interactions between the electron and μ^+ with the external magnetic field.

(b) Denote the state of $F = 1$, $M_F = +1$ with Ψ. As $\mathbf{F} = \mathbf{S}_\mu + \mathbf{S}_e$, we have

$$\mathbf{S}_\mu \cdot \mathbf{S}_e = \frac{1}{2}(\mathbf{F}^2 - \mathbf{S}_\mu^2 - \mathbf{S}_e^2) \, ,$$

and hence

$$\mathbf{S}_\mu \cdot \mathbf{S}_e \Psi = \frac{1}{2}(\mathbf{F}^2\Psi - \mathbf{S}_\mu^2\Psi - \mathbf{S}_e^2\Psi) = \frac{\hbar^2}{2}\left(2\Psi - \frac{3}{4}\Psi - \frac{3}{4}\Psi\right) = \frac{\hbar^2}{4}\Psi \, .$$

In the common eigenvector representation of $\mathbf{S}_e^z$, $\mathbf{S}_\mu^z$, the Ψ state is represented by the spinor

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_e \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_\mu \, .$$

Then

$$\mathbf{S}_e^z \Psi = \frac{\hbar}{2}\sigma_e^z \Psi = \frac{\hbar}{2}\Psi \, ,$$

$$\mathbf{S}_\mu^z \Psi = \frac{\hbar}{2}\sigma_\mu^z \Psi = \frac{\hbar}{2}\Psi \, ,$$

and so

$$\begin{aligned} H &= a\mathbf{S}_\mu \cdot \mathbf{S}_e \Psi + \frac{e}{m_e c} B S_e^z \Psi - \frac{e}{m_\mu c} B S_\mu^z \Psi \\ &= a\frac{\hbar^2}{4}\Psi + \frac{eB}{m_e c} \cdot \frac{\hbar}{2}\Psi - \frac{eB}{m_\mu c} \cdot \frac{\hbar}{2}\Psi \\ &= \left(\frac{1}{4}a\hbar^2 + \frac{eB}{2m_e c}\hbar - \frac{eB}{2m_\mu c}\hbar\right)\Psi \, . \end{aligned}$$

Hence the $(1, +1)$ state is an eigenstate of H with eigenvalue

$$\left(\frac{1}{4}a\hbar^2 + \frac{eB}{2m_e c}\hbar - \frac{eB}{2m_\mu c}\hbar\right) .$$

(c) The two particles in the state $(1, +1)$ have parallel spins, while those in the state (0,0) have anti-parallel spins. So relative to the direction of spin of the electron, the polarization directions of μ^+ in the two states are opposite. It follows that the spin of the positrons arising from the decay of μ^+ is opposite in direction to the spin of the electron. An (e^+e^-) pair annihilate to give rise to 3γ or 2γ in accordance with whether their spins are parallel or antiparallel. Therefore if it is observed that the (e^+e^-) pair arising from the decay $\mu^+ \to e^+\nu_e\tilde{\mu}_\mu$ annihilate to give 2γ, then it can be concluded that the transition is between the states $(1, +1)$ and (0,0).

1062

Muonic atoms consist of mu-mesons (mass $m_\mu = 206m_e$) bound to atomic nuclei in hydrogenic orbits. The energies of the mu mesic levels are shifted relative to their values for a point nucleus because the nuclear charge is distributed over a region with radius R. The effective Coulomb potential can be approximated as

$$V(r) = \begin{cases} -\dfrac{Ze^2}{r}, & (r \geq R) \\ -\dfrac{Ze^2}{R}\left(\dfrac{3}{2} - \dfrac{r^2}{2R^2}\right). & (r < R) \end{cases}$$

(a) State qualitatively how the energies of the $1s$, $2s$, $2p$, $3s$, $3p$, $3d$ muonic levels will be shifted absolutely and relative to each other, and explain physically any differences in the shifts. Sketch the unperturbed and perturbed energy level diagrams for these states.

(b) Give an expression for the first order change in energy of the $1s$ state associated with the fact that the nucleus is not point-like.

(c) *Estimate* the $2s$–$2p$ energy shift under the assumption that $R/a_\mu \ll 1$, where a_μ is the "Bohr radius" for the muon and show that this shift gives a measure of R.

(d) When is the method of part (b) likely to fail? Does this method underestimate or overestimate the energy shift. Explain your answer in physical terms.

Useful information:

$$\Psi_{1s} = 2N_0 \exp\left(-\frac{r}{a_\mu}\right) Y_{00}(\theta, \phi)\,,$$

$$\Psi_{2s} = \frac{1}{\sqrt{8}} N_0 \left(2 - \frac{r}{a_\mu}\right) \exp\left(-\frac{r}{2a_\mu}\right) Y_{00}(\theta, \phi)\,,$$

$$\Psi_{2p} = \frac{1}{\sqrt{24}} N_0 \frac{r}{a_\mu} \exp\left(-\frac{r}{2a_\mu}\right) Y_{1m}(\theta, \phi)\,,$$

$$N_0 = \frac{1}{a_\mu^{3/2}}\,.$$

(*Wisconsin*)

Solution:

(a) If nuclear charge is distributed over a finite volume, the intensity of the electric field at a point inside the nucleus is smaller than that at the same point if the nucleus is a point. Consequently the energy of the same state is higher in the former case. The probability of a $1s$ state electron staying in the nucleus is larger than that in any other state, so the effect of a finite volume of the nucleus on its energy level, i.e. the energy shift, is largest. Next come $2s$, $3s$, $2p$, $3p$, $3d$, etc. The energy levels are shown in Fig. 1.19.

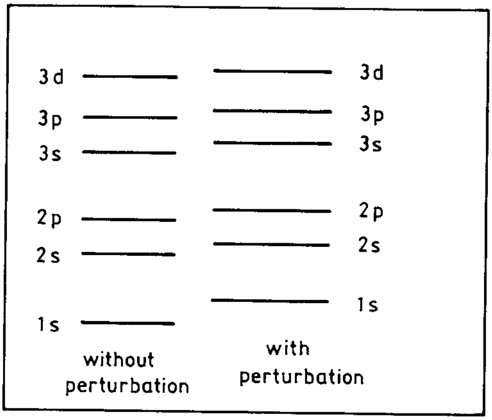

Fig. 1.19

(b) The perturbation potential due to the limited volume of nucleus has the form

$$\Delta V = \begin{cases} 0, & (r \geq R) \\ \dfrac{Ze^2}{R}\left(\dfrac{r^2}{2R^2} - \dfrac{3}{2} + \dfrac{R}{r}\right). & (r < R) \end{cases}$$

The first order energy correction of the $1s$ state with the approximation $\frac{R}{a_\mu} \ll 1$ is

$$\begin{aligned}
\Delta E_{1s} &= \int \Psi_{1s}^* \Delta V \Psi_{1s} d\tau \\
&= \frac{Ze^2}{R} 4N_0^2 \int_0^R \exp\left(-\frac{2r}{a_\mu}\right) \cdot \left(\frac{r^2}{2R^2} - \frac{3}{2} + \frac{R}{r}\right) r^2 dr \\
&\approx \frac{Ze^2}{R} 4N_0^2 \int_0^R \left(\frac{r^2}{2R^2} - \frac{3}{2} + \frac{R}{r}\right) r^2 dr \\
&= \frac{2Ze^2R^2}{5a_\mu^3} .
\end{aligned}$$

(c) The energy shifts for the $2s$ and $2p$ states are

$$\begin{aligned}
\Delta E_{2s} &= \int \Psi_{2s}^* \Delta V \Psi_{2s} d\tau \\
&= \frac{Ze^2N_0^2}{8R} \int_0^R \left(2 - \frac{r}{a_\mu}\right)^2 \exp\left(-\frac{r}{a_\mu}\right) \cdot \left(\frac{r^2}{2R^2} - \frac{3}{2} + \frac{R}{r}\right) r^2 dr \\
&\approx \frac{Ze^2N_0^2}{8R} \int_0^R 4\left(\frac{r^2}{2R^2} - \frac{3}{2} + \frac{R}{r}\right) r^2 dr \\
&= \frac{Ze^2R^2}{20a_\mu^3} , \\
\Delta E_{2p} &= \int \Psi_{2p}^* \Delta V \Psi_{2p} d\tau \\
&= \frac{Ze^2N_0^2}{24a_\mu^2 R} \int_0^R r^2 \exp\left(-\frac{r}{a_\mu}\right) \cdot \left(\frac{r^2}{2R^2} - \frac{3}{2} + \frac{R}{r}\right) r^2 dr
\end{aligned}$$

$$\approx \frac{Ze^2N_0^2}{24a_\mu^2 R}\int_0^R r^2\left(\frac{r^2}{2R^2}-\frac{3}{2}+\frac{R}{r}\right)r^2dr$$

$$=\frac{3Ze^2R^4}{3360a_\mu^5} \ll \Delta E_{2s}\,.$$

Hence the relative shift of $2s$–$2p$ is

$$\Delta E_{sp} \approx \Delta E_{2s} = \frac{Ze^2R^2}{20a_\mu^3}\,.$$

Thus R can be estimated from the relative shift of the energy levels.

(d) For large Z, $a_\mu = \frac{\hbar^2}{Zm_\mu e^2}$ becomes so small that $\frac{R}{a_\mu} \geq 1$. When $\frac{R}{a_\mu} \geq \frac{\sqrt{5}}{2}$, we have, using the result of (b),

$$\Delta E_{1s} = \frac{2Ze^2R^2}{5a_\mu^3} = \frac{4}{5}|E_{1s}^0|\left(\frac{R}{a_\mu}\right)^2 > |E_{1s}^0|\,,$$

where

$$E_{1s}^0 = -\frac{m_\mu Z^2e^4}{2\hbar^2}\,.$$

This means that $E_{1s} = E_{1s}^0 + \Delta E_{1s} > 0$, which is contradictory to the fact that E_{1s}, *a* bound state, is negative. Hence ΔE_{1s} as given by (b) is higher than the actual value. This is because we only included the zeroth order term in the expansion of $\exp(-\frac{2r}{a_\mu})$. Inclusion of higher order terms would result in more realistic values.

1063

Consider the situation which arises when a negative muon is captured by an aluminum atom (atomic number $Z = 13$). After the muon gets inside the "electron cloud" it forms a hydrogen-like muonic atom with the aluminum nucleus. The mass of the muon is 105.7 MeV.

(a) Compute the wavelength (in Å) of the photon emitted when this muonic atom decays from the $3d$ state. (Sliderule accuracy; neglect nuclear motion).

(b) Compute the mean life of the above muonic atom in the $3d$ state, taking into account the fact that the mean life of a hydrogen atom in the $3d$ state is 1.6×10^{-8} sec.

(*UC, Berkeley*)

Solution:

There are two energy levels in each of the $3d$, $3p$, $2p$ states, namely $3^2D_{5/2}$ and $3^2D_{3/2}$, $3^2P_{3/2}$ and $3^2P_{1/2}$, $2^2P_{3/2}$ and $2^2P_{1/2}$, respectively. There is one energy level each, $3^2S_{1/2}$, $2^2S_{1/2}$ and $1^2S_{1/2}$, in the $3s$, $2s$ and $1s$ states respectively.

The possible transitions are:

$$3^2D_{5/2} \to 3^2P_{3/2}, 3^2D_{5/2} \to 2^2P_{3/2}, 3^2D_{3/2} \to 3^2P_{1/2},$$
$$3^2D_{3/2} \to 2^2P_{3/2}, 3^2D_{3/2} \to 2^2P_{1/2}\,,$$
$$3^2P_{3/2} \to 3^2S_{1/2}, 3^2P_{3/2} \to 2^2S_{1/2}, 3^2P_{3/2} \to 1^2S_{1/2},$$
$$3^2P_{1/2} \to 2^2S_{1/2}, 3^2P_{1/2} \to 1^2S_{1/2}\,,$$
$$3^2S_{1/2} \to 2^2P_{3/2}, 3^2S_{1/2} \to 2^2P_{1/2}, 2^2P_{3/2} \to 2^2S_{1/2},$$
$$2^2P_{3/2} \to 1^2S_{1/2}, 2^2P_{1/2} \to 1^2S_{1/2}\,.$$

(a) The hydrogen-like mesic atom has energy

$$E = E_0 \left[\frac{1}{n^2} + \frac{\alpha^2 Z^2}{n^3}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right)\right],$$

where

$$E_0 = -\frac{2\pi^2 m_\mu e^4 Z^2}{(4\pi\varepsilon_0)^2 h^2} = -13.6 \times \frac{105.7}{0.511} \times 13^2 = -4.754 \times 10^5 \text{ eV}\,,$$

$\alpha = \frac{1}{137}$. Thus

$$\Delta E(3^2D_{5/2} \to 3^2P_{3/2}) = 26.42 \text{ eV}\,,$$

$$\Delta E(3^2D_{5/2} \to 2^2P_{3/2}) = 6.608 \times 10^4 \text{ eV}\,,$$

$$\Delta E(3^2D_{3/2} \to 3^2P_{1/2}) = 79.27 \text{ eV}\,,$$

$$\Delta E(3^2D_{3/2} \to 2^2P_{3/2}) = 6.596 \times 10^4 \text{ eV}\,,$$

$$\Delta E(3^2D_{3/2} \to 2^2P_{1/2}) = 6.632 \times 10^4 \text{ eV}\,,$$

$$\Delta E(3^2P_{3/2} \to 3^2S_{1/2}) = 79.27 \text{ eV}\,,$$

$$\Delta E(3^2P_{3/2} \to 2^2S_{1/2}) = 6.632 \times 10^4 \text{ eV}\,,$$

$$\Delta E(3^2P_{3/2} \to 1^2S_{1/2}) = 4.236 \times 10^5 \text{ eV}\,,$$

$$\Delta E(3^2P_{1/2} \to 2^2S_{1/2}) = 6.624 \times 10^4 \text{ eV}\,,$$

$$\Delta E(3^2P_{1/2} \to 1^2S_{1/2}) = 4.235 \times 10^5 \text{ eV}\,,$$

$$\Delta E(3^2S_{1/2} \to 2^2P_{3/2}) = 6.598 \times 10^5 \text{ eV}\,,$$

$$\Delta E(3^2S_{1/2} \to 2^2P_{1/2}) = 6.624 \times 10^4 \text{ eV}\,,$$

$$\Delta E(2^2P_{3/2} \to 2^2S_{1/2}) = 267.5 \text{ eV}\,,$$

$$\Delta E(2^2P_{3/2} \to 1^2S_{1/2}) = 3.576 \times 10^5 \text{ eV}\,,$$

$$\Delta E(2^2P_{1/2} \to 1^2S_{1/2}) = 3.573 \times 10^5 \text{ eV}\,.$$

Using the relation $\lambda = \frac{hc}{\Delta E} = \frac{12430}{\Delta E(\text{eV})}$Å, we obtain the wavelengths of the photons emitted in the decays of the $3d$ state: $\lambda = 470$ Å, 0.188 Å, 0.157 Å, 0.188 Å, 0.187 Å in the above order.

(b) The probability of a spontaneous transition is

$$P \propto \frac{e^2\omega^3}{\hbar c^3} R^2$$

with

$$\omega \propto \frac{m_\mu (Ze^2)^2}{\hbar^3}\,, \qquad R \propto \frac{\hbar^2}{m_\mu Ze^2}\,.$$

Thus

$$P \propto m_\mu (Ze^2)^4\,.$$

As the mean life of the initial state is

$$\tau = \frac{1}{P}\,,$$

the mean life of the $3d$ state of the μ mesic atom is

$$\tau = \frac{m_e \tau_0}{m_\mu Z^4} = 2.7 \times 10^{-15} \text{ s}\,.$$

where $\tau_0 = 1.6 \times 10^{-8}$ s is the mean life of a $3d$ state hydrogen atom.

1064

One method of measuring the charge radii of nuclei is to study the characteristic X-rays from exotic atoms.

(a) Calculate the energy levels of a μ^- in the field of a nucleus of charge Ze assuming a point nucleus.

(b) Now assume the μ^- is completely inside a nucleus. Calculate the energy levels assuming the nucleus is a uniform charge sphere of charge Ze and radius ρ.

(c) Estimate the energy of the K X-ray from muonic $^{208}Pb_{82}$ using the approximations in (a) or (b). Discuss the validity of these approximations.

NOTE: $m_\mu = 200 m_e$.

(*Princeton*)

Solution:

(a) The energy levels of μ^- in the field of a point nucleus with charge Ze are given by (**Problem 1035**)

$$E_n = Z^2 \frac{m_\mu}{m_e} E_n(H) = -Z^2 \times 200 \times \frac{13.6}{n^2}$$

$$= -\frac{2.72 \times 10^3}{n^2} Z^2 \text{ eV} ,$$

where $E_n(H)$ is the corresponding energy level of a hydrogen atom.

(b) The potential for μ^- moving in a uniform electric charge sphere of radius ρ is (**Problem 1050(a)**)

$$V(r) = -\frac{Ze^2}{\rho}\left(\frac{3}{2} - \frac{r^2}{2\rho^2}\right) = -\frac{3Ze^2}{2\rho} + \frac{1}{2}\left(\frac{Ze^2}{\rho^3}\right) r^2 .$$

The dependence of the potential on r suggests that the μ^- may be treated as an isotropic harmonic oscillator of eigenfrequency $\omega = \sqrt{\frac{Ze^2}{m_\mu \rho^3}}$. The energy levels are therefore

$$E_n = \hbar\omega\left(n + \frac{3}{2}\right) - \frac{3Ze^2}{2\rho} ,$$

where $n = 0, 1, 2, \ldots$, $\rho \approx 1.2 \times 10^{-13} A^{1/3}$ cm.

(c) K X-rays are emitted in the transitions of electron energy levels $n \geq 2$ to the $n = 1$ level.

The point-nucleus model (a) gives the energy of the X-rays as

$$\Delta E = E_2 - E_1 = -2.72 \times 10^3 \times 82^2 \left(\frac{1}{2^2} - 1 \right) = 1.37 \times 10^7 \text{ eV}.$$

The harmonic oscillator model (b) gives the energy of the X-rays as

$$\Delta E = E_2 - E_1 = \hbar\omega = \hbar \left(\frac{c}{\rho} \right) \sqrt{Z \frac{r_0}{\rho} \frac{m_e}{m_\mu}} = \frac{6.58 \times 10^{-16} \times 3 \times 10^{10}}{1.2 \times 10^{-13}}$$

$$\times \sqrt{\frac{82 \times 2.82 \times 10^{-13}}{208 \times 200 \times 1.2 \times 10^{-13}}} = 1.12 \times 10^7 \text{ eV},$$

where $r_0 = \frac{e^2}{m_e c^2} = 2.82 \times 10^{-13}$ cm is the classical radius of electron.

Discussion: As μ^- is much heavier than electron, it has a larger probability of staying inside the nucleus (first Bohr radius $a_0 \propto \frac{1}{m}$), which makes the effective nuclear charge $Z^* < Z$. Thus we may conclude that the energy of K X-rays as given by the point-nucleus model is too high. On the other hand, as the μ^- does have a finite probability of being outside the nucleus, the energy of the K X-rays as given by the harmonic oscillator model would be lower than the true value. As the probability of the μ^- being outside the nucleus decreases faster than any increase of Z, the harmonic oscillator model is closer to reality as compared to the point-nuclear model.

1065

A proposal has been made to study the properties of an atom composed of a π^+ ($m_{\pi^+} = 273.2\, m_e$) and a μ^- ($m_{\mu^-} = 206.77\, m_e$) in order to measure the charge radius of π^+ assuming that its charge is spread uniformly on a spherical shell of radius $r_0 = 10^{-13}$ cm and that the μ^- is a point charge. Express the potential as a Coulomb potential for a point charge plus a perturbation and use perturbation theory to calculate a numerical value for the percentage shift in the $1s$–$2p$ energy difference Δ (neglect spin orbit effects and Lamb shift). Given

$$a_0 = \frac{\hbar^2}{me^2},$$

$$R_{10}(r) = \left(\frac{1}{a_0}\right)^{3/2} 2\exp\left(-\frac{r}{a_0}\right),$$

$$R_{21}(r) = \left(\frac{1}{2a_0}\right)^{3/2} \frac{r}{a_0}\exp\left(-\frac{r}{a_0}\right)\cdot\frac{1}{\sqrt{3}}.$$

(*Wisconsin*)

Solution:

The potential function is

$$V(r) = \begin{cases} -e^2/r, & (r > r_0) \\ -e^2/r_0. & (r < r_0) \end{cases}$$

The Hamiltonian can be written as $H = H_0 + H'$, where H_0 is the Hamiltonian if π^+ is treated as a point charge, H' is taken as perturbation, being

$$H' = \begin{cases} 0, & (r > r_0) \\ e^2\left(\frac{1}{r} - \frac{1}{r_0}\right). & (r < r_0) \end{cases}$$

The shift of $1s$ level caused by H', to first order approximation, is

$$\Delta E_{1s} = \int \Psi^*_{1s}H'\Psi_{1s}d\tau = \int_0^{r_0} R^2_{10}(r)e^2\left(\frac{1}{r} - \frac{1}{r_0}\right)r^2dr \approx \frac{2e^2r_0^2}{3a_0^3},$$

assuming $r_0 \ll a_0$. The shift of $2p$ level is

$$\Delta E_{2p} = \int \Psi^*_{2p}H'\Psi_{2p}d\tau = \int_0^{r_0} R^2_{21}(r)e^2\left(\frac{1}{r} - \frac{1}{r_0}\right)r^2dr$$

$$\approx \frac{e^2r_0^4}{480a_0^5} \ll \Delta E_{1s},$$

using the same approximation. Thus

$$\Delta E_{1s} - \Delta E_{2p} \approx \Delta E_{1s} = \frac{2e^2r_0^2}{3a_0^3}.$$

Without considering the perturbation, the energy difference of $1s$–$2p$ is

$$\Delta = -\frac{me^4}{2\hbar^2}\left(\frac{1}{2^2} - 1\right) = \frac{3me^4}{8\hbar^2} = \frac{3e^2}{8a_0}.$$

Hence

$$\frac{\Delta E_{1s} - \Delta E_{2p}}{\Delta} \approx \frac{16}{9}\left(\frac{r_0}{a_0}\right)^2 .$$

As

$$m = \frac{m_{\mu^-} m_{\pi^+}}{m_{\mu^-} + m_{\pi^+}} = 117.7 m_e ,$$

we have

$$a_0 = \frac{\hbar^2}{me^2} = \left(\frac{\hbar^2}{m_e e^2}\right)\frac{m_e}{m} = \frac{0.53 \times 10^{-8}}{117.7} = 4.5 \times 10^{-11}\ \text{cm} ,$$

and hence

$$\frac{\Delta E_{1s} - \Delta E_{2p}}{\Delta} = \frac{16}{9} \times \left(\frac{10^{-3}}{4.5 \times 10^{-11}}\right)^2 = 8.8 \times 10^{-6} .$$

1066

A μ^- meson (a heavy electron of mass $M = 210 m_e$ with m_e the electron mass) is captured into a circular orbit around a proton. Its initial radius $R \approx$ the Bohr radius of an electron around a proton. Estimate how long (in terms of R, M and m_e) it will take the μ^- meson to radiate away enough energy to reach its ground state. Use classical arguments, including the expression for the power radiated by a nonrelativistic accelerating charged particle.

(CUSPEA)

Solution:

The energy of the μ^- is

$$E(r) = K(r) - \frac{e^2}{r} = -\frac{e^2}{2r} ,$$

where $K(r)$ is the kinetic energy.

The radiated power is $P = \frac{2e^2 a^2}{3c^3}$, where

$$a = \frac{F_{Coul}}{M} = \frac{e^2}{r^2 M}$$

is the centripetal acceleration. Energy conservation requires

$$\frac{dE}{dt} = -P\,,$$

i.e.,

$$\frac{e^2}{2r^2}\frac{dr}{dt} = -\frac{2e^2}{3c^3}\cdot\frac{e^4}{r^4M^2}\,.$$

Integration gives

$$R^3 - r^3 = \frac{4}{c^3}\cdot\frac{e^4}{M^2}t\,,$$

where R is the radius of the initial orbit of the μ mesion, being

$$R \approx \frac{\hbar^2}{me^2}\,.$$

At the μ ground state the radius of its orbit is the Bohr radius of the mesic atom

$$r_0 = \frac{\hbar^2}{Me^2}\,,$$

and the time t taken for the μ meson to spiral down to this state is given by

$$\left(\frac{\hbar^2}{e^2}\right)^3\left(\frac{1}{m^3} - \frac{1}{M^3}\right) = \frac{4e^4}{c^3M^2}t\,.$$

Since $M \gg m$, we have

$$\begin{aligned} t &\approx \frac{M^2c^3R^3}{4e^4} = \left(\frac{M}{m}\right)^2\left(\frac{mc^2}{e^2}\right)^2\frac{R^3}{4c} \\ &= 210^2 \times \left(\frac{5.3\times10^{-9}}{2.82\times10^{-13}}\right)^2 \times \frac{5.3\times10^{-9}}{4\times3\times10^{10}} = 6.9\times10^{-7}\ \text{s}\,. \end{aligned}$$

1067

Consider a hypothetical universe in which the electron has spin 3/2 rather than spin 1/2.

(a) Draw an energy level diagram for the $n = 3$ states of hydrogen in the absence of an external magnetic field. Label each state in spectroscopic notation and indicate which states have the same energy. Ignore hyperfine structure (interaction with the nuclear spin).

(b) Discuss qualitatively the energy levels of the two-electron helium atom, emphasizing the differences from helium containing spin 1/2 electrons.

(c) At what values of the atomic number would the first two inert gases occur in this universe?

(*Columbia*)

Solution:

(a) Consider a hydrogen atom having electron of spin 3/2. For $n = 3$, the possible quantum numbers are given in Table 1.1.

Table 1.1

n	l	j
	0	3/2
3	1	5/2, 3/2, 1/2
	2	7/2, 5/2, 3/2, 1/2

If fine structure is ignored, these states are degenerate with energy

$$E_n = -\frac{RhcZ^2}{n^2}$$

where $Z = 1, n = 3, R$ is the Rydberg constant, c is the speed of light.

If the relativistic effect and spin-orbit interactions are taken into account, the energy changes into $E = E_0 + \Delta E$ and degeneracy disappears, i.e., different states have different energies.

(1) For $l = 0$ and $j = 3/2$, there is only the correction ΔE_r arising from the relativistic effect, i.e.,

$$\Delta E = \Delta E_r = -A\left(\frac{1}{l+\frac{1}{2}} - \frac{3}{4n}\right) = -\frac{7}{4}A\,,$$

where $A = Rhc\alpha^2 Z^4/n^3$, α being the fine structure constant.

(2) For $l \neq 0$, in addition to ΔE_r there is also the spin-orbital coupling correction ΔE_{ls}, so that

$$\Delta E = \Delta E_r + \Delta E_{ls} = -A\left(\frac{1}{l+\frac{1}{2}} - \frac{3}{4n}\right)$$

$$+A\frac{1}{l\left(l+\frac{1}{2}\right)(l+1)}\cdot\frac{j(j+1)-l(l+1)-s(s+1)}{2}\,.$$

(i) For $l=1$,

$$\Delta E=\left[\frac{1}{6}j(j+1)-\frac{11}{8}\right]A\,,$$

Thus for

$$j=\frac{5}{2},\qquad \Delta E=\frac{1}{12}A\,,$$

$$j=\frac{3}{2},\qquad \Delta E=-\frac{3}{4}A\,,$$

$$j=\frac{1}{2},\qquad \Delta E=-\frac{5}{4}A\,.$$

(ii) For $l=2$,

$$\Delta E=\left[\frac{1}{30}j(j+1)-\frac{19}{40}\right]A\,,$$

Thus for

$$j=\frac{7}{2},\qquad \Delta E=\frac{1}{20}A\,,$$

$$j=\frac{5}{2},\qquad \Delta E=-\frac{11}{60}A\,,$$

$$j=\frac{3}{2},\qquad \Delta E=-\frac{7}{20}A\,.$$

$$j=\frac{1}{2},\qquad \Delta E=-\frac{9}{20}A\,.$$

The energy level scheme for $n=3$ of the hydrogen atom is shown in Fig. 1.20.

(b) Table 1.2 shows the single-electron energy levels of the helium atoms (electron spins 1/2 and 3/2).

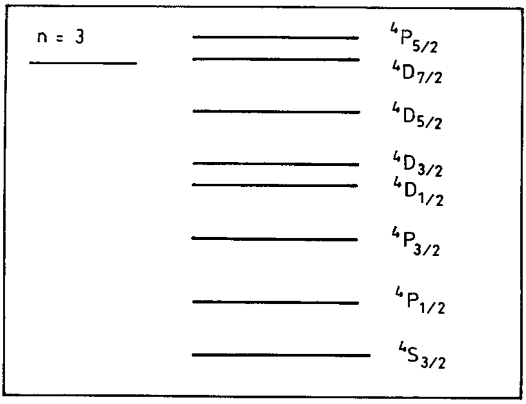

Fig. 1.20

Table 1.2

		He ($s = 3/2$)	He ($s = 1/2$)
$n_1 = 1$	Total electron spin	$S = 0, 2$	$S = 0$
$n_2 = 1$			
$l = 0$	energy level	$^1S_0$, $^5S_2$	$^1S_0$
$n_1 = 1$	Total electron spin	$S = 0, 1, 2, 3$	$S = 0, 1$
$n_2 = 2$			
$l_2 = 0, 1$	energy level	$l_2 = 0$: $^1S_0$, $^3S_1$, $^5S_2$, $^7S_3$	$l_2 = 0$: $^1S_0$, $^3S_1$
		$l_2 = 1$: $^1P_1$, $^3P_{2,1,0}$, $^5P_{3,2,1}$	$l_2 = 1$: $^1P_1$, $^3P_{2,1,0}$
		$^7P_{4,3,2}$	

(c) If the electron spin were 3/2, the atomic numbers Z of the first two inert elements would be 4 and 20.

1068

Figure 1.21 shows the ground state and first four excited states of the helium atom.

(a) Indicate on the figure the complete spectroscopic notation of each level.

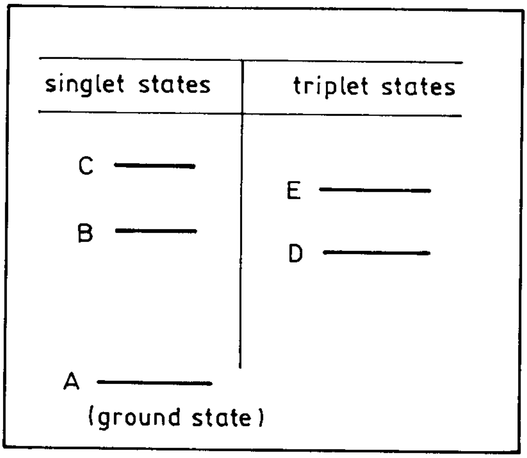

Fig. 1.21

(b) Indicate, with arrows on the figure, the allowed radiative dipole transitions.

(c) Give a qualitative reason why level B is lower in energy than level C.

(*Wisconsin*)

Solution:

(a) The levels in Fig. 1.21 are as follows:

A: 1^1S_0, constituted by $1s^2$,

B: 2^1S_0, constituted by $1s2s$,

C: 2^1P_1, constituted by $1s2p$,

D: 2^3S_1, constituted by $1s2s$,

E: $2^3P_{2,1,0}$, constituted by $1s2p$.

(b) The allowed radiative dipole transitions are as shown in Fig. 1.22. (Selection rules $\Delta L = \pm 1$, $\Delta S = 0$)

(c) In the C state constituted by $1s2p$, one of the electrons is excited to the $2p$ orbit, which has a higher energy than that of $2s$. The main reason is that the effect of the screening of the nuclear charge is larger for the p orbit.

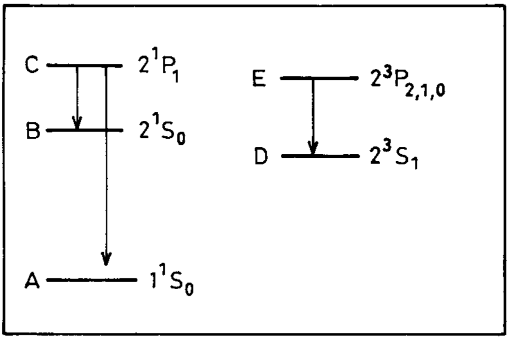

Fig. 1.22

1069

Figure 1.23 shows the ground state and the set of $n = 2$ excited states of the helium atom. Reproduce the diagram in your answer giving

(a) the spectroscopic notation for all 5 levels,

(b) an explanation of the source of ΔE_1,

(c) an explanation of the source of ΔE_2,

(d) indicate the allowed optical transitions among these five levels.

(Wisconsin)

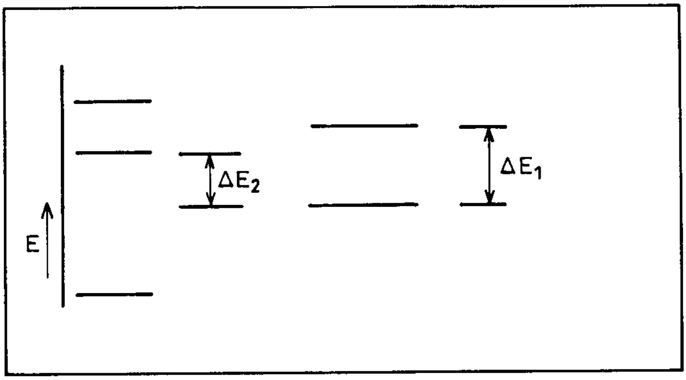

Fig. 1.23

Solution:

(a) See **Problem 1068(a)**.

(b) ΔE_1 is the difference in energy between different electronic configurations with the same S. The ^3P states belong to the configuration of $1s2p$, which has one electron in the $1s$ orbit and the other in the $2p$ orbit. The latter has a higher energy because the screening of the nuclear charge is greater for the p electron.

(c) ΔE_2 is the energy difference between levels of the same L in the same electronic configuration but with different S. Its origin lies in the Coulomb exchange energy.

(d) See **Problem 1068(b)**.

1070

Figure 1.24 is an energy level diagram for the ground state and first four excited states of a helium atom.

(a) On a copy of the figure, give the complete spectroscopic notation for each level.

(b) List the possible electric-dipole allowed transitions.

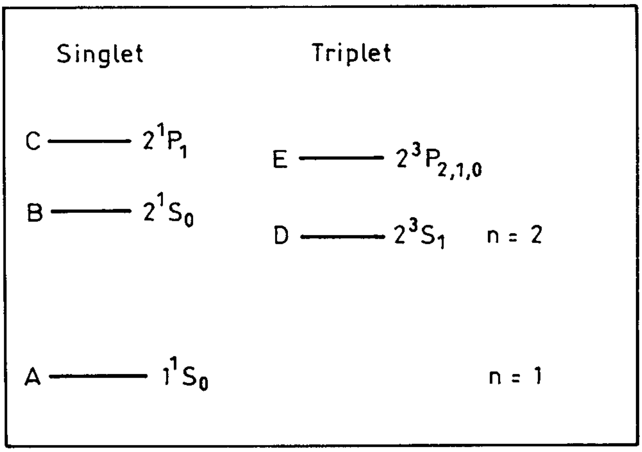

Fig. 1.24

(c) List the transitions between those levels that would be possible for an allowed 2-photon process (both photons electric dipole).

(d) Given electrons of sufficient energy, which levels could be populated as the result of electrons colliding with ground state atoms?

(*Wisconsin*)

Solution:

(a) (b) See **problem 1068**.

(c) The selection rule for a 2-photon process are

(1) conservation of parity,

(2) $\Delta J = 0, \pm 2$.

Accordingly the possible 2-photon process is

$$(1s2s)^1S_0 \to (1s^2)^1S_0 \,.$$

The transition $(1s2s)^3S$ to $(1s^2)^1S_0$ is also possible via the 2-photon process with a rate $10^{-8} \sim 10^{-9}$ s^{-1}. It has however been pointed out that the transition $2^3S_1 \to 1^1S_0$ could proceed with a rate $\sim 10^{-4}$ s via magnetic dipole radiation, attributable to some relativistic correction of the magnetic dipole operator relating to spin, which need not satisfy the condition $\Delta S = 0$.

(d) The $(1s2s)^1S_0$ and $(1s2s)^3S_1$ states are metastable. So, besides the ground state, these two levels could be populated by many electrons due to electrons colliding with ground state atoms.

1071

Sketch the low-lying energy levels of atomic He. Indicate the atomic configuration and give the spectroscopic notation for these levels. Indicate several transitions that are allowed in emission, several transitions that are allowed in absorption, and several forbidden transitions.

(*Wisconsin*)

Solution:

The energy levels of He are shown in Fig. 1.25.

According to the selection rules $\Delta S = 0$, $\Delta L = \pm 1$, $\Delta J = 0, \pm 1$ (except $0 \to 0$), the allowed transitions are: $3^1S_0 \to 2^1P_1$, $3^3S_1 \to 2^3P_{2,1,0}$, $2^1P_1 \to 1^1S_0$, $2^1P_1 \to 2^1S_0$, $3^3D_1 \to 3^3P_0$, $3^3D_{2,1} \to 3^3P_1$, $3^3D_{3,2,1} \to 3^3P_2$,

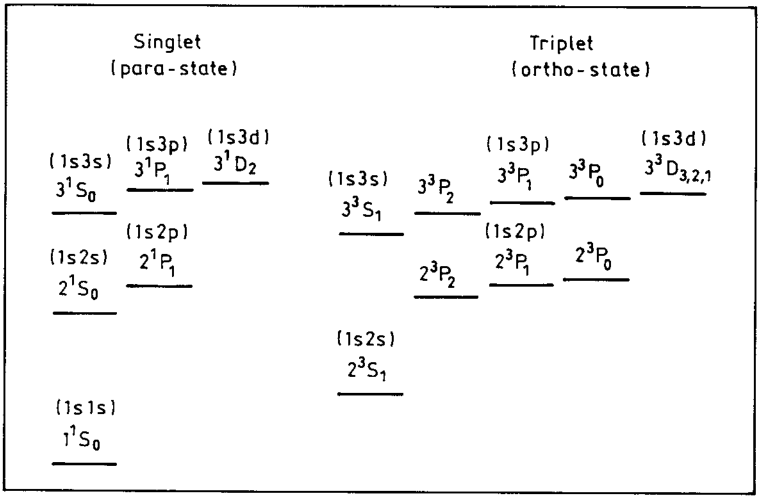

Fig. 1.25

$3^1D_2 \to 3^1P_1$, $3^1D_2 \to 2^1P_1$, $3^3D_1 \to 2^3P_{1,0}$, $3^3D_{3,2,1} \to 2^3P_2$, $3^3P_{2,1,0} \to 2^3S_1$. The reverse of the above are the allowed absorption transitions. Transitions between singlet and triplet states ($\Delta S \neq 0$) are forbidden, e.g. $2^3S_1 \to 1^1S_0$, $2^1P_1 \to 2^3S_1$.

1072

Sketch the energy level diagram for a helium atom in the $1s3d$ configuration, taking into account Coulomb interaction and spin-orbit coupling.

(UC, Berkeley)

Solution:

See **Problem 1100**.

1073

For helium atom the only states of spectroscopic interest are those for which at least one electron is in the ground state. It can be constructed from orthonormal orbits of the form

$$\Psi_{\pm}(1,2)=\frac{1}{\sqrt{2}}[\Phi_{1s}(1)\Phi_{nlm}(2)\pm\Phi_{nlm}(1)\Phi_{1s}(2)]\times \text{spin wave function}\,.$$

The para-states correspond to the + sign and the ortho-states to the − sign.

(a) Determine for which state the ortho- or the corresponding para-state has the lowest energy. (i.e. most negative).

(b) Present an argument showing for large n that the energy difference between corresponding ortho- and para-states should become small.

(*SUNY, Buffalo*)

Solution:

(a) For fermions like electrons the total wave function of a system must be antisymmetric.

If both electrons of a helium atom are in $1s$ orbit, Pauli's principle requires that their spins be antiparallel, i.e. the total spin function be antisymmetric. Then the spatial wave function must be symmetric and the state is the para-state 1^1S_0.

If only one electron is in $1s$ orbit, and the other is in the nlm-state, where $n\neq 1$, their spins may be either parallel or antiparallel and the spatial wave functions are, respectively,

$$\Psi_{\mp}=\frac{1}{\sqrt{2}}[\Phi_{1s}(1)\Phi_{nlm}(2)\mp\Phi_{nlm}(1)\Phi_{1s}(2)]\,.$$

Ignoring magnetic interactions, consider only the Coulomb repulsion between the electrons and take as perturbation $H'=e^2/r_{12}$, r_{12} being the distance between the electrons. The energy correction is then

$$\begin{aligned}W'_{\mp}=&\frac{1}{2}\iint[\Phi^*_{1s}(1)\Phi^*_{nlm}(2)\mp\Phi^*_{nlm}(1)\Phi^*_{1s}(2)]\\&\times\frac{e^2}{r_{12}}[\Phi_{1s}(1)\Phi_{nlm}(2)\mp\Phi_{nlm}(1)\Phi_{1s}(2)]d\tau_1d\tau_2\\=&J\mp K\end{aligned}$$

with

$$J=\iint\frac{e^2}{r_{12}}|\Phi_{1s}(1)\Phi_{nlm}(2)|^2d\tau_1d\tau_2\,,$$

$$K=\iint\frac{e^2}{r_{12}}\Phi^*_{1s}(1)\Phi_{nlm}(1)\Phi^*_{nlm}(2)\Phi_{1s}(2)d\tau_1d\tau_2\,.$$

Hence the ortho-state ($-$ sign above) has lower corrected energy. Thus para-helium has ground state 1^1S_0 and ortho-helium has ground state 2^3S_1, which is lower in energy than the 2^1S_0 state of para-helium (see Fig. 1.25).

(b) As n increases the mean distance r_{12} between the electrons increases also. This means that the energy difference 2K between the para- and ortho-states of the same electron configuration decreases as n increases.

1074

(a) Draw and qualitatively explain the energy level diagram for the $n = 1$ and $n = 2$ levels of helium in the nonrelativistic approximation.

(b) Draw and discuss a similar diagram for hydrogen, including all the energy splitting that are actually present.

(*CUSPEA*)

Solution:

(a) In the lowest energy level ($n = 1$) of helium, both electrons are in the lowest state $1s$. Pauli's principle requires the electrons to have antiparallel spins, so that the $n = 1$ level is a singlet. On account of the repulsion energy between the electrons, e^2/r_{12}, the ground state energy is higher than $2Z^2E_0 = 8E_0$, where $E_0 = -\frac{me^4}{2\hbar^2} = -13.6$ eV is the ground state energy of hydrogen atom.

In the $n = 2$ level, one electron is in $1s$ state while the other is in a higher state. The two electrons can have antiparallel or parallel spins (singlet or triplet states). As the probability for the electrons to come near each other is larger in the former case, its Coulomb repulsion energy between the electrons, e^2/r_{12}, is also larger. Hence in general a singlet state has higher energy than the corresponding triplet state (Fig. 1.26).

(b) The energy levels of hydrogen atom for $n = 1$ and $n = 2$ are shown in Fig. 1.27. If one considers only the Coulomb interaction between the nucleus and electron, the (Bohr) energy levels are given by

$$E_n = -\frac{m_e e^4}{2\hbar^2 n^2}\,,$$

which is a function of n only. If the relativistic effect and the spin-orbit interaction of the electron are taken into account, the $n = 2$ level splits into two levels with a spacing $\approx \alpha^2 E_2$, where α is the fine structure constant.

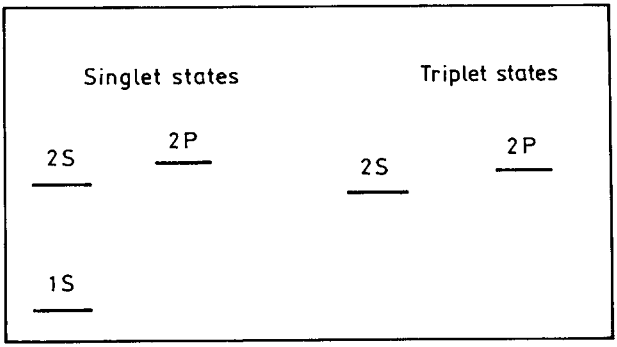

Fig. 1.26

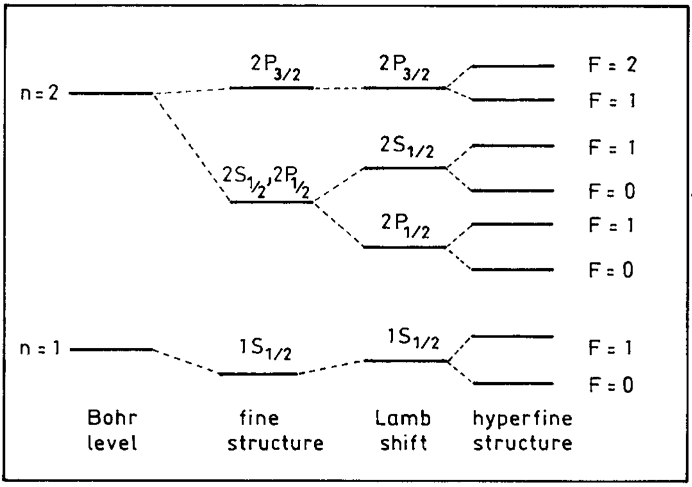

Fig. 1.27

If one considers, further, the interaction between the electron and its own magnetic field and vacuum polarization, Lamb shift results splitting the degenerate $2S_{1/2}$ and $2P_{1/2}$ states, the splitting being of the order $m_e c^2 \alpha^5$.

In addition, the levels split further on account of the interactions between the spin and orbital motions of the electron and the nuclear magnetic moment, giving rise to a hyperfine structure with spacing about 1/10 of the Lamb shift for the same n.

1075

(a) The $1s2s$ configuration of the helium atom has two terms $^3S_1$ and $^1S_0$ which lie about 20 eV above the ground state. Explain the meaning of the spectroscopic notation. Also give the reason for the energy splitting of the two terms and estimate the order of magnitude of the splitting.

(b) List the ground-state configurations and the lowest-energy terms of the following atoms: He, Li, Be, B, C, N, O, F and A.

Possible useful numbers:

$$a_B = 0.529\times 10^{-8}\ \text{cm}, \quad \mu_B = 9.27\times 10^{-21}\ \text{erg/gauss}, \quad e = 4.8\times 10^{-10}\ \text{esu}\,.$$

(*Princeton*)

Solution:

(a) The spectroscopic notation indicates the state of an atom. For example in $^3S_1$, the superscript 3 indicates the state is a triplet ($3 = 2S+1$), the subscript 1 is the total angular momentum quantum number of the atom, $J = S + L = 1$, S labels the quantum state corresponding to the orbital angular momentum quantum number $L = 0$ (S for $L = 0$, P for $L = 1$, D for $L = 2$, etc.).

The split in energy of the states $^1S_0$ and $^3S_1$ arises from the difference in the Coulomb interaction energy between the electrons due to their different spin states. In the $1s2s$ configuration, the electrons can have antiparallel or parallel spins, giving rise to singlet and triplet states of helium, the approximate energy of which can be obtained by perturbation calculations to be (**Problem 1073**)

$$E(\text{singlet}) = -\frac{Z^2e^2}{2a_0}\left(1+\frac{1}{2^2}\right) + J + K\,,$$

$$E(\text{triplet}) = -\frac{Z^2e^2}{2a_0}\left(1+\frac{1}{2^2}\right) + J - K\,,$$

where J is the average Coulomb energy between the electron clouds, K is the exchange energy. The splitting is

$$\Delta E = 2K$$

with

$$\begin{aligned} K &= e^2 \iint d^3x_1 d^3x_2 \frac{1}{r_{12}} \Psi^*_{100}(r_1)\Psi_{200}(r_1)\Psi_{100}(r_2)\Psi^*_{200}(r_2) \\ &= \frac{4Z^6e^2}{a_0^6}\left[\int_0^\infty r_1^2 \left(1 - \frac{Zr_1}{2a_0}\right)\exp\left(-\frac{3Zr_1}{2a_0}\right) dr_1\right]^2 \\ &\approx \frac{2^4 Ze^2}{3^6 a_0} . \end{aligned}$$

Thus

$$\begin{aligned} K &= \frac{2^5 e^2}{3^6 a_0} = \frac{2^5}{3^6}\frac{me^4}{\hbar^2} = \frac{2^5}{3^6}\left(\frac{e^2}{\hbar c}\right)^2 mc^2 \\ &= \frac{2^5}{3^6}\left(\frac{1}{137}\right)^2 \times 0.511 \times 10^6 = 1.2 \text{ eV} , \end{aligned}$$

and $\Delta E \approx 2$ eV.

(b)

Atom	Ground state configuration	Lowest-energy spectral term
He	$1s^2$	$^1S_0$
Li	$1s^2 2s^1$	$^2S_{1/2}$
Be	$1s^2 2s^2$	$^1S_0$
B	$1s^2 2s^2 2p^1$	$^2P_{1/2}$
C	$1s^2 2s^2 2p^2$	$^3P_0$
N	$1s^2 2s^2 2p^3$	$^4S_{3/2}$
O	$1s^2 2s^2 2p^4$	$^3P_2$
F	$1s^2 2s^2 2p^5$	$^2P_{3/2}$
A	$1s^2 2s^2 2p^6 3s^2 3p^6$	$^1S_0$

1076

Use a variational method, a perturbation method, sum rules, and/or other method to obtain crude estimates of the following properties of the helium atom:

(a) the minimum energy required to remove both electrons from the atom in its ground state,

(b) the minimum energy required to remove one electron from the atom in its lowest F state ($L = 3$), and

(c) the electric polarizability of the atom in its ground state. (The lowest singlet P state lies ~ 21 eV above the ground state.)

(*Princeton*)

Solution:

(a) In the *perturbation method*, the Hamiltonian of helium atom is written as

$$H = \frac{p_1^2}{2m_e} + \frac{p_2^2}{2m_e} - \frac{2e^2}{r_1} - \frac{2e^2}{r_2} + \frac{e^2}{r_{12}} = H_0 + \frac{e^2}{r_{12}}\,,$$

where

$$H_0 = \frac{p_1^2}{2m_e} + \frac{p_2^2}{2m_e} - \frac{2e^2}{r_1} - \frac{2e^2}{r_2}$$

is considered the unperturbed Hamiltonian, and the potential due to the Coulomb repulsion between the electrons as perturbation. The zero-order approximate wave function is then

$$\psi = \psi_{100}(r_1)\psi_{100}(r_2)\,,$$

where

$$\psi_{100}(r) = \frac{1}{\sqrt{\pi}}\left(\frac{2}{a}\right)^{3/2} e^{-2r/a}\,,$$

a being the Bohr radius. The zero-order (unperturbed) ground state energy is

$$E^{(0)} = 2\left(-\frac{2^2e^2}{2a}\right) = -\frac{4e^2}{a}\,,$$

where the factor 2 is for the two $1s$ electrons. The energy correction in first order perturbation is

$$E^{(1)} = \int |\psi_{100}|^2 \frac{e^2}{r_{12}} d\mathbf{r}_1 d\mathbf{r}_2 = \frac{5e^2}{4a}\,.$$

Hence the corrected ground state energy is

$$E = -\frac{4e^2}{a} + \frac{5e^2}{4a} = -\frac{11}{2}\cdot\frac{e^2}{2a} = -\frac{11}{2}\times 13.6 = -74.8\ \text{eV}\,,$$

and the ionization energy of ground state helium atom, i.e. the energy required to remove both electrons from the atom, is

$$E_I = -E = 74.8 \text{ eV} .$$

In the *variational method*, take as the trial wave function

$$\psi = \frac{\lambda^3}{\pi a^3} e^{-\lambda(r_1+r_2)/a} .$$

We then calculate

$$\begin{aligned}\langle H\rangle &= \iint \psi^* \left(-\frac{\hbar^2}{2m_e}\nabla_1^2 - \frac{\hbar^2}{2m_e}\nabla_2^2 - \frac{2e^2}{r_1} - \frac{2e^2}{r_2} + \frac{e^2}{r_{12}}\right)\psi d\mathbf{r}_1 d\mathbf{r}_2 \\ &= \left(2\lambda^2 - \frac{27}{4}\lambda\right) E_H ,\end{aligned}$$

where

$$E_H = \frac{e^2}{2a} = 13.6 \text{ eV} .$$

Minimizing $\langle H\rangle$ by taking

$$\frac{\partial\langle H\rangle}{\partial\lambda} = 0 ,$$

we find $\lambda = \frac{27}{16}$ and so

$$\langle H\rangle = \frac{27}{16}\left(\frac{27}{8} - \frac{27}{4}\right) E_H = -77.5 \text{ eV} .$$

The ionization energy is therefore $E_I = -\langle H\rangle = 77.5$ eV, in fairly good agreement with the perturbation calculation.

(b) In the lowest F state the electron in the $l = 3$ orbit is so far from the nucleus that the latter together with the $1s$ electron can be treated as a core of charge $+e$. Thus the excited atom can be considered as a hydrogen atom in the state $n = 4$. The ionizaion energy E_I, i.e. the energy required to remove one electron from the atom, is

$$E_I = -E = \frac{Ze^2}{2a4^2} = \frac{1}{16}\left(\frac{e^2}{2a}\right) = \frac{13.6}{16} = 0.85 \text{ eV} .$$

(c) Consider a perturbation u. The wave function and energy for the ground state, correct to first order, are

$$\Psi = \Psi_0 + \sum_{n\neq 0} \frac{u_{n0}}{E_0 - E_n}\Psi_n\,, \qquad E = E_0 + u_{00} + \sum_{n\neq 0} \frac{(u_{n0})^2}{E_0 - E_n}\,,$$

where Ψ_0, E_n are the unperturbed wave function and energy, and $u_{n0} \equiv \langle 0|u|n\rangle$. Write

$$\sum_{n\neq 0} u_{n0}\Psi_n = \sum_{n=0} u_{n0}\psi_n - u_{00}\psi_0 = u\psi_0 - u_{00}\psi_0\,,$$

with $u\psi_0 = \sum_{n=0} u_{n0}\psi_n$. Then

$$\Psi \approx \Psi_0 \left(1 + \frac{u - u_{00}}{E'}\right)\,,$$

E' being the average of $E_0 - E_n$.

The average total kinetic energy of the electrons is calculated using a variational method with $\psi = (1 + \lambda u)\psi_0$ as trial function:

$$\langle T\rangle = \frac{\int \Psi_0^*(1+\lambda u)\hat{T}\Psi_0(1+\lambda u)d\mathbf{r}}{\int \Psi_0^*\Psi_0(1+\lambda u)^2 d\mathbf{r}}\,,$$

where

$$\hat{T} = \frac{1}{2m_e}(p_1^2 + p_2^2) = -\frac{\hbar^2}{2m_e}(\nabla_1^2 + \nabla_2^2)\,,$$

or, in atomic units ($a_0 = \hbar = e = 1$),

$$\hat{T} = -\frac{1}{2}\sum_{i=1}^{2} \nabla_i^2\,.$$

Thus

$$\hat{T} \propto -\frac{1}{2}\sum_{i=1}^{2}\frac{1}{2}\int \{\Psi_0^*(1+\lambda u)\nabla_i^2(1+\lambda u)\Psi_0 + \Psi_0(1+\lambda u)$$

$$\times \nabla_i^2(1+\lambda u)\Psi_0^*\}d\mathbf{r}$$

$$= -\frac{1}{2}\sum_{i=1}^{2}\frac{1}{2}\int \{\Psi_0^*(1+\lambda u)^2\nabla_i^2\Psi_0 + \Psi_0(1+\lambda u)^2\nabla_i^2\Psi_0^*$$

$$+ 2\lambda\Psi_0\Psi_0^*(1+\lambda u)\nabla_i^2 u + 2\lambda(1+\lambda u)\nabla_i(\Psi_0\Psi_0^*)\cdot\nabla_i u\}d\mathbf{r}\,.$$

Consider

$$\sum_i \int \nabla_i \cdot [\psi_0 \psi_0^*(1+\lambda u)\nabla_i u] d\mathbf{r} = \oint_S \psi_0 \psi_0^*(1+\lambda u) \sum_i \nabla_i u \cdot d\mathbf{S} = 0$$

by virtue of Gauss' divergence theorem and the fact that $-\nabla_i u$ represents the mutual repulsion force between the electrons. As

$$\begin{aligned}\nabla_i \cdot [\Psi_0 \Psi_0^*(1+\lambda u)\nabla_i u] =& \Psi_0 \Psi_0^*(1+\lambda u)\nabla_i^2 u + (1+\lambda u)\nabla_i(\Psi_0 \Psi_0^*) \cdot \nabla_i u \\ & + \lambda \Psi_0 \Psi_0^* \nabla_i u \cdot \nabla_i u \,,\end{aligned}$$

we can write

$$\begin{aligned}&\int \{\Psi_0 \Psi_0^*(1+\lambda u)\nabla_i^2 u + (1+\lambda u)\nabla_i(\Psi_0 \Psi_0^*) \cdot \nabla_i u\} d\mathbf{r} \\ &\qquad = -\lambda \int \Psi_0 \Psi_0^* \nabla_i u \cdot \nabla_i u d\mathbf{r} \,.\end{aligned}$$

Hence

$$\begin{aligned}\langle T \rangle \propto & -\frac{1}{2} \sum_{i=1}^{2} \frac{1}{2} \int [\Psi_0^*(1+\lambda u)^2 \nabla_i^2 \Psi_0 + \Psi_0 (1+\lambda u)^2 \nabla_i^2 \Psi_0^*] d\mathbf{r} \\ & + \frac{\lambda^2}{2} \sum_{i=1}^{2} \int \Psi_0 \Psi_0^* \nabla_i u \cdot \nabla_i u d\mathbf{r} \,.\end{aligned}$$

The total energy E can be similarly obtained by considering the total Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{T} + u \,.$$

As $\hat{H}$ and $(1+\lambda u)$ commute, we have

$$\begin{aligned}\langle H \rangle &= \frac{\frac{1}{2} \int (1+\lambda u)^2 (\Psi_0^* \hat{H} \Psi_0 + \Psi_0 \hat{H} \Psi_0^*) d\mathbf{r} + \frac{\lambda^2}{2} \sum_{i=1}^{2} \int \Psi_0^* \Psi_0 \nabla_i u \cdot \nabla_i u d\mathbf{r}}{\int \Psi_0^* \Psi_0 (1+\lambda u)^2 d\mathbf{r}} \\ &= E_0 + \frac{\frac{1}{2} \int \Psi_0^* u (1+\lambda u)^2 \Psi_0 d\mathbf{r} + \frac{\lambda^2}{2} \sum_{i=1}^{2} \int \Psi_0^* \Psi_0 \nabla_i u \cdot \nabla_i u d\mathbf{r}}{\int \Psi_0^* \Psi_0 (1+\lambda u)^2 d\mathbf{r}}\end{aligned}$$

$$= E_0 + \frac{(u)_{00} + 2\lambda(u^2)_{00} + \lambda^2(u^3)_{00} + \dfrac{1}{2}\lambda^2 \displaystyle\sum_{i=1}^{2} \int [\nabla_i u \cdot \nabla_i u]_{00} d\mathbf{r}}{1 + 2\lambda(u)_{00} + \lambda^2(u^2)_{00}} ,$$

where E_0 is given by $\hat{H}\psi_0 = E_0\psi_0$, $(u)_{00} = \int \Psi_0^* u \Psi_0 d\mathbf{r}$, $(u^2)_{00} = \int \Psi_0^* u^2 \Psi_0 d\mathbf{r}$, etc. Neglecting the third and higher order terms, we have the energy correction

$$\Delta E \approx (u)_{00} + 2\lambda(u^2)_{00} - 2\lambda(u)_{00}^2 + \frac{1}{2}\lambda^2 \sum_{i=1}^{2} [(\nabla_i u) \cdot (\nabla_i u)]_{00} .$$

Minimizing ΔE by putting

$$\frac{d\Delta E}{d\lambda} = 0 ,$$

we obtain

$$2(u^2)_{00} - 2(u)_{00}^2 + \lambda \sum_{i=1}^{2} [(\nabla_i u) \cdot (\nabla_i u)]_{00} = 0 ,$$

or

$$\lambda = \frac{2[(u)_{00}^2 - (u^2)_{00}]}{\displaystyle\sum_{i=1}^{2} [\nabla_i u \cdot \nabla_i u]_{00}} .$$

This gives

$$\Delta E = (u)_{00} - \frac{2[(u)_{00}^2 - (u^2)_{00}]^2}{\displaystyle\sum_{i=1}^{2} [\nabla_i u \cdot \nabla_i u]_{00}} .$$

Consider a He atom in an electric field of strength ε whose direction is taken to be that of the z-axis. Then

$$u = -\varepsilon(z_1 + z_2) \equiv -\varepsilon z .$$

As the matrix element $(u)_{00}$ is zero for a spherically symmetric atom, we have

$$\Delta E \approx -\frac{2[(z^2)_{00}]^2 \varepsilon^4}{2\varepsilon^2} = -[(z^2)_{00}]^2 \varepsilon^2 .$$

The energy correction is related to the electric field by

$$\Delta E = -\frac{1}{2}\alpha\varepsilon^2 ,$$

where α is the polarizability. Hence

$$\alpha = 2[(z^2)_{00}]^2 = 2\langle(z_1 + z_2)^2\rangle^2 .$$

As $\langle z_1^2\rangle = \langle z_2^2\rangle \approx a'^2 = \frac{a_0^2}{Z^2}$, $\langle z_1 z_2\rangle = 0$, where a_0 is the Bohr radius, using $Z = 2$ for He we have

$$\alpha = \frac{8\hbar^2}{e^2 m_e}\frac{a_0^4}{2^4} \approx \frac{1}{2}a_0^3$$

in usual units. If the optimized $Z = \frac{27}{16}$ from (a) is used,

$$\alpha = 8\left(\frac{16}{27}\right)^4 a_0^3 = 0.98a_0^3 .$$

1077

Answer each of the following questions with a brief, and, where possible, quantitative statement. Give your reasoning.

(a) A beam of neutral atoms passes through a Stern-Gerlach apparatus. Five equally spaced lines are observed. What is the total angular momentum of the atom?

(b) What is the magnetic moment of an atom in the state $^3P_0$? (Disregard nuclear effects)

(c) Why are noble gases chemically inert?

(d) Estimate the energy density of black body radiation in this room in erg/cm^3. Assume the walls are black.

(e) In a hydrogen gas discharge both the spectral lines corresponding to the transitions $2^2P_{1/2} \to 1^2S_{1/2}$ and $2^2P_{3/2} \to 1^2S_{1/2}$ are observed. Estimate the ratio of their intensities.

(f) What is the cause for the existence of two independent term-level schemes, the singlet and the triplet systems, in atomic helium?

(*Chicago*)

Solution:

(a) The total angular momentum of an atom is

$$P_J = \sqrt{J(J+1)}\hbar .$$

As the neutral-atom beam splits into five lines, we have $2J+1=5$, or $J=2$. Hence

$$P_J=\sqrt{6}\hbar\,.$$

(b) The state has total angular momentum quantum number $J=0$. Hence its magnetic moment is $M=g\mu_B\sqrt{J(J+1)}=0$.

(c) The electrons of a noble gas all lie in completed shells, which cannot accept electrons from other atoms to form chemical bonds. Hence noble gases are chemically inert.

(d) The energy density of black body radiation is $u=4J_u/c$, where J_u is the radiation flux density given by the Stefan-Boltzmann's law

$$J_u=\sigma T^4\,,$$

$$\sigma=5.669\times10^{-5}\ \mathrm{erg\,cm^{-2}\,K^{-4}\,s^{-1}}\,.$$

At room temperature, $T=300$ K, and

$$\begin{aligned}u&=\frac{4}{3\times10^{10}}\times5.669\times10^{-5}\times300^4\\&=6.12\times10^{-5}\ \mathrm{erg\cdot cm^{-3}}\,.\end{aligned}$$

(e) The degeneracies of $2^2P_{1/2}$ and $2^2P_{3/2}$ are 2 and 4 respectively, while the energy differences between each of them and $1^2S_{1/2}$ are approximately equal. Hence the ratio of the intensities of the spectral lines $(2^2P_{1/2}\to 1^2S_{1/2})$ to $(2^2P_{3/2}\to 1^2S_{1/2})$ is 1:2.

(f) The LS coupling between the two electrons of helium produces $S=0$ (singlet) and $S=1$ (triplet) states. As the transition between them is forbidden, the spectrum of atomic helium consists of two independent systems (singlet and triplet).

1078

(a) Make a table of the atomic ground states for the following elements: H, He, Be, B, C, N, indicating the states in spectroscopic notation. Give J only for S states.

(b) State Hund's rule and give a physical basis for it.

(*Wisconsin*)

Solution:

(a) The atomic ground states of the elements are as follows:

element:	H	He	Li	Be	B	C	N
ground state:	$^2S_{1/2}$	$^1S_0$	$^2S_{1/2}$	$^1S_0$	$^2P_{1/2}$	$^3P_0$	$^4S_{3/2}$

(b) For a statement of Hund's rules see **Problem 1008**. Hund's rules are empirical rules based on many experimental results and their application is consequently restricted. First, they are reliable only for determining the lowest energy states of atoms, except those of very heavy elements. They fail in many cases when used to determine the order of energy levels. For example, for the electron configuration $1s^2 2s2p^3$ of Carbon, the order of energy levels is obtained experimentally as $^5S <^3 D <^1 D <^3 S <^1 P$. It is seen that although ^3S is a higher multiplet, its energy is higher than that of ^1D. For higher excited states, the rules may also fail. For instance, when one of the electrons of Mg atom is excited to d-orbital, the energy of ^1D state is lower than that of ^3D state.

Hund's rules can be somewhat understood as follows. On account of Pauli's exclusion principle, equivalent electrons of parallel spins tend to avoid each other, with the result that their Coulomb repulsion energy, which is positive, tends to be smaller. Hence energies of states with most parallel spins (with largest S) will be the smallest. However the statement regarding states of maximum angular momentum cannot be so readily explained.

1079

(a) What are the terms arising from the electronic configuration $2p3p$ in an (LS) Russell-Saunders coupled atom? Sketch the level structure, roughly show the splitting, and label the effect causing the splitting.

(b) What are the electric-dipole transition selection rules for these terms?

(c) To which of your forbidden terms could electric dipole transitions from a $^3P_1$ term be made?

(*Wisconsin*)

Solution:

(a) The spectroscopic terms arising from the electronic configuration $2p3p$ in LS coupling are obtained as follows.

As $l_1 = l_2 = 1$, $s_1 = s_2 = \frac{1}{2}$, $\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2$, $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$, $\mathbf{J} = \mathbf{L} + \mathbf{S}$, we can have $S = 1, 0$, $L = 2, 1, 0$, $J = 3, 2, 1, 0$.

For $S = 0$, $L = 2$, $J = 2$: ${}^1D_2$, $L = 1$, $J = 1$: ${}^1P_1$, $L = 0$, $J = 0$: ${}^1S_0$. For $S = 1$, $L = 2$, $J = 3, 2, 1, 0$: ${}^3D_{3,2,1}$, $L = 1$, $J = 2, 1, 0$: ${}^3P_{2,1,0}$, $L = 0$, $J = 1$: ${}^3S_1$. Hence the terms are

$$\begin{aligned} &\text{singlet}: {}^1S_0, \quad {}^1P_1, \quad {}^1D_2 \\ &\text{triplet}: {}^3S_1, \quad {}^3P_{2,1,0}, \quad {}^3D_{3,2,1} \end{aligned}$$

The corresponding energy levels are shown in Fig. 1.28.

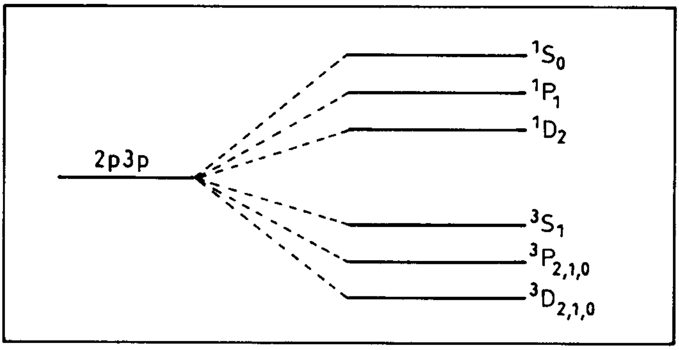

Fig. 1.28

Splitting of spectroscopic terms of different S is caused by the Coulomb exchange energy. Splitting of terms of the same S but different L is caused by the Coulomb repulsion energy. Splitting of terms of the same L, S but different J is caused by the coupling between orbital angular momentum and spin, i.e., by magnetic interaction.

(b) Selection rules for electric-dipole transitions are

(i) Parity must be reversed: even $\leftrightarrow$ odd.

(ii) Change in quantum numbers must satisfy

$$\Delta S = 0, \quad \Delta L = 0, \pm 1, \quad \Delta J = 0, \pm 1 \qquad (\text{excepting } 0 \to 0)\,.$$

Electric-dipole transition does not take place between these spectral terms because they have the same parity.

(c) If the $^3P_1$ state considered has odd parity, it can undergo transition to the forbidden spectral terms $^3S_0$, $^3P_{2,1,0}$, $^3D_{2,1}$.

1080

The atoms of lead vapor have the ground state configuration $6s^2 6p^2$.

(a) List the quantum numbers of the various levels of this configuration assuming LS coupling.

(b) State whether transitions between these levels are optically allowed, i.e., are of electric-dipole type. Explain why or why not.

(c) Determine the total number of levels in the presence of a magnetic field **B**.

(d) Determine the total number of levels when a weak electric field **E** is applied together with **B**.

(*Chicago*)

Solution:

(a) The two $6s$ electrons fill the first subshell. They must have anti-parallel spins, forming state $^1S_0$. Of the two $6p$ electrons, their orbital momenta can add up to a total $L = 0, 1, 2$. Their total spin quantum number S is determined by Pauli's exclusion principle for electrons in the same subshell, which requires $L + S =$ even (**Problem 2054(a)**). Hence $S = 0$ for $L = 0, 2$ and $S = 1$ for $L = 1$. The configuration thus has three "terms" with different L and S, and five levels including the fine structure levels with equal L and S but different J. The spectroscopic terms for configuration are therefore

$$^1S_0, ^3P_{0,1,2}, ^1D_2 \, .$$

(b) Electric-dipole transitions among these levels which have the same configuration are forbidden because the levels have the same parity.

(c) In a magnetic field each level with quantum number J splits into $2J + 1$ components with different M_J. For the $6p^2$ levels listed above the total number of sublevels is $1 + 1 + 3 + 5 + 5 = 15$.

(d) The electric field **E** perturbs the sublevels but causes no further splitting because the sublevels have no residual degeneracy. In other words, the applied electric field does not cause new splitting of the energy levels, whose total number is still 15.

1081

Consider a multi-electron atom whose electronic configuration is $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p 4d$.

(a) Is this element in the ground state? If not, what is the ground state?

(b) Suppose a Russell-Saunders coupling scheme applies to this atom. Draw an energy level diagram roughly to scale beginning with a single unperturbed configuration and then taking into account the various interactions, giving the perturbation term involved and estimating the energy split. Label the levels at each stage of the diagram with the appropriate term designation.

(c) What are the allowed transitions of this state to the ground state, if any?

(*Columbia*)

Solution:

(a) The atom is not in the ground state, which has the outermost-shell electronic configuration $4p^2$, corresponding to atomic states ${}^1D_2$, ${}^3P_{2,1,0}$ and ${}^1S_0$ (**Problem 1080**), among which ${}^3P_0$ has the lowest energy.

(b) The energy correction arising from LS coupling is

$$\begin{aligned}\Delta E =& a_1 \mathbf{s}_1 \cdot \mathbf{s}_1 + a_2 \mathbf{l}_1 \cdot \mathbf{l}_2 + A\mathbf{L} \cdot \mathbf{S} \\ =& \frac{a_1}{2}[S(S+1) - s_1(s_1+1) - s_2(s_2+1)] + \frac{a_2}{2}[L(L+1) \\ & - l_1(l_1+1) - l_2(l_2+1)] + \frac{A}{2}[J(J+1) - L(L+1) - S(S+1)]\,,\end{aligned}$$

where a_1, a_2, A can be positive or negative. The energy levels can be obtained in three steps, namely, by plotting the splittings caused by S, L and J successively. The energy levels are given in Fig. 1.29.

(c) The selection rules for electric-dipole transitions are:

$$\Delta S = 0, \Delta L = 0, \pm 1, \Delta J = 0, \pm 1$$

(except $0 \to 0$).

The following transitions are allowed:

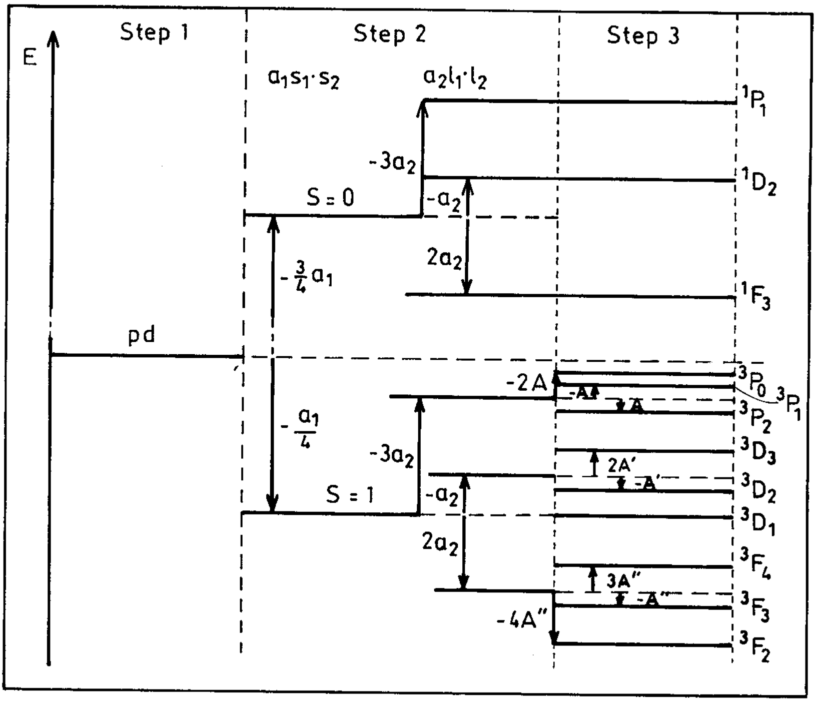

Fig. 1.29

$$(4p4d)^3P_1 \to (4p^2)^3P_0, \qquad (4p4d)^3P_1 \to (4p^2)^3P_1\,,$$

$$(4p4d)^3P_1 \to (4p^2)^3P_2\,, \qquad (4p4d)^3P_2 \to (4p^2)^3P_1\,,$$

$$(4p4d)^3P_2 \to (4p^2)^3P_2\,, \qquad (4p4d)^3P_0 \to (4p^2)^3P_1\,,$$

$$(4p4d)^3D_1 \to (4p^2)^3P_1\,, \qquad (4p4d)^3D_1 \to (4p^2)^3P_2\,,$$

$$(4p4d)^3D_2 \to (4p^2)^3P_1\,, \qquad (4p4d)^3D_2 \to (4p^2)^3P_2\,,$$

$$(4p4d)^3D_3 \to (4p^2)^3P_2\,, \qquad (4p4d)^1P_1 \to (4p^2)^1S_0\,,$$

$$(4p4d)^1P_1 \to (4p^2)^1D_2\,, \qquad (4p4d)^1D_2 \to (4p^2)^1D_2\,,$$

$$(4p4d)^1F_3 \to (4p^2)^1D_2\,.$$

1082

In the ground state of beryllium there are two $1s$ and two $2s$ electrons. The lowest excited states are those in which one of the $2s$ electrons is excited to a $2p$ state.

(a) List these states, giving all the angular momentum quantum numbers of each.

(b) Order the states according to increasing energy, indicating any degeneracies. Give a physical explanation for this ordering and estimate the magnitudes of the splitting between the various states.

(*Columbia*)

Solution:

(a) The electron configuration of the ground state is $1s^2 2s^2$. Pauli's principle requires $S = 0$. Thus the ground state has $S = 0$, $L = 0$, $J = 0$ and is a singlet ${}^1S_0$.

The lowest excited state has configuration $1s^2 2s2p$. Pauli's principle allows for both $S = 0$ and $S = 1$. For $S = 0$, as $L = 1$, we have $J = 1$ also, and the state is ${}^1P_1$. For $S = 1$, as $L = 1$, $J = 2, 1, 0$ and the states are ${}^3P_{2,1,0}$.

(b) In order of increasing energy, we have

$$^1S_0 <^3 P_0 <^3 P_1 <^3 P_2 <^1 P_1 \,.$$

The degeneracies of ${}^3P_2$, ${}^3P_1$ and ${}^1P_1$ are 5, 3, 3 respectively. According to Hund's rule (**Problem 1008(e)**), for the same configuration, the largest S corresponds to the lowest energy; and for a less than half-filled shell, the smallest J corresponds to the smallest energy. This roughly explains the above ordering.

The energy difference between ${}^1S_0$ and ${}^1P_1$ is of the order of 1 eV. The energy splitting between the triplet and singlet states is also ~ 1 eV. However the energy splitting among the triplet levels of a state is much smaller, $\sim 10^5$–10^{-4} eV.

1083

A characteristic of the atomic structure of the noble gasses is that the highest p-shells are filled. Thus, the electronic configuration in neon, for example, is $1s^22s^22p^6$. The total angular momentum **J**, total orbital angular momentum **L** and total spin angular momentum **S** of such a closed shell configuration are all zero.

(a) Explain the meaning of the symbols $1s^22s^22p^6$.

(b) The lowest group of excited states in neon corresponds to the excitation of one of the $2p$ electrons to a $3s$ orbital. The $(2p^5)$ core has orbital and spin angular momenta equal in magnitude but oppositely directed to these quantities for the electron which was removed. Thus, for its interaction with the excited electron, the core may be treated as a p-wave electron.

Assuming LS (Russell-Saunders) coupling, calculate the quantum numbers (L, S, J) of this group of states.

(c) When an atom is placed in a magnetic field H, its energy changes (from the $H = 0$ case) by ΔE:

$$\Delta E = \frac{e\hbar}{2mc} gHM\,,$$

where M can be J, $J-1, J-2, \dots, -J$. The quantity g is known as the Laudé g-factor. Calculate g for the $L = 1$, $S = 1$, $J = 2$ state of the $1s^22s^22p^53s$ configuration of neon.

(d) The structure of the $1s^22s^22p^53p$ configuration of neon is poorly described by Russell-Saunders coupling. A better description is provided by the "pair coupling" scheme in which the orbital angular momentum $\mathbf{L}_2$ of the outer electron couples with the total angular momentum $\mathbf{J}_c$ of the core. The resultant vector **K** $(\mathbf{K} = \mathbf{J}_c + \mathbf{L}_2)$ then couples with the spin $\mathbf{S}_2$ of the outer electron to give the total angular momentum **J** of the atom.

Calculate the J_c, K, J quantum numbers of the states of the $1s^22s^2$ $2p^53p$ configuration.

(*CUSPEA*)

Solution:

(a) In each group of symbols such as $1s^2$, the number in front of the letter refers to the principal quantum number n, the letter (s, p, etc.) determines the quantum number l of the orbital angular momentum (s for $l = 0$, p for $l = 1$, etc.), the superscript after the letter denotes the number of electrons in the subshell (n, l).

(b) The coupling is the same as that between a p- and an s-electron. Thus we have $l_1 = 1$, $l_2 = 0$ and so $L = 1 + 0 = 1$; $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$ and so

$S = \frac{1}{2} + \frac{1}{2} = 1$ or $S = \frac{1}{2} - \frac{1}{2} = 0$. Then $L = 1$, $S = 1$ give rise to $J = 2$, 1, or 0; $L = 1$, $S = 0$ give rise to $J = 1$. To summarize, the states of (L, S, J) are (1,1,2), (1,1,1), (1,1,0), (1,0,1).

(c) The g-factor is given by

$$g = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}\,.$$

For (1,1,2) we have

$$g = 1 + \frac{6+2-2}{2\times 6} = \frac{3}{2}\,.$$

(d) The coupling is between a core, which is equivalent to a p-electron, and an outer-shell p-electron, i.e. between $l_c = 1$, $s_c = \frac{1}{2}$; $l_2 = 1$, $s_2 = \frac{1}{2}$. Hence

$$J_c = \frac{3}{2}, \frac{1}{2}, L_2 = 1, S_2 = \frac{1}{2}\,.$$

For $J_c = \frac{3}{2}$, $L_2 = 1$, we have $K = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$.

Then for $K = \frac{5}{2}$, $J = 3$, 2; for $K = \frac{3}{2}$, $J = 2, 1$; for

$$K = \frac{1}{2}, J = 1, 0\,.$$

For $J_c = \frac{1}{2}$, $L_2 = 1$, we have $K = \frac{3}{2}, \frac{1}{2}$. Then for

$$K = \frac{3}{2}, J = 2, 1; \quad \text{for } K = \frac{1}{2}, J = 1, 0\,.$$

1084

A furnace contains atomic sodium at low pressure and a temperature of 2000 K. Consider only the following three levels of sodium:

$1s^2 2s^2 2p^6 3s$: ^2S, zero energy (ground state),

$1s^2 2s^2 2p^6 3p$: ^2P, 2.10 eV,

$1s^2 2s^2 2p^6 4s$: ^2S, 3.18 eV.

(a) What are the photon energies of the emission lines present in the spectrum? What are their relative intensities? (Give appropriate expressions and evaluate them approximately as time permits).

(b) Continuous radiation with a flat spectrum is now passed through the furnace and the absorption spectrum observed. What spectral lines are observed? Find their relative intensities.

(*UC, Berkeley*)

Solution:

(a) As $E_0 = 0$ eV, $E_1 = 2.10$ eV, $E_2 = 3.18$ eV, there are two electric-dipole transitions corresponding to energies

$$E_{10} = 2.10 \text{ eV}, \qquad E_{21} = 1.08 \text{ eV}.$$

The probability of transition from energy level k to level i is given by

$$A_{ik} = \frac{e^2\omega_{ki}^3}{3\hbar^2 c^3}\frac{1}{g_k}\sum_{m_k,m_i}|\langle im_i|\mathbf{r}|km_k\rangle|^2\,,$$

where $\omega_{ki} = (E_k - E_i)/\hbar$, i, k being the total angular momentum quantum numbers, m_k, m_i being the corresponding magnetic quantum numbers. The intensities of the spectral lines are

$$I_{ik} \propto N_k \hbar\omega_{ki} A_{ik}\,,$$

where the number of particles in the ith energy level $N_i \propto g_i \exp(-\frac{E_i}{kT})$. For ^2P, there are two values of $J : J = 3/2,\ 1/2$. Suppose the transition matrix elements and the spin weight factors of the two transitions are approximately equal. Then the ratio of the intensities of the two spectral lines is

$$\begin{aligned}\frac{I_{01}}{I_{12}} &= \left(\frac{\omega_{10}}{\omega_{21}}\right)^4 \exp\left(\frac{E_{21}}{kT}\right)\\ &= \left(\frac{2.10}{1.08}\right)^4 \exp\left(\frac{1.08}{8.6\times10^{-5}\times 2000}\right) = 8\times 10^3\,.\end{aligned}$$

(b) The intensity of an absorption line is

$$I_{ik} \propto B_{ik} N_k \rho(\omega_{ik})\hbar\omega_{ik}\,,$$

where

$$B_{ik} = \frac{4\pi^2 e^2}{3\hbar^2}\frac{1}{g_k}\sum_{m_k,m_i}|\langle im_i|\mathbf{r}|km_k\rangle|^2$$

is Einstein's coefficient. As the incident beam has a flat spectrum, $\rho(\omega)$ is constant. There are two absorption spectral lines: $E_0 \to E_1$ and $E_1 \to E_2$. The ratio of their intensities is

$$\frac{I_{10}}{I_{21}} = \frac{B_{10}N_0\omega_{10}}{B_{21}N_1\omega_{21}} \approx \left(\frac{\omega_{10}}{\omega_{21}}\right)\exp\left(\frac{E_{10}}{kT}\right)$$
$$= \left(\frac{2.10}{1.08}\right)\exp\left(\frac{2.10}{8.62\times 10^{-5}\times 2000}\right) = 4\times 10^5\,.$$

1085

For C $(Z = 6)$ write down the appropriate electron configuration. Using the Pauli principle derive the allowed electronic states for the 4 outermost electrons. Express these states in conventional atomic notation and order in energy according to Hund's rules. Compare this with a $(2p)^4$ configuration.

(*Wisconsin*)

Solution:

The electron configuration of C is $1s^2 2s^2 2p^2$. The two $1s$ electrons form a complete shell and need not be considered. By Pauli's principle, the two electrons $2s^2$ have total spin $S = 0$, and hence total angular momentum 0. Thus we need consider only the coupling of the two p-electrons. Then the possible electronic states are $^1S_0$, $^3P_{2,1,0}$, $^1D_2$ (**Problem 1088**). According to Hund's rule, in the order of increasing energy they are $^3P_0, {}^3P_1, {}^3P_2, {}^1D_2, {}^1S_0$.

The electronic configuration of $(2p)^4$ is the same as the above but the energy order is somewhat different. Of the ^3P states, $J = 0$ has the highest energy while $J = 2$ has the lowest. The other states have the same order as in the $2s^2 2p^2$ case.

1086

The atomic number of Mg is $Z = 12$.

(a) Draw a Mg atomic energy level diagram (not necessarily to scale) illustrating its main features, including the ground state and excited states arising from the configurations in which one valence electron is in the $3s$ state and the other valence electron is in the state nl for $n = 3$, 4 and $l = 0$, 1. Label the levels with conventional spectroscopic notation. Assuming LS coupling.

(b) On your diagram, indicate the following (give your reasoning):

(1) an allowed transition,

(2) a forbidden transition,

(3) an intercombination line (if any),

(4) a level which shows (1) anomalous and (2) normal Zeeman effect, if any.

(*Wisconsin*)

Solution:

(a) Figure 1.30 shows the energy level diagram of Mg atom.

(b) (1) An allowed transition:

$$(3s3p)^1P_1 \to (3s3s)^1S_0 \,.$$

(2) A forbidden transition:

$$(3s4p)^1P_1 \not\to (3s3p)^1P_1 \,.$$

($\Delta\pi = 0$, violating selection rule for parity)

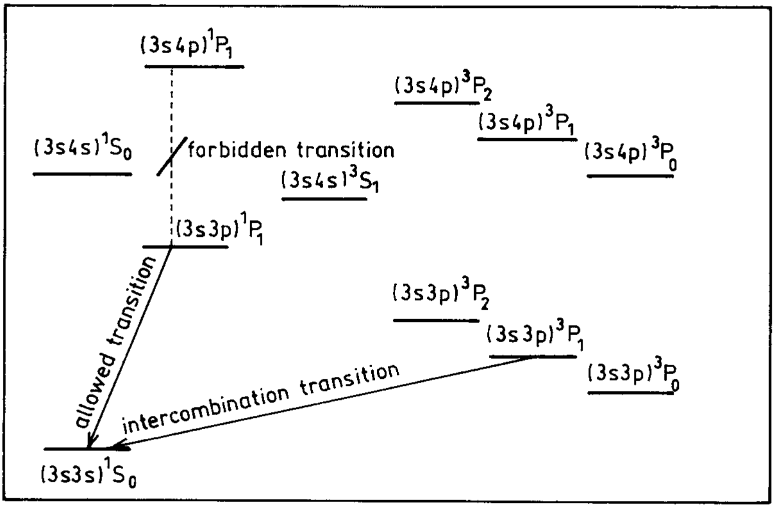

Fig. 1.30

(3) An intercombination line:

$$(3s3p)^3P_1 \to (3s3s)^1S_0 \,.$$

(4) In a magnetic field, the transition $(3s3p)^1P_1 \to (3s3s)^1S_0$ only produces three lines, which is known as normal Zeeman effect, as shown in Fig. 1.31(a). The transition $(3s4p)^3P_1 \to (3s4s)^3S_1$ produces six lines and is known as anomalous Zeeman effect. This is shown in Fig. 1.31(b). The spacings of the sublevels of $(3s3p)^1P_1$, $(3s4p)^3P_1$, and $(3s4s)^3S_1$ are $\mu_B B$, $3\mu_B B/2$ and $2\mu_B B$ respectively.

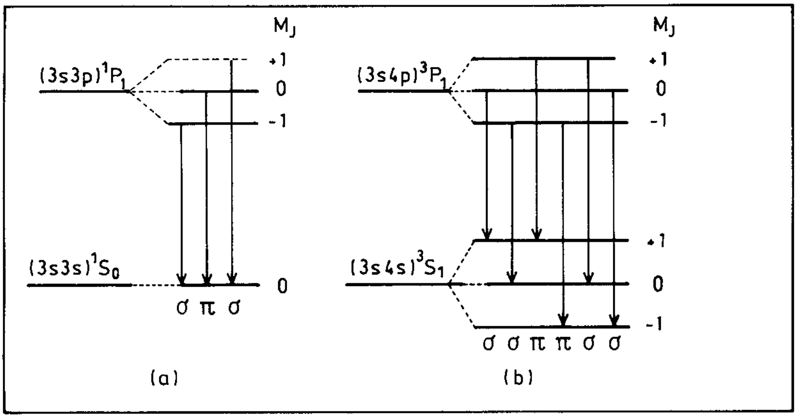

Fig. 1.31

1087

Give, in spectroscopic notation, the ground state of the carbon atom, and explain why this is the ground.

(*Wisconsin*)

Solution:

The electron configuration of the lowest energy state of carbon atom is $1s^2 2s^2 2p^2$, which can form states whose spectroscopic notations are $^1S_0$, $^3P_{0,1,2}$, $^1D_2$. According to Hund's rule, the ground state has the largest total spin S. But if there are more than one such states, the ground state corresponds to the largest total orbital angular momentum L among such states. If the number of electrons is less than that required to half-fill the shell, the lowest-energy state corresponds to the smallest total angular momentum J. Of the above states, $^3P_{0,1,2}$ have the largest S. As the p-shell is less than half-full, the state $^3P_0$ is the ground state.

1088

What is meant by the statement that the ground state of the carbon atom has the configuration $(1s)^2(2s)^2(2p)^2$?

Assuming that Russell-Saunders coupling applies, show that there are 5 spectroscopic states corresponding to this configuration: $^1S_0$, $^1D_2$, $^3P_1$, $^3P_2$, $^3P_0$.

(*Wisconsin*)

Solution:

The electronic configuration of the ground state of carbon being $(1s)^2$ $(2s)^2(2p)^2$ means that, when the energy of carbon atom is lowest, there are two electrons on the s-orbit of the first principal shell and two electrons each on the s- and p-orbits of the second principal shell.

The spectroscopic notations corresponding to the above electronic configuration are determined by the two equivalent electrons on the p-orbit.

For these two p-electrons, the possible combinations and sums of the values of the z-component of the orbital quantum number are as follows:

m_{l2}	m_{l1}	1	0	-1
1		2	1	0
0		1	0	-1
-1		0	-1	-2

For $m_{l1} = m_{l2}$, or $L = 2,\ 0$, Pauli's principle requires $m_{s1} \neq m_{s2}$, or $S = 0$, giving rise to terms $^1D_2$, $^1S_0$.

For $m_{s1} = m_{s2}$, or $S = 1$, Pauli's principle requires $m_{l1} \neq m_{l2}$, or $L = 1$, and so $J = 2,1,0$, giving rise to terms $^3P_{2,1,0}$. Hence corresponding to the electron configuration $1s^2 2s^2 2p^2$ the possible spectroscopic terms are

$$^1S_0, {}^1D_2, {}^3P_2, {}^3P_1, {}^3P_0 \,.$$

1089

Apply the Russell-Saunders coupling scheme to obtain all the states associated with the electron configuration $(1s)^2(2s)^2(2p)^5(3p)$. Label each state by the spectroscopic notation of the angular-momentum quantum numbers appropriate to the Russell-Saunders coupling.

(*Wisconsin*)

Solution:

The $2p$-orbit can accommodate $2(2l+1) = 6$ electrons. Hence the configuration $(1s)^2(2s)^2(2p)^5$ can be represented by its complement $(1s)^2(2s)^2$ $(2p)^1$ in its coupling with the $3p$ electron. In LS coupling the combination of the $2p$- and $3p$-electrons can be considered as follows. As $l_1 = 1$, $l_2 = 1$, $s_1 = \frac{1}{2}$, $s_2 = \frac{1}{2}$, we have $L = 2, 1, 0$; $S = 1, 0$. For $L = 2$, we have for $S = 1$: $J = 3, 2, 1$; and for $S = 0$: $J = 2$, giving rise to ${}^3D_{3,2,1}$, ${}^1D_2$. For $L = 1$, we have for $S = 1$: $J = 2, 1, 0$; and for $S = 0$: $J = 1$, giving rise to ${}^3P_{2,1,0}$, ${}^1P_1$. For $L = 0$, we have for $S = 1$: $J = 1$; for $S = 0$: $J = 0$, giving rise to ${}^3S_1$, ${}^1S_0$. Hence the given configuration has atomic states

$$^3S_1,^3P_{2,1,0},^3D_{3,2,1},^1S_0,^1P_1,^1D_2\,.$$

1090

The ground configuration of Sd (scandium) is $1s^2 2s^2 2p^6 3s^2 3p^6 3d 4s^2$.

(a) To what term does this configuration give rise?

(b) What is the appropriate spectroscopic notation for the multiplet levels belonging to this term? What is the ordering of the levels as a function of the energy?

(c) The two lowest (if there are more than two) levels of this ground multiplet are separated by 168 cm^{-1}. What are their relative population at $T = 2000$ K?

$$h = 6.6 \times 10^{-34}\ \mathrm{J\,sec}\,, \qquad c = 3 \times 10^8\ \mathrm{m/s}\,, \qquad k = 1.4 \times 10^{-23}\ \mathrm{J/K}\,.$$

(*Wisconsin*)

Solution:

(a) Outside completed shells there are one $3d$-electron and two $4s$-electrons to be considered. In LS coupling we have to combine $l = 2$,

$s = \frac{1}{2}$ with $L = 0$, $S = 0$. Hence $L = 2$, $S = \frac{1}{2}$, and the spectroscopic notations of the electron configuration are

$$^2D_{5/2}, ^2D_{3/2} .$$

(b) The multiplet levels are $^2D_{5/2}$ and $^2D_{3/2}$, of which the second has the lower energy according to Hund's rules as the D shell is less than half-filled.

(c) The ratio of particle numbers in these two energy levels is

$$\frac{g_1}{g_2} \exp\left(-\frac{\Delta E}{kT}\right) ,$$

where $g_1 = 2\times\frac{3}{2}+1 = 4$ is the degeneracy of $^2D_{3/2}$, $g_2 = 2\times\frac{5}{2}+1 = 6$ is the degeneracy of $^2D_{5/2}$, ΔE is the separation of these two energy levels. As

$$\begin{aligned}\Delta E &= hc\Delta\tilde{\nu} = 6.6 \times 10^{-34} \times 3 \times 10^8 \times 168 \times 10^2 \\ &= 3.3 \times 10^{-21} \text{ J} ,\end{aligned}$$

$$\frac{g_1}{g_2} \exp\left(-\frac{\Delta E}{kT}\right) = 0.6 .$$

1091

Consider the case of four equivalent p-electrons in an atom or ion. (Think of these electrons as having the same radial wave function, and the same orbital angular momentum $l = 1$).

(a) Within the framework of the Russell-Saunders (LS) coupling scheme, determine all possible configurations of the four electrons; label these according to the standard spectroscopic notation, and in each case indicate the values of L, S, J and the multiplicity.

(b) Compute the Landé g-factor for all of the above states for which $J = 2$.

(*UC, Berkeley*)

Solution:

(a) The p-orbit of a principal-shell can accommodate $2(2 \times 1 + 1) = 6$ electrons and so the terms for p^n and p^{6-n} are the same. Thus the situation of four equivalent p-electrons is the same as that of 2 equivalent p-electrons. In accordance with Pauli's principle, the spectroscopic terms are (**Problem 1088**)

$$\begin{aligned} &{}^1S_0 \qquad (S=0, L=0, J=0) \\ &{}^1D_2 \qquad (S=0, L=2, J=2) \\ &{}^3P_{2,1,0} \qquad (S=1, L=1, J=2,1,0)\,. \end{aligned}$$

(b) The Landé g-factors are given by

$$g = 1 + \frac{J(J+1)+S(S+1)-L(L+1)}{2J(J+1)}\,.$$

For ${}^1D_2$:

$$g = 1 + \frac{2\times 3 + 0\times 1 - 2\times 3}{2\times 2\times 3} = 1\,,$$

For ${}^3P_2$:

$$g = 1 + \frac{2\times 3 + 1\times 2 - 1\times 2}{2\times 2\times 3} = 1.5\,.$$

1092

For the sodium doublet give:

(a) Spectroscopic notation for the energy levels (Fig. 1.32).
(b) Physical reason for the energy difference E.
(c) Physical reason for the splitting ΔE.
(d) The expected intensity ratio

$$D_2/D_1 \quad \text{if } kT \gg \Delta E\,.$$

(*Wisconsin*)

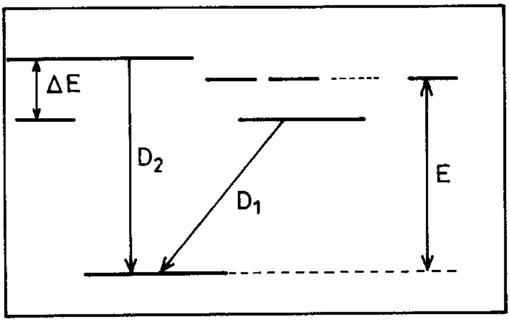

Fig. 1.32

Solution:

(a) The spectroscopic notations for the energy levels are shown in Fig. 1.33.

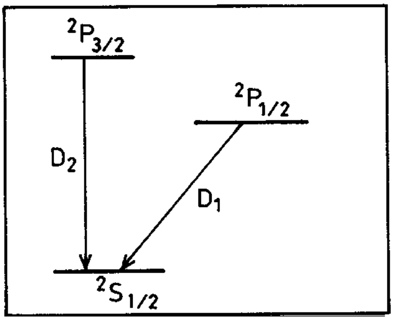

Fig. 1.33

(b) The energy difference E arises from the polarization of the atomic nucleus and the penetration of the electron orbits into the nucleus, which are different for different orbital angular momenta l.

(c) ΔE is caused by the coupling between the spin and orbit angular momentum of the electrons.

(d) When $kT \gg \Delta E$, the intensity ratio D_2/D_1 is determined by the degeneracies of $^2P_{3/2}$ and $^2P_{1/2}$:

$$\frac{D_2}{D_1} = \frac{2J_2+1}{2J_1+1} = \frac{3+1}{1+1} = 2\,.$$

1093

(a) What is the electron configuration of sodium ($Z = 11$) in its ground state? In its first excited state?

(b) Give the spectroscopic term designation (e.g. $^4S_{3/2}$) for each of these states in the LS coupling approximation.

(c) The transition between the two states is in the visible region. What does this say about kR, where k is the wave number of the radiation and R is the radius of the atom? What can you conclude about the multipolarity of the emitted radiation?

(d) What are the sodium "D-lines" and why do they form a doublet?

(*Wisconsin*)

Solution:

(a) The electron configuration of the ground state of Na is $1s^2 2s^2 2p^6 3s^1$, and that of the first excited state is $1s^2 2s^2 2p^6 3p^1$.

(b) The ground state: $^2S_{1/2}$.

The first excited state: $^2P_{3/2}, ^2P_{1/2}$.

(c) As the atomic radius $R \approx 1$ Å and for visible light $k \approx 10^{-4}$ Å^{-1}, we have $kR \ll 1$, which satisfies the condition for electric-dipole transition. Hence the transitions $^2P_{3/2} \to ^2S_{1/2}$, $^2P_{1/2} \to ^2S_{1/2}$ are electric dipole transitions.

(d) The D-lines are caused by transition from the first excited state to the ground state of Na. The first excited state is split into two energy levels $^2P_{3/2}$ and $^2P_{1/2}$ due to LS coupling. Hence the D-line has a doublet structure.

1094

Couple a p-state and an s-state electron via

(a) Russell-Saunders coupling,

(b) j, j coupling,

and identify the resultant states with the appropriate quantum numbers. Sketch side by side the energy level diagrams for the two cases and show which level goes over to which as the spin-orbit coupling is increased.

(*Wisconsin*)

Solution:

We have $s_1 = s_2 = 1/2$, $l_1 = 1$, $l_2 = 0$.

(a) In LS coupling, $\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2$, $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$, $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Thus $L = 1, S = 1, 0$.

For $S = 1$, $J = 2$, 1, 0, giving rise to ${}^3P_{2,1,0}$.

For $S = 0$, $J = 1$, giving rise to ${}^1P_1$.

(b) In jj coupling, $\mathbf{j}_1 = \mathbf{l}_1 + \mathbf{s}_1$, $\mathbf{j}_2 = \mathbf{l}_2 + \mathbf{s}_2$, $\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$. Thus $j_1 = \frac{3}{2}, \frac{1}{2}$, $j_2 = \frac{1}{2}$.

Hence the coupled states are

$$\left(\frac{3}{2}, \frac{1}{2}\right)_2, \left(\frac{3}{2}, \frac{1}{2}\right)_1, \left(\frac{1}{2}, \frac{1}{2}\right)_1, \left(\frac{1}{2}, \frac{1}{2}\right)_0,$$

where the subscripts indicate the values of J.

The coupled states are shown in Fig. 1.34.

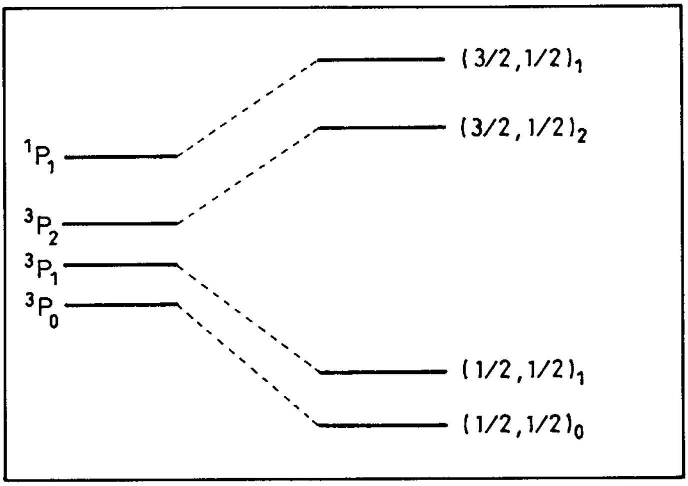

Fig. 1.34

1095

(a) State the ground state configuration of a carbon atom, and list the levels (labeled in terms of Russel-Saunders coupling) of this configuration.

(b) Which is the ground state level? Justify your answer.

(*Wisconsin*)

Solution:

(a) The electronic configuration of the ground state of carbon is $1s^2$ $2s^2 2p^2$. The corresponding energy levels are ${}^1S_0$, ${}^2P_{2,1,0}$, ${}^1D_2$.

(b) According to Hund's rules, the ground state is ${}^3P_0$.

1096

For each of the following atomic radiative transitions, indicate whether the transition is allowed or forbidden under the electric-dipole radiation selection rules. For the forbidden transitions, cite the selection rules which are violated.

(a) He: $(1s)(1p)\ {}^1P_1 \to (1s)^2\ {}^1S_0$
(b) C: $(1s)^2(2s)^2(2p)(3s)\,{}^3P_1 \to (1s)^2(2s)^2(2p)^2\ {}^3P_0$
(c) C: $(1s)^2(2s)^2(2p)(3s)\,{}^3P_0 \to (1s)^2(2s)^2(2p)^2\ {}^3P_0$
(d) Na: $(1s)^2(2s)^2(2p)^6(4d)\,{}^2D_{5/2} \to (1s)^2(2s)^2(2p)^6(3p)\ {}^2P_{1/2}$
(e) He: $(1s)(2p)\ {}^3P_1 \to (1s)^2\ {}^1S_0$

(*Wisconsin*)

Solution:

The selection rules for single electric-dipole transition are

$$\Delta l = \pm 1, \qquad \Delta j = 0, \pm 1\,.$$

The selection rules for multiple electric-dipole transition are

$$\Delta S = 0\,, \qquad \Delta L = 0\,, \pm 1, \qquad \Delta J = 0, \pm 1 (0 \not\leftrightarrow 0)\,.$$

(a) Allowed electric-dipole transition.

(b) Allowed electric-dipole transition.

(c) Forbidden as the total angular momentum J changes from 0 to 0 which is forbidden for electric-dipole transition.

(d) Forbidden as it violates the condition $\Delta J = 0, \pm 1$.

(e) Forbidden as it violates the condition $\Delta S = 0$.

1097

Consider a hypothetical atom with an electron configuration of two identical p-shell electrons outside a closed shell.

(a) Assuming LS (Russell-Saunders) coupling, identify the possible levels of the system using the customary spectroscopic notation, $^{(2S+1)}L_J$.

(b) What are the parities of the levels in part (a)?

(c) In the independent-particle approximation these levels would all be degenerate, but in fact their energies are somewhat different. Describe the physical origins of the splittings.

(*Wisconsin*)

Solution:

(a) The electronic configuration is p^2. The two p-electrons being equivalent, the possible energy levels are (**Problem 1088**)

$$^1S_0,\ ^3P_{2,1,0},\ ^1D_2\,.$$

(b) The parity of an energy level is determined by the sum of the orbital angular momentum quantum numbers: parity $\pi = (-1)^{\Sigma l}$. Parity is even or odd depending on π being $+1$ or -1. The levels $^1S_0$, $^3P_{2,1,0}$, $^1D_2$ have $\Sigma l = 2$ and hence even parity.

(c) See **Problem 1079(a)**.

1098

What is the ground state configuration of potassium (atomic number 19).

(*UC, Berkeley*)

Solution:

The ground state configuration of potassium is $1s^2 2s^2 2p^6 3s^2 3p^6 4s^1$.

1099

Consider the ^{17}O isotope ($I = 5/2$) of the oxygen atom. Draw a diagram to show the fine-structure and hyperfine-structure splittings of the levels described by $(1s^2 2s^2 2p^4)\,^3P$. Label the states by the appropriate angular-momentum quantum numbers.

(*Wisconsin*)

Solution:

The fine and hyperfine structures of the ^3P state of ^{17}O is shown in Fig. 1.35.

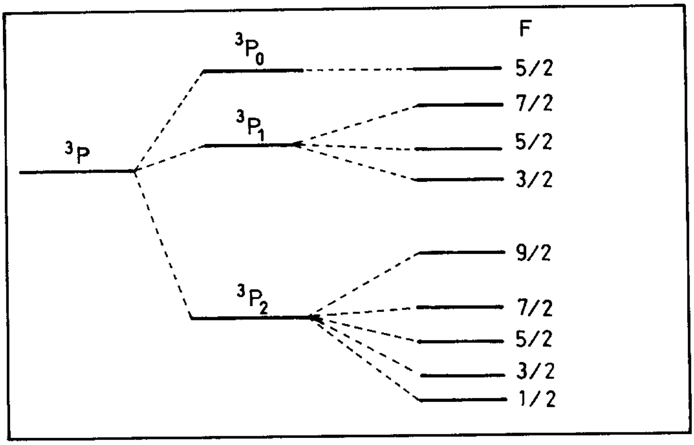

Fig. 1.35

1100

Consider a helium atom with a $1s3d$ electronic configuration. Sketch a series of energy-level diagrams to be expected when one takes successively into account:

(a) only the Coulomb attraction between each electron and the nucleus,
(b) the electrostatic repulsion between the electrons,
(c) spin-orbit coupling,
(d) the effect of a weak external magnetic field.

(*Wisconsin*)

Solution:

The successive energy-level splittings are shown in Fig. 1.36.

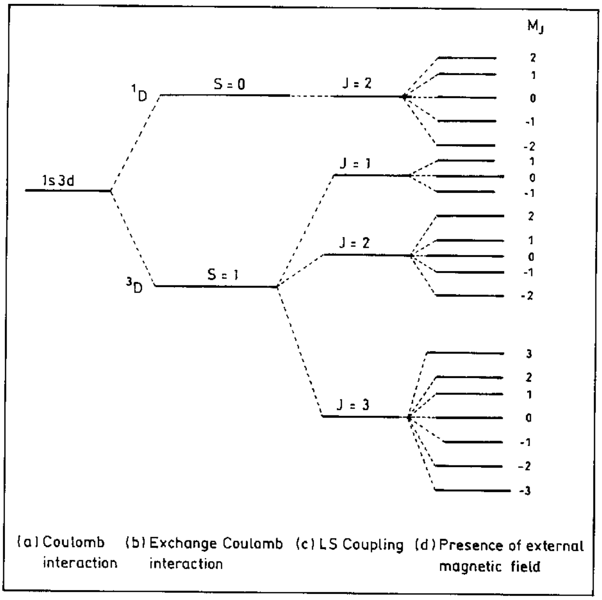

Fig. 1.36

1101

Sodium chloride forms cubic crystals with four Na and four Cl atoms per cube. The atomic weights of Na and Cl are 23.0 and 35.5 respectively. The density of NaCl is 2.16 gm/cc.

(a) Calculate the longest wavelength for which X-rays can be Bragg reflected.

(b) For X-rays of wavelength 4 Å, determine the number of Bragg reflections and the angle of each.

(*UC, Berkeley*)

Solution:

(a) Let V be the volume of the unit cell, N_A be Avogadro's number, ρ be the density of NaCl. Then

$$V\rho N_A = 4(23.0 + 35.5)\,,$$

giving

$$V = \frac{4 \times 58.5}{2.16 \times 6.02 \times 10^{23}} = 1.80 \times 10^{-22} \text{ cm}^3 ,$$

and the side length of the cubic unit cell

$$d =^3 \sqrt{V} = 5.6 \times 10^{-8} \text{ cm} = 5.6 \text{ Å} .$$

Bragg's equation $2d \sin\theta = n\lambda$, then gives

$$\lambda_{\max} = 2d = 11.2 \text{ Å} .$$

(b) For $\lambda = 4$ Å,

$$\sin\theta = \frac{\lambda n}{2d} = 0.357n .$$

Hence

$$\text{for } n = 1 : \sin\theta = 0.357, \theta = 20.9^\circ ,$$

$$\text{for } n = 2 : \sin\theta = 0.714, \theta = 45.6^\circ$$

For $n \geq 3$: $\sin\theta > 1$, and Bragg reflection is not allowed.

1102

(a) 100 keV electrons bombard a tungsten target ($Z = 74$). Sketch the spectrum of resulting X-rays as a function of $1/\lambda$ (λ = wavelength). Mark the K X-ray lines.

(b) Derive an approximate formula for λ as a function of Z for the K X-ray lines and show that the Moseley plot ($\lambda^{-1/2}$ vs. Z) is (nearly) a straight line.

(c) Show that the ratio of the slopes of the Moseley plot for K_α and K_β (the two longest-wavelength K-lines) is $(27/32)^{1/2}$.

(*Wisconsin*)

Solution:

(a) The X-ray spectrum consists of two parts, continuous and characteristic. The continuous spectrum has the shortest wavelength determined by the energy of the incident electrons:

$$\lambda_{\min} = \frac{hc}{E} = \frac{12.4}{100} \text{Å} = 0.124 \text{ Å} .$$

The highest energy for the K X-ray lines of W is 13.6×74^2 eV $=$ 74.5 keV, so the K X-ray lines are superimposed on the continuous spectrum as shown as Fig. 1.37.

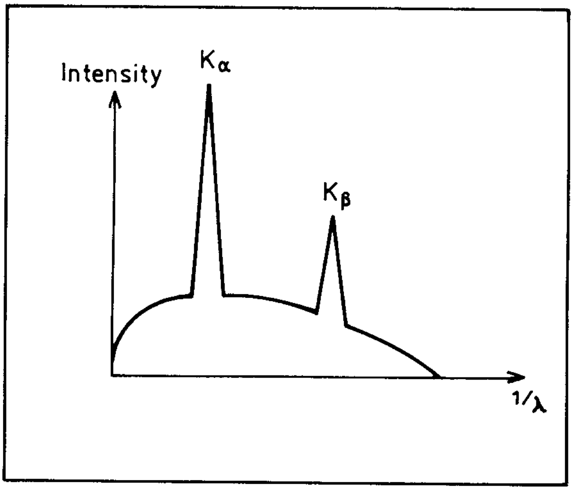

Fig. 1.37

(b) The energy levels of tungsten atom are given by

$$E_n = -\frac{RhcZ^{*2}}{n^2}\,,$$

where Z^* is the effective nuclear charge.

The K lines arise from transitions to ground state ($n \to 1$):

$$\frac{hc}{\lambda} = -\frac{RhcZ^{*2}}{n^2} + RhcZ^{*2}\,,$$

giving

$$\lambda = \frac{n^2}{(n^2-1)RZ^{*2}}\,,$$

or

$$\lambda^{-\frac{1}{2}} = Z^*\sqrt{\left(\frac{n^2-1}{n^2}\right)R} \approx Z\sqrt{\left(\frac{n^2-1}{n^2}\right)R}\,. \qquad (n = 1, 2, 3, \ldots)$$

Hence the relation between $\lambda^{-1/2}$ and Z is approximately linear.

(c) K_α lines are emitted in transitions $n = 2$ to $n = 1$, and K_β lines, from $n = 3$ to $n = 1$. In the Moseley plot, the slope of the K_α curve is $\sqrt{\frac{3}{4}R}$ and that of K_β is $\sqrt{\frac{8}{9}R}$, so the ratio of the two slopes is

$$\frac{\sqrt{(3/4)R}}{\sqrt{(8/9)R}} = \sqrt{\frac{27}{32}}\,.$$

1103

(a) If a source of continuum radiation passes through a gas, the emergent radiation is referred to as an absorption spectrum. In the optical and ultra-violet region there are absorption lines, while in the X-ray region there are absorption edges. Why does this difference exist and what is the physical origin of the two phenomena?

(b) Given that the ionization energy of atomic hydrogen is 13.6 eV, what would be the energy E of the radiation from the $n = 2$ to $n = 1$ transition of boron ($Z = 5$) that is 4 times ionized? (The charge of the ion is $+4e$.)

(c) Would the K_α fluorescent radiation from neutral boron have an energy E_k greater than, equal to, or less than E of part (b)? Explain why.

(d) Would the K absorption edge of neutral boron have an energy E_k greater than, equal to, or less than E_k of part (c)? Explain why.

(*Wisconsin*)

Solution:

(a) Visible and ultra-violet light can only cause transitions of the outer electrons because of their relatively low energies. The absorption spectrum consists of dark lines due to the absorption of photons of energy equal to the difference in energy of two electron states. On the other hand, photons with energies in the X-ray region can cause the ejection of inner electrons from the atoms, ionizing them. This is because in the normal state the outer orbits are usually filled. Starting from lower frequencies in the ultraviolet the photons are able to eject only the loosely bound outer electrons. As the frequency increases, the photons suddenly become sufficiently energetic to eject electrons from an inner shell, causing the absorption coefficient to increase suddenly, giving rise to an absorption edge. As the frequency is increased further, the absorption coefficient decreases approximately as ν^{-3} until the frequency becomes great enough to allow electron ejection from the next inner shell, giving rise to another absorption edge.

(b) The energy levels of a hydrogen-like atom are given by

$$E_n = -\frac{Z^2 e^2}{2n^2 a_0} = -\frac{Z^2}{n^2} E_0 \,,$$

where E_0 is the ionization energy of hydrogen. Hence

$$\begin{aligned} E_2 - E_1 &= -Z^2 E_0 \left(\frac{1}{2^2} - \frac{1}{1^2} \right) = 5^2 \times 13.6 \times \frac{3}{4} \\ &= 255 \text{ eV} \,. \end{aligned}$$

(c) Due to the shielding by the orbital electrons of the nuclear charge, the energy E_k of K_α emitted from neutral Boron is less than that given in (b).

(d) As the K absorption edge energy E_k correspond to the ionization energy of a K shell electron, it is greater than the energy given in (c).

1104

For Zn, the X-ray absorption edges have the following values in keV:

$$K \; 9.67, L_{\text{I}} \; 1.21, L_{\text{II}} \; 1.05, L_{\text{III}} \; 1.03 \,.$$

Determine the wavelength of the K_α line.

If Zn is bombarded by 5-keV electrons, determine

(a) the wavelength of the shortest X-ray line, and

(b) the wavelength of the shortest characteristic X-ray line which can be emitted.

Note: The K level corresponds to $n = 1$, the three L-levels to the different states with $n = 2$. The absorption edges are the lowest energies for which X-rays can be absorbed by ejection of an electron from the corresponding level. The K_α line corresponds to a transition from the lowest L level.

(*UC, Berkeley*)

Solution:

The K_α series consists of two lines, $K_{\alpha 1}(L_{\rm III} \to K)$, $K_{\alpha 2}(L_{\rm II} \to K)$:

$$E_{K_{\alpha_1}} = K_{L_{\rm III}} - E_K = 9.67 - 1.03 = 8.64 \text{ keV}\,,$$

$$E_{K_{\alpha_2}} = K_{L_{\rm II}} - E_K = 9.67 - 1.05 = 8.62 \text{ keV}\,.$$

Hence

$$\lambda_{K_{\alpha_1}} = \frac{hc}{E_{K_{\alpha_1}}} = \frac{12.41}{8.64} = 1.436 \text{ Å}\,,$$

$$\lambda_{K_{\alpha_2}} = \frac{hc}{E_{K_{\alpha_2}}} = 1.440 \text{ Å}\,.$$

(a) The minimum X-ray wavelength that can be emitted by bombarding the atoms with 5-keV electrons is

$$\lambda_{\min} = \frac{hc}{E_{\max}} = \frac{12.41}{5} = 2.482 \text{ Å}\,.$$

(b) It is possible to excite electrons on energy levels other than the K level by bombardment with 5-keV electrons, and cause the emission of characteristic X-rays when the atoms de-excite. The highest-energy X-rays have energy $0 - E_{\rm I} = 1.21$ keV, corresponding to a wavelength of 10.26 Å.

1105

The characteristic K_α X-rays emitted by an atom of atomic number Z were found by Morseley to have the energy $13.6 \times (1 - \frac{1}{4})(Z-1)^2$ eV.

(a) Interpret the various factors in this expression.

(b) What fine structure is found for the K_α transitions? What are the pertinent quantum numbers?

(c) Some atoms go to a lower energy state by an Auger transition. Describe the process.

(*Wisconsin*)

Solution:

(a) In this expression, 13.6 eV is the ground state energy of hydrogen atom, i.e., the binding energy of an 1s electron to unit nuclear charge, the factor $(1 - \frac{1}{4})$ arises from difference in principal quantum number between the states $n = 2$ and $n = 1$, and $(Z-1)$ is the effective nuclear charge. The K_α line thus originates from a transition from $n = 2$ to $n = 1$.

(b) The K_α line actually has a doublet structure. In LS coupling, the $n = 2$ state splits into three energy levels: $^2S_{1/2}$, $^2P_{1/2}$, $^2P_{3/2}$, while the $n = 1$ state is still a single state $^2S_{1/2}$. According to the selection rules $\Delta L = \pm 1$, $\Delta J = 0, \pm 1 (0 \not\leftrightarrow 0)$, the allowed transitions are

$$K_{\alpha 1}: \qquad 2^2P_{3/2} \to 1^2S_{1/2} \,,$$

$$K_{\alpha 2}: \qquad 2^2P_{1/2} \to 1^2S_{1/2} \,.$$

(c) The physical basis of the Auger process is that, after an electron has been removed from an inner shell an electron from an outer shell falls to the vacancy so created and the excess energy is released through ejection of another electron, rather than by emission of a photon. The ejected electron is called Auger electron. For example, after an electron has been removed from the K shell, an L shell electron may fall to the vacancy so created and the difference in energy is used to eject an electron from the L shell or another outer shell. The latter, the Auger electron, has kinetic energy

$$E = -E_L - (-E_k) - E_L = E_k - 2E_L \,,$$

where E_k and E_L are the ionization energies of K and L shells respectively.

1106

The binding energies of the two $2p$ states of niobium $(Z = 41)$ are 2370 eV and 2465 eV. For lead $(Z = 82)$ the binding energies of the $2p$ states are 13035 eV and 15200 eV. The $2p$ binding energies are roughly proportional to $(Z - a)^2$ while the splitting between the $2P_{1/2}$ and the $2P_{3/2}$ goes as $(Z - a)^4$. Explain this behavior, and state what might be a reasonable value for the constant a.

(*Columbia*)

Solution:

The $2p$ electron moves in a central potential field of the nucleus shielded by inner electrons. Taking account of the fine structure due to ls coupling, the energy of a $2p$ electron is given by

$$E = -\frac{1}{4}Rhc(Z-a_1)^2 + \frac{1}{8}Rhc\alpha^2(Z-a_2)^4\left(\frac{3}{8} - \frac{1}{j+\frac{1}{2}}\right)$$

$$= -3.4(Z-a_1)^2 + 9.06\times 10^{-5}(Z-a_2)^4\left(\frac{3}{8} - \frac{1}{j+\frac{1}{2}}\right),$$

as $Rhc = 13.6$ eV, $\alpha = 1/137$. Note that $-E$ gives the binding energy and that ${}^2P_{3/2}$ corresponds to lower energy according to Hund's rule. For Nb, we have $95 = 9.06\times 10^{-5}(41-a_2)^4 \times 0.5$, or $a_2 = 2.9$, which then gives $a_1 = 14.7$. Similarly we have for Pb: $a_1 = 21.4$, $a_2 = -1.2$.

1107

(a) Describe carefully an experimental arrangement for determining the wavelength of the characteristic lines in an X-ray emission spectrum.

(b) From measurement of X-ray spectra of a variety of elements, Moseley was able to assign an atomic number Z to each of the elements. Explain explicitly how this assignment can be made.

(c) Discrete X-ray lines emitted from a certain target cannot in general be observed as absorption lines in the same material. Explain why, for example, the K_α lines cannot be observed in the absorption spectra of heavy elements.

(d) Explain the origin of the continuous spectrum of X-rays emitted when a target is bombarded by electrons of a given energy. What feature of this spectrum is inconsistent with classical electromagnetic theory?

(*Columbia*)

Solution:

(a) The wavelength can be determined by the method of crystal diffraction. As shown in the Fig. 1.38, the X-rays collimated by narrow slits S_1, S_2, fall on the surface of crystal C which can be rotated about a vertical axis. Photographic film P forms an arc around C. If the condition $2d\sin\theta = n\lambda$, where d is the distance between neighboring Bragg planes and n is an integer, is satisfied, a diffraction line appears on the film at A. After rotating the crystal, another diffraction line will appear at A' which is symmetric to A. As $4\theta = \mathrm{arc}AA'/CA$, the wavelength λ can be obtained.

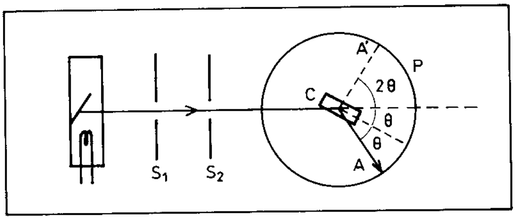

Fig. 1.38

(b) Each element has its own characteristic X-ray spectrum, of which the K series has the shortest wavelengths, and next to them the L series, etc. Moseley discovered that the K series of different elements have the same structure, only the wavelengths are different. Plotting $\sqrt{\tilde{\nu}}$ versus the atomic number Z, he found an approximate linear relation:

$$\tilde{\nu} = R(Z-1)^2 \left(\frac{1}{1^2} - \frac{1}{2^2} \right),$$

where $R = R_H c$, R_H being the Rydberg constant and c the velocity of light in free space.

Then if the wavelength or frequency of K_α of a certain element is found, its atomic number Z can be determined.

(c) The K_α lines represent the difference in energy between electrons in different inner shells. Usually these energy levels are all occupied and transitions cannot take place between them by absorbing X-rays with energy equal to the energy difference between such levels. The X-rays can only ionize the inner-shell electrons. Hence only absorption edges, but not absorption lines, are observed.

(d) When electrons hit a target they are decelerated and consequently emit bremsstrahlung radiation, which are continuous in frequency with the shortest wavelength determined by the maximum kinetic energy of the electrons, $\lambda = \frac{hc}{E_e}$. On the other hand, in the classical electromagnetic theory, the kinetic energy of the electrons can only affect the intensity of the spectrum, not the wavelength.

1108

In the X-ray region, as the photon energy decreases the X-ray absorption cross section rises monotonically, except for sharp drops in the cross section at certain photon energies characteristic of the absorbing material. For Zn ($Z = 30$) the four most energetic of these drops are at photon energies 9678 eV, 1236 eV, 1047 eV and 1024 eV.

(a) Identify the transitions corresponding to these drops in the X-ray absorption cross section.

(b) Identify the transitions and give the energies of Zn X-ray emission lines whose energies are greater than 5000 eV.

(c) Calculate the ionization energy of Zn^{29+} (i.e., a Zn atom with 29 electrons removed). (Hint: the ionization energy of hydrogen is 13.6 eV).

(d) Why does the result of part (c) agree so poorly with 9678 eV?

(*Wisconsin*)

Solution:

(a) The energies 9.768, 1.236, 1.047 and 1.024 keV correspond to the ionization energies of an 1s electron, a 2s electron, and each of two 2p electrons respectively. That is, they are energies required to eject the respective electrons to an infinite distance from the atom.

(b) X-rays of Zn with energies greater than 5 keV are emitted in transitions of electrons from other shells to the K shell. In particular X-rays emitted in transitions from L to K shells are

$$K_{\alpha 1}: \quad E = -1.024 - (-9.678) = 8.654 \text{ keV}\,, \qquad (L_{\text{III}} \to K)$$

$$K_{\alpha 2}: \quad E = -1.047 - (-9.678) = 8.631 \text{ keV}\,. \qquad (L_{\text{II}} \to K)$$

(c) The ionization energy of the Zn^{29+} (a hydrogen-like atom) is

$$E_{Zn} = 13.6\ Z^2 = 11.44 \text{ keV}\,.$$

(d) The energy 9.678 keV corresponds to the ionization energy of the 1s electron in the neutral Zn atom. Because of the Coulomb screening effect of the other electrons, the effective charge of the nucleus is $Z^* < 30$. Also the farther is the electron from the nucleus, the less is the nuclear charge Z^* it interacts with. Hence the ionization energy of a 1s electron of the neutral Zn atom is much less than that of the Zn^{29+} ion.

1109

Sketch a derivation of the "Landé g-factor", i.e. the factor determining the effective magnetic moment of an atom in weak fields.

(*Wisconsin*)

Solution:

Let the total orbital angular momentum of the electrons in the atom be $\mathbf{P}_L$, the total spin angular momentum be $\mathbf{P}_S$ ($\mathbf{P}_L$ and $\mathbf{P}_S$ being all in units of $\hbar$). Then the corresponding magnetic moments are $\boldsymbol{\mu}_L = -\mu_B \mathbf{P}_L$ and $\boldsymbol{\mu}_S = -2\mu_B \mathbf{P}_S$, where μ_B is the Bohr magneton. Assume the total magnetic moment is $\boldsymbol{\mu}_J = -g\mu_B \mathbf{P}_J$, where g is the Landé g-factor. As

$$\mathbf{P}_J = \mathbf{P}_L + \mathbf{P}_S\,,$$

$$\boldsymbol{\mu}_J = \boldsymbol{\mu}_L + \boldsymbol{\mu}_S = -\mu_B(\mathbf{P}_L + 2\mathbf{P}_S) = -\mu_B(\mathbf{P}_J + \mathbf{P}_S)\,,$$

we have

$$\begin{aligned}\boldsymbol{\mu}_J &= \frac{\boldsymbol{\mu}_J \cdot \mathbf{P}_J}{P_J^2}\mathbf{P}_J \\ &= -\mu_B \frac{(\mathbf{P}_J + \mathbf{P}_S)\cdot \mathbf{P}_J}{P_J^2}\mathbf{P}_J \\ &= -\mu_B \frac{P_J^2 + \mathbf{P}_S \cdot \mathbf{P}_J}{P_J^2}\mathbf{P}_J \\ &= -g\mu_B \mathbf{P}_J\,,\end{aligned}$$

giving

$$g = \frac{P_J^2 + \mathbf{P}_S \cdot \mathbf{P}_J}{P_J^2} = 1 + \frac{\mathbf{P}_S \cdot \mathbf{P}_J}{P_J^2}\,.$$

As

$$\mathbf{P}_L \cdot \mathbf{P}_L = (\mathbf{P}_J - \mathbf{P}_S)\cdot(\mathbf{P}_J - \mathbf{P}_S) = P_J^2 + P_S^2 - 2\mathbf{P}_J \cdot \mathbf{P}_S\,,$$

we have

$$\mathbf{P}_J \cdot \mathbf{P}_S = \frac{1}{2}(P_J^2 + P_S^2 - P_L^2)\,.$$

Hence

$$\begin{aligned}g &= 1 + \frac{P_J^2 + P_S^2 - P_L^2}{2P_J^2} \\ &= 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}\,.\end{aligned}$$

1110

In the spin echo experiment, a sample of a proton-containing liquid (e.g. glycerin) is placed in a steady but spatially inhomogeneous magnetic field of a few kilogauss. A pulse (a few microseconds) of a strong (a few gauss) rediofrequency field is applied perpendicular to the steady field. Immediately afterwards, a radiofrequency signal can be picked up from the coil around the sample. But this dies out in a fraction of a millisecond unless special precaution has been taken to make the field very spatially homogeneous, in which case the signal persists for a long time. If a second long radiofrequency pulse is applied, say 15 milliseconds after the first pulse, then a radiofrequency signal is observed 15 milliseconds after the second pulse (the echo).

(a) How would you calculate the proper frequency for the radiofrequency pulse?

(b) What are the requirements on the spatial homogeneity of the steady field?

(c) Explain the formation of the echo.

(d) How would you calculate an appropriate length of the first radiofrequency pulse?

(Princeton)

Solution:

(a) The radiofrequency field must have sufficiently high frequency to cause nuclear magnetic resonance:

$$\hbar\omega = \gamma_p \hbar H_0(\mathbf{r}) \, ,$$

or

$$\omega = \gamma_p \langle H_0(\mathbf{r}) \rangle \, ,$$

where γ_p is the gyromagnetic ratio, and $\langle H_0(\mathbf{r}) \rangle$ is the average value of the magnetic field in the sample.

(b) Suppose the maximum variation of H_0 in the sample is $(\Delta H)_m$. Then the decay time is $\frac{1}{\gamma_p (\Delta H)_m}$. We require $\frac{1}{\gamma_p}(\Delta H)_m > \tau$, where τ is the time interval between the two pulses. Thus we require

$$(\Delta H)_m < \frac{1}{\gamma_p \tau} \, .$$

(c) Take the z-axis along the direction of the steady magnetic field H_0. At $t = 0$, the magnetic moments are parallel to H_0 (Fig. 1.39(a)). After introducing the first magnetic pulse H_1 in the x direction, the magnetic moments will deviate from the z direction (Fig. 1.39(b)). The angle θ of the rotation of the magnetic moments can be adjusted by changing the width of the magnetic pulse, as shown in Fig. 1.39(c) where $\theta = 90°$.

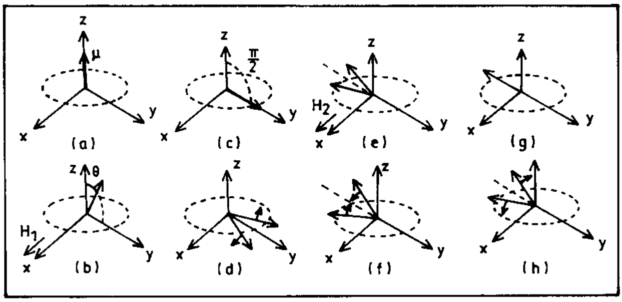

Fig. 1.39

The magnetic moments also processes around the direction of H_0. The spatial inhomogeneity of the steady magnetic field H_0 causes the processional angular velocity $\omega = \gamma_P H_0$ to be different at different points, with the result that the magnetic moments will fan out as shown in Fig. 1.39(d). If a second, wider pulse is introduced along the x direction at $t = \tau$ (say, at $t = 15$ ms), it makes all the magnetic moments turn 180° about the x-axis (Fig. 1.39(e)). Now the order of procession of the magnetic moments is reversed (Fig. 1.39(f)). At $t = 2\tau$, the directions of the magnetic moments will again become the same (Fig. 1.39(g)). At this instant, the total magnetic moment and its rate of change will be a maximum, producing a resonance signal and forming an echo wave (Fig. 1.40). Afterwards the magnetic moments scatter again and the signal disappears, as shown in Fig. 1.39(h).

(d) The first radio pulse causes the magnetic moments to rotate through an angle θ about the x-axis. To enhance the echo wave, the rotated magnetic moments should be perpendicular to H_0, i.e., $\theta \approx \pi/2$. This means that

$$\gamma_P H_1 t \approx \pi/2 \,,$$

i.e., the width of the first pulse should be $t \approx \frac{\pi}{2\gamma_P H_1}$.

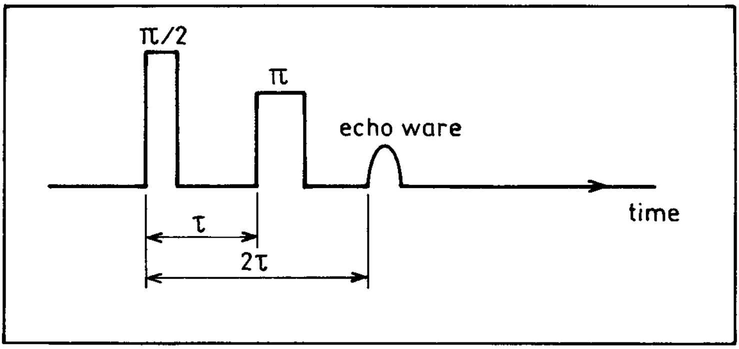

Fig. 1.40

1111

Choose only ONE of the following spectroscopes:

Continuous electron spin resonance

Pulsed nuclear magnetic resonance

Mössbauer spectroscopy

(a) Give a block diagram of the instrumentation required to perform the spectroscopy your have chosen.

(b) Give a concise description of the operation of this instrument.

(c) Describe the results of a measurement making clear what quantitative information can be derived from the data and the physical significance of this quantitative information.

(*SUNY, Buffalo*)

Solution:

(1) **Continuous electron spin resonance**

(a) The experimental setup is shown in Fig. 1.41.

(b) *Operation.* The sample is placed in the resonant cavity, which is under a static magnetic field B_0. Fixed-frequency microwaves B_1 created in the klystron is guided to the T-bridge. When the microwave power is distributed equally to the arms 1 and 2, there is no signal in the wave detector. As B_0 is varied, when the resonance condition is satisfied, the sample absorbs power and the balance between 1 and 2 is disturbed. The absorption

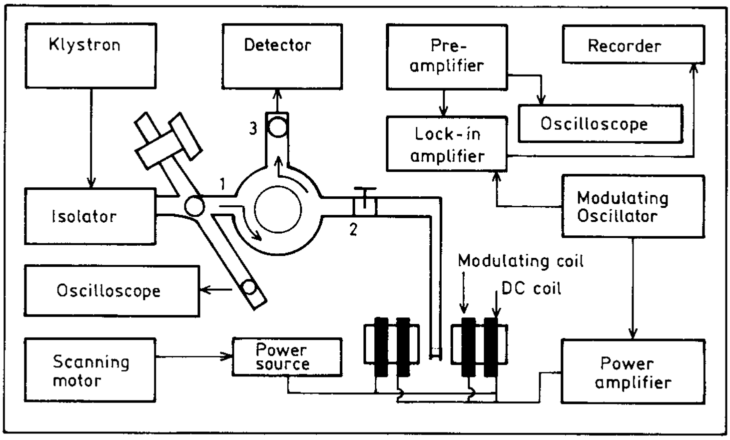

Fig. 1.41

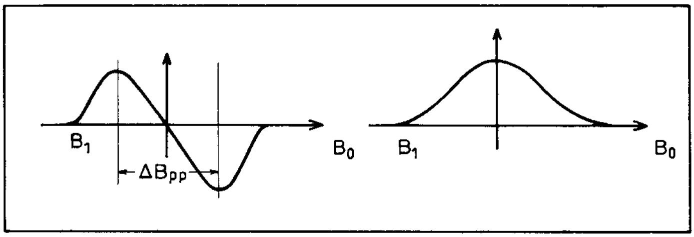

Fig. 1.42

signal is transmitted to the wave detector through arm 3, to be displayed or recorded.

(c) *Data analysis.* The monitor may show two types of differential graph (Fig. 1.42), Gaussian or Lagrangian, from which the following information may be obtained.

(i) The g-factor can be calculated from B_0 at the center and the microwave frequency.

(ii) The line width can be found from the peak-to-peak distance ΔB_{pp} of the differential signal.

(iii) The relaxation time T_1 and T_2 can be obtained by the saturation method, where T_1 and T_2 (Lorenzian profile) are given by

$$T_2(\text{spin–spin}) = \frac{1.3131 \times 10^{-7}}{g\Delta B_{pp}^0},$$

$$T_1 = \frac{0.9848 \times 10^{-7} \Delta B_{PP}^0}{g B_1^2} \left(\frac{1}{s} - 1 \right).$$

In the above g is the Landé factor, ΔB_{PP}^0 is the saturation peak-to-peak distance (in gauss), B_1 is the magnetic field corresponding to the edge of the spectral line, and s is the saturation factor.

(iv) The relative intensities.

By comparing with the standard spectrum, we can determine from the g-factor and the line profile to what kind of paramagnetic atoms the spectrum belongs. If there are several kinds of paramagnetic atoms present in the sample, their relative intensities give the relative amounts. Also, from the structure of the spectrum, the nuclear spin I may be found.

(2) **Pulsed nuclear magnetic resonance**

(a) Figure 1.43 shows a block diagram of the experimental setup.

(b) *Operation.* Basically an external magnetic field is employed to split up the spin states of the nuclei. Then a pulsed radiofrequency field is introduced perpendicular to the static magnetic field to cause resonant transitions between the spin states. The absorption signals obtained from the same coil are amplified, Fourier-transformed, and displayed on a monitor screen.

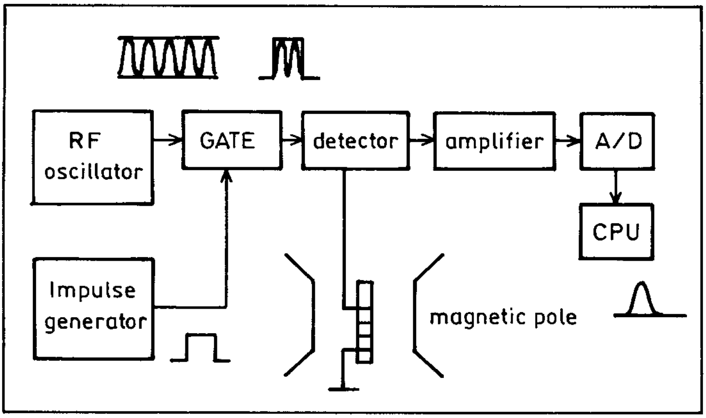

Fig. 1.43

(c) Information that can be deduced are positions and number of absorption peaks, integrated intensities of absorption peaks, the relaxation times T_1 and T_2.

The positions of absorption peaks relate to chemical displacement. From the number and integrated intensities of the peaks, the structure of the compound may be deduced as different kinds of atom have different ways of compounding with other atoms. For a given way of compounding, the integrated spectral intensity is proportional to the number of atoms. Consequently, the ratio of atoms in different combined forms can be determined from the ratio of the spectral intensities. The number of the peaks relates to the coupling between nuclei.

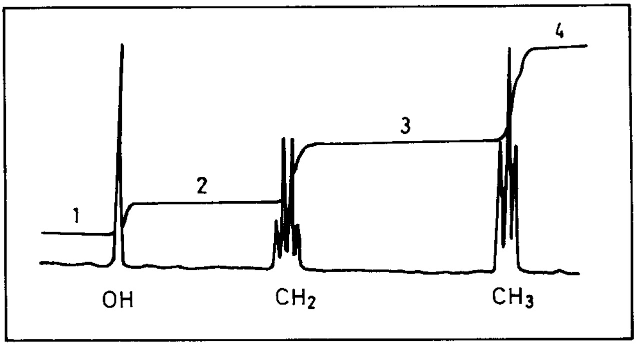

Fig. 1.44

For example Fig. 1.44 shows the nuclear magnetic resonance spectrum of H in alcohol. Three groups of nuclear magnetic resonance spectra are seen. The single peak on the left arises from the combination of H and O. The 4 peaks in the middle are the nuclear magnetic resonance spectrum of H in CH_2, and the 3 peaks on the right are the nuclear magnetic resonance spectrum of H in CH_3. The line shape and number of peaks are related to the coupling between CH_2 and CH_3. Using the horizontal line 1 as base line, the relative heights of the horizontal lines 2, 3, 4 give the relative integrated intensities of the three spectra, which are exactly in the ratio of 1:2:3.

(3) **Mössbauer spectroscopy**

(a) Figure 1.45 shows a block diagram of the apparatus.

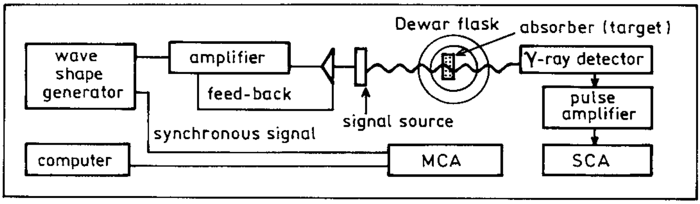

Fig. 1.45

(b) *Operation.* The signal source moves towards the fixed absorber with a velocity v modulated by the signals of the wave generator. During resonant absorption, the γ-ray detector behind the absorber produces a pulse signal, which is stored in the multichannel analyser MCA. Synchronous signals establish the correspondence between the position of a pulse and the velocity v, from which the resonant absorption curve is obtained.

(c) Information that can be obtained are the position δ of the absorption peak (Fig. 1.46), integrated intensity of the absorbing peak A, peak width Γ.

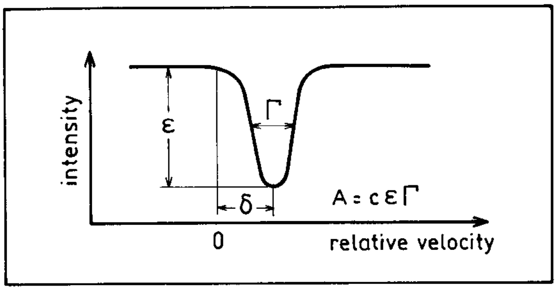

Fig. 1.46

Besides the effect of interactions among the nucleons inside the nucleus, nuclear energy levels are affected by the crystal structure, the orbital electrons and atoms nearby. In the Mössbauer spectrum the isomeric shift δ varies with the chemical environment. For instance, among the isomeric shifts of Sn^{2+}, Sn^{4+} and the metallic β-Sn, that of Sn^{2+} is the largest, that of β-Sn comes next, and that of the Sn^{4+} is the smallest.

The lifetime of an excited nuclear state can be determined from the width Γ of the peak by the uncertainty principle $\Gamma\tau \sim \hbar$.

The Mössbaur spectra of some elements show quadrupole splitting, as shown in Fig. 1.47. The quadrupole moment $Q = 2\Delta/e^2 q$ of the nucleus can be determined from this splitting, where q is the gradient of the electric field at the site of the nucleus, e is the electronic charge.

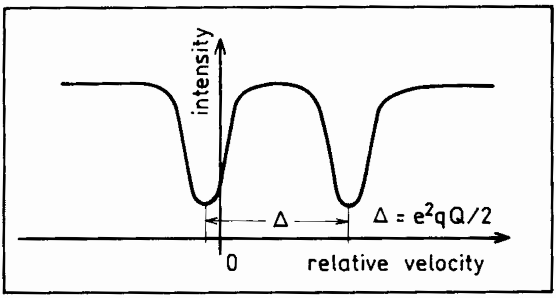

Fig. 1.47

1112

Pick ONE phenomenon from the list below, and answer the following questions about it:

(1) What is the effect? (e.g., "The Mössbauer effect is ...")

(2) How can it be measured?

(3) Give several sources of noise that will influence the measurement.

(4) What properties of the specimen or what physical constants can be measured by examining the effect?

Pick one:

(a) Electron spin resonance. (b) Mössbauer effect. (c) The Josephson effect. (d) Nuclear magnetic resonance. (e) The Hall effect.

(*SUNY, Buffalo*)

Solution:

(a) (b) (d) Refer to **Problem 1111**.

(c) *The Josephson effect*: Under proper conditions, superconducting electrons can cross a very thin insulation barrier from one superconductor into another. This is called the Josephson effect (Fig. 1.48). The

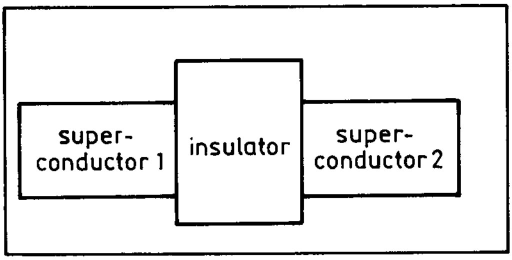

Fig. 1.48

Josephson effect is of two kinds, direct current Josephson effect and alternating current Josephson effect.

The direct current Josephson effect refers to the phenomenon of a direct electric current crossing the Josephson junction without the presence of any external electric or magnetic field. The superconducting current density can be expressed as $J_s = J_c \sin\varphi$, where J_c is the maximum current density that can cross the junction, φ is the phase difference of the wave functions in the superconductors on the two sides of the insulation barrier.

The alternating current Josephson effect occurs in the following situations:

1. When a direct current voltage is introduced to the two sides of the Josephson junction, a radiofrequency current $J_s = J_c \sin(\frac{2e}{\hbar}Vt + \varphi_0)$ is produced in the Josephson junction, where V is the direct current voltage imposed on the two sides of the junction.

2. If a Josephson junction under an imposed bias voltage V is exposed to microwaves of frequency ω and the condition $V = n\hbar\omega/2e$ $(n = 1, 2, 3, \dots)$ is satisfied, a direct current component will appear in the superconducting current crossing the junction.

Josephson effect can be employed for accurate measurement of $e/\hbar$. In the experiment the Josephson junction is exposed to microwaves of a fixed frequency. By adjusting the bias voltage V, current steps can be seen on the I–V graph, and $e/\hbar$ determined from the relation $\Delta V = \hbar\omega/2e$, where ΔV is the difference of the bias voltages of the neighboring steps.

The Josephson junction can also be used as a sensitive microwave detector. Furthermore, $\Delta V = \hbar\omega/2e$ can serve as a voltage standard.

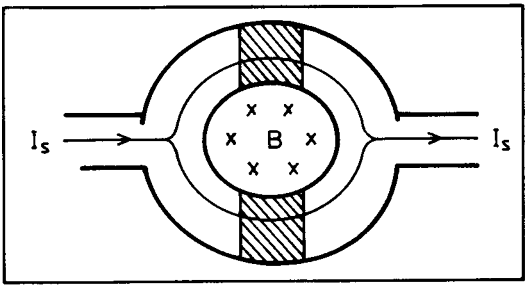

Fig. 1.49

Making use of the modulation effect on the junction current of the magnetic field, we can measure weak magnetic fields. For a ring structure consisting of two parallel Josephson junctions as shown in Fig. 1.49 ("double-junction quantum interferometer"), the current is given by

$$I_s = 2I_{s0} \sin \varphi_0 \cos \left(\frac{\pi \Phi}{\phi_0} \right) ,$$

where I_{s0} is the maximum superconducting current which can be produced in a single Josephson junction, $\phi_0 = \frac{h}{2e}$ is the magnetic flux quantum, Φ is the magnetic flux in the superconducting ring. Magnetic fields as small as 10^{-11} gauss can be detected.

(e) **Hall effect**. When a metallic or semiconductor sample with electric current is placed in a uniform magnetic field which is perpendicular to the current, a steady transverse electric field perpendicular to both the current and the magnetic field will be induced across the sample. This is called the Hall effect. The uniform magnetic field **B**, electric current density **j**, and the Hall electric field **E** have a simple relation: $\mathbf{E} = R_H \mathbf{B} \times \mathbf{j}$, where the parameter R_H is known as the Hall coefficient.

As shown in Fig. 1.50, a rectangular parallelepiped thin sample is placed in a uniform magnetic field **B**. The Hall coefficient R_H and the electric conductivity σ of the sample can be found by measuring the Hall voltage V_H, magnetic field B, current I, and the dimensions of the sample:

$$R_H = \frac{V_H d}{IB}, \qquad \sigma = \frac{Il}{Ubd},$$

where U is the voltage of the current source. From the measured R_H and σ, we can deduce the type and density N of the current carriers in a semiconductor, as well as their mobility μ.

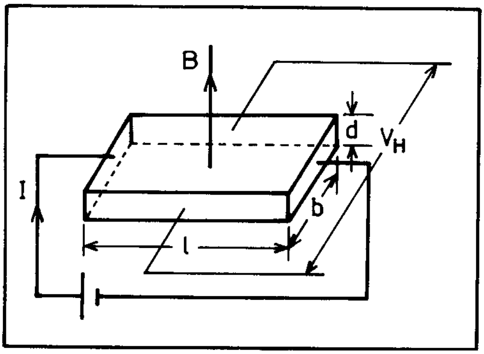

Fig. 1.50

The Hall effect arises from the action of the Lorentz force on the current carriers. In equilibrium, the magnetic force on the current carriers is balanced by the force due to the Hall electric field:

$$q\mathbf{E} = q\mathbf{v} \times \mathbf{B}\,,$$

giving

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} = \frac{1}{Nq}\mathbf{j} \times \mathbf{B}\,.$$

Hence $R_H = \frac{1}{Nq}$, where q is the charge of current carriers ($|q| = e$), from which we can determine the type of the semiconductor (p or n type in accordance with R_H being positive or negative). The carrier density and mobility are given by

$$N = \frac{1}{qR_H}\,,$$

$$\mu = \frac{\sigma}{Ne} = \sigma |R_H|\,.$$

1113

State briefly the importance of each of the following experiments in the development of atomic physics.

(a) Faraday's experiment on electrolysis.

(b) Bunsen and Kirchhoff's experiments with the spectroscope.

(c) J. J. Thomson's experiments on e/m of particles in a discharge.

(d) Geiger and Marsdens experiment on scattering of α-particles.

(e) Barkla's experiment on scattering of X-rays.

(f) The Frank-Hertz experiment.

(g) J. J. Thomson's experiment on e/m of neon ions.

(h) Stern-Gerlach experiment.

(i) Lamb-Rutherford experiment.

(*Wisconsin*)

Solution:

(a) Faraday's experiment on electrolysis was the first experiment to show that there is a natural unit of electric charge $e = F/N_a$, where F is the Faraday constant and N_a is Avogadro's number. The charge of any charged body is an integer multiple of e.

(b) Bunsen and Kirchhoff analyzed the Fraunhofer lines of the solar spectrum and gave the first satisfactory explanation of their origin that the lines arose from the absorption of light of certain wavelengths by the atmospheres of the sun and the earth. Their work laid the foundation of spectroscopy and resulted in the discovery of the elements rubidium and cesium.

(c) J. J. Thomson discovered the electron by measuring directly the e/m ratio of cathode rays. It marked the beginning of our understanding of the atomic structure.

(d) Geiger and Marsden's experiment on the scattering of α-particles formed the experimental basis of Rutherfold's atomic model.

(e) Barkla's experiment on scattering of X-rays led to the discovery of characteristic X-ray spectra of elements which provide an important means for studying atomic structure.

(f) The Frank-Hertz experiment on inelastic scattering of electrons by atoms established the existence of discrete energy levels in atoms.

(g) J. J. Thomson's measurement of the e/m ratio of neon ions led to the discovery of the isotopes ^{20}Ne and ^{22}Ne.

(h) The Stern-Gerlach experiment provided proof that there exist only certain permitted orientations of the angular momentum of an atom.

(i) The Lamb-Rutherford experiment provided evidence of interaction of an electron with an electromagnetic radiation field, giving support to the theory of quantum electrodynamics.

1114

In a Stern-Gerlach experiment hydrogen atoms are used.

(a) What determines the *number* of lines one sees? What features of the apparatus determine the magnitude of the *separation* between the lines?

(b) Make an *estimate* of the separation between the two lines if the Stern-Gerlach experiment is carried out with H atoms. Make any reasonable assumptions about the experimental setup. For constants which you do not know by heart, state where you would look them up and what units they should be substituted in your formula.

(*Wisconsin*)

Solution:

(a) A narrow beam of atoms is sent through an inhomogeneous magnetic field having a gradient $\frac{dB}{dz}$ perpendicular to the direction of motion of the beam. Let the length of the magnetic field be L_1, the flight path length of the hydrogen atoms after passing through the magnetic field be L_2 (Fig. 1.51).

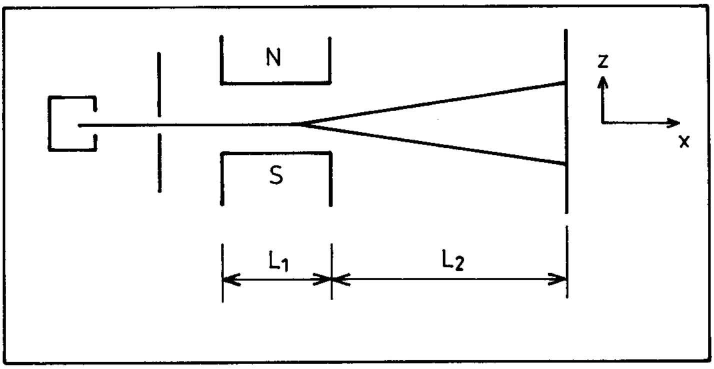

Fig. 1.51

The magnetic moment of ground state hydrogen atom is $\boldsymbol{\mu} = g\mu_B\mathbf{J} = 2\mu_B\mathbf{J}$. In the inhomogeneous magnetic field the gradient $\frac{\partial B}{\partial z}\mathbf{i}_z$ exerts a force on the magnetic moment $F_z = 2\mu_B M_J(\frac{\partial B}{\partial z})$. As $J = \frac{1}{2}$, $M_J = \pm\frac{1}{2}$ and the beam splits into two components.

After leaving the magnetic field an atom has acquired a transverse velocity $\frac{F_z}{m} \cdot \frac{L_1}{v}$ and a transverse displacement $\frac{1}{2}\frac{F_z}{m}(\frac{L_1}{v})^2$, where m and v are respectively the mass and longitudinal velocity of the atom. When the beam strikes the screen the separation between the lines is

$$\frac{\mu_B L_1}{mv^2}(L_1 + 2L_2)\left(\frac{1}{2} + \frac{1}{2}\right)\frac{\partial B}{\partial z}.$$

(b) Suppose $L_1 = 0.03\,\text{m}$, $L_2 = 0.10\,\text{m}$, $dB/dz = 10^3\ \text{T/m}$, $v = 10^3\ \text{m/s}$. We have

$$d = \frac{0.927 \times 10^{-23} \times 0.03}{1.67 \times 10^{-27} \times 10^6} \times (0.03 + 2 \times 0.10) \times 10^3$$

$$= 3.8 \times 10^{-2}\ \text{m} = 3.8\ \text{cm}.$$

1115

Give a brief description of the Stern-Gerlach experiment and answer the following questions:

(a) Why must the magnetic field be inhomogeneous?

(b) How is the inhomogeneous field obtained?

(c) What kind of pattern would be obtained with a beam of hydrogen atoms in their ground state? Why?

(d) What kind of pattern would be obtained with a beam of mercury atoms (ground state $^1S_0$)? Why?

(*Wisconsin*)

Solution:

For a brief description of the Stern-Gerlach experiment see **Problem 1114**.

(a) The force acting on the atomic magnetic moment μ in an inhomogeneous magnetic field is

$$F_z = -\frac{d}{dz}(\mu B \cos\theta) = -\mu \frac{dB}{dz}\cos\theta,$$

where θ is the angle between the directions of $\boldsymbol{\mu}$ and $\mathbf{B}$. If the magnetic field were uniform, there would be no force and hence no splitting of the atomic beam.

(b) The inhomogeneous magnetic field can be produced by non-symmetric magnetic poles such as shown in Fig. 1.52.

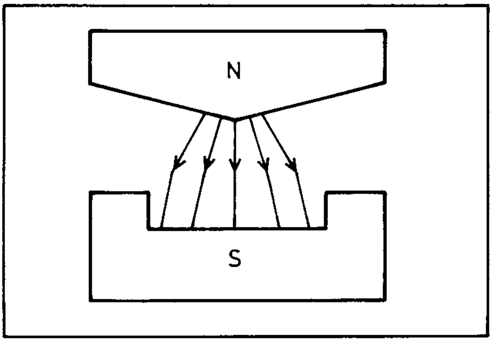

Fig. 1.52

(c) The ground state of hydrogen atom is $^2S_{1/2}$. Hence a beam of hydrogen atoms will split into two components on passing through an inhomogeneous magnetic field.

(d) As the total angular momentum J of the ground state of Hg is zero, there will be no splitting of the beam since $(2J + 1) = 1$.

1116

The atomic number of aluminum is 13.

(a) What is the electronic configuration of Al in its ground state?

(b) What is the term classification of the ground state? Use standard spectroscopic notation (e.g. $^4S_{1/2}$) and explain all superscripts and subscripts.

(c) Show by means of an energy-level diagram what happens to the ground state when a very strong magnetic field (Paschen-Back region) is applied. Label all states with the appropriate quantum numbers and indicate the relative spacing of the energy levels.

(*Wisconsin*)

Solution:

(a) The electronic configuration of the ground state of Al is

$$(1s)^2(2s)^2(2p)^6(3s)^2(3p)^1 .$$

(b) The spectroscopic notation of the ground state of Al is $^2P_{1/2}$, where the superscript 2 is the multiplet number, equal to $2S+1$, S being the total spin quantum number, the subscript 1/2 is the total angular momentum quantum number, the letter P indicates that the total orbital angular momentum quantum number $L = 1$.

(c) In a very strong magnetic field, LS coupling will be destroyed, and the spin and orbital magnetic moments interact separately with the external magnetic field, causing the energy level to split. The energy correction in the magnetic field is given by

$$\Delta E = -(\boldsymbol{\mu}_L + \boldsymbol{\mu}_s) \cdot \mathbf{B} = (M_L + 2M_s)\mu_B B ,$$

where

$$M_L = 1, 0, -1, \qquad M_S = 1/2, -1/2 .$$

The ^2P energy level is separated into 5 levels, the spacing of neighboring levels being $\mu_B B$. The split levels and the quantum numbers (L, S, M_L, M_S) are shown in Fig. 1.53.

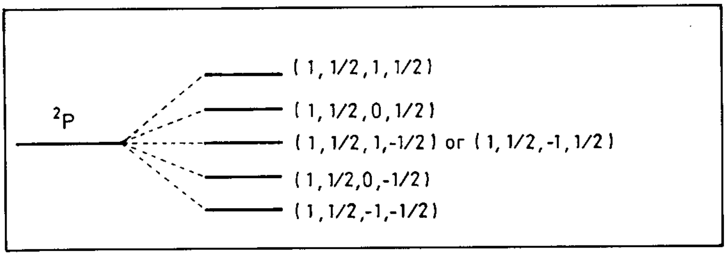

Fig. 1.53

1117

A heated gas of neutral lithium ($Z = 3$) atoms is in a magnetic field. Which of the following states lie lowest. Give brief physical reasons for your answers.

(a) 3 $^2P_{1/2}$ and 2 $^2S_{1/2}$.
(b) 5 $^2S_{1/2}$ and 5 $^2P_{1/2}$.

(c) $5\ ^2P_{3/2}$ and $5\ ^2P_{1/2}$.

(d) Substates of $5^2P_{3/2}$.

(*Wisconsin*)

Solution:

The energy levels of an atom will be shifted in an external magnetic field B by

$$\Delta E = M_J g \mu_B B \,,$$

where g is the Landé factor, M_J is the component of the total angular momentum along the direction of the magnetic field B. The shifts are only $\sim 5 \times 10^{-5}$ eV even in a magnetic field as strong as 1 T.

(a) $3^2P_{1/2}$ is higher than $2^2S_{1/2}$ (energy difference ~ 1 eV), because the principal quantum number of the former is larger. Of the $^2S_{1/2}$ states the one with $M_J = -\frac{1}{2}$ lies lowest.

(b) The state with $M_J = -1/2$ of $^2S_{1/2}$ lies lowest. The difference of energy between ^2S and ^2P is mainly caused by orbital penetration and is of the order ~ 1 eV.

(c) Which of the states $^2P_{3/2}$ and $^2P_{1/2}$ has the lowest energy will depend on the intensity of the external magnetic field. If the external magnetic field would cause a split larger than that due to LS-coupling, then the state with $M_J = -3/2$ of $^2P_{3/2}$ is lowest. Conversely, $M_J = -1/2$ of $^2P_{1/2}$ is lowest.

(d) The substate with $M_J = -3/2$ of $^2P_{3/2}$ is lowest.

1118

A particular spectral line corresponding to a $J = 1 \rightarrow J = 0$ transition is split in a magnetic field of 1000 gauss into three components separated by 0.0016 Å. The zero field line occurs at 1849 Å.

(a) Determine whether the total spin is in the $J = 1$ state by studying the g-factor in the state.

(b) What is the magnetic moment in the excited state?

(*Princeton*)

Solution:

(a) The energy shift in an external magnetic field B is

$$\Delta E = g\mu_B B\,.$$

The energy level of $J = 0$ is not split. Hence the splitting of the line due to the transition $J = 1 \to J = 0$ is equal to the splitting of $J = 1$ level:

$$\Delta E(J = 1) = hc\Delta\tilde{\nu} = hc\frac{\Delta\lambda}{\lambda^2}\,,$$

or

$$g = \frac{hc}{\mu_B B}\frac{\Delta\lambda}{\lambda^2}\,.$$

With

$$\Delta\lambda = 0.0016\ \text{Å}\,,$$

$$\lambda = 1849\ \text{Å} = 1849 \times 10^{-8}\ \text{cm}\,,$$

$$hc = 4\pi \times 10^{-5}\ \text{eV} \cdot \text{cm}\,,$$

$$\mu_B = 5.8 \times 10^{-9}\ \text{eV} \cdot \text{Gs}^{-1}\,,$$

$$B = 10^3\ \text{Gs}\,,$$

we find

$$g = 1\,.$$

As $J = 1$ this indicates (**Problem 1091(b)**) that $S = 0$, $L = 1$, i.e., only the orbital magnetic moment contributes to the Zeeman splitting.

(b) The magnetic moment of the excited atom is

$$\mu_J = g\mu_B P_J/\hbar = 1 \cdot \mu_B \cdot \sqrt{J(J+1)} = \sqrt{2}\mu_B\,.$$

1119

Compare the weak-field Zeeman effect for the $(1s3s)\ ^1S_0 \to (1s2p)\ ^1P_1$ and $(1s3s)\ ^3S_1 \to (1s2p)\ ^3P_1$ transitions in helium. You may be qualitative so long as the important features are evident.

(Wisconsin)

Solution:

In a weak magnetic field, each energy level of $^3P_1$, $^3S_1$ and $^1P_1$ is split into three levels. From the selection rules ($\Delta J = 0, \pm 1$; $M_J = 0, \pm 1$), we see that the transition $(1s3s)^1S_0 \to (1s2p)^1P_1$ gives rise to three spectral lines, the transition $(1s3s)^3S_1 \to (1s2p)^3P_1$ gives rise to six spectral lines, as shown in Fig. 1.54.

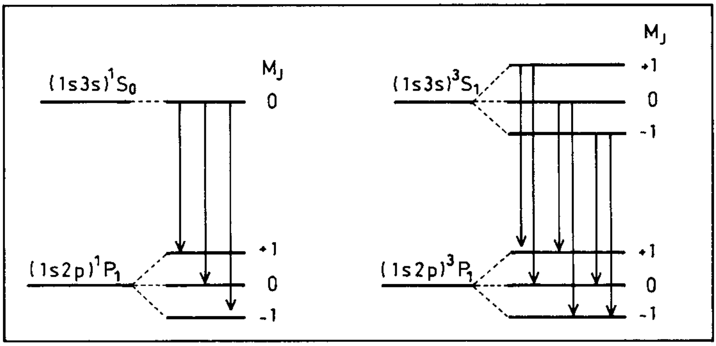

Fig. 1.54

The shift of energy in the weak magnetic field B is $\Delta E = gM_J\mu_B B$, where μ_B is the Bohr magneton, g is the Landé splitting factor given by

$$g = 1 + \frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}.$$

For the above four levels we have

Level	$(1s3s)^1S_0$	$(1s2p)^1P_1$	$(1s3s)^3S_1$	$(1s2p)^3P_1$
(JLS)	(000)	(110)	(101)	(111)
ΔE	0	$\mu_B B$	$2\mu_B B$	$3\mu_B B/2$

from which the energies of transition can be obtained.

1120

The influence of a magnetic field on the spectral structure of the prominent yellow light (in the vicinity of 6000 Å) from excited sodium vapor is being examined (Zeeman effect). The spectrum is observed for light emitted in a direction either along or perpendicular to the magnetic field.

(a) *Describe:* (i) The spectrum before the field is applied.

(ii) The change in the spectrum, for both directions of observation, after the field is applied.

(iii) What states of polarization would you expect for the components of the spectrum in each case?

(b) *Explain* how the above observations can be interpreted in terms of the characteristics of the atomic quantum states involved.

(c) If you have available a spectroscope with a resolution ($\lambda/\delta\lambda$) of 100000 what magnetic field would be required to resolve clearly the 'splitting' of lines by the magnetic field? (Numerical estimates to a factor of two or so are sufficient. You may neglect the line broadening in the source.)

(*Columbia*)

Solution:

(a) The spectra with and without magnetic field are shown in Fig. 1.55.

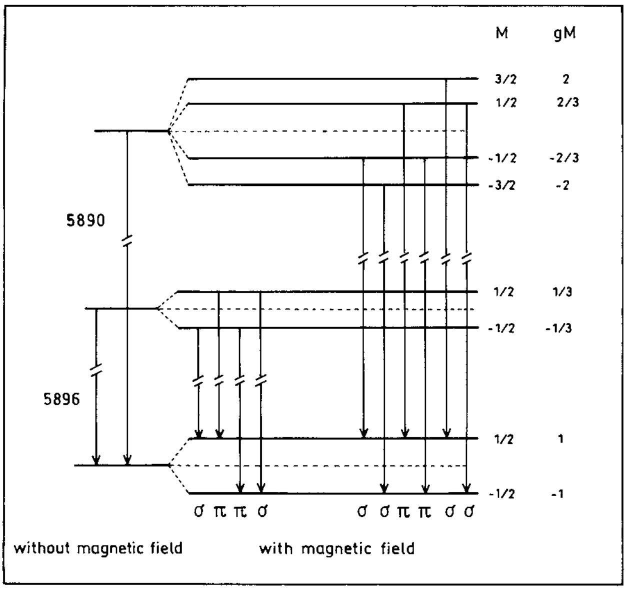

Fig. 1.55

(i) Before the magnetic field is introduced, two lines can be observed with wavelengths 5896 Å and 5890 Å in all directions.

(ii) After introducing the magnetic field, we can observe 6σ lines in the direction of the field and 10 lines, 4π lines and 6σ lines in a direction perpendicular to the field.

(iii) The σ lines are pairs of left and right circularly polarized light. The π lines are plane polarized light.

(b) The splitting of the spectrum arises from quantization of the direction of the total angular momentum. The number of split components is determined by the selection rule ($\Delta M_J = 1, 0, -1$) of the transition, while the state of polarization is determined by the conservation of the angular momentum.

(c) The difference in wave number of two nearest lines is

$$\Delta\tilde{\nu} = \frac{|g_1 - g_2|\mu_B B}{hc} = \frac{1}{\lambda_1} - \frac{1}{\lambda_2} \approx \frac{\delta\lambda}{\lambda^2}\,,$$

where g_1, g_2 are Landé splitting factors of the higher and lower energy levels. Hence the magnetic field strength required is of the order

$$B \sim \frac{hc\delta\lambda}{|g_1 - g_2|\mu_B\lambda^2} = \frac{12 \times 10^{-5} \times 10^8}{1 \times 6 \times 10^{-5}} \times \frac{10^{-5}}{6000} = 0.3\ T\,.$$

1121

Discuss qualitatively the shift due to a constant external electric field E_0 of the $n = 2$ energy levels of hydrogen. Neglect spin, but include the observed zero-field splitting W of the $2s$ and $2p$ states:

$$W = E_{2s} - E_{2p} \sim 10^{-5}\ \text{eV}\,.$$

Consider separately the cases $|e|E_0 a_0 \gg W$ and $|e|E_0 a_0 \ll W$, where a_0 is the Bohr radius.

(*Columbia*)

Solution:

Consider the external electric field E_0 as perturbation. Then $H' = e\mathbf{E}_0 \cdot \mathbf{r}$. Nonzero matrix elements exist only between states $|200\rangle$ and $|210\rangle$

among the four $|n=2\rangle$ states $|200\rangle, |211\rangle, |210\rangle, |21{-}1\rangle$. **Problem 1122(a)** gives

$$\langle 210|H'|200\rangle \equiv u = -3eE_0a_0 \,.$$

The states $|211\rangle$ and $|21\,-1\rangle$ remain degenerate.

(i) For $W \gg |e|E_0a_0$, or $W \gg |u|$, the perturbation is on nondegenerate states. There is nonzero energy correction only in second order calculation. The energy corrections are

$$E_+ = W + u^2/W, \;\; E_- = W - u^2/W \,.$$

(ii) For $W \ll |e|E_0a_0$, or $W \ll |u|$, the perturbation is among degenerate states and the energy corrections are

$$E_+ = -u = 3eE_0a_0, \;\; E_- = u = -3eE_0a_0 \,.$$

1122

A beam of excited hydrogen atoms in the $2s$ state passes between the plates of a capacitor in which a uniform electric field **E** exists over a distance L, as shown in the Fig. 1.56. The hydrogen atoms have velocity v along the x axis and the **E** field is directed along the z axis as shown.

All the $n=2$ states of hydrogen are degenerate in the absence of the **E** field, but certain of them mix when the field is present.

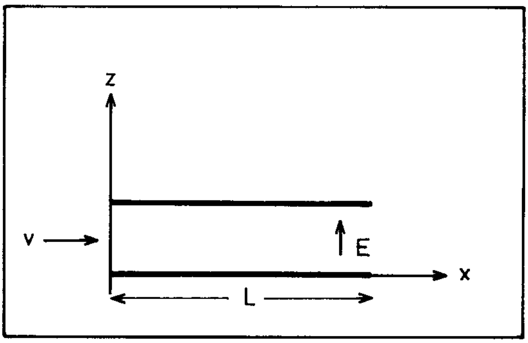

Fig. 1.56

(a) Which of the $n = 2$ states are connected in first order via the perturbation?

(b) Find the linear combination of $n = 2$ states which removes the degeneracy as much as possible.

(c) For a system which starts out in the $2s$ states at $t = 0$, express the wave function at time $t \leq L/v$.

(d) Find the probability that the emergent beam contains hydrogen in the various $n = 2$ states.

(MIT)

Solution:

(a) The perturbation Hamiltonian $H' = eEr\cos\theta$ commutes with $\hat{l}_z = -i\hbar\frac{\partial}{\partial\varphi}$, so the matrix elements of H' between states of different m vanish. There are 4 degenerate states in the $n = 2$ energy level:

$$2s: \quad l = 0, m = 0\,,$$

$$2p: \quad l = 1, m = 0, \pm 1\,.$$

The only nonzero matrix element is that between the $2s$ and $2p(m = 0)$ states:

$$\begin{aligned}\langle 210|eEr\cos\theta|200\rangle &= eE\int \psi_{210}(\mathbf{r})r\cos\theta\psi_{200}(\mathbf{r})d^3r\\ &= \frac{eE}{16a^4}\int_0^\infty\int_{-1}^1 r^4\left(2-\frac{r}{a}\right)e^{-r/a}\cos^2\theta d\cos\theta dr\\ &= -3eEa\,,\end{aligned}$$

where a is the Bohr radius.

(b) The secular equation determining the energy shift

$$\begin{vmatrix} -\lambda & -3eEa & 0 & 0\\ -3eEa & -\lambda & 0 & 0\\ 0 & 0 & -\lambda & 0\\ 0 & 0 & 0 & -\lambda \end{vmatrix} = 0$$

gives

$$\begin{aligned}\lambda &= 3eEa\,, & \Psi^{(-)} &= \frac{1}{\sqrt{2}}(\Phi_{200} - \Phi_{210})\,,\\ \lambda &= -3eEa\,, & \Psi^{(+)} &= \frac{1}{\sqrt{2}}(\Phi_{200} + \Phi_{210})\,,\\ \lambda &= 0\,, & \Psi &= \Phi_{211}, \Phi_{21-1}\,.\end{aligned}$$

(c) Let the energy of $n = 2$ state before perturbation be E_1. As at $t = 0$,

$$\Psi(t=0) = \Phi_{200} = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(\Phi_{200} - \Phi_{210}) + \frac{1}{\sqrt{2}}(\Phi_{200} + \Phi_{210})\right]$$

$$= \frac{1}{\sqrt{2}}(\Psi^{(-)} + \Psi^{(+)}),$$

we have

$$\Psi(t) = \frac{1}{\sqrt{2}}\left\{\Psi^{(-)} \exp\left[-\frac{i}{\hbar}(E_1 + 3eEa)t\right] + \Psi^{(+)} \exp\left[-\frac{i}{\hbar}(E_1 - 3eEa)t\right]\right\}$$

$$= \left[\Phi_{200}\cos\left(\frac{3eEat}{\hbar}\right) + \Phi_{210}\sin\left(\frac{3eEat}{\hbar}\right)\right]\exp\left(-\frac{i}{\hbar}E_1 t\right).$$

(d) When the beam emerges from the capacitor at $t = L/v$, the probability of its staying in $2s$ state is

$$\left|\cos\left(\frac{3eEat}{\hbar}\right)\exp\left(-\frac{i}{\hbar}E_1 t\right)\right|^2 = \cos^2\left(\frac{3eEat}{\hbar}\right) = \cos^2\left(\frac{3eEaL}{\hbar v}\right).$$

The probability of its being in $2p(m = 0)$ state is

$$\left|\sin\left(\frac{3eEat}{\hbar}\right)\exp\left(-\frac{i}{\hbar}E_1 t\right)\right|^2 = \sin^2\left(\frac{3eEat}{\hbar}\right) = \sin^2\left(\frac{3eEaL}{\hbar v}\right).$$

The probability of its being in $2p(m = \pm 1)$ state is zero.

2. MOLECULAR PHYSICS (1123–1142)

1123

(a) Assuming that the two protons of the H_2^+ molecule are fixed at their normal separation of 1.06 Å, sketch the potential energy of the electron along the axis passing through the protons.

(b) Sketch the electron wave functions for the two lowest states in H_2^+, indicating roughly how they are related to hydrogenic wave functions. Which wave function corresponds to the ground state of H_2^+, and why?

(c) What happens to the two lowest energy levels of H_2^+ in the limit that the protons are moved far apart?

(*Wisconsin*)

Solution:

(a) Take the position of one proton as the origin and that of the other proton at 1.06 Å along the x-axis. Then the potential energy of the electron is

$$V(r_1, r_2) = -\frac{e^2}{r_1} - \frac{e^2}{r_2},$$

where r_1 and r_2 are the distances of the electron from the two protons. The potential energy of the electron along the x-axis is shown in Fig. 1.57.

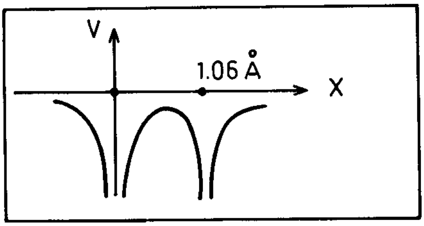

Fig. 1.57

(b) The molecular wave function of the H_2^+ has the forms

$$\Psi_S = \frac{1}{\sqrt{2}}(\Phi_{1s}(1) + \Phi_{1s}(2)),$$

$$\Psi_A = \frac{1}{\sqrt{2}}(\Phi_{1s}(1) - \Phi_{1s}(2)),$$

where $\Phi(i)$ is the wave function of an atom formed by the electron and the ith proton. Note that the energy of Ψ_S is lower than that of Ψ_A and so Ψ_S is the ground state of H_2^+; Ψ_A is the first excited state. Ψ_S and Ψ_A are linear combinations of $1s$ states of H atom, and are sketched in Fig. 1.58. The overlapping of the two hydrogenic wave functions is much larger in the case of the symmetric wave function Ψ_S and so the state is called a bonding state. The antisymmetric wave function Ψ_A is called an antibonding state. As Ψ_S has stronger binding its energy is lower.

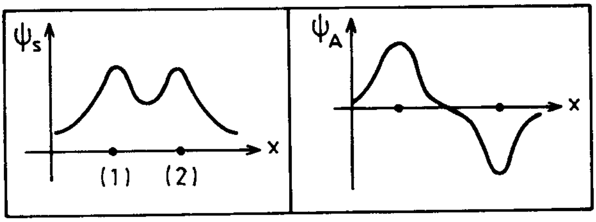

Fig. 1.58

(c) Suppose, with proton 1 fixed, proton 2 is moved to infinity, i.e. $\mathbf{r}_2 \to \infty$. Then $\Phi(2) \sim e^{-r_2/a} \to 0$ and $\Psi_S \approx \Psi_A \approx \Phi(1)$. The system breaks up into a hydrogen atom and a non-interacting proton.

1124

Given the radial part of the Schrödinger equation for a central force field $V(r)$:

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi(r)}{dr}\right)+\left[V(r)+\frac{l(l+1)\hbar^2}{2\mu r^2}\right]\Psi(r)=E\Psi(r)\,,$$

consider a diatomic molecule with nuclei of masses m_1 and m_2. A good approximation to the molecular potential is given by

$$V(r)=-2V_0\left(\frac{1}{\rho}-\frac{1}{2\rho^2}\right)\,,$$

where $\rho = r/a$, a with a being some characteristic length parameter.

(a) By expanding around the minimum of the effective potential in the Schrödinger equation, show that for small B the wave equation reduces to that of a simple harmonic oscillator with frequency

$$\omega=\left[\frac{2V_0}{\mu a^2(1+B)^3}\right]^{1/2}\,,\qquad \text{where } B=\frac{l(l+1)\hbar^2}{2\mu a^2V_0}\,.$$

(b) Assuming $\hbar^2/2\mu \gg a^2V_0$, find the rotational, vibrational and rotation-vibrational energy levels for small oscillations.

(*SUNY, Buffalo*)

Solution:

(a) The effective potential is

$$V_{\text{eff}} = \left[V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}\right] = -2V_0\left[\frac{a}{r} - \frac{a^2}{2r^2}(1+B)\right].$$

To find the position of minimum V_{eff}, let $\frac{dV_{\text{eff}}}{dr} = 0$, which gives $r = a(1+B) \equiv r_0$ as the equilibrium position. Expanding V_{eff} near $r = r_0$ and neglecting terms of orders higher than $(\frac{r-r_0}{a})^2$, we have

$$V_{\text{eff}} \approx -\frac{V_0}{1+B} + \frac{V_0}{(1+B)^3 a^2}[r-(1+B)a]^2\,.$$

The radial part of the Schrödinger equation now becomes

$$-\frac{\hbar^2}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi(r)}{dr}\right) + \left\{-\frac{V_0}{B+1} + \frac{V_0}{(1+B)^3a^2}[r-(1+B)a]^2\right\}\Psi(r)$$
$$= E\Psi(r)\,,$$

or, on letting $\Psi(r) = \frac{1}{r}\chi(r)$, $R = r - r_0$,

$$-\frac{\hbar^2}{2\mu}\frac{d^2}{dR^2}\chi(R) + \frac{V_0}{(1+B)^3a^2}R^2\chi(R) = \left(E + \frac{V_0}{1+B}\right)\chi(R)\,,$$

which is the equation of motion of a harmonic oscillator of angular frequency

$$\omega = \left[\frac{2V_0}{\mu a^2(1+B)^3}\right]^{1/2}.$$

(b) If $\hbar^2/2\mu \gg a^2V_0$, we have

$$B = \frac{l(l+1)\hbar^2}{2\mu a^2 V_0} \gg 1, \qquad r_0 \approx Ba\,,$$

$$\omega \approx \sqrt{\frac{2V_0}{\mu a^2 B^3}}\,.$$

The vibrational energy levels are given by

$$E_v = (n+1/2)\hbar\omega, n = 1,2,3\ldots.$$

The rotational energy levels are given by

$$E_r = \frac{l(l+1)\hbar^2}{2\mu r_0} \approx \frac{l(l+1)\hbar^2}{2\mu Ba} \, .$$

Hence, the vibration-rotational energy levels are given by

$$E = E_v + E_r \approx \left(n + \frac{1}{2}\right)\hbar\omega + \frac{l(l+1)\hbar^2}{2\mu Ba} \, .$$

1125

A beam of hydrogen molecules travel in the z direction with a kinetic energy of 1 eV. The molecules are in an excited state, from which they decay and dissociate into two hydrogen atoms. When one of the dissociation atoms has its final velocity perpendicular to the z direction its kinetic energy is always 0.8 eV. Calculate the energy released in the dissociative reaction.

(*Wisconsin*)

Solution:

A hydrogen molecule of kinetic energy 1 eV moving with momentum $\mathbf{p}_0$ in the z direction disintegrates into two hydrogen atoms, one of which has kinetic energy 0.8 eV and a momentum $\mathbf{p}_1$ perpendicular to the z direction. Let the momentum of the second hydrogen atom be $\mathbf{p}_2$, its kinetic energy be E_2. As $\mathbf{p}_0 = \mathbf{p}_1 + \mathbf{p}_2$, the momentum vectors are as shown in Fig. 1.59.

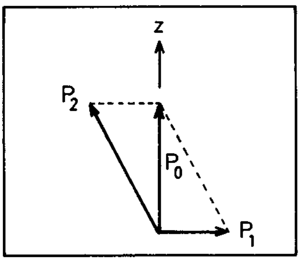

Fig. 1.59

We have

$$p_0 = \sqrt{2m(H_2)E(H_2)}$$
$$= \sqrt{2 \times 2 \times 938 \times 10^6 \times 1} = 6.13 \times 10^4 \text{ eV/c},$$
$$p_1 = \sqrt{2m(H)E(H)}$$
$$= \sqrt{2 \times 938 \times 10^6 \times 0.8} = 3.87 \times 10^4 \text{ eV/c}.$$

The momentum of the second hydrogen atom is then

$$p_2 = \sqrt{p_0^2 + p_1^2} = 7.25 \times 10^4 \text{ eV/c},$$

corresponding to a kinetic energy of

$$E_2 = \frac{p_2^2}{2m(H)} = 2.80 \text{ eV}.$$

Hence the energy released in the dissociative reaction is $0.8 + 2.8 - 1 = 2.6$ eV.

1126

Interatomic forces are due to:

(a) the mutual electrostatic polarization between atoms.
(b) forces between atomic nuclei.
(c) exchange of photons between atoms.

(CCT)

Solution:

The answer is (a).

1127

Which of the following has the smallest energy-level spacing?

(a) Molecular rotational levels,
(b) Molecular vibrational levels,
(c) Molecular electronic levels.

(CCT)

Solution:

The answer is (a). $\Delta E_e > \Delta E_v > \Delta E_r$.

1128

Approximating the molecule ${}^1{}_1\text{H}\ {}^{17}{}_{35}\text{Cl}$ as a rigid dumbbell with an internuclear separation of 1.29×10^{-10} m, calculate the frequency separation of its far infrared spectral lines. ($h = 6.6 \times 10^{-34}$ J sec, 1 amu $= 1.67 \times 10^{-27}$ kg).

(*Wisconsin*)

Solution:

The moment of inertia of the molecule is

$$I = \mu r^2 = \frac{m_{Cl} m_H}{m_{Cl} + m_H} r^2 = \frac{35}{36} \times 1.67 \times 10^{-27} \times (1.29 \times 10^{-10})^2$$
$$= 2.7 \times 10^{-47} \text{ kg} \cdot m^2$$

The frequency of its far infrared spectral line is given by

$$\nu = \frac{hcBJ(J+1) - hcBJ(J-1)}{h} = 2cBJ\,,$$

where $B = \hbar^2/(2Ihc)$. Hence

$$\nu = \frac{\hbar^2}{Ih} J\,, \text{ and so } \Delta\nu = \frac{\hbar^2}{hI} = \frac{h}{4\pi^2 I} = \frac{6.6 \times 10^{-34}}{4\pi^2 \times 2.7 \times 10^{-47}} = 6.2 \times 10^{11} \text{ Hz}\,.$$

1129

(a) Recognizing that a hydrogen nucleus has spin 1/2 while a deuterium nucleus has spin 1, enumerate the possible nuclear spin states for H_2, D_2 and HD molecules.

(b) For each of the molecules H_2, D_2 and HD, discuss the rotational states of the molecule that are allowed for each nuclear spin state.

(c) Estimate the energy difference between the first two rotational levels for H_2. What is the approximate magnitude of the contribution of the nuclear kinetic energy? The interaction of the two nuclear spins? The interaction of the nuclear spin with the orbital motion?

(d) Use your answer to (c) above to obtain the distribution of nuclear spin states for H_2, D_2 and HD at a temperature of 1 K.

(*Columbia*)

Solution:

(a) As $s(p) = \frac{1}{2}, s(d) = 1$, and $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$, the spin of H_2 is 1 or 0, the spin of D_2 is 2, 1 or 0, and the spin of HD is 1/2 or 3/2.

(b) The two nuclei of H_2 are identical, so are the nuclei of D_2. Hence the total wave functions of H_2 and D_2 must be antisymmetric with respect to exchange of particles, while there is no such rule for DH. The total wave function may be written as $\Psi_T = \Psi_e\Psi_v\Psi_r\Psi_s$, where Ψ_e, Ψ_v, Ψ_r, and Ψ_s are the electron wave function, nuclear vibrational wave function, nuclear rotational wave function, and nuclear spin wave function respectively. For the ground state, the Ψ_e, Ψ_v are exchange-symmetric. For the rotational states of H_2 or D_2, a factor $(-1)^J$ will occur in the wave function on exchanging the two nuclei, where J is the rotational quantum number. The requirement on the symmetry of the wave function then gives the following:

$$H_2: \quad \text{For } S = 1 \ (\Psi_s \text{ symmetric}), \ J = 1, 3, 5, \dots ;$$

$$\text{for } S = 0 \ (\Psi_s \text{ anitsymmetric}), \ J = 0, 2, 4, \dots .$$

$$D_2: \quad \text{For } S = 0, 2, J = 0, 2, 4, \dots ; \quad \text{for } S = 1, J = 1, 3, 5, \dots .$$

$$HD: \quad S = \frac{1}{2}, \frac{3}{2}; J = 1, 2, 3, \dots (\text{no restriction}) .$$

(c) For H_2, take the distance between the two nuclei as $a \approx 2a_0 \approx 1$ Å $a_0 = \frac{\hbar^2}{m_e e^2}$ being the Bohr radius. Then $I = 2m_p a_0^2 = \frac{1}{2} m_p a^2$ and the energy difference between the first two rotational states is

$$\Delta E = \frac{\hbar^2}{2I} \times [1 \times (1 + 1) - 0 \times (0 + 1)] \approx \frac{2\hbar^2}{m_p a^2} \approx \frac{m_e}{m_p} E_0 ,$$

where

$$E_0 = \frac{2\hbar^2}{m_e a^2} = \frac{e^2}{2a_0}$$

is the ionization potential of hydrogen. In addition there is a contribution from the nuclear vibrational energy: $\Delta E_v \approx \hbar\omega$. The force between the nuclei is $f \approx e^2/a^2$, so that $K = |\nabla f| \approx \frac{2e^2}{a^3}$, giving

$$\Delta E_0 = \hbar\omega \approx \sqrt{\frac{K}{m_p}} = \sqrt{\frac{2e^2\hbar^2}{m_p a^3}} = \sqrt{\frac{m_e}{m_p}}\frac{e^2}{2e_0} = \sqrt{\frac{m_e}{m_p}}E_0 \,.$$

Hence the contribution of the nuclear kinetic energy is of the order of $\sqrt{\frac{m_e}{m_p}}E_0$.

The interaction between the nuclear spins is given by

$$\Delta E \approx \mu_N^2/a^3 \approx \left(\frac{e\hbar}{2m_p c}\right)^2 \frac{1}{8a_0^3} = \frac{1}{16}\left(\frac{\hbar}{m_p c}\right)^2 \left(\frac{m_e e^2}{\hbar^2}\right)^2 \frac{e^2}{2a_0}$$

$$= \frac{1}{16}\left(\frac{m_e}{m_p}\right)^2 \left(\frac{e^2}{\hbar c}\right)^2 E_0 = \frac{1}{16}\left(\frac{m_e}{m_p}\right)^2 \alpha^2 E_0 \,,$$

where $\alpha = \frac{1}{137}$ is the fine structure constant, and the interaction between nuclear spin and electronic orbital angular momentum is

$$\Delta E \approx \mu_N \mu_B / a_0^3 \approx \frac{1}{2}\left(\frac{m_e}{m_p}\right)\alpha^2 E_0 \,.$$

(d) For H_2, the moment of inertia is $I = \mu a^2 = \frac{1}{2}m_p a^2 \approx 2m_p a_0^2$, so the energy difference between states $l = 0$ and $l = 1$ is

$$\Delta E_{H_2} = \frac{\hbar^2}{2I} \times (2 - 0) = \frac{2m_e}{m_p}E_0 \,.$$

For D_2, as the nuclear mass is twice that of H_2,

$$\Delta E_{D_2} = \frac{1}{2}\Delta E_{H_2} = \frac{m_e}{m_p}E_0 \,.$$

As $kT = 8.7 \times 10^{-5}$ eV for $T = 1$ K, $\Delta E \approx \frac{E_0}{2000} = 6.8 \times 10^{-3}$ eV, we have $\Delta E \gg kT$ and so for both H_2 and D_2, the condition $\exp(-\Delta E/kT) \approx 0$ is satisfied. Then from Boltzmann's distribution law, we know that the H_2 and D_2 molecules are all on the ground state.

The spin degeneracies $2S + 1$ are for H_2, $g_{s=1} : g_{s=0} = 3 : 1$; for D_2, $g_{s=2} : g_{s=1}$: $g_{s=0} = 5 : 3 : 1$; and for HD, $g_{s=2/3} : g_{s=1/2} = 2 : 1$. From the population ratio g_2/g_1, we can conclude that most of H_2 is in the state of $S = 1$; most of D_2 is in the states of $S = 2$ and $S = 1$, the relative ratio being 5:3. Two-third of HD is in the state $S = 3/2$ and one-third in $S = 1/2$.

1130

Consider the (homonuclear) molecule $^{14}N_2$. Use the fact that a nitrogen nucleus has spin $I = 1$ in order to derive the result that the ratio of intensities of adjacent rotational lines in the molecular spectrum is 2:1.

(*Chicago*)

Solution:

As nitrogen nucleus has spin $I = 1$, the total wave function of the molecule must be symmetric. On interchanging the nuclei a factor $(-1)^J$ will occur in the wave function. Thus when the rotational quantum number J is even, the energy level must be a state of even spin, whereas a rotational state with odd J must be associated with an antisymmetric spin state. Furthermore, we have

$$\frac{g_S}{g_A} = \frac{(I+1)(2I+1)}{I(2I+1)} = (I+1)/I = 2:1$$

where g_S is the degeneracy of spin symmetric state, g_A is the degeneracy of spin antisymmetric state. As a homonuclear molecule has only Raman spectrum for which $\Delta J = 0, \pm 2$, the symmetry of the wave function does not change in the transition. The same is true then for the spin function. Hence the ratio of intensities of adjacent rotational lines in the molecular spectrum is $2:1$.

1131

Estimate the lowest neutron kinetic energy at which a neutron, in a collision with a molecule of gaseous oxygen, can lose energy by exciting molecular rotation. (The bond length of the oxygen molecule is 1.2 Å).

(*Wisconsin*)

Solution:

The moment of inertia of the oxygen molecule is

$$I = \mu r^2 = \frac{1}{2} m r^2 \,,$$

where r is the bond length of the oxygen molecule, m is the mass of oxygen atom.

The rotational energy levels are given by

$$E_J = \frac{h^2}{8\pi^2 I} J(J+1), \qquad J = 0, 1, 2, \dots .$$

To excite molecular rotation, the minimum of the energy that must be absorbed by the oxygen molecule is

$$E_{\rm min} = E_1 - E_0 = \frac{h^2}{4\pi^2 I} = \frac{h^2}{2\pi^2 m r^2} = \frac{2(\hbar c)^2}{mc^2 r^2}$$

$$= \frac{2 \times (1.97 \times 10^{-5})^2}{16 \times 938 \times 10^6 \times (1.2 \times 10^{-8})^2} = 3.6 \times 10^{-4} \text{ eV} .$$

As the mass of the neutron is much less than that of the oxygen molecule, the minimum kinetic energy the neutron must possess is 3.6×10^{-4} eV.

1132

(a) Using hydrogen atom ground state wave functions (including the electron spin) write wave functions for the hydrogen molecule which satisfy the Pauli exclusion principle. Omit terms which place both electrons on the same nucleus. Classify the wave functions in terms of their total spin.

(b) Assuming that the only potential energy terms in the Hamiltonian arise from Coulomb forces discuss qualitatively the energies of the above states at the normal internuclear separation in the molecule and in the limit of very large internuclear separation.

(c) What is meant by an "exchange force"?

(*Wisconsin*)

Solution:

Figure 1.60 shows the configuration of a hydrogen molecule. For convenience we shall use atomic units in which a_0 (Bohr radius) $= e = \hbar = 1$.

(a) The Hamiltonian of the hydrogen molecule can be written in the form

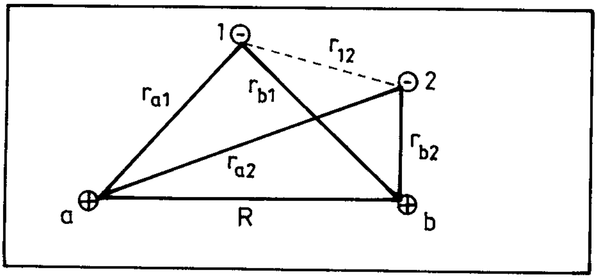

Fig. 1.60

$$\hat{H} = -\frac{1}{2}(\nabla_1^2 + \nabla_2^2) + \frac{1}{r_{12}} - \left(\frac{1}{r_{a1}} + \frac{1}{r_{a2}} + \frac{1}{r_{b1}} + \frac{1}{r_{b2}}\right) + \frac{1}{R}\,.$$

As the electrons are indistinguishable and in accordance with Pauli's principle the wave function of the hydrogen molecule can be written as

$$\Psi_S = [\Psi(r_{a1})\Psi(r_{b2}) + \Psi(r_{a2})\Psi(r_{b1})]\chi_0$$

or

$$\Psi_A = [\Psi(r_{a1})\Psi(r_{b2}) - \Psi(r_{a2})\Psi(r_{b1})]\chi_1\,,$$

where χ_0, χ_1 are spin wave functions for singlet and triplet states respectively, $\psi(r) = \frac{\lambda^{3/2}}{\sqrt{\pi}}e^{-\lambda r}$, the parameter λ being 1 for ground state hydrogen atom.

(b) When the internuclear separation is very large the molecular energy is simply the sum of the energies of the atoms.

If two electrons are to occupy the same spatial position, their spins must be antiparallel as required by Pauli's principle. In the hydrogen molecule the attractive electrostatic forces between the two nuclei and the electrons tend to concentrate the electrons between the nuclei, forcing them together and thus favoring the singlet state. When two hydrogen atoms are brought closer from infinite separation, the repulsion for parallel spins causes the triplet-state energy to rise and the attraction for antiparallel spins causes the singlet-state energy to fall until a separation of $\sim$ $1.5a_0$ is reached, thereafter the energies of both states will rise. Thus the singlet state has lower energy at normal internuclear separation.

(c) The contribution of the Coulomb force between the electrons to the molecular energy consists of two parts, one is the Coulomb integral arising from the interaction of an electron at location 1 and an electron at location 2. The other is the exchange integral arising from the fact that part of the time electron 1 spends at location 1 and electron 2 at location 2 and part of the time electron 1 spends at location 2 and electron 2 at location 1. The exchange integral has its origin in the identity of electrons and Pauli's principle and has no correspondence in classical physics. The force related to it is called exchange force.

The exchange integral has the form

$$\varepsilon = \iint d\tau_1 d\tau_2 \frac{1}{r_{12}} \psi^*(r_{a1})\psi(r_{b1})\psi(r_{a2})\psi^*(r_{b2})\,.$$

If the two nuclei are far apart, the electrons are distinguishable and the distinction between the symmetry and antisymmetry of the wave functions vanishes; so does the exchange force.

1133

(a) Consider the ground state of a dumbbell molecule: mass of each nucleus $= 1.7 \times 10^{-24}$ gm, equilibrium nuclear separation $= 0.75$ Å. Treat the nuclei as distinguishable. Calculate the energy difference between the first two rotational levels for this molecule. Take $\hbar = 1.05 \times 10^{-27}$ erg.sec.

(b) When forming H_2 from atomic hydrogen, 75% of the molecules are formed in the ortho state and the others in the para state. What is the difference between these two states and where does the 75% come from?

(*Wisconsin*)

Solution:

(a) The moment of inertia of the molecule is

$$I_0 = \mu r^2 = \frac{1}{2} m r^2 \,,$$

where r is the distance between the nuclei. The rotational energy is

$$E_J = \frac{\hbar^2}{2I_0} J(J+1)\,,$$

with

$$J = \begin{cases} 0, 2, 4, \dots & \text{for para-hydrogen}\,, \\ 1, 3, 5, \dots & \text{for ortho-hydrogen}\,. \end{cases}$$

As
$$\frac{\hbar^2}{2I} = \frac{\hbar^2}{mr^2} = \frac{(\hbar c)^2}{mc^2r^2} = \frac{1973^2}{9.4 \times 10^8 \times 0.75^2} = 7.6 \times 10^{-3} \text{ eV},$$
the difference of energy between the rotational levels $J = 0$ and $J = 1$ is
$$\Delta E_{0,1} = \frac{\hbar^2}{I_0} = 1.5 \times 10^{-2} \text{ eV}.$$

(b) The two nuclei of hydrogen molecule are protons of spin $\frac{1}{2}$. Hence the H_2 molecule has two nuclear spin states $I = 1, 0$. The states with total nuclear spin $I = 1$ have symmetric spin function and are known as ortho-hydrogen, and those with $I = 0$ have antisymmetric spin function and are known as para-hydrogen.

The ratio of the numbers of ortho H_2 and para H_2 is given by the degeneracies $2I + 1$ of the two spin states:
$$\frac{\text{degeneracy of ortho } H_2}{\text{degeneracy of para } H_2} = \frac{3}{1}.$$
Thus 75% of the H_2 molecules are in the ortho state.

1134

A $^7N_{14}$ nucleus has nuclear spin $I = 1$. Assume that the diatomic molecule N_2 can rotate but does not vibrate at ordinary temperatures and ignore electronic motion. Find the relative abundance of ortho and para molecules in a sample of nitrogen gas. (Ortho = symmetric spin state; para = antisymmetric spin state), What happens to the relative abundance as the temperature is lowered towards absolute zero?

(*SUNY, Buffalo*)

Solution:

The $^7N_{14}$ nucleus is a boson of spin $I = 1$, so the total wave function of a system of such nuclei must be symmetric. For the ortho-nitrogen, which has symmetric spin, the rotational quantum number J must be an even number for the total wave function to be symmetric. For the para-nitrogen, which has antisymmetric spin, J must be an odd number.

The rotational energy levels of N_2 are
$$E_J = \frac{\hbar^2}{2H} J(J + 1), \qquad J = 0, 1, 2, \dots.$$
where H is its moment of inertia. Statistical physics gives

$$\frac{\text{population of para-nitrogen}}{\text{population of ortho-nitrogen}} = \frac{\sum_{\text{even } J} (2J+1) \exp\left[-\frac{\hbar^2}{2HkT} J(J+1)\right]}{\sum_{\text{odd } J} (2J+1) \exp\left[-\frac{\hbar^2}{2HkT} J(J+1)\right]} \times \frac{I+1}{I} ,$$

where I is the spin of a nitrogen nucleus.

If $\hbar^2/HRT \ll 1$, the sums can be approximated by integrals:

$$\begin{aligned} &\sum_{\text{even } J} (2J+1) \exp\left[-\frac{\hbar^2}{2HkT} J(J+1)\right] \\ &\quad = \sum_{m=0}^{\infty} (4m+1) \exp\left[-\frac{\hbar^2}{2HkT} 2m(2m+1)\right] \\ &\quad = \frac{1}{2} \int_0^{\infty} \exp\left(-\frac{\hbar^2 x}{2HkT}\right) dx = \frac{HkT}{\hbar^2} , \end{aligned}$$

where $x = 2m(2m+1)$;

$$\begin{aligned} &\sum_{\text{odd } J} (2J+1) \exp\left[-\frac{\hbar^2}{2HkT} J(J+1)\right] \\ &\quad = \sum_{m=0}^{\infty} (4m+3) \exp\left[-\frac{\hbar^2}{2HkT} (2m+1)(2m+2)\right] \\ &\quad = \frac{1}{2} \int_0^{\infty} \exp\left(-\frac{\hbar^2 y}{2HkT}\right) dy = \frac{HkT}{\hbar^2} \exp\left(-\frac{\hbar^2}{HkT}\right) , \end{aligned}$$

where $y = (2m+1)(2m+2)$.

Hence

$$\begin{aligned} \frac{\text{population of para-nitrogen}}{\text{population of ortho-nitrogen}} &= \frac{I+1}{I} \exp\left(\frac{\hbar^2}{HkT}\right) \approx \frac{I+1}{I} \\ &= \frac{1+1}{1} = 2:1 . \end{aligned}$$

For $T \to 0$, $\hbar^2/HkT \gg 1$, then

$$\sum_{\text{even } J} (2J+1) \exp\left[-\frac{\hbar^2}{2HkT} J(J+1)\right]$$

$$= \sum_{m=0}^{\infty} (4m+1) \exp\left[-\frac{\hbar^2}{2HkT} 2m(2m+1)\right] \approx 1\,,$$

$$\sum_{\text{odd } J} (2J+1) \exp\left[-\frac{\hbar^2}{2HkT} J(J+1)\right]$$

$$= \sum_{m=0}^{\infty} (4m+3) \exp\left[-\frac{\hbar^2}{2HkT} (2m+1)(2m+2)\right]$$

$$\approx 3 \exp\left[-\frac{\hbar^2}{HkT}\right]\,,$$

retaining the lowest order terms only. Hence

$$\frac{\text{population of para-nitrogen}}{\text{population of ortho-nitrogen}} \approx \frac{I+1}{3I} \exp\left(\frac{\hbar^2}{HkT}\right) \to \infty\,,$$

which means that the N_2 molecules are all in the para state at 0 K.

1135

In HCl a number of absorption lines with wave numbers (in cm^{-1}) 83.03, 103.73, 124.30, 145.03, 165.51, and 185.86 have been observed. Are these vibrational or rotational transitions? If the former, what is the characteristic frequency? If the latter, what J values do they correspond to, and what is the moment of inertia of HCl? In that case, estimate the separation between the nuclei.

(Chicago)

Solution:

The average separation between neighboring lines of the given spectrum is 20.57 cm^{-1}. The separation between neighboring vibrational lines is of the order of 10^{-1} eV $= 10^3$ cm^{-1}. So the spectrum cannot originate from transitions between vibrational energy levels, but must be due to transitions between rotational levels.

The rotational levels are given by

$$E = \frac{\hbar^2}{2I} J(J+1)\,,$$

where J is the rotational quantum number, I is the moment of inertia of the molecule:

$$I = \mu R^2 = \frac{m_{Cl} m_H}{m_{Cl} + m_H} R^2 = \frac{35}{36} m_H R^2\,,$$

μ being the reduced mass of the two nuclei forming the molecule and R their separation. In a transition $J' \to J' - 1$, we have

$$\frac{hc}{\lambda} = \frac{\hbar^2}{2I} [J'(J'+1) - (J'-1)J'] = \frac{\hbar^2 J'}{I}\,,$$

or

$$\tilde{\nu} = \frac{1}{\lambda} = \frac{\hbar J'}{2\pi I c}\,.$$

Then the separation between neighboring rotational lines is

$$\Delta\tilde{\nu} = \frac{\hbar}{2\pi I c}\,,$$

giving

$$R = \left[\frac{\hbar c}{2\pi \left(\frac{35}{36}\right) m_H c^2 \Delta\tilde{\nu}} \right]^{\frac{1}{2}}$$

$$= \left[\frac{19.7 \times 10^{-12}}{2\pi \left(\frac{35}{36}\right) \times 938 \times 20.57} \right]^{\frac{1}{2}} = 1.29 \times 10^{-8}\ \text{cm} = 1.29\ \text{Å}\,.$$

As $J' = \frac{\tilde{\nu}}{\Delta\tilde{\nu}}$, the given lines correspond to $J' = 4, 5, 6, 7, 8, 9$ respectively.

1136

When the Raman spectrum of nitrogen ($^{14}N^{14}N$) was measured for the first time (this was before Chadwick's discovery of the neutron in 1932), scientists were very puzzled to find that the nitrogen nucleus has a spin of $I = 1$. Explain

(a) how they could find the nuclear spin $I = 1$ from the Raman spectrum;

(b) why they were surprised to find $I = 1$ for the nitrogen nucleus. Before 1932 one thought the nucleus contained protons and electrons.

(*Chicago*)

Solution:

(a) For a diatomic molecule with identical atoms such as $(^{14}\text{N})_2$, if each atom has nuclear spin I, the molecule can have symmetric and antisymmetric total nuclear spin states in the population ratio $(I+1)/I$. As the nitrogen atomic nucleus is a boson, the total wave function of the molecule must be symmetric. When the rotational state has even J, the spin state must be symmetric. Conversely when the rotational quantum number J is odd, the spin state must be antisymmetric. The selection rule for Raman transitions is $\Delta J = 0, \pm 2$, so Raman transitions always occur according to $J_{\text{even}} \to J_{\text{even}}$ or J_{odd} to J_{odd}. This means that as J changes by one successively, the intensity of Raman lines vary alternately in the ratio $(I+1)/I$. Therefore by observing the intensity ratio of Raman lines, I may be determined.

(b) If a nitrogen nucleus were made up of 14 protons and 7 electrons (nuclear charge = 7), it would have a half-integer spin, which disagrees with experiments. On the other hand, if a nitrogen nucleus is made up of 7 protons and 7 neutrons, an integral nuclear spin is expected, as found experimentally.

1137

A molecule which exhibits one normal mode with normal coordinate Q and frequency Ω has a polarizability $\alpha(Q)$. It is exposed to an applied incident field $E = E_0 \cos \omega_0 t$. Consider the molecule as a classical oscillator.

(a) Show that the molecule can scatter radiation at the frequencies ω_0 (Rayleigh scattering) and $\omega_0 \pm \Omega$ (first order Raman effect).

(b) For which $\alpha(Q)$ shown will there be no first order Raman scattering?

(c) Will O_2 gas exhibit a first order vibrational Raman effect? Will O_2 gas exhibit a first order infrared absorption band? Explain your answer briefly.

(*Chicago*)

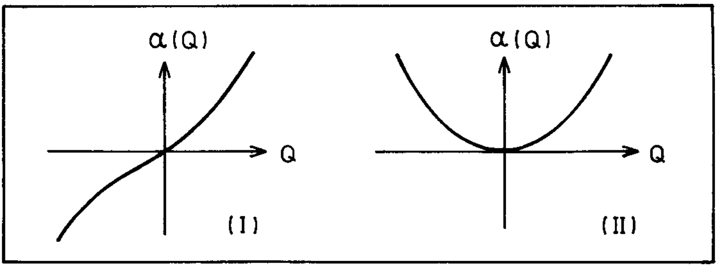

Fig. 1.61

Solution:

(a) On expanding $\alpha(Q)$ about $Q = 0$,

$$\alpha(Q) = \alpha_0 + \left(\frac{d\alpha}{dQ}\right)_{Q=0} Q + \frac{1}{2}\left(\frac{d^2\alpha}{dQ^2}\right)_{Q=0} Q^2 + \cdots .$$

and retaining only the first two terms, the dipole moment of the molecule can be given approximately as

$$P = \alpha E \approx \left[\alpha_0 + \left(\frac{d\alpha}{dQ}\right)_{Q=0} Q\cos\Omega t\right] E_0 \cos\omega_0 t$$

$$= \alpha_0 E_0 \cos\omega_0 t + QE_0 \left(\frac{d\alpha}{dQ}\right)_{Q=0} \left\{\frac{1}{2}[\cos(\omega_0+\Omega)t + \cos(\omega_0-\Omega)t]\right\} .$$

As an oscillating dipole radiates energy at the frequency of oscillation, the molecule not only scatters radiation at frequency ω_0 but also at frequencies $\omega_0 \pm \Omega$.

(b) The first order Raman effect arises from the term involving $(\frac{d\alpha}{dQ})_{Q=0}$. Hence in case (II) where $(\frac{d\alpha}{dQ})_{Q=0} = 0$ there will be no first order Raman effect.

(c) There will be first order Raman effect for O_2, for which there is a change of polarizability with its normal coordinate such that $(\frac{d\alpha}{dQ})_{Q=0} \neq 0$.

However, there is no first order infrared absorption band, because as the charge distribution of O_2 is perfectly symmetric, it has no intrinsic electric dipole moment, and its vibration and rotation cause no electric dipole moment change.

1138

Figure 1.62 shows the transmission of light through HCl vapor at room temperature as a function of wave number (inverse wavelength in units of cm^{-1}) decreasing from the left to the right.

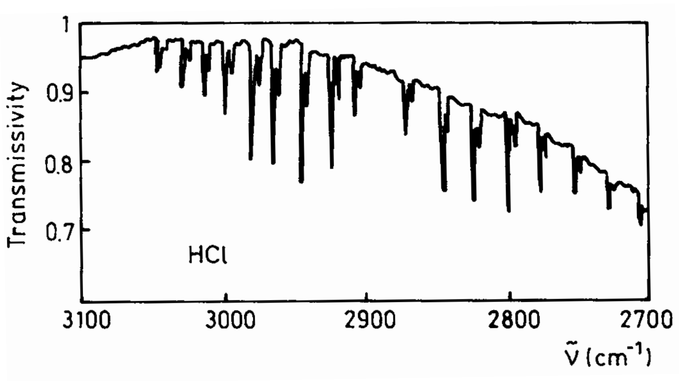

Fig. 1.62

Explain all the features of this transmission spectrum and obtain quantitative information about HCl. Sketch an appropriate energy level diagram labeled with quantum numbers to aid your explanation. Disregard the slow decrease of the top baseline for $\lambda^{-1} < 2900\ \mathrm{cm}^{-1}$ and assume that the top baseline as shown represents 100% transmission. The relative magnitudes of the absorption lines are correct.

(*Chicago*)

Solution:

Figure 1.62 shows the vibration-rotational spectrum of the molecules of hydrogen with two isotopes of chlorine, $H^{35}Cl$ and $H^{37}Cl$, the transition energy being

$$E_{v,k} = (v + 1/2)h\nu_0 + \frac{\hbar^2 k(k+1)}{2I},$$

where v, k are the vibrational and rotational quantum numbers respectively. The selection rules are $\Delta v = \pm 1, \Delta k = \pm 1$.

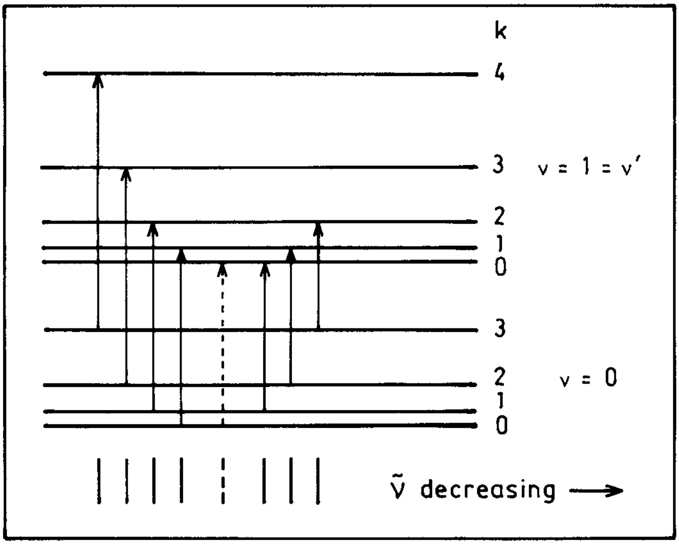

Fig. 1.63

The "missing" absorption line at the center of the spectrum shown in Fig. 1.63 corresponds to $k = 0 \to k' = 0$. This forbidden line is at $\lambda^{-1} = 2890\ \text{cm}^{-1}$, or $\nu_0 = c\lambda^{-1} = 8.67 \times 10^{13}\ \text{s}^{-1}$.

From the relation

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{K}{\mu}},$$

where K is the force constant, $\mu = \frac{35}{36}m_H = 1.62 \times 10^{-24}$ g is the reduced mass of HCl, we obtain $K = 4.8 \times 10^5\ \text{erg cm}^{-2} = 30\ \text{eV Å}^{-2}$.

Figure 1.64 shows roughly the potential between the two atoms of HCl. Small oscillations in r may occur about r_0 with a force constant $K = \frac{d^2V}{dr^2}|_{r=r_0}$. From the separation of neighboring rotational lines $\Delta\tilde{\nu} = 20.5\ \text{cm}^{-1}$, we can find the equilibrium atomic separation (**Problem 1135**)

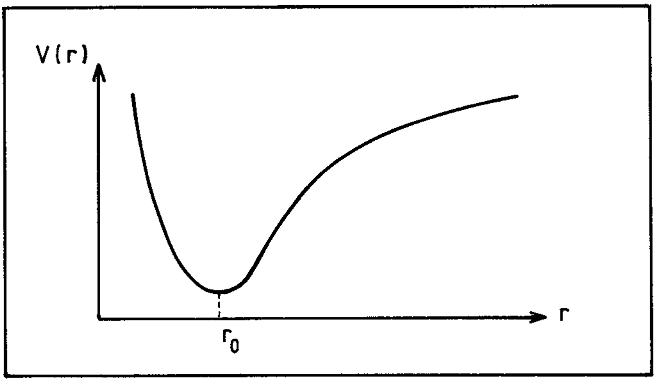

Fig. 1.64

$$r_0 = \left[\frac{\hbar c}{2\pi \left(\frac{36}{37} \right) m_H c^2 \Delta \tilde{\nu}} \right]^{\frac{1}{2}} = 1.30 \times 10^{-8} \text{ cm}$$

$$= 1.30 \text{ Å} .$$

The Isotope ratio can be obtained from the intensity ratio of the two series of spectra in Fig. 1.62. For $H^{35}Cl$, $\mu = \frac{35}{36} m_H$, and for $H^{37}Cl$, $\mu = \frac{37}{38} m_H$. As the wave number of a spectral line $\tilde{\nu} \propto \frac{1}{\mu}$, the wave number of a line of $H^{37}Cl$ is smaller than that of the corresponding line of $H^{35}Cl$. We see from Fig. 1.62 that the ratio of the corresponding spectral intensities is 3:1, so the isotope ratio of ^{35}Cl to ^{37}Cl is 3:1.

1139

(a) Using the fact that electrons in a molecule are confined to a volume typical of the molecule, estimate the spacing in energy of the excited states of the electrons (E_{elect}).

(b) As nuclei in a molecule move they distort electronic wave functions. This distortion changes the electronic energy. The nuclei oscillate about positions of minimum total energy, comprising the electron energy and the repulsive Coulomb energy between nuclei. Estimate the frequency and therefore the energy of these vibrations (E_{vib}) by saying that a nucleus is in a harmonic oscillator potential.

(c) Estimate the deviations from the equilibrium sites of the nuclei.

(d) Estimate the energy of the rotational excitations (E_{rot}).

(e) Estimate the ratio of $E_{\text{elect}} : E_{\text{vib}} : E_{\text{rot}}$ in terms of the ratio of electron mass to nuclear mass, m_e/m_n.

(*Columbia*)

Solution:

(a) The uncertainty principle $pd \approx \hbar$ gives the energy spacing between the excited states as $E_{\text{elect}} = \frac{p^2}{2m_e} \approx \frac{\hbar^2}{2m_e d^2}$, where d, the linear size of the molecule, is of the same order of magnitude as the Bohr radius $a_0 = \frac{\hbar^2}{m_e e^2}$.

(b) At equilibrium, the Coulomb repulsion force between the nuclei is $f \approx \frac{e^2}{d^2}$, whose gradient is $K \approx \frac{f}{d} \approx \frac{e^2}{d^3}$. The nuclei will oscillate about the equilibrium separation with angular frequency

$$\omega = \sqrt{\frac{K}{m}} \approx \sqrt{\frac{m_e}{m}}\sqrt{\frac{e^2 a_0}{m_e d^4}} = \sqrt{\frac{m_e}{m}}\frac{\hbar}{m_e d^2}\,,$$

where m is the reduced mass of the atomic nuclei.

Hence

$$E_{\text{vib}} = \hbar\omega \approx \sqrt{\frac{m_e}{m}} E_{\text{elect}}\,.$$

(c) As

$$E_{\text{vib}} = \frac{1}{2} m\omega^2 (\Delta x)^2 = \hbar\omega\,,$$

we have

$$\Delta x \approx \left(\frac{m_e}{m}\right)^{\frac{1}{4}} d\,.$$

(d) The rotational energy is of the order $E_{\text{rot}} \approx \frac{\hbar^2}{2I}$. With $I \approx md^2$, we have

$$E_{\text{rot}} \approx \frac{m_e}{m} E_{\text{elect}}\,.$$

(e) As $m \approx m_n$, the nuclear mass, we have

$$E_{\text{elect}} : E_{\text{vib}} : E_{\text{rot}} \approx 1 : \sqrt{\frac{m_e}{m_n}} : \frac{m_e}{m_n}\,.$$

1140

Sketch the potential energy curve $V(r)$ for the HF molecule as a function of the distance r between the centers of the nuclei, indicating the dissociation energy on your diagram.

(a) What simple approximation to $V(r)$ can be used near its minimum to estimate vibrational energy levels? If the zero-point energy of HF is 0.265 eV, use your approximation (without elaborate calculations) to estimate the zero-point energy of the DF molecule (D = deuteron, F = ^{19}F).

(b) State the selection rule for electromagnetic transitions between vibrational levels in HF within this approximation, and briefly justify your answer. What is the photon energy for these transitions?

(*Wisconsin*)

Solution:

(a) Figure 1.65 shows $V(r)$ and the dissociation energy E_d for the HF molecule. Near the minimum potential point r_0, we may use the approximation

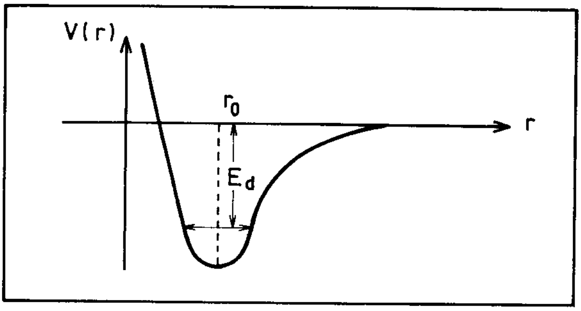

Fig. 1.65

$$V(r) \approx \frac{1}{2}k(r - r_0)^2 \, .$$

Thus the motion about r_0 is simple harmonic with angular frequency $\omega_0 = \sqrt{\frac{k}{\mu}}$, μ being the reduced mass of the nuclei. The zero-point energy is $E_0 = \frac{1}{2}\hbar\omega_0$.

As their electrical properties are the same, DF and HF have the same potential curve. However their reduced masses are different:

$$\mu(DF) = \frac{m(D)m(F)}{m(D)+m(F)} = \frac{2\times 19}{2+19}u = 1.81u\,,$$

$$\mu(HF) = \frac{m(H)m(F)}{m(H)+m(F)} = \frac{1\times 19}{1+19}u = 0.95u\,.$$

where u is the nucleon mass.

Hence

$$\frac{E_0(HF)}{E_0(DF)} = \sqrt{\frac{\mu(DF)}{\mu(HF)}}$$

and the zero-point energy of DF is

$$E_0(DF) = \sqrt{\frac{\mu(HF)}{\mu(DF)}}E_0(HF) = 0.192 \text{ eV}\,.$$

(b) In the harmonic oscillator approximation, the vibrational energy levels are given by

$$E_\nu = (\nu + 1/2)\hbar\omega, \quad \nu = 0, 1, 2, 3 \ldots\,.$$

The selection rule for electromagnetic transitions between these energy levels is

$$\Delta\nu = \pm 1, \pm 2, \pm 3, \ldots\,,$$

while the selection rule for electric dipole transitions is

$$\Delta\nu = \pm 1\,.$$

In general, the electromagnetic radiation emitted by a moving charge consists of the various electric and magnetic multipole components, each with its own selection rule $\Delta\nu$ and parity relationship between the initial and final states. The lowest order perturbation corresponds to electric dipole transition which requires $\Delta\nu = \pm 1$ and a change of parity.

For purely vibrational transitions, the energy of the emitted photon is approximately $\hbar\omega_0 \sim 0.1$ to 1 eV.

1141

Diatomic molecules such as HBr have excitation energies composed of electronic, rotational, and vibrational terms.

(a) Making rough approximations, estimate the magnitudes of these three contributions to the energy, in terms of fundamental physical constants such as $M, m_e, e, \ldots$, where M is the nuclear mass.

(b) For this and subsequent parts, assume the molecule is in its electronic ground state. What are the selection rules that govern radiative transitions? Justify your answer.

(c) An infrared absorption spectrum for gaseous HBr is shown in Fig. 1.66. (Infrared absorption involves no electronic transitions.) Use it to determine the moment of interia I and the vibrational frequency ω_0 for HBr.

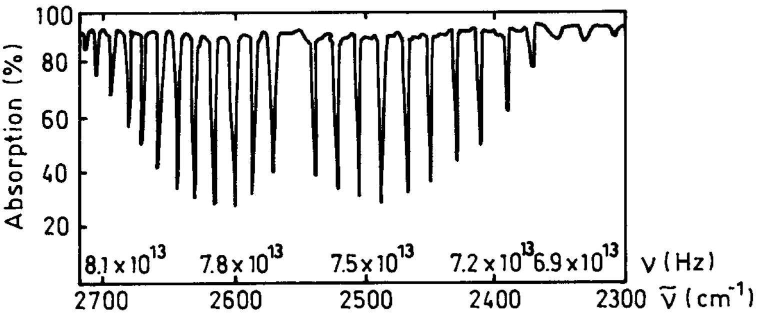

Fig. 1.66

(d) Note that the spacing between absorption lines increases with increasing energy. Why?

(e) How does this spectrum differ from that of a homonuclear molecule such as H_2 or D_2?

(*Princeton*)

Solution:

(a) Let a denote the linear dimension of the diatomic molecule. As the valence electron moves in an orbit of linear dimension a, the uncertainty of momentum is $\Delta p \approx \hbar/a$ and so the order of magnitude of the zero-point energy is

$$E_e \approx \frac{(\Delta p)^2}{m_e} \approx \frac{\hbar^2}{m_e a^2} .$$

A harmonic oscillator with mass m and coefficient of stiffness k is used as model for nuclear vibration. A change of the distance between the two nuclei will considerably distort the electronic wave function and thus relate to a change of the electronic energy, i.e. $ka^2 \approx E_e$.

Hence

$$E_{\text{vib}} \approx \hbar\omega \approx \hbar\sqrt{\frac{k}{M}} = \sqrt{\frac{m_e}{M}}\sqrt{\frac{\hbar^2}{m_e a^2}}\sqrt{ka^2} \approx \left(\frac{m_e}{M}\right)^{\frac{1}{2}} E_e$$

The molecular rotational energy levels are obtained by treating the molecule as a rotator of moment of inertia $I \approx Ma^2$. Thus

$$E_{\text{rot}} \approx \frac{\hbar^2}{I} \approx \frac{m_e}{M}\frac{\hbar^2}{m_e a^2} \approx \frac{m_e}{M}E_e \,.$$

(b) The selection rules for radiative transitions are $\Delta J = \pm 1, \Delta v = \pm 1$, where J is the rotational quantum number, v is the vibrational quantum number. As the electrons remain in the ground state, there is no transition between the electronic energy levels. The transitions that take place are between the rotational or the vibrational energy levels.

(c) From Fig. 1.66 we can determine the separation of neighboring absorption lines, which is about $\Delta\tilde{\nu} = 18\ \text{cm}^{-1}$. As (**Problem 1135**) $\Delta\tilde{\nu} = 2B$, where $B = \frac{\hbar}{4\pi Ic}$, the moment of inertia is

$$I = \frac{\hbar}{2\pi c\Delta\tilde{\nu}} = 3.1 \times 10^{-40}\ \text{g cm}^2 \,.$$

Corresponding to the missing spectral line in the middle we find the vibrational frequency $\nu_0 = 3 \times 10^{10} \times 2560 = 7.7 \times 10^{13}$ Hz.

(d) Actually the diatomic molecule is not exactly equivalent to a harmonic oscillator. With increasing vibrational energy, the average separation between the nuclei will become a little larger, or B_v a little smaller:

$$B_v = B_e - \left(\nu + \frac{1}{2}\right)\alpha_e \,,$$

where B_e is the value B when the nuclei are in the equilibrium positions, $\alpha_e > 0$ is a constant. A transition from E to $E'(E < E')$ produces an absorption line of wave number

$$\begin{aligned}\tilde{\nu} &= \frac{E' - E}{hc} = \frac{1}{hc}[(E'_{\text{vib}} + E'_{\text{rot}}) - (E_{\text{vib}} + E_{\text{rot}})] \\ &= \tilde{\nu}_0 + B'J'(J'+1) - BJ(J+1) \,.\end{aligned}$$

where $B' < B$. For the R branch, $J' = J + 1$, we have

$$\tilde{\nu}_R = \tilde{\nu}_0 + (B' + B)J' + (B' - B)J'^2 ,$$

and hence the spectral line separation

$$\Delta\tilde{\nu} = (B' + B) + (B' - B)(2J' + 1) ,$$

where $J' = 1, 2, 3 \ldots$. Hence, when the energy of spectral lines increases, i.e., J' increases, $\Delta\tilde{\nu}$ will decrease.

For the P branch, $J' = J - 1$,

$$\tilde{\nu}_P = \tilde{\nu}_0 - (B' + B)J + (B' - B)J^2 ,$$

$$\Delta\tilde{\nu} = (B' + B) - (B' - B)(2J + 1) ,$$

where $J = 1, 2, 3 \ldots$. Thus $\Delta\tilde{\nu}$ will decrease with increasing spectral line energy.

(e) Molecules formed by two identical atoms such as H_2 and D_2 have no electric dipole moment, so the vibration and rotation of these molecules do not relate to absorption or emission of electric-dipole radiation. Hence they are transparent in the infrared region.

1142

In a recent issue of Science Magazine, G. Zweig discussed the idea of using free quarks (if they should exist) to catalyze fusion of deuterium. In an ordinary negative deuterium molecule (ded) the two deuterons are held together by an electron, which spends most of its time between the two nuclei. In principle a neutron can tunnel from one proton to the other, making a tritium plus p + energy, but the separation is so large that the rate is negligible. If the electron is replaced with a massive quark, charge $-4e/3$, the separation is reduced and the tunneling rate considerably increased. After the reaction, the quark generally escapes and captures another deuteron to make a dQ atom, charge $-e/3$. The atom decays radiatively to the ground state, then captures another deuteron in a large-n orbit. This again decays down to the ground state. Fusion follows rapidly and the quark is released again.

(a) Suppose the quark is much more massive than the deuteron. What is the order of magnitude of the separation of the deuterons in the ground state of the dQd molecule?

(b) Write down an expression for the order of magnitude of the time for a deuteron captured at large radius (large n) in dQ to radiatively settle to the ground state. Introduce symbols like mass, charge, etc. as needed; do not evaluate the expression.

(c) Write down the expression for the probability of finding the neutron-proton separation in a deuteron being $r \geq r_0$, with $r_0 \gg 10^{-13}$ cm. Again, introduce symbols like deuteron binding energy as needed, and do not evaluate the expression.

(d) As a simple model for the tunneling rate suppose that if the neutron reaches a distance $r \geq r_0$ from the proton it certainly is captured by the other deuteron. Write down an order of magnitude expression for the halflife of dQd (but do not evaluate it).

(*Princeton*)

Solution:

(a) The dQd molecule can be considered as H_2^+ ion with the replacements $m_e \to m$, the deuteron mass, nuclear charge $e \to$ quark charge $-\frac{4}{3}e$.

Then by analogy with H_2^+ ion, the Hamiltonian for the dQd molecule can be written as

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} - \frac{4e^2}{3r_1} - \frac{4e^2}{3r_2} + \frac{e^2}{r_{12}} \, ,$$

where $r_{12} = |\mathbf{r}_1 - \mathbf{r}_2|$, $\mathbf{r}_1, \mathbf{r}_2$ being the radius vectors of the deuterons from the massive quark.

Assume the wave function of the ground state can be written as

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \Psi_{100}(\mathbf{r}_1)\Psi_{100}(\mathbf{r}_2) \, ,$$

where

$$\Psi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi}} a^{-3/2} \exp\left(-\frac{r}{a}\right) \, ,$$

with

$$a = \frac{3\hbar^2}{4me^2} \, .$$

The average separation of the deuterons in the ground state is

$$\bar{r}_{12} = \frac{1}{\pi^2 a^6} \iint r_{12} \exp\left[-\frac{2(r_1 + r_1)}{a}\right] d\mathbf{r}_1 d\mathbf{r}_1 = \frac{8}{5} a = \frac{6\hbar^2}{5me^2} \, .$$

(b) A hydrogen-like atom of nuclear charge Ze has energy $-\frac{Z^2e^4m_e}{2\hbar^2n^2}$. By analogy the dQd molecule has ground state energy

$$\begin{aligned}E &= -2\times\left(\frac{4}{3}\right)^2\frac{me^4}{2\hbar^2}+\frac{e^2}{r_{12}}\\&= -\frac{4}{3}\frac{e^2}{a}+\frac{5}{8}\frac{e^2}{a}=-\frac{17}{24}\frac{e^2}{a}\,.\end{aligned}$$

When n is very large, the molecule can be considered as a hydrogen-like atom with dQ as nucleus (charge $= -\frac{4e}{3}+e=-\frac{e}{3}$) and the second d taking the place of orbital electron (charge $= +e$). Accordingly the energy is

$$E_n=-\frac{4}{6}\frac{e^2}{a}-\left(\frac{1}{3}\right)^2\frac{me^4}{2\hbar^2n^2}=-\frac{4}{6}\frac{e^2}{a}-\frac{1}{6}\frac{e^2}{a'}\frac{1}{n^2}\,,$$

where $a'=\frac{3\hbar^2}{me^2}$. Hence when the system settles to the ground state, the energy emitted is

$$\Delta E=E_n-E_0=-\frac{4e^2}{6a}+\frac{17}{24}\frac{e^2}{a}-\frac{e^2}{6a'}\frac{1}{n^2}\approx\frac{e^2}{24a}\,.$$

The emitted photons have frequency

$$\omega=\frac{\Delta E}{\hbar}\approx\frac{e^2}{24\hbar a}\,.$$

The transition probability per unit time is given by

$$A_{n1}=\frac{4e^2\omega^3}{3\hbar c^3}|\mathbf{r}_{1n}|^2\,,$$

and so the time for deuteron capture is of the order

$$\tau=1/A_{n1}=\frac{3\hbar c^3}{4e^2\omega^3|\mathbf{r}_{1n}|^2}\,.$$

The wave function of the excited state is

$$\Psi_\pm=\frac{1}{\sqrt{2}}[\Psi_{100}(\mathbf{r}_1)\Psi_{nlm}(\mathbf{r}_2)\pm\Psi_{100}(\mathbf{r}_2)\Psi_{nlm}(\mathbf{r}_1)]\,,$$

which only acts on one d. As

$$\langle\Psi_{100}|\mathbf{r}|\Psi_{100}\rangle = 0\,,$$

we have

$$\mathbf{r}_{1n} = \frac{1}{\sqrt{2}}\langle\Psi_{nlm}|\mathbf{r}|\Psi_{100}\rangle$$

and hence

$$\tau = \frac{3\hbar c^3}{2e^2\omega^3|\langle\Psi_{nlm}|\mathbf{r}|\Psi_{100}\rangle|^2}\,.$$

(c) In a deuteron the interaction potential between the proton and neutron can be taken to be that shown in Fig. 1.67, where W is the binding energy and $a \approx 10^{-13}$ cm.

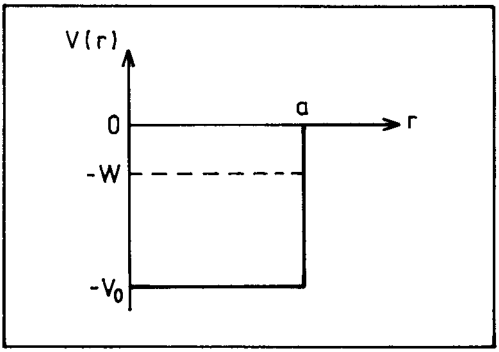

Fig. 1.67

The radial part of the wave function can be shown to satisfy the equation

$$R'' + \frac{1}{r}R' + \frac{M}{\hbar^2}[-W - V(r)]R = 0\,,$$

where M is the mass of the neutron. Let $rR = u$. The above becomes

$$u'' - \frac{M}{\hbar^2}[W + V(r)]u = 0\,.$$

As $V = -V_0$ for $0 \le r \le a$ and $V = 0$ otherwise, we have

$$\begin{cases} u'' - \dfrac{M}{\hbar^2}[W - V_0]u = 0, & (r \le a)\,, \\ u'' - \dfrac{MW}{\hbar^2}u = 0\,, & (r \ge a)\,. \end{cases}$$

The boundary conditions are $u|_{r=0} = 0$ and $u|_{r\to\infty}$ finite. Satisfying these the solutions are

$$u = \begin{cases} A\sin(k_1 r), & (r \le a) \\ B\exp(-k_2 r), & (r \ge a) \end{cases}$$

where $k_1 = \sqrt{\frac{M}{\hbar^2}(V_0 - W)}$, $k_2 = \sqrt{\frac{MW}{\hbar^2}}$. Continuity of the wave function at $r = a$ further requires

$$u = \begin{cases} A\sin(k_1 r), & (r \le a) \\ A\sin(k_1 a)\exp[-k_2(a - r)]\,. & (r \ge a) \end{cases}$$

Continuity of the first derivative of the wave function at $r = a$ gives

$$\cot(k_1 a) = -k_2/k_1\,.$$

Hence the probability of finding $r \ge r_0$ is

$$P = \frac{\int_{r_0}^{\infty} r^2 R^2(r)dr}{\int_0^{\infty} r^2 R^2(r)dr} = \frac{\sin^2(k_1 a)\exp[2k_2(a - r_0)]}{ak_2 - \dfrac{k_2}{2k_1}\sin(2k_1 a) + \sin^2(k_1 a)}$$

$$\approx \frac{\sin^2(k_1 a)}{ak_2}\exp(-2k_2 r_0)\,,$$

as $r_0 \gg a$.

A rough estimate of the probability can be obtained by putting $u \approx C\exp(-k_2 r)$, for which

$$P = \frac{\int_{r_0}^{\infty}\exp(-2k_2 r)dr}{\int_0^{\infty}\exp(-2k_2 r)dr} = \exp(-2k_2 r_0)\,.$$

(d) The neutron has radial velocity

$$v = \frac{p}{M} = \sqrt{\frac{2(V_0 - W)}{M}}$$

in the potential well. The transition probability per unit time is

$$\lambda = \frac{vP}{a}\,,$$

and so the halflife of dQd is given by

$$\tau = \frac{\ln 2}{\lambda} = \frac{a\ln 2}{vP} \approx a\ln 2\sqrt{\frac{M}{2(V_0 - W)}}\exp(2k_2 r_0)\,.$$

PART II

NUCLEAR PHYSICS

This Page Intentionally Left Blank

1. BASIC NUCLEAR PROPERTIES (2001–2023)

2001

Discuss 4 independent arguments against electrons existing inside the nucleus.

(*Columbia*)

Solution:

First argument - Statistics. The statistical nature of nuclei can be deduced from the rotational spectra of diatomic molecules. If a nucleus (A,Z) were to consist of A protons and (A-Z) electrons, the spin of an odd-odd nucleus or an odd-even nucleus would not agree with experimental results, Take the odd-odd nucleus ^{14}N as example. An even number of protons produce an integer spin while an odd number of electrons produce a half-integer spin, so the total spin of the ^{14}N nucleus would be a half-integer, and so it is a fermion. But this result does not agree with experiments. Hence, nuclei cannot be composed of protons and electrons.

Second argument - Binding energy. The electron is a lepton and cannot take part in strong interactions which bind the nucleons together. If electrons existed in a nucleus, it would be in a bound state caused by Coulomb interaction with the binding energy having an order of magnitude

$$E \approx -\frac{Ze^2}{r},$$

where r is the electromagnetic radius of the nucleus, $r = 1.2A^{1/3}$fm. Thus

$$E \approx -Z\left(\frac{e^2}{\hbar c}\right)\frac{\hbar c}{r} = -\frac{197Z}{137 \times 1.2A^{1/3}} \approx -1.20\frac{Z}{A^{1/3}}\ \text{MeV}.$$

Note that the fine structure constant

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}.$$

Suppose $A \approx 124$, $Z \approx A/2$. Then $E \approx -15$ MeV, and the de Brogile wavelength of the electron would be

$$\lambda = \hbar/p = c\hbar/cp = 197/15 = 13\ \text{fm}.$$

As $\lambda > r$ the electron cannot be bound in the nucleus.

Third argument - Nuclear magnetic moment. If the nucleus consists of neutrons and protons, the nuclear magnetic moment is the sum of the contributions of the two kinds of nucleons. While different coupling methods give somewhat different results, the nuclear magnetic moment should be of the same order of magnitude as that of a nucleon, μ_N. On the other hand, if the nucleus consisted of protons and electrons, the nuclear magnetic moment should be of the order of magnitude of the magnetic moment of an electron, $\mu_e \approx 1800\mu_N$. Experimental results agree with the former assumption, and contradict the latter.

Fourth argument - β-decay. Nucleus emits electrons in β-decay, leaving behind a daughter nucleus. So this is a two-body decay, and the electrons emitted should have a monoenergetic spectrum. This conflicts with the continuous β energy spectrum found in such decays. It means that, in a β-decay, the electron is accompanied by some third, neutral particle. This contracts the assumption that there were only protons and electrons in a nucleus.

The four arguments above illustrate that electrons do not exist in the nucleus.

2002

The size of the nucleus can be determined by (a) electron scattering, (b) energy levels of muonic atoms, or (c) ground state energies of the isotopic spin multiplet . Discuss what physical quantities are measured in two and only two of these three experiments and how these quantities are related to the radius of the nucleus.

(*SUNY, Buffalo*)

Solution:

(a) It is the nuclear form factor that is measured in electron scattering experiments:

$$F(q^2) = \frac{(d\sigma)_{\text{exp}}}{(d\sigma)_{\text{point}}}\,,$$

where $(d\sigma)_{\text{exp}}$ is the experimental value, $(d\sigma)_{\text{point}}$ is the theoretical value obtained by considering the nucleus as a point. With first order Born approximation, we have

$$F(q^2) = \int \rho(\mathbf{r}) e^{i\mathbf{q}\cdot\mathbf{r}} d^3\mathbf{r} \,.$$

Assuming $\rho(\mathbf{r}) = \rho(r)$ and $\mathbf{q}\cdot\mathbf{r} \ll 1$, we have

$$F(q^2) \approx \int \rho(r) \left[1 + \frac{1}{2}(i\mathbf{q}\cdot\mathbf{r})^2\right] d^3\mathbf{r} = 1 - \frac{1}{2}\int \rho(r)(\mathbf{q}\cdot\mathbf{r})^2 d^3\mathbf{r}$$

$$= 1 - \frac{1}{c2}\int \rho(r) q^2 r^2 \cdot 4\pi r^2 dr \int_0^\pi \frac{1}{2}\cos^2\theta \sin\theta d\theta$$

$$= 1 - \frac{1}{6} q^2 \langle r^2 \rangle$$

with $\langle r^2 \rangle = \int \rho(r) r^2 d^3\mathbf{r}$.

By measuring the angular distribution of elastically scattered electrons, we can deduce $F(q^2)$, and so obtain the charge distribution $\rho(r)$ as a function of r, which gives a measure of the nuclear size.

(b) We can measure the energy differences between the excited states and the ground state of the muonic atom. As the mass of a muon is $m_\mu \approx 210 m_e$, the first radius of the muonic atom is $a_\mu \approx (1/210)a_0$, where a_0 is the Bohr radius, so that the energy levels of muonic atom are more sensitive to its nuclear radius. Consider for example the s state, for which the Hamiltonian is

$$H = -\frac{1}{2m_\mu}\nabla^2 + V(r) \,.$$

If the nucleus can be considered as a point charge, then $V(r) = V_0(r) = -e^2/r$, r being the distance of the muon from the nucleus.

If on the other hand we consider the nuclear charge as being uniformly distributed in a sphere of radius R, then

$$V(r) = \begin{cases} -\dfrac{e^2}{2R^3}(3R^2 - r^2) \,, & 0 < r \le R \,, \\[2ex] -\dfrac{e^2}{r} \,, & r > R \,. \end{cases}$$

To obtain the energy shift of the ground state, ΔE, caused by the finite size of the nucleus, we take

$$H' = H - H_0 = V(r) - V_0(r) = \begin{cases} -\dfrac{e^2}{2R^3}(3R^2 - r^2) + \dfrac{e^2}{r} \,, & 0 < r \le R \,, \\[2ex] 0, & r > R \,, \end{cases}$$

as perturbation. Then

$$\Delta E = \langle \Phi_0 | H' | \Phi_0 \rangle = 4\pi \int_0^R |\Phi_0|^2 H' r^2 dr \,,$$

where $\Phi_0 = \left(\frac{1}{\pi a_\mu^3}\right)^{1/2} e^{-\frac{r}{a_\mu}}$. As $R \sim 10^{-12}$ cm, $a_\mu \sim 10^{-10}$ cm, we can take $\frac{R}{a_\mu} \ll 1$ and hence $e^{-2r/a_\mu} \approx \left(1 - \frac{2r}{a_\mu}\right)$. Then $\Delta E = \frac{2}{5}\left(\frac{e^2}{2a_\mu}\right)\left(\frac{R}{a_\mu}\right)^2$, neglecting terms of order $(\frac{R}{a_\mu})^3$ and higher.

We can measure the energy of the X-rays emitted in the transition from the first excited state to the ground state,

$$E_X = (E_1 - E_0) - \frac{2}{5}\left(\frac{e^2}{a_\mu}\right)\left(\frac{R}{a_\mu}\right)^2 \,,$$

where E_1 and E_0 are eigenvalues of H_0, i.e. E_1 is the energy level of the first excited state and E_0 is the energy level of the ground state (for a point-charge nucleus). If the difference between E_X and $(E_1 - E_0)$, is known, R can be deduced.

(c) The nuclear structures of the same isotopic spin multiplet are the same so that the mass difference in the multiplet arises from electromagnetic interactions and the proton-neutron mass difference. Thus (**Problem 2009**)

$$\begin{aligned}
\Delta E &\equiv [M(Z, A) - M(Z-1, A)]c^2 \\
&= \Delta E_e - (m_n - m_p)c^2 \\
&= \frac{3e^2}{5R}[Z^2 - (Z-1)^2] - (m_n - m_p)c^2 \,,
\end{aligned}$$

from which R is deduced

It has been found that $R \approx R_0 A^{\frac{1}{3}}$ with $R_0 = 1.2 - 1.4$ fm.

2003

To study the nuclear size, shape and density distribution one employs electrons, protons and neutrons as probes.

(a) What are the criteria in selecting the probe? Explain.

(b) Compare the advantages and disadvantages of the probes mentioned above.

(c) What is your opinion about using photons for this purpose?

(*SUNY, Buffalo*)

Solution:

(a) The basic criterion for selecting probes is that the de Broglie wavelength of the probe is less than or equal to the size of the object being studied. Thus $\lambda = h/p \leq d_n$, or $p \geq h/d_n$, where d_n is the linear size of the nucleus. For an effective study of the nuclear density distribution we require $\lambda \ll d_n$.

(b) Electrons are a suitable probe to study the nuclear electromagnetic radius and charge distribution because electrons do not take part in strong interactions, only in electromagnetic interactions. The results are therefore easy to analyze. In fact, many important results have been obtained from electron-nucleus scatterings, but usually a high energy electron beam is needed. For example, take a medium nucleus. As $d_n \approx 10^{-13}$ cm, we require

$$p_e \approx \hbar/d_n \approx 0.2 \text{ GeV}/c, \qquad \text{or} \qquad E_e \approx pc = 0.2 \text{ GeV}\,.$$

Interactions between protons and nuclei can be used to study the nuclear structure, shape and distribution. The advantage is that proton beams of high flux and suitable parameters are readily available. The disadvantage is that both electromagnetic and strong interactions are present in proton-nucleus scatterings and the results are rather complex to analyse.

Neutrons as a probe are in principle much 'neater' than protons, However, it is much more difficult to generate neutron beams of high energy and suitable parameters. Also detection and measurements are more difficult for neutrons.

(c) If photons are used as probe to study nuclear structure, the high energy photons that must be used to interest with nuclei would show a hadron-like character and complicate the problem.

2004

Consider a deformed nucleus (shape of an ellipsoid, long axis 10% longer than short axis). If you compute the electric potential at the first Bohr radius, what accuracy can you expect if you treat the nucleus as a point charge? Make reasonable estimate; do not get involved in integration.

(*Wisconsin*)

Solution:

Assume the charge distribution in the nucleus is uniform, ellipsoidal and axially symmetric. Then the electric dipole moment of the nucleus is zero, and the potential can be written as

$$V = V_p + V_q\,,$$

where $V_p = Q/r$ is the potential produced by the nucleus as a point charge, $V_q = MQ/r^3$, M being the electric quadrupole moment.

For the ellipsoid nucleus, let the long axis be $a = (1+\varepsilon)R$, the short axis be $b = (1-\varepsilon/2)R$, where ε is the deformed parameter, and R is the nuclear radius. As $a : b = 1.1$, we have $\frac{3\varepsilon}{2} = 0.1$, or $\varepsilon = 0.2/3$, and so

$$M = \frac{2}{5}(a^2 - b^2) = \frac{2}{5}(a-b)(a+b) = \frac{1.22}{15}R^2\,.$$

For a medium nucleus, take $A \sim 125$, for which $R = 1.2A^{1/3} = 6$ fm. Then

$$\Delta V = \frac{V_q}{V_p} = \frac{M}{r^2} = \frac{1.22}{15}\frac{R^2}{r^2} = \frac{1.22}{15} \times \left(\frac{6 \times 10^{-13}}{0.53 \times 10^{-8}}\right) \approx 1 \times 10^{-9}\,,$$

at the first Bohr radius $r = 0.53 \times 10^{-8}$ cm. Thus the relative error in the potential if we treat the nucleus as a point charge is about 10^{-9} at the first Bohr orbit.

2005

The precession frequency of a nucleus in the magnetic field of the earth is 10^{-1}, 10^1, 10^3, 10^5 $\sec^{-1}$.

(*Columbia*)

Solution:

The precession frequency is given by

$$\omega = \frac{geB}{2m_N c}\,.$$

With $g = 1$, $e = 4.8 \times 10^{-10}$ esu, $c = 3 \times 10^{10}$ cm/s, $B \approx 0.5$ Gs, $m_N \approx 10^{-23}$g for a light nucleus, $\omega = \frac{4.8 \times 0.5 \times 10^{-10}}{2 \times 10^{-23} \times 3 \times 10^{10}} = 0.4 \times 10^3 \mathrm{s}^{-1}$.

Hence the answer is 10^3 s^{-1}.

2006

Given the following information for several light nuclei (1 amu = 931.5 MeV) in Table 2.1.

(a) What are the approximate magnetic moments of the neutron, $^3H_1$, $^3He_2$, and $^6Li_3$?

(b) What is the maximum-energy β-particle emitted when $^3H_1$ decays to $^3He_2$?

(c) Which reaction produces more energy, the fusion of $^3H_1$ and $^3He_2$ or $^2H_1$ and $^4He_2$?

(*Wisconsin*)

Table 2.1

Nuclide	J^π	Nuclide mass (amu)	magnetic moment (μ_N)
$^1H_1$	$1/2^+$	1.00783	+2.79
$^2H_1$	1^+	2.01410	+0.86
$^3H_1$	$1/2^+$	3.01605	—
$^3He_2$	$1/2^+$	3.01603	—
$^4He_2$	0^+	4.02603	0
$^6Li_3$	1^+	6.01512	—

Solution:

The nuclear magnetic moment is given by $\boldsymbol{\mu} = g\mu_N\mathbf{J}$, where $\mathbf{J}$ is the nuclear spin, g is the Landé factor, μ_N is the nuclear magneton. Then from the table it is seen that

$$g(^1\mathrm{H}_1) = 2 \times 2.79 = 5.58, \qquad g(^2H_1) = 0.86, \qquad g(^4He_2) = 0\,.$$

When two particles of Landé factors g_1 and g_2 combine into a new particle of Landé factor g, (assuming the orbital angular momentum of relative motion is zero), then

$$g = g_1\frac{J(J+1)+j_1(j_1+1)-j_2(j_2+1)}{2J(J+1)}$$
$$+ g_2\frac{J(J+1)+j_2(j_2+1)-j_1(j_1+1)}{2J(J+1)}\,,$$

where J is the spin of the new particle, j_1 and j_2 are the spins of the constituent particles.

$^2H_1$ is the combination of a neutron and $^1H_1$, with $J = 1$, $j_1 = j_2 = 1/2$. Let $g_1 = g(n)$, $g_2 = g(^1H_1)$. Then $\frac{1}{2}g_1 + \frac{1}{2}g_2 = g(^2H_1)$, or

$$g(n) = g_1 = 2(0.86 - 2.79) = -3.86\,.$$

According to the single-particle shell model, the magnetic moment is due to the last unpaired nucleon. For ^3H, $j = 1/2$, $l = 0$, $s = 1/2$, same as for ^1H. Thus, $g(^3H) = g(^1H)$. Similarly ^3He has an unpaired n so that $g(^3He) = g(n)$. Hence

$$\mu(^3H) = 2.79\mu_N, \qquad \mu(^3He) = -1.93\mu_N\,.$$

$^6Li_3$ can be considered as the combination of $^4He_2$ and $^2H_1$, with $J = 1$, $j_1 = 0$, $j_2 = 1$. Hence

$$g = \left(\frac{2-2}{2 \times 2}\right) g_1 + \left(\frac{2+2}{2 \times 2}\right) g_2 = g_2\,,$$

or

$$g(^6Li_3) = g(^2H_1) = 0.86\,.$$

(a) The approximate values of the magnetic moments of neutron, $^3H_1$, $^3He_2$, $^6Li_3$ are therefore

$$\mu(n) = g(n)\mu_N/2 = -1.93\mu_N\,,$$

$$\mu(^3H_1) = 2.79\mu_N\,,$$

$$\mu(^3He_2) = -1.93\mu_N\,,$$

$$\mu(^6Li) = g(^6Li_3)\mu_N \times 1 = 0.86\mu_N\,.$$

(b) The β-decay from $^3H_1$ to ^3He is by the interaction

$$^3H_1 \to {}^3He_2 + e^- + \bar{\nu}_e\,,$$

where the decay energy is

$$Q = m(^3H_1) - m(^3He_2) = 3.01605 - 3.01603 = 0.00002 \text{ amu}$$

$$= 2 \times 10^{-5} \times 938 \times 10^3 \text{ keV} = 18.7 \text{ keV}\,.$$

Hence the maximum energy of the β-particle emitted is 18.7 keV.

(c) The fusion reaction of $^3\text{H}_1$ and $^3\text{He}_2$,

$$^3\text{H}_1 + {}^3\text{He}_2 \to {}^6\text{Li}_3\,,$$

releases an energy

$$Q = m(^3\text{H}_1) + m(^3\text{He}_2) - m(^6\text{Li}_3) = 0.01696 \text{ amu} = 15.9 \text{ MeV}\,.$$

The fusion reaction of $^2\text{H}_1$ and $^4\text{He}_2$,

$$^2\text{H}_1 + {}^4\text{He}_2 \to {}^6\text{Li}_3\,,$$

releases an energy

$$Q' = m(^2\text{H}_1) + m(^4\text{He}_2) - m(^6\text{Li}_3) = 0.02501 \text{ amu} = 23.5 \text{ MeV}\,.$$

Thus the second fusion reaction produces more energy.

2007

To penetrate the Coulomb barrier of a light nucleus, a proton must have a minimum energy of the order of

(a) 1 GeV.
(b) 1 MeV.
(c) 1 KeV.

(*CCT*)

Solution:

The Coulomb barrier of a light nucleus is $V = Q_1Q_2/r$. Let $Q_1 \approx Q_2 \approx e$, $r \approx 1$ fm. Then

$$V = e^2/r = \frac{\hbar c}{r}\left(\frac{e^2}{\hbar c}\right) = \frac{197}{1}\cdot\frac{1}{137} = 1.44 \text{ MeV}\,.$$

Hence the answer is (b).

2008

What is the density of nuclear matter in ton/cm^3?

(a) 0.004.
(b) 400.
(c) 10^9.

(*CCT*)

Solution:

The linear size of a nucleon is about 10^{-13} cm, so the volume per nucleon is about 10^{-39} cm^3. The mass of a nucleon is about 10^{-27} kg $= 10^{-30}$ ton, so the density of nuclear matter is $\rho = m/V \approx 10^{-30}/10^{-39} = 10^9 \text{ton/cm}^3$. Hence the answer is (c).

2009

(a) Calculate the electrostatic energy of a charge Q distributed uniformly throughout a sphere of radius R.

(b) Since $^{27}_{14}\text{Si}$ and $^{27}_{13}\text{Al}$ are "mirror nuclei", their ground states are identical except for charge. If their mass difference is 6 MeV, estimate their radius (neglecting the proton-neutron mass difference).

(*Wisconsin*)

Solution:

(a) The electric field intensity at a point distance r from the center of the uniformly charged sphere is

$$E(r) = \begin{cases} \dfrac{Qr}{R^3} & \text{for } r < R\,, \\[2ex] \dfrac{Q}{r^2} & \text{for } r > R\,. \end{cases}$$

The electrostatic energy is

$$\begin{aligned} W &= \int_0^\infty \frac{1}{8\pi} E^2 d\tau \\ &= \frac{Q^2}{8\pi}\left[\int_0^R \left(\frac{r}{R^3}\right)^2 4\pi r^2 dr + \int_R^\infty \frac{1}{r^4} 4\pi r^2 dr\right] \\ &= \frac{Q^2}{2}\left(\int_0^R \left(\frac{r^4}{R^6} dr + \int_R^\infty \frac{1}{r^2} dr\right)\right. \\ &= \frac{Q^2}{2}\left(\frac{1}{5R} + \frac{1}{R}\right) \\ &= \frac{3Q^2}{5R}\,. \end{aligned}$$

(b) The mass difference between the mirror nuclei $^{27}_{14}Si$ and $^{27}_{13}Al$ can be considered as due to the difference in electrostatic energy:

$$\Delta W = \frac{3e^2}{5R}(Z_1^2 - Z_2^2)\,.$$

Thus

$$\begin{aligned} R &= \frac{3e^2}{5\Delta W}(14^2 - 13^2) = \frac{3\hbar c}{5\Delta W}\left(\frac{e^2}{\hbar c}\right)(14^2 - 13^2) \\ &= \frac{3 \times 1.97 \times 10^{-11}}{5 \times 6} \times \frac{1}{137} \times (14^2 - 13^2) \\ &= 3.88 \times 10^{-11} \text{ cm} \\ &= 3.88 \text{ fm}\,. \end{aligned}$$

2010

The nucleus $^{27}_{14}Si$ decays to its "mirror" nucleus $^{27}_{13}Al$ by positron emission. The maximum (kinetic energy$+m_e c^2$) energy of the positron is 3.48 MeV. Assume that the mass difference between the nuclei is due to the Coulomb energy. Assume the nuclei to be uniformly charged spheres of charge Ze and radius R. Assuming the radius is given by $r_0 A^{1/3}$, use the above data to estimate r_0.

(*Princeton*)

Solution:

$$^{27}_{14}Si \to {}^{27}_{13}Al + \beta^+ + \nu\,.$$

If the recoil energy of the nucleus is neglected, the maximum energy of the positron equals roughly the mass difference between the nuclei minus $2m_e c^2$. The Coulomb energy of a uniformly charged sphere is (**Problem 2009**)

$$E_e = \frac{3e^2 Z^2}{5R} = \frac{3e^2}{5r_0} Z^2 A^{-1/3}\,.$$

For $^{27}_{14}Si$ and $^{27}_{13}Al$,

$$E_e = \frac{3e^2}{5r_0} 27^{-\frac{1}{3}}(14^2 - 13^2) = \frac{27e^2}{5r_0} = 3.48 + 1.02 = 4.5 \text{ MeV}\,,$$

or

$$r_0 = \frac{27e^2}{5 \times 4.5} = \frac{27\hbar c}{5 \times 4.5}\left(\frac{e^2}{\hbar c}\right) = \frac{27 \times 1.97 \times 10^{-11}}{5 \times 4.5 \times 137}$$
$$= 1.73 \times 10^{-13} \text{ cm} = 1.73 \text{ fm}\,.$$

2011

The binding energy of ${}^{90}_{40}\mathrm{Zr}_{50}$ is 783.916 MeV. The binding energy of ${}^{90}_{39}\mathrm{Y}_{51}$ is 782.410 MeV. Estimate the excitation energy of the lowest $T = 6$ isospin state in ${}^{90}\mathrm{Zr}$.

(*Princeton*)

Solution:

The energy difference between two members of the same isospin multiplet is determined by the Coulomb energies and the neutron-proton mass difference. Thus (**Problem 2009**)

$$\begin{aligned}\Delta E &= E(A, Z+1) - E(A, Z) = \Delta E_e - (m_n - m_p)c^2 \\ &= \frac{3e^2}{5R}(2Z+1) - 0.78 = \frac{3(2Z+1)c\hbar\alpha}{5R} - 0.78 \\ &= \frac{3(2 \times 39 + 1) \times 197}{5 \times 1.2 \times 90^{1/3} \times 137} - 0.78 \\ &= 11.89 \text{ MeV}\end{aligned}$$

using $R = 1.2A^{1/3}$ fm.

Hence the excitation energy of the $T = 6$ state of ${}^{90}\mathrm{Zr}$ is

$$E = -782.410 + 11.89 + 783.916 = 13.40 \text{ MeV}\,.$$

2012

The masses of a set of isobars that are members of the same isospin multiplet can be written as the expectation value of a mass operator with the form

$$M = a + bT_z + cT_z^2\,,$$

where a, b, c are constants and T_z is the operator for the z component of the isotopic spin.

(a) Derive this formula.

(b) How large must the isospin be in order to test it experimentally?

(*Princeton*)

Solution:

(a) Members of the same isospin multiplet have the same spin-parity J^p because of the similarity of their structures. Their mass differences are determined by the Coulomb energies and the neutron-proton mass difference. Let the nuclear mass number be A, neutron number be N, then $A = Z + N = 2Z - (Z - N) = 2Z - 2T_z$. As (**Problem 2009**)

$$\begin{aligned}
M &= \frac{3e^2Z^2}{5R} + (m_p - m_n)T_z + M_0 \\
&= B\left(\frac{A}{2} + T_z\right)^2 + CT_z + M_0 \\
&= \frac{BA^2}{4} + BAT_z + BT_z^2 + CT_z + M_0 \\
&= M_0 + \frac{BA^2}{4} + (C + BA)T_z + BT_z^2 \\
&= a + bT_z + cT_z^2
\end{aligned}$$

with $a = M_0 + BA^2/4$, $b = C + BA$, $c = B$.

The linear terms in the formula arise from the neutron-proton mass difference and the Coulomb energy, while the quadratic term is mainly due to the Coulomb energy.

(b) There are three constants a,b,c in the formula, so three independent linear equations are needed for their determination. As there are $(2T + 1)$ multiplets of an isospin T, in order to test the formula experimentally we require at least $T = 1$.

2013

Both nuclei $^{14}_{7}$N and $^{12}_{6}$C have isospin $T = 0$ for the ground state. The lowest $T = 1$ state has an excitation energy of 2.3 MeV in the case of

$^{14}_{7}$N and about 15.0 MeV in the case of $^{12}_{6}$C. Why is there such a marked difference? Indicate also the basis on which a value of T is ascribed to such nuclear states. (Consider other members of the $T = 1$ triplets and explain their relationship in terms of systematic nuclear properties.)

(*Columbia*)

Solution:

The excited states with $T = 1$ of $^{12}_{6}$C form an isospin triplet which consists of $^{12}_{5}$B, $^{12}_{6}$C and $^{12}_{7}$N. $^{12}_{5}$B and $^{12}_{7}$N have $|T_3| = 1$, so they are the ground states of the triplet. Likewise, $^{14}_{6}$C and $^{14}_{8}$O are the ground states of the isospin triplet of the $T = 1$ excited states of $^{14}_{7}$N. The binding energies $M - A$ are given in the table below.

Elements	M-A (MeV)
$^{12}_{6}C$	0
$^{12}_{5}B$	13.370
$^{14}_{7}N$	2.864
$^{14}_{6}C$	3.020

The energy difference between two nuclei of an isospin multiplet is

$$\begin{aligned}
\Delta E &= [M(Z,A) - M(Z-1,A)]c^2 \\
&= \frac{3e^2}{5R}(2Z-1) - (m_n - m_p)c^2 \\
&= \frac{3e^2}{5R_0A^{1/3}}(2Z-1) - 0.78 \\
&= \frac{3\hbar c}{5R_0A^{1/3}}\left(\frac{e^2}{\hbar c}\right)(2Z-1) - 0.78 \\
&= \frac{3\times 197}{5\times 137R_0A^{1/3}}(2Z-1) - 0.78 \text{ MeV}\,.
\end{aligned}$$

Taking $R_0 \approx 1.4$ fm and so

$$M(^{14}N, T=1) - M(^{14}C, T=1) = 2.5 \text{ MeV}/c^2\,,$$

$$M(^{12}C, T=1) - M(^{12}B, T=1) = 2.2 \text{ MeV}/c^2\,,$$

we have

$$\begin{aligned} M(^{14}N, T=1) - M(^{14}N, T=0) \\ &= M(^{14}N, T=1) - M(^{14}C, T=1) \\ &\quad + M(^{14}C, T=1) - M(^{14}N, T=0) \\ &= 2.5 + 3.02 - 2.86 \\ &= 2.66 \text{ MeV}/c^2 \,, \end{aligned}$$

$$\begin{aligned} M(^{12}C, T=1) - M(^{12}C, T=0) \\ &= M(^{12}C, T=1) - M(^{12}B, T=1) \\ &\quad + M(^{12}B, T=1) - M(^{12}C, T=0) \\ &= 2.2 + 13.37 \\ &= 15.5 \text{ MeV}/c^2 \,, \end{aligned}$$

which are in agreement with the experiment values 2.3 MeV and 15.0 MeV. The large difference between the excitation energies of ^{12}C and ^{14}N is due to the fact that the ground state of ^{12}C is of an α-group structure and so has a very low energy.

The nuclei of an isospin multiplet have similar structures and the same J^p. The mass difference between two isospin multiplet members is determined by the difference in the Coulomb energy of the nuclei and the neutron-proton mass difference. Such data form the basis of isospin assignment. For example ^{14}O, ^{14}N* and ^{14}C belong to the same isospin multiplet with $J^p = 0^+$ and ground states ^{14}C and ^{14}O, ^{14}N* being an exciting state. Similarly ^{12}C*, ^{12}C and ^{12}B form an isospin multiplet with $J^p = 1^+$, of which ^{14}N and ^{12}B are ground states while ^{12}C* is an excited state.

2014

(a) Fill in the missing entries in the following table giving the properties of the ground states of the indicated nuclei. The mass excess $\Delta M_{Z,A}$ is defined so that

$$M_{Z,A} = A(931.5 \text{ MeV}) + \Delta M_{Z,A} \,,$$

where $M_{Z,A}$ is the atomic mass, A is the mass member, T and T_z are the quantum members for the total isotopic spin and the third component of isotopic spin. Define your convention for T_z.

(b) The wave function of the isobaric analog state (IAS) in ^{81}Kr is obtained by operating on the ^{81}Br ground state wave function with the isospin upping operator T.

(i) What are J^π, T, and T_z for the IAS in ^{81}Kr?

(ii) Estimate the excitation energy of the IAS in ^{81}Kr.

(iii) Now estimate the decay energy available for decay of the IAS in ^{81}Kr by emission of a

neutron, γ-ray, α-particle, β^+-ray.

(iv) Assuming sufficient decay energy is available for each decay mode in (iii), indicate selection rules or other factors which might inhibit decay by that mode.

(*Princeton*)

Isotopes	Z	T_z	T	J^p	Mass excess (MeV)
n	0				8.07
^{1}H	1				7.29
^{4}He	2				2.43
^{77}Se	34			$1/2^-$	−74.61
^{77}Br	35			$3/2^-$	−73.24
^{77}Kr	36			$7/2^+$	−70.24
^{80}Br	35			1^+	−76.89
^{80}Kr	36				−77.90
^{81}Br	35			$3/2^-$	−77.98
^{81}Kr	36			$7/2^+$	−77.65
^{81}Rb	37			$3/2^-$	−77.39

Solution:

(a) The table is completed as shown in the next page.

(b) (i) The isobasic analog state (IAS) is a highly excited state of a nucleus with the same mass number but with one higher atomic number, i.e. a state with the same A, the same T, but with T_z increased by 1. Thus for ^{81}Br, $|T, T_z\rangle = |11/2, -11/2\rangle$, the quantum numbers of the IAS in ^{81}Kr are $T = 11/2$, $T_z = -9/2$, $J^p[^{81}\text{Kr(IAS)}] = J^p(^{81}\text{Br}) = 3/2^-$.

Isotopes	Z	T_z	T	J^p	Mass excess (MeV)
n	0	$-1/2$	$1/2$	$1/2^+$	8.07
^{1}H	1	$1/2$	$1/2$	$1/2^+$	7.29
^{4}He	2	0	0	0^+	2.43
^{77}Se	34	$-9/2$	$9/2$	$1/2^-$	-74.61
^{77}Br	35	$-7/2$	$7/2$	$3/2^-$	-73.24
^{77}Kr	36	$-5/2$	$5/2$	$7/2^+$	-70.24
^{80}Br	35	-5	5	1^+	-76.89
^{80}Kr	36	-4	4	0^+	-77.90
^{81}Br	35	$-11/2$	$11/2$	$3/2^-$	-77.98
^{81}Kr	36	$-9/2$	$9/2$	$7/2^+$	-77.65
^{81}Rb	37	$-7/2$	$7/2$	$3/2^-$	-77.39

(ii) The mass difference between ^{81}Br and ^{81}Kr(IAS) is due to the difference between the Coulomb energies of the nuclei and the neutron-proton mass difference:

$$\Delta M_{^{81}\text{Kr}(IAS)} = \Delta M_{^{81}\text{Br}} + \frac{3}{5} \times \frac{(2Z-1)e^2}{R_0 A^{1/3}} - [m(n) - M(^1\text{H})]$$

$$= \Delta M_{^{81}\text{Br}} + 0.719\left(\frac{2Z-1}{A^{\frac{1}{3}}}\right) - 0.78 \text{ MeV}\,,$$

as $R_0 = 1.2$ fm, $m_n - m_p = 0.78$ MeV. With $Z = 36$, $A = 81$, $\Delta M_{^{81}\text{Br}} = -77.98$ MeV, we have $\Delta M_{^{81}\text{Kr}(IAS)} = -67.29$ MeV.

Hence the excitation energy of ^{81}Kr(IAS) from the ground state of ^{81}Kr is

$$\Delta E = -67.29 - (-77.65) = 10.36 \text{ MeV}\,.$$

(iii) For the neutron-decay $^{81}\text{Kr(IAS)} \to n + ^{80}\text{Kr}$,

$$Q_1 = \Delta M_{^{81}\text{Kr}(IAS)} - \Delta(n) - \Delta M_{^{80}\text{Kr}}$$

$$= -67.29 - 8.07 + 77.90 = 2.54 \text{ MeV}\,.$$

For the γ-decay $^{81}\text{Kr}(IAS) \to {}^{81}\text{Kr} + \gamma$,

$$Q_2 = \Delta M_{^{81}\text{Kr}(IAS)} - \Delta M_{^{81}\text{Kr}} = -67.29 - (-77.65) = 10.36 \text{ MeV}\,.$$

For the α-decay $^{81}\text{Kr}(IAS) \to \alpha + ^{77}\text{Se}$,

$$\begin{aligned}Q_3 &= \Delta M_{^{81}\mathrm{Kr}(IAS)} - \Delta M_\alpha - \Delta M_{^{77}\mathrm{Se}} \\ &= -67.29 - 2.43 - (-74.61) = 4.89 \text{ MeV}.\end{aligned}$$

For the β^+-decay $^{81}\mathrm{Kr}(IAS) \to ^{81}\mathrm{Br} + \beta^+ + \nu_e$,

$$\begin{aligned}Q_4 &= \Delta M_{^{81}\mathrm{Kr}(IAS)} - \Delta M_{^{81}\mathrm{Br}} - 2m_e \\ &= -67.29 - (-77.98) - 1.02 = 9.67 \text{ MeV}.\end{aligned}$$

(iv)

$$\begin{array}{lccccc}\text{In the interaction} & ^{81}\mathrm{Kr}(IAS) \to & ^{81}\mathrm{Kr} & + & n \\ T: & 11/2 & 4 & & \frac{1}{2}\end{array}$$

$\Delta T \neq 0$. As strong interaction requires conservation of T and T_z, the interaction is inhibited.

$$\begin{array}{lcc}\text{In the interaction} & ^{81}\mathrm{Kr}(IAS) \to & ^{81}\mathrm{Kr} + \gamma \\ J^p: & \frac{3}{2}^- & \frac{7}{2}^+\end{array}$$

we have $\Delta J = \left|\frac{3}{2} - \frac{7}{2}\right| = 2$, $\Delta P = -1$; so it can take place through $E3$ or $M2$ type transition.

$$\begin{array}{lcccc}\text{The interaction} & ^{81}\mathrm{Kr}(IAS) \to & ^{77}\mathrm{Se} & + & \alpha \\ T: & 11/2 & 9/2 & & 0 \\ T_z: & -9/2 & -9/2 & & 0\end{array}$$

is inhibited as isospin is not conserved.

$$\begin{array}{lcccc}\text{The interaction} & ^{81}\mathrm{Kr}(IAS) \to & ^{81}\mathrm{Br} & + & \beta^+ + \nu_e \\ J^p: & 3/2^- & \frac{3}{2}^- & & \end{array}$$

is allowed, being a mixture of the Fermi type and Gamow–Teller type interactions.

2015

Isospin structure of magnetic dipole moment.

The magnetic dipole moments of the free neutron and free proton are $-1.913\mu_N$ and $+2.793\mu_N$ respectively. Consider the nucleus to be a collection of neutrons and protons, having their free moments.

(a) Write down the magnetic moment operator for a nucleus of A nucleons.

(b) Introduce the concept of isospin and determine the isoscalar and isovector operators. What are their relative sizes?

(c) Show that the sum of magnetic moments in nuclear magnetons of two $T = 1/2$ mirror nuclei is

$$J + (\mu_p + \mu_n - 1/2)\left\langle \sum_{i=l}^{A} \sigma_z^{(i)} \right\rangle ,$$

where J is the total nuclear spin and $\sigma_z^{(i)}$ is the Pauli spin operator for a nucleon.

(*Princeton*)

Solution:

(a) The magnetic moment operator for a nucleus of A nucleons is

$$\boldsymbol{\mu} = \sum_{i=l}^{A} (g_l^i \boldsymbol{l}_i + g_s^i \boldsymbol{S}_i) ,$$

where for neutrons: $g_l = 0$, $g_s = 2\mu_n$; for protons: $g_l = 1$, $g_s = 2\mu_p$ and $\boldsymbol{S}$ is the spin operator $\frac{1}{2}\boldsymbol{\sigma}$.

(b) Charge independence has been found to hold for protons and neutrons such that, if Coulomb forces are ignored, the $p-p$, $p-n$, $n-n$ forces are identical provided the pair of nucleons are in the same spin and orbital motions. To account for this, isospin T is introduced such that p and n have the same T while the z component T_z in isospin space is $T_z = \frac{1}{2}$ for p and $T_z = -\frac{1}{2}$ for n. There are four independent operators in isospin space:

scalar operator: unit matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

vector operators: Pauli matrices, $\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let the wave functions of proton and neutron be $\psi_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\psi_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ respectively, and define $\tau_\pm = \tau_1 \pm i\tau_2$, $T = \tau/2$. Then

$$T_3\Psi_p = \frac{1}{2}\Psi_p, \qquad \tau_3\Psi_p = \Psi_p\,,$$

$$T_3\Psi_n = -\frac{1}{2}\Psi_n, \qquad \tau_3\Psi_n = -\Psi_n\,,$$

$$T_+\Psi_n = \Psi_p, \qquad T_-\Psi_p = \Psi_n\,.$$

(c) The mirror nucleus is obtained by changing all the protons of a nucleus into neutrons and all the neutrons into protons. Thus mirror nuclei have the same T but opposite T_z. In other words, for mirror nuclei, if the isospin quantum numbers of the first nucleus are $\left(\frac{1}{2}, \frac{1}{2}\right)$, then those of the second are $\left(\frac{1}{2}, -\frac{1}{2}\right)$.

For the first nucleus, the magnetic moment operator is

$$\boldsymbol{\mu}_1 = \sum_{i=1}^{A}(g_l^i \boldsymbol{l}_1^i + g_s^i \boldsymbol{S}_1^i)\,.$$

We can write

$$g_l = \frac{1}{2}(1+\tau_3), \qquad g_s = (1+\tau_3)\mu_p + (1-\tau_3)\mu_n\,,$$

since $g_l\psi_p = \psi_p$, $g_l\psi_n = 0$, etc. Then

$$\begin{aligned}
\boldsymbol{\mu}_1 &= \sum_{i=1}^{A}\frac{(1+\tau_3^i)}{2}\boldsymbol{l}_1^i + \left[\sum_{i=1}^{A}(1+\tau_3^i)\mu_p + \sum_{i=1}^{A}(1-\tau_3^i)\mu_n\right]\boldsymbol{S}_l^i \\
&= \frac{1}{2}\sum_{i=1}^{A}(\boldsymbol{l}_1^i + \boldsymbol{S}_1^i) + \left(\mu_p + \mu_n - \frac{1}{2}\right)\sum_{i=1}^{A}\boldsymbol{S}_1^i \\
&\quad + \frac{1}{2}\sum_{i=1}^{A}\tau_3^i[\boldsymbol{l}_1^i + 2(\mu_p - \mu_n)\boldsymbol{S}_1^i]\,.
\end{aligned}$$

Similarly for the other nucleus we have

$$\boldsymbol{\mu}_2 = \frac{1}{2}\sum_{i=1}^{A}(\boldsymbol{l}_2^i + \boldsymbol{S}_2^i) + \left(\mu_p + \mu_n - \frac{1}{2}\right)\sum_{i=1}^{A}\boldsymbol{S}_2^i + \frac{1}{2}\sum_{i=1}^{A}\tau_3^i[\boldsymbol{l}_2^i + 2(\mu_p - \mu_n)\boldsymbol{S}_2^i]\,.$$

As $\boldsymbol{J}^i = \sum_{i=1}^{A}(\boldsymbol{l}^i + \boldsymbol{S}^i)$, the mirror nuclei have $\boldsymbol{J}^1 = \boldsymbol{J}^2$ but opposite T_3 values, where $T_3 = \frac{1}{2}\sum_{i=1}^{A}\tau_3^i$, $\boldsymbol{S} = \frac{1}{2}\boldsymbol{\sigma}$.

The observed magnetic moment is $\mu = \langle \mu_z \rangle = \langle JJ_zTT_3|\mu_z|JJ_zTT_3\rangle$. Then for the first nucleus:

$$\begin{aligned}\mu_1 &= \left\langle JJ_z\frac{1}{2}\frac{1}{2}\right|\frac{J_z}{2} + \left(\mu_p + \mu_n - \frac{1}{2}\right) \times \frac{1}{2}\sum_{i=1}^{A}(\sigma_1^i)_z \\ &\quad + \frac{1}{2}\sum_{i=1}^{A}\tau_3^i[l_{1z}^i + 2(\mu_p - \mu_n)S_{1z}^i]\left|JJ_z\frac{1}{2}\frac{1}{2}\right\rangle \\ &= \frac{J_z}{2} + \frac{1}{2}\left(\mu_p + \mu_n - \frac{1}{2}\right)\left\langle\sum_{i=1}^{A}(\sigma_1^i)_z\right\rangle \\ &\quad + \left\langle JJ_z\frac{1}{2}\frac{1}{2}\right|\frac{1}{2}\sum_{i=1}^{A}\tau_3^i[l_{1z}^i + 2(\mu_p - \mu_n)S_{1z}^i]\left|JJ_z\frac{1}{2}\frac{1}{2}\right\rangle,\end{aligned}$$

and for the second nucleus:

$$\begin{aligned}\mu_2 &= \frac{J_z}{2} + \frac{1}{2}\left(\mu_p + \mu_n - \frac{1}{2}\right)\left\langle\sum_{i=1}^{A}(\sigma_1^i)_z\right\rangle \\ &\quad + \left\langle JJ_z\frac{1}{2} - \frac{1}{2}\right|\frac{1}{2}\sum_{i=1}^{A}\tau_3^i[l_{2z}^i + 2(\mu_p - \mu_n)S_{2z}^i]\left|JJ_z\frac{1}{2} - \frac{1}{2}\right\rangle.\end{aligned}$$

The sum of the magnetic moments of the mirror nuclei is

$$\mu_1 + \mu_2 = J_z + \left(\mu_p + \mu_n - \frac{1}{2}\right)\left\langle\sum_{i=1}^{A}\sigma_z^i\right\rangle,$$

as the last terms in the expression for μ_1 and μ_2 cancel each other.

2016

Hard sphere scattering:

Show that the classical cross section for elastic scattering of point particles from an infinitely massive sphere of radius R is isotropic.

(*MIT*)

Solution:

In classical mechanics, in elastic scattering of a point particle from a fixed surface, the emergent angle equals the incident angle. Thus if a particle moving along the $-z$ direction impinges on a hard sphere of radius R at a surface point of polar angle θ, it is deflected by an angle $\alpha = 2\theta$. As the impact parameter is $b = R\sin\theta$, the differential scattering cross section is

$$\frac{d\sigma}{d\Omega} = \frac{2\pi b db}{2\pi \sin\alpha d\alpha} = \frac{R^2 \sin\theta \cos\theta d\theta}{4\sin\theta\cos\theta d\theta} = \frac{R^2}{4},$$

which is independent of θ, showing that the scattering is isotropic.

2017

A convenient model for the potential energy V of a particle of charge q scattering from an atom of nuclear charge Q is $V = qQe^{-\alpha r}/r$. Where α^{-1} represents the screening length of the atomic electrons.

(a) Use the Born approximation

$$f = -\frac{1}{4\pi}\int e^{-i\Delta\mathbf{k}\cdot\mathbf{r}}\frac{2m}{\hbar^2}V(r)d^3\mathbf{r}$$

to calculate the scattering cross section σ.

(b) How should α depend on the nuclear charge Z?

(*Columbia*)

Solution:

(a) In Born approximation

$$f = -\frac{m}{2\pi\hbar^2}\int V(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}}d^3\mathbf{r},$$

where $\mathbf{q} = \mathbf{k}-\mathbf{k}_0$ is the momentum transferred from the incident particle to the outgoing particle. We have $|\mathbf{q}| = 2k_0 \sin\frac{\theta}{2}$, where θ is the angle between the incident and outgoing particles. As $V(\mathbf{r})$ is spherically symmetric,

$$\begin{aligned} f(\theta) &= -\frac{m}{2\pi\hbar^2}\int_0^\infty \int_0^{2\pi}\int_0^\pi V(r)e^{-i\Delta kr\cos\theta}\sin\theta r^2 dr d\varphi d\theta \\ &= -\frac{2m}{\hbar^2\Delta k}\int_0^\infty V(r)\sin(\Delta kr)rdr \\ &= -\frac{2mQq}{\hbar^2}\cdot\frac{1}{\alpha^2+(\Delta k)^2}. \end{aligned}$$

The differential cross section is

$$d\sigma = |f(\theta)|^2 d\Omega = \frac{4m^2Q^2q^2}{\hbar^4} \cdot \frac{d\Omega}{[\alpha^2 + (\Delta k^2)]^2}$$

$$= \frac{m^2Q^2q^2}{4\hbar^4 k_0^4} \cdot \frac{d\Omega}{\left(\frac{\alpha^2}{4k_0^2} + \sin^2\frac{\theta}{2}\right)^2},$$

and the total cross-section is

$$\sigma = \int d\sigma = \frac{m^2Q^2q^2}{4\hbar^4 k_0^4} \int_0^{2\pi} \int_0^{\pi} \frac{\sin\theta d\theta d\varphi}{\left(\frac{\alpha^2}{4k_0^2} + \sin^2\frac{\theta}{2}\right)^2}$$

$$= \frac{16\pi m^2Q^2q^2}{\hbar^4\alpha^2(4k_0^2 + \alpha^2)}.$$

(b) α^{-1} gives a measure of the size of atoms. As Z increases, the number of electrons outside of the nucleus as well as the probability of their being near the nucleus will increase, enhancing the screening effect. Hence α is an increasing function of Z.

2018

Consider the scattering of a 1-keV proton by a hydrogen atom.

(a) What do you expect the angular distribution to look like? (Sketch a graph and comment on its shape).

(b) Estimate the total cross section. Give a numerical answer in cm^2, m^2 or barns, and a reason for your answer.

(*Wisconsin*)

Solution:

The differential cross section for elastic scattering is (**Problem 2017**)

$$\frac{d\sigma}{d\Omega} = \frac{m^2q^2Q^2}{4\hbar^4k_0^4} \frac{1}{\left(\frac{\alpha^2}{4k_0^4} + \sin\frac{\theta}{2}\right)^2}.$$

For proton and hydrogen nuclues $Q = q = e$. The screening length can be taken to be $\alpha^{-1} \approx R_0$, R_0 being the Bohr radius of hydrogen atom. For an incident proton of 1 keV; The wave length is

$$\lambda\!\!\!^{-}_0 = \frac{\hbar}{\sqrt{2\mu E}} = \frac{c\hbar}{\sqrt{2\mu c^2 E}} = \frac{197}{\sqrt{1 \times 938 \times 10^{-3}}} = 203 \text{ fm}\,.$$

With $\alpha^{-1} \approx R_0 = 5.3 \times 10^4$ fm, $\frac{\alpha^2}{4k_0^2} = \left(\frac{\lambda\!\!\!^{-}_0}{2\alpha^{-1}}\right)^2 \ll 1$ and so

$$\frac{d\sigma}{d\Omega} \approx \frac{m^2 e^4}{4\hbar^2 k_0^2} \frac{1}{\sin^4 \frac{\theta}{2}}\,,$$

which is the Rurthford scattering formula.

The scattering of 1-keV protons from hydrogen atom occurs mainly at small angles (see Fig. 2.1). The probability of large angle scattering (near head-on collision) is very small, showing that hydrogen atom has a very small nucleus.

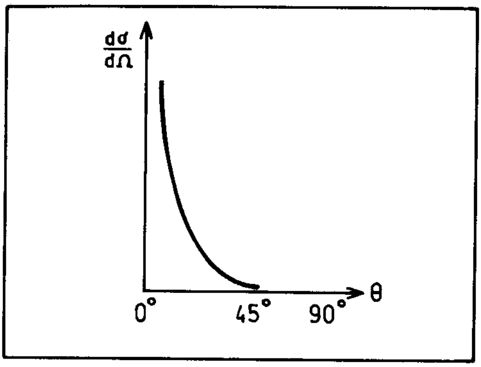

Fig. 2.1

(b) As given in **Problem 2017**,

$$\sigma = \frac{16\pi m^2 e^4}{\hbar^4 \alpha^2 (4k_0^2 + \alpha^3)} \approx \frac{16\pi m^2 e^4}{\hbar^4 \alpha^2 4k_0^2}$$

$$= 4\pi \left[\frac{mc^2 R_0 \lambda\!\!\!^{-}_0}{\hbar c}\left(\frac{e^2}{\hbar c}\right)\right]^2 = 4\pi \left(\frac{938 \times 5.3 \times 10^4 \times 203}{197 \times 137}\right)^2$$

$$= 1.76 \times 10^{12} \text{ fm}^2 = 1.76 \times 10^{-14} \text{ cm}^2\,.$$

2019

(a) At a center-of-mass energy of 5 MeV, the phase describing the elastic scattering of a neutron by a certain nucleus has the following values: $\delta_0 = 30^0$, $\delta_1 = 10^0$. Assuming all other phase shifts to be negligible, plot $d\sigma/d\Omega$ as a function of scattering angle. Explicitly calculate $d\sigma/d\Omega$ at 30^0, 45^0 and 90^0. What is the total cross section σ?

(b) What does the fact that all of the phase shifts $\delta_2, \delta_3 \ldots$ are negligible imply about the range of the potential? Be as quantitative as you can.

(*Columbia*)

Solution:

(a) The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)\right|^2 .$$

Supposing only the first and second terms are important, we have

$$\begin{aligned}\frac{d\sigma}{d\Omega} &\approx \frac{1}{k^2}|e^{i\delta_0}\sin\delta_0 + 3e^{i\delta_1}\sin\delta_1\cos\theta|^2 \\ &= \frac{1}{k^2}|(\cos\delta_0\sin\delta_0 + 3\cos\delta_1\sin\delta_1\cos\theta) + i(\sin^2\delta_0 + 3\sin^2\delta_1\cos\theta)|^2 \\ &= \frac{1}{k^2}[\sin^2\delta_0 + 9\sin^2\delta_1\cos^2\theta + 6\sin\delta_0\sin\delta_1\cos(\delta_1-\delta_0)\cos\theta] \\ &= \frac{1}{k^2}[0.25 + 0.27\cos^2\theta + 0.49\cos\theta]\,,\end{aligned}$$

where k is the wave number of the incident neutron in the center-of-mass frame. Assume that the mass of the nucleus is far larger than that of the neutron m_n. Then

$$\begin{aligned}k^2 &\approx \frac{2m_nE}{\hbar^2} = \frac{2m_nc^2E}{(\hbar c)^2} = \frac{2\times 938\times 5}{197^2\times 10^{-30}} \\ &= 2.4\times 10^{29}\ m^{-2} = 2.4\times 10^{25}\ cm^{-2}\,.\end{aligned}$$

The differential cross section for other angles are given in the following table. The data are plotted in Fig. 2.2 also.

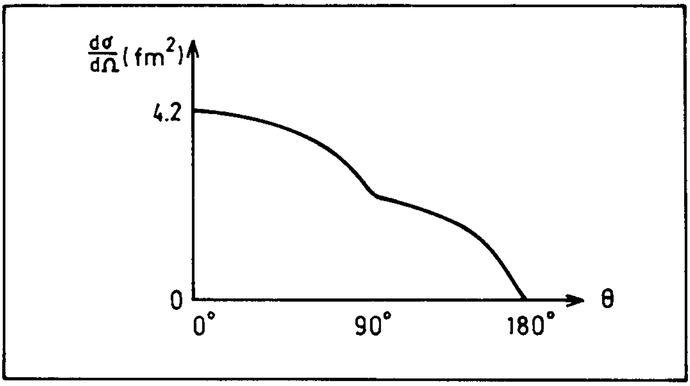

Fig. 2.2

θ	0^0	30^0	45^0	90^0	180^0
$k^2 \frac{d\sigma}{d\Omega}$	1	0.88	0.73	0.25	0
$\frac{d\sigma}{d\Omega} \times 10^{26}$ (cm^2)	4.2	3.7	3.0	1.0	0

The total cross section is

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{2\pi}{k^2} \int_0^\pi (0.25 + 0.49 \cos\theta + 0.27 \cos^2\theta) \sin\theta d\theta$$

$$= \frac{4\pi}{k^2} \left(0.25 + \frac{1}{3} \times 0.27\right) = 1.78 \times 10^{-25} \text{ cm}^2 \approx 0.18 \ b\,.$$

(b) The phase shift δ_l is given by

$$\delta_l \approx -\frac{2m_n k}{\hbar^2} \int_0^\infty V(r) J_l^2(kr) r^2 dr\,,$$

where J_l is a spherical Bessel function. As the maximum of $J_l(x)$ occurs nears $x = l$, for higher l values J_l in the region of potential $V(r)$ is rather small and can be neglected. In other words, $\delta_2, \delta_3 \ldots$ being negligible means that the potential range is within $R \approx 1/k$. Thus the range of the potential is $R \approx (2.4 \times 10^{25})^{-1/2} = 2 \times 10^{-13}$ cm $= 2$ fm.

2020

Neutrons of 1000 eV kinetic energy are incident on a target composed of carbon. If the inelastic cross section is 400×10^{-24} cm^2, what upper and lower limits can you place on the elastic scattering cross section?

(*Chicago*)

Solution:

At 1 keV kinetic energy, only s-wave scattering is involved. The phase shift δ must have a positive imaginary part for inelastic process to take place. The elastic and inelastic cross sections are respectively given by

$$\sigma_e = \pi \lambda\!\!\!^{-2} |e^{2i\delta} - 1|^2 ,$$

$$\sigma_{in} = \pi \lambda\!\!\!^{-2} (1 - |e^{2i\delta}|^2) .$$

The reduced mass of the system is

$$\mu = \frac{m_n m_c}{m_c + m_n} \approx \frac{12}{13} m_n .$$

For $E = 1000$ eV,

$$\lambda\!\!\!^{-} = \frac{\hbar}{\sqrt{2\mu E}} = \frac{\hbar c}{\sqrt{2\mu c^2 E}}$$

$$= \frac{197}{\sqrt{2 \times \frac{12}{13} \times 940 \times 10^{-3}}} = 150 \text{ fm} ,$$

$$\pi \lambda\!\!\!^{-2} = 707 \times 10^{-24} \text{ cm}^2 .$$

As

$$1 - |e^{2i\delta}|^2 = \frac{\sigma_{in}}{\pi \lambda\!\!\!^{-2}} = \frac{400}{707} = 0.566 ,$$

we have

$$|e^{2i\delta}| = \sqrt{1 - 0.566} = 0.659 ,$$

or

$$e^{2i\delta} = \pm 0.659 .$$

Hence the elastic cross section

$$\sigma_e = \pi \lambda\!\!\!^{-2} |e^{2i\delta} - 1|^2$$

has maximum and minimum values

$$(\sigma_e)_{\max} = 707 \times 10^{-24}(-0.659 - 1)^2 = 1946 \times 10^{-24} \text{ cm}^2 ,$$

$$(\sigma_e)_{\min} = 707 \times 10^{-24}(0.659 - 1)^2 = 82 \times 10^{-24} \text{ cm}^2 .$$

2021

The study of the scattering of high energy electrons from nuclei has yielded much interesting information about the charge distributions in nuclei and nucleons. We shall here consider a simple version in which the electrons are supposed to have zero spin. We also assume that the nucleus, of charge Ze, remains fixed in space (i.e., its mass is assumed infinite). Let $\rho(\mathbf{x})$ denote the charge density in the nucleus. The charge distribution is assumed to be spherically symmetric but otherwise arbitrary.

Let $f_c\ (\mathbf{p}_i, \mathbf{p}_f)$, where $\mathbf{p}_i$ is the initial and $\mathbf{p}_f$ the final momentum, be the scattering amplitude in the first Born approximation for the scattering of an electron from a point-nucleus of charge Ze. Let $f(\mathbf{p}_i, \mathbf{p}_f)$ be the scattering amplitude of an electron from a real nucleus of the same charge. Let $\mathbf{q} = \mathbf{p}_i - \mathbf{p}_f$ denote the momentum transfer. The quantity F defined by

$$f(\mathbf{p}_i, \mathbf{p}_f) = F(q^2) f_c(\mathbf{p}_i, \mathbf{p}_f)$$

is called the form factor. It is easily seen that F, in fact, depends on $\mathbf{p}_i$ and $\mathbf{p}_f$ only through the quantity q^2.

(a) The form factor $F(q^2)$ and the Fourier transform of the charge density $\rho(\mathbf{x})$ are related in a very simple manner. State and derive this relationship within the framework of the nonrelativistic Schrödinger theory. The assumption that the electrons are "nonrelativistic" is here made so that the problem will be simplified. However, on careful consideration it will probably be clear that the assumption is irrelevant: the same result applies in the "relativistic" case of the actual experiment. It is also the case that the neglect of the electron spin does not affect the essence of what we are here concerned with.

(b) Figure 2.3 shows some experimental results pertaining to the form factor for the proton, and we shall regard our theory as applicable to these data. On the basis of the data shown, compute the root-mean-square (charge) radius of the proton. Hint: Note that there is a simple relationship between the root-mean-square radius and the derivative of $F(q^2)$ with respect to q^2, at $q^2 = 0$. Find this relationship, and then compute.

(UC, Berkeley)

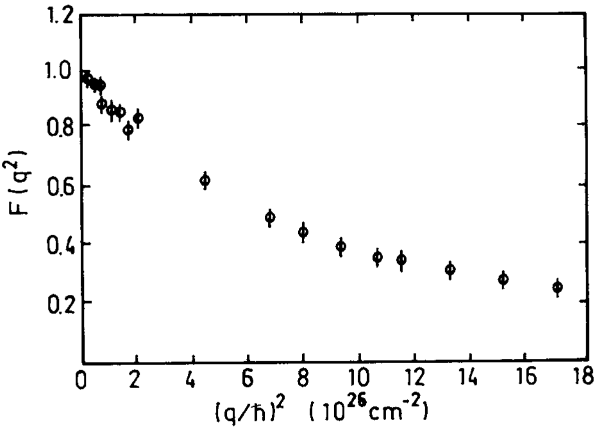

Fig. 2.3

Solution:

(a) In the first Born approximation, the scattering amplitude of a high energy electron from a nucleus is

$$f(\mathbf{p}_i,\mathbf{p}_f)=-\frac{m}{2\pi\hbar^2}\int V(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}d^3\mathbf{x}\,.$$

For a nucleus of spherically symmetric charge distribution, the potential at position $\mathbf{x}$ is

$$V(\boldsymbol{x})=\int\frac{\rho(r)Ze}{|\mathbf{x}-\mathbf{r}|}d^3\mathbf{r}\,.$$

Thus

$$\begin{aligned}f(\mathbf{p}_i,\mathbf{p}_f)&=-\frac{m}{2\pi\hbar^2}\int d^3\boldsymbol{x}e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}\int d^3\mathbf{r}\frac{\rho(r)Ze}{|\mathbf{x}-\mathbf{r}|}\\&=-\frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}\rho(r)e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}\int d^3\mathbf{x}\frac{Ze}{|\mathbf{x}-\mathbf{r}|}e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{r})/\hbar}\\&=-\frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}\rho(r)e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}\int d^3\mathbf{x}'\frac{Ze}{x'}e^{i\mathbf{q}\cdot\mathbf{x}'/\hbar}\,.\end{aligned}$$

On the other hand, for a point nucleus we have $V(\mathbf{x})=\frac{Ze}{x}$ and so

$$f_c(\mathbf{p}_i,\mathbf{p}_f)=-\frac{m}{2\pi\hbar^2}\int\frac{Ze}{x}e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}d^3\mathbf{x}\,.$$

Comparing the two equations above we obtain

$$f(\mathbf{p}_i, \mathbf{p}_f) = f_c(\mathbf{p}_i, \mathbf{p}_f) \int d^3\mathbf{r}\rho(r)e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}$$

and hence

$$F(q^2) = \int d^3\mathbf{r}\rho(r)e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}\,.$$

(b) When $q \approx 0$,

$$\begin{aligned}
F(q^2) &= \int \rho(r)e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}d^3\mathbf{r} \\
&\approx \int \rho(r)\left[1 + i\mathbf{q}\cdot\mathbf{r}/\hbar - \frac{1}{2}(\mathbf{q}\cdot\mathbf{r})^2/\hbar^2\right]d^3\mathbf{r} \\
&= \int \rho(r)d^3\mathbf{r} - \frac{1}{2}\int (\rho(r)q^2r^2\cos^2\theta/\hbar^2)\cdot r^2\sin\theta dr d\theta d\varphi \\
&= F(0) - \frac{2\pi}{3}\frac{q^2}{\hbar^2}\int r^4\rho(r)dr\,,
\end{aligned}$$

i.e.,

$$F(q^2) - F(0) = -\frac{2\pi}{3}\frac{q^2}{\hbar^2}\int r^4\rho(r)dr\,.$$

Note that $\frac{i}{\hbar}\int \rho(r)\mathbf{q}\cdot\mathbf{r}d^3\mathbf{r} = 0$ as $\int_0^\pi \cos\theta\sin\theta d\theta = 0$. The mean-square radius $\langle r^2\rangle$ is by definition

$$\begin{aligned}
\langle r^2\rangle &= \int d^3\mathbf{r}\rho(r)r^2 = 4\pi\int \rho(r)r^4dr \\
&= -6\hbar^2\frac{F(q^2) - F(0)}{q^2} = -6\hbar^2\left(\frac{\partial F}{\partial q^2}\right)_{q^2=0}\,.
\end{aligned}$$

From Fig. 2.3,

$$-\hbar^2\left(\frac{\partial F}{\partial q^2}\right)_{q^2=0} \approx -\frac{0.8 - 1.0}{2 - 0}\times 10^{-26} = 0.1\times 10^{-26}\ \text{cm}^2$$

Hence $\langle r^2\rangle = 0.6\times 10^{-26}$ cm^2, or $\sqrt{\langle r^2\rangle} = 0.77\times 10^{-13}$ cm, i.e., the root-mean-square proton radius is 0.77 fm.

2022

The total (elastic+inelastic) proton-neutron cross section at center-of-mass momentum $p = 10$ GeV/c is $\sigma = 40$ mb.

(a) Disregarding nucleon spin, set a lower bound on the elastic center-of-mass proton-neutron forward differential cross-section.

(b) Assume experiments were to find a violation of this bound. What would this mean?

(*Chicago*)

Solution:

(a) The forward $p - n$ differential cross section is given by

$$\left.\frac{d\sigma}{d\Omega}\right|_{0^0} = |f(0)|^2 \geq |\mathrm{Im} f(0)|^2 = \left(\frac{k}{4\pi}\sigma_t\right)^2 ,$$

where the relation between $\mathrm{Im} f(0)$ and σ_t is given by the optical theorem. As $k = p/\hbar$ we have

$$\left.\frac{d\sigma}{d\Omega}\right|_{0^0} \geq \left(\frac{pc}{4\pi\hbar c}\sigma_l\right)^2 = \left(\frac{10^4 \times 40 \times 10^{-27}}{4\pi \times 1.97 \times 10^{-11}}\right)^2$$

$$= 2.6 \times 10^{-24} \text{ cm}^2 = 2.6 \ b\,.$$

(b) Such a result would mean a violation of the optical theorem, hence of the unitarity of the S-matrix, and hence of the probabilistic interpretation of quantum theory.

2023

When a 300-GeV proton beam strikes a hydrogen target (see Fig. 2.4), the elastic cross section is maximum in the forward direction. Away from the exact forward direction, the cross section is found to have a (first) minimum.

(a) What is the origin of this minimum? Estimate at what laboratory angle it should be located.

(b) If the beam energy is increased to 600 GeV, what would be the position of the minimum?

(c) If the target were lead instead of hydrogen, what would happen to the position of the minimum (beam energy= 300 GeV)?

(d) For lead, at what angle would you expect the second minimum to occur?

(*Chicago*)

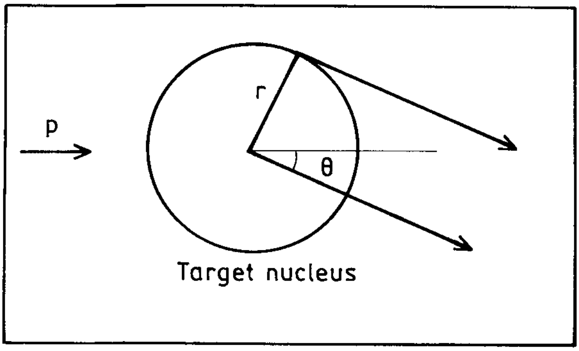

Fig. 2.4

Solution:

(a) The minimum in the elastic cross section arises from the destructive interference of waves resulting from scattering at different impact parameters. The wavelength of the incident proton, $\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{pc} = \frac{2\pi\times 1.97\times 10^{-11}}{300\times 10^3} = 4.1\times 10^{-16}$ cm, is much smaller than the size $\sim 10^{-13}$ cm of the target proton. The first minimum of the diffraction pattern will occur at an angle θ such that scattering from the center and scattering from the edge of the target proton are one-half wavelength out of phase, i.e.,

$$r\theta_{\min} = \lambda/2 = 2.1\times 10^{-16}\ \text{cm}\,.$$

Thus, if $r = 1.0\times 10^{-13}$ cm, the minimum occurs at 2.1×10^{-3}rad.

(b) If $E \to 600$ GeV/c, then $\lambda \to \lambda/2$ and $\theta_{\min} \to \theta_{\min}/2$ i.e., the minimum will occur at $\theta_{\min} = 1.05\times 10^{-3}$rad.

(c) For $Pb : A = 208$, $r = 1.1\times 208^{\frac{1}{3}} = 6.5$ fm, and we may expect the first minimum to occur at $\theta_{\min} = 3.2\times 10^{-4}$rad.

(d) At the second minium, scattering from the center and scattering from the edge are 3/2 wavelengths out of phase. Thus the second minimum will occur at $\theta_{\min} = 3\times 3.2\times 10^{-4} = 9.6\times 10^{-4}$rad.

2. NUCLEAR BINDING ENERGY, FISSION AND FUSION (2024–2047)

2024

The semiempirical mass formula relates the mass of a nucleus, $M(A, Z)$, to the atomic number Z and the atomic weight A. Explain and justify each of the terms, giving approximate values for the magnitudes of the coefficients or constants in each term.

(*Columbia*)

Solution:

The mass of a nucleus, $M(Z, A)$, is

$$M(Z,A) = ZM(^1\text{H}) + (A-Z)m_n - B(Z,A)\,,$$

where $B(Z, A)$ is the binding energy of the nucleus, given by the liquid-drop model as

$$B(Z,A) = B_v + B_s + B_e + B_a + B_p = a_v A - a_s A^{2/3} - a_e Z^2 A^{-1/3} - a_a\left(\frac{A}{2} - Z\right)^2 A^{-1} + a_p \delta A^{-1/2}\,,$$

where B_v, B_s, B_e are respectively the volume and surface energies and the electrostatic energy between the protons.

As the nuclear radius can be given as $r_0 A^{-1/3}$, r_0 being a constant, B_v, which is proportional to the volume of the nucleus, is proportional to A. Similarly the surface energy is proportional to $A^{2/3}$. The Coulomb energy is proportional to Z^2/R, and so to $Z^2A^{-1/3}$.

Note that B_s arises because nucleus has a surface, where the nucleons interact with only, on the average, half as many nucleons as those in the interior, and may be considered as a correction to B_v.

B_a arises from the symmetry effect that for nuclides with mass number A, nuclei with $Z = \frac{A}{2}$ is most stable. A departure from this condition leads to instability and a smaller binding energy.

Lastly, neutrons and protons in a nucleus each have a tendency to exist in pairs. Thus nuclides with proton number and neutron number being even-even are the most stable; odd-odd, the least stable; even-odd or odd-even, intermediate in stability. This effect is accounted for by the pairing energy $B_p = a_p \delta A^{-1/2}$, where

$$\delta = \begin{cases} 1 & \text{for even-even nucleus,} \\ 0 & \text{for odd-even or even-odd nucleus,} \\ -1 & \text{for odd-odd nucleus.} \end{cases}$$

The values of the coefficients can be determined by a combination of theoretical calculations and adjustments to fit the experimental binding energy values. These have been determined to be

$$a_v = 15.835 \text{ MeV}, \qquad a_s = 18.33 \text{ MeV}, \qquad a_e = 0.714 \text{ MeV},$$

$$a_a = 92.80 \text{ MeV}, \qquad a_p = 11.20 \text{ MeV}.$$

2025

The nuclear binding energy may be approximated by the empirical expression

$$\text{B.E.} = a_1 A - a_2 A^{2/3} - a_3 Z^2 A^{-1/3} - a_4 (A - 2Z)^2 A^{-1}.$$

(a) Explain the various terms in this expression.

(b) Considering a set of isobaric nuclei, derive a relationship between A and Z for naturally occurring nuclei.

(c) Use a Fermi gas model to estimate the magnitude of a_4. You may assume $A \neq 2Z$ and that the nuclear radius is $R = R_0 A^{1/3}$.

(*Princeton*)

Solution:

(a) The terms in the expression represent volume, surface, Coulomb and symmetry energies, as explained in **Problem 2024** (where $a_a = 4a_4$).

(b) For isobaric nuclei of the same A and different Z, the stable nuclides should satisfy

$$\frac{\partial(B.E.)}{\partial Z} = -2A^{-1/3} a_3 Z + 4a_4 A^{-1}(A - 2Z) = 0,$$

giving

$$Z = \frac{A}{2 + \frac{a_3}{2a_4} A^{2/3}}.$$

With $a_3 = 0.714$ MeV, $a_4 = 23.20$ MeV,

$$Z = \frac{A}{2 + 0.0154A^{2/3}} \, .$$

(c) A fermi gas of volume V at absolute temperature $T = 0$ has energy

$$E = \frac{2V}{h^3} \cdot \frac{4\pi}{5} \cdot \frac{p_0^5}{2m}$$

and particle number

$$N = \frac{2V}{h^3} \cdot \frac{4\pi}{3} \cdot p_0^3 \, ,$$

where we have assumed that each phase cell can accommodate two particles (neutrons or protons) of opposite spins. The limiting momentum is then

$$p_0 = h \left(\frac{3}{8\pi} \cdot \frac{N}{V} \right)^{\frac{1}{3}}$$

and the corresponding energy is

$$E = \frac{3}{40} \left(\frac{3}{\pi} \right)^{\frac{2}{3}} \frac{h^2}{m} V^{-\frac{2}{3}} N^{\frac{5}{3}} \, .$$

For nucleus (A, Z) consider the neutrons and protons as independent gases in the nuclear volume V. Then the energy of the lowest state is

$$\begin{aligned} E &= \frac{3}{40} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m} \frac{N^{5/3} + Z^{5/3}}{V^{2/3}} \\ &= \frac{3}{40} \left(\frac{9}{4\pi^2} \right)^{2/3} \frac{h^2}{mR_0^2} \frac{N^{5/3} + Z^{5/3}}{A^{2/3}} \\ &= C \frac{N^{5/3} + Z^{5/3}}{A^{2/3}} \, , \end{aligned}$$

where $V = \frac{4\pi}{3} R_0^3 A$, $R_0 \approx 1.2\,\text{fm}$, $C = \frac{3}{40} \left(\frac{9}{4\pi^2} \right)^{2/3} \frac{1}{mc^2} \left(\frac{hc}{R_0} \right)^2 = \frac{3}{40} \left(\frac{9}{4\pi^2} \right)^{\frac{2}{3}}$ $\times \frac{1}{940} \left(\frac{1238}{1.2} \right)^2 = 31.7 \text{ MeV}$.

For stable nuclei, $N + Z = A$, $N \approx Z$. Let $N = \frac{1}{2} A(1 + \varepsilon/A)$, $Z = \frac{1}{2} A(1 - \varepsilon/A)$, where $\frac{\varepsilon}{A} \ll 1$. As

$$\left(1 + \frac{\varepsilon}{A} \right)^{5/3} = 1 + \frac{5\varepsilon}{3A} + \frac{5\varepsilon^2}{9A^2} + \dots \, ,$$

$$\left(1 - \frac{\varepsilon}{A} \right)^{5/3} = 1 - \frac{5\varepsilon}{3A} + \frac{5\varepsilon^2}{9A^2} - \dots \, ,$$

we have

$$N^{\frac{5}{3}} + Z^{\frac{5}{3}} \approx 2\left(\frac{A}{2}\right)^{\frac{5}{3}}\left(1 + \frac{5\varepsilon^2}{9A^2}\right)$$

and

$$E \approx 2^{-2/3}CA\left[1 + \frac{5\varepsilon^2}{9A^2}\right] = 2^{-2/3}CA + \frac{5}{9} \times 2^{-2/3}C\frac{(N-Z)^2}{A}\,.$$

The second term has the form $a_4\frac{(N-Z)^2}{A}$ with

$$a_4 = \frac{5}{9} \times 2^{-2/3}C \approx 11 \text{ MeV}\,.$$

The result is smaller by a factor of 2 from that given in **Problem 2024**, where $a_4 = a_a/4 = 23.20$ MeV. This may be due to the crudeness of the model.

2026

The greatest binding energy per nucleon occurs near ^{56}Fe and is much less for ^{238}U. Explain this in terms of the semiempirical nuclear binding theory. State the semiempirical binding energy formula (you need not specify the values of the various coefficients).

(*Columbia*)

Solution:

The semiempirical formula for the binding energy of nucleus (A, Z) is

$$B(Z,A) = B_v + B_s + B_e + B_a + B_p = a_vA - a_sA^{2/3} - a_eZ^2A^{-1/3} - a_a\left(\frac{A}{2} - Z\right)^2 A^{-1} + a_p\delta A^{-1/2}\,.$$

The mean binding energy per nucleon is then

$$\varepsilon = B/A = a_v - a_sA^{-1/3} - a_eZ^2A^{-4/3} - a_a\left(\frac{1}{2} - \frac{Z}{A}\right)^2 + a_p\delta A^{-3/2}\,.$$

Consider the five terms that contribute to ε. The contribution of the pairing energy (the last term) for the same A may be different for different combinations of Z, N, though it generally decreases with increasing A. The contribution of the volume energy, which is proportional to A, is a constant. The surface energy makes a negative contribution whose absolute value decreases with increasing A. The Coulomb energy also makes a negative contribution whose absolute value increases with A as Z and A increase together. The symmetry energy makes a negative contribution too, its absolute value increasing with A because Z/A decreases when A increases. Adding together these terms, we see that the mean binding energy increases with A at first, reaching a flat maximum at $A \sim 50$ and then decreases gradually, as shown in Fig. 2.5.

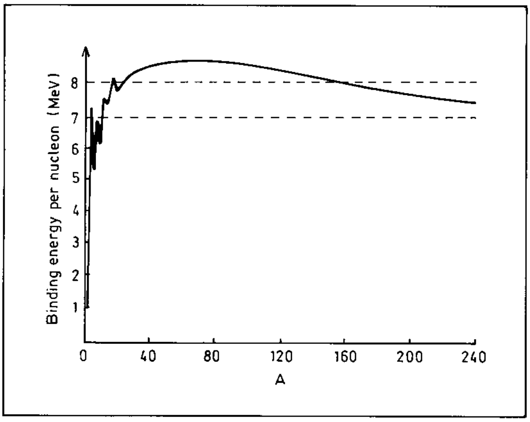

Fig. 2.5

2027

Draw a curve showing binding energy per nucleon as a function of nuclear mass. Give values in MeV, as accurately as you can. Where is the maximum of the curve? From the form of this curve explain nuclear fission and estimate the energy release per fission of ^{235}U. What force is principally responsible for the form of the curve in the upper mass region?

(*Wisconsin*)

Solution:

Figure 2.5 shows the mean binding energy per nucleon as a function of nuclear mass number A. The maximum occurs at $A \sim 50$. As A increases from 0, the curve rises sharply for $A < 30$, but with considerable fluctuations. Here the internucleon interactions have not reached saturation and there are not too many nucleons present so that the mean binding energy increases rapidly with the mass number. But because of the small number of nucleons, the pairing and symmetry effects significantly affect the mean binding energy causing it to fluctuate.

When $A > 30$, the mean binding energy goes beyond 8 MeV. As A increases further, the curve falls gradually. Here, with sufficient number of nucleons, internucleon forces become saturated and so the mean binding energy tends to saturate too. As the number of nucleons increases further, the mean binding energy decreases slowly because of the effect of Coulomb repulsion.

In nuclear fission a heavy nucleus dissociates into two medium nuclei. From the curve, we see that the products have higher mean binding energy. This excess energy is released. Suppose the fission of ^{235}U produces two nuclei of $A \sim 117$. The energy released is $235 \times (8.5 - 7.6) = 210$ MeV.

2028

Is the binding energy of nuclei more nearly proportional to $A(= N + Z)$ or to A^2? What is the numerical value of the coefficient involved (state units). How can this A dependence be understood? This implies an important property of nucleon-nucleon forces. What is it called? Why is a neutron bound in a nucleus stable against decay while a lambda particle in a hypernucleus is not?

(*Wisconsin*)

Solution:

The nuclear binding energy is more nearly proportional to A with a coefficient of 15.6 MeV. Because of the saturation property of nuclear forces, a nucleon can only interact with its immediate neighbors and hence with only a limited number of other nucleons. For this reason the binding energy is proportional to A, rather than to A^2, which would be the case if the nucleon interacts with all nucleons in the nuclues. Nuclear forces are therefore short-range forces.

The underlying cause of a decay is for a system to transit to a state of lower energy which is, generally, also more stable. A free neutron decays according to

$$n \to p + e + \bar{\nu}$$

and releases an energy

$$Q = m_n - m_p - m_e = 939.53 - 938.23 - 0.51 = 0.79 \text{ MeV}\,.$$

The decay of a bound neutron in a nucleus ^AX_N will result in a nucleus $^AX_{N-1}$. If the binding energy of $^AX_{N-1}$ is lower than that of ^AX_N and the difference is larger than 0.79 MeV, the decay would increase the system's energy and so cannot take place. Hence neutrons in many non-β-radioactive nuclei are stable. On the other hand, the decay energy of a Λ^0-particle, 37.75 MeV, is higher than the difference of nuclear binding energies between the initial and final systems, and so the Λ-particle in a hypernucleus will decay.

2029

Figure 2.5 shows a plot of the average binding energy per nucleon E vs. the mass number A. In the fission of a nucleus of mass number A_0 (mass M_0) into two nuclei A_1 and A_2 (masses M_1 and M_2), the energy released is

$$Q = M_0c^2 - M_1c^2 - M_2c^2\,.$$

Express Q in terms of $\varepsilon(A)$ and A. Estimate Q for symmetric fission of a nucleus with $A_0 = 240$.

(*Wisconsin*)

Solution:

The mass of a nucleus of mass number A is

$$M = Zm_p + (A - Z)m_n - B/c^2\,,$$

where Z is its charge number, m_p and m_n are the proton and neutron masses respectively, B is the binding energy. As $Z_0 = Z_1 + Z_2$, $A_0 = A_1 + A_2$, and so $M_0 = M_1 + M_2 + (B_1 + B_2)/c^2 - B_0/c^2$, we have

$$Q = M_0c^2 - M_1c^2 - M_2c^2 = B_1 + B_2 - B_0\,.$$

The binding energy of a nucleus is the product of the average binding energy and the mass number:

$$B = \varepsilon(A) \times A\,.$$

Hence

$$Q = B_1 + B_2 - B_0 = A_1\varepsilon(A_1) + A_2\varepsilon(A_2) - A_0\varepsilon(A_0)\,.$$

With $A_0 = 240$, $A_1 = A_2 = 120$ in a symmetric fission, we have from Fig. 2.5

$$\varepsilon(120) \approx 8.5 \text{ MeV}\,, \qquad \varepsilon(240) \approx 7.6 \text{ MeV}\,.$$

So the energy released in the fission is

$$Q = 120\varepsilon(120) + 120\varepsilon(120) - 240\varepsilon(240) \approx 216 \text{ (MeV)}\,.$$

2030

(a) Construct an energy-versus-separation plot which can be used to explain nuclear fission. Describe qualitatively the relation of the features of this plot to the liquid-drop model.

(b) Where does the energy released in the fission of heavy elements come from?

(c) What prevents the common elements heavier than iron but lighter than lead from fissioning spontaneously?

(*Wisconsin*)

Solution:

(a) Nuclear fission can be explained using the curve of specific binding energy ε vs. nuclear mass number A (Fig. 2.5). As A increases from 0, the binding energy per nucleon E, after reaching a broad maximium, decreases gradually. Within a large range of A, $\varepsilon \approx 8$ MeV/nucleon. The approximate linear dependence of the binding energy on A, which shows the saturation of nuclear forces (**Problems 2028**), agrees with the liquid-drop model.

(b) As a heavy nucleus dissociates into two medium nuclei in fission, the specific binding energy increases. The nuclear energy released is the difference between the binding energies before and after the fission:

$$Q = A_1\varepsilon(A_1) + A_2\varepsilon(A_2) - A\varepsilon(A)\,,$$

where A, A_1 and A_2 are respectively the mass numbers of the nuclei before and after fission, $\varepsilon(A_i)$ being the specific binding energy of nucleus A_i.

(c) Although the elements heavier than iron but lighter than lead can release energy in fission if we consider specific binding energies alone, the Coulomb barriers prevent them from fissioning spontaneously. This is because the fission barriers of these nuclei are so high that the probability of penetration is very small.

2031

Stable nuclei have N and Z which lie close to the line shown roughly in Fig. 2.6.

(a) Qualitatively, what features determine the shape of this curve.

(b) In heavy nuclei the number of protons is considerably less than the number of neutrons. Explain.

(c) $^{14}O(Z = 8, N = 6)$ has a lifetime of 71 sec. Give the particles in the final state after its decay.

(*Wisconsin*)

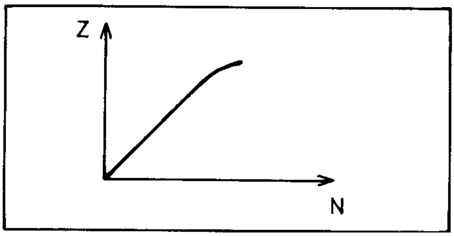

Fig. 2.6

Solution:

(a) Qualitatively, Pauli's exclusion principle allows four nucleons, 2 protons of opposite spins and 2 neutrons of opposite spins, to occupy the same energy level, forming a tightly bound system. If a nucleon is added, it would have to go to the next level and would not be so lightly bound. Thus the most stable nuclides are those with $N = Z$.

From binding energy considerations (**Problem 2025**), A and Z of a stable nuclide satisfy

$$Z = \frac{A}{2 + 0.0154A^{2/3}},$$

or, as $A = N + Z$,

$$N = Z(1 + 0.0154A^{2/3}).$$

This shows that for light nuclei, $N \approx Z$, while for heavy nuclei, $N > Z$, as shown in Fig. 2.6.

(b) For heavy nuclei, the many protons in the nucleus cause greater Coulomb repulsion. To form a stable nucleus, extra neutrons are needed to counter the Coulomb repulsion. This competes with the proton-neutron symmetry effect and causes the neutron-proton ratio in stable nuclei to increase with A. Hence the number of protons in heavy nuclei is considerably less than that of neutrons.

(c) As the number of protons in ^{14}O is greater than that of neutrons, and its half life is 71 s, the decay is a β^+ decay

$$^{14}O \to ^{14}N + e^+ + \nu_e,$$

the decay products being ^{14}N, e^+, and electron-neutrino. Another possible decay process is by electron capture. However, as the decay energy of ^{14}O is very large, ($E_{\max} > 4$ MeV), the branching ratio of electron capture is very small.

2032

The numbers of protons and neutrons are roughly equal for stable lighter nuclei; however, the number of neutrons is substantially greater than the number of protons for stable heavy nuclei. For light nuclei, the energy required to remove a proton or a neutron from the nucleus is roughly the same; however, for heavy nuclei, more energy is required to remove a proton than a neutron. Explain these facts, assuming that the specific nuclear forces are exactly equal between all pairs of nucleons.

(Columbia)

Solution:

The energy required to remove a proton or a neutron from a stable nucleus (Z, A) is

$$S_p = B(Z, A) - B(Z - 1, A - 1),$$

or

$$S_n = B(Z, A) - B(Z, A-1)\,.$$

respectively, where B is the binding energy per nucleon of a nuclues. In the liquid-drop model (**Problem 2024**), we have

$$B(Z, A) = a_v A - a_s A^{2/3} - a_c Z^2 A^{-1/3} - a_a \left(\frac{A}{2} - Z\right)^2 A^{-1} + a_p \delta A^{-1/2}\,.$$

Hence

$$S_p - S_n = -a_c(2Z-1)(A-1)^{-\frac{1}{3}} + a_a(A-2Z)(A-1)^{-1}\,,$$

where $a_c = 0.714$ MeV, $a_a = 92.8$ MeV. For stable nuclei (**Problem 2025**),

$$Z = \frac{A}{2 + \frac{2a_c}{a_a}A^{2/3}} \approx \frac{A}{2}\left(1 - \frac{a_c}{a_a}A^{2/3}\right)\,,$$

and so

$$S_p - S_n \approx \frac{a_c}{A-1}\left[A^{5/3} - (A-1)^{5/3} + \frac{a_c}{a_a}A^{5/3}(A-1)^{2/3}\right]\,.$$

For heavy nuclei, $A \gg 1$ and $S_p - S_n \approx 5.5 \times 10^{-3}\ A^{4/3}$. Thus $S_p - S_n$ increases with A, i.e., to dissociate a proton from a heavy nucleus needs more energy than to dissociate a neutron.

2033

All of the heaviest naturally-occurring radioactive nuclei are basically unstable because of the Coulomb repulsion of their protons. The mechanism by which they decrease their size is alpha-decay. Why is alpha-decay favored over other modes of disintegration (like proton-, deuteron-, or triton-emission, or fission)? Discuss briefly in terms of

(a) energy release, and

(b) Coulomb barrier to be penetrated.

(*Wisconsin*)

Solution:

(a) A basic condition for a nucleus to decay is that the decay energy is larger than zero. For heavy nuclei however, the decay energy of proton-, deuteron- or triton-emission is normally less than zero. Take the isotopes and isotones of ${}^{238}_{95}$Am as an example. Consider the ten isotopes of Am. The proton-decay energies are between −3.9 MeV and −5.6 MeV, the deuteron-decay energies are between −7.7 MeV and −9.1 MeV, the triton-decay energies are between −7.6 MeV and −8.7 MeV, while the α-decay energies are between 5.2 MeV and 6.1 MeV. For the three isotones of ${}^{238}_{95}Am$, the proton-, deuteron- and triton-decay energies are less than zero while their α-decay energies are larger than zero. The probability for fission of a heavy nucleus is less than that for α-decay also because of its much lower probability of penetrating the Coulomb barrier. Therefore α-decay is favored over other modes of disintegration for a heavy nucleus.

(b) Figure 2.7 shows the Coulomb potential energy of a nucleus of charge $Z_1 e$ and a fragment of charge $Z_2 e$.

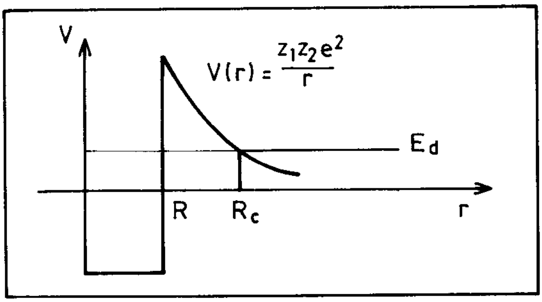

Fig. 2.7

Suppose a nucleus is to break up into two fragments of charges $Z_1 e$ and $Z_2 e$. The probability of penetrating the Coulomb barrier by a fragment of energy E_d is

$$\exp\left(-\frac{2}{\hbar}\int_R^{R_c}\left[2\mu\left(\frac{Z_1 Z_2 e^2}{r} - E_d\right)\right]^{1/2} dr\right) = \exp(-G)\,,$$

where μ is the reduced mass of the system,

$$R_c = \frac{Z_1 Z_2 e^2}{E_d}\,,$$

and

$$G = \frac{2\sqrt{2\mu E_d}}{\hbar}\int_R^{R_c}\left(\frac{R_c}{r} - 1\right)^{1/2} dr\,.$$

Integrating we have

$$\int_R^{R_c} \sqrt{\frac{R_c}{r} - 1}dr = R_c \int_1^{R_c/R} \frac{1}{p^2}\sqrt{p-1}dp$$

$$= R_c \left[-\frac{1}{p}\sqrt{p-1} + \tan^{-1}\sqrt{p-1}\right]_1^{R_c/R}$$

$$\approx R_c \left[\frac{\pi}{2} - \left(\frac{R}{R_c}\right)^{\frac{1}{2}}\right]$$

taking $\frac{R_c}{R} \gg 1$, and hence

$$G \approx \frac{2R_c\sqrt{2\mu E_d}}{\hbar}\left[\frac{\pi}{2} - \left(\frac{R}{R_c}\right)^{1/2}\right] \approx \frac{2Z_1Z_2e^2\sqrt{2\mu}}{\hbar\sqrt{E_d}}\left[\frac{\pi}{2} - \left(\frac{R}{R_c}\right)^{1/2}\right].$$

For fission, though the energy release is some 50 times larger than that of α-decay, the reduced mass is 20 times larger and Z_1Z_2 is 5 times larger. Then the value of G is 4 times larger and so the barrier penetrating probability is much lower than that for α-decay.

2034

Instability ('radioactivity') of atomic nuclei with respect to α-particle emission is a comparatively common phenomenon among the very heavy nuclei but proton-radioactivity is virtually nonexistent. Explain, with such relevant quantitative arguments as you can muster, this striking difference.

(*Columbia*)

Solution:

An explanation can be readily given in terms of the disintegration energies. In the α-decay of a heavy nucleus (A, Z) the energy release given by the liquid-drop model (**Problem 2024**) is

$$E_d = M(A,Z) - M(A-4, Z-2) - M(4,2)$$

$$= -B(A,Z) + B(A-4,Z-2) + B(4,2)$$

$$= -a_s[A^{2/3} - (A-4)^{2/3}] - a_c[Z^2A^{-\frac{1}{3}} - (Z-2)^2(A-4)^{-\frac{1}{3}}]$$

$$- a_a \left[\left(\frac{A}{2} - Z \right)^2 A^{-1} - \left(\frac{A-4}{2} - Z + 2 \right)^2 (A-4)^{-1} \right]$$

$$+ B(4,2) - 4a_v \, .$$

For heavy nuclei, $\frac{2}{Z} \ll 1$, $\frac{4}{A} \ll 1$, and the above becomes

$$E_d \approx \frac{8}{3} a_s A^{-1/3} + 4a_c Z A^{-\frac{1}{3}} \left(1 - \frac{Z}{3A} \right) - a_a \left(1 - \frac{2Z}{A} \right)^2 + 28.3 - 4a_v$$

$$= 48.88 A^{-1/3} + 2.856 Z A^{-1/3} \left(1 - \frac{Z}{3A} \right)$$

$$- 92.80 \left(1 - \frac{2Z}{A} \right)^2 - 35.04 \text{ MeV} \, .$$

For stable nuclei we have (**Problem 2025**)

$$Z = \frac{A}{2 + 0.0154 A^{2/3}} \, .$$

E_d is calculated for such nuclei and plotted as the dashed wave in Fig. 2.8.

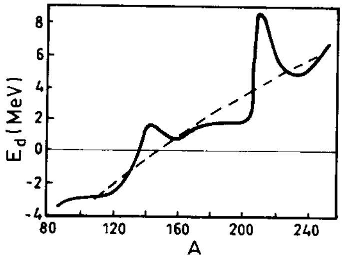

Fig. 2.8

For α-decay to take place, we require $E_d > 0$. It is seen that E_d increases generally with A and is positve when $A \geq 150$. Thus only heavy nuclei have α-decays. The actual values of E_d for naturally occurring nuclei are shown as the solid curve in the figure. It intersects the $E_d = 0$ line at $A \approx 140$, where α-radioactive isotopes $^{147}_{62}Sm$, $^{144}_{60}Nd$ are actually observed. For the proton-decay of a heavy nucleus, we have

$$\begin{aligned}
M(A,Z) - M(A-1,Z-1) - M(0,1) \\
&= -B(A,Z) + B(A-1,Z-1) + B(0,1) \\
&\approx -B(A,Z) + B(A-1,Z-1) = -\varepsilon < 0\,,
\end{aligned}$$

where ε is the specific binding energy and is about 7 MeV for heavy nuclei. As the decay energy is negative, proton-decay cannot take place. However, this consideration is for stable heavy nuclei. For those nuclei far from stability curve, the neutron-proton ratio may be much smaller so that the binding energy of the last proton may be negative and proton-emission may occur. Quite different from neutron-emission, proton-emission is not a transient process but similar to α-decay; it has a finite half-life due to the Coulomb barrier. As the proton mass is less than the α-particle mass and the height of the Coulomb barrier it has to penetrate is only half that for the α-particle, the half-life against p-decay should be much less than that against α-decay. All proton-emitters should also have β^+-radioactivity and orbital-electron capture, and their half-lives are related to the probabilities of such competing proceses. Instances of proton-radioactivity in some isomeric states have been observed experimentally.

2035

(a) Derive argument for why heavy nuclei are α-radioactive but stable against neutron-emission.

(b) What methods and arguments are used to determine nuclear radii?

(c) What are the properties that identify a system of nucleons in its lowest energy state? Discuss the nonclassical properties.

(d) The fission cross sections of the following uranium ($Z = 92$) isotopes for thermal neutrons are shown in the table below.

Isotope	σ (barns)
^{230}U	20
^{231}U	300
^{232}U	76
^{233}U	530
^{234}U	0
^{235}U	580
^{236}U	0

The fast-neutron fission cross sections of the same isotopes are all of the order of a few barns, and the even-odd periodicity is much less pronounced. Explain these facts.

(*Columbia*)

Solution:

(a) The reason why heavy nuclei only are α-radioactive has been discussed in **Problems 2033** and **2034**. For ordinary nuclei near the β-stability curve, the binding energy of the last neutron is positive so that no neutron-radioactivity exists naturally. However, for neutron-rich isotopes far from the β-stability curve, the binding energy may be negative for the last neutron, and so neutron-emission may occur spontaneously. As there is no Coulomb barrier for neutrons, emission is a transient process. Also, certain excited states arising from β-decays may emit neutrons. In such cases, as the neutron-emission follows a β-decay the emitted neutrons are called delayed neutrons. The half-life against delayed-neutron emission is the same as that against the related β-decay.

(b) There are two categories of methods for measuring nuclear radii. The first category makes use of the range of the strong interaction of nuclear forces by studying the scattering by nuclei of neutrons, protons or α-particles, particularly by measuring the total cross-section of intermediate-energy neutrons. Such methods give the nuclear radius as

$$R = R_0 A^{1/3}, \qquad R_0 \approx (1.4 \sim 1.5)\ \text{fm}\,.$$

The other category of methods makes use of the Coulomb interaction between charged particles and atomic nuclei or that among particles within a nucleus to get the electromagnetic nuclear radius. By studying the scattering between high energy electrons and atomic nuclei, the form factors of the nuclei may be deduced which gives the electromagnetic nuclear radius. Assuming mirror nuclei to be of the same structure, their mass difference is caused by Coulomb energy difference and the mass difference between neutron and proton. We have (**Problem 2010**)

$$\Delta E = \frac{3}{5}\frac{e^2}{R}(2Z-1) - (m_n - m_p)c^2$$

for the energy difference between the ground states of the mirror nuclei, which then gives the electromagnetic nuclear radius R. A more precise method is to study the deviation of μ-mesic atom from Bohr's model of hydrogen atom (**problem 1062**). Because the Bohr radius of the mesic atom is much smaller than that of the hydrogen atom, the former is more sensitive to the value of the electromagnetic nuclear radius, which, by this method, is

$$R = R_0 A^{1/3}, \qquad R_0 \approx 1.1 \text{ fm}.$$

High-energy electron scattering experiments show that charge distribution within a nucleus is nonuniform.

(c) The ground state of a system of nucleons is identified by its spin, parity and isospin quantum numbers.

Spin and parity are determined by those of the last one or two unpaired nucleons. For the ground state of an even-even nucleus, $J^p = 0^+$. For an even-odd nucleus, the nuclear spin and parity are determined by the last nucleon, and for an odd-odd nucleus, by the spin-orbit coupling of the last two nucleons.

The isospin of the nuclear ground state is $I = \frac{1}{2}|N - Z|$.

(d) There is a fission barrier of about 6 MeV for uranium so that spontaneous fission is unlikely and external inducement is required. At the same time, there is a tendency for neutrons in a nucleus to pair up so that isotopes with even numbers of neutrons, N, have higher binding energies. When an uranium isotope with an odd number of neutrons captures a neutron and becomes an isotope of even N, the excitation energy of the compound nucleus is large, sufficient to overcome the fission barrier, and fission occurs. On the other hand, when an even-N uranium isotope captures a neutron to become an isotope of odd N, the excitation energy of the compound nucleus is small, not sufficient to overcome the fission barrier, and fission does not take place. For example, in $^{235}U + n \rightarrow ^{236} U^*$ the excitation energy of the compound nucleus $^{236}\text{U}^*$ is 6.4 MeV, higher than the fission barrier of ^{236}U of 5.9 MeV, so the probability of this reaction results in a fission is large. In $^{238}U + n \rightarrow ^{239} U^*$, the excitation energy is only 4.8 MeV, lower than the fission barrier of 6.2 MeV of ^{239}U, and so the probability for fission is low. Such nuclides require neutrons of higher energies to achieve fission. When the neutron energy is higher than a certain threshold, fission cross section becomes large and fission may occur.

Thermal neutrons, which can cause fission when captured by odd-N uranium isotopes, have long wavelengths and hence large capture cross sections. Thus the cross sections for fission induced by thermal neutrons are large, in hundreds of barns, for uranium isotopes of odd N. They are small for isotope of even N.

If a fast neutron is captured by an uranium isotope the excitation energy of the compound nucleus is larger than the fission barrier and fission occurs irrespective of whether the isotope has an even or an odd number of neutrons. While fast neutrons have smaller probability of being captured their fission cross section, which is of the order of a few barns, do not change with the even-odd periodicity of the neutron number of the uranium isotope.

2036

The semiempirical mass formula modified for nuclear-shape eccentricity suggests a binding energy for the nucleus ${}^A_ZX$:

$$B = \alpha A - \beta A^{2/3}\left(1 + \frac{2}{5}\varepsilon^2\right) - \gamma Z^2 A^{-\frac{1}{3}}\left(1 - \frac{1}{5}\varepsilon^2\right),$$

where α, β, $\gamma = 14$, 13, 0.6 MeV and ε is the eccentricity.

(a) Briefly interpret this equation and find a limiting condition involving Z and A such that a nucleus can undergo prompt (unhindered) spontaneous fission. Consider ${}^{240}_{94}Pu$ as a specific example.

(b) The discovery of fission shape isomers and the detection of spontaneous fission of heavy isotopes from their ground state suggest a more complicated nuclear potential energy function $V(\varepsilon)$. What simple nuclear excitations can account for the two sets of states of ${}^{240}_{94}Pu$ shown below (Fig. 2.9). Discuss similarities and differences between the two. What are the implications for $V(\varepsilon)$? Draw a rough sketch of $V(\varepsilon)$.

(*Princeton*)

Solution:

(a) In the mass formula, the first term represents volume energy, the second term surface energy, in which the correction $\frac{2}{5}\varepsilon^2$ is for deformation from spherical shape of the nucleus, the third term, the Coulomb energy, in which the correction $\frac{1}{5}\varepsilon^2$ is also for nucleus deformation. Consequent to nuclear shape deformation, the binding energy is a function of the eccentricity ε. The limiting condition for stability is $\frac{dB}{d\varepsilon} = 0$. We have

$$\frac{dB}{d\varepsilon} = -\frac{4\beta}{5}A^{2/3}\varepsilon + \gamma\frac{Z^2}{A^{1/3}} \cdot \frac{2}{5}\varepsilon = \frac{2}{5}\varepsilon A^{2/3}\left(\frac{\gamma Z^2}{A} - 2\beta\right).$$

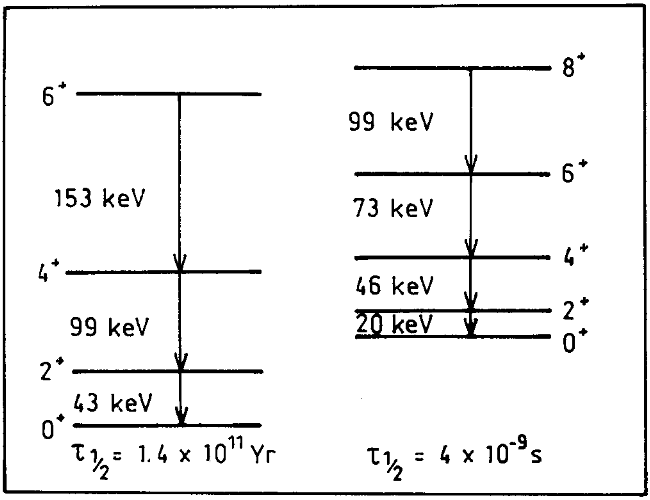

Fig. 2.9

If $\frac{dB}{d\varepsilon} > 0$, nuclear binding energy increases with ε so the deformation will keep on increasing and the nucleus becomes unstable. If $\frac{dB}{d\varepsilon} < 0$, binding energy decreases as ε increases so the nuclear shape will tend to that with a lower ε and the nucleus is stable. So the limiting condition for the nucleus to undergo prompt spontaneous fission is $\frac{d\beta}{d\varepsilon} > 0$, or

$$\frac{Z^2}{A} \geq \frac{2\beta}{\gamma} = 43.3 \,.$$

For ^{240}Pu, $\frac{Z^2}{A} = 36.8 < 43.3$ and so it cannot undergo prompt spontaneous fission; it has a finite lifetime against spontaneous fission.

(b) The two sets of energy levels of ^{240}Pu (see Fig. 2.9) can be interpreted in terms of collective rotational excitation of the deformed nucleus, as each set satisfies the rotational spectrum relation for the $K = 0$ rotational band

$$E_I = \frac{\hbar^2}{2M}[I(I+1)] \,.$$

Both sets of states show characteristics of the rotational spectrums of even-even nuclei; they differ in that the two rotational bands correspond to different rotational moments of inertia M. The given data correspond to

$\frac{\hbar^2}{2J} \approx 7$ MeV for the first set, $\frac{\hbar^2}{2J} \approx 3.3$ MeV for the second set. The different moments of inertia suggest different deformations. Use of a liquid-drop shell model gives a potential $V(\varepsilon)$ in the form of a two-peak barrier, as shown in Fig. 2.10. The set of states with the longer lifetime corresponds to the ground-state rotational band at the first minimum of the two-peak potential barrier. This state has a thicker fission barrier to penetrate and hence a longer lifetime ($T_{1/2} = 1.4 \times 10^{11}$ yr for ^{240}Pu). The set of rotational band with the shorter lifetime occurs at the second minimum of the potential barrier. In this state the fission barrier to penetrate is thinner, hence the shorter lifetime ($T_{1/2} = 4 \times 10^{-9}$s for ^{240}Pu). The difference between the two rotational bands arises from the different deformations; hence the phenomenon is referred to as nuclear shape isomerism.

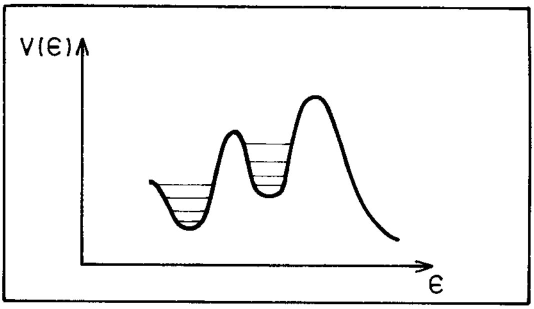

Fig. 2.10

2037

Assume a uranium nucleus breaks up spontaneously into two roughly equal parts. Estimate the reduction in electrostatic energy of the nuclei. What is the relationship of this to the total change in energy? (Assume uniform charge distribution; nuclear radius= 1.2×10^{-13} $A^{1/3}$ cm)

(Columbia)

Solution:

Uranium nucleus has $Z_0 = 92$, $A_0 = 236$, and radius $R_0 = 1.2 \times 10^{-13} A_0^{1/3}$ cm. When it breaks up into to two roughly equal parts, each part has

$$Z = \frac{1}{2}Z_0,\ A = \frac{1}{2}A_0,\ R = 1.2 \times 10^{-13}A^{1/3}\ \text{cm}\,.$$

The electrostatic energy of a sphere of a uniformly distributed charge Q is $\frac{3}{5}Q^2/R$, where R is the radius. Then for uranium fission, the electrostatic energy reduction is

$$\begin{aligned}
\Delta E &= \frac{3}{5}\left[\frac{(Z_0e)^2}{R_0} - 2 \times \frac{(Ze)^2}{R}\right] \\
&= \frac{3 \times Z_0^2 e^2}{5}\frac{1}{R_0}\left[1 - \frac{1}{2^{2/3}}\right] = 0.222 \times \frac{Z_0^2}{R_0}\left(\frac{e^2}{\hbar c}\right)\hbar c \\
&= \frac{0.222 \times 92^2}{1.2 \times 10^{-13} \times 236^{\frac{1}{3}}} \times \frac{1}{137} \times 1.97 \times 10^{-11} \\
&= 364\ \text{MeV}\,.
\end{aligned}$$

This reduction is the source of the energy released in uranium fission. However, to calculate the actual energy release, some other factors should also be considered such as the increase of surface energy on fission.

2038

Estimate (order of magnitude) the ratio of the energy released when 1 g of uranium undergoes fission to the energy released when 1 g of TNT explodes.

(*Columbia*)

Solution:

Fission is related to nuclear forces whose interaction energy is about 1 MeV/nucleon. TNT explosion is related to electromagnetic forces whose interaction energy is about 1 eV/molecule. As the number of nucleons in 1 g of uranium is of the same order of magnitude as the number of molecules in 1 g of TNT, the ratio of energy releases should be about 10^6.

A more precise estimate is as follows. The energy released in the explosion of 1 g of TNT is about 2.6×10^{22} eV. The energy released in the fission of a uranium nucleus is about 210 MeV. Then the fission of 1 g of uranium releases an energy $\frac{6.023\times10^{23}}{238} \times 210 = 5.3 \times 10^{23}$ MeV. Hence the ratio is about 2×10^7.

2039

The neutron density $\rho(\mathbf{x}, t)$ inside a block of U^{235} obeys the differential equation

$$\frac{\partial \rho(\boldsymbol{x}, t)}{\partial t} = A\nabla^2 \rho(\mathbf{x}, t) + B\rho(\mathbf{x}, t)\,,$$

where A and B are positive constants. Consider a block of U^{235} in the shape of a cube of side L. Assume that those neutrons reaching the cube's surface leave the cube immediately so that the neutron density at the U^{235} surface is always zero.

(a) Briefly describe the physical processes which give rise to the $A\nabla^2\rho$ and the $B\rho$ terms. In particular, explain why A and B are both positive.

(b) There is a critical length L_0 for the sides of the U^{235} cube. For $L > L_0$, the neutron density in the cube is unstable and increases exponentially with time — an explosion results. For $L < L_0$, the neutron density decreases with time — there is no explosion. Find the critical length L_0 in terms of A and B.

(*Columbia*)

Solution:

(a) The term $B\rho(\mathbf{x}, t)$, which is proportional to the neutron density, accounts for the increase of neutron density during nuclear fission. $B\rho(\mathbf{x}, t)$ represents the rate of increase of the number of neutrons, in a unit volume at location $\mathbf{x}$ and at time t, caused by nuclear fission. It is proportional to the number density of neutrons which induce the fission. As the fission of U^{235} increases the neutron number, B is positive. The term $A\nabla^2\rho(\mathbf{x}, t)$ describes the macroscopic motion of neutrons caused by the nonuniformity of neutron distribution. As the neutrons generally move from locations of higher density to locations of lower density, A is positive too.

(b) Take a vertex of the cube as the origin, and its three sides as the x-, y- and z-axes. Let $\rho(\mathbf{x}, t) = f(x, y, z)e^{-\alpha t}$. Then the differential equation becomes

$$A\nabla^2 f(x, y, z) + (\alpha + B) f(x, y, z) = 0$$

with boundary condition

$$f(x, y, z)|_{i=0,L} = 0\,, \qquad i = x, y, z\,.$$

Try a solution of the form $f = X(x)Y(y)Z(z)$. Substitution gives

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k_x^2 + k_y^2 + k_z^2 = 0\,,$$

where we have rewritten $\frac{\alpha+B}{A} = k_x^2 + k_y^2 + k_z^2$. The boundary condition becomes

$$X(x) = 0 \text{ at } x = 0, L;\ Y(y) = 0 \text{ at } y = 0, L;\ Z(z) = 0 \text{ at } z = 0, L\,.$$

The last differentiation equation can be separated into 3 equations:

$$\frac{d^2X}{dx^2} + k_x^2 X = 0\,, \quad \text{etc.}$$

The solutions of these equations are

$$X = C_{xi}\sin\left(\frac{n_{xi}\pi}{L}x\right)\,,$$

$$Y = C_{yj}\sin\left(\frac{n_{yj}\pi}{L}y\right)\,,$$

$$Z = C_{zk}\sin\left(\frac{n_{zk}\pi}{L}x\right)\,,$$

with $n_{xi}, n_{yj}, n_{zk} = \pm 1, \pm 2, \pm 3 \ldots$ and C_{xi}, C_{yj}, C_{zk} being arbitrary constants. Thus

$$f(x,y,z) = \sum_{ijk} C_{ijk}\sin\left(\frac{n_{xi}\pi}{L}x\right)\sin\left(\frac{n_{yj}\pi}{L}y\right)\sin\left(\frac{n_{zk}\pi}{L}z\right)\,,$$

with

$$\frac{\alpha+B}{A} = \left(\frac{\pi}{L}\right)^2 (n_{xi}^2 + n_{yj}^2 + n_{zk}^2)\,, \qquad C_{ijk} = C_{zi}C_{yi}C_{zk}\,.$$

If $\alpha < 0$, the neutron density will increase exponentially with time, leading to instability and possible explosion. Hence the critial length L_0 is given by

$$\alpha = \frac{A\pi^2}{L_0^2}(n_{xi}^2 + n_{yj}^2 + n_{zk}^2) - B = 0\,,$$

or

$$L_0 = \pi\sqrt{\frac{A}{B}(n_{xi}^2 + n_{yj}^2 + n_{zk}^2)}\,.$$

In particular, for $n_{xi} = n_{yj} = n_{zk} = 1$,

$$L_0 = \pi\sqrt{\frac{3A}{B}}\,.$$

2040

The half-life of U^{235} is $10^3, 10^6, 10^9, 10^{12}$ years.

(*Columbia*)

Solution:

10^9 years. (Half-life of U^{235} is 7×10^8 years)

2041

Number of fission per second in a 100-MW reactor is: 10^6, 10^{12}, 10^{18}, 10^{24}, 10^{30}.

(*Columbia*)

Solution:

Each fission of uranium nucleus releases about 200 MeV $= 320\times10^{-13}$ J. So the number of fissions per second in a 100-MW reactor is

$$N = \frac{100 \times 10^6}{320 \times 10^{-13}} = 3 \times 10^{18}\,.$$

Hence the answer is 10^{18}.

2042

Explain briefly the operation of a "breeder" reactor. What physical constant of the fission process is a prerequisite to the possibility of "breeding"? What important constraint is placed on the choice of materials in the reactor? In particular, could water be used as a moderator?

(*Wisconsin*)

Solution:

A breeder reactor contains a fissionable material and a nonfissionable one that can be made fissionable by absorbing a neutron. For example,

^{235}U and ^{238}U. Suppose 3 neutrons are emitted per fission. One is needed to induce a fission in another fuel atom and keep the chain reaction going. If the other two neutrons can be used to convert two nonfissionable atoms into fissionable ones, then two fuel atoms are produced when one is consumed, and the reactor is said to be a breeder.

In the example, neutrons from the fission of ^{235}U may be used to convert ^{238}U to fissionable ^{239}Pu:

$$\begin{aligned} n + {}^{238}U \to {}^{239}U &+ \gamma \\ &\hookrightarrow_{\beta^-} {}^{239}Np \underset{\beta^-}{\to} {}^{239}Pu \end{aligned}$$

A prerequisite to breeding is that η, the number of neutrons produced per neutron absorbed in the fuel, should be larger than 2. In the example, this is achieved by the use of fast neutrons and so no moderator is needed.

2043

(a) Describe briefly the type of reaction on which a nuclear fission reactor operates.

(b) Why is energy released, and roughly how much per reaction?

(c) Why are the reaction products radioactive?

(d) Why is a "moderator" necessary? Are light or heavy elements preferred for moderators, and why?

(*Wisconsin*)

Solution:

(a) In nuclear fission a heavy nucleus disassociates into two medium nuclei. In a reactor the fission is induced. It takes place after a heavy nucleus captures a neutron. For example

$$n +^{235} U \to X + Y + n + \cdots .$$

(b) The specific binding energy of a heavy nucleus is about 7.6 MeV per nucleon, while that of a medium nucleus is about 8.5 MeV per nucleon. Hence when a fission occurs, some binding energies will be released. The energy released per fission is about 210 MeV.

(c) Fission releases a large quantity of energy, some of which is in the form of excitation energies of the fragments. Hence fission fragments are in general highly excited and decay through γ emission. In addition, the neutron-to-proton ratios of the fragments, which are similar to that of the original heavy nucleus, are much larger than those of stable nuclei of the same mass. So the fragments are mostly unstable neutron-rich isotopes having strong β^- radioactivity.

(d) For reactors using ^{235}U, fission is caused mainly by thermal neutrons. However, fission reaction emits fast neutrons; so some moderator is needed to reduce the speed of the neutrons. Lighter nuclei are more suitable as moderator because the energy lost by a neutron per neutron-nucleus collision is larger if the nucleus is lighter.

2044

Give the three nuclear reactions currently considered for controlled thermonuclear fusion. Which has the largest cross section? Give the approximate energies released in the reactions. How would any resulting neutrons be used?

(*Wisconsin*)

Solution:

Reactions often considered for controlled thermonuclear fusion are

$$D + D \to {}^3\text{He} + n + 3.25 \text{ MeV},$$

$$D + D \to T + p + 4.0 \text{ MeV},$$

$$D + T \to {}^4\text{He} + n + 17.6 \text{ MeV}.$$

The cross section of the last reaction is the largest.

Neutrons resulting from the reactions can be used to induce fission in a fission-fusion reactor, or to take part in reactions like $^6\text{Li} + n \to {}^4\text{He} + T$ to release more energy.

2045

Discuss thermonuclear reactions. Give examples of reactions of importance in the sun, the H bomb and in controlled fusion attempts. Estimate roughly in electron volts the energy release per reaction and give the characteristic of nuclear forces most important in these reactions.

(*Wisconsin*)

Solution:

The most important thermonuclear reactions in the sun are the proton-proton chain

$$p + p \to d + e^+ + \nu_e \,,$$

$$d + p \to {}^3\text{He} + \gamma \,,$$

$${}^3\text{He} + {}^3\text{He} \to {}^4\text{He} + 2p \,,$$

the resulting reaction being

$$4p + 2d + 2p + 2^3\text{He} \to 2d + 2e^+ + 2\nu_e + 2^3\text{He} + {}^4\text{He} + 2p \,,$$

or

$$4p \to {}^4\text{He} + 2e^+ + 2\nu_e \,.$$

The energy released in this reaction is roughly

$$Q = [4M({}^1\text{H}) - M({}^4\text{He})]c^2 = 4 \times 1.008142 - 4.003860$$

$$= 0.02871 \text{ amu} = 26.9 \text{ MeV} \,.$$

The explosive in a H bomb is a mixture of deuterium, tritium and lithium in some condensed form. H bomb explosion is an uncontrolled thermonuclear reaction which releases a great quantity of energy at the instant of explosion. The reaction chain is

$${}^6\text{Li} + n \to {}^4\text{He} + t \,,$$

$$D + t \to {}^4\text{He} + n \,,$$

with the resulting reaction

$${}^6\text{Li} + d \to 2^4\text{He} \,.$$

The energy released per reaction is

$$Q = [M({}^6\text{Li}) + M({}^2\text{H}) - 2M({}^4\text{He})]c^2$$

$$= 6.01690 + 2.01471 - 2 \times 4.00388$$

$$= 0.02385 \text{ amu} = 22.4 \text{ MeV} \,.$$

An example of possible controlled fusion is

$$t + d \to {}^4\text{He} + n\,,$$

where the energy released is

$$\begin{aligned} Q &= [M(^3\text{H}) + M(^2\text{H}) - M(^4\text{He}) - M(n)]c^2 \\ &= 3.01695 + 2.01471 - 4.00388 - 1.00896 \\ &= 0.01882 \text{ amu} = 17.65 \text{ MeV}\,. \end{aligned}$$

The most important characteristic of nuclear forces in these reactions is saturation, which means that a nucleon interacts only with nucleons in its immediate neighborhood. So while the nuclear interactions of a nucleon in the interior of a nucleus are saturated, the interactions of a nucleon on the surface of the nucleus are not. Then as the ratio of the number of nucleons on the nucleus surface to that of those in the interior is larger for lighter nuclei, the mean binding energy per nucleon for a lighter nucleus is smaller than for a heavier nucleus. In other words nucleons in lighter nuclei are combined more loosely. However, because of the effect of the Coulomb energy of the protons, the mean binding energies of very heavy nuclei are less than those of medium nuclei.

2046

For some years now, R. Davis and collaborators have been searching for solar neutrinos, in a celebrated experiment that employs as detector a large tank of C_2Cl_4 located below ground in the Homestake mine. The idea is to look for argon atoms (A^{37}) produced by the inverse β-decay reaction $Cl^{37}(\nu, e^-)Ar^{37}$. This reaction, owing to threshold effects, is relatively insensitive to low energy neutrinos, which constitute the expected principal component of neutrinos from the sun. It is supposed to respond to a smaller component of higher energy neutrinos expected from the sun. The solar constant (radiant energy flux at the earth) is $\sim 1\ kW/m^2$.

(a) Outline the principal sequence of nuclear processes presumed to account for energy generation in the sun. What is the slow link in the chain? Estimate the mean energy of the neutrinos produced in this chain.

What is the expected number flux at the earth of the principal component of solar neutrinos?

(b) Outline the sequence of minor nuclear reactions that is supposed to generate the higher energy component of the neutrino spectrum, the component being looked for in the above experiment. Briefly discuss the experiment itself, and the findings to date.

(*Princeton*)

Solution:

(a) The principal sequence of nuclear processes presumed to generate solar energy is

(1) $p + p \to d + e^+ + \nu_e \,, E_\nu = 0 - 0.42$ MeV ,

(2) $d + p \to {}^3\text{He} + \gamma \,,$

(3) ${}^3\text{He} + {}^3\text{He} \to {}^4\text{He} + 2p \,,$

The resulting reaction being $4p \to {}^4\text{He} + 2e^+ + 2\nu_e + 26.7$ MeV.

The reaction (1) is the slow link. About 25 MeV of the energy changes into thermal energy in the sequence, the rest being taken up by the neutrinos. So the mean energy of a neutrino is

$$E_\nu \approx (26.7 - 25)/2 \approx 0.85 \text{ MeV} \,.$$

As each 25 MeV of solar energy arriving on earth is accompanied by 2 neutrions, the number flux of solar neutrinos at the earth is

$$I = 2\left(\frac{1 \times 10^3}{25 \times 1.6 \times 10^{-13}}\right) = 5 \times 10^{14} \; m^{-2} s^{-1} \,.$$

(b) The minor processes in the sequence are

(1) ${}^3\text{He} + {}^4\text{He} \to {}^7Be + \gamma \,,$

(2) ${}^7Be + e^- \to {}^7\text{Li} + \nu_e$, $E_\nu = 0.478$ MeV(12%) and 0.861 MeV (88%),

(3) ${}^7\text{Li} + p \to 2{}^4\text{He}$,

(4) ${}^7Be + p \to {}^8B + \gamma$,

(5) ${}^8B \to 2{}^4\text{He} + e^+ + \nu_e$, $E_\nu \approx 0 \sim 17$ MeV.

The high energy neutrinos produced in the ${}^8B$ decay are those being measured in the experiment

In the experiment of Davis et al, a tank of 390000 liters of C_2Cl_4 was placed in a mine 1.5 kilometers below ground, to reduce the cosmic-ray background. The threshold energy for the reaction between solar neutrino and Cl, $\nu_e + {}^{37}Cl \to e^- + {}^{37}Ar$, is 0.814 MeV. The Ar gas produced then decays by electron capture, $e^- + {}^{37}Ar \to \nu_e + {}^{37}Cl$, the energy of the Auger electron emitted following this process being 2.8 keV. The half-life of Ar against the decay is 35 days. When the Ar gas produced, which had accumulated in the tank for several months, was taken out and its radioactivity measured with a proportional counter, the result was only one-third of what had been theoretically expected. This was the celebrated case of the "missing solar neutrinos". Many possible explanations have been proposed, such as experimental errors, faulty theories, or "neutrinos oscillation", etc.

2047

In a crude, but not unreasonable, approximation, a neutron star is a sphere which consists almost entirely of neutrons which form a nonrelativistic degenerate Fermi gas. The pressure of the Fermi gas is counterbalanced by gravitational attraction.

(a) Estimate the radius of such a star to within an order of magnitude if the mass is 10^{33} g. Since only a rough numerical estimate is required, you need to make only reasonable simplifying assumptions like taking a uniform density, and estimate integrals you cannot easily evaluate, etc. (Knowing the answer is not enough here; you must derive it.)

(b) In the laboratory, neutrons are unstable, decaying according to $n \to p+e+\nu+1$ MeV with a lifetime of 1000 s. Explain briefly and qualitatively, but precisely, why we can consider the neutron star to be made up almost entirely of neutrons, rather than neutrons, protons, and electrons.

(*Columbia*)

Solution:

(a) Let R be the radius of the neutron star. The gravitational potential energy is

$$V_g = -\int_0^R \frac{4}{3}\pi r^3 \rho \left(\frac{G}{r}\right) 4\pi r^2 \rho dr = -\frac{3}{5}\frac{GM^2}{R},$$

where $\rho = \frac{3M}{4\pi R^3}$ is the density of the gas, M being its total mass, G is the gravitational constant. When R increases by ΔR, the pressure P of the gas does an external work $\Delta W = P\Delta V = 4\pi PR^2\Delta R$. As $\Delta W = -\Delta V_g$, we have

$$P = \frac{3GM^2}{20\pi R^4} \, .$$

The pressure of a completely degenerate Fermi gas is

$$P = \frac{2}{5} N E_f \, ,$$

where $N = \frac{\rho}{M_n}$ is the neutron number density, M_n being the neutron mass,

$$E_f = \frac{\hbar^2}{2M_n} \left(\frac{9\pi}{4} \frac{M}{M_n R^3} \right)^{2/3}$$

is the limiting energy. Equating the expressions for P gives

$$\begin{aligned} R &= \left(\frac{9\pi}{4} \right)^{\frac{2}{3}} \frac{\hbar^2}{G M_n^3} \left(\frac{M_n}{M} \right)^{\frac{1}{3}} \\ &= \left(\frac{9\pi}{4} \right)^{\frac{2}{3}} \times \frac{(1.05 \times 10^{-34})^2}{6.67 \times 10^{-11} \times (1.67 \times 10^{-27})^3} \times \left(\frac{1.67 \times 10^{-27}}{10^{30}} \right)^{\frac{1}{3}} \\ &= 1.6 \times 10^4 \ m \, . \end{aligned}$$

(b) Let d be the distance between neighboring neutrons. As $\frac{M}{M_n} \approx \left(\frac{2R}{d}\right)^3$, $d \approx 2R \left(\frac{M_n}{M}\right)^{\frac{1}{3}} = 4 \times 10^{-15}$ m. If electrons existed in the star, the magnitude of their mean free path would be of the order of d, and so the order of magnitude of the kinetic energy of an electron would be $E \approx cp \sim c\hbar/d \sim 50$ MeV. Since each neutron decay only gives out 1 MeV, and the neutron's kinetic energy is less than $E_f \approx 21$ MeV, it is unlikely that there could be electrons in the neutron star originating from the decay of neutrons, if energy conservation is to hold. Furthermore, because the neutrons are so close together, e and p from a decay would immediately recombine. Thus there would be no protons in the star also.

3. THE DEUTERON AND NUCLEAR FORCES (2048–2058)

2048

If the nuclear force is charge independent and a neutron and a proton form a bound state, then why is there no bound state for two neutrons? What information does this provide on the nucleon-nucleon force?

(*Wisconsin*)

Solution:

A system of a neutron and a proton can form either singlet or triplet spin state. The bound state is the triplet state because the energy level of the singlet state is higher. A system of two neutrons which are in the same energy level can form only singlet spin state, and no bound state is possible. This shows the spin dependency of the nuclear force.

2049

A deuteron of mass M and binding energy $B(B \ll Mc^2)$ is disintegrated into a neutron and a proton by a gamma ray of energy E_γ. Find, to lowest order in B/Mc^2, the minimum value of $(E_\gamma - B)$ for which the reaction can occur.

(*Wisconsin*)

Solution:

In the disintegration of the deuteron, $E_\gamma - B$ is smallest when E_γ is at threshold, at which the final particles are stationary in the center-of-mass system. In this case the energy of the incident photon in the center-of-mass system of the deuteron is $E^* = (m_n + m_p)c^2$.

Let M be the mass of the deuteron. As $E^2 - p^2c^2$ is Lorentz-invariant and $B = (m_n + m_p - M)c^2$, we have

$$(E_\gamma + Mc^2) - E_\gamma^2 = (m_n + m_p)^2 c^4 \,,$$

i.e.,

$$2E_\gamma Mc^2 = [(m_n + m_p)^2 - M^2]c^4 = (B + 2Mc^2)B \,,$$

or

$$E_\gamma - B = \frac{B^2}{2Mc^2} \,,$$

which is the minimum value of $E_\gamma - B$ for the reaction to occur.

2050

According to a simple-minded picture, the neutron and proton in a deuteron interact through a square well potential of width $b = 1.9 \times 10^{-15}$ m and depth $V_0 = 40$ MeV in an $l = 0$ state.

(a) Calculate the probability that the proton moves within the range of the neutron. Use the approximation that $m_n = m_p = M$, $kb = \frac{\pi}{2}$, where $k = \sqrt{\frac{M(V_0-\varepsilon)}{\hbar^2}}$ and ε is the binding energy of the deuteron.

(b) Find the mean-square radius of the deuteron.

(*SUNY, Buffalo*)

Solution:

The interaction may be considered as between two particles of mass M, so the reduced mass is $\mu = \frac{1}{2}M$. The potential energy is

$$V(r) = \begin{cases} -V_0, & r < b, \\ 0, & r > b, \end{cases}$$

where r is the distance between the proton and the neutron. The system's energy is $E = -\varepsilon$.

For $l = 0$ states, let the wave function be $\Psi = u(r)/r$. The radial Schrödinger equation

$$u'' + \frac{2\mu}{\hbar^2}(E - V)u = 0$$

can be written as

$$u'' + k^2 u = 0\,, \qquad r \le b\,,$$

$$u'' - k_1^2 u = 0\,, \qquad r > b\,,$$

where

$$k = \sqrt{\frac{M(V_0 - \varepsilon)}{\hbar^2}}\,,$$

$$k_1 = \sqrt{\frac{M\varepsilon}{\hbar^2}}\,.$$

With the boundary condition $\psi = 0$ at $r = 0$ and $\psi =$ finite at $r = \infty$, we get $u(r) = A\sin(kr)$, $r \le b$; $Be^{-k_1(r-b)}$, $r > b$.

The continuity of $\psi(r)$ and that of $\psi'(r)$ at $r = b$ require

$$A\sin(kb) = B\,,$$

$$kA\cos(kb) = -k_1 B\,,$$

which give

$$\cot(kb) = -\frac{k_1}{k} = -\sqrt{\frac{\varepsilon}{V_0 - \varepsilon}}\,.$$

If we take the approximation $kb = \frac{\pi}{2}$, then $A \approx B$ and $\cot(kb) \approx 0$. The latter is equivalent to assuming $V_0 \gg \varepsilon$, which means there is only one found state.

To normalize, consider

$$\begin{aligned} 1 &= \int_0^\infty |\psi(r)|^2 4\pi r^2 dr \\ &= 4\pi A^2 \int_0^b \sin^2(kr)dr + 4\pi B^2 \int_b^\infty e^{-2k_1(r-b)} d\gamma \\ &\approx 2\pi A^2 b \left(1 + \frac{1}{bk_1}\right)\,. \end{aligned}$$

Thus

$$A \approx B \approx \left[2\pi b \left(1 + \frac{1}{bk_1}\right)\right]^{-\frac{1}{2}}\,.$$

(a) The probability of the proton moving within the range of the force of the neutron is

$$P = 4\pi A^2 \int_0^b \sin^2(kr)dr = \left(1 + \frac{1}{k_1 b}\right)^{-1}\,.$$

As

$$k = \frac{\sqrt{M(V_0 - \varepsilon)}}{\hbar} \approx \frac{\pi}{2b}\,,$$

i.e.

$$\begin{aligned} \varepsilon &\approx V_0 - \frac{1}{Mc^2}\left(\frac{\pi\hbar c}{2b}\right)^2 \\ &= 40 - \frac{1}{940}\left(\frac{\pi \times 1.97 \times 10^{-13}}{2 \times 1.9 \times 10^{-15}}\right)^2 = 11.8 \text{ MeV}\,, \end{aligned}$$

and

$$k_1 = \frac{\sqrt{Mc^2\varepsilon}}{\hbar c} = \frac{\sqrt{940 \times 11.8}}{1.97 \times 10^{-13}} = 5.3 \times 10^{14}\ m^{-1}\,,$$

we have

$$P = \left(1 + \frac{1}{5.3 \times 10^{14} \times 1.9 \times 10^{-15}}\right)^{-1} = 0.50\,.$$

(b) The mean-square radius of the deuteron is

$$\begin{aligned}
\overline{r^2} &= \langle \Psi | r^2 | \Psi \rangle_{r<b} + \langle \Psi | r^2 | \Psi \rangle_{r>b} \\
&= 4\pi A^2 \left[\int_0^b \sin^2(kr) r^2 dr + \int_b^\infty e^{-2k_1(r-b)} r^2 dr\right] \\
&= \frac{b^2}{1 + \frac{1}{k_1 b}} \left[\left(\frac{1}{3} + \frac{4}{\pi^2}\right) + \frac{1}{k_1 b} + \frac{1}{(k_1 b)^2} + \frac{1}{2(k_1 b)^3}\right] \\
&\approx \frac{b^2}{2}\left(\frac{1}{3} + \frac{4}{\pi^2} + 2.5\right) = 5.8 \times 10^{-30} m^2\,.
\end{aligned}$$

Hence

$$(\overline{r^2})^{\frac{1}{2}} = 2.4 \times 10^{-15}\ m\,.$$

2051

(a) A neutron and a proton can undergo radioactive capture at rest: $p + n \to d + \gamma$. Find the energy of the photon emitted in this capture. Is the recoil of the deuteron important?

(b) Estimate the energy a neutron incident on a proton at rest must have if the radioactive capture is to take place with reasonable probability from a p-state ($l = 1$). The radius of the deuteron is $\sim 4 \times 10^{-13}$ cm.

$m_p = 1.00783$ amu, $m_n = 1.00867$ amu, $m_d = 2.01410$ amu, 1 amu $= 1.66 \times 10^{-24}$ g $= 931$ MeV, 1 MeV $= 1.6 \times 10^{-13}$ joule $= 1.6 \times 10^{-6}$ erg, $\hbar = 1.05 \times 10^{-25}$ erg.s.

(*Wisconsin*)

Solution:

(a) The energy released in the radioactive capture is

$$Q = [m_p + m_n - m_d]c^2 = 1.00783 + 1.00867 - 2.01410 \text{ amu} = 2.234 \text{ MeV}\,.$$

This energy appears as the kinetic energies of the photon and recoil deuteron. Let their respective momenta be $\boldsymbol{p}$ and $-\boldsymbol{p}$. Then

$$Q = pc + \frac{p^2}{2m_d}\,,$$

or

$$(pc)^2 + 2m_d c^2 (pc) - 2m_d c^2 Q = 0\,.$$

Solving for pc we have

$$pc = m_d c^2 \left(-1 + \sqrt{1 + \frac{2Q}{m_d c^2}}\right)\,.$$

As $Q/m_d c^2 \ll 1$, we can take the approximation

$$p \approx m_d c \left(-1 + 1 + \frac{Q}{m_d c^2}\right) \approx \frac{Q}{c}\,.$$

Thus the kinetic energy of the recoiling deuteron is

$$E_{\text{recoil}} = \frac{p^2}{2m_d} = \frac{Q^2}{2m_d c^2} = \frac{2.234^2}{2 \times 2.0141 \times 931} = 1.33 \times 10^{-3}\ \text{MeV}\,.$$

Since

$$\frac{\Delta E_{\text{recoil}}}{E_\gamma} = \frac{1.34 \times 10^{-3}}{2.234} = 6.0 \times 10^{-4}\,,$$

the recoiling of the deuteron does not significantly affect the energy of the emitted photon, its effect being of the order 10^{-4}.

(b) Let the position vectors of the neutron and proton be $\mathbf{r}_1$, $\mathbf{r}_2$ respectively. The motion of the system can be treated as that of a particle of mass $\mu = \frac{m_p m_n}{m_p + m_n}$, position vector $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, having momentum $\mathbf{p}' = \mu\dot{\mathbf{r}}$ and kinetic energy $T' = \frac{p'^2}{2\mu}$ in the center-of-mass frame. The laboratory energy is

$$T = T' + \frac{1}{2}(m_p + m_n)\dot{\mathbf{R}}^2\,,$$

where $\dot{\mathbf{R}} = (m_n \dot{\mathbf{r}}_1 + m_p \dot{\mathbf{r}}_2)/(m_n + m_p)$.

To a good approximation we can take $m_p \simeq m_n$. Initially $\dot{\mathbf{r}}_2 = 0$, so that $\dot{\mathbf{R}} = \frac{1}{2}\dot{\mathbf{r}}_1$, $T = \frac{m_n}{2}\dot{\mathbf{r}}_1^2 = \frac{p^2}{2m_n}$, where $\mathbf{p} = m_n \dot{\mathbf{r}}_1$ is the momentum of the neutron in the laboratory. Substitution in the energy equation gives

$$\frac{p^2}{2m_n} = \frac{p'^2}{m_n} + \frac{p^2}{4m_n}\,,$$

or

$$p^2 = 4p'^2 .$$

The neutron is captured into the p-state, which has angular momentum eigenvalue $\sqrt{1(1+1)}\hbar$. Using the deuteron radius a as the radius of the orbit, we have $p'a \approx \sqrt{2}\hbar$ and hence the kinetic energy of the neutron in the laboratory

$$T = \frac{p^2}{2m_n} = \frac{2p'^2}{m_n} = \frac{4}{m_n c^2}\left(\frac{\hbar c}{a}\right)^2 = \frac{4}{940}\left(\frac{1.97 \times 10^{-11}}{4 \times 10^{-13}}\right)^2 = 10.32 \text{ MeV} .$$

2052

Consider the neutron-proton capture reaction leading to a deuteron and photon, $n + p \to d + \gamma$. Suppose the initial nucleons are unpolarized and that the center of mass kinetic energy T in the initial state is very small (thermal). Experimental study of this process provides information on s-wave proton-neutron scattering, in particular on the singlet scattering length a_s. Recall the definition of scattering length in the terms of phase shift: $k \cot\delta \to -1/a_s$, as $k \to 0$. Treat the deuteron as being a pure s-state .

(a) Characterize the leading multipolarity of the reaction (electric dipole? magnetic dipole? etc.?). Give your reason.

(b) Show that the capture at low energies occurs from a spin singlet rather than spin triplet initial state.

(c) Let B be the deuteron binding energy and let $m = m_p = m_n$ be the nucleon mass. How does the deuteron spatial wave function vary with neutron-proton separation r for large r?

(d) In the approximation where the neutron-proton force is treated as being of very short range, the cross section σ depends on T, B, a_s, m and universal parameters in the form $\sigma = \sigma_0(T, B, m) f(a_s, B, m)$, where f would equal unity if $a_s = 0$. Compute the factor f for $a_s \neq 0$.

(Princeton)

Solution:

(a) As the center-of-mass kinetic energy of the $n-p$ system is very small, the only reaction possible is s-wave capture with $l = 0$. The possible initial states are $^1S_0$ state: $\mathbf{s}_p + \mathbf{s}_n = 0$. As $P(^1S_0) = 1$, we have $J^p = 0^+$; $^3S_1$ state: $\mathbf{s}_p + \mathbf{s}_n = 1$. As $P(^3S_1) = 1$, we have $J^p = 1^+$. The final state is a deuteron, with $J^p = 1^+$, and thus $S = 1$, $l = 0, 2$ (**Problem 2058(b)**). The initial states have $l = 0$. Hence there are two possible transitions with $\Delta l = 0, 2$ and no change of parity. Therefore the reactions are of the $M1, E2$ types.

(b) Consider the two transitions above: $^1S_0 \to {}^3S_1$, and $^3S_1 \to {}^3S_1$. As both the initial and final states of each case have $l = 0$, only those interaction terms involving spin in the Hamiltonian can cause the transition. For such operators, in order that the transition matrix elements do not vanish the spin of one of the nucleons must change during the process. Since

$$\text{for} \quad {}^3S_1 \to {}^3S_1, \qquad \Delta l = 0, \qquad \Delta S = 0\,,$$

$$\text{for} \quad {}^1S_0 \to {}^3S_1, \qquad \Delta l = 0, \qquad \Delta S \neq 0\,,$$

the initial state which satisfies the transition requirement is the spin-singlet $^1S_0$ state of the $n - p$ system.

(c) Let the range of neutron-proton force be a. The radial part of the Schrödinger equation for the system for s waves is

$$\frac{d^2u}{dr^2} + \frac{2\mu}{\hbar^2}(T - V)u = 0\,,$$

where $u = rR(r), R(r)$ being the radial spatial wave function, $\mu = \frac{m}{2}$, and V can be approximated by a rectangular potential well of depth B and width a:

$$V = \begin{cases} -B & \text{for} \quad 0 \leq r \leq a\,, \\ 0 & \text{for} \quad a < r\,. \end{cases}$$

The solution for large r gives the deuteron spatial wave function as

$$R(r) = \frac{A}{r}\sin(kr + \delta)$$

where $k = \frac{\sqrt{mT}}{\hbar}$, A and δ are constants.

(d) The solutions of the radial Schrödinger equation for s waves are

$$u = \begin{cases} A\sin(kr + \delta), & \text{with} \quad k = \dfrac{\sqrt{mT}}{\hbar}\,, \quad \text{for } r \geq a\,, \\[2ex] A'\sin Kr, & \text{with} \quad K = \dfrac{\sqrt{m(T + B)}}{\hbar}\,, \quad \text{for } r \leq a\,. \end{cases}$$

The continuity of the wave function and its first derivative at $r = a$ gives

$$\tan(ka + \delta) = \frac{k}{K}\tan Ka\,, \tag{1}$$

and hence

$$\delta = \arctan\left(\frac{k}{K}\tan Ka\right) - ka\,. \tag{2}$$

The scattering cross section is then

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta\,.$$

Consider the case of $k \to 0$. We have $\delta \to \delta_0$, $K \to K_0 = \frac{\sqrt{mB}}{\hbar}$, and, by definition, $a_s = -\frac{\tan\delta_0}{k}$.

With $k \to 0$, Eq. (1) gives

$$ka + \tan\delta_0 \approx \frac{k}{K_0}\tan K_0 a(1 - ka\tan\delta_0) \approx \frac{k}{K_0}\tan K_0 a\,,$$

or

$$ka - ka_s \approx \frac{k}{K_0}\tan K_0 a\,,$$

i.e.,

$$a_s \approx -a\left(\frac{\tan K_0 a}{K_0 a} - 1\right)\,.$$

If $a_s = -\frac{\tan\delta_0}{k} \to 0$, then $\delta_0 \to 0$ also (k is small but finite). The corresponding scattering cross section is

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 \approx \frac{4\pi}{k^2}\delta_0^2 = \frac{4\pi}{k^2}k^2 a_s^2 = 4\pi a^2\left(\frac{\tan K_0 a}{K_0 a} - 1\right)^2\,.$$

Hence

$$f(a_s, B, m) = \frac{\sigma}{\sigma_0} \approx \frac{\sin^2[\arctan(\frac{k}{K}\tan Ka) - ka]}{k^2 a^2(\frac{\tan K_0 a}{K_0 a} - 1)^2}$$

$$\approx \frac{\sin^2[\arctan(\frac{k}{K}\tan Ka) - ka]}{k^2 a_s^2}\,.$$

2053

The only bound two-nucleon configuration that occurs in nature is the deuteron with total angular momentum $J = 1$ and binding energy -2.22 MeV.

(a) From the above information alone, show that the $n - p$ force must be spin dependent.

(b) Write down the possible angular momentum states for the deuteron in an LS coupling scheme. What general liner combinations of these states are possible? Explain.

(c) Which of the states in (b) are ruled out by the existence of the quadrupole moment of the deuteron? Explain. Which states, in addition, are ruled out if the deuteron has pure isospin $T = 0$?

(d) Calculate the magnetic moment of the deuteron in each of the allowed states in part (c), and compare with the observed magnetic moment $\mu_d = 0.875\mu_N, \mu_N$ being the nuclear magneton.

(NOTE: $\mu_p = 2.793\mu_N$ and $\mu_n = -1.913\mu_N$)

The following Clebsch–Gordan coefficients may be of use:

[Notation; $\langle J_1 J_2 M_1 M_2 | J_{\text{TOT}} M_{\text{TOT}} \rangle$]

$$\langle 2, 1; 2, -1 | 1, 1 \rangle = (3/5)^{1/2} ,$$

$$\langle 2, 1; 1, 0 | 1, 1 \rangle = -(3/10)^{1/2} ,$$

$$\langle 2, 1; 0, 1 | 1, 1 \rangle = (1/10)^{1/2} .$$

(*Princeton*)

Solution:

(a) The spin of naturally occurring deuteron is $J = 1$. As $\mathbf{J} = \mathbf{s}_n + \mathbf{s}_p + \mathbf{l}_p$, we can have

$$\text{for } |\mathbf{s}_n + \mathbf{s}_p| = 1 , \quad l = 0, 1, 2, \text{ possible states } {}^3S_1 \cdot {}^3P_1, {}^3D_1 ,$$

$$\text{for } |\mathbf{s}_n + \mathbf{s}_p| = 0 , \quad l = 1, \text{ possible state } {}^1P_1 .$$

However, as no stable singlet state ${}^1S_0$, where n, p have antiparallel spins and $l = 0$, is found, this means that when n, p interact to form $S = 1$ and $S = 0$ states, one is stable and one is not, indicating the spin dependence of nuclear force.

(b) As shown above, in LS coupling the possible configurations are ${}^3S_1$, ${}^3D_1$ of even party and ${}^3P_1$, ${}^1P_1$ of odd parity.

As the deuteron has a definite parity, only states of the same parity can be combined. Thus

$$\Psi(n,p) = a^3S_1 + b^3D_1 \text{ or } c^3P_1 + d^1P_1 \,,$$

where a, b, c, d are constants, are the general linear combinations possible.

(c) $l = 1$ in the P state corresponds to a translation of the center of mass of the system, and does not give rise to an electric quadrupole moment. So the existence of an electric quadrupole moment of the deuteron rules out the combination of P states. Also, in accordance with the generalized Pauli's principle, the total wave function of the $n-p$ system must be antisymmetric. Thus, in

$$\Psi(n,p) = \Psi_l(n,p)\Psi_s(n,p)\Psi_T(n,p) \,,$$

where l, s, T label the space, spin and isospin wave functions, as $T = 0$ and so the isospin wave function is exchange antisymmetric, the combined space and spin wave function must be exchange symmetric. It follows that if $l = 1$, then $S = 0$, if $l = 0, 2$ then $S = 1$. This rules out the ${}^3P_1$ state. Hence, considering the electric quadrupole moment and the isospin, the deuteron can only be a mixed state of ${}^3S_1$ and ${}^3D_1$.

(d) For the ${}^3S_1$ state, $l = 0$, and the orbital part of the wave function has no effect on the magnetic moment; only the spin part does. As $S = 1$, the n and p have parallel spins, and so

$$\mu({}^3S_1) = \mu_p + \mu_n = (2.793 - 1.913)\mu_N = 0.88\mu_N \,.$$

For the ${}^3D_1$ state, when $m = 1$, the projection of the magnetic moment on the z direction gives the value of the magnetic moment. Expanding the total angular momentum $|1,1\rangle$ in terms of the D states we have

$$|1,1\rangle = \sqrt{\frac{3}{5}}|2,2,1,-1\rangle - \sqrt{\frac{3}{10}}|2,1,1,0\rangle + \sqrt{\frac{1}{10}}|2,0,1,1\rangle \,.$$

The contribution of the D state to the magnetic moment is therefore

$$\begin{aligned}\mu(^3D_1) &= \left[\frac{3}{5}(g_l m_{l1} + g_s m_{s1}) + \frac{3}{10}(g_l m_{l2} + g_s m_{s2}) \right.\\ &\quad \left. + \frac{1}{10}(g_l m_{l3} + g_s m_{s3})\right]\mu_N \\ &= \left[\left(\frac{3}{5}m_{l1} + \frac{3}{10}m_{l2} + \frac{1}{10}m_{l3}\right) \times \frac{1}{2} \right.\\ &\quad \left. + \left(\frac{3}{5}m_{s1} + \frac{3}{10}m_{s2} + \frac{1}{10}m_{s3}\right) \times 0.88\right]\mu_N \\ &= 0.31\mu_N\,.\end{aligned}$$

Note that g_l is 1 for p and 0 for n, g_s is 5.5855 for p and -3.8256 for n, and so g_l is $\frac{1}{2}$ and g_s is 0.88 for the system (**Problem 2056**).

As experimentally $\mu_d = 0.857\mu_N$, the deuteron must be a mixed state of S and D. Let the proportion of D state be x, and that of S state be $1 - x$. Then

$$0.88(1 - x) + 0.31x = 0.857\,,$$

giving $x \approx 0.04$, showing that the deuteron consists of 4% $^3D_1$ state and 96% $^3S_1$ state.

2054

Consider a nonrelativistic two-nucleon system. Assume the interaction is charge independent and conserves parity.

(a) By using the above assumptions and the Pauli principle, show that $\mathbf{S}^2$, the square of the two-nucleon spin, is a good quantum number.

(b) What is the isotopic spin of the deuteron? Justify your answer!

(c) Specify all states of a two-neutron system with total angular momentum $J \le 2$. Use the notation $^{2S+1}X_J$ where X gives the orbital angular momentum.

(SUNY Buffalo)

Solution:

(a) Let the total exchange operator of the system be $P = P'P_{12}$, where P' is the space reflection, or parity, operator, P_{12} is the spin exchange operator

$$P_{12} = \frac{1}{2}(1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) = \mathbf{S}^2 - 1\,,$$

where $\boldsymbol{\sigma}_i = 2\mathbf{s}_i (i = 1, 2)$, $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$, using units where $\hbar = 1$. Pauli's principle gives $[P, H] = 0$, and conservation of parity gives $[P', H] = 0$. As

$$\begin{aligned} 0 &= [P, H] = [P'P_{12}, H] = P'[P_{12}, H] + [P', H]P_{12} \\ &= P'[P_{12}, H] = P'[\mathbf{S}^2 - 1, H] = P'[\mathbf{S}^2, H]\,, \end{aligned}$$

we have $[\mathbf{S}^2, H] = 0$, and so $\mathbf{S}^2$ is a good quantum number.

(b) The isospin of the nuclear ground state always takes the smallest possible value. For deuteron,

$$\mathbf{T} = \mathbf{T}_p + \mathbf{T}_n, \qquad T_z = T_{pz} + T_{nz} = \frac{1}{2} - \frac{1}{2} = 0\,.$$

For ground state $T = 0$.

(c) As $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$ and $s_1 = s_2 = \frac{1}{2}$ the quantum number S can be 1 or 0. The possible states with $J \le 2$ are

$$S = 0\,, \qquad l = 0: \quad {}^1S_0\,,$$

$$S = 0\,, \qquad l = 1: \quad {}^1P_1\,,$$

$$S = 0\,, \qquad l = 2: \quad {}^1D_2\,,$$

$$S = 1\,, \qquad l = 0: \quad {}^3S_1\,,$$

$$S = 1\,, \qquad l = 1: \quad {}^3P_2, {}^3P_1, {}^3P_0\,,$$

$$S = 1\,, \qquad l = 2: \quad {}^3D_2, {}^3D_1\,,$$

$$S = 1\,, \qquad l = 3: \quad {}^3F_2\,,$$

However, a two-neutron system is required to be antisymmetric with respect to particle exchange. Thus $(-1)^{l+S+1} = -1$, or $l + S =$ even. Hence the possible states are ${}^1S_0, {}^1D_2, {}^3P_2, {}^3P_1, {}^3P_0, {}^3F_2$.

2055

Consider the potential between two nucleons. Ignoring velocity-dependent terms, derive the most general form of the potential which is consistent with applicable conservation laws including that of isotopic spin. Please list each conservation law and indicate its consequences for the potential.

(*Chicago*)

Solution:

(a) Momentum conservation – invariance in space translation.

This law means that the potential function depends only on the relative position between the two nucleons $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$.

(b) Angular momentum conservation – invariance in continuous space rotation: $\boldsymbol{x}' = \hat{R}\boldsymbol{x}$, $\boldsymbol{J}^{(i)\prime} = \hat{R}\boldsymbol{J}^{(i)}$, $i = 1, 2$, where $\hat{R}$ is the rotational operator.

The invariants in the rotational transformation are 1, $\boldsymbol{x}^2$ $\boldsymbol{J}^{(i)} \cdot \boldsymbol{x}$, $\boldsymbol{J}^{(1)} \cdot \boldsymbol{J}^{(2)}$ and $[\boldsymbol{J}^{(1)} \times \boldsymbol{J}^{(2)}] \cdot \boldsymbol{x}$. Terms higher than first order in $\boldsymbol{J}^{(1)}$ or in $\boldsymbol{J}^{(2)}$ can be reduced as $\boldsymbol{J}_i\boldsymbol{J}_j = \delta_{ij} + i\varepsilon_{ijk}\boldsymbol{J}_k$. Also $(\boldsymbol{J}^{(1)} \times \boldsymbol{x}) \cdot (\boldsymbol{J}^{(2)} \times \boldsymbol{x}) = (\boldsymbol{J}^{(1)} \times \boldsymbol{x}) \times \boldsymbol{J}^{(2)} \cdot \boldsymbol{x} = (\boldsymbol{J}^{(1)} \cdot \boldsymbol{J}^{(2)})\boldsymbol{x}^2 - (\boldsymbol{J}^{(1)} \cdot \boldsymbol{x})(\boldsymbol{J}^{(2)} \cdot \boldsymbol{x})$.

(c) Parity conservation – invariance in space reflection: $\boldsymbol{x}' = -\boldsymbol{x}$, $\boldsymbol{J}^{(i)\prime} = \boldsymbol{J}^{(i)}$, $i = 1, 2$.

Since $\boldsymbol{x}$ is the only polar vector, in the potential function only terms of even power in $\boldsymbol{x}$ are possible. Other invariants are 1, $\boldsymbol{x}^2$, $\boldsymbol{J}^{(1)} \cdot \boldsymbol{J}^{(2)}$, $(\boldsymbol{J}^{(1)} \cdot \boldsymbol{x})(\boldsymbol{J}^{(2)} \cdot \boldsymbol{x})$.

(d) Isotopic spin conservation – rotational invariance in isotopic spin space:

$$\boldsymbol{I}^{(i)\prime} = R_J \boldsymbol{I}^{(i)}, \qquad i = 1, 2\,.$$

The invariants are 1 and $\boldsymbol{I}^{(1)} \cdot \boldsymbol{I}^{(2)}$.

(e) Conservation of probability – Hamiltonian is hermitian: $V^+ = V$.

This implies the realness of the coefficient of the potential function, i.e., $V_{sk}(r)$, where $r = |\boldsymbol{x}|$, is real. Thus in

$$V(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{J}^{(1)}, \boldsymbol{J}^{(2)}, \boldsymbol{I}^{(1)}, \boldsymbol{I}^{(2)}) = V_a + \boldsymbol{J}^{(1)} \cdot \boldsymbol{J}^{(2)} V_b\,,$$

where V_a and V_b are of the form

$$V_0(r) + V_1(r)\boldsymbol{J}^{(1)} \cdot \boldsymbol{J}^{(2)} + V_2(r)\frac{(\boldsymbol{J}^{(1)} \cdot \boldsymbol{x})(\boldsymbol{J}^{(2)} \cdot \boldsymbol{x})}{\boldsymbol{x}^2}\,,$$

as the coefficients $V_{sk}(r)$ $(s = a, b;\ k = 0, 1, 2)$ are real functions.

(f) Time reversal (inversion of motion) invariance:

$$V = U^{-1}V^*U, \qquad U^{-1}\boldsymbol{J}^*U = -\boldsymbol{J}\,.$$

This imposes no new restriction on V.

Note that V is symmetric under the interchange $1 \leftrightarrow 2$ between two nucleons.

2056

The deuteron is a bound state of a proton and a neutron of total angular momentum $J = 1$. It is known to be principally an $S(l = 0)$ state with a small admixture of a $D(l = 2)$ state.

(a) Explain why a P state cannot contribute.

(b) Explain why a G state cannot contribute.

(c) Calculate the magnetic moment of the pure D state $n - p$ system with $J = 1$. Assume that the n and p spins are to be coupled to make the total spin $\mathbf{S}$ which is then coupled to the orbital angular momentum $\mathbf{L}$ to give the total angular momentum $\mathbf{J}$. Express your result in nuclear magnetons. The proton and neutron magnetic moments are 2.79 and -1.91 nuclear magnetons respectively.

(*CUSPEA*)

Solution:

(a) The P state has a parity opposite to that of S and D states. As parity is conserved in strong interactions states of opposite parities cannot be mixed. Hence the P state cannot contribute to a state involving S and D states

(b) The orbital angular momentum quantum number of G state is $l = 4$. It cannot be coupled with two $1/2$ spins to give $J = 1$. Hence the G state cannot contribute to a state of $J = 1$.

(c) We have $\mathbf{J} = \mathbf{L} + \mathbf{S}$,

$$\boldsymbol{\mu} = \frac{[(g_L\mathbf{L} + g_s\mathbf{S}) \cdot \mathbf{J}]}{J(J+1)}\mathbf{J}\mu_0\,,$$

where μ_0 is the nuclear magneton. By definition,

$$\mathbf{S} = \mathbf{s}_p + \mathbf{s}_n \,,$$

$$\boldsymbol{\mu}_s = \frac{[(g_p \mathbf{s}_p + g_n \mathbf{s}_n) \cdot \mathbf{S}]}{S(S+1)} \mathbf{S}\mu_0 \equiv g_s \mathbf{S}\mu_0 \,,$$

or

$$g_s = \frac{g_p \mathbf{s}_p \cdot \mathbf{S} + g_n \mathbf{s}_n \cdot \mathbf{S}}{S(S+1)} \,.$$

Consider $\mathbf{s}_n = \mathbf{S} - \mathbf{s}_p$. As $\mathbf{s}_n^2 = \mathbf{S}^2 + \mathbf{s}_p^2 - 2\mathbf{S} \cdot \mathbf{s}_p$, we have

$$\mathbf{S} \cdot \mathbf{s}_p = \frac{S(S+1) + s_p(s_p+1) - s_n(s_n+1)}{2} = 1 \,,$$

since $s_p = s_n = \frac{1}{2}$, $S = 1$ (for $J = 1$, $l = 2$). Similarly $\mathbf{S} \cdot \mathbf{s}_n = 1$. Hence

$$g_s = \frac{1}{2}(g_p + g_n) \,.$$

As the neutron, which is uncharged, makes no contribution to the orbital magnetic moment, the proton produces the entire orbital magnetic moment, but half the orbital angular momentum. Hence $g_L = \frac{1}{2}$.

Substitution of g_s and g_L in the expression for $\boldsymbol{\mu}$ gives

$$\frac{\boldsymbol{\mu}}{\mu_0} = \frac{\frac{1}{2}(\mathbf{L} \cdot \mathbf{J}) + \frac{1}{2}(g_p + g_n)(\mathbf{S} \cdot \mathbf{J})}{J(J+1)} \mathbf{J} \,.$$

As

$$\begin{aligned} \mathbf{L} \cdot \mathbf{J} &= \frac{1}{2}[J(J+1) + L(L+1) - S(S+1)] \\ &= \frac{1}{2}(1 \times 2 + 2 \times 3 - 1 \times 2) = 3 \,, \\ \mathbf{S} \cdot \mathbf{J} &= \frac{1}{2}[J(J+1) + S(S+1) - L(L+1)] \\ &= \frac{1}{2}(1 \times 2 + 1 \times 2 - 2 \times 3) = -1 \,, \\ \frac{\boldsymbol{\mu}}{\mu_0} &= \frac{1}{2}\left(\frac{3}{2} - \frac{g_p + g_n}{2}\right) \mathbf{J} \,. \end{aligned}$$

with $\mu_p = g_p s_p \mu_0 = \frac{1}{2} g_p \mu_0$, $\mu_n = g_n s_n \mu_0 = \frac{1}{2} g_n \mu_0$, we have

$$\mu = \left(\frac{3}{4} - \frac{\mu_p + \mu_n}{2}\right) \mu_0 = \left(\frac{3}{4} - \frac{2.79 - 1.91}{2}\right) \mu_0 = 0.31 \mu_0 \,.$$

2057

(a) The deuteron ($^2_1$H) has $J = 1\hbar$ and a magnetic moment ($\mu = 0.857\mu_N$) which is approximately the sum of proton and neutron magnetic moments ($\mu_p = 2.793\mu_N$, and $\mu_n = -1.913\mu_N$). From these facts what can one infer concerning the orbital motion and spin alignment of the neutron and proton in the deuteron?

(b) How might one interpret the lack of exact equality of μ and $\mu_n + \mu_p$?

(c) How can the neutron have a nonzero magnetic moment?

(*Wisconsin*)

Solution:

(a) As $\mu \approx \mu_n + \mu_p$, the orbital motions of proton and neutron make no contribution to the magnetic moment of the deuteron. This means that the orbital motion quantum number is $l = 0$. As $J = 1$ the spin of the deuteron is 1 and it is in the $^3S_1$ state formed by proton and neutron of parallel-spin alignment.

(b) The difference between μ and $\mu_n + \mu_p$ cannot be explained away by experimental errors. It is interpreted as due to the fact that the neutron and proton are not in a pure $^3S_1$ state, but in a mixture of $^3S_1$ and $^3D_1$ states. If a proportion of the latter of about 4% is assumed, agreement with the experimental value can be achieved.

(c) While the neutron has net zero charge, it has an inner structure. The current view is that the neutron consists of three quarks of fractional charges. The charge distribution inside the neutron is thus not symmetrical, resulting in a nonzero magnetic moment.

2058

The deuteron is a bound state of a proton and a neutron. The Hamiltonian in the center-of-mass system has the form

$$H = \frac{\mathbf{p}^2}{2\mu} + V_1(r) + \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_n V_2(r) + \left[\left(\boldsymbol{\sigma}_p \cdot \frac{\boldsymbol{x}}{r}\right)\left(\boldsymbol{\sigma}_n \cdot \frac{\boldsymbol{x}}{r}\right) - \frac{1}{3}(\boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_n)\right] V_3(r) \,,$$

where $\boldsymbol{x} = \boldsymbol{x}_n - \boldsymbol{x}_p$, $r = |\boldsymbol{x}|$, $\boldsymbol{\sigma}_p$ and $\boldsymbol{\sigma}_n$ are the Pauli matrices for the spins of the proton and neutron, μ is the reduced mass, and $\mathbf{p}$ is conjugate to $\mathbf{x}$.

(a) Total angular momentum ($\mathbf{J}^2 = J(J+1)$) and parity are good quantum numbers. Show that if $V_3 = 0$, total orbital angular momentum ($\mathbf{L}^2 = L(L+1)$) and total spin ($\mathbf{S}^2 = S(S+1)$) are good quantum numbers, where $\mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_p + \boldsymbol{\sigma}_n)$. Show that if $V_3 \neq 0$, S is still a good quantum number. [It may help to consider interchange of proton and neutron spins.]

(b) The deuteron has $J = 1$ and positive parity. What are the possible values of L? What is the value of S?

(c) Assume that V_3 can be treated as a small perturbation. Show that in zeroth order ($V_3 = 0$) the wave function of the state with $J_z = +1$ is of the form $\Psi_0(r)|\alpha,\alpha\rangle$, where $|\alpha,\alpha\rangle$ is the spin state with $s_{pz} = s_{nz} = 1/2$. What is the differential equation satisfied by $\Psi_0(r)$?

(d) What is the first order shift in energy due to the term in V_3? Suppose that to first order the wave function is

$$\Psi_0(r)|\alpha,\alpha\rangle + \Psi_1(\mathbf{x})|\alpha,\alpha\rangle + \Psi_2(\mathbf{x})(|\alpha,\beta\rangle + |\beta,\alpha\rangle) + \Psi_3(\mathbf{x})|\beta,\beta\rangle\,,$$

where $|\beta\rangle$ is a state with $s_z = -\frac{1}{2}$ and Ψ_0 is as defined in part (c). By selecting out the part of the Schördinger equation that is first order in V_3 and proportional to $|\alpha,\alpha\rangle$, find the differential equation satisfied by $\Psi_1(\mathbf{x})$. Separate out the angular dependence of $\Psi_1(\mathbf{x})$ and write down a differential equation for its radial dependence.

(*MIT*)

Solution:

(a) We have $[\mathbf{L}^2, \boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n] = 0$, $[\mathbf{L}^2, V_i(r)] = 0$, $[\mathbf{S}^2, V_i(r)] = 0$, $[\mathbf{S}^2, \mathbf{p}^2] = 0$; $[\mathbf{S}^2, \boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n] = [\mathbf{S}^2, 2\mathbf{S}^2 - 3] = 0$ as $\mathbf{S}^2 = \mathbf{s}_p^2 + \mathbf{s}_n^2 + 2\mathbf{s}_p\cdot\mathbf{s}_n = \frac{3}{4} + \frac{3}{4} + \frac{1}{2}\boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n$;

$$\begin{aligned}
&\left[\mathbf{S}^2, 3\left(\boldsymbol{\sigma}_p\cdot\frac{\boldsymbol{x}}{r}\right)\left(\boldsymbol{\sigma}_n\cdot\frac{\boldsymbol{x}}{r}\right) - \boldsymbol{\sigma}_p\cdot\boldsymbol{\sigma}_n\right] \\
&\quad = \left[\mathbf{S}^2, \frac{12(\mathbf{s}\cdot\mathbf{x})^2}{r^2} - 2\mathbf{S}^2 + 3\right] = \left[\mathbf{S}^2, \frac{12(\mathbf{s}\cdot\mathbf{x})^2}{r^2}\right] \\
&\quad = \frac{12(\mathbf{s}\cdot\mathbf{x})}{r^2}[\mathbf{S}^2, \mathbf{s}\cdot\mathbf{x}] + [\mathbf{S}^2, \mathbf{s}\cdot\mathbf{x}]\frac{12(\mathbf{s}\cdot\mathbf{x})}{r^2} = 0
\end{aligned}$$

as

$$\frac{(\boldsymbol{\sigma}_p\cdot\mathbf{x})}{r}\frac{(\boldsymbol{\sigma}_n\cdot\mathbf{x})}{r} = \frac{4}{r^2}(\mathbf{s}_p\cdot\mathbf{x})(\mathbf{s}_n\cdot\mathbf{x}) = \frac{4}{r^2}(\mathbf{s}\cdot\mathbf{x})^2\,;$$

$[\mathbf{L}^2, \mathbf{p}^2] = \mathbf{L}[\mathbf{L}, \mathbf{p}^2] + [\mathbf{L}, \mathbf{p}^2]\mathbf{L} = 0$ as $[l_\alpha, \mathbf{p}^2] = 0$.

Hence if $V_3 = 0$, $[\mathbf{L}^2, H] = 0$, $[\mathbf{S}^2, H] = 0$, and the total orbital angular momentum and total spin are good quantum numbers. If $V_3 \neq 0$, as $[\mathbf{S}^2, H] = 0$, S is still a good quantum number.

(b) The possible values of L are 0,2 for positive parity, and so the value of S is 1.

(c) If $V_3 = 0$, the Hamiltonian is centrally symmetric. Such a symmetric interaction potential between the proton and neutron gives rise to an S state ($L = 0$). The S state of deuteron would have an admixture of D-state if the perturbation V_3 is included.

In the case of $V_3 = 0$, $L = 0$, $S = 1$ and $S_z = 1$, so $J_z = +1$ and the wave function has a form $\Psi_0(r)|\alpha, \alpha\rangle$. Consider

$$\begin{aligned} H\Psi_0(r)|\alpha,\alpha\rangle &= \left[-\frac{\nabla^2}{2\mu} + V_1(r) + (2\mathbf{S}^2 - 3)V_2(r)\right]\Psi_0(r)|\alpha,\alpha\rangle \\ &= \left[-\frac{\nabla^2}{2\mu} + V_1(r) + V_2(r)\right]\Psi_0(r)|\alpha,\alpha\rangle \\ &= E_c\Psi_0(r)|\alpha,\alpha\rangle \end{aligned}$$

noting that $2S^2 - 3 = 2.1.2 - 3 = 1$. Thus $\Psi_0(r)$ satisfies

$$\left[-\frac{\nabla^2}{2\mu} + V_1(r) + V_2(r) - E_c\right]\Psi_0(r) = 0\,,$$

or

$$-\frac{1}{2\mu}\frac{1}{r^2}\frac{d}{dr}[r^2\Psi_0'(r)] + [V_1(r) + V_2(r) - E_c]\Psi_0(r) = 0\,,$$

i.e.,

$$-\frac{1}{2\mu}\Psi_0''(r) - \frac{1}{\mu r}\Psi_0'(r) + [V_1(r) + V_2(r) - E_c]\Psi_0(r) = 0\,.$$

(d) Now, writing S_{12} for the coefficient of $V_3(r)$,

$$\begin{aligned} H &= -\frac{\nabla^2}{2\mu} + V_1(r) + (2\mathbf{S}^2 - 3)V_2(r) + S_{12}V_3(r) \\ &= -\frac{\nabla^2}{2\mu} + V_1(r) + V_2(r) + S_{12}V_3(r)\,, \end{aligned}$$

so

$$\begin{aligned}H\Psi &= \left(-\frac{\nabla^2}{2\mu}+V_1+V_2\right)\Psi_0(r)|\alpha,\alpha\rangle + \left(-\frac{\nabla^2}{2\mu}+V_1+V_2\right)[\Psi_1|\alpha,\alpha\rangle \\ &\quad + \Psi_2(|\alpha,\beta\rangle + |\beta,\alpha\rangle) + \Psi_3|\beta,\beta\rangle] + S_{12}V_3\Psi_0|\alpha,\alpha\rangle \\ &= E_c\Psi_0(r)|\alpha,\alpha\rangle + E_c[\Psi_1|\alpha,\alpha\rangle + \Psi_2(|\alpha,\beta\rangle \\ &\quad + |\beta,\alpha\rangle) + \Psi_3|\beta,\beta\rangle] + \Delta E\Psi_0(r)|\alpha,\alpha\rangle\,,\end{aligned}$$

where

$$\begin{aligned}S_{12}V_3\Psi_0(r)|\alpha,\alpha\rangle &= [(\sigma_{pz}\cos\theta\cdot\sigma_{nz}\cos\theta)|\alpha,\alpha\rangle - \frac{1}{3}|\alpha,\alpha\rangle]V_3\Psi_0(r)+\cdots \\ &= \left(\cos^2\theta-\frac{1}{3}\right)V_3\Psi_0(r)|\alpha,\alpha\rangle+\cdots\,,\end{aligned}$$

terms not proportional to $|\alpha,\alpha\rangle$ having been neglected.

Selecting out the part of the Schrödinger equation that is first order in V_3 and proportional to $|\alpha,\alpha\rangle$, we get

$$\left(-\frac{\nabla^2}{2\mu}+V_1+V_2\right)\Psi_1(\mathbf{x})+\left(\cos^2\theta-\frac{1}{3}\right)V_3\Psi_0(r)=E_c\Psi_1(\mathbf{x})+\Delta E\Psi_0(r)\,.$$

Thus the angular-dependent part of $\Psi_1(\mathbf{x})$ is

$$Y_{20}=3\left(\frac{5}{16\pi}\right)^{\frac{1}{2}}\left(\cos^2\theta-\frac{1}{3}\right)\,,$$

since for the state $|\alpha,\alpha\rangle$, $S_z=1$ and so $L_z=0$, i.e. the angular part of the wave function is Y_{20}. Therefore we have

$$\begin{aligned}&-\frac{1}{2\mu}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi_1(r)}{dr}\right)+V_1(r)\Psi_1(r)+V_2(r)\Psi_2(r)\\ &\qquad+\frac{l(l+1)}{r^2}\Psi_1(r)+\frac{1}{3}\sqrt{\frac{16\pi}{5}}V_3\Psi_0(r)\\ &\qquad=E_c\Psi_1(r)+\Delta E\Psi_0(r)\end{aligned}$$

with $\Psi_1(\mathbf{x}) = \Psi_1(r)Y_{20}$, $l = 2$, or

$$-\frac{1}{2\mu}\Psi_1''(r) - \frac{1}{\mu r}\Psi_1'(r) + \left[V_1(r) + V_2(r) + \frac{6}{r^2} - E_c\right]\Psi_1(r)$$

$$+\left(\frac{1}{3}\sqrt{\frac{16\pi}{5}}V_3 - \Delta E\right)\Psi_0(r) = 0$$

with

$$\Delta E = \left(\cos^2\theta - \frac{1}{3}\right)V_3 \,.$$

4. NUCLEAR MODELS (2059–2075)

2059

What are the essential features of the liquid-drop, shell, and collective models of the nucleus? Indicate what properties of the nucleus are well predicted by each model, and how the model is applied.

(*Columbia*)

Solution:

It is an empirical fact that the binding energy per nucleon, B, of a nucleus and the density of nuclear matter are almost independent of the mass number A. This is similar to a liquid-drop whose heat of evaporation and density are independent of the drop size. Add in the correction terms of surface energy, Coulomb repulsion energy, pairing energy, symmetry energy and we get the liquid-drop model. This model gives a relationship between A and Z of stable nuclei, i.e., the β-stability curve, in agreement with experiment. Moreover, the model explains why the elements ^{43}Te, ^{61}Pm have no β-stable isobars. If we treat the nucleus's radius as a variable parameter in the mass-formula coefficients a_{surface} and a_{volume} and fit the mass to the experimental value, we find that the nuclear radius so deduced is in good agreement with those obtained by all other methods. So the specific binding energy curve is well explained by the liquid-drop model.

The existence of magic numbers indicates that nuclei have internal structure. This led to the nuclear shell model similar to the atomic model, which could explain the special stability of the magic-number nuclei. The shell model requires:

(1) the existence of an average field, which for a spherical nucleus is a central field,

(2) that each nucleon in the nucleus moves independently,

(3) that the number of nucleons on an energy level is limited by Pauli's principle,

(4) that spin-orbit coupling determines the order of energy levels.

The spin and parity of the ground state can be predicted using the shell model. For even-even nuclei the predicted spin and parity of the ground state, 0^+, have been confirmed by experiment in all cases. The prediction is based on the fact that normally the spin and parity are 0^+ when neutrons and protons separately pair up. The predictions of the spin and parity of the ground state of odd-A nuclei are mostly in agreement with experiment. Certain aspects of odd-odd nuclei can also be predicted. In particular it attributes the existence of magic numbers to full shells.

The shell model however cannot solve all the nuclear problems. It is quite successfull in explaining the formation of a nucleus by adding one or several nucleons to a full shell (spherical nucleus), because the nucleus at this stage is still approximately spherical. But for a nucleus between two closed shells, it is not spherical and the collective motion of a number of nucleons become much more important. For example, the experimental values of nuclear quadrupole moment are many times larger than the values calculated from a single particle moving in a central field for a nucleus between full shells. This led to the collective model, which, by considering the collective motion of nucleons, gives rise to vibrational and rotational energy levels for nuclides in the ranges of $60 < A < 150$ and $190 < A < 220$, $150 < A < 190$ and $A > 220$ respectively:

2060

Discuss briefly the chief experimental systematics which led to the shell model description for nuclear states. Give several examples of nuclei which correspond to closed shells and indicate which shells are closed.

(*Wisconsin*)

Solution:

The main experimental evidence in support of the nuclear shell model is the existence of magic numbers. When the number of the neutrons or of the protons in a nucleus is 2, 8, 20, 28, 50, 82 and 126 (for neutrons only), the nucleus is very stable. In nature the abundance of nuclides with such magic numbers are larger than those of the nearby numbers. Among all the stable nuclides, those of neutron numbers 20, 28, 50 and 82 have more isotones, those of proton numbers 8, 20, 28, 50 and 82 have more stable isotopes, than the nearby nuclides. When the number of neutrons or protons in a nuclide is equal to a magic number, the binding energy measured experimentally is quite different from that given by the liquid-drop model. The existence of such magic numbers implies the existence of shell structure inside a nucleus similar to the electron energy levels in an atom.

^4He is a double-magic nucleus; its protons and neutrons each fill up the first main shell. ^{16}O is also a double-magic nucleus, whose protons and neutrons each fill up the first and second main shells. ^{208}Pb is a double-magic nucleus, whose protons fill up to the sixth main shell, while whose neutrons fill up to the seventh main shell. Thus these nuclides all have closed shells.

2061

(a) Discuss the standard nuclear shell model. In particular, characterize the successive shells according to the single-particle terms that describe the shell, i.e., the principal quantum number n, the orbital angular momentum quantum number l, and the total angular momentum quantum number j (spectroscopic notation is useful here, e.g., $2s_{1/2}, 1p_{3/2}$, etc..). Discuss briefly some of the basic evidence in support of the shell model.

(b) Consider a nuclear level corresponding to a closed shell plus a single proton in a state with the angular momentum quantum numbers l and j. Of course $j = l \pm 1/2$. Let g_p be the empirical gyromagnetic ratio of the free proton. Compute the gyromagnetic ratio for the level in question, for each of the two cases $j = l + 1/2$ and $j = l - 1/2$.

(*Princeton*)

Solution:

(a) The basic ideas of the nuclear shell model are the following. Firstly we assume each nucleon moves in an average field which is the sum of the actions of the other nucleons on it. For a nucleus nearly spherically in shape, the average field is closely represented by a central field. Second, we assume that the low-lying levels of a nucleus are filled up with nucleons in accordance with Pauli's principle. As collisions between nucleons cannot cause a transition and change their states, all the nucleons can maintain

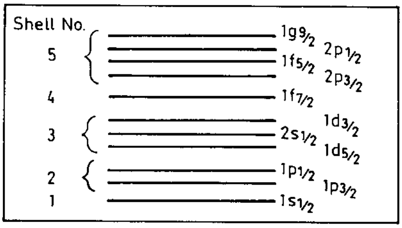

Fig. 2.11

their states of motion, i.e., they move independently in the nucleus. We can take for the average central field a Woods–Saxon potential well compatible with the characteristics of the interaction between nucleons, and obtain the energy levels by quantum mechanical methods. Considering the spin-orbital interaction, we get the single-particle energy levels (Fig. 2.11), which can be filled up with nucleons one by one. Note that each level has a degeneracy $2j+1$. So up to the first 5 shells as shown, the total number of protons or neutrons accommodated are 2, 8, 20, 28 and 50.

The main experimental evidence for the shell model is the existence of magic numbers. Just like the electrons outside a nucleus in an atom, if the numbers of neutrons on protons in a nucleus is equal to some 'magic number' (8,20,28,50 or 82), the nucleus has greater stability, larger binding energy and abundance, and many more stable isotopes.

(b) According to the shell model, the total angular momentum of the nucleons in a closed shell is zero, so is the magnetic moment. This means that the magnetic moment and angular momentum of the nucleus are determined by the single proton outside the closed shell.

As

$$\boldsymbol{\mu}_j = \boldsymbol{\mu}_l + \boldsymbol{\mu}_s \,,$$

i.e.,

$$g_j\mathbf{j} = g_l\mathbf{l} + g_s\mathbf{s} \,,$$

we have

$$g_j\mathbf{j}\cdot\mathbf{j} = g_l\mathbf{l}\cdot\mathbf{j} + g_s\mathbf{s}\cdot\mathbf{j} \,.$$

With

$$\mathbf{l}\cdot\mathbf{j} = \frac{1}{2}(\mathbf{j}^2+\mathbf{l}^2-\mathbf{s}^2) = \frac{1}{2}[j(j+1)+l(l+1)-s(s+1)] \,,$$

$$\mathbf{s}\cdot\mathbf{j} = \frac{1}{2}(\mathbf{j}^2+\mathbf{s}^2-\mathbf{l}^2) = \frac{1}{2}[j(j+1)+s(s+1)-l(l+1)] \,,$$

we have

$$g_j = g_l \frac{j(j+1)+l(l+1)-s(s+1)}{2j(j+1)} + g_s \frac{j(j+1)+s(s+1)-l(l+1)}{2j(j+1)}.$$

For proton, $g_l = 1$, $g_s = g_p$, the gyromagnetic ratio for free proton ($l = 0, j = s$), $s = \frac{1}{2}$. Hence we have

$$g_j = \begin{cases} \dfrac{2j-1}{2j} + \dfrac{g_p}{2j} & \text{for } j = l + 1/2\,, \\ \dfrac{1}{j+1}\left(j + \dfrac{3}{2} - \dfrac{g_p}{2}\right) & \text{for } j = l - 1/2\,. \end{cases}$$

2062

The energy levels of the three-dimensional isotropic harmonic oscillator are given by

$$E = (2n + l + 3/2)\hbar\omega = \left(N + \frac{3}{2}\right)\hbar\omega\,.$$

In application to the single-particle nuclear model $\hbar\omega$ is fitted as $44A^{-\frac{1}{3}}$ MeV.

(a) By considering corrections to the oscillator energy levels relate the levels for $N \leq 3$ to the shell model single-particle level scheme. Draw an energy level diagram relating the shell model energy levels to the unperturbed oscillator levels.

(b) Predict the ground state spins and parities of the following nuclei using the shell model:

$$^3_2He,\ ^{17}_{8}O,\ ^{34}_{19}K,\ ^{41}_{20}Ca\,.$$

(c) Strong electric dipole transitions are not generally observed to connect the ground state of a nucleus to excited levels lying in the first 5 MeV of excitation. Using the single-particle model, explain this observation and predict the excitation energy of the giant dipole nuclear resonance.

(*Princeton*)

Solution:

(a) Using LS coupling, we have the splitting of the energy levels of a harmonic oscillator as shown in Fig. 2.12.

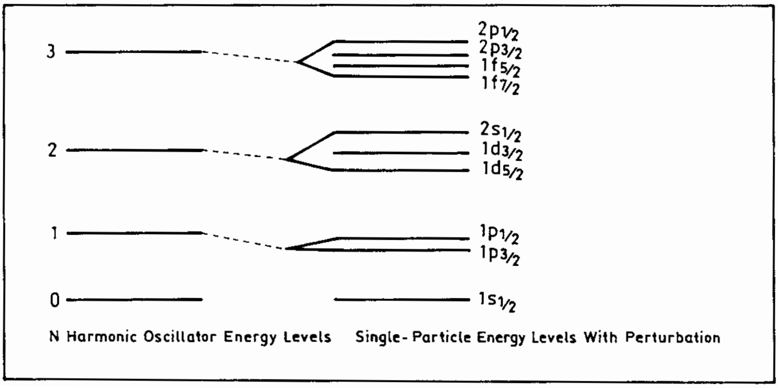

Fig. 2.12

(b) According to Fig. 2.12 we have the following:

$^3_2$He: The last unpaired nucleon is a neutron of state $1s_{\frac{1}{2}}$, so $J^\pi = (1/2)^+$.

$^{17}_8$O: The last unpaired nucleon is a neutron of state $1d_{5/2}$, so $J^\pi = (5/2)^+$.

$^{34}_{19}$K: The last two unpaired nucleons are a proton of state $2s_{\frac{1}{2}}$ and a neutron of state $1d_{3/2}$, so $J^\pi = 1^+$.

$^{41}_{20}$Ca: The last unpaired nucleon is a neutron of state $1f_{7/2}$, so $J^\pi = (7/2)^-$.

(c) The selection rules for electric dipole transition are

$$\Delta J = J_f - J_i = 0, 1, \qquad \Delta\pi = -1 \,,$$

where J is the nuclear spin, π is the nuclear parity. As $\hbar\omega = 44A^{-\frac{1}{3}}$ MeV, $\hbar\omega > 5$ MeV for a nucleus. When N increases by 1, the energy level increases by $\Delta E = \hbar\omega > 5$ MeV. This means that excited states higher than the ground state by less than 5 MeV have the same N and parity as the latter. As electric dipole transition requires $\Delta\pi = -1$, such excited states cannot connect to the ground state through an electric dipole transition. However, in LS coupling the energy difference between levels of different N can be smaller than 5 MeV, especially for heavy nuclei, so that electric dipole transition may still be possible.

The giant dipole nuclear resonance can be thought of as a phenomenon in which the incoming photon separates the protons and neutrons in the nucleus, increasing the potential energy, and causing the nucleus to vibrate.

Resonant absorption occurs when the photon frequency equals resonance frequency of the nucleus.

2063

To some approximation, a medium weight nucleus can be regarded as a flat-bottomed potential with rigid walls. To simplify this picture still further, model a nucleus as a cubical box of length equal to the nuclear diameter. Consider a nucleus of iron-56 which has 28 protons and 28 neutrons. Estimate the kinetic energy of the highest energy nucleon. Assume a nuclear diameter of 10^{-12} cm.

(*Columbia*)

Solution:

The potential of a nucleon can be written as

$$V(x,y,z) = \begin{cases} \infty, & |x|,|y|,|z| > \dfrac{a}{2}\,, \\ 0, & |x|,|y|,|z| < \dfrac{a}{2}\,, \end{cases}$$

where a is the nuclear diameter. Assume the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\Psi(x,y,z) + V(x,y,z)\Psi(x,y,z) = E\Psi(x,y,z)$$

can be separated in the variables by letting $\Psi(x,y,z) = \Psi(x)\Psi(y)\Psi(z)$. Substitution gives

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx_i^2}\Psi(x_i) + V(x_i)\Psi(x_i) = E_i\Psi(x_i)\,,$$

with

$$V(x_i) = \begin{cases} \infty, & |x_i| > \dfrac{a}{2}\,, \\ 0, & |x_i| < \dfrac{a}{2}\,, \end{cases}$$

$i = 1, 2, 3$; $x_1 = x$, $x_2 = y$, $x_3 = z$, $E = E_1 + E_2 + E_3$.

Solving the equations we have

$$\Psi(x_i) = A_i \sin(k_i x_i) + B_i \cos(k_i x_i)$$

with $k_i = \frac{\sqrt{2mE_i}}{\hbar}$. The boundary condition $\Psi(x_i)|_{x_i=\pm\frac{a}{2}} = 0$ gives

$$\Psi(x_i) = \begin{cases} A_i \sin\left(\frac{n\pi}{a}x_i\right), & \text{with } n \text{ even}, \\ B_i \cos\left(\frac{n\pi}{a}x_i\right), & \text{with } n \text{ odd}, \end{cases}$$

and hence

$$E_{xi} = \frac{k_{xi}^2\hbar^2}{2m} = \frac{\pi^2 n_{xi}^2 \hbar^2}{2ma^2}, \quad n_x = 1, 2, 3, \dots,$$

$$E = E_0(n_x^2 + n_y^2 + n_z^2),$$

where

$$E_0 = \frac{\pi^2\hbar^2}{2ma^2} = \frac{\pi^2(c\hbar)^2}{2mc^2 \cdot a^2} = \frac{\pi^2(1.97\times 10^{-11})^2}{2\times 939\times 10^{-24}} = 2.04\ \text{MeV}.$$

(n_x, n_y, n_z)	Number of states	Number of nucleons	E
(111)	1	4	$3E_0$
(211) (121) (112)	3	12	$6E_0$
(221) (122) (212)	3	12	$9E_0$
(311) (131) (113)	3	12	$11E_0$
(222)	1	4	$12E_0$
(123) (132) (231) (213) (312) (321)	6	24	$14E_0$

According to Pauli's principle, each state can accommodate one pair of neutrons and one pair of protons, as shown in the table.

For ^{56}Fe, $E_{\max} = 14E_0 = 2.04 \times 14 = 28.6$ MeV.

2064

Light nuclei in the shell model.

(a) Using the harmonic-oscillator shell model, describe the expected configurations for the ground states of the light stable nuclei with $A \leq 4$, specifying also their total L, S, J and T quantum numbers and parity.

(b) For ^{4}He, what states do you expect to find at about one oscillator quantum of excitation energy?

(c) What radioactive decay modes are possible for each of these states?

(d) Which of these states do you expect to find in ^{4}H? Which do you expect to find in ^{4}Be?

(e) Which of the excited states of ^{4}He do you expect to excite in α-particle inelastic scattering? Which would you expect to be excited by proton inelastic scattering?

(*Princeton*)

Solution:

(a) According to Fig. 2.11 we have

$A = 1$: The stable nucleus ^{1}H has configuration: $p(1s_{1/2})^1$,

$$L = 0,\ S = 1/2,\ J^p = 1/2^+,\ T = 1/2\,.$$

$A = 2$: The stable nucleus ^{2}H has configuration: $p(1s_{1/2})^1, n(1s_{1/2})^1$,

$$L = 0,\ S = 1,\ J^p = 1^+,\ T = 0\,.$$

$A = 3$: The stable nucleus ^{3}He has configuration: $p(1s_{1/2})^2, n(1s_{1/2})^1$,

$$L = 0,\ S = 1/2,\ J^p = 1/2^+,\ T = 1/2\,.$$

$A = 4$: The stable nucleus ^{4}He has configuration: $p(1s_{1/2})^2, n(1s_{1/2})^2$,

$$L = 0,\ S = 0,\ J^p = 0^+,\ T = 0\,.$$

(b) Near the first excited state of the harmonic oscillator, the energy level is split into two levels $1p_{3/2}$ and $1p_{1/2}$ because of the LS coupling of the p state. The isospin of ^{4}He is $T_z = 0$, $T = 0$ for the ground state. So the possible excitated states are the following:

(i) When a proton (or neutron) is of $1p_{3/2}$ state, the other of $1s_{1/2}$ state, the possible coupled states are $1^-, 2^-$ ($T = 0$ or $T = 1$).

(ii) When a proton (or neutron) is of $1p_{1/2}$ state, the other of $1s_{1/2}$ state, the possible coupled states are $0^-, 1^-$ ($T = 0$ or 1).

(iii) When two protons (or two neutrons) are of $1p_{1/2}$ (or $1p_{3/2}$) state, the possible coupled state is 0^+ ($T = 0$).

(c) The decay modes of the possible states of ^{4}He are:

	J^p	T	Decay modes
Ground state:	0^+	0	*Stable*
Excited states:	0^+	0	p
	0^-	0	p, n
	2^-	0	p, n
	2^-	1	p, n
	1^-	1	$p, n\gamma$
	0^-	1	p, n
	1^-	1	$p, n\gamma$
	1^-	0	p, n, d

(d) ^{4}H has isospin $T = 1$, so it can have all the states above with $T = 1$, namely $2^-, 1^-, 0^-$.

The isospin of ^{4}Be is $T \geq 2$, and hence cannot have any of the states above.

(e) $\alpha - \alpha$ scattering is between two identical nuclei, so the total wave function of the final state is exchange symmetric and the total angular momentum is conserved

In the initial state, the two α-particles have $L = 0, 2, \ldots$

In the final state, the two α-particles are each of 0^- state, $L = 0, 2 \ldots$

Thus an α-particle can excite ^{4}He to 0^- state while a proton can excite it to 2^-, or 0^- states.

2065

Explain the following statements on the basis of physical principles:

(a) The motion of individual nucleons inside a nucleus may be regarded as independent from each other even though they interact very strongly.

(b) All the even-even nuclei have 0^+ ground state.

(c) Nuclei with outer shells partially filled by odd number of nucleons tend to have permanent deformation.

(*SUNY, Buffalo*)

Solution:

(a) The usual treatment is based on the assumption that the interaction among nucleons can be replaced by the action on a nucleon of the mean field produced by the other nucleons. The nucleons are considered to move independently of one another. Despite the high nucleon density inside a nucleus it is assumed that the individual interactions between nucleons do not manifest macroscopically. Since nucleons are fermions, all the low energy levels of the ground state are filled up and the interactions among nucleons cannot excite a nucleon to a higher level. We can then employ a model of moderately weak interaction to describe the strong interactions among nucleons.

(b) According to the nuclear shell model, the protons and neutrons in an even-even nucleus tend to pair off separately, i.e., each pair of neutrons or protons are in the same orbit and have opposite spins, so that the total angular momentum and total spin of each pair of nucleons are zero. It follows that the total angular momentum of the nucleus is zero. The parity of each pair of nucleons is $(-1)^{2l} = +1$, and so the total parity of the nucleus is positive. Hence for an even-even nucleus, $J^p = 0^+$.

(c) Nucleons in the outermost partially-filled shell can be considered as moving around a nuclear system of zero spin. For nucleons with $l \neq 0$, the orbits are ellipses. Because such odd nucleons have finite spins and magnetic moments, which can polarize the nuclear system, the nucleus tends to have permanent deformation.

2066

Explain the following:

(a) The binding energy of adding an extra neutron to a ^{3}He nucleus (or of adding an extra proton to a ^{3}H nucleus) to form ^{4}He is greater than 20 MeV. However neither a neutron nor a proton will bind stably to ^{4}He.

(b) Natural radioactive nuclei such as ^{232}Th and ^{238}U decay in stages, by α- and β-emissions, to isotopes of Pb. The half-lives of ^{232}Th and ^{238}U are greater than 10^9 years and the final Pb-isotopes are stable; yet the intermediate α-decay stages have much shorter half-lives – some less than 1 hour or even 1 second – and successive stages show generally a decrease in half-life and an increase in α-decay energy as the final Pb-isotope is approached.

(*Columbia*)

Solution:

(a) ^{4}He is a double-magic nucleus in which the shells of neutrons and protons are all full. So it is very stable and cannot absorb more neutrons or protons. Also, when a ^{3}He captures a neutron, or a ^{3}H captures a proton to form ^{4}He, the energy emitted is very high because of the high binding energy.

(b) The reason that successive stages of the decay of ^{232}Th and ^{238}U show a decrease in half-life and an increase in α-decay energy as the final Pb-isotopes are approached is that the Coulomb barrier formed between the α-particle and the daughter nucleus during α-emission obstructs the decay. When the energy of the α-particle increases, the probability of its penetrating the barrier increases, and so the half-life of the nucleus decreases. From the Geiger–Nuttall formula for α-decays

$$\log \lambda = A - BE_d^{-1/2} ,$$

where A and B are constants with A different for different radioactivity series, λ is the α-decay constant and E_d is the decay energy, we see that a small change in decay energy corresponds to a large change in half-life.

We can deduce from the liquid-drop model that the α-decay energy E_d increases with A. However, experiments show that for the radioactive family ^{232}Th and ^{238}U, E_d decreases as A increases. This shows that the liquid-drop model can only describe the general trend of binding energy change with A and Z, but not the fluctuation of the change, which can be explained only by the nuclear shell model.

2067

(a) What spin-parity and isospin would the shell model predict for the ground states of ${}^{13}_{5}$B, ${}^{13}_{6}$C, and ${}^{13}_{7}$N? (Recall that the $p_{3/2}$ shell lies below the $p_{1/2}$.)

(b) Order the above isobaric triad according to mass with the lowest-mass first. Briefly justify your order.

(c) Indicate how you could estimate rather closely the energy difference between the two lowest-mass members of the above triad.

(Wisconsin)

Solution:

(a) The isospin of the ground state of a nucleus is $I = |Z - N|/2$, where N, Z are the numbers of protons and neutrons inside the nucleus respectively. The spin-parity of the ground state of a nucleus is decided by that of the last unpaired nucleon. Thus (Fig. 2.11)

$$ {}^{13}_{5}B : J^p = \left(\frac{3}{2}\right)^- , \quad \text{as the unpaired proton is in } 1p_{\frac{3}{2}} \text{ state} , $$

$$ I = \frac{3}{2} ; $$

$$ {}^{13}_{6}C : J^p = \left(\frac{1}{2}\right)^- , \quad \text{as the unpaired neutron is in } 1p_{1/2} \text{ state} , $$

$$ I = \frac{1}{2} ; $$

$$ {}^{13}_{7}N : J^p = \left(\frac{1}{2}\right)^- , \quad \text{as the unpaired proton is in } 1p_{1/2}\text{state} , $$

$$ I = \frac{1}{2} . $$

(b) Ordering the nuclei with the lowest-mass first gives ${}^{13}_{6}$C, ${}^{13}_{7}$N, ${}^{13}_{5}$B. ${}^{13}_{6}$C and ${}^{13}_{7}$N belong to the same isospin doublet. Their mass difference arises from the difference in Coulomb energy and the mass difference between neutron and proton, with the former being the chiefly cause. ${}^{13}_{7}$N has one more proton than ${}^{13}_{6}$C, and so has greater Coulomb energy and hence larger mass. Whereas ${}^{13}_{5}$B has fewer protons, it has more neutrons and is far from the line of stable nuclei and so is less tightly formed. Hence it has the largest mass.

(c) Consider the two lowest-mass members of the above triad, ${}^{13}_{6}$C and ${}^{13}_{6}$N. If the nuclei are approximated by spheres of uniform charge, each will have electrostatic (Coulomb) energy $W = 3Q^2/5R$, R being the nuclear radius $R \approx 1.4A^{1/2}$ fm. Hence the mass difference is

$$\begin{aligned}[M({}^{13}_{7}N) - M({}^{13}_{6}C)]c^2 &= \frac{3}{5R}(Q_N^2 - Q_C^2) - [M_n - M({}^1H)]c^2 \\ &= \frac{3\hbar c}{5R}\left(\frac{e^2}{\hbar c}\right)(7^2 - 6^2) - 0.78 \\ &= 0.6 \times \frac{197}{137} \times \frac{49 - 36}{1.4 \times 13^{1/3}} - 0.78 \\ &= 2.62 \text{ MeV}\,.\end{aligned}$$

2068

In the nuclear shell model, orbitals are filled in the order

$$1s_{1/2},\ 1p_{3/2},\ 1p_{1/2},\ 1d_{5/2},\ 2s_{1/2},\ 1d_{3/2},\ \text{etc.}$$

(a) What is responsible for the splitting between the $p_{3/2}$ and $p_{1/2}$ orbitals?

(b) In the model, ^{16}O ($Z = 8$) is a good closed-shell nucleus and has spin and parity $J^\pi = 0^+$. What are the predicted J^π values for ^{15}O and ^{17}O?

(c) For odd-odd nuclei a range of J^π values is allowed. What are the allowed values for ^{18}F ($Z = 9$)?

(d) For even-even nuclei (e.g. for ^{18}O) J^π is always 0^+. How is this observation explained?

(*Wisconsin*)

Solution:

(a) The splitting between $p_{3/2}$ and $p_{1/2}$ is caused by the spin-orbit coupling of the nucleons.

(b) Each orbital can accommodate $2j+1$ protons and $2j+1$ neutrons. Thus the proton configuration of ^{15}O is $(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2$, and its neutron configuration is $(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^1$. As the protons all pair up but the neutrons do not, the spin-parity of ^{15}O is determined by the angular momentum and parity of the unpaired neutron in the $1p_{\frac{1}{2}}$ state. Hence the spin-parity of ^{15}O of $J^p = 1/2^-$.

The proton configuration of ^{17}O is the same as that of ^{15}O, but its neutron configuration is $(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^1$. So the spin-parity of ^{17}O is that of the neutron in the $1d_{5/2}$ state, $J^p = 5/2^+$.

(c) The neutron configuration of ^{18}F is $(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^1$, its proton configuration is $(1s_{1/2})^2(1p_{3/2})^4(1p_{1/2})^2(1d_{5/2})^1$. As there are two unpaired nucleons, a range of J^p values are allowed, being decided by the neutron and proton in the $1d_{5/2}$ states. As $l_n = 2, l_p = 2$, the parity is $\pi = (-1)^{l_n+l_p} = +1$. As $j_n = 5/2, j_p = 5/2$, the possible spins are $J = 0, 1, 2, 3, 4, 5$. Thus the possible values of the spin-parity of ^{18}F are $0^+, 1^+, 2^+, 3^+, 4^+, 5^+$. (It is in fact 1^+.)

(d) For an even-even nucleus, as an even number of nucleons are in the lowest energy levels, the number of nucleons in every energy level is even. As an even number of nucleons in the same energy level have angular momenta of the same absolute value, and the angular momenta of paired nucleons are aligned oppositely because of the pairing force, the total angular momentum of the nucleons in an energy level is zero. Since all the proton shells and neutron shells have zero angular momentum, the spin of an even-even nucleus is zero. As the number of nucleons in every energy level of an even-even nucleus is even, the parity of the nucleus is positive.

2069

The single-particle energies for neutrons and protons in the vicinity of $^{208}_{82}\text{Pb}_{126}$ are given in Fig. 2.13. Using this figure as a guide, estimate or evaluate the following.

(a) The spins and parities of the ground state and the first two excited states of ^{207}Pb.

(b) The ground state quadrupole moment of ^{207}Pb.

(c) The magnetic moment of the ground state of ^{209}Pb.

(d) The spins and parities of the lowest states of $^{208}_{83}$Bi (nearly degenerate). What is the energy of the ground state of ^{208}Bi relative to ^{208}Pb?

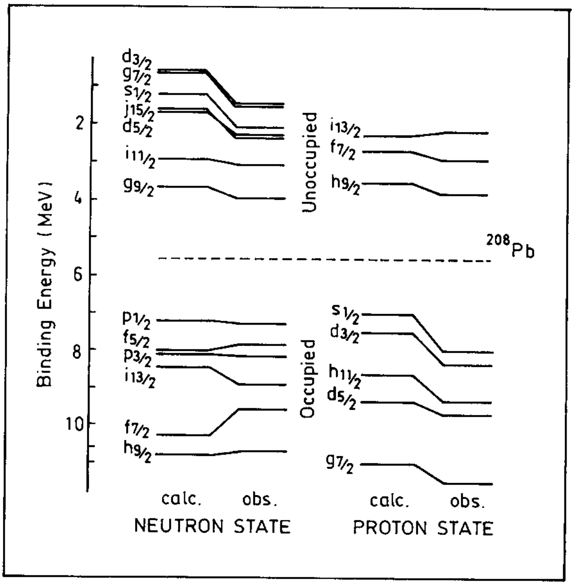

Fig. 2.13

(e) The isobaric analog state in ^{208}Bi of the ground state of ^{208}Pb is defined as

$$T_+|^{208}\text{Pb (ground state)}\rangle$$

with $T_+ = \sum_i t_+(i)$, where t_+ changes a neutron into a proton. What are the quantum numbers (spin, parity, isospin, z component of isospin) of the isobaric analog state? Estimate the energy of the isobaric analog state above the ground state of ^{208}Pb due to the Coulomb interaction.

(f) Explain why one does not observe super-allowed Fermi electron or positron emission in heavy nuclei.

(*Princeton*)

Solution:

(a) $^{207}_{82}$Pb consists of full shells with a vacancy for a neutron in $p_{1/2}$ level. The spin-parity of the ground state is determined by that of the unpaired neutron in $p_{1/2}$ and so is $(1/2)^-$. The first excited state is formed by a $f_{5/2}$ neutron transiting to $p_{1/2}$. Its J^p is determined by the single neutron vacancy left in $f_{5/2}$ level and is $(5/2)^-$. The second excited state is formed by a $p_{3/2}$ neutron refilling the $f_{5/2}$ vacancy (that is to say a $p_{3/2}$ neutron goes to $p_{1/2}$ directly). J^p of the nucleus in the second excited state is then determined by the single neutron vacancy in $p_{3/2}$ level and is $\left(\frac{3}{2}\right)^-$. Hence the ground and first two excited states of ^{207}Pb have $J^p = (\frac{1}{2})^-, (\frac{5}{2})^-, (\frac{3}{2})^-$.

(b) The nucleon shells of $^{207}_{82}$Pb are full except there is one neutron short in $p_{1/2}$ levels. An electric quadrupole moment can arise from polarization at the nuclear center caused by motion of neutrons. But as $J = 1/2$, the electric quadrupole moment of ^{207}Pb is zero.

(c) $^{209}_{82}$Pb has a neutron in $g_{9/2}$ outside the full shells. As the orbital motion of a neutron makes no contribution to the nuclear magnetic moment, the total magnetic moment equals to that of the neutron itself:

$\mu(^{209}Pb) = -1.91\mu_N, \mu_N$ being the nuclear magneton.

(d) For $^{208}_{83}$Bi, the ground state has an unpaired proton and an unpaired neutron, the proton being in $h_{9/2}$, the neutron being in $p_{1/2}$. As $J = 1/2 + 9/2 = 5$ (since both nucleon spins are antiparallel to l), $l_p = 5$, $l_n = 1$ and so the parity is $(-1)^{l_p+l_n} = +$, the states has $J^p = 5^+$. The first excited state is formed by a neutron in $f_{5/2}$ transiting to $p_{1/2}$ and its spin-parity is determined by the unpaired $f_{5/2}$ neutron and $h_{9/2}$ proton. Hence $J = 5/2 + 9/2 = 7$, parity is $(-1)^{1+5} = +$, and so $J^p = 7^+$. Therefore, the two lowest states have spin-parity 5^+ and 7^+.

The energy difference between the ground states of ^{208}Bi and ^{208}Pb can be obtained roughly from Fig. 2.13. As compared with ^{208}Pb, ^{208}Bi has one more proton at $h_{9/2}$ and one less neutron at $p_{1/2}$ we have

$$\Delta E = E(Bi) - E(Pb) \approx 7.2 - 3.5 + 2\Delta \approx 3.7 + 1.5 = 5.2 \text{ MeV},$$

where $\Delta = m_n - m_p$, i.e., the ground state of ^{208}Bi is 5.2 MeV higher than that of ^{208}Pb.

(e) As T_+ only changes the third component of the isospin,

$$T_+|T, T_3\rangle = A|T, T_3 + 1\rangle .$$

Thus the isobaric analog state should have the same spin, parity and isospin, but a different third component of the isospin of the original nucleus. As ^{208}Pb has $J^p = 0^+$, $T = 22$, $T_3 = -22$, ^{208}Bi, the isobaric analog state of ^{208}Pb, has the same J^p and T but a different $T_3 = -21$. The energy difference between the two isobaric analog states is

$$\Delta E \approx \frac{6}{5}\frac{Ze^2}{R} + (m_H - m_n)c^2 = \frac{6}{5}\frac{Z\hbar c}{R}\left(\frac{e^2}{\hbar c}\right) - 0.78$$

$$= \frac{6 \times 82 \times 197}{5 \times 1.2 \times 208^{1/3} \times 137} - 0.78 = 19.1 \text{ MeV}.$$

(f) The selection rules for super-allowed Fermi transition are $\Delta J = 0$, $\Delta P = +$, $\Delta T = 0$, so the wave function of the daughter nucleus is very similar to that of the parent. As the isospin is a good quantum number super-allowed transitions occur generally between isospin multiplets. For a heavy nucleus, however, the difference in Coulomb energy between isobaric analog states can be 10 MeV or higher, and so the isobaric analogy state is highly excited. As such, they can emit nucleons rather than undergo β-decay.

2070

The simplest model for low-lying states of nuclei with N and Z between 20 and 28 involves only $f_{7/2}$ nucleons.

(a) Using this model predict the magnetic dipole moments of $^{41}_{20}\mathrm{Ca}_{21}$ and $^{41}_{21}\mathrm{Sc}_{20}$. Estimate crudely the electric quadrupole moments for these two cases as well.

(b) What states are expected in $^{42}_{20}\mathrm{Ca}$ according to an application of this model? Calculate the magnetic dipole and electric quadrupole moments for these states. Sketch the complete decay sequence expected experimentally for the highest spin state.

(c) The first excited state in $^{43}_{21}\mathrm{Ca}_{23}$ is shown below in Fig. 2.14 with a half-life of 34 picoseconds for decay to the ground state. Estimate the lifetime expected for this state on the basis of a single-particle model. The

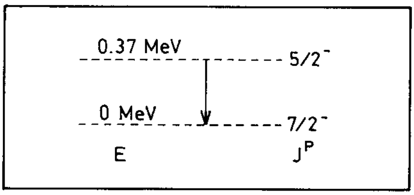

Fig. 2.14

experimental values are

$$\mu_n = -1.91\mu_N, \qquad \mu(^{41}Ca) = -1.59\mu_N$$

$$\mu_p = 2.79\mu_N, \qquad \mu(^{41}Sc) = 5.43\mu_N$$

(*Princeton*)

Solution:

(a) ^{41}Ca has a neutron and ^{41}Sc has a proton outside closed shells in state $1f_{7/2}$. As closed shells do not contribute to the nuclear magnetic moment, the latter is determined by the extra-shell nucleons. The nuclear magnetic moment is given by

$$\boldsymbol{\mu} = g\mathbf{j}\mu_N \,,$$

where $\mathbf{j}$ is the total angular momentum, μ_N is the nuclear magneton. For a single nucleon in a central field, the g-factor is (**Problem 2061**)

$$g = \frac{(2j-1)g_l + g_s}{2j} \qquad \text{for } j = l + \frac{1}{2} \,,$$

$$g = \frac{(2j+3)g_l - g_s}{2(j+1)} \qquad \text{for } j = l - \frac{1}{2} \,.$$

For neutron, $g_l = 0$, $g_s = g_n = -\frac{1.91}{\frac{1}{2}} = -3.82$. As $l = 3$ and $j = \frac{7}{2} = 3 + \frac{1}{2}$, we have for ^{41}Ca

$$\mu(^{41}Ca) = -\frac{3.82}{2j} \times j\mu_N = -1.91\mu_N \,.$$

For proton, $g_l = 1$, $g_s = g_p = \frac{2.79}{1/2} = 5.58$. As $j = \frac{7}{2} = 3 + \frac{1}{2}$, we have for ^{41}Sc

$$\mu(^{41}Sc) = \frac{(7-1) + 5.58}{7} \times \frac{7}{2}\mu_N = 5.79\mu_N \,.$$

Note that these values are only in rough agreement with the given experimental values.

The electric quadrupole moment of ^{41}Sc, which has a single proton outside closed shells, is given by

$$Q(^{41}Sc) = -e^2\langle r^2\rangle \frac{2j-1}{2(j+1)} = -\langle r^2\rangle \frac{2j-1}{2(j+1)} \,,$$

where $\langle r^2 \rangle$ is the mean-square distance from the center and the proton charge is taken to be one. For an order-of-magnitude estimate take $\langle r^2 \rangle = (1.2 \times A^{1/3})^2\ fm^2$. Then

$$Q(^{41}Sc) = -\frac{6}{9} \times (1.2 \times 41^{\frac{1}{3}})^2 = -1.14 \times 10^{-25}\ cm^2\,.$$

^{41}Ca has a neutron outside the full shells. Its electric quadrupole moment is caused by the polarization of the neutron relative to the nucleus center and is

$$Q(^{41}Ca) \approx \frac{Z}{(A-1)^2} |Q(^{41}Sc)| = 1.43 \times 10^{-27}\ cm^2\,.$$

(b) As shown in Fig. 2.15 the ground state of ^{42}Ca nucleus is 0^+. The two last neutrons, which are in $f_{7/2}$ can be coupled to form levels of $J = 7, 6, 5 \ldots, 0$ and positive parity. Taking into account the antisymmetry for identical particles, the possible levels are those with $J = 6, 4, 2, 0$. (We require $L + S =$ even, see **Problem 2054**. As $S = 0$, $J =$ even.)

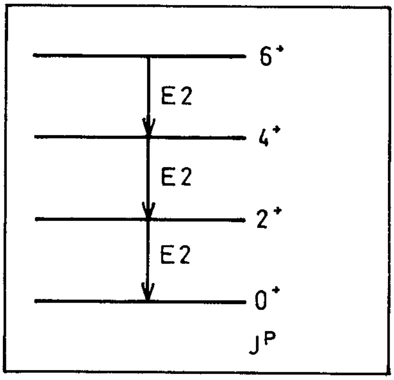

Fig. 2.15

The magnetic dipole moment $\boldsymbol{\mu}$ of a two-nucleon system is given by

$$\boldsymbol{\mu} = g\mathbf{J}\mu_N = (g_1\mathbf{j}_1 + g_2\mathbf{j}_2)\mu_N$$

with $\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$. As

$$g\mathbf{J}^2 = g_1\mathbf{j}_1 \cdot \mathbf{J} + g_2\mathbf{j}_2 \cdot \mathbf{J}\,,$$

$$\mathbf{j}_1 \cdot \mathbf{J} = \frac{1}{2}(\mathbf{J}^2 + \mathbf{j}_1^2 - \mathbf{j}_2^2)\,,$$

$$\mathbf{j}_2 \cdot \mathbf{J} = \frac{1}{2}(\mathbf{J}^2 + \mathbf{j}_2^2 - \mathbf{j}_1^2)\,,$$

we have

$$g\mathbf{J}^2 = \frac{1}{2}(g_1 + g_2)\mathbf{J}^2 + \frac{1}{2}(g_1 - g_2)(\mathbf{j}_1^2 - \mathbf{j}_2^2)\,.$$

or

$$g = \frac{1}{2}(g_1 + g_2) + \frac{1}{2}(g_1 - g_2)\frac{j_1(j_1+1) - j_2(j_2+1)}{J(J+1)}\,.$$

For ^{42}Ca, the two nucleons outside full shells each has $j = 7/2$. As

$$g_1 = g_2 = \frac{-3.82}{j_1}\,, \qquad j_1 = \frac{7}{2}\,,$$

we have $\mu(^{42}Ca) = g_1 J\mu_N = -1.09 J\mu_N$ with $J = 0, 2, 4, 6$.

The ground-state quadrupole moment of ^{42}Ca is $Q = 0$. One can get the excited state quadrupole moment using the reduced transition rate for γ-transition

$$B(E2, 2^+ \to 0^+) = \frac{e^2 Q_0^2}{16\pi}$$

where Q_0 is the intrinsic electric quadrupole moment. The first excited state 2^+ of ^{42}Ca has excitation energy 1.524 MeV and

$$B(E2 : 2^+ \to 0^+) = 81.5e^2\ fm^4\,,$$

or

$$Q_0 = \sqrt{16\pi \times 81.5} = 64\ fm^2\,.$$

For other states the quadrupole moments are given by

$$Q = \frac{K^2 - J(J+1)}{(J+1)(2J+3)}Q_0 = -\frac{J(J+1)Q_0}{(J+1)(2J+3)} = \frac{-J}{2J+3}Q_0$$

as $K = 0$. Thus $Q = 18.3\ fm^2$ for $J = 2$, $23.3\ fm^2$ for $J = 4$, and $25.6\ fm^2$ for $J = 6$.

(c) The selection rule for the γ-transition $(\frac{5}{2})^- \to (\frac{7}{2})^-$ is $(\frac{5}{2} - \frac{7}{2}) \leq L \leq \frac{5}{2} + \frac{7}{2}$, i.e. $L = 1, 2, 3, 4, 5, 6$, with the lowest order having the highest probability, for which parity is conserved. Then the most probable are magnetic dipole transition M_1 for which $\Delta P = -(-1)^{1+1} = +$, or electric quadrupole transition $E2$ for which $\Delta P = (-1)^2 = +$. According to the single-particle model (**Problem 2093**),

$$\lambda_{M1} = \frac{1.9(L+1)}{L[(2L+1)!!]^2}\left(\frac{3}{L+3}\right)^2\left(\frac{E_\gamma}{197}\right)^{2L+1} \times (1.4 \times A^{1/3})^{2L-2} \times 10^{21}$$

$$= \frac{1.9 \times 2}{3^2}\left(\frac{3}{4}\right)^2\left(\frac{0.37}{197}\right)^3 (1.4 \times 43^{1/3})^0 \times 10^{21}$$

$$= 1.57 \times 10^{12}\ s^{-1}\,,$$

$$\lambda_{E2} = \frac{4.4(L+1)}{L[(2L+1)!!]^2}\left[\frac{3}{L+3}\right]^2\left(\frac{E_\gamma}{197}\right)^{2L+1} \times (1.4 \times A^{1/3})^{2L} \times 10^{21}$$

$$= \frac{4.4 \times 3}{2 \times (5 \times 3)^2}\left(\frac{3}{L+3}\right)^2\left(\frac{0.37}{197}\right)^5 (1.4 \times 43^{1/3})^4 \times 10^{21}$$

$$= 1.4 \times 10^8\ s^{-1}\,.$$

As $\lambda_{E2} \ll \lambda_{M1}$, $E2$ could be neglected, and so

$$T_{1/2} \approx \frac{\ln 2}{\lambda_{M1}} = \frac{\ln 2}{1.57 \times 10^{12}} = 4.4 \times 10^{-13}\ s\,.$$

This result from the single-particle model is some 20 times smaller than the experimental value. The discrepancy is probably due to γ-transition caused by change of the collective motion of the nucleons.

2071

The variation of the binding energy of a single neutron in a "realistic" potential model of the neutron-nucleus interaction is shown in Fig. 2.16.

(a) What are the neutron separation energies for $^{40}_{20}$Ca and $^{208}_{82}$Pb?

(b) What is the best neutron magic number between those for ^{40}Ca and ^{208}Pb?

(c) Draw the spectrum including spins, parities and approximate relative energy levels for the lowest five states you would expected in ^{210}Pb and explain.

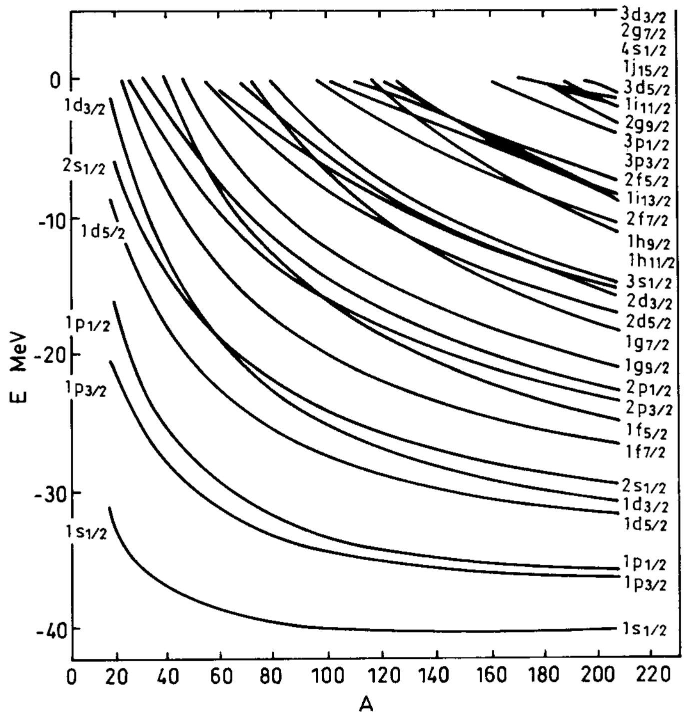

Fig. 2.16

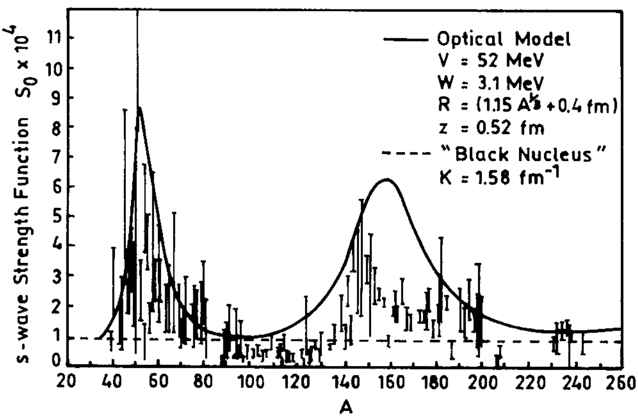

Fig. 2.17

(d) The s-wave neutron strength function S_0 is defined as the ratio of the average neutron width $\langle\Gamma_n\rangle$ to the average local energy spacing $\langle D\rangle$:

$$S_0 = \langle\Gamma_n\rangle/\langle D\rangle\,.$$

Figure 2.17 shows the variation of the thermal neutron strength function S_0 with mass number A. Explain the location of the single peak around $A \approx 50$, and the split peak around $A \approx 160$. Why is the second peak split?

(*Princeton*)

Solution:

(a) The outermost neutron of ^{40}Ca is the twentieth one. Figure 2.16 gives for $A = 40$ that the last neutron is in $1d_{3/2}$ shell with separation energy of about 13 MeV.

^{208}Pb has full shells, the last pair of neutrons being in $3p_{1/2}$ shell. From Fig. 2.16 we note that for $A = 208$, the separation energy of each neutron is about 3 MeV.

(b) The neutron magic numbers between ^{40}Ca and ^{208}Pb are 28, 50 and 82. For nuclei of $N = Z$, at the neutron magic number $N = 28$ the separation energies are about 13 MeV. At neutron number $N = 50$, the separation energies are also about 13 MeV. At N=82, the separation energies are about 12 MeV. However, for heavy nuclei, there are more neutrons than protons, so $A < 2N$. On account of this, for the nuclei of magic numbers 50 and 82, the separation energies are somewhat less than those given above. At the magic number 28 the separation energy is highest, and so this is the best neutron magic number.

(c) The last two neutrons of ^{210}Pb are in $2g_{9/2}$ shell, outside of the double-full shells. As the two nucleons are in the same orbit and will normally pair up to $J = 0$, the even-even nucleus has ground state 0^+.

The two outermost neutrons in $2g_{9/2}$ of ^{210}Pb can couple to form states of $J = 9, 8, 7\ldots$. However a two-neutron system has isospin $T = 1$. As the antisymmetry of the total wave function requires $J + T =$ odd, the allowed J are 8, 6, 4, 2, 0 and the parity is positive. Thus the spin-parities of the lowest five states are $8^+, 6^+, 4^+, 2^+, 0^+$. Because of the residual interaction, the five states are of different energy levels as shown in Fig. 2.18.

(d) Near $A = 50$ the s-wave strength function has a peak. This is because when $A = 50$ the excitation energy of $3s$ energy level roughly equals the neutron binding energy. A calculation using the optical model gives the

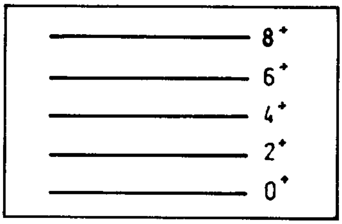

Fig. 2.18

shape of the peak as shown in Fig. 2.17 (solid curve). When $150 < A < 190$, the s-wave strength function again peaks due to the equality of excitation energy of $4s$ neutron and its binding energy. However, nuclear deformation in this region is greater, particularly near $A = 160$ to 170, where the nuclei have a tendency to deform permanently. Here the binding energies differ appreciably from those given by the single-particle model: the peak of the s-wave strength function becomes lower and splits into two smaller peaks.

2072

Figure 2.19 gives the low-lying states of ^{18}O with their spin-parity assignments and energies (in MeV) relative to the 0^+ ground state.

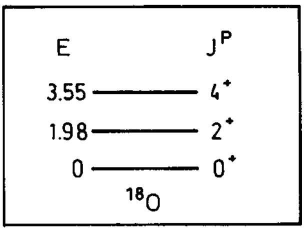

Fig. 2.19

(a) Explain why these J^p values are just what one would expect in the standard shell model.

(b) What J^p values are expected for the low-lying states of ^{19}O?

(c) Given the energies (relative to the ground state) of these ^{18}O levels, it is possible within the shell model, ignoring interconfiguration interactions, to compute the energy separations of the ^{19}O levels. However, this requires familiarity with complicated Clebsch–Gordon coefficients. To simplify matters, consider a fictitious situation where the 2^+ and 4^+ levels of ^{18}O have the energies 2 MeV and $6\frac{2}{3}$ MeV respectively. For this fictitious world, compute the energies of the low-lying ^{19}O levels.

(*Princeton*)

Solution:

(a) In a simple shell model, ignoring the residual interactions between nucleons and considering only the spin-orbit coupling, we have for a system of A nucleons,

$$H = \Sigma H_i \, ,$$

with

$$\begin{aligned} H_i &= T_i + V_i \, , \\ V_i &= V_0^i(r) + f(r)\boldsymbol{S}_i \cdot \boldsymbol{l}_i \, , \\ H_i\Psi_i &= E_i\Psi_i \, , \\ \Psi &= \prod_{i=1}^{A} \psi_i \, . \end{aligned}$$

When considering residual interactions, the difference of energy between different interconfigurations of the nucleons in the same level must be taken into account.

For ^{18}O nucleus, the two neutrons outside the full shells can fill the $1d_{5/2}$, $2s_{1/2}$ and $1d_{3/2}$ levels (see Fig. 2.16). When the two nucleons are in the same orbit, the antisymmetry of the system's total wave function requires $T + J =$ odd. As $T = 1, J$ is even. Then the possible ground and excited states of ^{18}O are:

$$\begin{aligned} (1d_{5/2})^2: &\quad J = 0^+, 2^+, 4^+, \quad T = 1 \, , \\ (1d_{5/2}2s_{1/2}): &\quad J = 2^+, \quad T = 1 \, , \\ (2s_{1/2})^2: &\quad J = 0^+, \quad T = 1 \, , \\ (1d_{3/2})^2: &\quad J = 0^+, 2^+, \quad T = 1 \, . \end{aligned}$$

The three low-lying states of ^{18}O as given in Fig. 2.19, $0^+, 2^+, 4^+$, should then correspond to the configuration $(1d_{5/2})^2$. However, when considering the energies of the levels, using only the $(d_{5/2})^2$ configuration does not agree well with experiment. One must also allow mixing the configurations $1d_{5/1}, 2s_{1/2}, 1d_{3/2}$, which gives fairly good agreement with the experimental values, as shown in Fig. 2.20.

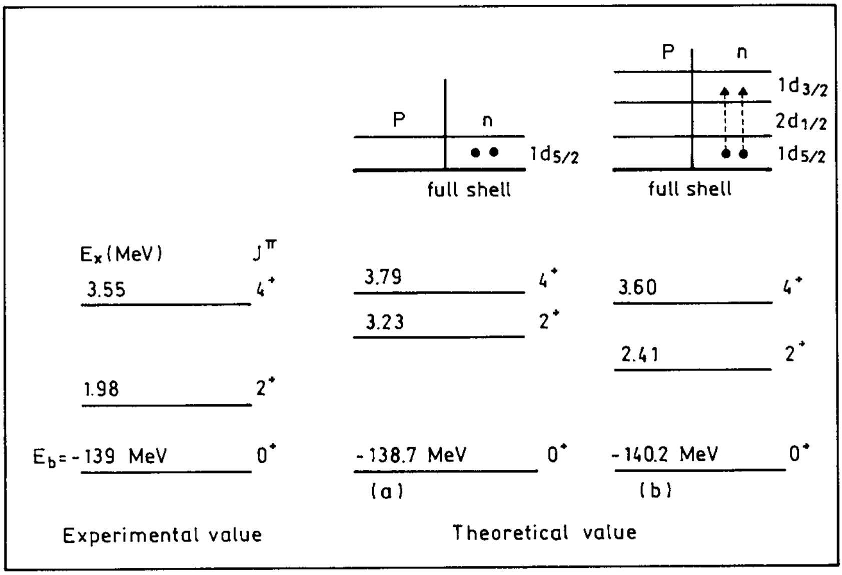

Fig. 2.20

(b) To calculate the lowest levels of ^{19}O using the simple shell model and ignoring interconfiguration interactions, we consider the last unpaired neutron. According to Fig. 2.16, it can go to $1d_{5/2}, 2s_{1/2}$, or $1d_{3/2}$. So the ground state is $\left(\frac{5}{2}\right)^+$, the first excited state $\left(\frac{1}{2}\right)^+$, and the second excited state $\left(\frac{3}{2}\right)^+$.

If interconfiguration interactions are taken into account, the three neutrons outside the full shells can go into the $1d_{5/2}$ and $2s_{1/2}$ orbits to form the following configurations:

$$[(d_{5/2})^3]_{5/2,m}, \quad [(d_{5/2})^2 s_{1/2}]_{5/2,m}, \quad [d_{5/2}(s_{1/2})_0^2]_{5/2,m}, J^p = \left(\frac{5}{2}\right)^+ ,$$

$$[(d_{5/2})_0^2 s_{1/2}]_{1/2,m}, J^p = \left(\frac{1}{2}\right)^+ ,$$

$$[(d_{5/2})^3]_{3/2,m}, \quad [(d_{5/2})_2^2 s_{1/2}]_{3/2,m}, J^p = \left(\frac{3}{2}\right)^+ .$$

Moreover, states with $J^p = \frac{7^+}{2}, \frac{9^+}{2}$ are also possible.

(c) In the fictitious case the lowest excited states of ^{18}O are $0^+, 2^+, 4^+$ with energies $0, 2, 6\frac{2}{3}$ MeV as shown in Fig. 2.21.

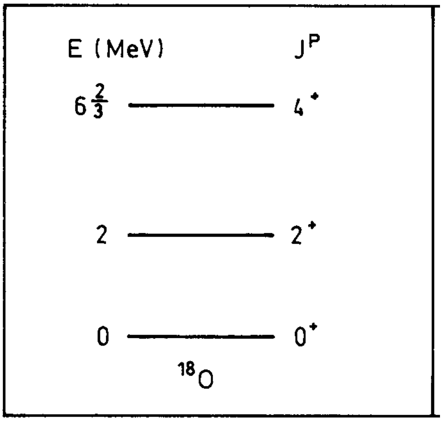

Fig. 2.21

This fictitious energy level structure corresponds to the rotational spectrum of an even-even nucleus, for in the latter we have

$$\frac{E_2}{E_1} = \frac{J_2(J_2+1)}{J_1(J_1+1)} = \frac{4(4+1)}{2(2+1)} = \frac{6\frac{2}{3}}{2} .$$

Taking this assumption as valid, one can deduce the moment of inertia I of ^{18}O. If this assumption can be applied to ^{19}O also, and if the moments of inertia of ^{19}O, ^{18}O can be taken to be roughly equal, then one can estimate the energy levels of ^{19}O. As $E_J = \frac{\hbar^2}{2I}J(J+1)$, we have for ^{18}O

$$\frac{\hbar^2}{2I} = \frac{E_J}{J(J+1)} = \frac{2}{2(2+1)} = \frac{1}{3} \text{ MeV} .$$

Assume that I is the same for ^{19}O. From (b) we see that the three lowest rotational levels of ^{19}O correspond to $J = \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$. Hence

$$E_{5/2} = 0, \text{ being the ground state of } {}^{19}O\,,$$

$$E_{7/2} = \frac{1}{3}\left[\frac{7}{2}\left(\frac{7}{2}+1\right) - \frac{5}{2}\left(\frac{5}{2}+1\right)\right] = 2\frac{1}{3} \text{ MeV}\,,$$

$$E_{9/2} = \frac{1}{3} \times \frac{1}{4}(9 \times 11 - 5 \times 7) = 5\frac{1}{3} \text{ MeV}\,.$$

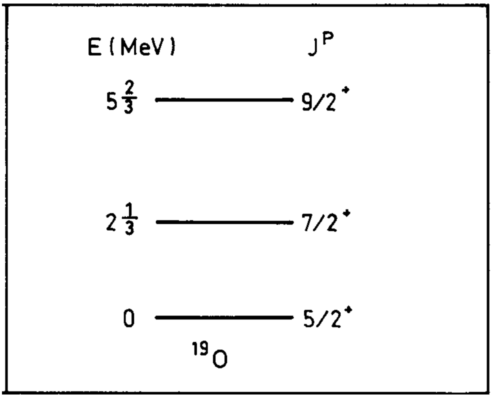

Fig. 2.22

2073

The following nonrelativistic Hamiltonians can be used to describe a system of nucleons:

$$H_0 = \sum_i \frac{\boldsymbol{p}_i^2}{2m} + \frac{1}{2}m\omega_0^2 \mathbf{r}_i^2\,,$$

$$H_1 = H_0 - \sum_i \beta \hat{\mathbf{l}}_i \cdot \mathbf{s}_i\,,$$

$$H_2 = H_1 - \sum_i \frac{1}{2}m\omega^2(2z_i^2 - x_i^2 - y_i^2)\,,$$

where $\hbar\omega_0 \gg \beta \gg \hbar\omega$.

(a) For each Hamiltonian H_0, H_1, H_2, identify the exactly and approximately conserved quantities of the system. For the ground state of each model, give the appropriate quantum numbers for the last filled single-particle orbital when the number n of identical nucleons is 11, 13 and 15.

(b) What important additional features should be included when the low-lying states of either spherical or deformed nucleons are to be described?

(c) The known levels of Aluminum 27, $^{27}_{13}Al_{14}$, below 5 MeV are shown in Fig. 2.23. Which states correspond to the predictions of the spherical and of the deformed models?

(*Princeton*)

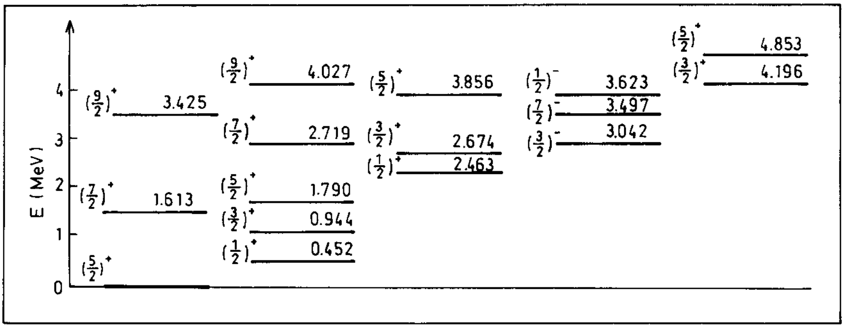

Fig. 2.23

Solution:

(a) For H_0 the exactly conserved quantities are energy E, orbital angular momentum L, total spin S, total angular momentum J, and parity.

For H_1 the exactly conserved quantities are E, J and parity, the approximately conserved ones are L and S.

For H_2 the exactly conserved quantities are E, the third component of the total angular momentum J_z, and parity, the approximately conserved ones are J, L, S.

As H_0 is an isotropic harmonic oscillator field, $E_N = \left(N + \frac{3}{2}\right)\hbar\omega$. The low-lying states are as follows (Figs. 2.12 and 2.16):

$N = 0$ gives the ground state $1s_{1/2}$.

$N = 1$ gives the p states, $1p_{3/2}$ and $1p_{1/2}$ which are degenerate.

$N = 2$ gives $2s$ and $1d$ states, $1d_{5/2}$, $2s_{1/2}$, $1d_{3/2}$, which are degenerate.

When the number of identical nucleons is $n = 11, 13, 15$, the last filled nucleons all have $N = 2$.

H_1 can be rewritten as

$$H_1 = H_0 - \sum_i \beta(\mathbf{l}_i \cdot \mathbf{s}_i) = H_0 - \sum_i \frac{1}{2}\beta[j_i(j_i+1) - l_i(l_i+1) - s_i(s_i+1)] .$$

The greater is j_i, the lower is the energy. For this Hamiltonian, some of the degeneracy is lost: $1p_{3/2}$ and $1p_{1/2}$ are separated, so are $1d_{3/2}$ and $1d_{5/2}$. 11 or 13 identical nucleons can fill up to the $1d_{5/2}$ state, while for $n = 15$, the last nucleon well go into the $2s_{1/2}$ state.

H_2 can be rewritten as

$$H_2 = H_1 - \sum_i \frac{1}{2} m\omega^2 r_i^2 (3\cos^2\theta - 1) ,$$

which corresponds to a deformed nucleus. For the Hamiltonain, $1p_{3/2}$, $1d_{3/2}$, and $1d_{5/2}$ energy levels are split further:

$1d_{5/2}$ level is split into $\left(\frac{1}{2}\right)^+, \left(\frac{3}{2}\right)^+, \left(\frac{5}{2}\right)^+$,
$1d_{3/2}$ level is split into $\left(\frac{1}{2}\right)^+, \left(\frac{3}{2}\right)^+$,
$1p_{3/2}$ level is split into $\left(\frac{1}{2}\right)^-, \left(\frac{3}{2}\right)^-$,

Let the deformation parameter be ε. The order of the split energy levels well depend on ε. According to the single-particle model of deformed nuclei, when $\varepsilon \approx 0.3$ (such as for ^{27}Al), the orbit of the last nucleon is

$\left(\frac{3}{2}\right)^+$ of the $1d_{5/2}$ level if $n = 11$,
$\left(\frac{5}{2}\right)^+$ of the $1d_{5/2}$ level if $n = 13$,
$\left(\frac{1}{2}\right)^+$ of the $2s_{1/2}$ level if $n = 15$.

(b) For a spherical nucleus, when considering the ground and low excited states, pairing effect and interconfiguration interactions are to be included. For a deformed nucleus, besides the above, the effect of the deforming field on the single-particle energy levels as well as the collective vibration and rotation are to be taken into account also.

(c) ^{27}Al is a deformed nucleus with $\varepsilon \approx 0.3$. The configurations of the 14 neutrons and 13 protons in a spherical nucleus are

$$n : (1s_{1/2})^2 (1p_{3/2})^4 (1p_{1/2})^2 (1d_{5/2})^6 ,$$

$$n : (1s_{1/2})^2 (1p_{3/2})^4 (1p_{1/2})^2 (1d_{5/2})^5 .$$

The ground state is given by the state of the last unpaired nucleon $(1d_{5/2})$: $J^p = \left(\frac{5}{2}\right)^+$.

If the nucleus is deformed, not only are energy levels like $1p_{3/2}$, $1d_{5/2}$, $1d_{3/2}$ split, the levels become more crowded and the order changes. Strictly speaking, the energy levels of ^{27}Al are filled up in the order of single-particle energy levels of a deformed nucleus. In addition, there is also collective motion, which makes the energy levels very complicated. Comparing the energy levels with theory, we have, corresponding to the levels of a spherical nucleus of the same J^p, the levels,

$$\begin{aligned}
\text{ground state :} \qquad & J^p = \left(\frac{5}{2}\right)^+, E = 0\,, \\
\text{excited states :} \qquad & J^p = \left(\frac{1}{2}\right)^+, E = 2.463 \text{ MeV}\,, \\
& J^p = \left(\frac{3}{2}\right)^+, E = 4.156 \text{ MeV}\,;
\end{aligned}$$

corresponding to the single-particle energy levels of a deformed nucleus the levels

$$\begin{aligned}
\text{ground state :} \qquad & K^p = \left(\frac{5}{2}\right)^+, E = 0\,, \\
\text{excited states :} \qquad & K^p = \left(\frac{1}{2}\right)^+, E = 0.452 \text{ MeV}\,, \\
& K^p = \left(\frac{1}{2}\right)^+, E = 2.463 \text{ MeV}\,, \\
& K^p = \left(\frac{1}{2}\right)^-, E = 3.623 \text{ MeV}\,, \\
& K^p = \left(\frac{3}{2}\right)^+, E = 4.196 \text{ MeV}\,,
\end{aligned}$$

Also, every K^p corresponds to a collective-rotation energy band of the nucleus given by

$$E_J = \frac{\hbar^2}{2I}[J(J+1) - K(K+1)]\,,$$

where $\quad K \neq 1/2, J = K, K+1, \ldots .$

$$E_J = \frac{\hbar^2}{2I}\left[J(J+1) - \frac{3}{4} + a - a(-1)^{J+1/2}\left(J+\frac{1}{2}\right)\right]\,,$$

where $\quad K = 1/2, J = K, K+1, \ldots .$

For example, for rotational bands $\left(\frac{5}{2}\right)^+(0)$, $\left(\frac{7}{2}\right)^+(1.613)$, $\left(\frac{9}{2}\right)^+(3.425)$, we have $K = \frac{5}{2}$,

$$\left(\frac{\hbar^2}{2I}\right)[(K+1)(K+2) - K(K+1)] = 1.613 \text{ MeV}\,,$$

$$\left(\frac{\hbar^2}{2I}\right)[(K+2)(K+3) - K(K+1)] = 3.425 \text{ MeV}\,.$$

giving $\frac{\hbar^2}{2I} \approx 0.222$ MeV. For rotational bands $\left(\frac{1}{2}\right)^+(0.452)$, $\left(\frac{3}{2}\right)^+(0.944)$, $\left(\frac{5}{2}\right)^+(1.790)$, $\left(\frac{7}{2}\right)^+(2.719)$, $\left(\frac{9}{2}\right)^+(4.027)$, we have

$$\frac{\hbar^2}{2I} \approx 0.150 \text{ MeV}, \qquad a \approx -3.175 \times 10^2\,.$$

Similarly for $\left(\frac{1}{2}\right)^-(3.623)$, $\left(\frac{7}{2}\right)^-(3.497)$ and $\left(\frac{3}{2}\right)^-(3.042)$ we have

$$\frac{\hbar^2}{2I} \approx 0.278 \text{ MeV}, \qquad a \approx 5.092\,.$$

2074

A recent model for collective nuclear states treats them in terms of interacting bosons. For a series of states that can be described as symmetric superposition of S and D bosons (i.e. of spins 0 and 2 respectively), what are the spins of the states having $N_d = 0, 1, 2$ and 3 bosons? If the energy of the S bosons is E_s and the energy of the D bosons is E_d, and there is a residual interaction between pairs of D bosons of constant strength α, what is the spectrum of the states with $N_s + N_d = 3$ bosons?

(*Princeton*)

Solution:

When $N_d = 0$, spin is 0,
$N_d = 1$, spin is 2,
$N_d = 2$, spin is 4,2,0,
$N_d = 3$, spin is 6, 4, 2, 0.
For states of $N_s + N_d = 3$, when

$$\begin{aligned}
N_d = 0 &: \ N_s = 3, \qquad E = 3E_s \,, \\
N_d = 1 &: \ N_s = 2, \qquad E = E_d + 2E_s \,, \\
N_d = 2 &: \ N_s = 1, \qquad E = 2E_d + E_s + \alpha \,, \\
N_d = 3 &: \ N_s = 0, \qquad E = 3E_d + 3\alpha \,.
\end{aligned}$$

2075

A simplified model of the complex nuclear interaction is the pairing force, specified by a Hamiltonian of the form

$$H = -g \begin{pmatrix} 1 & 1 & \cdot & \cdot & 1 \\ 1 & 1 & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & \cdot & \cdot & 1 \end{pmatrix} ,$$

in the two-identical-particle space for a single j orbit, with the basic states given by $(-1)^{j-m}|jm\rangle|j-m\rangle$. This interaction has a single outstanding eigenstate. What is its spin? What is its energy? What are the spins and energies of the rest of the two-particle states?

(*Princeton*)

Solution:

Suppose H is a $(j+\frac{1}{2}) \times (j+\frac{1}{2})$ matrix. The eigenstate can be written in the form

$$\Psi^{N=2} = \left(j+\frac{1}{2}\right)^{-1/2} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix},$$

where the column matrix has rank $(j+\frac{1}{2}) \times 1$. Then

$$\hat{H}\Psi^{N=2} = -g\left(j+\frac{1}{2}\right)\Psi^{N=2}.$$

Thus the energy eigenvalue of $\Psi^{N=2}$ is $-g\left(j+\frac{1}{2}\right)$. As the pairing force acts on states of $J=0$ only, the spin is zero.

As the sum of the energy eigenvalues equals the trace of the $\hat{H}$ matrix, $-g\left(j+\frac{1}{2}\right)$, and H is a negative quantity, all the eigenstates orthogonal to $\Psi^{N=2}$ have energy eigenvalues zero, the corresponding angular momenta being $J = 2, 4, 6\ldots$, etc.

5. NUCLEAR DECAYS (2076–2107)

2076

In its original (1911) form the Geiger–Nuttall law expresses the general relationship between α-particle range (R_α) and decay constant (λ) in natural α-radioactivity as a linear relation between $\log\lambda$ and $\log R$. Subsequently this was modified to an approximate linear relationship between $\log\lambda$ and some power of the α-particle energy, $E^x(\alpha)$.

Explain how this relationship between decay constant and energy is explained quantum-mechanically. Show also how the known general features of the atomic nucleus make it possible to explain the extremely rapid dependence of λ on $E(\alpha)$. (For example, from $E(\alpha) = 5.3$ MeV for Po^{210} to $E(\alpha) = 7.7$ MeV for Po^{214}, λ increases by a factor of some 10^{10}, from a half-life of about 140 days to one of 1.6×10^{-4} sec.)

(*Columbia*)

Solution:

α-decay can be considered as the transmission of an α-particle through the potential barrier of the daughter nucleus. Similar to that shown in

Fig. 2.7, where R is the nuclear radius, r_1 is the point where the Coulomb repulsive potential $V(r) = Zze^2/r$ equals the α-particle energy E. Using a three-dimensional potential and neglecting angular momentum, we can obtain the transmission coefficient T by the W.K.B. method:

$$T = e^{-2G},$$

where

$$G = \frac{1}{\hbar}\int_R^{r_1}(2m|E-V|)^{1/2}dr,$$

with $V = zZe^2/r$, $E = zZe^2/r_1$, $z = 2$, Ze being the charge of the daughter nucleus. Integration gives

$$\begin{aligned} G =& \frac{1}{\hbar}(2mzZe^2r_1)^{1/2}\left[\arccos\left(\frac{R}{r_1}\right) - \left(\frac{R}{r_1} - \frac{R^2}{r_1^2}\right)^{1/2}\right] \\ &\xrightarrow{\frac{R}{r_1}\to 0} \frac{1}{\hbar}(2mzZe^2r_1)^{1/2}\left[\frac{\pi}{2} - \left(\frac{R}{r_1}\right)^{1/2}\right]. \end{aligned}$$

Suppose the α-particle has velocity v_0 in the potential well. Then it collides with the walls $\frac{v_0}{R}$ times per unit time and the probability of decay per unit time is $\lambda = v_0T/R$. Hence

$$\ln\lambda = -\frac{\sqrt{2m}BR\pi}{\hbar}\left(E^{-\frac{1}{2}} - \frac{2}{\pi}B^{-\frac{1}{2}}\right) + \ln\frac{v_0}{R},$$

where $B = zZe^2/R$. This is a linear relationship between $\log\lambda$ and $E^{-1/2}$ for α-emitters of the same radioactive series.

For $_{84}$Po,

$$\begin{aligned} \log_{10}\frac{T(^{210}Po)}{T(^{214}Po)} =& 0.434[\ln\lambda(^{214}Po) - \ln\lambda(^{210}Po)] \\ =& 0.434\times\sqrt{2mc^2}zZ\left(\frac{e^2}{\hbar c}\right)\left(\frac{1}{\sqrt{E_{210}}} - \frac{1}{\sqrt{E_{214}}}\right) \\ =& \frac{0.434\times\sqrt{8\times 940}\times 2\times(84-2)}{137}\left(\frac{1}{\sqrt{5\cdot 3}} - \frac{1}{\sqrt{7\cdot 7}}\right) \\ \approx& 10. \end{aligned}$$

Thus the life-times differ by 10 orders of magnitude.

2077

The half-life of a radioactive isotope is strongly dependent on the energy liberated in the decay. The energy dependence of the half-life, however, follows quite different laws for α- and β-emitters.

(a) Derive the specific law for α-emitters.

(b) Indicate why the law for β-emitters is different by discussing in detail the difference between the two processes.

(*Columbia*)

Solution:

(a) For a quantum-mechanical derivation of the Geiger–Nuttall law for α-decays see **Problem 2076**.

(b) Whereas α-decay may be considered as the transmission of an α-particle through a Coulomb potential barrier to exit the daughter nucleus, β-decay is the result of the disintegration of a neutron in the nucleus into a proton, which remains in the nucleus, an electron and an antineutrino, which are emitted. Fermi has obtained the β-particle spectrum using a method similar to that for γ-emission. Basically the transition probability per unit time is given by Fermi's golden rule No. 2,

$$\omega = \frac{2\pi}{\hbar}|H_{fi}|^2\rho(E)\,,$$

where E is the decay energy, H_{fi} is the transition matrix element and $\rho(E) = \frac{dN}{dE}$ is the number of final states per unit energy interval.

For decay energy E, the number of states of the electron in the momentum interval p_e and $p_e + dp_e$ is

$$dN_e = \frac{V4\pi p_e^2 dp_e}{(2\pi\hbar)^3}\,,$$

where V is the volume of normalization. Similarly for the antineutrino we have

$$dN_\nu = \frac{4\pi p_\nu^2 dp_\nu}{(2\pi\hbar)^3}\,,$$

and so $dN = dN_e dN_\nu$. However p_e and p_ν are not independent. They are related through $E_e = \sqrt{p_e^2c^2 + m_e^2c^4}$, $E_\nu = p_\nu c$ by $E = E_e + E_\nu$. We can write $p_\nu = \frac{E-E_e}{c}$, and for a given E_e, $dp_\nu = \frac{dE_\nu}{c} = \frac{dE}{c}$. Thus

$$\frac{dN}{dE} = \int \frac{dN_e dN_\nu}{dE} = \frac{V^2}{4\pi^4\hbar^6c^3}\int_0^{p_{\max}} (E - E_e)^2 p_e^2 dp_e\,,$$

where $p_{\max}$ corresponds to the end-point energy of the β-particle spectrum $E_0 \approx E$, and hence

$$\lambda = \frac{2\pi}{\hbar}|H_{fi}|^2 \frac{dN}{dE} = \frac{g^2|M_{fi}|^2}{2\pi^3\hbar^7c^3}\int_0^{p_{\max}}(E - \sqrt{p_e^2c^2 + m_e^2c^4})^2 p_e^2 dp_e\,,$$

where $M_{fi} = \frac{VH_{fi}}{g}$ and g is the coupling constant.

In terms of the kinetic energy T, as

$$E_e = T + m_ec^2 = \sqrt{p_e^2c^2 + m_e^2c^4}\,, \qquad E = T_0 + m_ec^2\,,$$

the above integral can be written in the form

$$\int_0^{T_0}(T + m_ec^2)(T^2 + 2m_ec^2T)^{\frac{1}{2}}(T_0 - T)^2 dT\,.$$

This shows that for β-decays

$$\lambda \sim T_0^5\,,$$

which is the basis of the Sargent curve.

This relation is quite different from that for α-decays,

$$\lambda \sim \exp\left(-\frac{c}{\sqrt{E}}\right)\,,$$

where E is the decay energy and C is a constant.

2078

Natural gold $^{197}_{79}\mathrm{Au}$ is radioactive since it is unstable against α-decay with an energy of 3.3 MeV. Estimate the lifetime of $^{197}_{79}\mathrm{Au}$ to explain why gold does not burn a hole in your pocket.

(*Princeton*)

Solution:

The Geiger–Nuttall law

$$\log_{10}\lambda = C - DE_\alpha^{-1/2}\,,$$

where C, D are constants depending on Z, which can be calculated using quantum theory, E_α is the α-particle energy, can be used to estimate the life-time of ^{197}Au. For a rough estimate, use the values of C, D for Pb, $C \approx 52$, $D \approx 140$ (MeV)$^{\frac{1}{2}}$. Thus

$$\lambda \approx 10^{(52-140E^{-1/2})} \approx 10^{-25}\ s^{-1}$$

and so

$$T_{1/2} = \frac{1}{\lambda}\ln 2 \approx 6.9 \times 10^{24}\ s \approx 2.2 \times 10^{17}\ yr\,.$$

Thus the number of decays in a human's lifetime is too small to worry about.

2079

The half-life of ^{239}Pu has been determined by immersing a sphere of ^{239}Pu of mass 120.1 gm in liquid nitrogen of a volume enough to stop all α-particles and measuring the rate of evaporation of the liquid. The evaporation rate corresponded to a power of 0.231 W. Calculate, to the nearest hundred years, the half-life of ^{239}Pu, given that the energy of its decay alpha-particles is 5.144 MeV. (Take into account the recoil energy of the product nucleus.) Given conversion factors:

$$1 \text{ MeV} = 1.60206 \times 10^{-13} \text{ joule}\,,$$

$$1 \text{ atomic mass unit} = 1.66 \times 10^{-24} \text{ gm}\,.$$

(*SUNY, Buffalo*)

Solution:

The decay takes place according to $^{239}Pu \to \alpha + ^{235}U$.

The recoil energy of ^{235}U is

$$E_u = \frac{p_u^2}{2M_u} = \frac{p_\alpha^2}{2M_u} = \frac{2M_\alpha E_\alpha}{2M_u} = \frac{4}{235}E_\alpha\,.$$

The energy released per α-decay is

$$E = E_u + E_\alpha = \frac{239}{235}E_\alpha = 5.232 \text{ MeV}\,.$$

The decay rate is

$$\frac{dN}{dt} = \frac{0.231}{5.232 \times 1.60206 \times 10^{-13}} = 2.756 \times 10^{11}\ s^{-1}\,.$$

The number of ^{239}Pu is

$$N = \frac{120.1 \times 5.61 \times 10^{26}}{239 \times 939} = 3.002 \times 10^{23}\,.$$

The half-life is

$$T_{1/2} = \frac{\ln 2}{\lambda} = \frac{N \ln 2}{\frac{dN}{dt}} = \frac{3.002 \times 10^{23} \times \ln 2}{2.756 \times 10^{11}} = 7.55 \times 10^{11}\ s = 2.39 \times 10^{4}\ yr\,.$$

2080

^{8}Li is an example of a β-delayed particle emitter. The ^{8}Li ground state has a half-life of 0.85 s and decays to the 2.9 MeV level in Be as shown in Fig. 2.24. The 2.9 MeV level then decays into 2 alpha-particles with a half-life of 10^{-22} s.

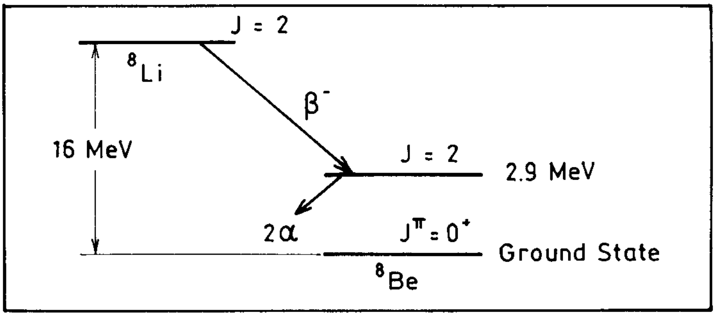

Fig. 2.24

(a) What is the parity of the 2.9 MeV level in ^{8}Be? Give your reasoning.

(b) Why is the half-life of the ^{8}Be 2.9 MeV level so much smaller than the half life of the ^{8}Li ground state?

(c) Where in energy, with respect to the ^{8}Be ground state, would you expect the threshold for ^{7}Li neutron capture? Why?

(*Wisconsin*)

Solution:

(a) The spin-parity of α-particle is $J^p = 0^+$. In $^8Be \to \alpha + \alpha$, as the decay final state is that of two identical bosons, the wave function is required to be exchange-symmetric. This means that the relative orbital quantum number l of the α-particles is even, and so the parity of the final state of the two α-particle system is

$$\pi_f = (+1)^2(-1)^l = +1 \, .$$

As the α-decay is a strong-interaction process, (extremely short half-life), parity is conserved. Hence the parity of the 2.9 MeV excited state of ^{8}Be is positive.

(b) The β-decay of the ^{8}Li ground state is a weak-interaction process. However, the α-decay of the 2.9 MeV excited state of ^{8}Be is a strong-interaction process with a low Coulomb barrier. The difference in the two interaction intensities leads to the vast difference in the lifetimes.

(c) The threshold energy for ^{7}Li neutron capture is higher than the ^{8}Be ground state by

$$\begin{aligned} M(^7\mathrm{Li}) + m(n) - M(^8Be) =& M(^7\mathrm{Li}) + m(n) - M(^8\mathrm{Li}) \\ &+ M(^8\mathrm{Li}) - M(^8Be) = S_n(^8\mathrm{Li}) + 16 \text{ MeV} \, . \end{aligned}$$

where $S_n(^8\mathrm{Li})$ is the energy of dissociation of ^{8}Li into ^{7}Li and a neutron. As

$$\begin{aligned} S_n(^8\mathrm{Li}) =& M(^7\mathrm{Li}) + M_n(n) - M(^8\mathrm{Li}) = 7.018223 + 1.00892 - 8.025018 \\ =& 0.002187 \text{ amu} = 2.0 \text{ MeV} \, , \end{aligned}$$

the threshold of neutron capture by ^{7}Li is about 18 MeV higher than the ground state of ^{8}Be. Note that as ^{8}Li is outside the stability curve against β-decay, the energy required for removal of a neutron from it in rather small.

2081

The following atomic masses have been determined (in amu):

(1) $^7_3\mathrm{Li}$ 7.0182
 ^7_4Be 7.0192

(2) ${}^{13}_{6}C$ 13.0076
${}^{13}_{7}N$ 13.0100

(3) ${}^{19}_{9}F$ 19.0045
${}^{19}_{10}Ne$ 19.0080

(4) ${}^{34}_{15}P$ 33.9983
${}^{34}_{16}S$ 33.9978

(5) ${}^{35}_{16}S$ 34.9791
${}^{35}_{17}Cl$ 34.9789

Remembering that the mass of the electron is 0.00055 amu, indicate which nuclide of each pair is unstable, its mode(s) of decay, and the approximate energy released in the disintegration. Derive the conditions for stability which you used.

(*Columbia*)

Solution:

As for each pair of isobars the atomic numbers differ by one, only β-decay or orbital electron capture is possible between them.

Consider β-decay. Let M_x, M_y, m_e represent the masses of the original nucleus, the daughter nucleus, and the electron respectively. Then the energy release in the β-decay is $E_d(\beta^-) = [M_x(Z, A) - M_y(Z+1, A) - m_e]c^2$. Expressing this relation in amu and neglecting the variation of the binding energy of the electrons in different atoms and shells, we have

$$\begin{aligned} E_d(\beta^-) &= [M_x(Z, A) - Zm_e - M_y(Z+1, A) + (Z+1)m_e - m_e]c^2 \\ &= [M_x(Z, A) - M_y(Z+1, A)]c^2 \,, \end{aligned}$$

where M indicates atomic mass. Thus β-decay can take place only if $M_x > M_y$. Similarly for β^+-decay, we have

$$E_d(\beta^+) = [M_x(Z, A) - M_y(Z-1, A) - 2m_e]c^2 \,,$$

and so β^+-decay can take place only if $M_x - M_y > 2m_e = 0.0011$ amu. In the same way we have for orbital electron capture (usually from the K shell)

$$E_d(i) = [M_x(Z, A) - M_y(Z-1, A)]c^2 - W_i \,.$$

where W_i is the binding energy of an electron in the ith shell, ~ 10 eV or 1.1×10^{-8} amu for K-shell, and so we require $M_x - M_y > W_i/c^2$

Let $\Delta = M(Z+1, A) - M(Z, A)$.

Pair (1), $\Delta = 0.001$ amu < 0.0011 amu, ${}^7_4$Be is unstable against K-electron capture.

Pair (2), $\Delta = 0.0024$ amu > 0.0011 amu, ${}^{13}_7$N is unstable against β-decay and K-electron capture.

Pair (3), $\Delta = 0.0035$ amu > 0.0011 amu, ${}^{19}_{10}$Ne is unstable against β^+-decay and K-electron capture.

Pair (4), $\Delta = -0.0005$ amu, ${}^{34}_{15}$P is unstable against β^--decay.

Pair (5), $\Delta = -0.0002$ amu, ${}^{35}_{16}$S is unstable against β^--decay.

2082

^{34}Cl positron-decays to ^{34}S. Plot a spectrum of the number of positrons emitted with momentum p as a function of p. If the difference in the masses of the neutral atoms of ^{34}Cl and ^{34}S is 5.52 MeV/c^2, what is the maximum positron energy?

(*Wisconsin*)

Solution:

^{34}Cl decays according to

$$^{34}Cl \to {}^{34}S + e^+ + \nu\,.$$

The process is similar to β^--decay and the same theory applies. The number of decays per unit time that emit a positron of momentum between p and $p + dp$ is (**Problem 2077(b)**)

$$I(p)dp = \frac{g^2|M_{fi}|^2}{2\pi^3\hbar^7c^3}(E_m - E)^2p^2dp\,,$$

where E_m is the end-point (total) energy of the β^+-spectrum. Thus $I(p)$ is proportional to $(E_m - E)^2p^2$, as shown in Fig. 2.25. The maximum β^+-particle energy is

$$\begin{aligned} E_{\max\beta^+} &= [M(^{34}Cl) - M(^{34}S) - 2m_e]c^2 = 5.52 \text{ MeV} - 1.022 \text{ MeV} \\ &= 4.50 \text{ MeV}\,. \end{aligned}$$

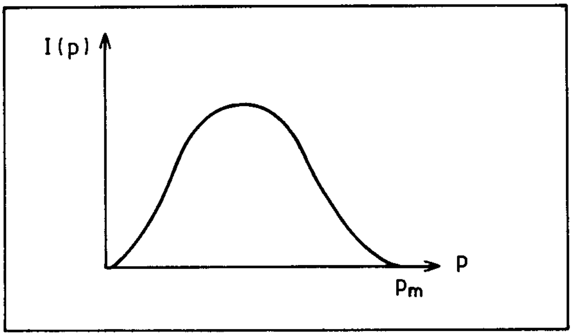

Fig. 2.25

2083

Both ^{161}Ho and ^{163}Ho decay by allowed electron capture to Dy isotopes, but the Q_{EC} values are about 850 keV and about 2.5 keV respectively. (Q_{EC} is the mass difference between the final ground state of nucleus plus atomic electrons and the initial ground state of nucleus plus atomic electrons.) The Dy orbital electron binding energies are listed in the table bellow. The capture rate for $3p_{1/2}$ electrons in ^{161}Ho is about 5% of the 3s capture rate. Calculate the $3p_{1/2}$ to 3s relative capture rate in ^{161}Ho. How much do the $3p_{1/2}$ and 3s capture rates change for both ^{161}Ho and ^{163}Ho if the Q_{EC} values remain the same, but the neutrino, instead of being massless, is assumed to have a mass of 50 eV?

Orbital	Binding Energy (keV)
$1s$	54
$2s$	9
$2p_{1/2}$	8.6
$3s$	2.0
$3p_{1/2}$	1.8

(Princeton)

Solution:

As ^{161}Ho and ^{163}Ho have the same nuclear charge Z, their orbital-electron wave functions are the same, their 3s and $3p_{1/2}$ waves differing only in phase. So the transition matrix elements for electron capture are also the same.

The decay constant is given by

$$\lambda \approx A|M_{if}|^2\rho(E)\,,$$

where M_{if} is the transition matrix element, $\rho(E)$ is the density of states, and A is a constant. For electron capture, the nucleus emits only a neutrino, and so the process is a two-body one. We have

$$\rho(E) \propto E_\nu^2 \approx (Q_{EC}-B)^2\,,$$

where B is the binding energy of an electron in s or p state. As

$$\frac{\lambda(3p_{1/2})}{\lambda(3s)} = \frac{|M(3p_{1/2})|^2(Q_{EC}-B_p)^2}{|M(3s)|^2(Q_{EC}-B_s)^2} = 0.05,$$

$$\frac{|M(3p_{1/2})^2}{|M(3s)|^2} = 0.05 \times \left(\frac{850-2.0}{850-1.8}\right)^2 = 0.04998\,.$$

Hence for ^{163}Ho,

$$\frac{\lambda(3p_{1/2})}{\lambda(3s)} = \frac{|M(3p_{1/2})|^2(Q_{EC}-B_p)^2}{|M(3s)|^2(Q_{EC}-B_s)^2}$$

$$= 0.04998 \times \left(\frac{2.5-1.8}{2.5-2.0}\right)^2 \approx 9.8\%\,.$$

If $m_\nu = 50$ eV, then, as $E_\nu^2 = p_\nu^2 + m_\nu^2$, the phase-space factor in $P(E)$ changes:

$$p_\nu^2\frac{dp_\nu}{dE_\nu} = (E_\nu^2 - m_\nu^2)\frac{E_\nu}{p_\nu} = E_\nu\sqrt{E_\nu^2 - m_\nu^2} \approx E_\nu^2\left(1-\frac{m_\nu^2}{2E_\nu^2}\right)\,.$$

Hence the decay constant for every channel for ^{161}Ho and ^{163}Ho changes from λ_0 to λ:

$$\lambda \approx \lambda_0\left(1-\frac{1}{2}\frac{m_\nu^2}{E_\nu^2}\right)\,,$$

or

$$\frac{\lambda_0-\lambda}{\lambda_0} \approx \frac{1}{2}\frac{m_\nu^2}{E_\nu^2}\,.$$

Thus for ^{161}Ho, 3s state:

$$\frac{\lambda_0 - \lambda}{\lambda_0} = \frac{1}{2} \times \frac{50^2}{848^2 \times 10^6} = 1.74 \times 10^{-9} ,$$

$3p_{1/2}$ state:

$$\frac{\lambda_0 - \lambda}{\lambda_0} = \frac{1}{2} \times \frac{50^2}{848.2^2 \times 10^6} = 1.74 \times 10^{-9} ;$$

for ^{163}Ho, 3s state:

$$\frac{\lambda_0 - \lambda}{\lambda_0} = \frac{1}{2} \times \frac{50^2}{0.5 \times 10^6} = 5 \times 10^{-3} ,$$

$3p_{1/2}$ state:

$$\frac{\lambda_0 - \lambda}{\lambda_0} = \frac{1}{2} \times \frac{50^2}{0.7^2 \times 10^6} = 2.25 \times 10^{-3} .$$

2084

An element of low atomic number Z can undergo allowed positron β-decay. Let p_0 be the maximum possible momentum of the positron, supposing $p_0 \ll mc$ (m =positron mass); and let Γ_β be the beta-decay rate. An alternative process is K-capture, the nucleus capturing a K-shell electron and undergoing the same nuclear transition with emission of a neutrino. Let Γ_K be the decay rate for this process. Compute the ratio Γ_K/Γ_β. You can treat the wave function of the K-shell electron as hydrogenic, and can ignore the electron binding energy.

(*Princeton*)

Solution:

The quantum perturbation theory gives the probability of a β^+-decay per unit time with decay energy E as

$$\omega = \frac{2\pi}{\hbar} \left| \int \psi_f^* H \psi_i d\tau \right|^2 \frac{dn}{dE} ,$$

where ψ_i is the initial wave function, ψ_f is the final wave function and $\frac{dn}{dE}$ is the number of final states per unit interval of E. As the final state has three particles (nucleus, β^+ and ν), $\psi_f = u_f \phi_\beta \phi_\nu$ (assuming no interaction among the final particles or, if there is, the interaction is very weak), where u_f is the wave function of the final nucleus, ϕ_β, ϕ_ν are respectively the wave functions of the positron and neutrino.

In Fermi's theory of β-decay, H is taken to be a constant. Let it be g. Furthermore, the β^+-particle and neutrino are considered free particles and represented by plane waves:

$$\phi_\beta^* = \frac{1}{\sqrt{V}} e^{-i\mathbf{k}_\beta \cdot \mathbf{r}}, \qquad \phi_\nu^* = \frac{1}{\sqrt{V}} e^{-i\mathbf{k}_\nu \cdot \mathbf{r}},$$

where V is the volume of normalization, $\mathbf{k}_\beta$ and $\mathbf{k}_\nu$ are respectively the wave vectors of the β^+-particle and neutrino. Let

$$\int \psi_i u_f^* e^{-i(\mathbf{k}_\beta + \mathbf{k}_\nu)\cdot \mathbf{r}} d\tau = M_{fi}.$$

The final state is a three-particle state, and so dn is the product of the numbers of state of the final nucleus, the β^+-particle and neutrino. For β^+-decay, the number of states of the final nucleus is 1, while the number of states of β^+-particle with momentum between p and $p + dp$ is

$$dn_\beta = \frac{4\pi p^2 dp}{(2\pi\hbar)^3} V,$$

and that of the neutrino is

$$dn_\nu = \frac{4\pi p_\nu^2 dp_\nu}{(2\pi\hbar)^3} V.$$

Hence

$$\frac{dn}{dE} = \frac{dn_\beta dn_\nu}{dE} = \frac{p^2 p_\nu^2 dp dp_\nu}{4\pi^4 \hbar^6 dE} V^2.$$

The sum of the β^+-particle and neutrino energies equals the decay energy E (neglecting nuclear recoil):

$$E_e + E_\nu \approx E,$$

and so for a given positron energy E_e, $dE_\nu = dE$. Then as the rest mass of neutrino is zero or very small, $E_\nu = cp_\nu$, and

$$p_\nu = (E - E_e)/c, \qquad dp_\nu = \frac{dE}{c}.$$

Therefore

$$\frac{dn}{dE} = \frac{(E-E_e)^2 p^2 dp}{4\pi^4 \hbar^6 c^3} V^2 .$$

On writing

$$\omega = \int I(p) dp ,$$

the above gives

$$I(p)dp = \frac{g^2 |M_{fi}|^2}{2\pi^3 \hbar^7 c^3} (E-E_e)^2 p^2 dp .$$

The β^+-decay rate Γ_β is

$$\Gamma_\beta = \int_0^{p_0} I(p)dp = B \int_0^{p_0} (E-E_e)^2 p^2 dp$$

where

$$B = \frac{g^2 |M_{fi}|^2}{2\pi^3 \hbar^7 c^3}$$

and p_0 is the maximum momentum of the positron, corresponding to a maximum kinetic energy $E_0 \approx E$. As $E_0 \ll m_e c^2$, and so $E_0 = \frac{p_0^2}{2m_e}$, $E_e \approx \frac{p^2}{2m_e}$, we have

$$\begin{aligned}
\Gamma_\beta &= B \int_0^{p_0} \frac{1}{(2m_e)^2} (p_0^4 + p^4 - 2p_0^2 p^2) p^2 dp \\
&= \frac{Bp_0^7}{4m_e^2} \left(\frac{1}{3} + \frac{1}{7} - \frac{2}{5} \right) \\
&\approx 1.9 \times 10^{-2} \frac{Bp_0^7}{m_e^2} .
\end{aligned}$$

In K-capture, the final state is a two-body system, and so monoenergetic neutrinos are emitted. Consider

$$\Gamma_K = \frac{2\pi}{\hbar} \left| \int \psi_f^* H \psi_i d\tau \right|^2 \frac{dn}{dE} .$$

The final state wave function ψ_f^* is the product of the daughter nucleus wave function u_f^* and the neutrino wave function ϕ_ν^*. The neutrino can be considered a free particle and its wave a plane wave

$$\phi_\nu^* = \frac{1}{\sqrt{V}} e^{-i\mathbf{k}_\nu \cdot \mathbf{r}} .$$

The initial wave function can be taken to be approximately the product of the wave functions of the parent nucleus and K-shell electron:

$$\phi_K = \frac{1}{\sqrt{\pi}}\left(\frac{Zm_e e^2}{\hbar^2}\right)^{3/2} e^{-Zm_e e^2 r/\hbar^2}\,.$$

Then as

$$\begin{aligned}\left|\int \psi_f^* H\psi_i d\tau\right| &= \frac{g}{\sqrt{V\pi}}\left(\frac{Zm_e e^2}{\hbar^2}\right)^{\frac{3}{2}}\left|\int u_f^* u_i e^{-i\mathbf{k}_\nu\cdot\mathbf{r}} e^{-\frac{Zm_e e^2}{\hbar^2}r} d\tau\right| \\ &\approx \frac{g}{\sqrt{V\pi}}\left(\frac{Zm_e e^2}{\hbar^2}\right)^{3/2} |M_{fi}|,\end{aligned}$$

$$\frac{dn}{dE} = \frac{4\pi V p_\nu^2 dp_\nu}{(2\pi\hbar)^3 dE} = \frac{4\pi V}{(2\pi\hbar)^3}\frac{E_\nu^2}{c^3}\,,$$

taking $E_\nu \approx E$ and neglecting nuclear recoil, we have

$$\Gamma_K = \frac{m_e^3 g^2 |M_{fi}|^2}{\pi^2\hbar^7 e^3}\left(\frac{Ze^2}{\hbar}\right)^3 E_\nu^2 = 2\pi m_e^3 B\left(\frac{Ze^2}{\hbar}\right)^3 E_\nu^2\,.$$

Ignoring the electron binding energy, we can take $E_\nu \approx E_0 + 2m_e c^2 \approx 2m_e c^2$, and hence

$$\frac{\Gamma_K}{\Gamma_\beta} = \frac{8\pi Z^3}{1.9\times 10^{-2}}\left(\frac{e^2}{\hbar c}\right)^3\left(\frac{m_e c}{p_0}\right)^7 = 5.1\times 10^{-4} Z^3\left(\frac{m_e c}{p_0}\right)^7\,.$$

Thus $\frac{\Gamma_k}{\Gamma_\beta} \propto \frac{1}{p_0^7}$. It increases rapidly with decreasing p_0.

2085

Tritium, the isotope ^{3}H, undergoes beta-decay with a half-life of 12.5 years. An enriched sample of hydrogen gas containing 0.1 gram of tritium produces 21 calories of heat per hour.

(a) For these data calculate the average energy of the β-particles emitted.

(b) What specific measurements on the beta spectrum (including the decay nucleus) indicate that there is an additional decay product and specifically that it is light and neutral.

(c) Give a critical, quantitative analysis of how a careful measurement of the beta spectrum of tritium can be used to determine (or put an upper limit on) the mass of the electron's neutrino.

(*Columbia*)

Solution:

(a) The decay constant is

$$\lambda = \frac{\ln 2}{T_{\frac{1}{2}}} = \frac{\ln 2}{12.5 \times 365 \times 24} = 6.33 \times 10^{-6} hr^{-1} .$$

Hence

$$-\frac{dN}{dt} = \lambda N = \frac{0.1 \times 6.023 \times 10^{23}}{3} \times 6.33 \times 10^{-6} = 1.27 \times 10^{17}$$

decay per hour and the average energy of the β-particles is

$$\bar{E} = \frac{21 \times 4.18}{1.27 \times 10^{17}} = 6.91 \times 10^{-16}\ J = 4.3 \text{ keV} .$$

(b) Both α- and β-decays represent transitions between two states of definite energies. However, the former is a two-body decay (daughter nucleus $+\alpha$-particle) and the conservation laws of energy and momentum require the α-particles to be emitted monoenergetic, whereas β-transition is a three-body decay (daughter nucleus + electron or position + neutrino) and so the electrons emitted have a continuous energy distribution with a definite maximum approximately equal to the transition energy. Thus the α-spectrum consists of a vertical line (or peak) while the β-spectrum is continuous from zero to a definite end-point energy. Thus a measurement of the β spectrum indicates the emission of a third, neutral particle. Conservation of energy indicates that it is very light.

(c) Pauli suggested that β-decay takes place according to

$${}^A_Z X \to {}^A_{Z+1} Y + \beta^- + \bar{\nu}_e .$$

As shown in Fig. 2.25, β^- has a continuous energy spectrum with a maximum energy E_m. When the kinetic energy of $\bar{\nu}_e$ trends to zero, the energy of β^- trends to E_m. Energy conservation requires

$$M({}^A_Z X) = M({}^A_{Z+1} Y) + \frac{E_m}{c^2} + m_\nu ,$$

or, for the process under consideration,

$$m_\nu = M(^3_1H) - M(^3_2He) - E_m/c^2 \, .$$

If E_m is precisely measured, the neutrino mass can be calculated. It has been found to be so small that only an upper limit can be given.

2086

(a) Describe briefly the energy spectra of alpha- and beta-particles emitted by radioactive nuclei. Emphasize the differences and qualitatively explain the reasons for them.

(b) Draw a schematic diagram of an instrument which can measure one of these spectra. Give numerical estimates of essential parameters and explain how they are chosen.

(*UC, Berkeley*)

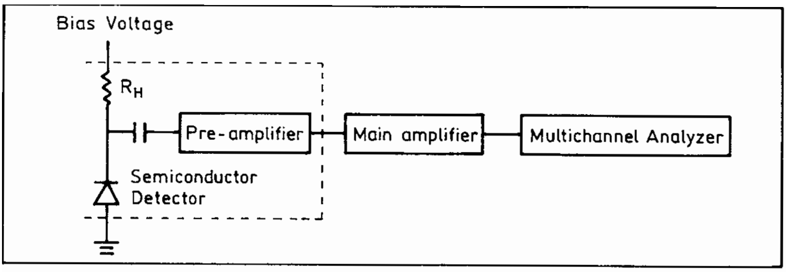

Fig. 2.26

Solution:

(a) α-particles from a radioactive nuclide are monoenergetic; the spectrum consists of vertical lines. β-particles have a continuous energy spectrum with a definite end-point energy. This is because emission of a β-particle is accompanied by a neutrino which takes away some decay energy.

(b) Figure 2.26 is a schematic sketch of a semiconductor α-spectrometer.

The energy of an α-particle emitted in α-decay is several MeV in most cases, so a thin-window, gold-silicon surface-barrier semiconductor detector is used which has an energy resolution of about 1 percent at room-temperature. As the α-particle energy is rather low, a thick, sensitive layer is not needed and a bias voltage from several tens to 100 V is sufficient. For good measurements the multichannel analyzer should have more than 1024 channels, using about 10 channels for the full width at half maximum of a peak.

2087

The two lowest states of scandium-42, ${}^{42}_{21}Sc_{21}$, are known to have spins 0^+ and 7^+. They respectively undergo positron-decay to the first 0^+ and 6^+ states of calcium-42, ${}^{42}_{20}Ca_{22}$, with the positron reduced half-lives $(\text{ft})_{0^+} = 3.2 \times 10^3$ seconds, $(\text{ft})_{7^+} = 1.6 \times 10^4$ seconds. No positron decay has been detected to the 0^+ state at 1.84 MeV. (See Fig. 2.27.)

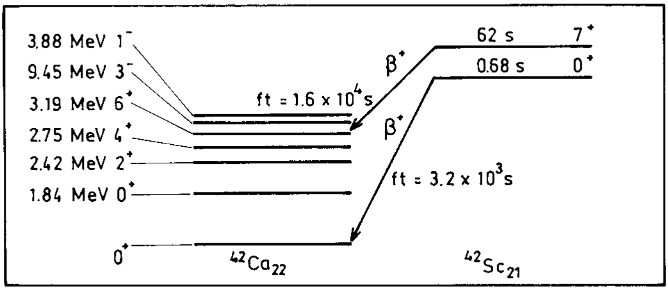

Fig. 2.27

(a) The two states of ^{42}Sc can be simply accounted for by assuming two valence nucleons with the configuration $(f_{7/2})^2$. Determine which of the indicated states of ^{42}Ca are compatible with this configuration. Briefly outline your reasoning. Assuming charge independence, assign isospin quantum numbers $|T, T_Z\rangle$ for all $(f_{7/2})^2$ states. Classify the nature of the two beta-transitions and explain your reasoning.

(b) With suitable wave functions for the $|J, M_J\rangle = |7, 7\rangle$ state of scandium-42 and the $|6, 6\rangle$ state of calcium-42, calculate the ratio $(ft)_{7^+}/(ft)_{0^+}$ expected for the two positron-decays.

For $j = l + \frac{1}{2}$:

$$\hat{S}_-|j, j\rangle = \frac{1}{(2j)^{1/2}}|j, j-1\rangle + \left(\frac{2j-1}{2j}\right)^{1/2} |j-1, j-1\rangle\,,$$

$$\hat{S}_z|j,j\rangle = \frac{1}{2}|j,j\rangle\,,$$

$$G_v = 1.4 \times 10^{-49}\ \mathrm{erg\,cm^3}\,,$$

$$G_A = 1.6 \times 10^{-49}\ \mathrm{erg\,cm^3}\,.$$

(*Princeton*)

Solution:

(a) For ^{42}S, $T_z = \frac{1}{2}(Z-N) = 0$. As the angular momenta of the two nucleons are $7/2$ each and the isospins are $1/2$ each, vector addition gives for the nuclear spin an integer from 0 to 7, and for the nuclear isospin 0 or 1. The generalized Pauli's principle requires the total wave function to be antisymmetric, and so $J + T =$ odd. Hence the states compatible with the configuration $(f_{7/2})^2$ are $J = 0^+$, 2^+, 4^+, 6^+ when $T = 1$, and $J = 1^+, 3^+, 5^+, 7^+$ when $T = 0$.

The transition $7^+ \to 6^+$ is a Gamow–Teller transition as for such transitions $\Delta J = 0$ or 1 ($J_i = 0$ to $J_f = 0$ is forbidden), $\Delta T = 0$ or 1, $\pi_i = \pi_f$.

The transition $0^+ \to 0^+$ is a Fermi transition as for such transitions $\Delta J = 0$, $\Delta T = 0$, $\pi_i = \pi_f$.

(b) The probability per unit time of β-transition is $\Gamma(\beta) \propto G_v^2\langle M_F\rangle^2 + G_A^2\langle M_{GT}\rangle^2$, where $\langle M_F\rangle^2$ and $\langle M_{GT}\rangle^2$ are the squares of the spin-averaged weak interaction matrix elements:

$$\langle M_F\rangle^2 = \frac{1}{2J_i+1}\sum_{M_i,M_f}\langle J_f M_f T_f T_{fz}|1\cdot\sum_{k=1}^{A} t_\pm(k)|J_i M_i T_i T_{iz}\rangle^2$$

$$= \langle J_f M T_f T_{fz}|1\cdot\sum_{k=1}^{A} t_\pm(k)|J_i M T_i T_{iz}\rangle^2\,,$$

$$\langle M_{GT}\rangle^2 = \frac{1}{2J_i+1}\sum_{m,M_i,M_f}|\langle J_f M_f T_f T_{fz}|\sum_{k=1}^{A}\sigma_m(k)t_\pm(k)|J_i M_i T_i T_{iz}\rangle|^2\,,$$

where m takes the values $+1, 0, -1$, for which

$$\sigma_{+1} = \sigma_x + i\sigma_y\,,\quad \sigma_0 = \sigma_z\,,\quad \sigma_{-1} = \sigma_x - i\sigma_y\,.$$

Then the half-life is

$$ft = \frac{K}{G_v^2\langle M_F\rangle^2 + G_A^2\langle M_{GT}\rangle^2}\,,$$

where $K = 2\pi^3\hbar^7 \ln 2/m^5c^4$, a constant. Hence

$$\frac{ft(7^+ \to 6^+)}{ft(0^+ \to 0^+)} = \frac{G_v^2 \langle M_F\rangle_{0^+}^2}{G_A^2 \langle M_{GT}\rangle_{7^+}^2} .$$

Consider

$$\langle M_F\rangle = \langle JMTT_{fz}|1 \cdot \sum_{k=1}^{A} t_\pm(k)|JMTT_{iz}\rangle = \langle JMTT_{fz}|T_\pm|JMTT_{iz}\rangle$$

$$= \sqrt{T(T+1) - T_{iz}T_{fz}} ,$$

replacing the sum of the z components of the isospins of the nucleons by the z-component of the total isospin. Taking $T = 1$, $T_{iz} = 0$, we have

$$\langle M_F\rangle^2 = 2 .$$

Consider

$$\langle M_{GT}\rangle^2 = \sum_m |\langle 6,6,1,-1|\{\sigma_m(1)t_\pm(1) + \sigma_m(2)t_\pm(2)\}|7,7,1,0\rangle|^2 ,$$

where only the two nucleons outside full shells, which are identical, are taken into account. Then

$$\langle M_{GT}\rangle^2 = 4\sum_m |\langle 6,6,1,-1|\sigma_m(1)t_\pm(1)|7,7,1,0\rangle|^2 .$$

Writing the wave functions as combinations of nucleon wave functions:

$$|7,7\rangle = \left|\frac{7}{2},\frac{7}{2};\frac{7}{2},\frac{7}{2}\right\rangle ,$$

$$|7,6\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{7}{2},\frac{6}{2};\frac{7}{2},\frac{7}{2}\right\rangle + \left|\frac{7}{2},\frac{7}{2};\frac{7}{2},\frac{6}{2}\right\rangle\right) ,$$

$$|6,6\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{7}{2},\frac{6}{2};\frac{7}{2},\frac{7}{2}\right\rangle - \left|\frac{7}{2},\frac{7}{2};\frac{7}{2},\frac{6}{2}\right\rangle\right) ,$$

we have

$$\langle M_{GT}\rangle^2 = 4\left|\left\langle \frac{7}{2},\frac{6}{2};\frac{7}{2},\frac{7}{2},;1,-1\left|\frac{\sigma_-(1)t_\pm(1)}{2}\right|\frac{7}{2},\frac{7}{2};\frac{7}{2},\frac{7}{2};1,0\right\rangle\right|^2 = 2 .$$

Thus

$$\frac{(ft)_{7^+}}{(ft)_{0^+}} = \frac{G_v^2}{G_A^2} \approx \left(\frac{1.4}{1.6}\right)^2 \approx 0.77\,.$$

2088

The still-undetected isotope copper-57 ($^{57}_{29}Cu_{28}$) is expected to decay by positron emission to nickel-57 ($^{57}_{28}Ni_{29}$).

(a) Suggest shell-model spin-parity assignments for the ground and first excited states of these nuclei.

(b) Estimate the positron end-point energy for decay from the ground state of copper-57 to the ground state of nickel-57. Estimate the half-life for this decay (order of magnitude).

(c) Discuss what one means by Fermi and by Gamow–Teller contributions to allowed β-decays, and indicate the corresponding spin-parity selection rules. For the above decay process, estimate the ratio Γ_F/Γ_{GT} of the two contributions to the decay rate. Does one expect appreciable β^+-decay from the copper-57 ground state to the first excited state of nickel-57? Explain.

(d) Nickel-58 occurs naturally. Briefly sketch an experimental arrangement for study of copper-57 positron-decay.

(*Princeton*)

Solution:

(a) ^{57}Cu and ^{57}Ni are mirror nuclei with the same energy-level structure of a single nucleon outside of double-full shells. The valence nucleon is proton for ^{57}Cu and neutron for ^{57}Ni, the two nuclei having the same features of ground and first excited states.

For the ground state, the last nucleon is in state $2p_{3/2}$ (Fig. 2.11), and so $J^\pi = (\frac{3}{2})^-$; for the first excited state, the nucleon is in state $1f_{5/2}$, and so $J^\pi = (\frac{5}{2})^-(E_1 = 0.76 \text{ MeV})$.

(b) As ^{57}Cu and ^{57}Ni are mirror nuclei, their mass difference is (**Problem 2067(c)**)

$$\begin{aligned}\Delta E &= M(Z+1, A)c^2 - M(Z, A)c^2 \\ &= \frac{3e^2}{5R}[(Z+1)^2 - Z^2] - (m_n - M_H)c^2\end{aligned}$$

$$= \frac{3c\hbar}{5R}\left(\frac{e^2}{c\hbar}\right)(2Z+1) - (m_n - M_H)c^2$$

$$= \frac{3 \times 197 \times (2 \times 28 + 1)}{5 \times 1.2 \times 57^{1/3} \times 137} - 0.78$$

$$= 9.87 \text{ MeV}.$$

Thus the ground state of ^{57}Cu is 9.87 MeV higher than that of ^{57}Ni. The positron end-point energy for decay from the ground state of ^{57}Cu to that of ^{57}Ni is

$$E_0 = \Delta E - 2m_e c^2 \approx 9.87 - 1.02 \approx 8.85 \text{ MeV}.$$

As the β^+-decay is from $(\frac{3}{2})^-$ to $(\frac{3}{2})^-$, $\Delta J = 0$, $\Delta\pi = +$, $\Delta T = 0$, $\Delta T_z = -1$, the decay is of a superallowed type. To simplify calculation take $F(Z, E) = 1$. Then (**Problem 2084**)

$$\lambda_\beta \approx \int_0^{p_0} I(p)dp \approx B\int_0^{E_0}(E_0 - E)^2 E^2 dE$$

$$= BE_0^5\left(\frac{1}{3} + \frac{1}{5} - \frac{1}{2}\right) = \frac{1}{30}BE_0^5,$$

where

$$B = \frac{g^2|M_{fi}|^2}{2\pi^3 c^6 \hbar^7} = 3.36 \times 10^{-3} \text{ MeV}^{-5} s^{-1},$$

with $|M_{fi}|^2 \approx 1$, $g = 8.95 \times 10^{-44}$ MeV $\cdot$ cm^3 (experimental value). Hence

$$\tau_{1/2} = \ln 2/\lambda = \frac{30 \ln 2}{BE_0^5} = 0.114 \ s.$$

(c) In β^+-decay between mirror nuclei ground states $\frac{3}{2}^- \to \frac{3}{2}^-$, as the nuclear structures of the initial and final states are similar, the transition is of a superallowed type. Such transitions can be classified into Fermi and Gamow–Teller types. For the Fermi type, the selection rules are $\Delta J = 0$, $\Delta\pi = +$, the emitted neutrino and electron have antiparallel spins. For the Gamow–Teller type, the selection rules are $\Delta J = 0, \pm 1$, $\Delta\pi = +$, the emitted neutrino and electron have parallel spins.

For transition $\frac{3}{2}^- \to \frac{3}{2}^-$ of the Fermi type,

$$|M_F|^2 = T(T+1) - T_{iz}T_{fz} = \frac{1}{2}\left(\frac{1}{2} + 1\right) + \frac{1}{2} \times \frac{1}{2} = 1.$$

For transition $\frac{3}{2}^- \to \frac{3}{2}^-$ of the Gamow–Teller type,

$$|M_{GT}|^2 = \frac{J_f + 1}{J_f} = \frac{3/2 + 1}{3/2} = \frac{5}{3}\,.$$

The coupling constants for the two types are related roughly by $|g_{GT}| \approx 1.24|g_F|$. So the ratio of the transition probabilities is

$$\frac{\lambda_F}{\lambda_{GT}} = \frac{g_F^2|M_F|^2}{g_{GT}^2|M_{GT}|^2} = \frac{1}{1.24^2 \times 5/3} = 0.39\,.$$

The transition from ^{57}Cu to the first excited state of ^{57}Ni is a normal-allowed transition because $\Delta J = 1$, $\Delta\pi = +$. As the initial and final states are $2p_{3/2}$ and $1f_{5/2}$, and so the difference in nuclear structure is greater, the fT of this transition is larger than that of the superallowed one by 2 to 3 orders of magnitude. In addition, there is the space phase factor $\left(\frac{8.85-0.76}{8.85}\right)^5 = 0.64$. Hence the branching ratio is very small, rendering such a transition difficult to detect.

(d) When we bombard ^{58}Ni target with protons, the following reaction may occur:

$$^{58}Ni + p \to ^{57} Cu + 2n$$

As the mass-excess $\Delta = (M - A)$ values (in MeV) are

$$\Delta(n) \approx 8.071, \quad \Delta(^1H) = 7.289\,,$$

$$\Delta(^{58}Ni) = -60.235, \quad \Delta(^{57}Cu) \approx -46.234\,.$$

We have

$$\begin{aligned} Q &= \Delta(^{58}Ni) + \Delta(^1H) - \Delta(^{57}Cu) - 2\Delta(n) \\ &= -60.235 + 7.289 + 46.234 - 2 \times 8.071 = -22.854 \text{ MeV}\,. \end{aligned}$$

Hence the reaction is endoergic and protons of sufficient energy are needed. The neutrons can be used to monitor the formation of ^{57}Cu, and measuring the delay in β^+ emission relative to n emission provides a means to study β^+-decay of ^{57}Cu.

2089

Suppose a search for solar neutrinos is to be mounted using a large sample of lithium enriched in the isotope $^7_3$Li. Detection depends on production, separation and detection of the electron-capturing isotope $^7_4$Be with a half-life of 53 days. The low lying levels of these two nuclei are shown below in Fig. 2.28. The atomic mass of $^7_4$Be in its ground state lies 0.86 MeV above the atomic mass of $^7_3$Li in its ground state.

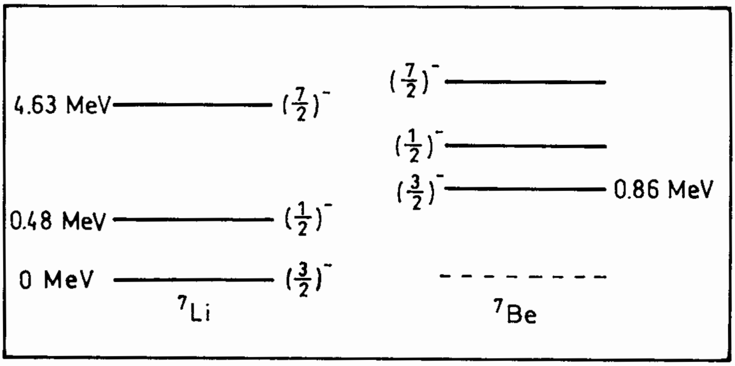

Fig. 2.28

(a) Discuss the electron-capture modes of the ground state of beryllium-7 by providing estimates for the branching ratios and relative decay probabilities (ft ratios).

(b) To calibrate this detector, a point source emitting 10^{17} monochromatic neutrinos/sec with energy 1.5 MeV is placed in the center of a one metric ton sphere of lithium-7. Estimate the total equilibrium disintegration rate of the beryllium-7, given

$$G_V = 1.42 \times 10^{-49} \text{ erg cm}^3 ,$$

$$G_A = 1.60 \times 10^{-49} \text{ erg cm}^3 ,$$

$$\rho_{\text{Li}} = 0.53 \text{ gm/cm}^3 .$$

(*Princeton*)

Solution:

(a) Two modes of electron capture are possible:

$$\left(\frac{3^-}{2}\right)^- \to \left(\frac{3}{2}\right)^- : \quad \Delta J = 0, \Delta P = + ,$$

which is a combination of F and G-T type transitions;

$$\left(\frac{3}{2}\right)^- \to \left(\frac{1}{2}\right)^- \ : \ \Delta J = 1, \Delta P = +\,,$$

which is a pure G-T type transition.

$^7_3$Li and $^7_4$Be are mirror nuclei with $T = \frac{1}{2}$, and $T_z = \frac{1}{2}$ and $-\frac{1}{2}$ respectively.

For the F-type transition $\left(\frac{3}{2}\right)^- \to \left(\frac{3}{2}\right)^-$ the initial and final wave functions are similar and so

$$\langle M_F \rangle^2 = T(T+1) - T_{zi}T_{zf} = \frac{1}{2}\cdot\frac{3}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4} + \frac{1}{4} = 1\,.$$

For the G-T-type transition $\frac{3^-}{2} \to \frac{3^-}{2}$, the single-particle model gives

$$\langle M_{G-T} \rangle^2 = \frac{J_f + 1}{J_f} = \frac{3/2+1}{3/2} = \frac{5}{3}\,.$$

For the G-T-type transition $\left(\frac{3}{2}\right)^- \to \left(\frac{1}{2}\right)^-$, the transition is form $l + \frac{1}{2}$ to $l - \frac{1}{2}$ with $l = 1$, and the single-particle model gives

$$\langle M_{G-T} \rangle^2 = \frac{4l}{2l+1} = \frac{4}{3}\,.$$

As $\lambda_K(M^2, W_\nu) = |M|^2 W_\nu^2$, where W_ν is the decay energy,

$$\begin{aligned}\frac{\lambda_K\left(\frac{3^-}{2} \to \frac{3^-}{2}\right)}{\lambda_K\left(\frac{3^-}{2} \to \frac{1^-}{2}\right)} &= \frac{\langle M_{G-T}\rangle^2_{3/2} + \frac{G_V^2}{G_A^2}\langle M_F\rangle^2}{\langle M_{G-T}\rangle^2_{1/2}} \cdot \frac{W_{\nu_1}^2}{W_{\nu_2}^2} \\ &= \frac{\frac{5}{3} + \left(\frac{1.42}{1.60}\right)^2}{\frac{4}{3}} \times \left(\frac{0.86}{0.86-0.48}\right)^2 \\ &= \frac{(5 + 0.79 \times 3) \times 0.86^2}{4 \times (0.86-0.48)^2} = 9.43\,.\end{aligned}$$

Hence the branching ratios are $B(\frac{3^-}{2} \to \frac{3^-}{2}) = \frac{9.43}{10.43} = 90.4\%$,

$$B\left(\frac{3^-}{2} \to \frac{1^-}{2}\right) = \frac{1}{10.43} = 9.6\%\,.$$

The fT ratio of the two transitions is

$$\frac{(fT)_{3/2^-}}{(fT)_{1/2^-}} = \frac{\langle M_{G-T}\rangle_{1/2}^2}{\langle M_{G-T}\rangle_{3/2}^2 + \frac{G_V^2}{G_A^2}\langle M_F\rangle^2} = \frac{4}{3\times 0.79+5} = 0.543\,.$$

(b) When irradiating ^{7}Li with neutrinos, ^{7}Li captures neutrino and becomes ^{7}Be. On the other hand, ^{7}Be undergoes decay to ^{7}Li. Let the number of ^{7}Be formed per unit time in the irradiation be ΔN_1. Consider a shell of ^{7}Li of radius r and thickness dr. It contains

$$\frac{4\pi r^2 \rho n dr}{A}$$

^{7}Li nuclei, where $n =$ Avogadro's number, $A =$ mass number of ^{7}Li. The neutrino flux at r is $\frac{I_0}{4\pi r^2}$. If $\sigma =$ cross section for electron-capture by ^{7}Li, $a =$ activity ratio of ^{7}Li for forming ^{7}Be, $R =$ radius of the sphere of ^{7}Li, the number of ^{7}Be nuclei produced per unit time is

$$\Delta N_1 = \int \frac{I_0}{4\pi r^2}\rho n\sigma a\cdot 4\pi r^2 dr/A = I_0\rho n\sigma a R/A\,.$$

With $a = 0.925$, $\rho = 0.53$ g cm^{-3}, $A = 7$, $n = 6.023\times 10^{23}$, $R = \left(\frac{3\times 10^6}{4\pi\rho}\right)^{\frac{1}{3}} = 76.7$ cm, $I_0 = 10^{17}$ s^{-1}, $\sigma \approx 10^{-43}$ cm^2, we have

$$\begin{aligned}\Delta N_1 &= \frac{10^{17}\times 0.53\times 6.023\times 10^{23}\times 10^{-43}\times 0.925\times 76.7}{7}\\ &= 3.2\times 10^{-2}\ s^{-1}\,.\end{aligned}$$

At equilibrium this is also the number of ^{7}Be that decay to ^{7}Li.

Hence the rate of disintegration of ^{7}Be at equilibrium is $3.2\times 10^{-2}\ s^{-1}$. Note that the number of ^{7}Li produced in ^{7}Be decays is negligible compared with the total number present.

2090

It is believed that nucleons (N) interact directly through the weak interaction and that the latter violates parity conservation. One way to study the nature of the N-N weak interaction is by means of α-decay, as typified by the decays of the 3^+, $T = 1$ and 3^-, $T = 0$ states of ^{20}Ne (Fig. 2.29).

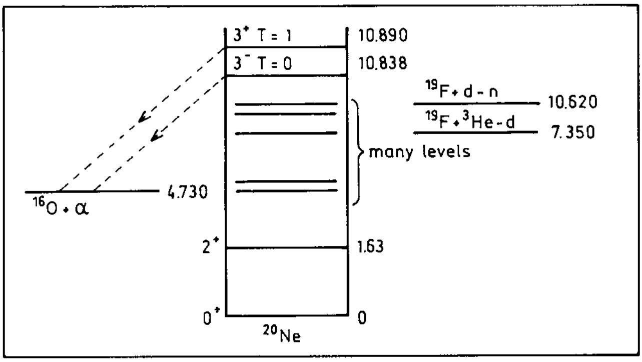

Fig. 2.29

In the following you will be asked to explain the principles of an experiment to measure the weak-interaction matrix element between these states, $\langle 3^+|H_{\text{weak}}|3^-\rangle$.

(a) The N-N weak interaction has isoscalar, isovector, and isotensor components (i.e., ranks 0,1, and 2 in isospin). Which components contribute to the matrix element $\langle 3^+|H_{\text{weak}}|3^-\rangle$?

(b) Explain the parity and isospin selection rules for α-decay. In particular, explain which of the two ^{20}Ne states would decay to the ground state of $^{16}\text{O} + \alpha$ if there were no parity-violating N-N interaction.

(c) Allowing for a parity-violating matrix element $\langle 3^+|H_{\text{weak}}|3^-\rangle$ of 1 eV, estimate the α width of the parity-forbidden transition, Γ_α (forbidden), in terms of the α width of the parity-allowed transition, Γ_α (allowed). Assume Γ_α (allowed) is small compared with the separation energy between the 3^+, 3^- states.

(d) The α width of the parity-allowed transition is Γ_α (allowed) = 45 keV, which is not small compared with the separation energy. Do you expect the finite width of this state to modify your result of part (c) above? Discuss.

(e) The direct reaction $^{19}\text{F}(^3\text{He},\text{d})^{20}\text{Ne}^*$ populates one of the excited states strongly. Which one do you expect this to be and why?

(f) There is also a $1^+/1^-$ parity doublet at $\sim$ 11.23 MeV. Both states have $T = 1$.

(i) In this case which state is parity-forbidden to α-decay?

(ii) As in part(a), which isospin components of the weak N-N interaction contribute to the mixing matrix element? (Note that ^{20}Ne is self-conjugate) Which would be determined by a measurement of the parity-forbidden α width?

(*Princeton*)

Solution:

(a) As $T = 1$ for the 3^+ state and $T = 0$ for the 3^- state, only the isovector component with $\Delta T = 1$ contributes to $\langle 3^+|H_{\text{weak}}|3^-\rangle$.

(b) α-decay is a strong interaction for which isospin is conserved. Hence $\Delta T = 0$. As the isospin of α-particle is zero, the isospin of the daughter nucleus should equal that of the parent. As ^{16}O has $T = 0$, only the 3^-, $T = 0$ state can undergo α-decay to $^{16}\text{O} + \alpha$. As both the spins of ^{16}O and α are zero, and the total angular momentum does not change in α-decay, the final state orbital angular momentum is $l = 3$ and so the parity is $(-1)^3 = -1$. As it is the same as for the initial state, the transition is parity-allowed.

(c) Fermi's golden rule gives the first order transition probability per unit time as

$$\lambda = \frac{2\pi}{\hbar}|V_{fi}|^2\rho(E_f)\,,$$

where V_{fi} is the transition matrix element and $\rho(E_f)$ is the final state density. Then the width of the parity-allowed transition ($3^-, T = 0$ to $^{16}O + \alpha$) is

$$\Gamma_\alpha = \frac{2\pi}{\hbar}|V_{3^-\to{}^{16}\text{O}}|^2\rho(E_f)\,.$$

The parity-forbidden transition ($3^+, T = 1$ to $^{16}\text{O} + \alpha$) is a second order process, for which

$$\lambda = \frac{2\pi}{\hbar}\left|\sum_{n\neq i}\frac{V_{fn}V_{ni}}{E_i - E_n + i\varepsilon}\right|^2\rho(E_f)\,,$$

where 2ε is the width of the intermediate state, and the summation is to include all intermediate states. In this case, the only intermediate state is that with 3^-, $T = 0$. Hence

$$\begin{aligned}\Gamma'_\alpha &= \frac{2\pi}{\hbar}|V_{3^-\to{}^{16}\text{O}}|^2\frac{1}{(E_i - E_n)^2 + \varepsilon^2}|\langle 3^+|H_{\text{weak}}|3^-\rangle|^2\rho(E_f)\\ &= \Gamma_\alpha\frac{|\langle 3^+|H_{\text{weak}}|3^-\rangle|^2}{(\Delta E)^2 + (\Gamma_\alpha/2)^2}\,,\end{aligned}$$

where ΔE is the energy spacing between the $3^+, 3^-$ states, Γ_α is the width of the parity-allowed transition. If $\Gamma_\alpha \ll \Delta E$, as when $\langle 3^+|H_{\text{weak}}|3^-\rangle = 1$ eV, $\Delta E = 0.052$ MeV $= 52 \times 10^3$ eV, we have

$$\Gamma'_\alpha \approx \frac{|\langle 3^+|H_{\text{weak}}|3^-\rangle|^2}{(\Delta E)^2}\Gamma_\alpha = \frac{\Gamma_\alpha}{52^2 \times 10^6} = 3.7 \times 10^{-10}\Gamma_\alpha\,.$$

(d) As $\Gamma_\alpha = 45$ keV, $(\Gamma_\alpha/2)^2$ cannot be ignored when compared with $(\Delta E)^2$. Hence

$$\Gamma'_\alpha = \frac{10^{-6}}{52^2 + \frac{45^6}{4}}\Gamma_\alpha = 3.1 \times 10^{-10}\Gamma_\alpha = 1.4 \times 10^{-5}\text{ eV}\,.$$

(e) Consider the reaction $^{19}F(^3He, d)^{20}Ne^*$. Let the spins of ^{19}F, ^3He, d, ^{20}Ne, and the captured proton be $\mathbf{J}_A$, $\mathbf{J}_a$, $\mathbf{J}_b$, $\mathbf{J}_B$, J_p, the orbital angular momenta of ^3He, d and the captured proton be $\mathbf{l}_a$, $\mathbf{l}_b$, $\mathbf{l}_p$, respectively. Then

$$\mathbf{J}_A + \mathbf{J}_a + \mathbf{l}_a = \mathbf{J}_B + \mathbf{J}_b + \mathbf{l}_b\,.$$

As

$$\mathbf{J}_A = \mathbf{J}_p + \mathbf{l}_b, \quad \mathbf{l}_a = \mathbf{l}_p + \mathbf{l}_b, \quad \mathbf{J}_A + \mathbf{s}_p + \mathbf{l}_p = \mathbf{J}_B\,,$$

and $J_A = \frac{1}{2}$, $J_B = 3$, $J_b = 1$, $l_b = 0$, $s_p = \frac{1}{2}$, we have $J_p = \frac{1}{2}$, $l_p = 2, 3, 4$. Parity conservation requires $P(^{19}F)P(p)(-1)^{l_p} = P(^{20}Ne^*)$, $P(^{20}Ne^*) = (-1)^{l_p}$.

Experimentally l_p is found from the angular distribution to be $l_p = 2$. Then $P(^{20}Ne^*) = +$, and so the reaction should populate the 3^+ state of Ne^*, not the 3^- state.

(f) (i) The 1^+ state is parity-forbidden to α-decay. On the other hand, in the α-decay of the 1^- state, $l_f + J_\alpha + J_{^{16}O} = 1$, $P_f = P(\alpha)P(^{16}O)(-1)^{l_f} = -1$, so that its α-decay is parity-allowed

(ii) As ^{20}Ne is a self-conjugate nucleus, $T_3 = 0$ because $\langle 1,0|1,0;1,0\rangle = 0$. So only the components of $T = 0, 2$ can contribute. However in weak interaction, $|\Delta T| \leq 1$, and so only the component with $\Delta T = 0$ can contribute to the experiment result.

2091

Consider the following energy level structure (Fig. 2.30):

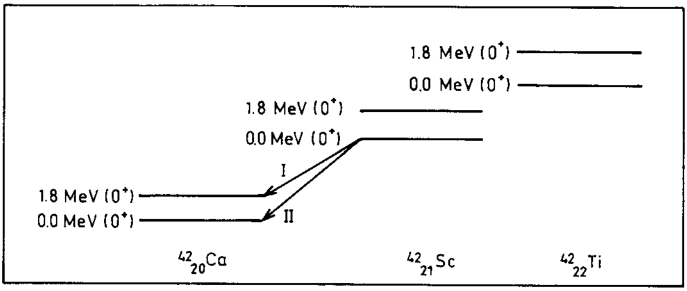

Fig. 2.30

The ground states form an isotriplet as do the excited states (all states have a spin-parity of 0^+). The ground state of $^{42}_{21}$Sc can β-decay to the ground state of $^{42}_{20}$Ca with a kinetic end-point energy of 5.4 MeV (transition II in Fig. 2.30).

(a) Using phase space considerations only, calculate the ratio of rates for transitions I and II.

(b) Suppose that the nuclear states were, in fact, pure (i.e. unmixed) eigenstates of isospin. Why would the fact that the Fermi matrix element is an isospin ladder operator forbid transition I from occurring?

(c) Consider isospin mixing due to electromagnetic interactions. In general

$$H_{EM} = H_0 + H_1 + H_2 \,,$$

where the subscripts refer to the isospin tensor rank of each term. Write the branching ratio $\frac{\Gamma_I}{\Gamma_{II}}$ in terms of the reduced matrix elements of each part of H_{EM} which mixes the states.

(d) Using the results of parts (a) and (c), ignoring H_2, and given that $\frac{\Gamma_I}{\Gamma_{II}} = 6 \times 10^{-5}$, calculate the value of the reduced matrix element which mixes the ground and excited states of $^{42}_{20}$Ca.

(Princeton)

Solution:

(a) From phase space consideration only, for β-decay of $E_0 \gg m_e c^2$, $\Gamma \approx E_0^5$ (**Problem 2077**). Thus

$$\frac{\Gamma_I}{\Gamma_{II}} = \frac{(5.4 - 1.8)^5}{(5.4 - 0)^5} \approx 0.13 \,.$$

(b) For Fermi transitions within the same isospin multiplet, because the structures of the initial and final states are similar, the transition probability is large. Such transitions are generally said to be superallowed. For $0^+ \to 0^+ (T = 1)$, there is only the Fermi type transition, for which

$$\begin{aligned} \langle M_F \rangle^2 &= \langle \alpha, T_f, T_{f3} | \sum_{K=1}^{A} t_\pm(K) | \alpha', T_i, T_{i3} \rangle^2 \\ &= \left(\delta_{\alpha\alpha'} \delta_{T_i T_f} \sqrt{T(T+1) - T_{i3} T_{f3}} \right)^2 \\ &= \begin{cases} T(T+1) - T_{i3} T_{f3}, & \text{if } \alpha = \alpha',\ T_f = T_i\,, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

ignoring higher order corrections to the Fermi matrix element. Here α is any nuclear state quantum number other than isospin. From this we see that channel II is a transition within the same isospin multiplet, i.e., a superallowed one, channel I is a transition between different isospin multiplets, i.e., a Fermi-forbidden transition.

(c) We make use of the perturbation theory. Let the ground and excited states of ^{42}Ca be $|1\rangle$ and $|2\rangle$ respectively. Because of the effect of H_{EM}, the states become mixed. Let the mixed states be $|1\rangle'$ and $|2\rangle'$, noting that the mixing due to H_{EM} is very small. We have

$$H^0 |1\rangle = E_1 |1\rangle\,,$$

$$H^0 |2\rangle = E_2 |2\rangle\,,$$

where E_1 and E_2 are the energies of the two states ($E_1 \approx E_0$, $E_2 - E_1 = 1.8$ MeV).

Consider

$$H = H^0 + H_{EM}\,,$$

where $H_{EM} = H_0 + H_1 + H_2$. As the index refers to isospin tensor rank, we write H_0, H_1 H_2 as $P_{0,0}$, $P_{1,0}$, $P_{2,0}$ and define

$$\langle J_1 m_1 | P_{\mu\nu} | J_2 m_2 \rangle = C^{J_1 m_1}_{\mu\nu J_2 m_2} \langle J_1 || P_{\mu\nu} || J_2 \rangle\,.$$

Then

$$
\begin{aligned}
H_{EM} &= P_{0,0} + P_{1,0} + P_{2,0}\,, \\
\langle 1|H_{EM}|2\rangle &= \langle \alpha, 1, -1|(P_{0,0} + P_{1,0} + P_{2,0})|\alpha', 1, -1\rangle \\
&= \left(\langle \alpha, 1||P_0||\alpha', 1\rangle - \sqrt{\frac{1}{2}}\langle \alpha, 1||P_1||\alpha', 1\rangle\right. \\
&\quad \left. + \sqrt{\frac{1}{10}}\langle \alpha, 1||P_2||\alpha', 1\rangle\right),
\end{aligned}
$$

$$
\begin{aligned}
\langle 1|H_{EM}|1\rangle = \langle 2|H_{EM}|2\rangle &= \langle \alpha, 1, -1|(P_{0,0} + P_{1,0} + P_{2,0})|\alpha, 1, -1\rangle \\
&= \langle \alpha, 1||P_0||\alpha, 1\rangle - \sqrt{\frac{1}{2}}\langle \alpha, 1||P_1||\alpha, 1\rangle + \sqrt{\frac{1}{10}}\langle \alpha, 1||P_2||\alpha, 1\rangle\,.
\end{aligned}
$$

In the above equations, α and α' denote the quantum numbers of $|1\rangle$ and $|2\rangle$ other than the isospin, and $\langle \alpha, 1||P||\alpha, 1\rangle$ denote the relevant part of the reduced matrix element. Thus

$$
\frac{\Gamma_I}{\Gamma_{II}} = \frac{E_1^5|M_1|^2}{E_2^5|M_2|^2} = \frac{(5.4 - 1.8 - \langle 2|H_{EM}|2\rangle)^5}{(5.4 - \langle 1|H_{EM}|1\rangle)^5}\frac{\langle 1|H_{EM}|2\rangle^2}{(E_2 - E_1)^2}\,.
$$

If energy level corrections can be ignored, then $\langle 1|H_{EM}|1\rangle \ll E_1, E_2$, and

$$
\begin{aligned}
\frac{\Gamma_I}{\Gamma_{II}} &= \frac{E_{10}^5}{E_{20}^5(E_2 - E_1)^2}|\langle 1|H_{\mathrm{EM}}|2\rangle|^2 \\
&= \frac{(5.4 - 1.8)^5}{5.4^5 \times 1.8^2}\left(\langle 1||P_0||2\rangle - \sqrt{\frac{1}{2}}\langle 1||P_1||2\rangle + \sqrt{\frac{1}{10}}\langle 1||P_2||2\rangle\right)^2\,.
\end{aligned}
$$

If we ignore the contribution of H_2 and assume $\langle 1||P_0||2\rangle = 0$, then the isoscalar H does not mix the two isospin states and we have

$$
\frac{\Gamma_I}{\Gamma_{II}} = \frac{E_{10}^5}{E_{20}^5(E_2 - E_1)^2}|\langle \alpha, 1||P_1||\alpha', 1\rangle|^2\,.
$$

(d) In the simplified case above,

$$
\frac{\Gamma_I}{\Gamma_{II}} = \frac{(5.4 - 1.8)^5}{5.4^5 \times 1.8^2}|\langle \alpha, 1||P_1||\alpha', 1\rangle|^2 = 6 \times 10^{-5}
$$

gives

$$|\langle\alpha, 1||P_1||\alpha', 1\rangle|^2 = 24.6 \times 6 \times 10^{-5} = 1.48 \times 10^{-3}\ \text{MeV}^2\,,$$

or

$$|\langle\alpha, 1||P_1||\alpha', 1\rangle| = 38\ \text{keV}\,.$$

2092

"Unlike atomic spectroscopy, electric dipole (E1) transitions are not usually observed between the first few nuclear states".

(a) For light nuclei, give arguments that support this statement on the basis of the shell model. Indicate situations where exceptions might be expected.

(b) Make an order-of-magnitude "guesstimate" for the energy and radioactive lifetime of the lowest-energy electric dipole transition expected for ${}^{17}_{9}F_8$, outlining your choice of input parameters.

(c) Show that for nuclei containing an equal number of neutrons and protons ($N = Z$), no electric dipole transitions are expected between two states with the same isospin T.

The following Clebch–Gordan coefficient may be of use:

Using notation $\langle J_1 J_2 M_1 M_2 | J_{TOT} M_{TOT}\rangle$, $\langle J100|J0\rangle = 0$.

(*Princeton*)

Solution:

(a) Based on single-particle energy levels given by shell model, we see that levels in the same shell generally have the same parity, especially the lowest-lying levels like $1s, 1p, 1d, 2s$ shells, etc. For light nuclei, γ-transition occurs mainly between different single-nucleon levels. In transitions between different energy levels of the same shell, parity does not change. On the other hand, electric dipole transition $E1$ follows selection rules $\Delta J = 0$ or 1, $\Delta P = -1$. Transitions that conserve parity cannot be electric dipole in nature. However if the ground and excited states are not in the same shell, parity may change in a transition. For example in the transition $1p_{3/2} \to 1s_{1/2}$, $\Delta J = 1$, $\Delta P = -1$. This is an electric dipole transition.

(b) In the single-particle model, the lowest-energy electric dipole transition $E1$ of ^{17}F is $2s_{1/2} \to 1p_{1/2}$. The transition probability per unit time can be estimated by (**Problem 2093** with $L = 1$)

$$\lambda \approx \frac{c}{4}\left(\frac{e^2}{\hbar c}\right)\left(\frac{E_\gamma}{\hbar c}\right)^3 \langle r\rangle^2 ,$$

where E_γ is the transition energy and $\langle r\rangle \sim R = 1.4 \times 10^{-13}\ A^{1/3}$ cm. Thus

$$\lambda \approx \frac{3\times 10^{10}\times(1.4\times 10^{-13})^2}{4\times 137\times(197\times 10^{-13})^3}\ A^{2/3}E_\gamma^3 = 1.4\times 10^{14}A^{2/3}E_\gamma^3 ,$$

where E_γ is in MeV. For ^{17}F we may take $E_\gamma \approx 5$ MeV, $A = 17$, and so

$$\lambda = 1.2 \times 10^{17}\ s ,$$

or

$$\tau = \lambda^{-1} = 9 \times 10^{-18}\ s .$$

(c) For light or medium nuclei, the isospin is a good quantum number. A nucleus state can be written as $|JmTT_z\rangle$, where J, m refer to angular momentum, T, T_z refer to isospin. The electric multipole transition operator between two states is

$$\begin{aligned} O_E(L,E) &= \sum_{K=1}^{A}\left[\frac{1}{2}(1+\tau_z(K))e_p + \frac{1}{2}(1-\tau_z(K))e_n\right] r^L(K)Y_{LM}(r(K)) \\ &= \sum_{K=1}^{A} S(L,M,K)\cdot 1 + \sum_{K=1}^{A} V(L,M,K)\tau_z(K) \end{aligned}$$

with

$$S(L,M,K) = \frac{1}{2}(e_p+e_n)r^L(K)Y_{LM}(r(K)) ,$$

$$V(L,M,K) = \frac{1}{2}(e_p-e_n)r^L(K)Y_{LM}(r(K)) ,$$

where τ_z is the z component of the isospin matrix, for which $\tau_z\phi_n = -\phi_n$, $\tau_z\phi_p = +\phi_p$.

The first term is related to isospin scalar, the second term to isospin vector. An electric multipole transition from J, T, T_z to J', T', T_z' can be written as

$$B_E(L : J_iT_iT_z \to J_fT_fT_z) = \langle J_fT_fT_z|O_E(L)|J_iT_iT_z\rangle^2/(2J_i+1)$$

$$= \frac{1}{(2J_i+1)(2T_f+1)}[\delta_{T_iT_f}\langle J_fT_f|\sum_{K=1}^{A} S(L,K)\cdot 1|J_iT_i\rangle$$

$$+ \langle T_iT_z10|T_fT_z\rangle\langle J_fT_f|\sum_{K=1}^{A} V(L,K)\tau_z(K)|J_iT_i\rangle]^2 \,.$$

From the above equation, we see that for electric multipole transitions between two states the isospin selection rule is $\Delta T \leq 1$. When $\Delta T = 0$, $\delta'_{TT} \neq 0$, there is an isospin scalar component; when $\Delta T = 1$, the scalar component is zero.

For electric dipole transition,

$$\sum_{K=1}^{A} S(L,K)\cdot 1 = \sum_{K=1}^{A} \frac{1}{2}(e_p+e_n)r(K)Y_{LM}(r(K))$$

$$= \frac{1}{2}(e_p+e_n)\sum_{K=1}^{A} r(K)Y_{LM}(r(K))\,,$$

r being nucleon coordinate relative to the center of mass of the nucleus.

For spherically or axially symmetric nuclei, as$\sum_{K=1}^{A} rY_{LM}(r(K))$ is zero, the isospin scalar term makes no contribution to electric dipole transition. For the isospin vector term, when $T_i = T_f = T$,

$$\langle T_iT_z10|T_fT_z\rangle = \frac{T_z}{\sqrt{T(T+1)}}\,.$$

Then for nuclei with $Z = N$, in transitions between two levels of $\Delta T = 0$, as $T_z = 0$,

$$\langle T_iT_z10|T_fT_z\rangle = 0\,.$$

and so both the isospin scalar and vector terms make no contribution. Thus for self-conjugate nuclei, states with $T_i = T_f$ cannot undergo electric dipole transition.

2093

(a) Explain why electromagnetic E_λ radiation is emitted predominantly with the lowest allowed multipolarity L. Give an estimate for the ratios $E_1 : E_2 : E_3 : E_4 : E_5$ for the indicated transitions in ^{16}O (as shown in Fig. 2.31).

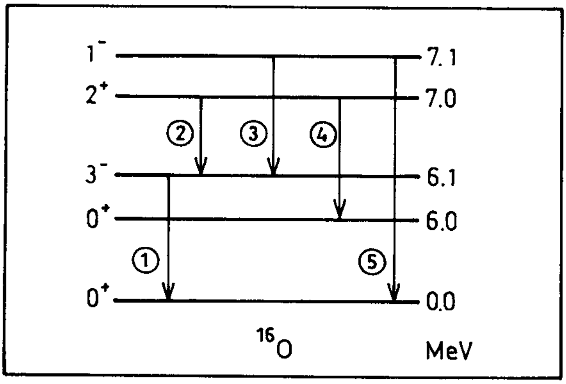

Fig. 2.31

(b) Estimate the lifetime of the 7.1 MeV state. Justify your approximations.

(c) List the possible decay modes of the 6.0 MeV state.

(*Princeton*)

Solution:

(a) In nuclear shell theory, γ-ray emission represents transition between nucleon energy states in a nucleus. For a proton moving in a central field radiation is emitted when it transits from a higher energy state to a lower one in the nucleus. If L is the degree of the electric multipole radiation, the transition probability per unit time is given by

$$\lambda_E(L) \approx \frac{2(L+1)}{L[(2L+1)!!]^2}\left(\frac{3}{L+3}\right)^2\left(\frac{e^2}{\hbar}\right)k^{2L+1}\langle r^L\rangle^2 ,$$

where $k = \frac{w}{c} = \frac{E_\gamma}{\hbar c}$ is the wave number of the radiation, E_γ being the transition energy, and $\langle \gamma^L\rangle^2 \approx R^{2L}$, $R = 1.4 \times 10^{-13} A^{1/3}$ cm being the nuclear radius. Thus

$$\begin{aligned}\lambda_E(L) \approx & \frac{2(L+1)}{L[(2L+1)!!]^2}\left(\frac{3}{L+3}\right)^2\left(\frac{e^2}{R\hbar c}\right)\left(\frac{E_\gamma c}{\hbar c}\right)\left(\frac{E_\gamma R}{\hbar c}\right)^{2L} \\ = & \frac{2(L+1)}{L[(2L+1)!!]^2}\left(\frac{3}{L+3}\right)^2\frac{1}{137}\left(\frac{3\times 10^{10} E_\gamma}{197\times 10^{-13}}\right)\end{aligned}$$

$$\times \left(\frac{E_\gamma \times 1.4 \times 10^{-13} A^{1/3}}{197 \times 10^{-13}}\right)^{2L}$$

$$= \frac{4.4(L+1)}{L[(2L+1)!!]^2} \left(\frac{3}{L+3}\right)^2 \left(\frac{E_\gamma}{197}\right)^{2L+1} (1.4A^{1/3})^{2L} \times 10^{21}\ s^{-1}$$

with E_γ in MeV. Consider ^{16}O. If $E_\gamma \sim 1$ MeV, we have

$$\frac{\lambda_E(L+1)}{\lambda_E(L)} \sim (kR)^2 = \left(\frac{E_\gamma R}{\hbar c}\right)^2 = \left(\frac{1.4 \times 10^{-13} \times 16^{1/3}}{197 \times 10^{-13}}\right)^2 \approx 3 \times 10^{-4}.$$

Hence $\lambda_E(L)$ decreases by a factor 10^{-4} as L increases by 1. This means that E_L radiation is emitted predominantly with the lowest allowed multipolarity L.

The tranistions of ^{16}O indicated in Fig. 2.31 can be summarized in the table below.

Transition	$\Delta\pi$	Δl	Type	L	E_γ (MeV)
E_1	yes	3	octopole	3	6.1
E_2	yes	1	dipole	1	0.9
E_3	no	2	quadrupole	2	1.0
E_4	no	2	quadrupole	2	1.0
E_5	yes	1	dipole	1	7.1

Thus we have

$$\begin{aligned}\lambda_{E_1} : \lambda_{E_2} : \lambda_{E_3} : \lambda_{E_4} : \lambda_{E_5} &= \frac{4}{3(7!!)^2} \left(\frac{1}{2}\right)^2 \left(\frac{6.1}{197}\right)^7 (1.4A^{1/3})^6 \\ &: \frac{2}{(3!!)^2} \left(\frac{3}{4}\right)^2 \left(\frac{0.9}{197}\right)^3 (1.4A^{1/3})^2 \\ &: \frac{3}{2(5!!)^2} \left(\frac{3}{5}\right)^2 \left(\frac{1}{197}\right)^5 (1.4A^{1/3})^4 \\ &: \frac{3}{2(5!!)^2} \left(\frac{3}{5}\right)^2 \left(\frac{1}{197}\right)^5 (1.4A^{1/3})^4\end{aligned}$$

$$\begin{aligned}
&: \frac{2}{(3!!)^2}\left(\frac{3}{4}\right)^2\left(\frac{7.1}{197}\right)^3(1.4A^{1/3})^2 \\
&= 1.59\times10^{-12} : 1.48\times10^{-7} : 1.25\times10^{-12} \\
&\quad : 1.25\times10^{-12} : 7.28\times10^{-5} \\
&= 2.18\times10^{-8} : 2.03\times10^{-3} \\
&\quad : 1.72\times10^{-8} : 1.72\times10^{-8} : 1
\end{aligned}$$

Thus the transition probability of E_5 is the largest, that of E_2 is the second, those of E_3, E_4 and E_1 are the smallest.

(b) The half-life of the 7.1 MeV level can be determined from λ_{E_5}:

$$\lambda_{E_5} = \frac{4.4\times 2}{(3!!)^2}\left(\frac{3}{4}\right)^2\left(\frac{7.1}{197}\right)^3(1.4\times16^{1/3})^2\times10^{21} = 3.2\times10^{17}\ s^{-1},$$

giving

$$T_{1/2}(7.1\ \text{MeV}) = \ln 2/\lambda_{E_5} = 2.2\times10^{-18}\ s\,.$$

Neglecting transitions to other levels is justified as their probabilities are much smaller, e.g.,

$$\lambda_{E_3} : \lambda_{E_5} = 1.7\times10^{-8} : 1\,.$$

In addition, use of the single-particle model is reasonable as it assumes the nucleus to be spherically symmetric, the initial and final state wave functions to be constant inside the nucleus and zero outside which are plausible for ^{16}O.

(c) The γ-transition $0^+ \to 0^+$ from the 6.0 MeV states to the ground state of ^{16}O is forbidden. However, the nucleus can still go to the ground state by internal conversion.

2094

The γ-ray total nuclear cross section σ_{total} (excluding e^+e^- pair production) on neodymium 142 is given in Fig. 2.32

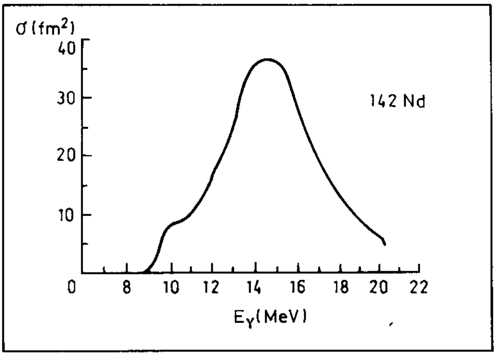

Fig. 2.32

(a) Which electric or magnetic multipole is expected to dominate the cross section and why?

(b) Considering the nucleus simply as two fluids of nucleons (protons and neutrons), explain qualitatively the origin of the resonance shown in the figure.

(c) Using a simple model of the nucleus as A particles bound in an harmonic oscillator potential, estimate the resonance energy as a function of A. Does this agree with the observed value in the figure for A = 142?

(d) Discuss the role of residual two-body interactions in modifying the estimate in (c).

(e) What are the physical processes responsible for the width of the resonance? Make rough estimates of the width due to different mechanisms.

(*Princeton*)

Solution:

(a) The excitation curves of reactions (γ, n) and (γ, p) show a broad resonance of several MeV width from $E_\gamma = 10$ to 20 MeV. This can be explained as follows. When the nuclear excitation energy increases, the density of states increases and the level widths become broader. When the level spacing and level width become comparable, separate levels join together, so that γ-rays of a wide range of energy can excite the nucleus, thus producing a broad resonance. If $E_\gamma \approx 15$ MeV, greater than the nucleon harmonic oscillator energy $\hbar\omega \approx 44/A^{1/3}$ MeV, dipole transition can occur. The single-particle model gives (**Problem 2093(a)**)

$$\frac{\Gamma(E2orM1)}{\Gamma(E1)} \approx (kR)^2 = \left(\frac{15 \times 1.4 \times 10^{-13} \times 142^{1/3}}{197 \times 10^{-13}}\right)^2 = 0.3\,.$$

Hence the nuclear cross section is due mainly to electric dipole absorption. We can also consider the collective absorption of the nucleus. We see that absorption of γ-rays causes the nucleus to deform and when the γ-ray energy equals the nuclear collective vibrational energy levels, resonant absorption can take place. As $E_\gamma \approx 15$ MeV, for ^{142}Nd nucleus, electric dipole, quadrupole, octopole vibrations are all possible. However as the energy is nearest to the electric dipole energy level, $E1$ resonant absorption predominates.

(b) Consider the protons and neutrons inside the nucleus as liquids that can seep into each other but cannot be compressed. Upon impact of the incoming photon, the protons and neutrons inside the nucleus tend to move to different sides, and their centers of mass become separated. Consequently, the potential energy of the nucleus increases, which generates restoring forces resulting in dipole vibration. Resonant absorption occurs if the photon frequency equals the resonant frequency of the harmonic oscillator.

(c) In a simple harmonic-oscillator model we consider a particle of mass $M = Am_N$, m_N being the nucleon mass, moving in a potential $V = \frac{1}{2}Kx^2$, where K, the force constant, is proportional to the nuclear cross-sectional area. The resonant frequency is $f \approx \sqrt{K/M}$. As $K \propto R^2 \propto A^{2/3}$, $M \propto A$, we have

$$f \propto A^{-1/6} \approx A^{-0.17}\,.$$

This agrees with the experimental result $E_\gamma \propto A^{-0.19}$ fairly well.

(d) The residual two-body force is non-centric. It can cause the nucleus to deform and so vibrate more easily. The disparity between the rough theoretical derivation and experimental results can be explained in terms of the residual force. In particular, for a much deformed nucleus double resonance peaks may occur. This has been observed experimently.

(e) The broadening of the width of the giant resonance is due mainly to nuclear deformation and resonance under the action of the incident photons. First, the deformation and restoring force are related to many factors and so the hypothetical harmonic oscillator does not have a "good" quality (Q value is small), correspondingly the resonance width is broad. Second, the photon energy can pass on to other nucleons, forming a compound nucleus and redistribution of energy according to the degree of freedom. This may generate a broad resonance of width from several to 10 MeV. In addition there are other broadening effects like the Doppler effect of an order of magnitude of several keV. For a nucleus of $A = 142$, the broadening due to Doppler effect is

$$\Delta E_D \approx \frac{E_\gamma^2}{Mc^2} \approx \frac{15^2}{142 \times 940} = 1.7 \times 10^{-3}\ MeV = 1.7\ \text{keV}\,.$$

2095

The total cross section for the absorption of γ-rays by ^{208}Pb (whose ground state has spin-parity $J^\pi = 0^+$) is shown in Fig. 2.33. The peak at 2.6 MeV corresponds to a $J^\pi = 3^-$ level in ^{208}Pb which γ-decays to a 1^- level at 1.2 MeV (see Fig. 2.34).

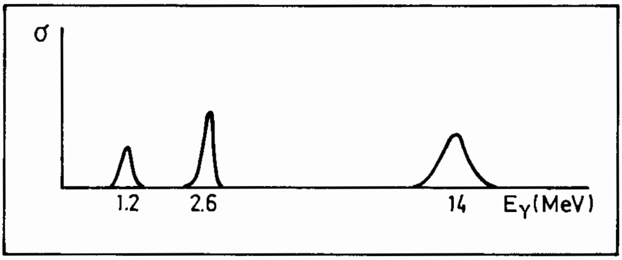

Fig. 2.33

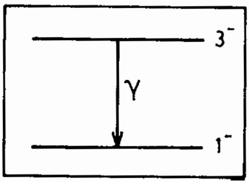

Fig. 2.34

(a) What are the possible electric and/or magnetic multipolarities of the γ-rays emitted in the transition between the 2.6 MeV and 1.2 MeV levels? Which one do you expect to dominate?

(b) The width of the 2.6 MeV level is less than 1 eV, whereas the width of the level seen at 14 MeV is 1 MeV. Can you suggest a plausible reason for this large difference? What experiment might be done to test your conjecture?

(*Wisconsin*)

Solution:

(a) In the transition $3^- \to 1^-$, the emitted photon can carry away an angular momentum $l = 4, 3, 2$. As there is no parity change, $l = 4, 2$. Hence the possible multipolarities of the transition are $E4$, $M3$ or $E2$. The electric quadrupole transition $E2$ is expected to dominate.

(b) The width of the 2.6 MeV level, which is less than 1 eV, is typical of an electromagnetic decay, whereas the 14 MeV obsorption peak is a giant dipole resonance (**Problem 2094**). As the resonance energy is high, the processes are mostly strong interactions with emission of nucleons, where the single-level widths are broader and many levels merge to form a broad, giant resonance. Thus the difference in decay mode leads to the large difference in level width.

Experimentally, only γ-rays should be found to be emitted from the 2.6 MeV level while nucleons should also be observed to be emitted from the 14 MeV level.

2096

Gamma-rays that are emitted from an excited nuclear state frequently have non-isotropic angular distribution with respect to the spin direction of the excited nucleus. Since generally the nuclear spins are not aligned, but their directions distributed at random, this anisotropy cannot be measured. However, for nuclides which undergo a cascade of γ-emissions (e.g., ^{60}Ni which is used for this problem-see Fig. 2.35), the direction of one of the cascading γ-rays can be used as a reference for the orientation of a specific nucleus. Thus, assuming a negligible half-life for the intermediate state, a measurement of the coincidence rate between the two γ-rays can give the angular correlation which may be used to determine the nuclear spins.

In the case of ^{60}Ni we find such a cascade, namely $J^p = 4^+ \to J^p = 2^+ \to J^p = 0^+$. The angular correlation function is of the form $W(\theta) \sim 1 + 0.1248 \cos^2 \theta + 0.0418 \cos^4 \theta$.

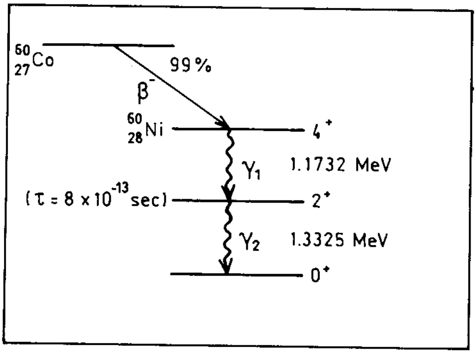

Fig. 2.35

(a) Of what types are the transitions?

(b) Why are the odd powers of $\cos\theta$ missing? Why is $\cos^4\theta$ the highest power?

(c) Draw a schematic diagram of an experimental setup showing how you would make the measurements. Identify all components. (Give block diagram.)

(d) Describe the γ-ray detectors.

(e) How do you determine the coefficients in the correlation function which would prove that ^{60}Ni undergoes the transition $4 \to 2 \to 0$?

(f) Accidental coincidences will occur between the two γ-ray detectors. How can you take account of them?

(g) How would a source of ^{22}Na be used to calibrate the detectors and electronics? (^{22}Na emits 0.511 MeV gammas from β^+ annihilation.)

(h) How would Compton scattering of γ-rays within the ^{60}Co source modify the measurements?

(*Chicago*)

Solution:

(a) Each of the two gamma-ray cascading emissions subtracts 2 from the angular momentum of the excited nucleus, but does not change the parity. Hence the two emissions are of electric-quadrupole $E2$ type.

(b) The angular correlation function for cascading emission can be written as

$$W(\theta) = \sum_{K=0}^{K_{\max}} A_{2K} P_{2K}(\cos\theta)\,,$$

where $0 \leq K_{\max} \leq \min(J_b, L_1, L_2)$,

$$A_{2K} = F_{2K}(L_1, J_a, J_b) F_{2K}(L_2, J_c, J_b)\,,$$

L_1, L_2 being the angular momenta of the two γ-rays, J_a, J_b, J_c being respectively the initial, intermediate and final nuclear spins, $P_{2K}(\cos\theta)$ are Legendre polynomials.

Since $W(\theta)$ depends on $P_{2K}(\cos\theta)$ only, it consists of even powers of $\cos\theta$. For the $4^+ \to 2^+ \to 0^+$ transition of ^{60}Ni, $K_{\max}$ is 2. Hence the highest power of $\cos\theta$ in $P_4(\cos\theta)$ is 4, and so is in $W(\theta)$.

(c) Figure 2.36 shows a block diagram of the experimental apparatus to measure the angular correlation of the γ-rays. With probe 1 fixed, rotate probe 2 in the plane of the source and probe 1 about the source to change the angle θ between the two probes, while keeping the distance between the probes constant. A fast-slow-coincidence method may be used to reduce spurious coincidences and multiscattering.

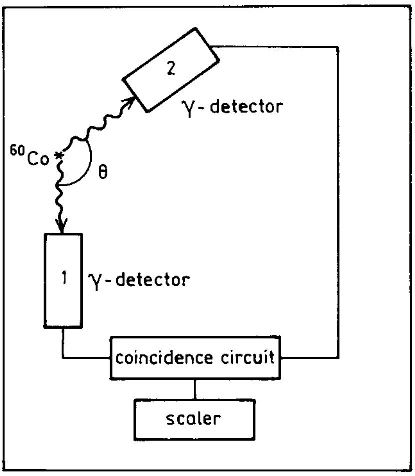

Fig. 2.36

(d) A γ-ray detector usually consists of a scintillator, a photomultiplier, and a signal-amplifying high-voltage circuit for the photomultiplier. When the scintillator absorbs a γ-ray, it fluoresces. The fluorescent photons hit the cathode of the photomultiplier, causing emission of primary photoelectrons, which are multiplied under the high voltage, giving a signal on the anode. The signal is then amplified and processed.

(e) The coincidence counting rate $W(\theta)$ is measured for various θ. Fitting the experimental data to the angular correlation function we can deduce the coefficients.

(f) We can link a delay line to one of the γ-detectors. If the delay time is long compared to the lifetime of the intermediate state the signals from the two detectors can be considered independent, and the coincidence counting rate accidental. This may then be used to correct the observed data.

(g) The two γ-photons of 0.511 MeV produced in the annihilation of β^+ from ^{22}Na are emitted at the same time and in opposite directions. They can be used as a basis for adjusting the relative time delay between the two detectors to compensate for any inherent delays of the probes and electronic circuits to get the best result.

(h) The Compton scattering of γ-rays in the ^{60}Co source will increase the irregularity of the γ-emission and reduce its anisotropy, thereby reducing the deduced coefficients in the angular correlation function.

2097

A nucleus of mass M is initially in an excited state whose energy is ΔE above the ground state of the nucleus. The nucleus emits a gamma-ray of energy $h\nu$ and makes a transition to its ground state.

Explain why the gamma-ray $h\nu$ is not equal to the energy level difference ΔE and determine the fractional change $\frac{h\nu-\Delta E}{\Delta E}$. (You may assume $\Delta E < Mc^2$)

(*Wisconsin*)

Solution:

The nucleus will recoil when it emits a γ-ray because of the conservation of momentum. It will thereby acquire some recoil energy from the excitation energy and make $h\nu$ less than ΔE.

Let the total energy of the nucleus be E and its recoil momentum be p. The conservation of energy and of momentum give

$$p = p_\gamma, \qquad E + E_\gamma = Mc^2 + \Delta E\,.$$

As

$$E_\gamma = P_\gamma c = h\nu, \qquad E = \sqrt{p^2c^2 + M^2c^4}\,,$$

we have

$$E_\gamma = \frac{1}{2Mc^2} \cdot \frac{(\Delta E)^2 + 2Mc^2\Delta E}{\left(1 + \frac{\Delta E}{Mc^2}\right)} \approx \Delta E - \frac{(\Delta E)^2}{2Mc^2}\,,$$

or

$$\frac{h\nu - \Delta E}{\Delta E} = -\frac{\Delta E}{2Mc^2}\,.$$

2098

A (hypothetical) particle of rest mass m has an excited state of excitation energy ΔE, which can be reached by γ-ray absorption. It is assumed that $\Delta E/c^2$ is not small compared to m.

Find the resonant γ-ray energy, E_γ, to excite the particle which is initially at rest.

(*Wisconsin*)

Solution:

Denote the particle by A. The reactions is $\gamma + A \to A^*$. Let E_γ and p_γ be the energy and momentum of the γ-ray, p be the momentum of A, initially at rest, after it absorbs the γ-ray. Conservation of energy requires

$$E_\gamma + mc^2 = \sqrt{\left(m + \frac{\Delta E}{c^2}\right)^2 c^4 + p^2c^2}\,.$$

Momentum conservation requires

$$p = p_\gamma\,,$$

or

$$pc = p_\gamma c = E_\gamma\,.$$

Its substitution in the energy equation gives

$$E_\gamma = \Delta E + \frac{(\Delta E)^2}{2mc^2}\,.$$

Thus the required γ-ray energy is higher than ΔE by $\frac{\Delta E^2}{2mc^2}$, which provides for the recoil energy of the particle.

2099

(a) Use the equivalence principle and special relativity to calculate, to first order in y, the frequency shift of a photon which falls straight down through a distance y at the surface of the earth. (Be sure to specify the sign.)

(b) It is possible to measure this frequency shift in the laboratory using the Mössbauer effect.

Describe such an experiment — specifically:

What is the Mössbauer effect and why is it useful here?

What energy would you require the photons to have?

How would you generate such photons?

How would you measure such a small frequency shift?

Estimate the number of photons you would need to detect in order to have a meaningful measurement.

(*Columbia*)

Solution:

(a) Let the original frequency of the photon be ν_0, and the frequency it has after falling a distance y in the earth's gravitational field be ν. Then the equivalent masses of the photon are respectively $h\nu_0/c^2$ and $h\nu/c^2$. Suppose the earth has mass M and radius R. Conservation of energy requires

$$h\nu_0 - G\frac{M \cdot \frac{h\nu_0}{c^2}}{R+y} = h\nu - G\frac{M \cdot \frac{h\nu}{c^2}}{R},$$

where G is the gravitational constant, or, to first order in y,

$$\frac{\nu - \nu_0}{\nu_0} = \frac{GM}{c^2}\left(\frac{1}{R} - \frac{1}{R+y}\right) \approx \frac{gy}{c^2} = 1.09 \times 10^{-16} y,$$

where g is the acceleration due to gravity and y is in meters. For example, taking $y = 20\ m$ we have

$$\frac{\nu - \nu_0}{\nu_0} = 2.2 \times 10^{-15}.$$

(b) In principle, photons emitted by a nucleus should have energy E_γ equal to the excitation energy E_0 of the nucleus. However, on account of the recoil of the nucleus which takes away some energy, $E_\gamma < E_0$, or more precisely (**Problem 2097**),

$$E_\gamma = E_0 - \frac{E_0^2}{2Mc^2} \,,$$

where M is the mass of the nucleus. Likewise, when the nucleus absorbs a photon by resonant absorption the latter must have energy (**Problem 2098**)

$$E_\gamma = E_0 + \frac{E_0^2}{2Mc^2} \,.$$

As $\frac{E_0^2}{2Mc^2}$ is usually larger than the natural width of the excited state, γ-rays emitted by a nucleus cannot be absorbed by resonant absorption by the same kind of nucleus.

However, when both the γ source and the absorber are fixed in crystals, the whole crystal recoils in either process, $M \to \infty$, $\frac{E_0^2}{2Mc^2} \to 0$. Resonant absorption can now occur for absorber nuclei which are the same as the source nuclei. This is known as the Mössbauer effect. It allows accurate measurement of γ-ray energy, the precision being limited only by the natural width of the level.

To measure the frequency shift $\frac{\Delta\nu}{\nu_0} = 2.2 \times 10^{-15}$, the γ source used must have a level of natural width Γ/E_γ less than $\Delta\nu/\nu_0$. A possible choice is ^{67}Zn which has $E_\gamma = 93$ keV, $\Gamma/E_\gamma = 5.0 \times 10^{-16}$. Crystals of ^{67}Zn are used both as source and absorber. At $y = 0$, both are kept fixed in the same horizontal plane and the resonant aborption curve is measured. Then move the source crystal to 20 m above the absorber. The frequency of the photons arriving at the fixed absorber is $\nu_0 + \Delta\nu$ and resonant absorption does not occur. If the absorber is given a downward velocity of v such that by the Doppler effect the photons have frequency ν_0 as seen by the absorber, resonant absorption can take place. As

$$\nu_0 = (\nu_0 + \Delta\nu)\left(1 - \frac{v}{c}\right) \approx \nu_0 + \Delta\nu - \nu_0\left(\frac{v}{c}\right) \,,$$

$$\begin{aligned} v &\approx c\left(\frac{\Delta\nu}{\nu_0}\right) = 3 \times 10^{10} \times 2.2 \times 10^{-15} \\ &= 6.6 \times 10^{-5}\ cm\ s^{-1} \,, \end{aligned}$$

which is the velocity required for the absorber.

Because the natural width for γ-emission of ^{67}Zn is much smaller than $\Delta\nu/\nu_0$, there is no need for a high counting rate. A statistical error of 5% at the spectrum peak is sufficient for establishing the frequency shift, corresponding to a photon count of 400.

2100

A parent isotope has a half-life $\tau_{1/2} = 10^4$ yr$= 3.15 \times 10^{11}$ s. It decays through a series of radioactive daughters to a final stable isotope. Among the daughters the greatest half-life is 20 yr. Others are less than a year. At $t = 0$ one has 10^{20} parent nuclei but no daughters.

(a) At $t = 0$ what is the activity (decays/sec) of the parent isotope?

(b) How long does it take for the population of the 20 yr isotope to reach approximately 97% of its equilibrium value?

(c) At $t = 10^4$ yr how many nuclei of the 20 yr isotope are present? Assume that none of the decays leading to the 20 yr isotope is branched.

(d) The 20 yr isotope has two competing decay modes: α, 99.5%; β, 0.5%. At $t = 10^4$ yr, what is the activity of the isotope which results from the β-decay?

(e) Among the radioactive daughters, could any reach their equilibrium populations much more quickly (or much more slowly) than the 20 yr isotope?

(*Wisconsin*)

Solution:

(a) The decay constant of the parent isotope is

$$\lambda_1 = \frac{\ln 2}{\tau_{1/2}} = 6.93 \times 10^{-5}\ yr^{-1} = 2.2 \times 10^{-12}\ s^{-1} .$$

When $t = 0$, the activity of the parent isotope is

$$A_1(0) = \lambda_1 N_1(t=0) = \frac{2.2 \times 10^{-12} \times 10^{20}}{3.7 \times 10^7} = 5.95 \text{ millicurie} .$$

(b) Suppose the 20 yr isotope is the nth-generation daughter in a radioactive series. Then its population is a function of time:

$$N_n(t) = N_1(0)(h_1 e^{-\lambda_1 t} + h_2 e^{-\lambda_2 t} + \cdots + h_n e^{-\lambda_n t}) ,$$

where

$$h_1 = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1) \cdots (\lambda_n - \lambda_1)},$$

$$h_2 = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_2) \cdots (\lambda_n - \lambda_2)},$$

$$\vdots$$

$$h_n = \frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{(\lambda_1 - \lambda_n)(\lambda_2 - \lambda_n) \cdots (\lambda_{n-1} - \lambda_n)},$$

where $N_1(0)$ is the number of the parent nuclei at $t = 0$, λ_i is the decay constant of the ith-generation daughter. For secular equilibrium we require $\lambda_1 \ll \lambda_j$, $j = 2, 3, \ldots, n, \ldots$. As the nth daugther has the largest half-life of 10^{20} yr, we also have $\lambda_n \ll \lambda_j$, $j = 2, 3, \ldots, (j \neq n)$, $\lambda_n = \ln 2/\tau_{1/2} = 3.466 \times 10^{-2}\ yr^{-1}$. Thus

$$h_1 \approx \frac{\lambda_1}{\lambda_n}, \qquad h_n \approx -\frac{\lambda_1}{\lambda_n}.$$

After a sufficiently long time the system will reach an equilibrium at which $\lambda_n N_n^e(t) = \lambda_1 N_1^e(t)$, the superscript e denoting equilibrium values, or

$$N_n^e(t) = \frac{\lambda_1}{\lambda_n} N_1^e(t) = \frac{\lambda_1}{\lambda_n} N_1(0) e^{-\lambda_1 t}.$$

At time t before equilibrium is reached we have

$$N_n(t) \approx N_1(0) \left(\frac{\lambda_1}{\lambda_n} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_n} e^{-\lambda_n t} \right).$$

When $N_n(t) = 0.97 N_n^e(t)$, or

$$0.97 \frac{\lambda_1}{\lambda_n} N_1(0) e^{-\lambda_1 t} \approx N_1(0) \left(\frac{\lambda_1}{\lambda_n} e^{-\lambda_1 t} - \frac{\lambda_1}{\lambda_n} e^{-\lambda_n t} \right),$$

the time is $t = t_0$ given by

$$t_0 = \frac{\ln 0.03}{\lambda_1 - \lambda_n} \approx 101\ yr.$$

Hence after about 101 years the population of the 20 yr isotope will reach 97% of its equilibrium value.

(c) At $t = 10^4$ yr, the system can be considered as in equilibrium. Hence the population of the 20 yr isotope at that time is

$$N_n(10^4) = \frac{\lambda_1}{\lambda_n} N_1(0) e^{-\lambda_1 t} = 10^{17} .$$

(d) After the system has reached equilibrium, all the isotopes will have the same activity. At $t = 10^4$ years, the activity of the parent isotope is

$$\begin{aligned} A_1(10^4) &= \lambda_1 N(0) e^{-\lambda_1 t} = 6.93 \times 10^{-5} \times 10^{20} \times \exp(-6.93 \times 10^{-5} \times 10^4) \\ &= 3.47 \times 10^{15}\ yr^{-1} = 3.0\ mc . \end{aligned}$$

The activity of the β-decay product of the 20 yr isotope is

$$A_\beta = 3 \times 0.05 = 0.15\ mc .$$

(e) The daughter nuclei ahead of the 20 yr isotope will reach their equilibrium populations more quickly than the 20 yr isotope, while the daughter nuclei after the 20 yr isotope will reach their equilibrium populations approximately as fast as the 20 yr isotope.

2101

A gold foil 0.02 cm thick is irradiated by a beam of thermal neutrons with a flux of 10^{12} neutrons/cm^2/s. The nuclide ^{198}Au with a half-life of 2.7 days is produced by the reaction $^{197}\text{Au}(n, \gamma)^{198}\text{Au}$. The density of gold is 19.3 gm/cm^3 and the cross section for the above reaction is 97.8×10^{-24} cm^2. ^{197}Au is 100% naturally abundant.

(a) If the foil is irradiated for 5 minutes, what is the ^{198}Au activity of the foil in decays/cm^2/s?

(b) What is the maximum amount of ^{198}Au/cm^2 that can be produced in the foil?

(c) How long must the foil be irradiated if it is to have 2/3 of its maximum activity?

(Columbia)

Solution:

(a) Initially the number of ^{197}Au nuclei per unit area of foil is

$$N_1(0) = \frac{0.02 \times 19.3}{197} \times 6.023 \times 10^{23} = 1.18 \times 10^{21}\ \text{cm}^{-2} .$$

Let the numbers of ^{197}Au and ^{198}Au nuclei at time t be N_1, N_2 respectively, σ be the cross section of the (n, γ) reaction, I be flux of the incident neutron beam, and λ be the decay constant of ^{198}Au. Then

$$\frac{dN_1}{dt} = -\sigma I N_1 ,$$

$$\frac{dN_2}{dt} = \sigma I N_1 - \lambda N_2 .$$

Integrating we have

$$N_1 = N_1(0) e^{-\sigma I t,} ,$$

$$N_2 = \frac{\sigma I}{\lambda - \sigma I} N_1(0)(\bar{e}^{\sigma I t} - e^{-\lambda t}) .$$

As

$$\lambda = \frac{\ln 2}{2.7 \times 24 \times 3600} = 2.97 \times 10^{-6}\ s^{-1} ,$$

$$\sigma I = 9.78 \times 10^{-23} \times 10^{12} = 9.78 \times 10^{-11}\ s^{-1} \ll \lambda ,$$

at $t = 5\ min = 300\ s$ the activity of ^{198}Au is

$$\begin{aligned} A(300s) = \lambda N_2(t) &= \frac{\lambda \sigma I N_1(0)}{\lambda - \sigma I}(e^{-\sigma I t} - e^{-\lambda t}) \approx \sigma I N_1(0)(1 - e^{-\lambda t}) \\ &= 9.78 \times 10^{-11} \times 1.18 \times 10^{21} \times [1 - \exp(-2.97 \times 10^{-6} \times 300)] \\ &= 1.03 \times 10^{8}\ \text{cm}^{-2}\ s^{-1} . \end{aligned}$$

(b) After equilibrium is attained, the activity of a nuclide, and hence the number of its nuclei, remain constant. This is the maximum amount of ^{198}Au that can be produced. As

$$\frac{dN_2}{dt} = 0 ,$$

we have

$$\lambda N_2 = \sigma I N_1 \approx \sigma I N_1(0)$$

giving

$$N_2 = \frac{\sigma I}{\lambda} N_1(0) = \frac{9.78 \times 10^{-11}}{2.97 \times 10^{-6}} \times 1.18 \times 10^{21}$$
$$= 3.89 \times 10^{16} \text{ cm}^{-2} .$$

(c) As

$$A = \frac{2}{3} A_{\max} \approx \sigma I N_1(0)(1 - e^{-\lambda t}) ,$$

$$t = -\frac{1}{\lambda} \ln \left(1 - \frac{2}{3} \frac{A_{\max}}{\sigma I N_1(0)} \right) = -\frac{1}{\lambda} \ln \left(1 - \frac{2}{3} \right) = 3.70 \times 10^5 \ s = 4.28 \text{ day} .$$

2102

In the fission of ^{235}U, 4.5% of the fission lead to ^{133}Sb. This isotope is unstable and is the parent of a chain of β-emitters ending in stable ^{133}Cs:

$$^{133}Sb \xrightarrow{10 \text{ min}} {}^{133}Te \xrightarrow{60 \text{ min}} {}^{133}I \xrightarrow{22\text{hours}} {}^{133}Xe \xrightarrow{5.3\text{days}} {}^{133}Cs .$$

(a) A sample of 1 gram of uranium is irradiated in a pile for 60 minutes. During this time it is exposed to a uniform flux of 10^{11} neutrons/cm^2 sec. Calculate the number of atoms of Sb, Te, and I present upon removal from the pile. Note that uranium consists of 99.3% ^{238}U and 0.7% ^{235}U, and the neutron fission cross section of ^{235}U is 500 barns. (You may neglect the shadowing of one part of the sample by another.)

(b) Twelve hours after removal from the pile the iodine present is removed by chemical separation. How many atoms of iodine would be obtained if the separation process was 75% efficient?

(*Columbia*)

Solution:

(a) The number of Sb atoms produced in the pile per second is

$$C = N_0 \cdot f \cdot \sigma \cdot 4.5\%$$
$$= \frac{1 \times 0.007}{235} \times 6.023 \times 10^{23} \times 10^{11} \times 500 \times 10^{-24} \times 0.045$$
$$= 4.04 \times 10^7 \ s^{-1} .$$

Let the numbers of atoms of Sb, Te, I present upon removal from the pile be N_1, N_2, N_3 and their decay constants be $\lambda_1, \lambda_2, \lambda_3$ respectively. Then $\lambda_1 = \frac{\ln 2}{600} = 1.16 \times 10^{-3}\ s^{-1}$, $\lambda_2 = 1.93 \times 10^{-4}\ s^{-1}$, $\lambda_3 = 8.75 \times 10^{-6}\ s^{-1}$, and $\frac{dN_1}{dt} = C - \lambda_1 N_1$, with $N_1 = 0$ at $t = 0$, giving for $T = 3600\ s$,

$$N_1(T) = \frac{C}{\lambda_1}(1 - e^{-\lambda_1 T}) = 3.43 \times 10^{10}\,,$$

$\frac{dN_2}{dt} = \lambda_1 N_1 - \lambda_2 N_2$, with $N_2 = 0$, at $t = 0$, giving

$$N_2(T) = \frac{C}{\lambda_2}\left(1 + \frac{\lambda_2}{\lambda_1 - \lambda_2}e^{-\lambda_1 T} - \frac{\lambda_1}{\lambda_1 - \lambda_2}e^{-\lambda_2 T}\right) = 8.38 \times 10^{10}\,,$$

$\frac{dN_3}{dt} = \lambda_2 N_2 - \lambda_3 N_3$, with $N_3 = 0$, at $t = 0$, giving

$$\begin{aligned} N_3(T) = &\frac{C}{\lambda_3}\left[1 - \frac{\lambda_2\lambda_3 e^{-\lambda_1 T}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} - \frac{\lambda_3\lambda_1 e^{-\lambda_2 T}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)}\right] \\ &+ \frac{C}{\lambda_3}\left[\frac{\lambda_2\lambda_3}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} - \frac{\lambda_1\lambda_3}{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)} - 1\right]e^{-\lambda_3 T} \\ = &\frac{C}{\lambda_3}\left[1 - \frac{\lambda_2\lambda_3 e^{-\lambda_1 T}}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} - \frac{\lambda_3\lambda_1 e^{-\lambda_2 T}}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_1)}\right. \\ &\left. - \frac{\lambda_1\lambda_2 e^{-\lambda_3 T}}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}\right] \approx \frac{C}{\lambda_3}(1 - e^{-\lambda_3 T}) \\ = &\frac{C}{\lambda_3}(1 - 0.969) = 2.77 \times 10^{10}\,. \end{aligned}$$

(b) After the sample is removed from the pile, no more Sb is produced, but the number of Sb atoms will decrease with time. Also, at the initial time $t = T, N_1, N_2, N_3$ are not zero. We now have

$$\begin{aligned} N_1(t) = &N_1(T)e^{-\lambda_1 t}\,, \\ N_2(t) = &\frac{\lambda_1}{\lambda_2 - \lambda_1}N_1(T)e^{-\lambda_1 t} + \left[N_2(T) + \frac{\lambda_1 N_1(T)}{\lambda_1 - \lambda_2}e^{-\lambda_2 t}\right]\,, \\ N_3(t) = &\frac{\lambda_1\lambda_2 N_1(T)}{(\lambda_2 - \lambda_1)(\lambda_3 - \lambda_1)}e^{-\lambda_1 t} + \frac{\lambda_2}{\lambda_3 - \lambda_2}\left[N_2(T) + \frac{\lambda_1 N_1(T)}{\lambda_1 - \lambda_2}\right]e^{-\lambda_2 t} \\ &+ \left[N_3(T) + \frac{\lambda_2}{\lambda_2 - \lambda_3}N_2(T) + \frac{\lambda_1\lambda_2 N_1(T)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\right]e^{-\lambda_3 t}\,. \end{aligned}$$

For $t = 12$ hours, as $t \gg \tau_1, \tau_2$,

$$N_3(12 \text{ hours}) \approx \left[N_3(T) + \frac{\lambda_2}{\lambda_2 - \lambda_3}N_2(T) + \frac{\lambda_1\lambda_2 N_1(T)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3)}\right] e^{-\lambda_3 t}$$

$$= 10^{10} \times [2.77 + 8.80 + 3.62]$$

$$\times \exp(-8.75 \times 10^{-6} \times 12 \times 3600)$$

$$= 1.04 \times 10^{11} .$$

The number of atoms of I isotope obtained is

$$N = 0.75 \times N_3 = 7.81 \times 10^{10} .$$

2103

A foil of ^{7}Li of mass 0.05 gram is irradiated with thermal neutrons (capture cross section 37 milllibars) and forms ^{8}Li, which decays by β^--decay with a half-life of 0.85 sec. Find the equilibrium activity (number of β-decays per second) when the foil is exposed to a steady neutron flux of 3×10^{12} neutrons/sec·cm^2.

(*Columbia*)

Solution:

Let the ^{7}Li population be $N_1(t)$, the ^{8}Li population be $N_2(t)$. Initially

$$N_1(0) = \frac{0.05}{7} \times 6.023 \times 10^{23} = 4.3 \times 10^{21}, \quad N_2(0) = 0 .$$

During the neutron irradiation, $N_1(t)$ changes according to

$$\frac{dN_1}{dt} = -\sigma\phi N_1 ,$$

where σ is the neutron capture cross section and ϕ is the neutron flux, or

$$N_1(t) = N_1(0)e^{-\sigma\phi t} .$$

$N_2(t)$ changes according to

$$\frac{dN_2}{dt} = -\frac{dN_1}{dt} - \lambda N_2(t) = N_1(0)\sigma\phi e^{-\sigma\phi t} - \lambda N_2(t) ,$$

where λ is the β-decay constant of ^{8}Li. Integration gives

$$N_2(t) = \frac{\sigma\phi}{\lambda - \sigma\phi}(e^{-\sigma\phi t} - e^{-\lambda t})N_1(0) .$$

At equilibrium, $\frac{dN_2}{dt} = 0$, which gives the time t it takes to reach equilibrium:

$$t = \frac{1}{\lambda - \sigma\phi} \ln\left(\frac{\lambda}{\sigma\phi}\right) .$$

As $\lambda = \frac{\ln 2}{0.85} = 0.816\ s^{-1}$, $\sigma\phi = 3.7 \times 10^{-26} \times 3 \times 10^{12} = 1.11 \times 10^{-13}\ s^{-1}$,

$$t \approx \frac{1}{\lambda} \ln\left(\frac{\lambda}{\sigma\phi}\right) = 3.63\ s .$$

The equilibrium activity is

$$A = \lambda N_2 \approx \frac{\lambda\sigma\phi N_1(0)}{\lambda - \sigma\phi} \approx \sigma\phi N_1(0) = 4.77 \times 10^8\ Bq = 12.9\ mc .$$

2104

In a neutron-activation experiment, a flux of 10^8 neutrons/cm^2·sec is incident normally on a foil of area 1 cm^2, density 10^{22} atoms/cm^3, and thickness 10^{-2} cm (Fig. 2.37). The target nuclei have a total cross section for neutron capture of 1 barn (10^{-24} cm^2), and the capture leads uniquely to a nuclear state which β-decays with a lifetime of 10^4 sec. At the end of 100 sec of neutron irradiation, at what rate will the foil be emitting β-rays?

(*Wisconsin*)

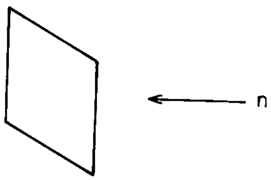

Fig. 2.37

Solution:

Let the number of target nuclei be $N(t)$, and that of the unstable nuclei resulting from neutron irradiation be $N_\beta(t)$. As the thickness of the target is 10^{-2} cm, it can be considered thin so that

$$\frac{dN(t)}{dt} = -\sigma\phi N(t)\,,$$

where ϕ is the neutron flux, σ is the total neutron capture cross section of the target nuclei. Integration gives $N(t) = N(0)e^{-\sigma\phi t}$. As $\sigma\phi = 10^{-24} \times 10^8 = 10^{-16}\ s^{-1}$, $\sigma\phi t = 10^{-14} \ll 1$ and we can take $N(t) \approx N(0)$, then

$$\frac{dN}{dt} \approx -\sigma\phi N(0)\,,$$

indicating that the rate of production is approximately constant.

Consider the unstable nuclide. We have

$$\frac{dN_\beta(t)}{dt} \approx \sigma\phi N(0) - \lambda N_\beta(t)\,,$$

where λ is the β-decay constant. Integrating we have

$$N_\beta(t) = \frac{\sigma\phi N(0)}{\lambda}(1 - e^{-\lambda t})\,,$$

and so

$$A = N_\beta(t)\lambda = \sigma\phi N(0)(1 - e^{-\lambda t})\,.$$

At $t = 100\ s$, the activity of the foil is

$$A = 10^{-16} \times 10^{22} \times 1 \times 10^{-2} \times (1 - e^{-10^{-2}}) = 99.5\ s^{-1}$$

as

$$\lambda = \frac{1}{10^4} = 10^{-4}\ s\,.$$

2105

Radioactive dating is done using the isotope
(a) ^{238}U.
(b) ^{12}C.
(c) ^{14}C.

(CCT)

Solution:

^{14}C. The radioactive isotope ^{14}C maintains a small but fixed proportion in the carbon of the atomsphere as it is continually produced by bombardment of cosmic rays. A living entity, by exchanging carbon with the atmosphere, also maintains the same isotopic proportion of ^{14}C. After it dies, the exchange ceases and the isotopic proportion attenuates, thus providing a means of dating the time of death. ^{12}C is stable and cannot be used for this purpose. ^{238}U has a half-life of 4.5×10^9 years, too long for dating.

2106

^{14}C decays with a half-life of about 5500 years.

(a) What would you guess to be the nature of the decay, and what are the final products? Very briefly explain.

(b) If no more ^{14}C enters biological systems after their death, estimate the age of the remains of a tree whose radioactivity (decays/sec) of the type given in (a) is 1/3 of that of a comparable but relatively young tree.

(*Wisconsin*)

Solution:

(a) ^{14}C is a nuclide with excess neutrons, and so it will β^--decay to ^{14}N according to

$$^{14}C \rightarrow ^{14}N + e^- + \bar{\nu}_e \,.$$

(b) The number of ^{14}C of a biological system attenuates with time after death according to $N(t) = N(0)e^{-\lambda t}$, which gives the activity of ^{14}C as

$$A(t) = \lambda N(t) = A(0)e^{-\lambda t} \,.$$

Thus the age of the dead tree is

$$t = \frac{1}{\lambda} \ln \frac{A(0)}{A(t)} = \frac{\tau_{1/2}}{\ln 2} \ln \frac{A(0)}{A(t)}$$

$$= \frac{5500}{\ln 2} \ln \left(\frac{3}{1}\right) = 8717 \text{ years} \,.$$

2107

Plutonium (^{238}Pu, $Z = 94$) has been used as power source in space flights. ^{238}Pu has an α-decay half-life of 90 years (2.7×10^9 sec).

(a) What are the Z and N of the nucleus which remains after α-decay?

(b) Why is ^{238}Pu more likely to emit α's than deuterons as radiation?

(c) Each of the α-particles is emitted with 5.5 MeV. What is the power released if there are 238 gms of ^{238}Pu (6×10^{23} atoms)? (Use any units you wish but specify.)

(d) If the power source in (c) produces 8 times the minimum required to run a piece of apparatus, for what period will the source produce sufficient power for that function.

(*Wisconsin*)

Solution:

(a) The daughter nucleus has $N = 142$, $Z = 92$.

(b) This is because the binding energy of α-particle is higher than that of deuteron and so more energy will be released in an α-decay. For ^{238}Pu,

$$^{238}_{94}Pu \to ^{234}_{92}U + \alpha, \quad Q = 46.186 - 38.168 - 2.645 \approx 5.4 \text{ MeV},$$

$$^{238}_{94}Pu \to ^{236}_{93} Np + d, \quad Q = 46.186 - 43.437 - 13.136 \approx -10.4 \text{ MeV}.$$

Deuteron-decay is not possible as $Q < 0$.

(c) Because of the recoil of ^{234}U, the decay energy per ^{238}Pu is

$$E_d = E_\alpha + E_U = \frac{p_\alpha^2}{2m_\alpha} + \frac{p_\alpha^2}{2m_U} = E_\alpha \left(1 + \frac{m_\alpha}{m_U}\right) = 5.5 \left(\frac{238}{234}\right) = 5.6 \text{ MeV}.$$

As the half-life of ^{238}Pu is $T_{1/2} = 90\ yr = 2.7 \times 10^9\ s$, the decay constant is

$$\lambda = \ln 2/T_{1/2} = 2.57 \times 10^{-10}\ s^{-1}.$$

For 238 g of ^{238}Pu, the energy released per second at the beginning is

$$\frac{dE}{dt} = E_d \frac{dN}{dt} = E_d \lambda N_0 = 5.6 \times 2.57 \times 10^{-10} \times 6 \times 10^{23} = 8.6 \times 10^{14} \text{ MeV}/s.$$

(d) As the amount of ^{238}Pu nuclei attenuates, so does the power output:

$$W(t) = W(0) e^{-\lambda t}.$$

When $W(t_0) = W(0)/8$,

$$t_0 = \ln 8/\lambda = 3 \ln 2/\lambda = 3T_{1/2} = 270\ yr.$$

Thus the apparatus can run normally for 270 years.

6. NUCLEAR REACTIONS (2108–2120)

2108

Typical nuclear excitation energies are about 10^{-2}, 10^1, 10^3, 10^5 MeV.

(*Columbia*)

Solution:

10^1 MeV.

2109

The following are atomic masses in units of u ($1\ u = 932$ MeV/c^2).

Electron	0.000549	$^{152}_{62}Sm$	151.919756
Neutron	1.008665	$^{152}_{63}Eu$	151.921749
$^1_1$H	1.007825	$^{152}_{64}Gd$	151.919794

(a) What is the Q-value of the reaction ^{152}Eu(n,p)?

(b) What types of weak-interaction decay can occur for ^{152}Eu?

(c) What is the maximum energy of the particles emitted in each of the processes given in (b)?

(*Wisconsin*)

Solution:

(a) The reaction $^{152}Eu + n \to {}^{152}Sm + p$ has Q-value

$$\begin{aligned} Q &= [m(^{152}Eu) + m(n) - m(^{152}Sm) - m(p)]c^2 \\ &= [M(^{152}Eu) + m(n) - M(^{152}Sm) - M(^1H)]c^2 \\ &= 0.002833\ u = 2.64\ \text{MeV}\,, \end{aligned}$$

where m denotes nuclear masses, M denotes atomic masses. The effects of the binding energy of the orbiting electrons have been neglected in the calculation.

(b) The possible weak-interaction decays for ^{152}Eu are β-decays and electron capture:

$$\beta^-\text{-decay} : {}^{152}Eu \to {}^{152} Gd + e^- + \bar{\nu}_e \,,$$

$$\beta^+\text{-decay} : {}^{152}Eu \to {}^{152} Sm + e^+ + \nu_e \,,$$

$$\text{orbital electron capture} : {}^{152}Eu + e^- \to {}^{152} Sm + \nu_e \,.$$

Consider the respective Q-values:

$$\beta^-\text{-decay} : \; E_d(\beta^-) = [M({}^{152}Eu) - M({}^{152}Gd)]c^2 = 1.822 \text{ MeV} > 0 \,,$$

energetically possible.

$$\begin{aligned}\beta^+\text{-decay} : \; E_d(\beta^+) &= [M({}^{152}Eu) - M({}^{152}Sm) - 2m(e)]c^2 \\ &= 0.831 \text{ MeV} > 0 \,,\end{aligned}$$

energetically possible.

Orbital electron capture:

$$E_d(EC) = [M({}^{152}Eu) - M({}^{152}Sm)]c^2 - W_j = 1.858 \text{ MeV} - W_j \,,$$

where W_j is the electron binding energy in atomic orbits, the subscript j indicating the shell K, L, M, etc., of the electron. Generally $W_j \ll 1$ MeV, and orbital electron capture is also energetically possible for ^{152}Eu.

(c) As the mass of electron is much smaller than that of the daughter nucleus, the latter's recoil can be neglected. Then the maximum energies of the particles emitted in the processes given in (b) are just the decay energies. Thus

for β^--decay, the maximum energy of electron is 1.822 MeV,

for β^+-decay, the maximum energy of positron is 0.831 MeV.

For orbital electron capture, the neutrinos are monoenergetic, their energies depending on the binding energies of the electron shells from which they are captured. For example, for K capture, $W_k \approx 50$ keV, $E_\nu \approx 1.8$ MeV.

2110

(a) Consider the nuclear reaction

$$^1H + {}^A X \to {}^2 H + {}^{A-1} X \,.$$

For which of the following target nuclei ${}^AX$ do you expect the reaction to be the strongest, and why?

$$^AX = ^{39}Ca,\ ^{40}Ca,\ ^{41}Ca\,.$$

(b) Use whatever general information you have about nuclei to estimate the temperature necessary in a fusion reactor to support the reaction

$$^2H + ^2H \to {}^3He + n\,.$$

(*Wisconsin*)

Solution:

(a) The reaction is strongest with a target of ^{41}Ca. In the reaction the proton combines with a neutron in ^{41}Ca to form a deuteron. The isotope ^{41}Ca has an excess neutron outside of a double-full shell, which means that the binding energy of the last neutron is lower than those of ^{40}Ca, ^{39}Ca, and so it is easier to pick up.

(b) To facilitate the reaction $^2H+^2H \to ^3H+n$, the two deuterons must be able to overcome the Coulomb barrier $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$, where r is the distance between the deuterons. Take the radius of deuteron as 2 fm. Then $r_{\min} = 4\times 10^{-15}\ m$, and $V_{\max} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_{\min}}$. The temperature required is

$$T \gtrsim \frac{V_{\max}}{k} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_{\min}}\frac{1}{k} = \left(\frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c}\right)\left(\frac{\hbar c}{r_{\min}}\right)\frac{1}{k}$$

$$= \frac{1}{137} \times \left(\frac{197\times 10^{-15}}{4\times 10^{-15}}\right)\frac{1}{8.6\times 10^{-11}} = 4\times 10^9\ K\,.$$

In the above k is Boltzmann's constant. Thus the temperature must be higher than 4×10^9 K for the fusion reaction $^2H+^2H \to {}^3He+n$ to occur.

2111

(a) Describe one possible experiment to determine the positions (excitation energies) of the excited states (energy levels) of a nucleus such as ^{13}C. State the target, reaction process, and detector used.

(b) In the proposed experiment, what type of observation relates to the angular momentum of the excited state?

(*Wisconsin*)

Solution:

(a) Bombard a target of ^{12}C with deuterons and detect the energy spectrum of the protons emitted in the reaction ^{12}C(d,p)^{13}C with a gold-silicon surface-barrier semiconductor detector. This, combined with the known energy of the incident deuterons, then gives the energy levels of the excited states of ^{13}C. One can also use a Ge detector to measure the energy of the γ-rays emitted in the de-excitation of $^{13}C^*$ and deduce the excited energy levels.

(b) From the known spin-parity of ^{12}C and the measured angular distribution of the reaction product p we can deduced the spin-parity of the resultant nucleus ^{13}C.

2112

Given the atomic mass excess $(M - A)$ in keV:

$$^1n = 8071 \text{ keV}, \ ^1H = 7289 \text{ keV}, \ ^7Li = 14907 \text{ keV}\,, \ ^7Be = 15769 \text{ keV}\,,$$

and for an electron $m_0c^2 = 511$ keV.

(a) Under what circumstances will the reaction ^{7}Li(p,n)^{7}Be occur?

(b) What will be the laboratory energy of the neutrons at threshold for neutron emission?

(*Wisconsin*)

Solution:

(a) In $^7Li + p \rightarrow ^7 Be + n + Q$ the reaction energy Q is

$$\begin{aligned} Q &= \Delta M(^7Li) + \Delta M(^1H) - \Delta M(^7Be) - \Delta M(n) \\ &= 14907 + 7289 - 15769 - 8071 = -1644 \text{ keV}\,. \end{aligned}$$

This means that in the center-of-mass system, the total kinetic energy of ^{7}Li and p must reach 1644 keV for the reaction to occur. Let E, P be the total energy and momentum of the proton in the laboratory system. We require

$$(E + m_{Li}c^2)^2 - P^2c^2 = (|Q| + m_{Li}c^2 + m_pc^2)^2\,.$$

As $E^2 = m_p^2c^4 + P^2c^2$, $E \approx T + m_pc^2$, $|Q| \ll m_{Li}$, m_p, we have $2(E - m_pc^2)m_{Li}c^2 \approx 2|Q|(m_{Li} + m_p)c^2$, or

$$T = \frac{m_p + m_{Li}}{m_{Li}} \times |Q| \approx \frac{1+7}{7} \times 1644 = 1879 \text{ keV}.$$

Thus the kinetic energy T of the incident proton must be higher than 1879 keV.

(b) The velocity of the center of mass in the laboratory is

$$V_c = \frac{m_p}{m_p + m_{Li}} V_p .$$

As at threshold the neutron is produced at rest in the center-of-mass system, its velocity the laboratory is V_c. Its laboratory kinetic energy is therefore

$$\frac{1}{2} m_n V_c^2 = \frac{1}{2} \frac{m_n m_p^2}{(m_p + m_{Li})^2} \cdot \frac{2T}{m_p} = \frac{m_n m_p T}{(m_p + m_{Li})^2} \approx \frac{T}{64} = 29.4 \text{ keV}.$$

2113

The nucleus ^{8}Be is unstable with respect to dissociation into two α-particles, but experiments on nuclear reactions characterize the two lowest unstable levels as

$J = 0$, even parity, $\sim$95 keV above the dissociation level,

$J = 2$, even parity, $\sim$3 MeV above the dissociation level.

Consider how the existence of these levels influence the scattering of α-particles from helium gas. Specifically:

(a) Write the wave function for the elastic scattering in its partial wave expansion for $r \to \infty$.

(b) Describe qualitatively how the relevant phase shifts vary as functions of energy in the proximity of each level.

(c) Describe how the variation affects the angular distribution of α-particles.

(*Chicago*)

Solution:

(a) The wave function for elastic scattering of α-particle (He^{++}) by a helium nucleus involves two additive phase shifts arising from Coulomb interaction (δ_l) and nuclear forces (η_l). To account for the identity of the two (spinless) particles, the spatial wave function must be symmetric with an even value of l. Its partial wave at $r \to \infty$ is

$$\sum_{l=0}^{\infty} \frac{1+(-1)^l}{2}(2l+1)i^l e^{i(\delta_l+\eta_l)}\frac{1}{kr}$$

$$\times \sin\left[kr - \frac{l\pi}{2} - \gamma \ln(2kr) + \delta_l + \eta_l\right] P_l(\cos\theta)\,,$$

where k is the wave number in the center-of-mass system and $\gamma = (2e)^2/\hbar v$, v being the relative velocity of the α-particles.

(b) The attractive nuclear forces cause each η_l to rise from zero as the center-of-mass energy increases to moderately high values. Specifically each η_l rises rather rapidly, by nearly π radians at each resonance, as the energy approaches and then surpasses any unstable level of a definite l of the compound nucleus, e.g., near 95 keV for $l = 0$ and near 3 MeV for $l = 2$ in the case of ^{8}Be.

However, the effect of nuclear forces remains generally negligible at energies lower than the Coulomb barrier, or whenever the combination of Coulomb repulsion and centrifugal forces reduces the amplitude of the relevant partial wave at values of r within the range of nuclear forces. Thus η_l remains ~ 0 (or $\sim n\pi$) except when very near a resonance, where η_l, rises by π anyhow. Taking $R \sim 1.5$ fm as the radius of each He^{++} nucleus, the height of the Coulomb barrier when two such nuclei touch each other is $B \sim (2e)^2/2R \sim 2$ MeV. Therefore the width of the $l = 0$ resonance at 95 keV is greatly suppressed by the Coulomb barrier, while the $l = 2$ resonance remains broad.

(c) To show the effect of nuclear forces on the angular distribution one may rewrite the partial wave expansion as

$$\sum_{l=0}^{\infty} \frac{1+(-1)^l}{2}(2l+1)i^l e^{i\delta_l}\frac{1}{kr}\left\{ \sin\left(kr - \frac{l\pi}{2} - \gamma\ln(2kr) + \delta_l\right)\right.$$

$$\left. + \left(\frac{e^{2i\eta_l}-1}{2i}\right)\exp\left[i\left(kr - \frac{l\pi}{2} - \gamma\ln(2kr) + \delta_l\right]\right\} P_l(\cos\theta)\,.$$

Here the first term inside the brackets represents the Coulomb scattering wave function unaffected by nuclear forces. The contribution of this term can be summed over l to give

$$\exp i\{kr\cos\theta - \gamma\ln[kr(1-\cos\theta)] + \delta_0\} - \gamma(kr)^{-1}\exp i\{kr\cos\theta$$
$$-\gamma\ln(kr) + \delta_0\}\cdot\frac{1}{\sqrt{2}}\left[\frac{e^{-i\gamma\ln(1-\cos\theta)}}{1-\cos\theta} + \frac{e^{-i\gamma\ln(1+\cos\theta)}}{1+\cos\theta}\right].$$

The second term represents the scattering wave due to nuclear forces, which interferes with the Coulomb scattering wave in each direction. However, it is extremely small for η_l very close to $n\pi$, as for energies below the Coulomb barrier. Accordingly, detection of such interference may signal the occurence of a resonance at some lower energy.

An experiment in 1956 showed no significant interference from nuclear scattering below 300 keV center-of-mass energy, at which energy it was found $\eta_0 = (178 \pm 1)$ degrees.

2114

A 3-MV Van de Graaff generator is equipped to accelerate protons, deuterons, doubly ionized ^{3}He particles, and alpha-particles.

(a) What are the maximum energies of the various particles available from this machine?

(b) List the reactions by which the isotope ^{15}O can be prepared with this equipment.

(c) List at least six reactions in which ^{15}N is the compound nucleus.

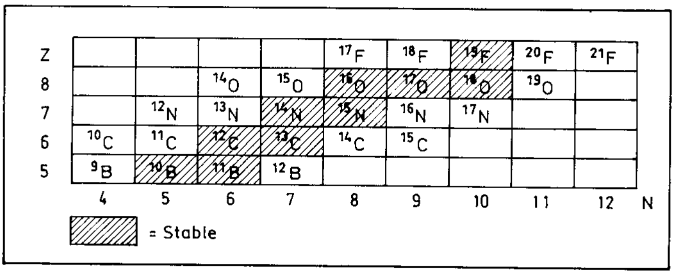

Fig. 2.38

(d) Describe two types of reaction experiment which can be carried out with this accelerator to determine energy levels in ^{15}N. Derive any equations needed. (Assume all masses are known. Figure 2.38 shows the isotopes of light nuclei.)

(*Columbia*)

Solution:

(a) The available maximum energies of the various particles are: 3 MeV for proton, 3 MeV for deuteron, 6 MeV for doubly ionized ^{3}He, 6 MeV for α-particle.

(b) Based energy consideration, the reactions that can produce ^{15}O are

$$p +^{14} N \to^{15} O + \gamma, \qquad Q = 7.292 \text{ MeV},$$

$$d +^{14} N \to^{15} O + n, \qquad Q = 5.067 \text{ MeV},$$

$$^3He +^{13} C \to^{15} O + n, \qquad Q = 6.476 \text{ MeV}.$$

^{15}O cannot be produced with α-particles because of their high binding energy and small mass, which result in $Q = -8.35$ MeV.

(c) The reactions in which ^{15}N is the compound nucleus are

$$\alpha +^{11} B \to^{15} N^* \to^{14} N + n, \qquad Q = 0.158 \text{ MeV},$$

$$\to^{15} N^* \to^{14} C + p, \qquad Q = 0.874 \text{ MeV},$$

$$\to^{15} N^* \to^{15} N + \gamma, \qquad Q = 10.991 \text{ MeV},$$

$$d +^{13} C \to^{15} N^* \to^{14} N + n, \qquad Q = 5.325 \text{ MeV},$$

$$\to^{15} N^* \to^{11} B + \alpha, \qquad Q = 5.168 \text{ MeV},$$

$$\to^{15} N^* \to^{14} C + p, \qquad Q = 5.952 \text{ MeV}.$$

(d) (1) For the reaction $\alpha +^{11} B \to^{15} N^* \to^{15} N + \gamma$, measure the γ-ray yield curve as a function of the energy E_α of the incoming α-particles. A resonance peak corresponds to an energy level of the compound nucleus $^{15}N^*$, which can be calculated:

$$E^* = \frac{11}{15} E_\alpha + m(^4He)c^2 + m(^{11}B)c^2 - m(^{15}N)c^2 .$$

(2) With incoming particles of known energy, measuring the energy spectrums of the produced particles enables one to determine the energy levels of $^{15}N^*$. For instance, the reaction

$$^3He + ^{14}N \rightarrow ^{15}N + d$$

has $Q = 4.558$ MeV for ground state ^{15}N. If the incoming ^{3}He has energy E_0, the outgoing deuteron has energy E' and angle of emission θ, the excitation energy E^* is given by

$$E^* = Q - Q',$$

where

$$\begin{aligned}
Q' &= \left[1 + \frac{m(d)}{m(^{15}N)}\right] E' - \left[1 - \frac{m(^3He)}{m(^{15}N)}\right] E_0 - \frac{2\sqrt{m(^3He)m(d)E_0E'}}{m(^{15}N)} \cos\theta \\
&= \left(1 + \frac{2}{15}\right) E' - \left(1 - \frac{3}{15}\right) E_0 - 2\frac{\sqrt{3 \times 2E_0E'}}{15} \cos\theta \\
&= \frac{1}{15}(17E' - 12E_0 - 2\sqrt{6E_0E'}\cos\theta).
\end{aligned}$$

2115

When Li^6 (whose ground state has $J = 1$, even parity) is bombarded by deuterons, the reaction rate in the reaction $Li^6 + d \rightarrow \alpha + \alpha$ shows a resonance peak at E (deuteron)$= 0.6$ MeV. The angular distribution of the α-particle produced shows a $(1 + A\cos^2\theta)$ dependence where θ is the emission angle relative to the direction of incidence of the deuterons. The ground state of the deuteron consists of a proton and a neutron in $^3S_1$ configuration. The masses of the relevant nuclides are

$$m_d = 2.0147 \text{ amu}, \ m_\alpha = 4.003 \text{ amu},$$

$$m_{Li} = 6.0170 \text{ amu}, \ m_{Be} = 8.0079 \text{ amu},$$

where 1 amu = 938.2 MeV.

From this information alone, determine the energy, angular momentum, and parity of the excited level in the compound nucleus. What partial wave deuterons (s,p,d, etc.) are effective in producing this excited level? (explain)

(Columbia)

Solution:

The excitation energy of the compound nucleus ^{8}Be* in the reaction $d +^6 Li \rightarrow^8 Be^*$ is

$$E(^8Be^*) = [m(^2H) + m(^6Li) - m(^8Be)] + E_d \frac{m(^6Li)}{m(^6Li) + m(^2H)}$$

$$= (2.0147 + 6.0170 - 8.0079) \times 938.2 + 0.6 \times \frac{6}{8} = 22.779 \text{ MeV}.$$

In the decay $^8Be^* \rightarrow \alpha + \alpha$, as J^π of α is 0^+, the symmetry of the total wave function of the final state requires that l_f, the relative orbital angular momentum of the two α-particles, be even and the decay, being a strong interaction, conserve parity, the parity of $^8Be^*$ is $\pi(^8Be^*) = (-1)^{l_f}(+1)^2 = +1$.

As the angular distribution of the final state α-particles is not spherically symmetric but corresponds to $l_f = 2$, we have

$$J^\pi(^8Be^*) = 2^+ .$$

Then the total angular momentum of the initial state $d+^6 Li$ is also $J_i = 2$. As $\mathbf{J}_i = \mathbf{J}_d + \mathbf{J}_{Li} + \mathbf{l}_i = \mathbf{1} + \mathbf{1} + \mathbf{l}_i$ and as

$$\mathbf{1} + \mathbf{1} = \begin{cases} \mathbf{0} \\ \mathbf{1}, \\ \mathbf{2} \end{cases} \quad \text{the possible values of } l_i \text{ are 0,1,2,3,4.}$$

However, the ground state parities of ^{6}Li and d are both positive, l_i must be even. As the angular distribution of the final state is not isotropic, $l_i \neq 0$ and the possible values of l_i are 2,4. So d-waves produce the main effect.

2116

Fast neutrons impinge on a 10-cm thick sample containing 10^{21} ^{53}Cr atoms/cm^3. One-tenth of one percent of the neutrons are captured into a spin-parity $J^\pi = 0^+$ excited state in ^{54}Cr. What is the neutron capture cross section for this state? The excited ^{54}Cr sometimes γ-decays as shown in Fig. 2.39. What is the most likely J^π for the excited state at 9.2 MeV? What are the multipolarities of the γ-rays?

(Wisconsin)

Solution:

Let the number of neutrons impinging on the sample be n and the neutron capture cross section for forming the 0^+ state be σ. Then $10 \times 10^{21} n\sigma = 10^{-3}\ n$, or

$$\sigma = 10^{-25}\ \text{cm}^2 = 0.1\ b\,.$$

Let the spin-parity of the 9.2 MeV level be J^p. As ^{54}Cr only occasionally γ-decays, the transitions are probably not of the $E1$ type, but correspond to the next lowest order. Consider $0^+ \to J^p$. If $\Delta J = 2$, the electric multipole field has parity $(-1)^{\Delta J} = +$, i.e. $J^p = 2^+$, and the transition is of the $E2$ type. The transitions γ_2, γ_3 are also between 0^+ and 2^+ states, so they are probably of the $E2$ type too. For $\gamma_4 : 2^+ \to 2^+$, we have $\Delta L = 1, 2, 3$ or 4. For no parity change between the initial and final states, γ_4 must be $E2$, $E4$ or $M1$, $M3$. Hence most probably $\gamma_4 = E2$, or $M1$, or both.

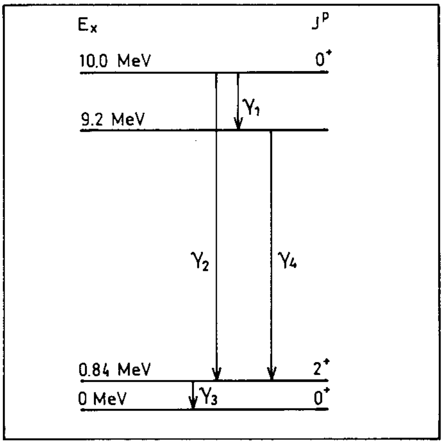

Fig. 2.39

2117

The surface of a detector is coated with a thin layer of a naturally fissioning heavy nuclei. The detector area is 2 cm^2 and the mean life of the fissioning isotope is $\frac{1}{3} \times 10^9$ years (1 $yr = 3 \times 10^7$ sec.). Twenty fissions are detected per second. The detector is then placed in a uniform neutron flux of 10^{11} neutrons/cm^2/sec. The number of fissions detected in the neutron flux is 120 per second. What is the cross section for neutron-induced fission?

(*Wisconsin*)

Solution:

Let the number of the heavy nuclei be N. Then the number of natural fissions taking place per second is

$$\frac{dN}{dt} = -\lambda N \approx -\lambda N_0 \,,$$

where $N_0 = N|_{t=0}$, as $\lambda = \frac{1}{\frac{1}{3}\times 10^9 \times 3 \times 10^7} = 10^{-16} \ll 1$.

The number of induced fissions per second is $\sigma N\phi \approx \sigma N_0 \phi$, where ϕ is the neutron flux, σ is the cross section for neutron-induced fission. As

$$\frac{\sigma N_0 \phi + \lambda N_0}{\lambda N_0} = \frac{120}{20} \,,$$

or

$$\frac{\sigma\phi}{\lambda} = \frac{100}{20} = 5 \,,$$

we have

$$\sigma = \frac{5\lambda}{\phi} = \frac{5 \times 10^{-16}}{10^{11}} = 5 \times 10^{-27} \text{ cm}^2 = 5 \text{ mb} \,.$$

2118

(a) How do you expect the neutron elastic scattering cross section to depend on energy for very low energy neutrons?

(b) Assuming nonresonant scattering, estimate the thermal neutron elastic cross section for ^{3}He.

(c) Use the information in the partial level scheme for $A = 4$ shown in Fig. 2.40 to estimate the thermal neutron absorption cross section for ^{3}He.

Resonant scattering may be important here.

(*Princeton*)

Solution:

(a) For thermal neutrons of very low energies, the elastic scattering cross section of light nuclei does not depend on the neutron energy, but is constant

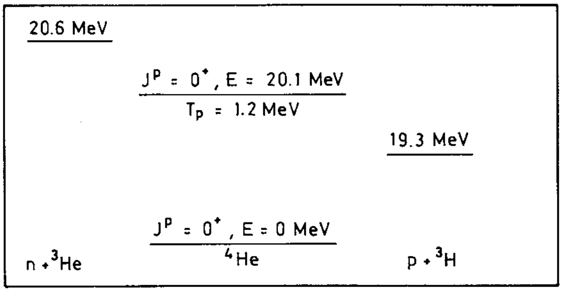

Fig. 2.40

for a large range of energy. But for heavier nuclei, resonant scattering can occur in some cases at very low neutron energies. For instance, resonant scattering with ^{157}Gd occurs at $E_n = 0.044$ eV.

(b) The thermal neutron nonresonant scattering cross section for nuclei is about $4\pi R_0^2$, where R_0 is the channel radius, which is equal to the sum of the radii of the incoming particle and the target nucleus. Taking the nuclear radius as

$$R \approx 1.5 \times 10^{-13}\ A^{1/3}\,,$$

the elastic scattering cross section of ^{3}He for thermal neutron is

$$\sigma = 4\pi R_0^2 \approx 4\pi[1.5 \times 10^{-13}(3^{1/3} + 1)]^2 = 1.7 \times 10^{-24}\ \text{cm}^2 = 1.7\ b\,.$$

(c) The Breit–Wigner formula

$$\sigma_{nb} = \pi \lambda\!\!\!^{-}{}^2 \frac{\Gamma_n \Gamma_b}{(E' - E_0)^2 + \Gamma^2/4}$$

can be used to calculate the neutron capture cross section for ^{3}He in the neighborhood of a single resonance. Here $\lambda\!\!\!^{-}$ is the reduced wavelength of the incident particle, E' is the energy and E_0 is the energy at resonance peak of the compound nucleus $A = 4$, Γ_n and Γ_b are the partial widths of the resonant state for absorption of neutron and for emission of b respectively, and Γ is the total level width.

For laboratory thermal neutrons, $E_n \approx 0.025\ eV$,

$$\lambda\!\!\!^{-} = \frac{\hbar}{\sqrt{2\mu E_n}} = \frac{\hbar}{\sqrt{\frac{2m_n m_{He}}{m_n+m_{He}}E_n}} = \frac{\hbar c}{\sqrt{\frac{3}{2}E_n m_n c^2}}$$

$$= \frac{197\times 10^{-13}}{\sqrt{\frac{3}{2}\times 2.5\times 10^{-8}\times 940}} = 3.3\times 10^{-9}\ \text{cm}\,.$$

As both the first excited and ground states of ^{4}He have 0^+, $\Gamma_\gamma = 0$, and the only outgoing channel is for the excited state of ^{4}He to emit a proton. The total width is $\Gamma = \Gamma_n + \Gamma_p$. With $\Gamma_n \approx 150\ eV$, $\Gamma \approx \Gamma_p = 1.2$ MeV, $E' = 20.6$ MeV, $E = 20.1$ MeV, we obtain

$$\sigma = \pi\lambda\!\!\!^{-\,2}\frac{\Gamma_n\Gamma_p}{(E'-E_0)^2+\Gamma^2/4} = 1\times 10^{-20}\ \text{cm}^2 = 1\times 10^4\ b\,.$$

2119

Typical cross section for low energy neutron-nucleus scattering is 10^{-16}, 10^{-24}, 10^{-32}, 10^{-40} cm^2.

(Columbia)

Solution:

10^{-24} cm^2. The radius of the sphere of action of nuclear forces is $\sim 10^{-12} - 10^{-13}$ cm, and a typical scattering cross-section can be expected to be of the same order of magnitude as its cross-sectional area.

2120

In experiments on the reaction ^{21}Ne(d, ^{3}He)^{20}F with 26 MeV deuterons, many states in ^{20}F are excited. The angular distributions are characteristic of the direct reaction mechanism and therefore are easily sorted into those for which the angular momentum of the transferred proton is $l_p = 0$ or 1 or 2.

The lowest energy levels of ^{21}Ne and the known negative-parity states of ^{20}F below 4 MeV are as shown in Fig. 2.41 (the many positive-parity excited states of ^{20}F are omitted).

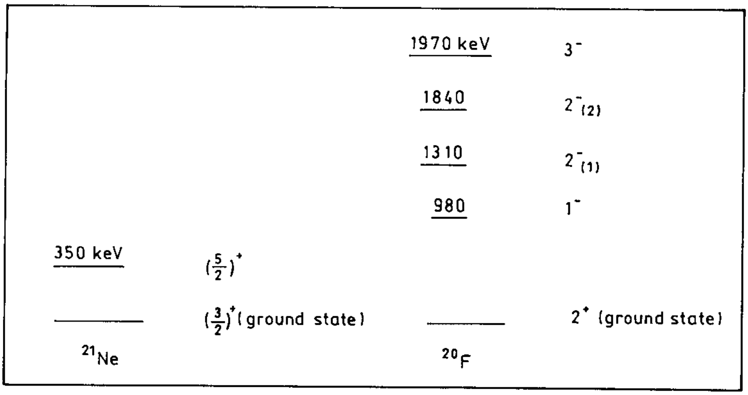

Fig. 2.41

The relative $l_p = 1$ strengths $S(J^\pi)$ observed in the (d, ^{3}He) reaction are approximately

$$S(1^-) = 0.84\,,$$

$$S(2_1^-) = 0.78\,,$$

$$S(2_2^-) = 0.79\,,$$

$$S(3^-) = 0.00\,.$$

(a) If the ^{21}Ne target and a ^{20}F state both have (1s-0d) configuration, they both have positive parity and therefore some $l_p = 0$ or $l_p = 2$ transitions are expected. On the other hand, the final states of ^{20}F with negative parity are excited with $l_p = 1$. Explain.

(b) In order to explain the observed negative-parity states in ^{20}F, one can try a coupling model of a hole weakly coupled to states of ^{21}Ne. With this model of a ^{21}Ne nucleus with an appropriate missing proton and level diagrams as given above, show how one can account for the negative-parity states in ^{20}F.

(c) In the limit of weak coupling; i.e., with no residual interaction between the hole and the particles, what would be the (relative) energies of the 4 negative-parity states?

(d) What would be the effect if now a weak particle-hole interaction were turned on? Do the appropriate centroids of the reported energies of the 1^-, 2^-, 2^-, 3^- states conform to this new situation?

(e) The weak coupling model and the theory of direct reactions lead to specific predictions about the relative cross sections (strengths) for the various final states. Compare these predictions with the observed S-factors given above. Show how the latter can be used to obtain better agreement with the prediction in part (d).

(*Princeton*)

Solution:

(a) The reactions are strong interactions, in which parity is conserved. So the parity change from initial to final state must equal the parity of the proton that is emitted as part of ^{3}He:

$$P(^{21}Ne) = P(^{20}F)P(p) = P(^{20}F)(-1)^{l_p} .$$

When both ^{20}F and ^{21}Ne have even parity, $(-1)^{l_p} = 1$ and so $l_p = 0, 2 \cdots$. As conservation of the total angular momentum requires that l_p be 0, 1, 2, we have $l_p = 0, 2$. Similarly, for the negative-parity states of ^{20}F, the angular momentum that the proton takes away can only be $1, 3 \cdots$. In particular for 1^- and 2^- states of ^{20}F, $l_p = 1$.

(b) In the weak coupling model, ^{20}F can be considered as consisting of ^{21}Ne and a proton hole (p^-). J^p of ^{20}F is then determined by a neutron in $1d_{3/2}$, $1d_{5/2}$, or $2s_{1/2}$ and a proton hole in $1p_{1/2}$, $1p_{3/2}$ or $2s_{1/2}$, etc., outside of full shells (Fig. 2.16). For example, the 1^- state of ^{20}F can be denoted as

$$\begin{aligned} |1M\rangle &= |1p^{-1}_{1/2},\ 1d_{3/2};\ 1, M\rangle \\ &= \sum_{m_1,m_2} \left\langle \frac{1}{2}, \frac{3}{2}, m_1, m_2 \middle| 1, M \right\rangle \psi_{1/2m}\psi_{3/2m} . \end{aligned}$$

where $^1p^{-1}_{1/2}$ means a proton hole in $1p_{1/2}$ state, $1d_{3/2}$ means a neutron in $1d_{3/2}$ state. In the same way, the 2^- can be denoted as

$$|1p^{-1}_{1/2}, 1d_{3/2}; 2, M\rangle \quad \text{and} \quad |1p^{-1}_{1/2}, 1d_{5/2}; 2, M\rangle ,$$

the 3^- state can be denoted as

$$|1p^{-1}_{1/2}, 1d_{5/2}; 3, M\rangle .$$

(c) We have $H = H_p + H_h + V_{ph}$, where H_p and H_h are respectively the Hamiltonian of the nuclear center and the hole, and V_{ph} is the potential due to the interaction of the hole and the nuclear center. In the limit of weak coupling,

$$V_{ph} = 0\,,$$

$$H_p\psi(a_1, j_1, m_1) = E_{a_1,j_1,m_1}\psi(a_1, j_1, m_1)\,,$$

$$H_h\phi(a_2, j_2, m_2) = E_{a_2,j_2,m_2}\phi(a_2, j_2, m_2)\,.$$

Then for the four negative-parity states we have

$$3^- : E_{3^-} = E_p(1d_{5/2}) + E_h(1p_{1/2})\,,$$

$$2_1^- : E_{2_1^-} = E_p(1d_{5/2}) + E_h(1p_{1/2})\,,$$

$$2_2^- : E_{2_2^-} = E_p(1d_{3/2}) + E_h(1p_{1/2})\,,$$

$$1^- : E_{1^-} = E_p(1d_{3/2}) + E_h(1p_{1/2})\,.$$

Thus $E_{3^-} = E_{2_1^-}$, $E_{2_2^-} = E_{1^-}$, as shown in Fig. 2.42, with values

$$E_{3^-} = E_{2_1^-} = 1230 \text{ keV}, \qquad E_{2_2^-} = E_{1^-} = 890 \text{ keV}\,.$$

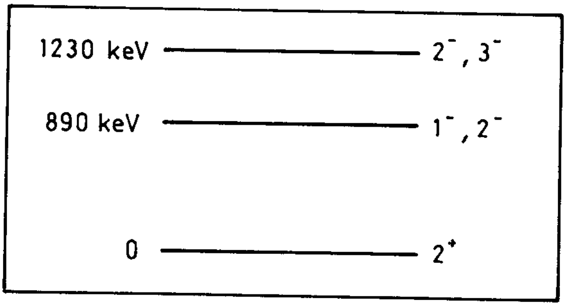

Fig. 2.42

(d) If $V_{ph} \neq 0$, i.e., coupling exists, then

$$E_{3^-} = H_p(1d_{5/2}) + H_h(1p_{1/2}) + \langle 1p_{1/2}^{-1}, 1d_{5/2}, 3|V_{ph}|1p_{1/2}^{-1}, 1d_{5/2}, 3\rangle\,,$$

$$E_{1^-} = H_p(1d_{3/2}) + H_h(1p_{1/2}) + \langle 1p_{1/2}^{-1}, 1d_{3/2}, 1|V_{ph}|1p_{1/2}^{-1}, 1d_{3/2}, 1\rangle\,.$$

As

$$\langle 1p_{1/2}^{-1}, 1d_{5/2}, 3^-|V_{ph}|1p_{1/2}^{-1}, 1d_{5/2}, 3^-\rangle \approx 0.7 \text{ MeV},$$

$$\langle 1p_{1/2}^{-1}, 1d_{3/2}, 1^-|V_{ph}|1p_{1/2}^{-1}, 1d_{3/2}, 1^-\rangle \approx 0.1 \text{ MeV},$$

$$\langle 1p_{1/2}^{-1}, 1d_{5/2}, 2^-|V_{ph}|1p_{1/2}^{-1}, 1d_{5/2}, 2^-\rangle = 0.45 \text{ MeV},$$

$$\langle 1p_{1/2}^{-1}, 1d_{3/2}, 2^-|V_{ph}|1p_{1/2}^{-1}, 1d_{3/2}, 2^-\rangle = 0.25 \text{ MeV},$$

$$\begin{aligned}&\langle 1p_{1/2}^{-1}, 1d_{5/2}, 2^-|V_{ph}|1p_{1/2}^{-1}, 1d_{3/2}, 2^-\rangle \\ &\quad = \langle 1p_{1/2}^{-1}, 1d_{3/2}, 2^-|V_{ph}|1p_{1/2}^{-1}, 1d_{5/2}, 2^-\rangle \\ &\quad = 0.3 \text{ MeV}.\end{aligned}$$

the above gives

$$\begin{aligned}E'_{3^-} &= 0.9 + 0.35 + 0.7 = 1.95 \text{ MeV} \\ E'_{1^-} &= 0.9 + 0.1 = 1.0 \text{ MeV}.\end{aligned}$$

$E'_{2_1^-}$ and $E'_{2_2^-}$ are the eigenvalues of the matrix

$$\begin{pmatrix}\langle 1p_{1/2}^{-1}; 1d_{5/2}, 2^-|H|1p_{1/2}^{-1}, 1d_{5/2}, 2^-\rangle & \langle 1p_{1/2}^{-1}; 1d_{5/2}, 2^-|H|1p_{1/2}^{-1}, 1d_{3/2}, 2^-\rangle \\ \langle 1p_{1/2}^{-1}; 1d_{3/2}, 2^-|H|1p_{1/2}^{-1}, 1d_{5/2}, 2^-\rangle & \langle 1p_{1/2}^{-1}; 1d_{3/2}, 2^-|H|1p_{1/2}^{-1}, 1d_{3/2}, 2^-\rangle\end{pmatrix}.$$

The secular equation

$$\begin{pmatrix}\lambda - 1.95 & -0.3 \\ -0.3 & \lambda - 1.1\end{pmatrix} = 0$$

gives $E'_{2_1^-} = \lambda_1 = 1.80$ MeV, $E'_{2_2^-} = \lambda_2 = 1.26$ MeV.

The energy levels are shown in Fig. 2.43.

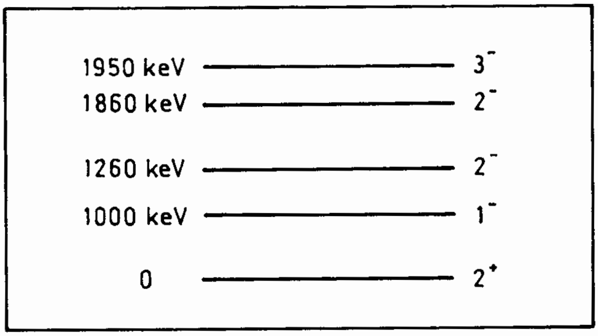

Fig. 2.43

(e) The relative strengths of the various final states as given by different theories are compared in the table below:

	Nilson model	PHF	Shell model	Experimental
$S(1^-)$	0.70	0.76	0.59	0.84
$S(2_1^-)$	0.93	0.20	0.72	0.78
$S(2_2^-)$	0.28	0.20	0.23	0.79
$S(3^-)$			0.002	0.00

It is noted in particular that for $S(2_2^-)$, the theoretical values are much smaller than the experimental values.