PART III

PARTICLES PHYSICS

This Page Intentionally Left Blank

1. INTERACTIONS AND SYMMETRIES (3001–3037)

3001

The interactions between elementary particles are commonly classified in order of decreasing strength as strong, electromagnetic, weak and gravitational.

(a) Explain, as precisely and quantitatively as possible, what is meant by 'strength' in this context, and how the relative strengths of these interactions are compared.

(b) For each of the first three classes state what conservation laws apply to the interaction. Justify your answers by reference to experimental evidence.

(*Columbia*)

Solution:

(a) The interactions can be classified according to the value of a characteristic dimensionless constant related through a coupling constant to the interaction cross section and interaction time. The stronger the interaction, the larger is the interaction cross section and the shorter is the interaction time.

Strong interaction: Range of interaction $\sim 10^{-13}$ cm. For example, the interaction potential between two nuclei has the from

$$V(r) = \frac{g_h}{r} \exp\left(-\frac{r}{R}\right),$$

where $R \approx \hbar/m_\pi c$ is the Compton wavelength of pion. Note the exponential function indicates a short interaction length. The dimensionless constant

$$g_h^2/\hbar c \approx 1 \sim 10$$

gives the interaction strength.

Electromagnetic interaction: The potential for two particles of charge e at distance r apart has the form

$$V_e(r) = e^2/r\,.$$

The dimensionless constant characteristic of interaction strength is the fine structure constant

$$\alpha = e^2/\hbar c \approx 1/137\,.$$

Weak interaction: Also a short-range interaction, its strength is represented by the Fermi coupling constant for β-decay

$$G_F = 1.4 \times 10^{-49} \text{ erg cm}^3 .$$

The potential of weak interaction has the form

$$V_w(r) = \frac{g_w}{r} \exp\left(- \frac{r}{R_w} \right),$$

where it is generally accepted that $R_w \approx 10^{-16}$ cm. The dimensionless constant characteristic of its strength is

$$g_w^2/\hbar c = G_F m_p^2 c/\hbar^3 \approx 10^{-5} .$$

Gravitational interaction: For example the interaction potential between two protons has the form

$$Gm_p^2/r .$$

The dimensionless constant is

$$Gm_p^2/\hbar c \approx 6 \times 10^{-39} .$$

As the constants are dimensionless they can be used to compare the interaction strengths directly. For example, the ratio of the strengths of gravitational and electromagnetic forces between two protons is

$$Gm_p^2/e^2 \approx 10^{-36} .$$

Because of its much smaller strength, the gravitational force can usually be neglected in particle physics. The characteristics of the four interactions are listed in Table 3.1.

Table 3.1

Interaction	Characteristic constant	Strength	Range of interaction	Typical cross section	Typical lifetime
Strong	$\frac{g_h^2}{\hbar c}$	$1 \sim 10$	10^{-13} cm	10^{-26} cm^2	10^{-23} s
Electromagnetic	$\frac{e^2}{\hbar c}$	$\frac{1}{137}$	∞	10^{-29} cm^2	10^{-16} s
Weak	$\frac{g_\omega^2}{\hbar c} = \frac{G_F m_p^2 c}{\hbar^3}$	10^{-5}	10^{-16} cm	10^{-38} cm^2	10^{-10} s
Gravitational	$\frac{Gm_p^2}{\hbar c}$	10^{-39}	∞		

Table 3.2

Quantity	E	J	P	Q	B	$L_e(L_\mu)$	I	I_3	S	P	C	T	CP	G
Strong	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
Electromagnetic	Y	Y	Y	Y	Y	Y	N	Y	Y	Y	Y	Y	Y	N
Weak	Y	Y	Y	Y	Y	Y	N	N	N	N	N	N	N	N

(b) The conservation laws valid for strong, electromagnetic, and weak interactions are listed in Table 3.2, where y =conserved, N =not conserved.

The quantities listed are all conserved in strong interaction. This agrees well with experiment. For example nucleon-nucleus and pion-nucleus scattering cross sections calculated using isospin coupling method based on strong forces agree well with observations.

In electromagnetic interaction I is not conserved, e.g. $\Delta I = 1$ in electromagnetic decay of Σ^0 ($\Sigma^0 \to \Lambda^0 + \gamma$).

In weak interaction I, I_3, S, P, C, T, PC are not conserved, e.g. 2π-decay of K_L^0. The process $K_L^0 \to \pi^+\pi^-$ violates PC conservation. As PCT is conserved, time-reversal invariance is also violated. All these agree with experiment.

3002

The electrostatic force between the earth and the moon can be ignored

(a) because it is much smaller than the gravitational force.
(b) because the bodies are electrically neutral.
(c) because of the tidal effect.

(*CCT*)

Solution:

For electrostatic interaction the bodies should be electrically charged. As the earth and the moon are both electrically neutral, they do not have electrostatic interaction. Thus answer is (b).

3003

(a) Explain the meaning of the terms: boson, fermion, hadron, lepton, baryon,

(b) Give one example of a particle for each of the above.

(c) Which of the above name is, and which is not, applicable to the photon?

(*Wisconsin*)

Solution:

(a) Fermion: All particles of half-integer spins.

Boson: All particles of integer spins.

Hardron: Particles which are subject to strong interaction are called hadrons.

Lepton: Particles which are not subject to strong interaction but to weak interaction are called leptons.

Baryon: Hadrons of half-integer spins are called baryons.

(b) Boson: π meson;

Fermion: proton;

Hardron: proton;

Lepton: neutrino;

Baryon: proton;

(c) The name boson is applicable to photon, but not the other names.

3004

Why does the proton have a parity while the muon does not? Because

(a) parity is not conserved in electromagnetism.

(b) the proton is better known.

(c) parity is defined from reactions relative to each other. Therefore, it is meaningful for the proton but not for the muon.

(*CCT*)

Solution:

The answer is (c).

3005

What is the G-parity operator and why was it introduced in particle physics? What are the eigenvalues of the G-operator for pions of different charges, and for a state of n pions?

What are the G values for ρ, ω, ϕ, and η mesons?

(*Buffalo*)

Solution:

The G-operator can be defined as $G = Ce^{i\pi I_2}$ where I_2 is the second component of isospin I, and C is the charge conjugation operator.

As the C-operator has eigenvalues only for the photon and neutral mesons and their systems, it is useful to be able to extend the operation to include charged particles as well. The G-parity is so defined that charged particles can also be eigenstates of G-parity. Since strong interaction is invariant under both isospin rotation and charge conjugation, G-parity is conserved in strong interaction, which indicates a certain symmetry in strong interaction. This can be used as a selection rule for certain charged systems.

For an isospin multiplet containing a neutral particle, the eigenvalue of G-operator is

$$G = C(-1)^I ,$$

where C is the C eigenvalue of the neutral particle, I is the isospin. For π meson, $C(\pi^0) = +1$, $I = 1$, so $G = -1$; for a system of n π-mesons, $G(n\pi) = (-1)^n$. Similarly for

$$\rho: \ C(\rho^0) = -1, \quad I(\rho) = 1, \quad G(\rho) = +1 ;$$

$$\omega: \ C(\omega^0) = -1, \quad I(\omega^0) = 0, \quad G(\omega) = -1 ;$$

$$\phi: \ C(\phi) = -1, \quad I(\phi) = 0, \quad G(\phi) = -1 ;$$

$$\eta^0: \ C(\eta^0) = +1, \quad I(\eta^0) = 0, \quad C(\eta^0) = +1 .$$

ρ, ω, ϕ decay by strong interaction. As G-parity is conserved in strong interaction, their G-parities can also be deduced from the decays. Thus as

$$\rho^0 \to \pi^+\pi^- , \qquad G(\rho) = (-1)^2 = 1 ;$$

$$\omega \to 3\pi , \qquad G(\omega) = (-1)^3 = -1 ;$$

$$\phi \to 3\pi , \qquad G(\phi) = (-1)^3 = -1 .$$

Note as η^0 decays by electromagnetic interaction, in which G-parity is not conserved, its G-parity cannot be deduced from the decay.

3006

Following is a list of conservation laws (or symmetries) for interactions between particles. For each indicate by S,E,W those classes of interactions — strong, electromagnetic, weak — for which no violation of the symmetry or conservation law has been observed. For any one of these conservation laws, indicate an experiment which established a violation.

(a) I-spin conservation
(b) I_3 conservation
(c) strangeness conservation
(d) invariance under CP

(*Wisconsin*)

Solution:

(a) I-spin conservation — S.
(b) I_3 conservation — S, E.
(c) Strangeness conservation — S, E.
(d) CP invariance — S, E, and W generally. CP violation in weak interaction is found only in K_L decay. Isospin nonconservation can be observed in the electromagnetic decay $\Sigma^0 \to \Lambda^0 + \gamma$. I_3 nonconservation can be observed in the weak decay $\pi^- \to \mu^- + \bar{\nu}_\mu$.

Strangeness nonconservation is found in the weak decay of strange particles. For example, in $\Lambda^0 \to \pi^- + p$, $S = -1$ for the initial state, $S = 0$ for the final state, and so $\Delta S = -1$.

The only observed case of CP violation is the K_L^0 decay, in which the 3π and 2π decay modes have the ratio

$$\eta = \frac{B(K_L^0 \to \pi^+\pi^-)}{B(K_L^0 \to \text{all charged particles})} \approx 2 \times 10^{-3}\,.$$

It shows that CP conservation is violated in K_L^0 decay, but only to a very small extent.

3007

A state containing only one strange particle

(a) can decay into a state of zero strangeness.

(b) can be created strongly from a state of zero strangeness.
(c) cannot exist.

(*CCT*)

Solution:

Strange particles are produced in strong interaction but decay in weak interaction, and the strangeness number is conserved in strong interaction but not in weak interaction. Hence the answer is (a).

3008

A particle and its antiparticle

(a) must have the same mass.
(b) must be different from each other.
(c) can always annihilate into two photons.

(*CCT*)

Solution:

Symmetry requires that a particle and its antiparticle must have the same mass. Hence the answer is (a).

3009

Discuss briefly four of the following:

(1) J/ψ particle.
(2) Neutral K meson system, including regeneration of K_s.
(3) The two types of neutrino.
(4) Neutron electric dipole moment.
(5) Associated production.
(6) Fermi theory of beta-decay.
(7) Abnormal magnetic moment of the muon.

(*Columbia*)

Solution:

(1) *J/ψ particle.* In 1974, C. C. Ting, B. Richter and others, using different methods discovered a heavy meson of mass $M = 3.1$ GeV/c^2. Its

lifetime was $3 \sim 4$ orders of magnitude larger than mesons of similar masses, which makes it unique in particle physics. Named J/ψ particle, it was later shown to be the bound state of a new kind of quark, called the charm quark, and its antiquark. The J/ψ particle decays into charmless particles via the OZI rule or into a lepton pair via electromagnetic interaction, and thus has a long lifetime. Some of its quantum numbers are

$$m(J/\psi) = (3096.9 \pm 0.1)\ \text{MeV}/c^2, \qquad \Gamma = (63 \pm 9)\text{keV}\,,$$

$$I^G(J^P)C = 0^-(1^-) - \,.$$

All of its decay channels have been fully studied. J/ψ particle and other charmed mesons and baryons make up the family of charmed particles, which adds significantly to the content of particle physics.

(2) *Neutral K mesons* Detailed discussions are given in **Problems 3056–3058**.

(3) *Two kinds of neutrino.* Experiments have shown that there are two types of neutrino: one (ν_e) is associated with electron (as in β-decay), the other (ν_μ) with muon (as in $\pi \to \mu$ decay). Also a neutrino and its antineutrino are different particles.

The scattering of high energy neutrinos can lead to the following reactions:

$$\nu_e + n \to p + e^-, \qquad \bar{\nu}_e + p \to n + e^+\,,$$

$$\nu_\mu + n \to p + \mu^-, \qquad \bar{\nu}_\mu + p \to n + \mu^+\,.$$

Suppose a neutrino beam from a certain source is scattered and it contains $\nu_\mu(\bar{\nu}_\mu)$. If $\nu_e(\bar{\nu}_e)$ and $\nu_\mu(\bar{\nu}_\mu)$ are the same, approximately the same numbers of $e^\mp$ and $\mu^\mp$ should be observed experimentally. If they are not the same, the reactions producing $e^\mp$ are forbidden and no electrons should be observed. An experiment carried out in 1962 used a proton beam of energy > 20 GeV to bombard a target of protons to produce energetic pions and kaons. Most of the secondary particles were emitted in a cone of very small opening angle and decayed with neutrinos among the final products. A massive shielding block was used which absorbed all the particles except the neutrinos. The resulting neutrino beam (98–99% ν_μ, 1–2% ν_e) was used to bombard protons to produce muons or electrons. Experimentally, 51 muon events, but not one confirmed electron event, were observed. This proved that $\nu_e(\bar{\nu}_e)$ and $\nu_\mu(\bar{\nu}_\mu)$ are different particles.

That ν and $\bar{\nu}$ are different can be proved by measuring the reaction cross section for neutrinos in ^{37}Cl. Consider the electron capture process

$$^{37}\mathrm{Ar} + e^- \to {}^{37}\mathrm{Cl} + \nu \,.$$

The reverse process can also occur:

$$\nu + {}^{37}\mathrm{Cl} \to {}^{37}\mathrm{Ar} + e^- \,.$$

If $\bar{\nu}$ and ν are the same, so can the process below:

$$\bar{\nu}_e + {}^{37}\mathrm{Cl} \to {}^{37}\mathrm{Ar} + e^- \,.$$

In an experiment by R. Davis and coworkers, 4000 liters of CCl_4 were placed next to a nuclear reactor, where $\bar{\nu}$ were generated. Absorption of the antineutrinos by ^{37}Cl produced ^{37}Ar gas, which was separated from CCl_4 and whose rate of K-capture radioactivity was measured. The measured cross section was far less than the theoretical value $\sigma \approx 10^{-43}$ cm^2 expected if ν_e and $\bar{\nu}_e$ were the same. This showed that $\bar{\nu}$ is different from ν.

(4) *Electric dipole moment of neutron*

Measurement of the electric dipole moment of the neutron had been of considerable interest for a long time as it offered a means of directly examining time reversal invariance. One method for this purpose is described in Fig. 3.1, which makes use of nuclear magnetic resonance and electrostatic

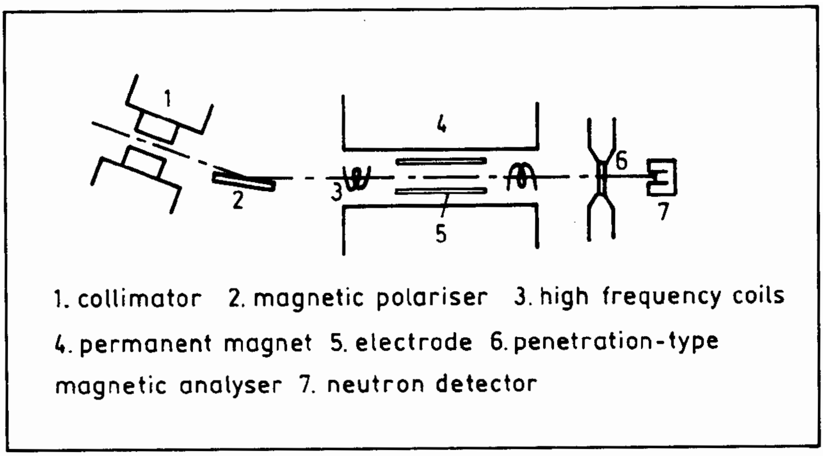

Fig. 3.1

deflection. It gave $P_n = eD$, where $D = (-1 \pm 4) \times 10^{-21}$ cm is the effective length of the dipole moment and e is the electron charge. Later, an experiment with cold neutrons gave $D = (0.4 \pm 1.1) \times 10^{-24}$ cm. This means that, within the experimental errors, no electric dipole moment was observed for the neutron.

(5) *Associated production*

Many new particles were discovered in cosmic rays around 1950 in two main categories — mesons and baryons. One peculiar characteristics of these particles was that they were produced in strong interaction (interaction time $\sim 10^{-23}$ s) but decayed in weak interaction ($\tau \sim 10^{-10} \sim 10^{-8}$ s). Also, they were usually produced in pairs. This latter phenomenon is called associated production and the particles are called strange particles. To account for the "strange" behavior a new additive quantum number called strangeness was assigned to all hadrons and the photon. The strangeness number S is zero for γ and the "ordinary" particles and is a small, positive or negative, integer for the strange particles K, Λ, Σ etc. A particle and its antiparticle have opposite strangeness numbers. S is conserved for strong and electromagnetic interactions but not for weak interaction. Thus in production by strong interaction from ordinary particles, two or more strange particles must be produced together to conserve S. This accounts for the associated production. In the decay of a strange particle into ordinary particles it must proceed by weak interaction as S is not conserved. The basic reason for the strange behavior of these particles is that they contain strange quarks or antiquarks.

(6) The Fermi theory of β-decay

Fermi put forward a theory of β-decay in 1934, which is analogous to the theory of electromagnetic transition. The basic idea is that just as γ-ray is emitted from an atom or nucleus in an electromagnetic transition, an electron and a neutrino are produced in the decay process. Then the energy spectrum of emitted electrons can be derived in a simple way to be

$$\left[\frac{dI(p_e)}{p_e^2 F dp_e}\right]^{1/2} = C|M_{ij}|^2(E_0 - E_e)\,,$$

where $dI(p_e)$ is the probability of emitting an electron of momentum between p_e and $p_e + dp_e$, E_e is the kinetic energy corresponding to p_e, E_0 is the maximum kinetic energy of the electrons, C is a constant, M_{ij} is the matrix element for weak interaction transition, $F(Z, E_e)$ is a factor which takes account of the effect of the Coulomb field of the nucleus on the emission of the electron. The theory, which explains well the phenomenon of β-decay, had been used for weak interaction processes until nonconservation of parity in weak interaction was discovered, when it was replaced by a revised version still close to the original form. Thus the Fermi theory may be considered the fundamental theory for describing weak interaction processes.

(7) *Abnormal magnetic moment of muon*

According to Dirac's theory, a singly-charged exact Dirac particle of spin J and mass m has a magnetic moment given by

$$\boldsymbol{\mu} = \frac{\mathbf{J}}{me} = g\frac{\mathbf{J}}{2mc},$$

where $g = -2$ for muon. However muon is not an exact Dirac particle, nor its g-factor exactly -2. It is said to have an abnormal magnetic moment, whose value can be calculated using quantum electrodynamics (QED) in accordance with the Feynman diagrams shown in Fig. 3.2. Let $\alpha = \frac{|g|-2}{2}$. QED gives

$$\begin{aligned}\alpha_\mu^{\rm th} &= \alpha/(2\pi) + 0.76578(\alpha/\pi)^2 + 2.55(\alpha/\pi)^3 + \cdots \\ &= (116592.1 \pm 1.0) \times 10^{-8},\end{aligned}$$

in excellent agreement with the experimental value

$$\alpha_\mu^{\rm exp} = (116592.2 \pm 0.9) \times 10^{-8}.$$

This has been hailed as the most brilliant achievement of QED.

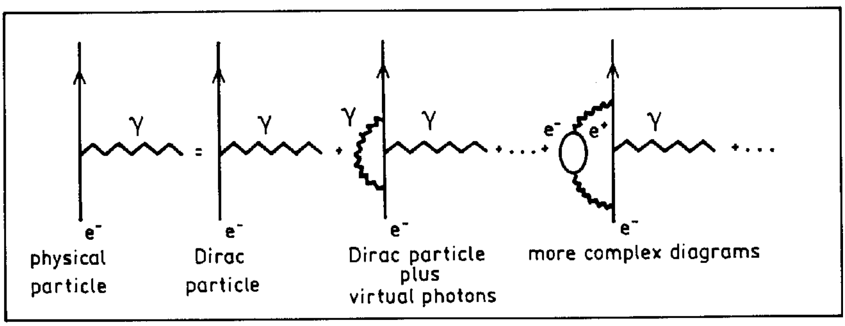

Fig. 3.2

3010

The lifetime of the muon is 10^9, 10^2, 10^{-2}, 10^{-6} second.

(Columbia)

Solution:

10^{-6} s (more precisely $\tau_\mu = 2.2 \times 10^{-6}$ s).

3011

List all of the known leptons. How does μ^+ decay? Considering this decay and the fact that $\nu_\mu + n \to e^- + p$ is found to be forbidden, discuss possible lepton quantum number assignments that satisfy additive quantum number conservation laws. How could ν_μ produce a new charged "heavy lepton"?

(Wisconsin)

Solution:

Up to now 10 kinds of leptons have been found. These are e^-, ν_e, μ^-, ν_μ, τ^- and their antiparticles e^+, $\bar{\nu}_e$, μ^+, $\bar{\nu}_\mu$, τ^+. ν_τ and $\bar{\nu}_\tau$ have been predicted theoretically, but not yet directly observed.

μ^+ decays according to $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$. It follows that $\bar{\nu}_e + \mu^+ \to e^+ + \bar{\nu}_\mu$. On the other hand the reaction $\nu_\mu + n \to e^- + p$ is forbidden. From these two reactions we see that for allowed reactions involving leptons, if there is a lepton in the initial state there must be a corresponding lepton in the final state. Accordingly we can define an electron lepton number L_e and a muon lepton number L_μ such that

$$L_e = 1 \qquad \text{for} \quad e^-, \nu_e \,,$$

$$L_\mu = 1 \qquad \text{for} \quad \mu^-, \nu_\mu \,,$$

with the lepton numbers of the antiparticles having the opposite sign, and introduce an additional conservation rule that the electron lepton number and the μ lepton number be separately conserved in a reaction.

It follows from a similar rule that to produce a charged heavy lepton, the reaction must conserve the corresponding lepton number. Then a new charged " heavy lepton" A^+ can be produced in a reaction

$$\nu_\mu + n \to A^+ + \nu_A + \mu^- + X \,,$$

where ν_A is the neutrino corresponding to A^+, X is a baryon. For example, $A^+ = \tau^+$, $\nu_A = \nu_\tau$.

3012

Give a non-trivial (rate greater than 5%) decay mode for each particle in the following list. If you include neutrinos in the final state, be sure to specify their type.

$$n \to, \pi^+ \to, \rho^0 \to, K^0 \to, \Lambda^0 \to, \Delta^{++} \to, \mu^- \to, \phi \to, \Omega^- \to, J/\Psi \to \ .$$

(*Wisconsin*)

Solution:

$n \to pe^-\bar{\nu}_e$; $\pi^+ \to \mu^+\nu_\mu$; $\rho^0 \to \pi^+\pi^-$; $K^0 \to \pi^+\pi^-$, $\pi^0\pi^0$, $\pi^0\pi^0\pi^0$, $\pi^+\pi^-\pi^0$, $\pi^\pm\mu^\mp\nu_\mu$, $\pi^0\mu^\pm e^\mp\nu_e$; $\Lambda^0 \to p\pi^-$, $n\pi^0$; $\Delta^{++} \to p\pi^+$; $\mu^- \to e^-\bar{\nu}_e\nu_\mu$; $\phi \to K^+K^-$, $K_L^0K_S^0$, $\pi^+\pi^-\pi^0$; $\Omega^- \to \Lambda K^-$, $\Xi^0\pi^-$, $\Xi^-\pi^0$; $J/\psi \to e^+e^-$, $\mu^+\mu^-$, hadrons.

3013

Consider the following high-energy reactions or particle decays:

(1) $\pi^- + p \to \pi^0 + n$
(2) $\pi^0 \to \gamma + \gamma + \gamma$
(3) $\pi^0 \to \gamma + \gamma$
(4) $\pi^+ \to \mu^+ + \nu_\mu$
(5) $\pi^+ \to \mu^+ + \bar{\nu}_\mu$
(6) $p + \bar{p} \to \Lambda^0 + \Lambda^0$
(7) $p + \bar{p} \to \gamma$.

Indicate for each case:

(a) allowed or forbidden,
(b) reason if forbidden,
(c) type of interaction if allowed (i.e., strong, weak, electromagnetic, etc.)

(*Wisconsin*)

Solution:

(1) $\pi^- + p \to \pi^0 + n$: All quantum numbers conserved, allowed by strong interaction.

(2) $\pi^0 \to \gamma + \gamma + \gamma : C(\pi^0) = +1$, $C(3\gamma) = (-1)^3 \neq C(\pi^0)$, forbidden as C-parity is not conserved.

(3) $\pi^0 \to \gamma + \gamma$: electromagnetic decay allowed.

(4) $\pi^+ \to \mu^+ + \nu_\mu$: weak decay allowed.

(5) $\pi^+ \to \mu^+ + \bar{\nu}_\mu$: left-hand side $L_\mu = 0$, right-hand side $L_\mu = -2$, forbidden as μ-lepton number is not conserved.

(6)

	$p+$	$\bar{p} \to$	$\Lambda^0 +$	Λ^0	
B	1	-1	1	1	$\Delta B = +2$
S	0	0	-1	-1	$\Delta S = -2$

it is forbidden as baryon number is not conserved.

(7) $p + \bar{p} \to \gamma$ is forbidden, for the angular momentum and parity cannot both be conserved. Also the momentum and energy cannot both be conserved, for

$W^2(p, \bar{p}) = (E_p + E_{\bar{p}})^2 - (\mathbf{p}_p + \mathbf{p}_{\bar{p}})^2 = m_p^2 + m_{\bar{p}}^2 + 2(E_p E_{\bar{p}} - \mathbf{p}_p \cdot \mathbf{p}_{\bar{p}}) \geq 2m_p^2 > 0$, as $E^2 = p^2 + m^2$, $E_p E_{\bar{p}} \geq p_p p_{\bar{p}} \geq \mathbf{p}_p \cdot \mathbf{p}_{\bar{p}}$, $W^2(\gamma) = E_\gamma^2 - p_\gamma^2 = E_\gamma^2 - E_\gamma^2 = 0$, and so $W(p, \bar{p}) \neq W^2(\gamma)$.

3014

For each of the following decays state a conservation law that forbids it:

$$n \to p + e^-$$

$$n \to \pi^+ + e^-$$

$$n \to p + \pi^-$$

$$n \to p + \gamma$$

(*Wisconsin*)

Solution:

$n \to p + e^-$: conservation of angular momentum and conservation of lepton number are both violated.

$n \to \pi^+ + e^-$: conservation of baryon number and conservation lepton number are both violated.

$n \to p + \pi^-$: conservation of energy is violated.

$n \to p + \gamma$: conservation of electric charge is violated.

3015

What conservation laws, invariance principles, or other mechanisms account for the suppressing or forbidding of the following processes?

(1) $p + n \to p + \Lambda^0$
(2) $K^+ \to \pi^+ + \pi^- + \pi^+ + \pi^- + \pi^+ + \pi^0$
(3) $\bar{K}^0 \to \pi^- + e^+ + \nu_e$
(4) $\Lambda^0 \to K^0 + \pi^0$
(5) $\pi^+ \to e^+ + \nu_e$ (relative to $\pi^+ \to \mu^+ + \nu_\mu$)
(6) $K_L^0 \to e^+ + e^-$
(7) $K^- \to \pi^0 + e^-$
(8) $\pi^0 \to \gamma + \gamma + \gamma$
(9) $K_L^0 \to \pi^+ + \pi^-$
(10) $K^+ \to \pi^+ + \pi^+ + \pi^0$

(*Wisconsin*)

Solution:

(1) Conservation of strangeness number and conservation of isospin are violated.

(2) Conservation of energy is violated.

(3) $\Delta S = 1$, $\Delta Q = 0$, the rule that if $|\Delta S| = 1$ in weak interaction, ΔS must be equals to ΔQ is violated

(4) Conservation of baryon number is violated.

(5) The process go through weak interaction and the ratio of rates is (**Problem 3040**)

$$\frac{\Gamma(\pi^+ \to e^+ + \nu_e)}{\Gamma(\pi^+ \to \mu^+ + \nu_\mu)} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.2 \times 10^{-4} \, .$$

Hence the $\pi \to e\nu$ mode is quite negligible.

(6) $\Delta S = -1$, $\Delta Q = 0$, same reason as for (3).

(7) Conservation of lepton number is violated.

(8) Conservation of C-parity is violated.
(9) CP parity conservation is violated.
(10) Conservation of electric charge is violated.

3016

Which of the following reactions violate a conservation law?
Where there is a violation, state the law that is violated.

$\mu^+ \to e^+ + \gamma$
$e^- \to \nu_e + \gamma$
$p + p \to p + \Sigma^+ + K^-$
$p \to e^+ + \nu_e$
$p \to e^+ + n + \nu_e$
$n \to p + e^- + \bar{\nu}_e$
$\pi^+ \to \mu^+ + \nu_\mu$

(*Buffalo*)

Solution:

$\mu^+ \to e^+ + \gamma$ is forbidden because it violates the conservation of lepton number, which must hold for any interaction.

$e^- \to \nu_e + \gamma$, $p + p \to p + \Sigma^+ + K^-$ are forbidden because they violate electric charge conservation.

$p \to e^+ + \nu_e$ is forbidden because it violates baryon number conservation.

$p \to e^+ + n + \nu_e$ is forbidden because it violates energy conservation.

$n \to p + e^- + \bar{\nu}_e$, $\pi^+ \to \mu^+ + \nu_\mu$ are allowed.

3017

(a) Explain why the following reactions are not observed, even if the kinetic energy of the first proton is several BeV:

(1) $p + p \to K^+ + \Sigma^+$
(2) $p + n \to \Lambda^0 + \Sigma^+$
(3) $p + n \to \Xi^0 + p$
(4) $p + n \to \Xi^- + K^+ + \Sigma^+$

(b) Explain why the following decay processes are not observed:

(1) $\Xi^0 \to \Sigma^0 + \Lambda^0$
(2) $\Sigma^+ \to \Lambda^0 + K^+$

(3) $\Xi^- \to n + \pi^-$
(4) $\Lambda^0 \to K^+ + K^-$
(5) $\Xi^0 \to p + \pi^-$

(*Columbia*)

Solution:

(a) The reactions involve only strongly interacting particles and should obey all the conservation laws. If some are violated then the process is forbidden and not observed. Some of the relevant data and quantum numbers are given in Table 3.3.

(1) $p + p \to K^+ + \Sigma^+$, the baryon number, the isospin and its third component are not conserved.

(2) $p + n \to \Lambda^0 + \Sigma^+$, the strangeness number ($\Delta S = -2$) and the third component of isospin are not conserved.

(3) $p + n \to \Xi^0 + p$, for the same reasons as for (2).

(4) $p + n \to \Xi^- + K^+ + \Sigma^+$, for the same reasons as for (2).

(b) All the decays are nonleptonic weak decays of strange particles, where the change of strangeness number S, isospin I and its third component I_3 should obey the rules $|\Delta S| = 1$, $|\Delta I_3| = 1/2$, $|\Delta I| = 1/2$.

(1) $\Xi^0 \to \Sigma^0 + \Lambda^0$, the energy and the baryon number are not conserved.

(2) $\Sigma^+ \to \Lambda^0 + K^+$, the energy is not conserved.

(3) $\Xi^- \to n + \pi^-$, $|\Delta S| = 2 > 1$, $|\Delta I_3| = 1 > 1/2$.

(4) $\Lambda^0 \to K^+ + K^-$, the baryon number is not conserved.

(5) $\Xi^0 \to p + \pi^-$, $|\Delta S| = 2 > 1$, $|\Delta I_3| = 1 > 1/2$.

Table 3.3

Particle	Lifetime(s)	Mass(MeV/c^2)	Spin J	Strangeness number S	Isospin I
$\pi^\pm$	2.55×10^{-8}	139.58	0	0	1
K	1.23×10^{-8}	493.98	0	± 1	1/2
p	*stable*	938.21	1/2	0	1/2
n	1.0×10^3	939.51	1/2	0	1/2
Λ^0	2.52×10^{-10}	1115.5	1/2	-1	0
Σ^+	0.81×10^{-10}	1189.5	1/2	-1	1
Σ^0	$< 10^{-14}$	1192.2	1/2	-1	1
Ξ^-	1.7×10^{-10}	1321	1/2	-2	1/2
Ξ^0	2.9×10^{-10}	1315	1/2	-2	1/2

3018

Listed below are a number of decay processes.

(a) Which do not occur in nature? For each of these specify the conservation law which forbids its occurrence.

(b) Order the remaining decays in order of increasing lifetime. For each case name the interaction responsible for the decay and give an order-of-magnitude estimate of the lifetime. Give a brief explanation for your answer.

$p \to e^+ + \pi^0$
$\Omega^- \to \Xi^0 + K^-$
$\rho^0 \to \pi^+ + \pi^-$
$\pi^0 \to \gamma + \gamma$
$D^0 \to K^- + \pi^+$
$\Xi^- \to \Lambda^0 + \pi^-$
$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$

Table 3.4

particle	mass (MeV/c^2)	J	B	L	I	S	G
γ	0	1	0	0	0	0	0
ν_e	0	1/2	0	1	0	0	0
ν_μ	0	1/2	0	1	0	0	0
e^-	0.5	1/2	0	1	0	0	0
μ^-	106	1/2	0	1	0	0	0
π^0	135	0	0	0	1	0	0
κ^-	494	0	0	0	1/2	-1	0
ρ^0	770	1	0	0	1	0	0
p	938	1/2	1	0	1/2	0	0
Λ^0	1116	1/2	1	0	0	-1	0
Ξ^-	1321	1/2	1	0	1/2	-2	0
Ω^-	1672	3/2	1	0	0	-3	0
D^0	1865	0	0	0	1/2	0	1

(*Columbia*)

Solution:

(a) $p \to e^+ + \pi^0$, forbidden as the lepton number and the baryon number are not conserved.

$\Omega^- \to \Xi^0 + K^-$, forbidden because the energy is not conserved as $m_\Omega < (m_\Xi + m_K)$.

(b) The allowed decays are arranged below in increasing order of lifetime:

$\rho^0 \to \pi^+ + \pi^-$, lifetime $\approx 10^{-24}$ s, strong decay,
$\pi^0 \to \gamma + \gamma$, lifetime $\approx 10^{-16}$ s, electromagnetic decay,
$D^0 \to K^- + \pi^+$, lifetime $\approx 10^{-13}$ s, weak decay,
$\Xi^- \to \Lambda^0 + \pi^-$, lifetime $\approx 10^{-10}$ s, weak decay,
$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$, lifetime $\approx 10^{-6}$ s, weak decay.

The first two decays are typical of strong and electromagnetic decays, the third and fourth are weak decays in which the strangeness number and the charm number are changed, while the last is the weak decay of a non-strange particle.

3019

An experiment is performed to search for evidence of the reaction $pp \to HK^+K^+$.

(a) What are the values of electric charge, strangeness and baryon number of the particle H? How many quarks must H contain?

(b) A theoretical calculation for the mass of this state H yields a predicted value of $m_H = 2150$ MeV.

What is the minimum value of incident-beam proton momentum necessary to produce this state? (Assume that the target protons are at rest)

(c) If the mass prediction is correct, what can you say about the possible decay modes of H? Consider both strong and weak decays.

(*Princeton*)

Solution:

(a) As K^+ has $S = 1$, $B = 0$, H is expected to have electric charge $Q = 0$, strangeness number $S = -2$, baryon number $B = 2$. To satisfy these requirements, H must contain at least six quarks (uu dd ss).

(b) At minimum incident energy, the particles are produced at rest in the center-of-mass frame. As $(\Sigma E)^2 - (\Sigma \mathbf{p})^2$ is invariant, we have

$$(E_0 + m_p)^2 - p_0^2 = (m_H + 2m_K)^2 ,$$

giving

$$E_0 = \frac{(m_H + 2m_K)^2 - 2m_p^2}{2m_p}$$
$$= \frac{(2.15 + 2 \times 0.494)^2 - 2 \times 0.938^2}{2 \times 0.938} = 4.311 \text{ GeV}\,,$$

and hence the minimum incident momentum

$$p_0 = \sqrt{E_0^2 - m_p^2} = 4.208 \text{ GeV}/e\,.$$

(c) As for strong decays, $\Delta S = 0$, $\Delta B = 0$, the possible channels are $H \to \Lambda^0\Lambda^0$, $\Lambda^0\Sigma^0$, $\Xi^- p$, $\Xi^0 n$.

However they all violate the conservation of energy and are forbidden. Consider possible weak decays. The possible decays are nonleptonic decays $H \to \Lambda + n$, $\Sigma^0 + n$, $\Sigma^- + p$, and semi-leptonic decays

$$H \to \Lambda + p + e^- + \bar{\nu}, \quad \Sigma^0 + p + e^- + \bar{\nu}\,.$$

3020

Having 4.5 GeV free energy, what is the most massive isotope one could theoretically produce from nothing?

(a) ^{2}D.
(b) ^{3}He.
(c) ^{3}T.

(*CCT*)

Solution:

With a free energy of 4.5 GeV, one could create baryons with energy below 2.25 GeV (To conserve baryon number, the same number of baryons and antibaryons must be produced together. Thus only half the energy is available for baryon creation). Of the three particles only ^{2}D has rest energy below this. Hence the answer is (a).

3021

(i) The decay $K \to \pi\gamma$ is absolutely forbidden by a certain conservation law, which is believed to hold exactly. Which conservation law is this?

(ii) There are no known mesons of electric charge two. Can you give a simple explanation of this?

(iii) Explain how the parity of pion can be measured by observation of the polarizations of the photons in $\pi^0 \to \gamma\gamma$.

(iv) To a very high accuracy, the cross section for e^-p scattering equals the cross section for e^+p scattering. Is this equality a consequence of a conservation law? If so, which one? If not, explain the observed equality. To what extent (if any at all) do you expect this equality to be violated?

(v) It has recently been observed that in inclusive Λ production (Fig. 3.3), for example $\pi p \to \Lambda$+anything, the Λ is produced with a surpisingly high polarization. Do you believe this polarization is

(a) along (or opposite to) the direction of the incident beam,
(b) along (or opposite to) the direction of motion of the outgoing Λ, or
(c) perpendicular to both?

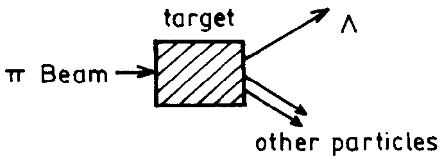

Fig. 3.3

(*Princeton*)

Solution:

(i) The decay is forbidden by the conservation of strangeness number, which holds exactly in electromagnetic interaction.

(ii) According to the prevailing theory, a meson consists of a quark and an antiquark. The absolute value of a quark's charge is not more than 2/3. So it is impossible for the charge of a meson consisting of two quarks to be equal to 2.

(iii) Let the wave vectors of the two photons be $\mathbf{k}_1$, $\mathbf{k}_2$, the directions of the polarization of their electric fields be $\mathbf{e}_1$, $\mathbf{e}_2$, and let $\mathbf{k} = \mathbf{k}_1 - \mathbf{k}_2$. Since the spin of π^0 is 0, the possible forms of the decay amplitude are $A\mathbf{e}_1 \cdot \mathbf{e}_2$ and $B\mathbf{k} \cdot (\mathbf{e}_1 \times \mathbf{e}_2)$, which, under space inversion, respectively does not and does change sign. Thus the former form has even parity, and the latter, odd parity. These two cases stand for the two different relative polarizations of the photons. The former describes mainly parallel polarizations, while the latter describes mainly perpendicular polarizations between the two photons. It is difficult to measure the polarization of high energy photons ($E \sim 70$ MeV) directly. But in π^0 decays, in a fraction α^2 of the cases the two photons convert directly to two electron-positron pairs. In such cases the relative polarization of the photons can be determined by measuring the angle between the two electron-positron pairs. The experimental results tend to favor the perpendicular polarization. Since parity is conserved in electromagnetic interaction, the parity of π^0 is odd.

(iv) No. To first order accuracy, the probability of electromagnetic interaction is not related to the sign of the charge of the incident particle. Only when higher order corrections are considered will the effect of the sign of the charge come in. As the strength of each higher order of electromagnetic interaction decreases by a factor α^2, this equality is violated by a fraction $\alpha^2 \approx 5.3 \times 10^{-5}$.

(v) The polarization $\boldsymbol{\sigma}$ of Λ is perpendicular to the plane of interaction. As parity is conserved in strong interaction, $\boldsymbol{\sigma}$ is perpendicular to the plane of production, i.e.,

$$\boldsymbol{\sigma} \propto \mathbf{p}_\pi \times \mathbf{p}_\Lambda$$

3022

Recently a stir was caused by the reported discovery of the decay $\mu^+ \to e^+ + \gamma$ at a branching ratio of $\sim 10^{-9}$.

(a) What general principle is believed to be responsible for the suppression of this decay?

(b) The apparatus consists of a stopping μ^+ beam and two NaI crystals, which respond to the total energy of the positrons or gamma rays. How would you arrange the crystals relative to the stopping target and beam, and what signal in the crystals would indicate that an event is such a μ decay?

(c) The background events are the decays $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu + \gamma$ with the neutrinos undetected. Describe qualitatively how one would distinguish events of this type from the $\mu^+ \to e^+ + \gamma$ events of interest.

(*Wisconsin*)

Solution:

(a) This decay is suppressed by the separate conservation of electron-lepton number and μ-lepton number,

(b) $\mu^+ \to e^+ + \gamma$ is a two-body decay. When the muon decays at rest into e^+ and γ, we have $E_e \approx E_\gamma = \frac{m_\mu c^2}{2}$. As e^+ and γ are emitted in opposite directions the two crystals should be placed face to face. Also, to minimize the effect of any directly incident mesons they should be placed perpendicular to the μ beam (see Fig. 3.4). The coincidence of e^+ and γ signals gives the μ decay events, including the background events given in (c).

(c) $\mu^+ \to e^+ + \gamma$ is a two-body decay and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu + \gamma$ is a four-body decay. In the former e^+ and γ are monoenergetic, while in the latter e^+ and γ have continuous energies up to a maximum. We can separate them by the amplitudes of the signals from the two crystals. For $\mu^+ \to e^+ + \gamma$, $(E_e + E_\gamma) = m_\mu$, while for $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu + \gamma$, $(E_e + E_\gamma) < m_\mu$.

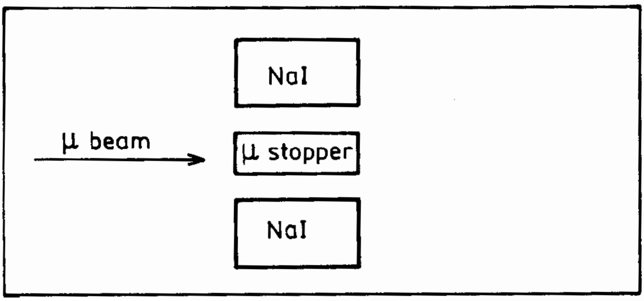

Fig. 3.4

3023

Describe the properties of the various types of pion and discuss in detail the experiments which have been carried out to determine their spin, parity, and isospin.

(*Buffalo*)

Solution:

There are three kinds of pion: $\pi^0, \pi^+\pi^-$, with π^+ being the antiparticle of π^- and π^0 its own antiparticle, forming an isospin triplet of $I = 1$. Their main properties are listed in Table 3.5.

Table 3.5

Particle	Mass(MeV)	Spin	Parity	C-Parity	Isospin	I_3	G
π^+	139.6	0	$-$		1	1	-1
π^0	135	0	$-$	$+$	1	0	-1
π^-	139.6	0	$-$		1	-1	-1

To determine the spin of π^+, we apply the principle of detailed balance to the reversible reaction $\pi^+ + d \rightleftarrows p + p$, where the forward reaction and its inverse have the same transition matrix element. Thus

$$\frac{d\sigma}{d\Omega}(pp \to d\pi^+) = \frac{d\sigma}{d\Omega}(d\pi^+ \to pp) \times 2\frac{p_\pi^2(2J_\pi + 1)(2J_d + 1)}{p_p^2(2J_p + 1)^2},$$

where p_π, p_p are the momenta of π and p, respectively, in the center-of-mass frame. Experimental cross sections give $2J_\pi + 1 = 1.00 \pm 0.01$, or $J_\pi = 0$.

The spin of π^- can be determined directly from the hyperfine structure of the π-mesic atom spectrum. Also the symmetry of particle and antiparticle requires π^+ and π^- to have the same spin. So the spin of π^- is also 0.

The spin of π^0 can be determined by studying the decay $\pi^0 \to 2\gamma$. First we shall see that a particle of spin 1 cannot decay into 2 γ's. Consider the decay in the center-of-mass frame of the 2 γ's, letting their momenta be $\mathbf{k}$ and $-\mathbf{k}$, their polarization vectors be $\boldsymbol{\varepsilon}_1$ and $\boldsymbol{\varepsilon}_2$ respectively. Because the spin of the initial state is 1, the final state must have a vector form. As a real photon has only transverse polarization, only the following vectors can be constructed from $\mathbf{k}$, $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$:

$$\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2, \quad (\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_2)\mathbf{k}, \quad (\varepsilon \times \varepsilon_2 \cdot \mathbf{k})\mathbf{k}.$$

All the three vector forms change sign when the 2 γ's are exchanged. However the 2γ system is a system of two bosons which is exchange-symmetric and so none of three forms can be the wave function of the system. Hence the spin of π^0 cannot be 1. On the other hand, consider the reaction $\pi^- + p \to \pi^0 + n$ using low energy (s-wave) π^-. The reaction is forbidden for $J_{\pi^0} \geq 2$. Experimentally, the cross section for the charge-exchange reaction is very large. The above proves that $J_{\pi^0} = 0$.

The parity of π^- can be determined from the reaction $\pi^- + d \to n + n$, employing low energy (s-wave) π^-. It is well known that $J_d^P = 1^+$, so $P(\pi^-) = P^2(n)(-1)^l$, l being the orbital angular momentum of the relative motion of the two neutrons. Since an n–n system is a Fermion system and so is exchange antisymmetric, $l = 1$, $J = 1$, giving $P(\pi^-) = -1$.

The parity of π^+ can be determined by studying the cross section for the reaction $\pi^+ + d \to p + p$ as a function of energy of the incident low energy (s-wave) π^+. This gives $P(\pi^+) = -1$.

The parity of π^0 can be determined by measuring the polarization of the decay $\pi^0 \to 2\gamma$. As $J(\pi^0) = 0$, and the 2γ system in the final state is exchange symmetric, possible forms of the decay amplitude are

$$\boldsymbol{\varepsilon}_1 \cdot \boldsymbol{\varepsilon}_2, \text{ corresponding to } P(\pi^0) = +1\,,$$

$$\mathbf{k} \cdot (\boldsymbol{\varepsilon}_1 \times \boldsymbol{\varepsilon}_2), \text{ corresponding to } P(\pi^0) = -1\,,$$

where $\mathbf{k}$ is the momentum of a γ in the π^0 rest frame. The two forms respectively represent the case of dominantly parallel polarizations and the case of dominantly perpendicular polarizations of the two photons. Consider then the production of electron-positron pairs by the 2 γ's:

$$
\begin{array}{l}
\pi^0 \to \gamma + \gamma \\
\qquad\ \ |\qquad \hookrightarrow e^+ + e^- \\
\qquad\ \ \hookrightarrow e^+ + e^-
\end{array}
$$

An electron-positron pair is created in the plane of the electric vector of the γ ray. As the experimental results show that the planes of the two pairs are mainly perpendicular to each other, the parity of π^0 is -1.

The isospin of π can be deduced by studying strong interaction processes such as

$$n + p \to d + \pi^0, \qquad p + p \to d + \pi^+\,.$$

Consider the latter reaction. The isospin of the initial state $(p+p)$ is $|1, 1\rangle$, the isospin of the final state is also $|1, 1\rangle$. As isospin is conserved, the transition to the final state $(d + \pi^+)$ has a probability of 100%. Whereas, in the former reaction, the isospin of the initial state is $\frac{1}{\sqrt{2}}(|1,0\rangle - |0,0\rangle)$, of which only the state $|1,0\rangle$ can transit to the $(d + \pi^0)$ system of isospin $|1,0\rangle$. Hence the probability for the transition from $(n + p)$ to $(d + \pi^0)$ is only 50%. In other words, if $I(\pi) = 1$, we would have

$$\sigma(pp \to d\pi^+) = 2\sigma(pn \to d\pi^0) .$$

As this agrees with experiment, $I(\pi) = 1$.

3024

The electrically neutral baryon Σ^0 (1915) (of mass 1915 MeV/c^2) has isospin $I = 1$, $I_3 = 0$. Call Γ_{K^-p}, $\Gamma_{\bar{K}^0 n}$, Γ_{π^-p}, $\Gamma_{\pi^+\pi^-}$ respectively the rates for the decays $\Sigma^0(1915) \to K^-p$, $\Sigma^0(1915) \to \bar{K}^0 n$, $\Sigma^0(1915) \to \pi^-p$, $\Sigma^0(1915) \to \pi^+\pi^-$. Find the ratios

$$\frac{\Gamma_{\bar{K}^0 n}}{\Gamma_{K^-p}}, \quad \frac{\Gamma_{\pi^-p}}{\Gamma_{K^-p}}, \quad \frac{\Gamma_{\pi^+\pi^-}}{\Gamma_{K^-p}} .$$

(The masses of the nucleons, K^-, and π^- mesons are such that all these decays are kinetically possible. You can disregard the small mass splitting within an isospin multiplet.)

(*Chicago*)

Solution:

n, p form an isospin doublet, π^+, π^0, π^- form an isospin triplet, and K^+, K^0 form an isospin doublet. K^- and $\bar{K}^0$, the antiparticles of K^+ and K^0 respectively, also form an isospin doublet. Write the isospin state of $\Sigma^0(1915)$ as $|1,0\rangle$, those of p and n as $|1/2,1/2\rangle$ and $|1/2,-1/2\rangle$, and those of $\bar{K}^0$ and K^- as $|1/2,1/2\rangle$ and $|1/2,-1/2\rangle$, respectively. As

$$\Psi(\bar{K}^0 n) = \left|\frac{1}{2},\frac{1}{2}\right\rangle \left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}}(|1,0\rangle + |0,0\rangle) ,$$

$$\Psi(\bar{K}^- p) = \left|\frac{1}{2},-\frac{1}{2}\right\rangle \left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}}(|1,0\rangle - |0,0\rangle) ,$$

$\Sigma^0(1915) \to \bar{K}^0 n$ and $\Sigma^0(1915) \to K^- p$ are both strong decays, the partial widths are

$$\Gamma_{\bar{K}^0 n} \propto |\langle \Psi(\Sigma^0)|H|\Psi(\bar{K}^0 n)\rangle|^2 = \left(\frac{a_1}{\sqrt{2}}\right)^2 = \frac{a_1^2}{2}\,,$$

$$\Gamma_{K^- p} \propto |\langle \Psi(\Sigma^0)|H|\Psi(K^- p)\rangle|^2 = \left(\frac{a_1}{\sqrt{2}}\right)^2 = \frac{a_1^2}{2}\,,$$

where $a_1 = \langle 1|H|1\rangle$. Note $\langle 1|H|0\rangle = 0$ and, as strong interaction is charge independent, a_1 only depends on I but not on I_3. Hence

$$\frac{\Gamma_{\bar{K}^0 n}}{\Gamma_{K^- p}} = 1\,.$$

$\Sigma^0(1915) \to p\pi^-$ is a weak decay ($\Delta I_3 = -\frac{1}{2} \neq 0$) and so

$$\frac{\Gamma_{\pi^- p}}{\Gamma_{K^- p}} \ll 1$$

(actually $\sim 10^{-10}$).

In the $\Sigma^0(1915) \to \pi^+\pi^-$ mode baryon number is not conserved, and so the reaction is forbidden. Thus

$$\Gamma_{\pi^+\pi^-} = 0\,,$$

or

$$\frac{\Gamma_{\pi^+\pi^-}}{\Gamma_{K^- p}} = 0\,.$$

3025

Which of the following reactions are allowed? If forbidden, state the reason.

(a) $\pi^- + p \to K^- + \Sigma^+$
(b) $d + d \to {}^4He + \pi^0$
(c) $K^- + p \to \Xi^- + K^+$

What is the ratio of reaction cross sections $\sigma(p+p \to \pi^+ + d)/\sigma(n+p \to \pi^0 + d)$ at the same center-of-mass energy?

(*Chicago*)

Solution:

(a) Forbidden as $\Delta I_3 = (-1/2) + (+1) - (-1) - 1/2 = 1 \neq 0$, $\Delta S = (-1) + (-1) - 0 - 0 = -2 \neq 0$.

(b) Forbidden as $I(d) = I(^4He) = 0$, $I(\pi^0) = 1$, $\Delta I = 1 \neq 0$

(c) Allowed by strong interaction as Q, I, I_3, and S are all conserved.

The difference in cross section between $pp \to \pi^+ d$ and $np \to \pi^0 d$ relates to isospin only. Using the coupling presentation for isospins and noting the orthogonality of the isospin wave functions, we have

$$\begin{aligned}
|pp\rangle &= \left|\frac{1}{2}, \frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle = |1, 1\rangle\,, \\
|\pi^+ d\rangle &= |1, 1\rangle\ |0, 0\rangle = |1, 1\rangle\,, \\
|np\rangle &= \left|\frac{1}{2}, -\frac{1}{2}\right\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle = \frac{1}{\sqrt{2}}|1, 0\rangle - \frac{1}{\sqrt{2}}|0, 0\rangle\,, \\
|\pi^0 d\rangle &= |1, 0\rangle\ |0, 0\rangle = |1, 0\rangle\,.
\end{aligned}$$

Hence the matrix element of $pp \to \pi^+ d$ is

$$\langle \pi^+ d|\hat{H}|pp\rangle \propto \langle 1, 1|\hat{H}|1, 1\rangle = \langle 1|\hat{H}|1\rangle = a_1\,.$$

Similarly, the matrix element of $np \to \pi^0 d$ is

$$\begin{aligned}
\langle \pi^0 d|\hat{H}|np\rangle &\propto \frac{1}{\sqrt{2}}\langle 1, 0|\hat{H}|1, 0\rangle - \frac{1}{\sqrt{2}}\langle 1, 0|\hat{H}|0, 0\rangle \\
&\propto \frac{1}{\sqrt{2}}\langle 1, 0|\hat{H}|1, 0\rangle = \frac{1}{\sqrt{2}}\langle 1|\hat{H}|1\rangle = \frac{a_1}{\sqrt{2}}\,,
\end{aligned}$$

as $\langle 1, 0|\hat{H}|0, 0\rangle = 0$ and strong interaction is independent of I_3. Therefore,

$$\frac{\sigma(pp \to \pi^+ d)}{\sigma(np \to \pi^0 d)} = \frac{|\langle \pi^+ d|\hat{H}|pp\rangle|^2}{|\langle \pi^0 d|\hat{H}|np\rangle|^2} = \frac{a_1^2}{\frac{1}{2}a_1^2} = 2\,.$$

3026

Given two angular momenta $\mathbf{J}_1$ and $\mathbf{J}_2$ (for example $\mathbf{L}$ and $\mathbf{S}$) and the corresponding wave functions.

(a) Compute the Clebsch–Gordan coefficients for the states with $\mathbf{J} = \mathbf{j}_1 + \mathbf{j}_2$, $M = m_1 + m_2$, where $j_1 = 1$ and $j_2 = 1/2$, $J = 3/2$, $M = 1/2$, for the various possible m_1 and m_2 values.

(b) Consider the reactions

(1) $\pi^+ p \to \pi^+ p$,

(2) $\pi^- p \to \pi^- p$,

(3) $\pi^- p \to \pi^0 n$.

These reactions, which conserve isospin, can occur in the isospin $I = 3/2$ state (Δ resonance) or $I = 1/2$ state (N^* resonance). Calculate the ratio of these cross sections $\sigma_1 : \sigma_2 : \sigma_3$ for an energy corresponding to a Δ resonance and to an N^* resonance . At a resonance energy you can neglect the effect due to the other isospin state. Note that the pion is an isospin $I_\pi = 1$ state and the nucleon an isospin $I_n = 1/2$ state.

(*UC, Berkeley*)

Solution:

(a) First consider

$$\left|\frac{3}{2}, \frac{3}{2}\right\rangle = |1, 1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle .$$

Applying the operator

$$L_- = J_x - iJy = (j_{1x} - ij_{1y}) + (j_{2x} - ij_{2y}) \equiv L_-^{(1)} + L_-^{(2)}$$

to the above:

$$L_-\left|\frac{3}{2}, \frac{3}{2}\right\rangle = L_-^{(1)}|1, 1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle + L_-^{(2)}|1, 1\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle ,$$

as

$$L_-|J, M\rangle = \sqrt{J(J+1) - M(M-1)}|J, M-1\rangle ,$$

we have

$$\sqrt{3}\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{2}|1, 0\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle + |1, 1\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle ,$$

or

$$\left|\frac{3}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}|1, 0\rangle \left|\frac{1}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}|1, 1\rangle \left|\frac{1}{2}, -\frac{1}{2}\right\rangle .$$

(b) We couple each initial pair in the isospin space:

$$|\pi^+p\rangle = |1,1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = \left|\frac{3}{2},\frac{3}{2}\right\rangle,$$

$$|\pi^-p\rangle = |1,-1\rangle\left|\frac{1}{2},\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle,$$

$$|\pi^0n\rangle = |1,0\rangle\left|\frac{1}{2},-\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle.$$

Because of charge independence in strong interaction, we can write

$$\left\langle\frac{3}{2},m_j|\hat{H}|\frac{3}{2},m_i\right\rangle = a_1\,,$$

$$\left\langle\frac{1}{2},m_j|\hat{H}|\frac{1}{2},m_i\right\rangle = a_2\,,$$

independent of the value of m. Furthermore the orthogonality of the wave functions requires

$$\left\langle\frac{1}{2}|\hat{H}|\frac{3}{2}\right\rangle = 0\,.$$

Hence the transition cross sections are

$$\sigma_1(\pi^+p \to \pi^+p) \propto \left|\left\langle\frac{3}{2},\frac{3}{2}\right|\hat{H}\left|\frac{3}{2},\frac{3}{2}\right\rangle\right|^2 = |a_1|^2\,,$$

$$\sigma_2(\pi^-p \to \pi^-p) \propto \left|\left(\sqrt{\frac{2}{3}}\left\langle\frac{1}{2},-\frac{1}{2}\right| + \sqrt{\frac{1}{3}}\left\langle\frac{3}{2},-\frac{1}{2}\right|\right)\right.$$

$$\left.\hat{H}\left(\sqrt{\frac{2}{3}}\left|\frac{1}{2},-\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{3}{2},-\frac{1}{2}\right\rangle\right)\right|^2$$

$$= \left|\frac{2}{3}a_2 + \frac{1}{3}a_1\right|^2,$$

$$\sigma_3(\pi^- p \to \pi^0 n) \propto \left| \left(\sqrt{\frac{2}{3}} \left\langle \frac{1}{2}, -\frac{1}{2} \right| + \sqrt{\frac{1}{3}} \left\langle \frac{3}{2}, -\frac{1}{2} \right| \right) \right.$$

$$\left. \hat{H} \left(\sqrt{\frac{2}{3}} \left| \frac{3}{2}, -\frac{1}{2} \right\rangle - \sqrt{\frac{1}{3}} \left| \frac{1}{2}, -\frac{1}{2} \right\rangle \right) \right|^2$$

$$= \left| -\frac{\sqrt{2}}{3} a_2 + \frac{\sqrt{2}}{3} a_1 \right|^2 ,$$

When Δ resonance takes place, $|a_1| \gg |a_2|$, and the effect of a_2 can be neglected. Hence

$$\sigma_1 \propto |a_1|^2 ,$$

$$\sigma_2 \propto \frac{1}{9} |a_1|^2 ,$$

$$\sigma_3 \propto \frac{2}{9} |a_1|^2 ,$$

and $\sigma_1 : \sigma_2 : \sigma_3 = 9 : 1 : 2$.

When N^* resonance occurs, $|a_1| \ll |a_2|$, and we have

$$\sigma_1 \approx 0 ,$$

$$\sigma_2 \propto \frac{4}{9} |a_2|^2 ,$$

$$\sigma_3 \propto \frac{2}{9} |a_2|^2 ,$$

$$\sigma_1 : \sigma_2 : \sigma_3 = 0 : 2 : 1 .$$

3027

Estimate the ratios of decay rates given below, stating clearly the selection rules ("fundamental" or phenomenological) which are operating. Also state whether each decay (regardless of the ratio) is strong, electromagnetic or weak. If at all possible, express your answer in terms of the fundamental constants G, α, θ_c, m_K, etc. Assume that the strong interactions have unit strength (i.e. , unit dimensionless coupling constant).

(a) $\dfrac{K^+ \to \pi^+\pi^0}{K_s^0 \to \pi^+\pi^-}$

(b) $\dfrac{\rho^0 \to \pi^0\pi^0}{\rho^0 \to \pi^+\pi^-}$

(c) $\dfrac{K_L^0 \to \mu^+\mu^-}{K_L^0 \to \pi^0\pi^0}$

(d) $\dfrac{K^+ \to \pi^+\pi^+e^-\nu}{K^- \to \pi^+\pi^-e^-\nu}$

(e) $\dfrac{\Omega^- \to \Sigma^-\pi^0}{\Omega^- \to \Xi^0\pi^-}$

(f) $\dfrac{\eta^0 \to \pi^+\pi^-}{\eta^0 \to \pi^+\pi^-\pi^0}$

(g) $\dfrac{\Lambda^0 \to K^-\pi^+}{\Lambda^0 \to p\pi^-}$

(h) $\dfrac{\theta^0 \to \pi^+\pi^-\pi^0}{\omega^0 \to \pi^+\pi^-\pi^0}$

(i) $\dfrac{\Sigma^- \to \Lambda^0\pi^-}{\Sigma^- \to n\pi^-}$

(j) $\dfrac{\pi^- \to e^-\nu}{K^+ \to \mu^+\nu}$

(*Princeton*)

Solution:

(a) Consider $K^+ \to \pi^+\pi^0$. For nonleptonic weak decays $\Delta I = 1/2$. As $I(K) = 1/2$, the isospin of the 2π system must be 0 or 1. The generalized Pauli's principle requires the total wave function of the 2π system to be symmetric. As the spin of K is 0, conservation of the total angular momentum requires $J(2\pi) = J(K) = 0$. Then as the spin of π is 0, $l(2\pi) = 0$. Thus the spatial and spin parts of the wave function of the 2π system are both symmetric, so the isospin wave function must also be symmetric. It follows that the isospin of the 2π system has two possible values, 0 or 2. Hence $I(\pi^+\pi^0) = 0$. However, $I_3(\pi^+\pi^0) = 1 + 0 = 1$. As the rule $I_3 \leq I$ is violated, the decay is forbidden. On the other hand, $K_s^0 \to \pi^+\pi^-$ is allowed as it satisfies the rule $\Delta I = 1/2$.

Therefore,

$$\frac{K^+ \to \pi^+\pi^-}{K_s^0 \to \pi^+\pi^-} \ll 1\,.$$

Note the ratio of the probability amplitudes for $\Delta I = 1/2, 3/2$ in K-decay, A_0 and A_2, can be deduced from

$$\frac{\Gamma(K^+ \to \pi^+\pi^0)}{\Gamma(K_s^0 \to \pi^+\pi^-)} = \frac{3}{4}\left(\frac{A_2}{A_0}\right)^2 \approx 1.5 \times 10^{-3}\,,$$

giving

$$\frac{A_2}{A_0} \approx 4.5\%\,.$$

(b) Consider the decay modes $\rho^0 \to \pi^+\pi^-$, $\pi^0\pi^0$. $\rho^0 \to \pi^+\pi^-$ is an allowed strong decay, while for $\rho^0 \to \pi^0\pi^0$, the C-parities are $C(\rho^0) = -1$, $C(\pi^0\pi^0) = 1$, and the decay is forbidden by conservation of C-parity. Hence

$$\frac{\rho^0 \to \pi^0\pi^0}{\rho^0 \to \pi^+\pi^-} \approx 0\,.$$

(c) As K_L^0 is not the eigenstate of CP, $K_L^0 \to \pi^0\pi^0$ has a nonzero branching ratio, which is approximately 9.4×10^{-4}. The decay $K_L^0 \to \mu^+\mu^-$, being a second order weak decay, has a probability even less than that of $K_L^0 \to \pi^0\pi^0$. It is actually a flavor-changing neutral weak current decay. Thus

$$1 \gg \frac{K_L^0 \to \mu^+\mu^-}{K_L^0 \to \pi^0\pi^0} \approx 0\,.$$

Experimentally, the ratio $\approx 10^{-8}/10^{-3} = 10^{-5}$.

(d) $K^+ \to \pi^+\pi^+e^-\bar{\nu}$ is a semileptonic weak decay and so ΔQ should be equal to ΔS, where ΔQ is the change of hadronic charge. As $\Delta S = 1$, $\Delta Q = -1$, it is forbidden. But as $K^- \to \pi^+\pi^-e^-\bar{\nu}$ is an allowed decay,

$$\frac{K^+ \to \pi^+\pi^+e^-\bar{\nu}}{K^- \to \pi^+\pi^-e^-\bar{\nu}} = 0\,.$$

(e) In $\Omega^- \to \Sigma^-\pi^0$, $\Delta S = 2$. Thus it is forbidden. As $\Omega^- \to \Xi^0\pi^-$ is allowed by weak interaction,

$$\frac{\Omega^- \to \Sigma^-\pi^0}{\Omega^- \to \Xi^0\pi^-} = 0\,.$$

(f) Consider $\eta^0 \to \pi^+\pi^-$. η^0 has $J^P = 0^-$ and decays electromagnetically ($\Gamma = 0.83\,\text{keV}$). As J^P of $\pi^\pm$ is 0^-, a $\pi^+\pi^-$ system can only form states

$0^+, 1^-, 2^+$. Since parity is conserved in electromagnetic decay, this decay mode is forbidden. On the other hand, $\eta^0 \to \pi^+\pi^-\pi^0$ is an electromagnetic decay with all the required conservation rules holding. Hence

$$\frac{\eta^0 \to \pi^+\pi^-}{\eta^0 \to \pi^+\pi^-\pi^0} = 0\,.$$

(g) $\Lambda^0 \to K^-\pi^+$ is a nonleptonic decay mode. As $\Delta I_3 = 1/2$, $\Delta S = 0$, it is forbidden. $\Lambda^0 \to p\pi^-$ is also a nonleptonic weak decay satisfying $|\Delta S| = 1$, $|\Delta I| = 1/2$, $|\Delta I_3| = 1/2$ and is allowed. Hence

$$\frac{\Lambda^0 \to K^-\pi^+}{\Lambda^0 \to p\pi^-} = 0\,.$$

(h) Consider $\theta^0 \to \pi^+\pi^-\pi^0$. θ^0 has strong decays ($\Gamma = 180$ MeV) and $I^G J^{PC} = 0^+2^{++}$. As $G(\pi^+\pi^-\pi^0) = (-1)^3 = -1$, $G(\theta^0) = +1$, G-parity is not conserved and the decay mode is forbidden. Consider $\omega^0 \to \pi^+\pi^-\pi^0$. As $I^G J^{PC}$ of ω^0 is 0^-1^{--}, it is allowed. Hence

$$\frac{\theta^0 \to \pi^+\pi^-\pi^0}{\omega^0 \to \pi^+\pi^-\pi^0} = 0\,.$$

(i) Consider $\Sigma^- \to \Lambda^0\pi^-$. As $\Delta S = 0$, it is forbidden. $\Sigma^- \to n\pi^-$ is an allowed nonleptonic weak decay. Hence

$$\frac{\Sigma^- \to \Lambda^0\pi^-}{\Sigma^- \to n\pi^-} = 0\,.$$

(j) $\pi^- \to e^-\bar{\nu}$ and $K^+ \to \mu^+\nu$ are both semileptonic two-body decays. For the former, $\Delta S = 0$ and the coupling constant is $G\cos\theta_c$, for the latter $\Delta S = 1$ and the coupling constant is $G\sin\theta_c$, where θ_c is the Cabbibo angle. By coupling of axial vectors we have

$$\omega'(\varphi \to l\nu) = \frac{f_\varphi^2 m_l^2 (m_\varphi^2 - m_l^2)^2}{4\pi m_\varphi^3}\,,$$

where f_φ is the coupling constant. Hence

$$\begin{aligned}
\frac{\pi^- \to e^-\bar{\nu}}{K^+ \to \mu^+\nu} &= \frac{f_\pi^2 m_e^2 (m_\pi^2 - m_e^2)^2 m_K^3}{f_K^2 m_\mu^2 (m_K^2 - m_\mu^2)^2 m_\pi^3} \\
&= \frac{m_K^3 m_e^2 (m_\pi^2 - m_e^2)^2}{m_\pi^3 m_\mu^2 (m_K^2 - m_\mu^2)^2} \cot^2\theta_c \\
&= 1.35 \times 10^{-4}\,,
\end{aligned}$$

using $\theta_c = 13.1^0$ as deduced from experiment.

3028

The Σ^* is an unstable hyperon with mass $m = 1385$ MeV and decay width $\Gamma = 35$ MeV, with a branching ratio into the channel $\Sigma^{*+} \to \pi^+\Lambda$ of 88%. It is produced in the reaction $K^-p \to \pi^-\Sigma^{*+}$, but the reaction $K^+p \to \pi^+\Sigma^{*+}$ does not occur.

(a) What is the strangeness of the Σ^*? Explain on the basis of the reactions given.

(b) Is the decay of the Σ^* strong or weak? Explain.

(c) What is the isospin of the Σ^*? Explain using the information above.

(*Wisconsin*)

Solution:

(a) As Σ^{*+} is produced in the strong interaction $K^-p \to \pi\Sigma^{*+}$, which conserves strangeness number, the strangeness number of Σ^{*+} is equal to that of K^-, namely, -1. As $S(K^+) = +1$, the reaction $K^+p \to \pi^+\Sigma^{*+}$ violates the conservation of strangeness number and is forbidden.

(b) The partial width of the decay $\Sigma^{*+} \to \Lambda\pi^+$ is

$$\Gamma_{\Lambda\pi} = 88\% \times 35 = 30.8 \text{ MeV},$$

corresponding to a lifetime

$$\tau_{\Lambda\pi} \approx \frac{\hbar}{\Gamma_{\Lambda\pi}} = \frac{6.62 \times 10^{-22}}{30.8} = 2.15 \times 10^{-23}\ s\,.$$

As its order of magnitude is typical of the strong interaction time, the decay is a strong decay.

(c) Isospin is conserved in strong interaction. The strong decay $\Sigma^{*+} \to \Lambda\pi^+$ shows that, as $I(\Lambda) = I_3(\Lambda) = 0$,

$$I(\Sigma^*) = I(\pi) = 1\,.$$

3029

A particle X has two decay modes with partial decay rates $\gamma_1(\text{sec}^{-1})$ and $\gamma_2(\text{sec}^{-1})$.

(a) What is the inherent uncertainty in the mass of X?

(b) One of the decay modes of X is the strong interaction decay

$$X \to \pi^+ + \pi^+ .$$

What can you conclude about the isotopic spin of X?

(*Wisconsin*)

Solution:

(a) The total decay rate of particle X is

$$\lambda = \gamma_1 + \gamma_2 .$$

So the mean lifetime of the particle is

$$\tau = \frac{1}{\lambda} = \frac{1}{\gamma_1 + \gamma_2} .$$

The inherent uncertainty in the mass of the particle, Γ, is given by the uncertainty principle $\Gamma\tau \sim \hbar$. Hence

$$\Gamma \sim \frac{\hbar}{\tau} = \hbar(\gamma_1 + \gamma_2) .$$

(b) As $X \to \pi^+\pi^+$ is a strong decay, isospin is conserved. π^+ has $I = 1$ and $I_3 = +1$. Thus final state has $I = 2$ and so the isospin of X is 2.

3030

Suppose that π^- has spin 0 and negative intrinsic parity. If it is captured by a deuterium nucleus from a p orbit in the reaction

$$\pi^- + d \to n + n ,$$

show that the two neutrons must be in a singlet state. The deuteron's spin-parity is 1^+.

(*Wisconsin*)

Solution:

The parity of the initial state $\pi^- d$ is

$$P_i = P(\pi^-)P(d)(-1)^l = (-1) \times (+1) \times (-1)^1 = +1 .$$

As the reaction is by strong interaction, parity is conserved, and so the parity of the final state is $+1$.

As the intrinsic parity of the neutron is $+1$, the parity of the final state nn is $P_f = (+1)^2(-1)^l = P_i = (-1)^1(-1)(+1)$, where l is the orbital momentum quantum number of the relative motion of the two neutrons in the final state. Thus $l = 0, 2, 4, \ldots$. However, the total wave function of the final state, which consists of two identical fermions, has to be exchange-antisymmetric. Now as l is even, i.e., the orbital wave function is exchange-symmetric, the spin wave function has to be exchange-antisymmetric. Hence the two neutrons must be in a singlet spin state.

3031

A negatively charged π-meson (a pseudoscalar particle: zero spin and odd parity) is initially bound in the lowest-energy Coulomb wave function around a deuteron. It is captured by the deuteron (a proton and neutron in $^3S_1$ state), which is converted into a pair of neutrons:

$$\pi^- + d \to n + n\,.$$

(a) What is the orbital angular momentum of the neutron pair?

(b) What is their total spin angular momentum?

(c) What is the probability for finding both neutron spins directed opposite the spin of the deuteron?

(d) If the deuteron's spin is initially 100% polarized in the $\mathbf{k}$ direction, what is the angular dependence of the neutron emission probability (per unit solid angle) for a neutron whose spin is opposite to that of the initial deuteron? (See Fig. 3.5) You may find some of the first few (not normalized) spherical harmonics useful:

$$\begin{aligned} Y_0^0 &= 1\,, \\ Y_1^{\pm 1} &= \mp \sin\theta e^{\pm i\phi}\,, \\ Y_1^0 &= \cos\theta\,, \\ Y_2^{\pm 1} &= \mp \sin 2\theta e^{\pm i\phi}\,. \end{aligned}$$

(*CUSPEA*)

Solution:

(a) As $J^P(d) = 1^+$, $J^P(\pi^-) = 0^-$, $J^P(n) = \frac{1}{2}^+$, angular momentum conservation demands $J = 1$, parity conservation demands $(+1)^2(-1)^L$

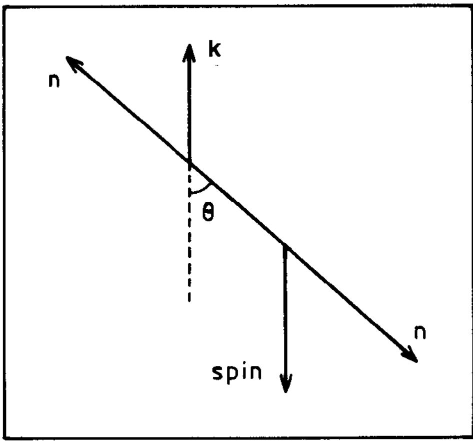

Fig. 3.5

$= (-1)(+1)(-1)^0$, or $(-1)^L = -1$ for the final state. As neutrons are fermions the total wave function of the final state is antisymmetric. Thus $(-1)^L(-1)^{S+1} = -1$, and $L+S$ is an even number. For a two-neutron system $S = 0, 1$. If $S = 0$, then $L = 0, 2, 4, \ldots$. But this would mean $(-1)^L = +1$, which is not true. If $S = 1$, the $L = 1, 3, 5, \ldots$, which satisfies $(-1)^L = -1$. Now if $L \geq 3$, then J cannot be 1. Hence the neutron pair has $L = 1$.

(b) The total spin angular momentum is $S = 1$.

(c) If the neutrons have spins opposite to the deuteron spin, $S_z = -\frac{1}{2} - \frac{1}{2} = -1$. Then $J_z = L_z + S_z = L_z - 1$. As $L = 1$, $L_z = 0, \pm 1$. In either case, $|\langle 1, L_z - 1|1, 1\rangle|^2 = 0$, i.e. the proabability for such a case is zero.

(d) The wave function for the neutron-neutron system is

$$\Psi = |1, 1\rangle = C_1 Y_1^1 \chi_{10} + C_2 Y_1^0 \chi_{11} ,$$

where C_1, C_2 are constants such that $|C_1|^2 = |C_2|^2 = 1/2$, and

$$\chi_{10} = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \qquad \chi_{11} = (\uparrow\uparrow) .$$

From the symmetry of the above wave function and the normalization condition, we get

$$\frac{dP}{d\Omega} = |C_1|^2 (Y_1^1 \chi_{10})^* (Y_1^1 \chi_{10})$$
$$= \frac{1}{2}(Y_1^1)^* Y_1^1$$
$$= \frac{3}{8\pi} \sin^2\theta \, .$$

3032

(a) The η^0-particle can be produced by s-waves in the reaction

$$\pi^- + p \to \eta^0 + n \, .$$

(Note no corresponding process $\pi^- + p \to \eta^- + p$ is observed)

(b) In the η^0 decay the following modes are observed, with the probabilities as indicated:

$$\eta^0 \to 2\gamma (38\% \text{ of total})$$
$$\to 3\pi (30\% \text{ of total})$$
$$\to 2\pi (< 0.15\% \text{ of total}) \, .$$

(c) The rest mass of the η^0 is 548.8 MeV.

Describe experiments/measurements from which the above facts (a) (b) (c) may have been ascertained. On the basis of these facts show, as precisely as possible, how the spin, isospin, and charge of the η^0 can be inferred.

(*Columbia*)

Solution:

An experiment for this purpose should consist of a π^- beam with variable momentum, a hydrogen target, and a detector system with good spatial and energy resolutions for detecting γ-rays and charged particles. The π^- momentum is varied to obtain more 2γ and 3π events. The threshold energy E_0 of the reaction is given by

$$(E_0 + m_p)^2 - P_0^2 = (m_\eta + m_n)^2 \, .$$

where P_0 is the threshold momentum of the incident π^-, or

$$E_0 = \frac{(m_\eta + m_n)^2 - m_p^2 - m_\pi^2}{2m_p}$$

$$= \frac{(0.5488 + 0.94)^2 - 0.938^2 - 0.14^2}{2 \times 0.938}$$

$$= 0.702 \text{ GeV} = 702 \text{ MeV},$$

giving

$$P_0 = \sqrt{E_0^2 - m_\pi^2} \approx 0.688 \text{ GeV}/c = 688 \text{ MeV}/c.$$

Thus η^0 can be produced only if the π^- momentum is equal to or larger than 688 MeV/c.

Suppose the center of mass of the $\pi^- p$ system moves with velocity $\beta_c c$ and let $\gamma_c = (1 - \beta_c^2)^{-\frac{1}{2}}$. Indicate quantities in the center-of-mass system (cms) by a bar. Lorentz transformation gives

$$\bar{P}_0 = \gamma_c (P_0 + \beta_c E_0).$$

As $\bar{P}_0 = \bar{P}_p = m_p \gamma_c \beta_c$, we have

$$\beta_c = \frac{P_0}{m_p + E_0} = \frac{688}{702 + 938} = 0.420,$$

$$\gamma_c = 1.10,$$

and hence

$$\bar{P}_0 = \gamma_c (P_0 - \beta_c E_0) = 433 \text{ Mev}/c.$$

The de Broglie wavelength of the incident π^- meson in cms is

$$\lambda = \frac{\hbar c}{\bar{P}_0 C} = \frac{197 \times 10^{-13}}{433} = 0.45 \times 10^{-13} \text{ cm}.$$

As the radius of proton $\approx 0.5 \times 10^{-13}$ cm, s-waves play the key role in the $\pi^- p$ interaction.

Among the final products, we can measure the invariant-mass spectrum of 2γ's. If we find an invariant mass peak at 548.8 MeV, or for 6γ events, 3 pairs of γ's with invariant mass peaking at m_π^0,or the total invariant mass of 6 γ's peaking at 548.8 MeV, we can conclude that η^0 particles have been created. One can also search for $\pi^+\pi^-\pi^0$ events. All these show the occurrence of

$$\begin{array}{l} \pi^- + p \to n + \eta^0 \\ \qquad\qquad\qquad \hookrightarrow 2\gamma \\ \qquad\qquad\qquad \hookrightarrow 3\pi^0, \pi^+\pi^-\pi^0 \end{array}$$

If the reaction $\pi^- + p \to p + \eta^-$ did occur, one would expect η^- to decay via the process

$$\eta^- \to \pi^+\pi^-\pi^- \, .$$

Experimentally no $\pi^+\pi^-\pi^-$ events have been observed.

The quantum numbers of η^0 can be deduced as follows.

Spin: As η^0 can be produced using s-waves, conservation of angular momentum requires the spin of η^0 to be either 0 or 1. However since a vector meson of spin 1 cannot decay into 2 γ's, $J(\eta^0) = 0$.

Parity: The branching ratios suggest η^0 can decay via electromagnetic interaction into 2 γ's, via strong interaction into 3 π's, but the branching ratio of 2π-decay is very small. From the 3π-decay we find

$$P(\eta^0) = P^3(\pi)(-1)^{l+l'} \, ,$$

where l and l' are respectively the orbital angular momentum of a 2π system and the relative orbital angular momentum of the third π relative to the 2π system. As $J(\eta^0) = 0$, conservation of total angular momentum requires $l' = -l$ and so

$$P(\eta^0) = (-1)^3 = -1 \, .$$

Isospin: Because η^- is not observed, η^0 forms an isospin singlet. Hence $I(\eta^0) = 0$.

Charge: Conservation of charge shows $Q(\eta^0) = 0$. In addition, from the 2γ-decay channel we can further infer that $C(\eta^0) = +1$.

To summarize, the quantum numbers of η^0 are $I(\eta^0) = 0$, $Q(\eta^0) = 0$, $J^{PC}(\eta^0) = 0^{-+}$. Like π and K mesons, η^0 is a pseudoscalar meson, and it forms an isospin singlet.

3033

A beam of K^+ or K^- mesons enters from the left a bubble chamber to which a uniform magentic field of $B \approx 12$ kGs is applied perpendicular to the observation window.

(a) Label with symbols (π^+, π^-, p, etc.) all the products of the decay of the K^+ in the bubble chamber pictures in Fig. 3.6 and give the complete reaction equation for K^+ applicable to each picture.

(b) In Fig. 3.7 the K^- particles come to rest in the bubble chamber. Label with symbols all tracks of particles associated with the K^- particle and identify any neutral particle by a dashed-line "track". Give the complete reaction equation for the K^- interaction applicable to each picture.

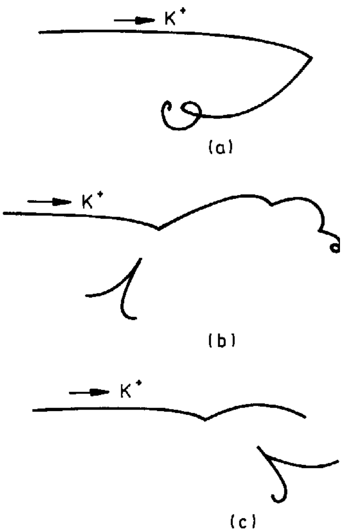

Fig. 3.6

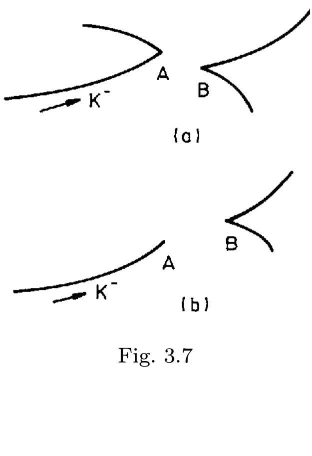

Fig. 3.7

(c) Assuming that tracks in Fig. 3.7(a) and Fig. 3.7(b) above all lie in the plane of the drawing determine the expressions for the lifetime of the neutral particle and its mass.

(*Chicago*)

Solution:

(a) The modes and branching ratios of K^+ decay are as follows:

$$\begin{aligned} K^+ \to\ & \mu^+\nu_\mu && 63.50\%\,, \\ & \pi^+\pi^0 && 21.16\%\,, \\ & \pi^+\pi^+\pi^- && 5.59\%\,, \\ & \pi^+\pi^0\pi^0 && 1.73\%\,, \\ & \mu^+\nu_\mu\pi^0 && 3.20\%\,, \\ & e^+\nu_e\pi^0 && 4.82\%\,. \end{aligned}$$

The products from decays of K^+ consist of three kinds of positively charged particle π^+, μ^+, e^+, one kind of negatively charged particle π^-, plus some neutral particles π^0, ν_μ, ν_e. Where π^+ is produced, there should be four linearly connected tracks of positively charged particles arising from $K^+ \to \pi^+ \to \mu^+ \to e^+$. Where μ^+ or e^+ is produced there should be three or two linearly connected tracks of positively charged particles in the picture arising from $K^+ \to \mu^+ \to e^+$ or $K^+ \to e^+$, respectively. Where π^0 is produced, because of the decay $\pi^0 \to 2\gamma (\tau \approx 10^{-16}\ s)$ and the subsequent electron-positron pair production of the γ-rays, we can see the e^+, e^- tracks starting out as a fork.

Analysing Fig. 3.6(a) we have Fig. 3.8. The decay of K^+ could produce either $\mu^+\nu$ or $\mu^+\gamma\pi^0$. As the probability is much larger for the former we assume that it was what actually happened. Then the sequence of events is as follows:

$$\begin{aligned} K^+ \to\ & \mu^+ + \nu_\mu \\ & \downarrow \\ & e^+\bar{\nu}_\mu\nu_e \\ & \hookrightarrow e^+ + e^- \to \gamma_1 + \gamma_2 \end{aligned}$$

Note the sudden termination of the e^+ track, which is due to the annihilation of the positron with an electron of the chamber producing two oppositely directed γ-rays.

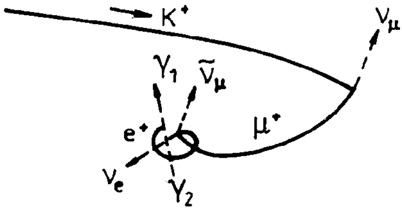

Fig. 3.8

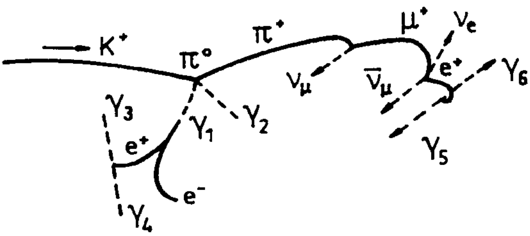

Fig. 3.9

Analysing Fig. 3.6(b) we have Fig. 3.9. The sequence of events is as follows:

$$
\begin{array}{l}
K^+ \to \pi^0 + \pi^+ \\
\qquad\qquad\quad \pi^+ \longrightarrow \mu^+ + \nu_\mu \\
\qquad\qquad\qquad\qquad \mu^+ \longrightarrow \underbrace{e^+ + \nu_e + \bar{\nu}_\mu\,, \quad e^+}_{} + e^- \to \gamma_5 + \gamma_6 \\
\qquad\quad \pi^0 \longrightarrow \gamma_1 + \gamma_2 \\
\qquad\qquad\qquad\qquad \gamma_1 \longrightarrow \underbrace{e^+ + e^-, \quad e^+}_{} + e^- \to \gamma_3 + \gamma_4
\end{array}
$$

or

$$
\begin{array}{l}
K^+ \to \pi^0 + \pi^0 + \pi^+ \\
\qquad\qquad\qquad\quad \pi^+ \longrightarrow \mu^+ + \nu_\mu \\
\qquad\qquad\quad \pi^0 \longrightarrow \gamma + \gamma \\
\qquad \pi^0 \longrightarrow \gamma + \gamma
\end{array}
$$

with the subsequent μ^+ decay and pair production of the γ-rays.

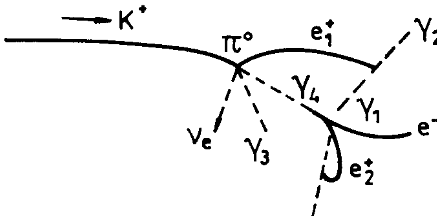

Fig. 3.10

Note that because of its short lifetime, π^0 decays almost immediately as it is produced. From Fig. 3.6(c) we have Fig. 3.10.

The sequence of events is as follows:

$$K^+ \to \nu_e + e_1^+ + \pi^0$$
$$\pi^0 \to \gamma_3 + \gamma_4$$
$$\gamma_4 \to e_2^+ + e^-, \quad e_2^+ + e^- \to \gamma_5 + \gamma_6$$
$$e_1^+ + e^- \to \gamma_1 + \gamma_2$$

Figure 3.7(a) is interpreted as follows:

$$K^- + n \to \Lambda^0 + \pi^-$$
$$\Lambda^0 \to p + \pi^-$$

The tracks are labelled in Fig. 3.11 below:

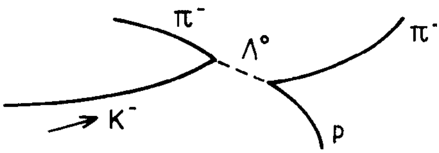

Fig. 3.11

Figure 3.7(b), is interpreted as

$$K^- + p \to \Lambda^0 + \pi^0$$
$$\pi^0 \to \gamma + \gamma$$
$$\Lambda^0 \to p + \pi^-$$

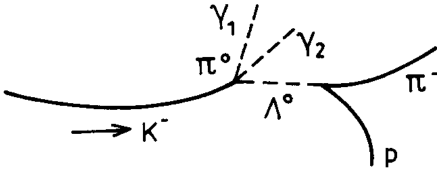

Fig. 3.12

Figure 3.12 shows the tracks with labels. Note that Λ^0 has a lifetime $\sim 10^{-10}$ s, sufficient to travel an appreciable distance in the chamber.

(c) To determine the mass and lifetime of the neutral particle Λ^0, we measure the length of the track of the neutral particle and the angles it makes with the tracks of p and π^-, θ_p and θ_π, and the radii of curvature, R_p and R_π, of the tracks of p and π^-. Force considerations give the momentum of a particle of charge e moving perpendicular to a magnetic field of flux density B as

$$P = eBR\,,$$

where R is the radius of curvature of its track. With e in C, B in T, R in m, we have

$$\begin{aligned} P &= eBRc\left(\frac{\text{joule}}{c}\right) = \left(\frac{1.6\times 10^{-19}\times 3\times 10^{8}}{1.6\times 10^{-19}\times 10^{9}}\right) BR\left(\frac{\text{GeV}}{c}\right) \\ &= 0.3BR\left(\frac{\text{GeV}}{c}\right). \end{aligned}$$

The momenta P_p, P_π of p and π^- from Λ^0 decay can then be determined from the radii of curvature of their tracks.

As $(\Sigma E)^2 - (\Sigma P)^2$ is invariant, we have

$$m_\Lambda^2 = (E_p + E_\pi)^2 - (\mathbf{P}_p + \mathbf{P}_\pi)^2\,,$$

where m_Λ is the rest mass of Λ^0.

As

$$E_p^2 = P_p^2 + m_p^2\,,$$

$$E_\pi^2 = P_\pi^2 + m_\pi^2\,,$$

we have

$$m_\Lambda = \sqrt{m_p^2 + m_\pi^2 + 2E_pE_\pi - 2P_\pi P_p \cos(\theta_p + \theta_\pi)}\,.$$

The energy and momentum of the Λ^0 particle are given by

$$E_\Lambda = E_p + E_\pi\,,$$

$$P_\Lambda = P_p \cos\theta_p + P_\pi \cos\theta_\pi\,.$$

If the path length of Λ is l, its laboratory lifetime is $\tau = \frac{l}{\beta c}$, and its proper lifetime is

$$\tau_0 = \frac{l}{\gamma\beta\tau} = \frac{l m_\Lambda}{P_\Lambda} = l(P_p\cos\theta_p + P_\pi\cos\theta_\pi)^{-1}$$
$$\times\,[m_p^2 + m_\pi^2 + 2E_pE_\pi - 2P_\pi P_p\cos(\theta_p + \theta_\pi)]^{1/2}\,.$$

3034

The invariant-mass spectrum of Λ^0 and π^+ in the reaction $K^- + p \to \Lambda^0 + \pi^+ + \pi^-$ shows a peak at 1385 MeV with a full width of 50 MeV. It is called Y_1^*. The $\Lambda^0\pi^-$ invariant-mass spectrum from the same reaction (but different events) shows a similar peak.

(a) From these data determine the strangeness, hypercharge and isospin of Y_1^*.

(b) Evidence indicates that the product $\Lambda^0+\pi^+$ from a Y_1^* is in a relative p state of angular momentum. What spin assignments J are possible for the Y_1^*? What is its intrinsic parity? (Hint: the intrinsic parity of Λ^0 is $+$ and that of π^+ is $-$)

(c) What (if any) other strong decay modes do you expect for Y_1^*?

(*Columbia*)

Solution:

(a) The resonance state Y_1^* with full width $\Gamma = 50$ MeV has a lifetime $\tau = \hbar/\Gamma = 6.6 \times 10^{-22}/50 = 1.3 \times 10^{-23}$ s. The time scale means that Y_1^* decays via strong interaction, and so the strangeness number S, hypercharge

Y, isospin I and its z-component I_3 are conserved. Hence

$$\begin{aligned} S(Y_1^*) &= S(\Lambda^0) + S(\pi^+) = -1 + 0 = -1\,, \\ Y(Y_1^*) &= Y(\Lambda^0) + Y(\pi^+) = 0 + 0 = 0\,, \\ I(Y_1^*) &= I(\Lambda^0) + I(\pi^+) = 0 + 1 = 1\,, \\ I_3(Y_1^*) &= I_3(\Lambda^0) + I_3(\pi^+) = 0 + 1 = 1\,. \end{aligned}$$

Y_1^* is actually an isospin triplet, its three states being Y_1^{*+}, Y_1^{*0}, and Y_1^{*-}. The resonance peak of $\Lambda^0\pi^-$ corresponds to Y_1^{*-}.

(b) Λ^0 has spin $J_\Lambda = 1/2$, π^+ has spin $J_\pi = 0$. The relative motion is a p state, so $l = 1$. Then $J_{Y_1^*} = \mathbf{1/2+1}$, the possible values being 1/2 and 3/2. The intrinsic parity of Y_1^* is $P(Y_1^*) = P(\pi)P(\Lambda)(-1)^l = (-1)(1)(-1) = 1$.

(c) Another possible strong decay channel is

$$Y_1^* \to \Sigma\pi\,.$$

As the intrinsic parity of Σ is $(+1)$, that of π, (-1), the particles emitted are in a relative p state

3035

Consider the hyperon nonleptonic weak decays:

$$\Lambda^0 \to p\pi^-$$

$$\Lambda^0 \to n\pi^0$$

$$\Sigma^- \to n\pi^-$$

$$\Sigma^+ \to p\pi^0$$

$$\Sigma^+ \to n\pi^+$$

$$\Xi^- \to \Lambda^0\pi^-$$

$$\Xi^0 \to \Lambda^0\pi^0$$

On assuming that these $\Delta S = 1$ weak decays satisfy the $\Delta I = 1/2$ rule, use relevant tables to find the values of x, y, z, as defined below:

$$x = \frac{A(\Lambda^0 \to p\pi^-)}{A(\Lambda^0 \to n\pi^0)} \,,$$

$$y = \frac{A(\Sigma^+ \to \pi^+ n) - A(\Sigma^- \to \pi^- n)}{A(\Sigma^+ \to \pi^0 p)} \,,$$

$$z = \frac{A(\Xi^0 \to \Lambda^0\pi^0)}{A(\Xi^- \to \Lambda^0\pi^-)} \,,$$

where A denotes the transition amplitude.

(*Columbia*)

Solution:

As nonleptonic decays of hyperon require $\Delta I = 1/2$, we can introduce an "imaginary particle" a having $I = \frac{1}{2}$, $I_3 = -\frac{1}{2}$, and combine the hyperon with a in isospin compling:

$$|\Lambda^0, a\rangle = |0,0\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle,$$

$$|\Sigma^-, a\rangle = |1,-1\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle,$$

$$|\Sigma^+, a\rangle = |1,1\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle,$$

$$|\Xi^0, a\rangle = \left|\frac{1}{2}, \frac{1}{2}\right\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{2}}|1,0\rangle + \sqrt{\frac{1}{2}}|0,0\rangle \,,$$

$$|\Xi^-, a\rangle = \left|\frac{1}{2}, -\frac{1}{2}\right\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = |1,-1\rangle \,.$$

Similarly, we find the isospin wave functions for the final states:

$$|\pi^-, p\rangle = |1,-1\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{2}{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle,$$

$$|\pi^0, p\rangle = |1,0\rangle\left|\frac{1}{2}, \frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}, \frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle,$$

$$
\begin{aligned}
|\pi^+, n\rangle &= |1,1\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{1}{3}}\left|\frac{3}{2}, \frac{1}{2}\right\rangle + \sqrt{\frac{2}{3}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle, \\
|\pi^0, n\rangle &= |1,0\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \sqrt{\frac{2}{3}}\left|\frac{3}{2}, -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle, \\
|\pi^-, n\rangle &= |1,-1\rangle\left|\frac{1}{2}, -\frac{1}{2}\right\rangle = \left|\frac{3}{2}, -\frac{3}{2}\right\rangle, \\
|\Lambda^0, \pi^0\rangle &= |0,0\rangle|1,0\rangle = |1,0\rangle\,, \\
|\Lambda^0, \pi^-\rangle &= |0,0\rangle|1,-1\rangle = |1,-1\rangle\,.
\end{aligned}
$$

The coefficients have been obtained from Clebsch–Gordan tables. The transition amplitudes are thus

$$
\begin{aligned}
A_1(\Lambda^0 \to n\pi^0) &= \sqrt{\frac{1}{3}}M_{1/2} \\
A_2(\Lambda^0 \to p\pi^-) &= -\sqrt{\frac{2}{3}}M_{1/2}\,,
\end{aligned}
$$

with

$$M_{1/2} = \left\langle\frac{1}{2}\middle|H_w\middle|\frac{1}{2}\right\rangle.$$

Hence

$$x = \frac{A_2}{A_1} = -\sqrt{2}\,.$$

Similarly,

$$
\begin{aligned}
A_3(\Sigma^- \to \pi^- n) &= M_{3/2}\,, \\
A_4(\Sigma^+ \to \pi^0 p) &= \sqrt{\frac{1}{3}}\sqrt{\frac{2}{3}}M_{3/2} - \sqrt{\frac{2}{3}}\sqrt{\frac{1}{3}}M_{1/2} = \frac{\sqrt{2}}{3}(M_{3/2} - M_{1/2})\,, \\
A_5(\Sigma^+ \to \pi^+ n) &= \sqrt{\frac{1}{3}}\sqrt{\frac{1}{3}}M_{3/2} + \sqrt{\frac{2}{3}}\sqrt{\frac{2}{3}}M_{1/2} = \frac{1}{3}(M_{3/2} + 2M_{1/2})\,,
\end{aligned}
$$

with

$$M_{3/2} = \left\langle\frac{3}{2}\middle|H_\omega\middle|\frac{3}{2}\right\rangle.$$

Hence

$$y = \frac{A_5 - A_3}{A_4} = \frac{M_{3/2} + 2M_{1/2} - 3M_{3/2}}{\sqrt{2}(M_{3/2} - M_{1/2})} = -\sqrt{2}\,.$$

Also,

$$A_6(\Xi^0 \to \Lambda^0\pi^0) = \sqrt{\frac{1}{2}}M_1\,,$$

$$A_7(\Xi^- \to \Lambda^0\pi^-) = M_1$$

with

$$M_1 = \langle 1|H_\omega|1\rangle\,.$$

Hence

$$z = \frac{A_6}{A_7} = \frac{1}{\sqrt{2}}\,.$$

3036

(a) The principle of detailed balance rests on the validity of time reversal invariance and serves to relate the cross section for a given reaction $a+b \to c+d$ to the cross section for the inverse reaction $c+d \to a+b$. Let $\sigma_I(W)$ be cross section for

$$\gamma + p \to \pi^+ + n$$

at total center-of-mass energy W, where one integrates over scattering angle, sums over final spins, and averages over initial spins. Let $\sigma_{II}(W)$ be the similarly defined cross section, at the same center-of-mass energy, for

$$\pi^+ + n \to \gamma + p\,.$$

Let μ be the pion mass, m the nucleon mass (neglect the small difference between the n and p masses). Given $\sigma_I(W)$, what does detailed balance predict for $\sigma_{II}(W)$?

(b) For reaction II, what is the threshold value W_{thresh} and how does $\sigma_{II}(W)$ vary with W just above threshold?

(*Princeton*)

Solution:

(a) For simplicity denote the state (a,b) by α and the state (c,d) by β. Let $\sigma_{\alpha\beta}$ be the cross section of the process

$$a + b \to c + d$$

and $\sigma_{\beta\alpha}$ be the cross section of the inverse process

$$c + d \to a + b\,.$$

If T invariance holds true, then when the forward and inverse reactions have the same energy W in the center-of-mass frame, $\sigma_{\alpha\beta}$ and $\sigma_{\beta\alpha}$ are related by

$$\frac{\sigma_{\alpha\beta}}{\sigma_{\beta\alpha}} = \frac{P_\beta^2(2I_c+1)(2I_d+1)}{P_\alpha^2(2I_a+1)(2I_b+1)}\,,$$

which is the principle of detailed balance. Here P_α is the relative momentum of the incident channel of the reaction $a + b \to c + d$, P_β is the relative momentum of the incident channel of the inverse reaction, I_a, I_b, I_c, I_d are respectively the spins of a, b, c, d.

For the reaction $\gamma + p \to \pi^+ + n$, in the center-of-mass frame of the incident channel let the momentum of the γ be P_γ, the energy of the proton be E_p. Then $W = E_\gamma + E_p$. As the γ has zero rest mass,

$$E_\gamma^2 - P_\gamma^2 = 0\,,$$

or

$$(W - E_p)^2 - P_\gamma^2 = 0\,.$$

With $P_\gamma = P_p$, $E_p^2 - P_p^2 = m^2$,

$$E_p = \frac{W^2 + m^2}{2W}\,.$$

Hence the relative momentum is

$$P_\alpha^2 = P_\gamma^2 = E_p^2 - m^2 = \frac{W^2 - m^2}{2W}\,.$$

For the inverse reaction $\pi^+ + n \to \gamma + p$, in the center-of-mass frame let the energy of π^+ be E_π, its momentum be P_π, and the energy of the neutron be E_n, then as $W = E_\pi + E_n$,

$$(W - E_n)^2 - E_\pi^2 = 0\,.$$

With $P_\pi = P_n$, $E_n^2 = P_\pi^2 + m^2$, $E_\pi^2 = P_\pi^2 + \mu^2$, we have

$$E_n = \frac{W^2 + m^2 - \mu^2}{2W},$$

and hence

$$P_\beta^2 = P_\pi^2 = E_n^2 - m^2 = \frac{(W^2 + m^2 - \mu^2)^2 - 4W^2 m^2}{4W^2}.$$

We have $I_\gamma = 1$, $I_p = 1/2$, $I_n = 1/2$, $I_\pi = 0$. However as photon has only left and right circular polarizations, $2I_\gamma + 1$ should be replaced by 2. Hence

$$\frac{\sigma_I(W)}{\sigma_{II}(W)} = \frac{P_\beta^2 (2I_\pi + 1)(2I_n + 1)}{P_\alpha^2 (2I_\gamma + 1)(2I_p + 1)} = \frac{P_\beta^2}{2P_\alpha^2},$$

or

$$\sigma_{II}(W) = \frac{(W^2 - m^2)^2}{(W^2 + m^2 - \mu^2)^2 - 4W^2 m^2} \sigma_I(W).$$

(b) At threshold all the final particles are produced at rest in the center-of-mass frame. The energy of the center of mass is $W^{th*} = m + \mu$. In the laboratory let the energy of the photon be E_γ. As the proton is at rest, at the threshold

$$(E_\gamma + m)^2 - P_\gamma^2 = (m + \mu)^2,$$

or, since $E_\gamma = P_\gamma$,

$$E_\gamma^{th} = \mu \left(1 + \frac{\mu}{2m}\right) = 150 \text{ MeV}.$$

When $E_\gamma > E_\gamma^{th}$, $\sigma(\gamma + p \to \pi^+ + n)$ increases rapidly with increasing E_γ. When $E_\gamma = 340$ MeV, a wide resonance peak appears, corresponding to an invariant mass

$$E^* = \sqrt{(E_\gamma + m_p)^2 - P_\gamma^2} = \sqrt{2m_p E_\gamma + m_p^2} = 1232 \text{ MeV}.$$

It is called the Δ particle. The width $\Gamma = 115$ MeV and $\sigma \approx 280$ μb at the peak.

3037

The following questions require rough, qualitative, or magnitude answers.

(a) How large is the cross section for $e^+e^- \to \mu^+\mu^-$ at a center-of-mass energy of 20 GeV? How does it depend on energy?

(b) How large is the neutrino-nucleon total cross section for incident neutrinos of 100 GeV (in the nucleon rest frame)? How does it depend on energy? At what energy is this energy dependence expected to change, according to the Weinberg–Salam theory?

(c) How long is the lifetime of the muon? Of the tau lepton? If a new lepton is discovered ten times heavier than tau, how long-lived is it expected to be, assuming it decays by the same mechanism as the muon and tau?

(d) How large is the nucleon-nucleon total cross section at accelerator energies?

(e) In pion-nucleon elastic scattering, a large peak is observed in the forward direction (scattering through small angles). A smaller but quite distinct peak is observed in the backward direction (scattering through approximately 180° in the center-of-mass frame). Can you explain the backward peak? A similar backward peak is observed in K^+p elastic scattering; but in K^-p scattering it is absent. Can you explain this?

(*Princeton*)

Solution:

(a) The energy dependence of the cross section for $e^+e^- \to \mu^+\mu^-$ can be estimated by the following method. At high energies $s^{\frac{1}{2}} \gg m_e, m_\mu$, where $s = E_{cm}^2$, and we can take $m_e \approx m_\mu \approx 0$. As there are two vertexes in the lowest order electromagnetic interaction, we have

$$\sigma = f(s)\alpha^2 \, .$$

where α is the fine structure constant $\frac{e^2}{\hbar c} = \frac{1}{137}$. Dimensionally $\sigma = [M]^{-2}$, $s = [M]^2$, $\alpha = [0]$, and so

$$f(s) \approx \frac{1}{s} \, ,$$

or

$$\sigma \approx \frac{\alpha^2}{s} \, .$$

A calculation using quantum electrodynamics without taking account of radiation correction gives

$$\sigma \approx \frac{4\pi\alpha^2}{3s} \, .$$

At $E_{cm} = 20$ GeV,

$$\sigma = \frac{4\pi\alpha^2}{3 \times 20^2} = 5.6 \times 10^{-7}\ \text{GeV}^{-2} = 2.2 \times 10^{-34}\ \text{cm}^2 = 220\ \text{pb} \, ,$$

as 1 $\text{MeV}^{-1} = 197 \times 10^{-13}$ cm.

(b) We can estimate the neutrino-nucleon total cross section in a similar manner. In the high energy range $s^{\frac{1}{2}} \gg m_p$, ν and p react by weak interaction, and

$$\sigma \approx G_F^2 f(s) \,.$$

Again using dimensional analysis, we have $G_F = [M]^{-2}$, $s = [M]^2$, $\sigma = [M]^{-2}$, and so $f(s) = [M]^2$, or

$$f(s) \approx s \,,$$

i.e.,

$$\sigma \approx G_F^2 s \,.$$

Let the energy of the neutrino in the neutron's rest frame be E_ν. Then

$$s = (E_\nu + m_p)^2 - p_\nu^2 = m_p^2 + 2m_p E_\nu \approx 2m_p E_\nu \,,$$

or

$$\sigma \approx G_F^2 s \approx G_F^2 m_p E_\nu \,.$$

For weak interaction (**Problem 3001**)

$$G_F m_p^2 = 10^{-5} \,.$$

With $m_p \approx 1$ GeV, at $E_\nu = 100$ GeV.

$$\begin{aligned}\sigma &\approx 10^{-10} E_\nu \text{ GeV}^{-2} \\ &= 10^{-10} \times 10^2 \times 10^{-6} \text{ MeV}^{-2} = 10^{-14} \times (197 \times 10^{-13})^2 \text{ cm}^2 \\ &= 4 \times 10^{-36} \text{ cm}^2 \,.\end{aligned}$$

Experimentally, $\sigma \approx 0.6 \times 10^{-38}$ cm^2. According to the Weinberg–Salam theory, σ changes greatly in the neighborhood of $s \approx m_W^2$, where m_W is the mass of the intermediate vector boson W, 82 GeV.

(c) μ has lifetime $\tau_\mu \approx 2.2 \times 10^{-6}$ s and τ has lifetime $\tau_\tau \approx 2.86 \times 10^{-13}$ s.

Label the new lepton by H. Then $m_H = 10 m_\tau$. On assuming that it decays by the same mechanism as muon and tau, its lifetime would be

$$\tau_H = \left(\frac{m_\tau}{m_H}\right)^5 \tau_\tau \approx 10^{-5} \tau_\tau = 2.86 \times 10^{-18} \ s \,.$$

(d) Nucleons interact by strong interaction. In the energy range of presentday accelerators the interaction cross section between nucleons is

$$\sigma_{NN} \approx \pi R_N^2 ,$$

R_N being the radius of the nucleon. With $R_N \approx 10^{-13}$ cm,

$$\sigma_{NN} \approx 3 \times 10^{-26} \text{ cm}^2 = 30 \text{ mb} .$$

Experimentally, $\sigma_{pp} \approx 30 \sim 50$ mb for $E_p = 2 \sim 10 \times 10^3$ GeV,

$$\sigma_{np} \approx 30 \sim 50 \text{ mb for } E_p = 5 \sim 10 \times 10^2 \text{ GeV} .$$

(e) Analogous to the physical picture of electromagnetic interaction, the interaction between hadrons can be considered as proceeding by exchanging virtual hadrons. Any hadron can be the exchanged particle and can be created by other hadrons, so all hadrons are equal. It is generally accepted that strong interaction arises from the exchange of a single particle, the effect of multiparticle exchange being considered negligible. This is the single-particle exchange model.

Figure 3.13(a) shows a t channel, where $t = -(P_{\pi^+} - P_{\pi^{+\prime}})^2$ is the square of the 4-momentum transfer of π^+ with respect to $\pi^{+\prime}$. Figure 3.13(b) shows

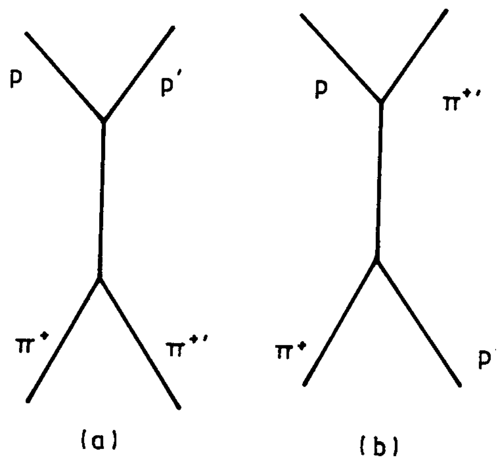

Fig. 3.13

a u channel, where $u = -(P_{\pi^+} - P_{p'})^2$ is the square of the 4-momentum transfer of π^+ with respect to p'. Let θ be the angle of the incident π^+ with respect to the emergent π^+. When $\theta = 0, |t|$ is very small; when $\theta = 180^0, |u|$ is very small. The former corresponds to the π^+ being scattered forwards and the latter corresponds to the π^+ being scattered backwards. As quantum numbers are conserved at each vertex, for the t channel the virtual exchange particle is a meson, for the u channel it is a baryon. This means that there is a backward peak for baryon-exchange scattering. Generally speaking, the amplitude for meson exchange is larger. Hence the forward peak is larger. For example, in $\pi^+ p$ scattering there is a u channel for exchanging n, and so there is a backward peak. In $K^+ p$ scattering, a virtual baryon $(S = -1, B = 1)$ or Λ^0 is exchanged. But in $K^- p$ scattering, if there is a baryon exchanged, it must have $S = 1, B = 1$. Since there is no such a baryon, $K^- p$ scattering does not have a backward peak.

2. WEAK AND ELECTROWEAK INTERACTIONS, GRAND UNIFICATION THEORIES(3038–3071)

3038

Consider the leptonic decays:

$$\mu^+ \to e^+ \nu \bar{\nu} \quad \text{and} \quad \tau^+ \to e^+ \nu \bar{\nu}$$

which are both believed to proceed via the same interaction.

(a) If the μ^+ mean life is 2.2×10^{-6} s, estimate the τ^+ mean life given that the experimental branching ratio for $\tau^+ \to e^+ \nu \bar{\nu}$ is 16%

Note that:

$$m_\mu = 106 \text{ MeV}/c^2\,,$$

$$m_\tau = 1784 \text{ MeV}/c^2\,,$$

$$m_e = 0.5 \text{ MeV}/c^2\,,$$

$$m_\nu = 0 \text{ MeV}/c^2\,,$$

(b) If the τ^+ is produced in a colliding beam accelerator (like PEP), $e^+e^- \to \tau^+\tau^-$ at $E_{em} = 29$ GeV (e^+ and e^- have equal and opposite momenta), find the mean distance (in the laboratory) the τ^+ will travel before decay.

(*UC, Berkeley*)

Solution:

(a) The theory of weak interaction gives the decay probabilities per unit time as

$$\lambda_\mu = \tau_\mu^{-1} = \frac{G_\mu^2 m_\mu^5}{192\pi^3}, \qquad \lambda_\tau = \frac{G_\tau^2 m_\tau^5}{192\pi^3}.$$

As the same weak interaction constant applies, $G_\mu = G_\tau$ and

$$\lambda_\tau/\lambda_\mu = m_\tau^5/m_\mu^5.$$

If λ is the total decay probability per unit time of τ^+, the branching ratio is $R = \lambda_\tau(\tau^+ \to e^+\nu\bar{\nu})/\lambda$.

Hence $\tau = \lambda^{-1} = R/\lambda_\tau(\tau^+ \to e^+\nu\bar{\nu}) = R\left(\frac{m_\mu}{m_\tau}\right)^5 \tau_\mu = 16\% \times \left(\frac{106}{1784}\right)^5 \times 2.2 \times 10^{-6} = 2.6 \times 10^{-13}\ s$.

(b) In the center-of-mass system, τ^+ and τ^- have the same energy. Thus

$$E_\tau = E_{\rm cm}/2 = 14.5 \text{ GeV}.$$

As the collision is between two particles of equal and opposite momenta, the center-of-mass frame coincides with the laboratory frame. Hence the laboratory Lorentz factor of τ is

$$\gamma = E_\tau/m_\tau = 14.5 \times 10^3/1784 = 8.13,$$

giving

$$\beta = \sqrt{1-\gamma^{-2}} = \sqrt{1-8.13^{-2}} = 0.992.$$

Hence the mean flight length in the laboratory is

$$L = \beta c \gamma \tau = 0.992 \times 3 \times 10^{10} \times 8.13 \times 2.6 \times 10^{-13} = 6.29 \times 10^{-2} \text{ cm}.$$

3039

Assume that the same basic weak interaction is responsible for the beta decay processes $n \to pe^-\bar{\nu}$ and $\Sigma^- \to \Lambda e^-\bar{\nu}$, and that the matrix elements describing these decays are the same. Estimate the decay rate of the process $\Sigma^- \to \Lambda e^- \bar{\nu}$ given the lifetime of a free neutron is about 10^3 seconds.

Given:

$$m_n = 939.57 \text{ MeV}/c^2\,, \qquad m_\Sigma = 1197.35 \text{ MeV}/c^2\,,$$

$$m_p = 938.28 \text{ MeV}/c^2\,, \qquad m_\Lambda = 1116.058 \text{ MeV}/c^2\,,$$

$$m_e = 0.51 \text{ MeV}/c^2\,, \qquad m_\nu = 0\,.$$

(*UC, Berkeley*)

Solution:

β-decay theory gives the transition probability per unit time as $W = 2\pi G^2 |M|^2 dN/dE_0$ and the total decay rate as $\lambda \propto E_0^5$, where E_0 is the maximum energy of the decay neutrino. For two decay processes of the same transition matrix element and the same coupling constant we have

$$\frac{\lambda_1}{\lambda_2} = \left(\frac{E_{01}}{E_{02}}\right)^5 .$$

Hence

$$\begin{aligned}
\lambda(\Sigma^- \to \Lambda e \bar{\nu}) &= \left[\frac{E_0(\Sigma^- \to \Lambda e^- \bar{\nu})}{E_0(n \to p e^- \bar{\nu})}\right]^5 \lambda_n \\
&= \left(\frac{m_\Sigma - m_\Lambda - m_e}{m_n - m_p - m_e}\right)^5 \frac{1}{\tau_n} \\
&= \left(\frac{1197.35 - 1116.058 - 0.51}{939.57 - 938.28 - 0.51}\right)^5 \times 10^{-3} \\
&= 1.19 \times 10^7 \ s^{-1} .
\end{aligned}$$

3040

Although the weak interaction coupling is thought to be universal, different weak processes occur at vastly different rates for kinematics reasons.

(a) Assume a universal V-A interaction, compute (or estimate) the ratio of rates:

$$\gamma = \frac{\Gamma(\pi^- \to \mu^- \bar{\nu})}{\Gamma(\pi^- \to e^- \bar{\nu})} .$$

Be as quantitative as you can.

(b) How would this ratio change (if the universal weak interaction coupling were scalar? Pseudoscalar?

(c) What would you expect (with V-A) for

$$\gamma' = \frac{\Gamma(\Lambda \to p\mu^- \bar{\nu})}{\Gamma(\Lambda \to pe^- \bar{\nu})} .$$

Here a qualitative answer will do.

Data:

$$J^P(\pi^-) = 0^-; \qquad M_\Lambda = 1190 \text{ MeV}/c^2;$$

$$M_\mu = 105 \text{ MeV}/c^2; \qquad M_e = 0.5 \text{ MeV}/c^2; \qquad M_p = 938 \text{ MeV}/c^2 .$$

(*Princeton*)

Solution:

(a) The weak interaction reaction rate is given by

$$\Gamma = 2\pi G^2 |M|^2 \frac{dN}{dE_0} ,$$

where $\frac{dN}{dE_0}$ is the number of the final states per unit energy interval, M is the transition matrix element, G is the weak interaction coupling constant.

Consider the two decay modes of π^-:

$$\pi^- \to \mu^- \bar{\nu}_\mu , \qquad \pi^- \to e^- \bar{\nu} .$$

Each can be considered as the interaction of four fermions through an intermediate nucleon-antinucleon state as shown in Fig. 3.14:

$$\pi^- \xrightarrow{\text{Strong-interaction}} \bar{p} + n \xrightarrow{\text{Weak-interaction}} e^- + \bar{\nu}_e \text{ or } \mu^- + \bar{\nu}_\mu .$$

From a consideration of parities and angular momenta, and basing on the V–A theory, we can take the coupling to be of the axial vector (A) type.

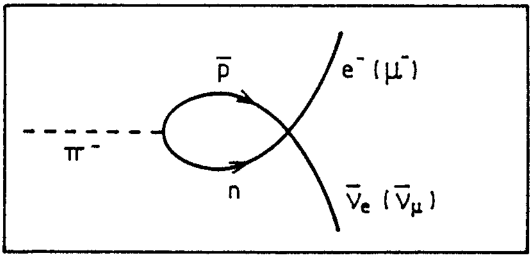

Fig. 3.14

For A coupling, $M^2 \approx 1 - \beta$, where β is the velocity of the charged lepton. The phase space factor is

$$\frac{dN}{dE_0} = Cp^2 \frac{dp}{dE_0} \,,$$

where C is a constant, p is the momentum of the charged lepton in the rest frame of the pion. The total energy of the system is

$$E_0 = m_\pi = p + \sqrt{p^2 + m^2} \,,$$

where m is the rest mass of the charged lepton, and the neutrino is assumed to have zero rest mass. Differentiating we have

$$\frac{dp}{dE_0} = \frac{E_0 - p}{E_0} \,.$$

From

$$m_\pi = p + \sqrt{p^2 + m^2}$$

we have

$$p = \frac{m_\pi^2 - m^2}{2m_\pi} \,.$$

Combining the above gives

$$\frac{dp}{dE_0} = \frac{m_\pi^2 + m^2}{2m_\pi^2} \,.$$

We also have

$$\beta = \frac{p}{\sqrt{p^2 + m^2}} = \frac{p}{m_\pi - p} \,,$$

and so

$$1-\beta=\frac{2m^2}{m_\pi^2+m^2}\,.$$

Thus the decay rate is proportional to

$$(1-\beta)p^2\frac{dp}{dE_0}=\frac{1}{4}\left(\frac{m}{m_\pi}\right)^2\left(\frac{m_\pi^2-m^2}{m_\pi}\right)^2\,.$$

Hence the ratio is

$$\gamma=\frac{\Gamma(\pi^-\to\mu^-\bar{\nu}_\mu)}{\Gamma(\pi^-\to e^-\bar{\nu}_e)}=\frac{m_\mu^2(m_\pi^2-m_\mu^2)^2}{m_e^2(m_\pi^2-m_e^2)^2}=8.13\times 10^3\,.$$

(b) For scalar coupling, $M^2\approx 1-\beta$ also and the ratio R would not change.

For pseudoscalar coupling, $M^2\approx 1+\beta$, and the decay rate would be proportional to

$$(1+\beta)p^2\frac{dp}{dE}=\frac{1}{4}\left(\frac{m_\pi^2-m^2}{m_\pi}\right)^2\,.$$

Then

$$\gamma=\frac{\Gamma(\pi^-\to\mu^-\bar{\nu}_\mu)}{\Gamma(\pi^-\to e^-\bar{\nu}_e)}=\frac{(m_\pi^2-m_\mu^2)^2}{(m_\pi^2-m_e^2)^2}=0.18\,.$$

These may be compared with the experimental result

$$\gamma^{\exp}=8.1\times 10^3\,.$$

(c) For the semileptonic decay of Λ^0 the ratio

$$\gamma'=\frac{\Gamma(\Lambda\to p\mu^-\bar{\nu}_\mu)}{\Gamma(\Lambda\to pe^-\bar{\nu}_e)}$$

can be estimated in the same way. We have

$$\gamma'_{th}=0.164\qquad \gamma'_{\exp}=0.187\pm 0.042\,,$$

which means Λ decay can be described in terms of the V-A coupling theory.

3041

List the general properties of neutrinos and antineutrinos. What was the physical motivation for the original postulate of the existence of the neutrino? How was the neutrino first directly detected?

(*Wisconsin*)

Solution:

Table 3.6 lists some quantum numbers of neutrino and antineutrino.

Table 3.6

	Charge	Spin	Helicity	Lepton number
neutrino	0	1/2	-1	$+1$
antineutrino	0	1/2	$+1$	-1

Both neutrino and antineutrino are leptons and are subject to weak interaction only. Three kinds of neutrinos and their antiparticles are at present believed to exist in nature. These are electron-neutrino, muon-neutrino, τ-neutrino, and their antiparticles. (ν_τ and $\bar{\nu}_\tau$ have not been detected experimentally).

Originally, in order to explain the conflict between the continuous energy spectrum of electrons emitted in β-decays and the discrete nuclear energy levels, Pauli postulated in 1933 the emission in β-decay also of a light neutral particle called neutrino. As it is neutral the neutrino cannot be detected, but it takes away a part of the energy of the transition. As it is a three-body decay the electron has continuous energy up to a definite cutoff given by the transition energy.

As neutrinos take part in weak interaction only, their direct detection is very difficult. The first experimental detection was carried out by Reines and Cowan during 1953–1959, who used $\bar{\nu}$ from a nuclear reactor to bombard protons. From the neutron decay $n \to p + e^- + \bar{\nu}$ we expect $\bar{\nu} + p \to n + e^+$ to occur. Thus if a neutron and a positron are detected simultaneously the existence of $\bar{\nu}$ is proved. It took the workers six years to get a positive result.

3042

(a) How many neutrino types are known to exist? What is the spin of a neutrino?

(b) What properties of neutrinos are conserved in scattering processes? What is the difference between a neutrino and an antineutrino? Illustrate this by filling in the missing particle:

$$\nu_\mu + e^- \to \mu^- + ?\,.$$

(c) Assume the neutrino mass is exactly zero. Does the neutrino have a magnetic moment? Along what direction(s) does the neutrino spin point? Along what direction(s) does the antineutrino spin point?

(d) What is the velocity of a 3°K neutrino in the universe if the neutrino mass is 0.1 eV?

(*Wisconsin*)

Solution:

(a) Two kinds of neutrino have been found so far. These are electron-neutrinos and muon-neutrinos and their antiparticles. Theory predicts the existence of a third kind of neutrino, τ-neutrino and its antiparticle. The neutrino spin is 1/2.

(b) In a scattering process, the lepton number of each kind of neutrino is conserved. The difference between a neutrino and the corresponding antineutrino is that they have opposite lepton numbers. Furthermore if the neutrino mass is zero, the helicities of neutrino and antineutrino are opposite. The unknown particle in the reaction is ν_e:

$$\nu_\mu + e^- \to \mu^- + \nu_e\,.$$

(c) If the neutrino masses are strictly zero, they have no magnetic moment. The neutrino spin points along a direction opposite to its motion, while the antineutrino spin does the reverse.

(d) The average kinetic energy of a neutrino in a gas of temperature T is $E_k = 3kT/2$, where k is Boltzmann's constant. The velocity of the neutrino is then

$$\beta = \sqrt{2E_k/m} = \sqrt{3kT/m} = \sqrt{3 \times 8.62 \times 10^{-5} \times 3/0.1} = 0.088\,,$$

corresponding to 2.6×10^7 m/s.

3043

(a) Describe the experiments that prove

(1) there are two kinds of neutrino,

(2) the interaction cross section is very small.

(b) Write down the reactions in which an energetic neutrino may produce a single pion with

(1) a proton, and with

(2) a neutron.

(c) Define helicity and what are its values for neutrino and antineutrino.

(d) Can the following modes of μ^+ decay proceed naturally? Why?

(1) $\mu^+ \to e^+ + \gamma$,

(2) $\mu^+ \to e^+ + e^- + e^+$.

(*SUNY Buffalo*)

Solution:

(a) (1) For two-neutrino experiment see **Problem 3009(3)**.

(2) The first observation of the interaction of free neutrinos was made by Reines and Cowan during 1953–1959, who employed $\bar{\nu}_e$ from a nuclear reactor, which have a broad spectrum centered around 1 MeV, as projectiles and cadmium chloride ($CdCl_2$) and water as target to initiate the reaction

$$\bar{\nu}_e + p \to n + e^+ .$$

The e^+ produced in this reaction rapidly comes to rest due to ionization loss and forms a positronium which annihilates to give two γ-rays, each of energy 0.511 MeV. The time scale for this process is of the order 10^{-9} s. The neutron produced, after it has been moderated in the water, is captured by cadmium, which then radiates a γ-ray of $\sim$ 9.1 MeV after a delay of several μs. A liquid scintillation counter which detects both rays gives two differential pulses with a time differential of about 10^{-5} s. The 200-litre target was sandwiched between two layers of liquid scintillator, viewed by banks of photomultipliers. The experiment gave $\sigma_\nu \sim 10^{-44}$ cm^2, consistent with theoretical expectation. Compared with the cross section σ_h of a hardon, $10^{-24\sim-26}$ cm^2, σ_ν is very small indeed.

(b) (1) $\nu_\mu + p \to \mu^- + p + \pi^+$.

(2)
$$\begin{aligned} \nu_\mu + n &\to \mu^- + n + \pi^+ \\ &\to \mu^- + p + \pi^0 . \end{aligned}$$

(c) The helicity of a particle is defined as $H = \frac{\mathbf{P}\cdot\boldsymbol{\sigma}}{|\mathbf{P}||\boldsymbol{\sigma}|}$, where $\mathbf{P}$ and $\boldsymbol{\sigma}$ are the momentum and spin of the particle. The neutrino has $H = -1$ and is said to be left-handed, the antineutrino has $H = +1$ and is right-handed.

(d) $\mu^+ \to e^+ + \gamma$, $\mu^+ \to e^+ + e^- + e^+$.

Neither decay mode can proceed because they violate the conservation of electron-lepton number and of muon-lepton number.

3044

A sensitive way to measure the mass of the electron neutrino is to measure

(a) the angular distribution in electron-neutrino scattering.
(b) the electron energy spectrum in beta-decay.
(c) the neutrino flux from the sun.

(*CCT*)

Solution:

In the Kurie plot of a β spectrum, the shape at the tail end depends on the neutrino mass. So the answer is (b).

3045

How many of one million 1-GeV neutrinos interact when traversing the earth? ($\sigma = 0.7 \times 10^{-38}$ cm^2/n, where n means a nucleon, $R = 6000$ km, $\rho \approx 5$ g/cm^2, $\langle A \rangle = 20$)

(a) all.
(b) ≈ 25.
(c) none.

(*CCT*)

Solution:

Each nucleon can be represented by an area σ. The number of nucleons encountered by a neutrino traversing earth is then

$$N_n = \frac{2R\sigma\rho N_A}{\langle A \rangle}\langle A \rangle\,,$$

where N_A =Avogadro's number. The total number of encounters (collisions) is

$$\begin{aligned} N &= N_\nu N_n = 2R\sigma\rho N_A N_\nu \\ &= 2 \times 6 \times 10^8 \times 0.7 \times 10^{-38} \times 5 \times 6.02 \times 10^{23} \times 10^6 = 25.2 \end{aligned}$$

So the answer is (b).

3046

The cross section rises linearly with E_ν. How long must a detector ($\rho \approx 5$ g/cm^3, $\langle A \rangle = 20$) be so that 1 out of 10^6 neutrinos with $E_\nu = 1000$ GeV interacts?

(a) 6 km.
(b) 480 m.
(c) 5 m.

(*CCT*)

Solution:

Write $L = 2R$ in **Problem 3045**, then $N \propto L\sigma$. As $\sigma' = 1000\sigma$, we have

$$\frac{1}{25.2} = \frac{10^3 L}{2R},$$

or

$$L = \frac{2 \times 6000}{25.2 \times 10^3} = 0.476\ km\,.$$

Hence the anser is (b).

3047

An experiment in a gold mine in South Dakota has been carried out to detect solar neutrinos using the reaction

$$\nu + Cl^{37} \to Ar^{37} + e^-\,.$$

The detector contains approximately 4×10^5 liters of tetrachlorethylene ($\mathrm{CCl_4}$). Estimate how many atoms of $\mathrm{Ar^{37}}$ would be produced per day. List your assumptions. How can you improve the experiment?

(*Columbia*)

Solution:

The threshold for the reaction $\nu + \mathrm{Cl}^{37} \to \mathrm{Ar}^{37} + e^-$ is $(M_{\mathrm{Ar}} - M_{\mathrm{Cl}})c^2 = 0.000874 \times 937.9 = 0.82$ MeV, so only neutrinos of $E_\nu > 0.82$ MeV can be detected. On the assumption that the density ρ of CCl_4 is near that of water, the number of Cl nuclei per unit volume is

$$n = \frac{4\rho N_0}{A} = (4/172) \times 6.02 \times 10^{23} = 1.4 \times 10^{22}\ \mathrm{cm}^{-3}\,,$$

where $A = 172$ is the molecular weight of $\mathrm{CCl_4}$.

In general the interaction cross section of neutrino with matter is a function of E_ν. Suppose $\bar{\sigma} \approx 10^{-42}$ cm^2/Cl. The flux of solar neutrinos on the earth's surface depends on the model assumed for the sun. Suppose the flux with $E_\nu > 0.82$ MeV is $F = 10^9$ cm^{-2} s^{-1}. Then the number of neutrinos detected per day is $N_\nu = nV\bar{\sigma}Ft = 1.4 \times 10^{22} \times 4 \times 10^5 \times 10^3 \times 10^{-42} \times 10^9 \times 24 \times 3600 = 4.8 \times 10^2$.

However only neutrinos with energies $E_\nu > 0.82$ MeV can be detected in this experiment, whereas solar neutrinos produced in the main process in the sun $p + p \to {}^2H + e^+ + \nu_e$ have maximum energy 0.42 MeV. Most solar neutrinos will not be detected in this way. On the other hand, if Ga or In are used as the detection medium, it would be possible to detect neutrinos of lower energies.

3048

It has been suggested that the universe is filled with heavy neutrinos ν_H (mass m_H) which decay into a lighter neutrino ν_L (mass m_L) and a photon, $\nu_H \to \nu_L + \gamma$, with a lifetime similar to the age of the universe. The ν_H were produced at high temperatures in the early days, but have since cooled and, in fact, they are now so cold that the decay takes place with the ν_H essentially at rest.

(a) Show that the photons produced are monoenergetic and find their energy.

(b) Evaluate your expression for the photon energy in the limit $m_L \ll m_H$. If the heavy neutrinos have a mass of 50 eV as has been suggested by recent terrestrial experiments, and $m_L \ll m_H$, in what spectral regime should one look for these photons?

(c) Suppose the lifetime of the heavy neutrinos were short compared to the age of the universe, but that they were still "cold" (in the above sense) at the time of decay. How would this change your answer to part (b) (qualitatively)?

(*Columbia*)

Solution:

(a) As it is a two-body decay, conservation of energy and conservation of momentum determine uniquely the energy of each decay particle. Thus the photons are monoenergetic. The heavy neutrinos can be considered as decaying at rest. Thus

$$m_H = E_L + E_\gamma , \qquad P_L = P_\gamma .$$

As $E_\gamma = P_\gamma$, $E_L^2 = P_L^2 + m_L^2$, these give

$$E_\gamma = \frac{1}{2m_H}(m_H^2 - m_L^2) .$$

(b) In the limit $m_H \gg m_L$, $E_\gamma \approx \frac{1}{2}m_H$. If $m_H = 50$ eV, $E_\gamma = 25$ eV. The photons emitted have wavelength

$$\lambda = \frac{h}{P_\gamma} = \frac{2\pi\hbar c}{P_\gamma c} = \frac{2\pi \times 197 \times 10^{-13}}{25 \times 10^{-6}} = 495 \times 10^{-8} \text{ cm} = 495 \text{ Å} .$$

This is in the regime of ultraviolet light. Thus one would have to look at extraterrestrial ultraviolet light for the detection of such photons.

(c) If the lifetime of the heavy neutrinos is far smaller than that of the universe, they would have almost all decayed into the lighter neutrinos. This would make their direct detection practically impossible.

3049

The particle decay sequence

$$\pi^+ \to \mu^+ + \nu_\mu , \qquad \mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$$

shows evidence of parity nonconservation.

(a) What observable quantity is measured to show this effect? Sketch or give a formula for the distribution of this observable.

(b) Does the process show that both decay processes violate parity conservation, or only one? Explain why.

(*Wisconsin*)

Solution:

(a) Suppose the pions decay in flight. We can study the forward muons μ^+ which stop and decay inside a carbon absorber. The angular distribution of the e^+ produced in the μ^+ decay can determine if parity is conserved. Relative to the initial direction of μ^+ the e^+ have angular distribution $dN/d\Omega = 1 - \frac{1}{3}\cos\theta$, which changes under space reflection $\theta \to \pi - \theta$. Hence parity is not conserved.

(b) Both the decay processes violate parity conservation since both proceed via weak interaction.

3050

Consider the following decay scheme:

$$\begin{aligned} \pi^+ \to\ & \mu^+ + \nu_1 \\ & \llcorner\!\!\longrightarrow e^+ + \nu_2 + \bar{\nu}_3 \end{aligned}$$

(a) If the pion has momentum p, what is the value of the minimum (and maximum) momentum of the muon? Express the answer in terms of m_μ, m_π and p ($m_{\nu_1} = m_{\nu_2} = m_{\bar{\nu}_3} = 0$) and assume $p \gg m_\mu, m_\pi$.

(b) If the neutrino in π decay has negative helicity, what is the helicity of the muon for this decay?

(c) Given that ν_2 and $\bar{\nu}_3$ have negative and positive helicities respectively, what is the helicity of the positron?

(d) What conserved quantum number indicates that ν_1 and $\bar{\nu}_3(\nu_2)$ are associated with the muon (electron) respectively?

(e) The pion decays to an electron: $\pi^+ \to e^+ + \nu_e$. Even though the kinematics for the electron and muon decay modes are similar, the rate of muon decay is 10^4 times the rate of electron decay. Explain.

(*Princeton*)

Solution:

(a) Let γ be the Lorentz factor of π^+. Then $\beta\gamma = \frac{p}{m_\pi}$,

$$\gamma = \frac{\sqrt{p^2 + m_\pi^2}}{m_\pi} \approx \left(1 + \frac{m_\pi^2}{2p^2}\right)\frac{p}{m_\pi}\,.$$

In the rest system of π^+, $p_\mu^* = p_\nu^* = E_\nu^*$, $m_\pi = E_\mu^* + E_\nu^* = E_\mu^* + p_\mu^*$, giving

$$p_\mu^* = p_\nu^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi},$$

$$E_\mu^* = \sqrt{p_\mu^{*2} + m_\mu^2} = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}.$$

Transforming to the laboratory system we have

$$p_\mu \cos\theta = \gamma p_\mu^* \cos\theta^* + \gamma\beta E_\mu^*.$$

In the direction of p $(\theta = 0)$, p_μ has extreme values

$$\begin{aligned}
(p_\mu)_{\max} &\approx p\left(1 + \frac{m_\pi^2}{2p^2}\right)\frac{m_\pi^2 - m_\mu^2}{2m_\pi^2} + p\frac{m_\pi^2 - m_\mu^2}{2m_\pi^2} \\
&= p + \frac{m_\pi^2 - m_\mu^2}{4p}, \qquad (\theta^* = 0) \\
(p_\mu)_{\min} &\approx -p\left(1 + \frac{m_\pi^2}{2p^2}\right)\frac{m_\pi^2 - m_\mu^2}{2m_\pi^2} + p\frac{m_\pi^2 + m_\mu^2}{2m_\pi^2} \\
&= \left(\frac{m_\mu^2}{m_\pi^2}\right)p - \frac{m_\pi^2 - m_\mu^2}{4p}. \qquad (\theta^* = \pi)
\end{aligned}$$

(b) If the neutrino in π^+ decay has negative helicity, from the fact that π^+ has zero spin and the conservation of total angular momentum and of momentum, we can conclude that μ^+ must have negative helicity in the rest system of π^+.

(c) Knowing that $\bar{\nu}_3$ and ν_2 respectively have positive and negative helicities, one still cannot decide on the helicity of e^+. If we moderate the decay muons and study their decay at rest, a peak is found at 53 MeV in the energy spectrum of the decay electrons. This means the electron and $\nu_2\bar{\nu}_3$ move in opposite directions. If the polarization direction of μ^+ does not change in the moderation process, the angular distribution of e^+ relative to p_μ,

$$\frac{dN_{e^+}}{d\Omega} \approx 1 - \frac{\alpha}{3}\cos\theta,$$

where $\alpha \approx 1$, shows that e^+ has a maximum probability of being emitted in a direction opposite to p_μ $(\theta = \pi)$. Hence the helicity of e^+ is positive.

The longitudinal polarization of the electron suggests that parity is not conserved in π and μ decays.

(d) The separate conservation of the electron- and muon-lepton numbers indicates that ν_1 and $\bar{\nu}_3$ are associated with muon and that ν_2 is associated with electron since the electron-lepton numbers of ν_1, ν_2 and $\bar{\nu}_3$ are 0,1,0, and their muon-lepton numbers are 1,0, -1 respectively.

(e) See **Problem 3040**.

3051

A beam of unpolarized electrons

(a) can be described by a wave function that is an equal superposition of spin-up and spin-down wave functions.

(b) cannot be described by a wave function.

(c) neither of the above.

(*CCT*)

Solution:

The answer is (a).

3052

Let $\mathbf{s}, \mathbf{p}$ be the spin and linear momentum vectors of an elementary particle respectively.

(a) Write down the transformations of $\mathbf{s}, \mathbf{p}$ under the parity operator $\hat{P}$ and the time reversal operator $\hat{T}$.

(b) In view of the answers to part (a), suggest a way to look for time reversal violation in the decay $\Lambda \to N + \pi$. Are any experimental details or assumptions crucial to this suggestion?

(*Wisconsin*)

Solution:

(a) Under the operation of the parity operator, $\mathbf{s}$ and $\mathbf{p}$ are transformed according to

$$\hat{P}\mathbf{s}\hat{P}^{-1} = \mathbf{s}, \qquad \hat{P}\mathbf{p}\hat{P}^{-1} = -\mathbf{p} \,.$$

Under the time-reversal operator $\hat{T}$, $\mathbf{s}$ and $\mathbf{p}$ are transformed according to

$$\hat{T}\mathbf{s}\hat{T}^{-1} = -\mathbf{s}, \qquad \hat{T}\mathbf{p}\hat{T}^{-1} = -\mathbf{p}\,.$$

(b) Consider the angular correlation in the decay of polarized Λ particles. Define

$$Q = \mathbf{s}_\Lambda \cdot (\mathbf{p}_N \times \mathbf{p}_\pi)\,,$$

where $\mathbf{s}_\Lambda$ is the spin of the Λ particle, $\mathbf{p}_N$ and $\mathbf{p}_\pi$ are the linear momenta of the nucleon and the pion respectively. Time reversal operation gives

$$\hat{T}Q\hat{T}^{-1} = \hat{T}\mathbf{s}_\Lambda\hat{T}^{-1} \cdot (\hat{T}\mathbf{p}_N\hat{T}^{-1} \times \hat{T}\mathbf{p}_\pi\hat{T}^{-1}) = -\mathbf{s}_\Lambda \cdot [(-\mathbf{p}_N) \times (-\mathbf{p}_\pi)] = -Q\,,$$

or

$$\bar{Q} = \langle\alpha|Q|\alpha\rangle = \langle\alpha|\hat{T}^{-1}\hat{T}Q\hat{T}^{-1}\hat{T}|\alpha\rangle = -\langle\alpha_T|Q|\alpha_T\rangle\,.$$

If time reversal invariance holds true, $|\alpha_T\rangle$ and $|\alpha\rangle$ would describe the same state and so

$$\bar{Q} = \langle\alpha|Q|\alpha\rangle = -\langle\alpha_T|Q|\alpha_T\rangle = -\bar{Q}\,,$$

or

$$\bar{Q} = 0\,.$$

To detect possible time reversal violation, use experimental setup as in Fig. 3.15. The pion and nucleon detectors are placed perpendicular to each other with their plane perpendicular to the Λ-particle spin. Measure the number of Λ decay events $N(\uparrow)$. Now reverse the polarization of the Λ-particles and under the same conditions measure the Λ decay events $N(\downarrow)$.

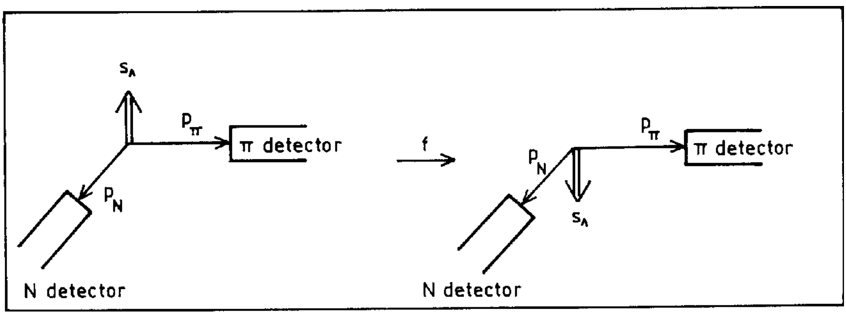

Fig. 3.15

A result $N(\uparrow) \neq N(\downarrow)$ would indicate time reversal violation in the decay $\Lambda \to \pi + N$.

This experiment requires all the Λ-particles to be strictly polarized.

3053

Consider the decay $\Lambda^0 \to p+\pi^-$. Describe a test for parity conservation in this decay. What circumstances may prevent this test from being useful?

(*Wisconsin*)

Solution:

$\Lambda^0 \to p + \pi^-$ is a nonleptonic decay. It is known Λ^0 and p both have spin 1/2 and positive parity, and π^- has spin 0 and negative parity.

As the total angular momentum is conserved, the final state may have relative orbital angular momentum 0 or 1. If

$l = 0$, the final-state parity is $P(p)P(\pi^-)(-1)^0 = -1$; if

$l = 1$, the final-state parity is $P(p)P(\pi^-)(-1)^1 = +1$.

Thus if parity is conserved in Λ^0 decay, $l = 0$ is forbidden. If parity is not conserved in Λ^0 decay, both the l values are allowed and the final-state proton wave function can be written as

$$\Psi = \Psi_s + \Psi_p = a_s Y_{0,0}\left|\frac{1}{2},\frac{1}{2}\right\rangle + a_p\left(\sqrt{\frac{2}{3}}Y_{1,1}\left|\frac{1}{2},-\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}Y_{1,0}\left|\frac{1}{2},\frac{1}{2}\right\rangle\right),$$

where $|\frac{1}{2},\frac{1}{2}\rangle$ and $|\frac{1}{2},-\frac{1}{2}\rangle$ are respectively the spin wave functions of the proton for $m = \pm\frac{1}{2}$, a_s and a_p are the amplitudes of the s and p waves.

Substitution of $Y_{1,1}$, $Y_{1,0}$, $Y_{0,0}$ gives

$$\begin{aligned}\Psi^*\Psi \propto & |a_s - a_p\cos\theta|^2 + |a_p|^2\sin^2\theta \\ & = |a_s|^2 + |a_p|^2 - 2\mathrm{Re}(a_s a_p^*)\cos\theta \propto 1 + \alpha\cos\theta\,,\end{aligned}$$

where $\alpha = 2\mathrm{Re}(a_a a_p^*)/(|a_s|^2 + |a_p|^2)$.

If the Λ^0-particles are polarized, the angular distribution of p or π^- will be of the form $1 + \alpha\cos\theta$, (in the rest frame of Λ^0, p and π^- move in opposite directions). If Λ^0 are not fully polarized, let the polarizability be P. Then the angular distribution of π^- or p is $(1 + \alpha P\cos\theta)$. In the above θ is the angle between the direction of π^- or p and the polarization direction of Λ^0.

Measurement of the angular distribution can be carried out using the polarized Λ^0 arising from the associated production

$$\pi^- + p \to \Lambda^0 + K^0 .$$

Parity conservation in the associated production, which is a strong interaction, requires the Λ^0-particles to be transversally polarized with the spin direction perpendicular to the reaction plane. Experimentally if the momentum of the incident π^- is slightly larger than 1 GeV/c, the polarizability of Λ^0 is about 0.7. Take the plane of production of Λ^0, which is the plane containing the directions of the incident π^- and the produced Λ^0 (K^0 must also be in this plane to satisfy momentum conservation) and measure the counting rate disparity of the π^- (or p) emitted in Λ^0 decay between the spaces above and below this plane ($\theta = 0$ to $\pi/2$ and $\theta = \pi/2$ to π). A disparity would show that parity is not conserved in Λ^0 decay. An experiment by Eister in 1957 using incident π^- of momenta $910 \sim 1300$ MeV/c resulted in $P = 0.7$. Note in the above process, the asymmetry in the emission of π^- originates from the polarization of Λ^0. If the Λ^0 particles has $P = 0$ the experiment could not be used to test parity conservation.

3054

The Λ and p particles have spin 1/2, the π has spin 0.

(a) Suppose the Λ is polarized in the z direction and decays at rest, $\Lambda \to p + \pi^-$. What is the most general allowed angular distribution of π^-? What further restriction would be imposed by parity invariance?

(b) By the way, how does one produce polarized Λ's?

(*Princeton*)

Solution:

(a) The initial spin state of Λ-particle is $|\frac{1}{2}, \frac{1}{2}\rangle$. Conservation of angular momentum requires the final-state πp system orbital angular momentum quantum number to be $l = 0$ or 1 (**Problem 3053**).

If $l = 0$, the final-state wave function is $\Psi_s = a_s Y_{00} |\frac{1}{2}, \frac{1}{2}\rangle$, where a_s is the s-wave amplitude in the decay, $|\frac{1}{2}, \frac{1}{2}\rangle$ is the proton spin state, Y_{00} is the orbital angular motion wave function.

If $l = 1$, the final-state wave function is

$$\Psi_p = a_p\left(\sqrt{\frac{2}{3}}Y_{11}\left|\frac{1}{2}, -\frac{1}{2}\right\rangle - \sqrt{\frac{1}{3}}Y_{10}\left|\frac{1}{2}, \frac{1}{2}\right\rangle\right),$$

where a_p is the p-wave amplitude in the decay, $\sqrt{\frac{2}{3}}, -\sqrt{\frac{1}{3}}$ are Clebsch–Gordan coefficients.

With $Y_{00} = \frac{1}{\sqrt{4\pi}}$, $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$, $Y_{11} = \sqrt{\frac{3}{8\pi}}e^{i\varphi}\sin\theta$, we have

$$\Psi_s = \frac{a_s}{\sqrt{4\pi}}\left|\frac{1}{2}, \frac{1}{2}\right\rangle,$$

$$\Psi_p = -\frac{a_p}{\sqrt{4\pi}}\left(e^{i\varphi}\sin\theta\left|\frac{1}{2}, -\frac{1}{2}\right\rangle + \cos\theta\left|\frac{1}{2}, \frac{1}{2}\right\rangle\right).$$

and the finalstate total wave function

$$\Psi = \frac{1}{\sqrt{4\pi}}\left((a_s - a_p\cos\theta)\left|\frac{1}{2}, \frac{1}{2}\right\rangle - a_p e^{i\varphi}\sin\theta\left|\frac{1}{2}, -\frac{1}{2}\right\rangle\right).$$

The probability distribution is then

$$\Psi^*\Psi \propto |a_s - a_p\cos\theta|^2 + |a_p\sin\theta|^2 = |a_s|^2 + |a_p|^2 - 2\mathrm{Re}(a_s a_p^*)\cos\theta.$$

Hence the pion angular distribution has the form

$$I(\theta) = C(1 + \alpha\cos\theta),$$

where α, C are constants.

The particles Λ, p, π have parities $+, +, -$ respectively. If parity is conserved in the decay, $l = 0$ is forbidden, i.e. $a_s = 0$, and the angular distribution of the pion is limited by the space-reflection symmetry to be symmetric above and below the decay plane. If observed otherwise, parity is not conserved.

(b) Polarized Λ^0-particles can be created by bombarding a proton target with pions:

$$\pi^- + p \to \Lambda^0 + K^0.$$

The Λ^0-particles are produced polarized perpendicular to the plane of production.

3055

(a) As is well known, parity is violated in the decay $\Lambda \to p + \pi^-$. This is reflected, for example, in the following fact. If the Λ-particle is fully polarized along, say, the z-axis, then the angular distribution of the proton obeys

$$\frac{d\Gamma}{d\Omega} = A(1 + \lambda \cos\theta) .$$

Given the parameter λ, what is the longitudinal polarization of the proton if the Λ is unpolarized?

(b) For strangeness-changing hyperon decays in general, e.g., $\Lambda \to p\pi^-$, $\Lambda \to n\pi^0$, $\Sigma^+ \to n\pi^+$, $\Sigma^+ \to p\pi^0$, $\Sigma^- \to n\pi^-$, $K^+ \to \pi^+\pi^0$, $K_s^0 \to \pi^+\pi^-$, $K_s^0 \to \pi^0\pi^0$, etc., there is ample evidence for the approximate validity of the so-called $\Delta I = \frac{1}{2}$ rule (the transition Hamiltonian acts like a member of an isotopic spin doublet). What does the $\Delta I = \frac{1}{2}$ rule predict for the relative rates of $K^+ \to \pi^+\pi^0$, $K_s^0 \to \pi^+\pi^-$, $K_s^0 \to \pi^0\pi^0$?

(*Princeton*)

Solution:

(a) Parity is violated in Λ^0 decay and the decay process is described with s and p waves of amplitudes a_s and a_p (**Problem 3053**). According to the theory on decay helicity, a hyperon of spin 1/2 decaying at rest and emitting a proton along the direction $\Omega = (\theta, \phi)$ has decay amplitude

$$f_{\lambda M}(\theta, \phi) = (2\pi)^{-\frac{1}{2}} \mathcal{D}^{1/2}_{M\lambda'}(\phi, \theta, 0) a_{\lambda'} ,$$

where M and λ' are respectively the spin projection of Λ^0 and the proton helicity. We use a_+ and a_- to represent the decay amplitudes of the two different helicities. Conservation of parity would require $a_+ = -a_-$. The total decay rate is

$$W = |a_+|^2 + |a_-|^2 .$$

The angular distribution of a particle produced in the decay at rest of a Λ^0 hyperon polarized along the z-axis is

$$\begin{aligned}
\frac{dP}{d\Omega} &= \frac{1}{W}\sum_{\lambda'}|f_{\lambda',1/2}(\theta,\phi)|^2 \\
&= (2\pi W)^{-1}\sum_{\lambda'}|a_{\lambda'}|^2[d^{1/2}_{1/2,\lambda'}(\theta)]^2 \\
&= (2\pi W)^{-1}\left(|a_+|^2\cos^2\frac{\theta}{2}+|a_-|^2\sin^2\frac{\theta}{2}\right) \\
&= A(1+\lambda\cos\theta)\,,
\end{aligned}$$

where $d^{1/2}_{1/2,\lambda'}(\theta)=\mathcal{D}^{1/2}_{M\lambda'}(\phi,\theta,0)$, $A=\frac{1}{4\pi}$, $\lambda=\frac{|a_+|^2-|a_-|^2}{|a_+|^2+|a_-|^2}$. Note $\lambda=0$ if parity is conserved.

The expectation value of the helicity of the protons from the decay of unpolarized Λ^0 particles is

$$\begin{aligned}
P &= (2W)^{-1}\sum_M\int\left(\frac{1}{2}|f_{1/2,M}|^2-\frac{1}{2}|f_{-1/2,M}|^2\right)d\Omega \\
&= (2W)^{-1}\sum_M\int\sum_{\lambda'}\lambda'|f_{\lambda',M}|^2 d\Omega \\
&= (2W)^{-1}\sum_M\sum_{\lambda'}\lambda'|a_{\lambda'}|^2(2\pi)^{-1}\int|d^{1/2}_{M\lambda'}(\theta)|^2 d\Omega \\
&= W^{-1}\sum_{\lambda'}\lambda'|a_{\lambda'}|^2\,,
\end{aligned}$$

where we have used

$$\sum_{M'}(d^J_{MM'}(\theta))^2=\sum_{M'}d^J_{MM'}(-\theta)d^J_{M'M}(\theta)=d^J_{MM}(\theta)\,.$$

Hence

$$P=\frac{1}{2}\frac{|a_+|^2-|a_-|^2}{|a_+|^2+|a_-|^2}=\frac{1}{2}\lambda\,.$$

(b) In the decays

$$K^+\to\pi^+\pi^0\,,$$

$$K^0_s\to\pi^+\pi^-\,,$$

$$K^0_s\to\pi^0\pi^0\,,$$

the final states consist of two bosons and so the total wave functions should be symmetric. As the spin of K is zero, the final-state angular momentum is zero. Then as pions have spin zero, $l = 0$ for the final state, i.e., the space wave function is symmetric. Hence the symmetry of the total wave function requires the final-state isospin wave function to be symmetric, i.e., $I = 0, 2$ as pions have isospin 1. Weak decays require $\Delta I = \frac{1}{2}$. As K has isospin $\frac{1}{2}$, the two-π system must have $I = 0, 1$. Therefore, $I = 0$.

For $K^+ \to \pi^+\pi^0$, the final state has $I_3 = 0 + 1 = 1$. As $I = 0$ or 2 and $I \geq I_3$, we require $I = 2$ for the final state. This violates the $\Delta I = 1/2$ rule and so the process is forbidden. Experimentally we find

$$\sigma(K_S^0 \to \pi^+\pi^-)/\sigma(K^+ \to \pi^+\pi^0) \approx 455 \gg 1\,.$$

On the other hand, in $K_S^0 \to \pi^+\pi^-$ or $\pi^0\pi^0$, as K^0, π^+, π^0, π^- have $I_3 = -\frac{1}{2}, 1, 0, -1$ respectively the final state has $I_3 = 0$, $I = 0$ or 2. The symmetry of the wave function requires $I = 0$. Hence the $\Delta I = \frac{1}{2}$ rule is satisfied and the final spin state is $|I, I_3\rangle = |0, 0\rangle$. Expanding the spin wave function we have

$$\begin{aligned}|I, I_3\rangle = |0,0\rangle &= \sqrt{\frac{1}{3}}(|1,1;1,-1\rangle + |1,-1;1,1\rangle - |1,0,1,0\rangle) \\ &= \sqrt{\frac{1}{3}}(|\pi^+\pi^-\rangle + |\pi^-\pi^+\rangle - |\pi^0\pi^0\rangle)\,.\end{aligned}$$

Therefore

$$\frac{K_s^0 \to \pi^+\pi^-}{K_s^0 \to \pi^0\pi^0} = 2\,.$$

3056

(a) Describe the CP violation experiment in K^0 decay and explain why this experiment is particularly appropriate.

(b) Find the ratio of K_S (K short) to K_L (K long) in a beam of 10 GeV/c neutral kaons at a distance of 20 meters from where the beam is produced.

$$(\tau_{K_L} = 5 \times 10^{-8}\ \text{sec}, \quad \tau_{K_s} = 0.86 \times 10^{-10}\ \text{sec})$$

(SUNY, Buffalo)

Solution:

(a) J. W. Cronin *et al.* observed in 1964 that a very few K^0 mesons decayed into 2 π's after a flight path of 5.7 feet from production. As the K_S^0 lifetime is short almost all K_S^0 should have decayed within centimeters from production. Hence the kaon beam at 5.7 feet from production should consist purely of K_L^0. If CP is conserved, K_L^0 should decay into 3 π's. The observation of 2π decay means that CP conservation is violated in K_L^0 decay. CP violation may be studied using K^0 decay because K^0 beam is a mixture of K_1^0 with $\eta_{CP} = 1$ and K_2^0 with $\eta_{CP} = -1$, which have different CP eigenvalues manifesting as 3π and 2π decay modes of different lifetimes. The branching ratio

$$R = \frac{K_L^0 \to \pi^+\pi^-}{K_L^0 \to all} \approx 2 \times 10^{-3}$$

quantizes the CP violation. K_L^0 corresponds to K_2^0 which has $\eta_{CP} = -1$ and should only decay into 3 π's. Experimentally it was found that K_L^0 also decays to 2 π's, i.e., $R \neq 0$. As $\eta_{CP}(\pi^+\pi^+) = +1$, CP violation occurs in K_L^0 decay.

(b) Take $M_{K^0} \approx 0.5$ GeV/c^2. Then $P_{K^0} \approx 10$ GeV/c gives

$$\beta\gamma \approx P_{K^0}/M_{K^0} = 20\,.$$

When K^0 are generated, the intensities of the long-lived K_L^0 and the short-lived K_S^0 are equal:

$$I_{L0} = I_{S0}\,.$$

After 20 meters of flight,

$$I_L = I_{L0}e^{-t/\gamma\tau_L} = I_{L0}e^{-20/\beta\gamma c\tau_L}\,,$$

$$I_S = I_{S0}e^{-t/\gamma\tau_S} = I_{L0}e^{-20/\beta\gamma c\tau_S}\,,$$

and so

$$I_S/I_L = e^{-\frac{20}{\beta\gamma c}\left(\frac{1}{\tau_S} - \frac{1}{\tau_L}\right)} \approx e^{-38.7} \approx 1.6 \times 10^{-17}\,.$$

Hence after 20 meters, the 2π decays are due entirely to K_L^0.

3057

The neutral K-meson states $|K^0\rangle$ and $|\bar{K}^0\rangle$ can be expressed in terms of states $|K_L\rangle$, $|K_S\rangle$:

$$|K^0\rangle = \frac{1}{\sqrt{2}}(|K_L\rangle + |K_S\rangle)\,,$$

$$|\bar{K}^0\rangle = \frac{1}{\sqrt{2}}(|K_L\rangle - |K_S\rangle)\,.$$

$|K_L\rangle$ and $|K_S\rangle$ are states with definite lifetimes $\tau_L \equiv \frac{1}{\gamma_L}$ and $\tau_S \equiv \frac{1}{\gamma_S}$, and distinct rest energies $m_Lc^2 \neq m_Sc^2$. At time $t=0$, a meson is produced in the state $|\psi(t=0)\rangle = |K^0\rangle$. Let the probability of finding the system in state $|K^0\rangle$ at time t be $P_0(t)$ and that of finding the system in state $|\bar{K}^0\rangle$ at time t be $\bar{P}_0(t)$. Find an expression for $P_0(t) - \bar{P}_0(t)$ in terms of γ_L, γ_S, m_Lc^2 and m_Sc^2. (Neglect CP violation)

(*Columbia*)

Solution:

We have at time t

$$\begin{aligned}
|\Psi(t)\rangle &= e^{-iHt}|\psi(0)\rangle = e^{-iHt}|K^0\rangle \\
&= e^{-iHt}\frac{1}{\sqrt{2}}(|K_L\rangle + |K_S\rangle) \\
&= \frac{1}{\sqrt{2}}[e^{-im_Lt-\gamma_Lt/2}|K_L\rangle + e^{-im_St-\gamma_St/2}|K_S\rangle]\,,
\end{aligned}$$

where the factors $\exp(-\gamma_Lt/2)$, $\exp(-\gamma_St/2)$ take account of the attenuation of the wave functions (particle number $\propto \bar{\Psi}\Psi$). Thus

$$\begin{aligned}
|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\left\{e^{-im_Lt-\gamma_Lt/2}\frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle)\right. \\
&\quad \left. + e^{-im_St-\gamma_St/2}\frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle)\right\} \\
&= \frac{1}{2}\{[e^{-im_Lt-\gamma_Lt/2} + e^{-im_St-\gamma_St/2}]|K^0\rangle \\
&\quad + [e^{-im_Lt-\gamma_Lt/2} - e^{-im_St-\gamma_St/2}]|\bar{K}^0\rangle\}\,,
\end{aligned}$$

and hence

$$\langle\psi(t)|\psi(t)\rangle = P_0(t) + \bar{P}_0(t)\,,$$

where

$$P_0(t) = \frac{1}{4}\{e^{-\gamma_L t} + e^{-\gamma_S t} + 2e^{-(\gamma_L+\gamma_S)t/2}\cos[(m_L - m_S)t]\},$$

$$\bar{P}_0(t) = \frac{1}{4}\{e^{-\gamma_L t} + e^{-\gamma_S t} - 2e^{-(\gamma_L+\gamma_S)t/2}\cos[(m_L - m_S)t]\}.$$

Thus we have

$$P_0(t) - \bar{P}_0(t) = e^{-(\gamma_L+\gamma_S)t/2}\cos[(m_L - m_S)t].$$

3058

(a) Explain how the dominance of one of the following four reactions can be used to produce a neutral kaon beam that is "pure" (i.e., uncontaminated by the presence of its antiparticle).

$$\pi^- p \to (\Lambda^0 \text{ or } K^0)(K^0 \text{ or } \bar{K}^0).$$

(b) A pure neutral kaon beam is prepared in this way. At time $t = 0$, what is the value of the charge asymmetry factor δ giving the number of $e^+\pi^-\nu$ decays relative to the number of $e^-\pi^+\nu^-$ decays as

$$\delta = \frac{N(e^+\pi^-\nu) - N(e^-\pi^+\bar{\nu})}{N(e^+\pi^-\nu) + N(e^-\pi^+\bar{\nu})}.$$

(c) In the approximation that CP is conserved, calculate the behavior of the charge asymmetry factor δ as a function of proper time. Explain how the observation of the time dependence of δ can be used to extract the mass difference Δm between the short-lived neutral kaon K_S^0 and the long-lived K_L^0.

(d) Now show the effect of a small nonconservation of CP on the proper time dependence of δ.

(*Princeton*)

Solution:

(a) The reaction $\pi^- p \to \Lambda^0 K^0$ obeys all the conservation laws, including $\Delta S = 0$, $\Delta I_z = 0$, for it to go by strong interaction. K^0 cannot be replaced by $\bar{K}^0$ without violating the rule $\Delta I_z = 0$. Hence it can be used to create a pure K^0 beam.

(b) When $t = 0$, the beam consists of only K^0. Decays through weak interaction obey selection rules

$$|\Delta S| = 1, \qquad |\Delta I| = |\Delta I_3| = \frac{1}{2}\,.$$

Then as $K^0 \to \pi^- e^+ \nu$ is allowed and $K^0 \to \pi^+ e^- \bar{\nu}$ is forbidden,

$$\delta(t=0) = \frac{N(e^+\pi^-\nu) - N(e^-\pi^+\bar{\nu})}{N(e^+\pi^-\nu) + N(e^-\pi^+\bar{\nu})} = 1\,.$$

(c) At time $t = 0$,

$$|K_L^0(0)\rangle = \frac{1}{\sqrt{2}}|K^0(0)\rangle\,,$$

$$|K_S^0(0)\rangle = \frac{1}{\sqrt{2}}|K^0(0)\rangle\,.$$

At time t

$$|K_L^0(t)\rangle = \frac{1}{\sqrt{2}}|K^0(0)\rangle e^{-(im_L t + \Gamma_L t/2)}\,,$$

$$|K_S^0(t)\rangle = \frac{1}{\sqrt{2}}|K^0(0)\rangle e^{-(im_S t + \Gamma_S t/2)}\,.$$

Hence

$$\begin{aligned} K^0(t)\rangle &= \frac{1}{\sqrt{2}}(|K_S^0(t)\rangle + |K^0(t)\rangle) \\ &= \frac{1}{2}|K^0(0)\rangle[e^{-(im_S t + \Gamma_S t/2)} + e^{-(im_L t + \Gamma_L t/2)}]\,, \\ \bar{K}^0(t)\rangle &= \frac{1}{2}|K^0(0)\rangle[e^{-(im_S t + \Gamma_S t/2)} - e^{-(im_L t + \Gamma_L t/2)}]\,. \end{aligned}$$

Note that the term $\Gamma t/2$ in the exponents accounts for the attenuation of K_S^0 and K_L^0 due to decay.

If the decay probabilities $N(K^0 \to \pi^- e^+ \nu) = N(\bar{K}^0 \to \pi^+ e^- \bar{\nu})$, then

$$\delta(t) = \frac{|e^{-(im_S+\Gamma_S/2)t} + e^{-(im_L+\Gamma_L/2)t}|^2 - |e^{-(im_S+\Gamma_S/2)t} - e^{-(im_L+\Gamma_L/2)t}|^2}{|e^{-(im_S+\Gamma_S/2)t} + e^{-(im_L+\Gamma_L/2)t}|^2 + |e^{-(im_S+\Gamma_S/2)t} - e^{-(im_L+\Gamma_L/2)t}|^2}$$

$$= \frac{2e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m t)}{e^{-\Gamma_L t} + e^{-\Gamma_S t}} \,.$$

Thus from the oscillation curve of $\delta(t)$, $\Delta m \equiv |m_L - m_S|$ can be deduced.

(d) If there is a small nonconservation of CP, let it be a small fraction ε. Then

$$\begin{aligned} |K^0(t)\rangle &= \frac{1}{\sqrt{2}}[(1+\varepsilon)|K_S^0(t)\rangle + (1-\varepsilon)|K_L^0(t)\rangle] \\ &= \frac{1}{2}|K^0(0)\rangle\{(e^{-(im_S t+\Gamma_S t/2)} + e^{-(im_L t+\Gamma_L t/2)})\} \\ &\quad + \varepsilon(e^{-(im_S t+\Gamma_S t/2)} - e^{-(im_L t+\Gamma_L t/2)})\,, \\ |\bar{K}^0(t)\rangle &= \frac{1}{2}|K^0(0)\rangle\{(e^{-(im_S t+\Gamma_S t/2)} - e^{-(im_L t+\Gamma_L t/2)}) \\ &\quad + \varepsilon(e^{-(im_S t+\Gamma_S t/2)} + e^{-(im_L t+\Gamma_L t/2)})\}\,, \end{aligned}$$

and so

$$\begin{aligned} \delta(t) &= \frac{\langle K^0(t)|K^0(t)\rangle - \langle \bar{K}^0(t)|\bar{K}^0(t)\rangle}{\langle K^0(t)|K^0(t)\rangle + \langle \bar{K}^0(t)|\bar{K}^0(t)\rangle} \\ &\approx \frac{2e^{-(\Gamma_L+\Gamma_S)t/2}\cos(\Delta m t)}{e^{-\Gamma_L t} + e^{-\Gamma_S t}} + Re(\varepsilon)\,. \end{aligned}$$

3059

In the Weinberg–Salam model, weak interactions are mediated by three heavy vector bosons, W^+, W^- and Z^0, with masses given by

$$M_W^2 = (\pi\alpha/\sqrt{2})G\sin^2\theta\,,$$

$$M_Z^2 = M_W^2/\cos^2\theta\,,$$

where α is the fine structure constant, θ is the "weak mixing angle" or the "Weinberg angle", and G is the Fermi constant. The interaction Lagrangian between electrons, positrons, electron-neutrinos and W's, Z^0 is

$$L_{\text{INT}} = \frac{\sqrt{\pi\alpha}}{\sin\theta}\left\{\frac{1}{\sqrt{2}}W_+^\mu \bar{\nu}\gamma_\mu(1-\gamma_5)e + \frac{1}{\sqrt{2}}W_-^\mu \bar{e}\gamma_\mu(1-\gamma_5)\nu\right.$$
$$\left. + \frac{1}{2\cos\theta}Z^\mu[\bar{\nu}\gamma_\mu(1-\gamma_5)\nu - \bar{e}\gamma_\mu(1-\gamma_5)e + 4\sin^2\theta\bar{e}\gamma_\mu e]\right\},$$

where ν and e are Dirac fields. Consider the elastic scattering of electron-antineutrinos off electrons

$$\bar{\nu}e^- \to \bar{\nu}e^- .$$

(a) Draw the lowest order Feynman diagram(s) for this process. Label each line.

(b) If the energies of the electron and antineutrino are small compared to M_W, the interaction between them can be represented by a four-fermion effective Lagrangian. Write down a correct effective Lagrangian, and put it into the form

$$L_{\text{eff}} = \frac{G}{\sqrt{2}}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu][\bar{e}\gamma_\mu(A-B\gamma_5)e],$$

where A and B are definite functions of θ.

NOTE: if ψ_1 and ψ_2 are anticommuting Dirac fields, then

$$[\bar{\psi}_1\gamma^\mu(1-\gamma_5)\psi_2][\bar{\psi}_2\gamma_\mu(1-\gamma_5)\psi_1] = [\bar{\psi}_1\gamma^\mu(1-\gamma_5)\psi_1][\bar{\psi}_2\gamma_\mu(1-\gamma_5)\psi_2].$$

(c) What experiments could be used to determine A and B?

(*Princeton*)

Solution:

(a) Elastic $\bar{\nu}e$ scattering can take place by exchanging W^- or Z^0. The respective lowest order Feynman diagrams are shown in Fig. 3.16.

(b) From the given Lagrangian, we can write down the Lagrangians for the two diagrams. For Fig. 3.16(a):

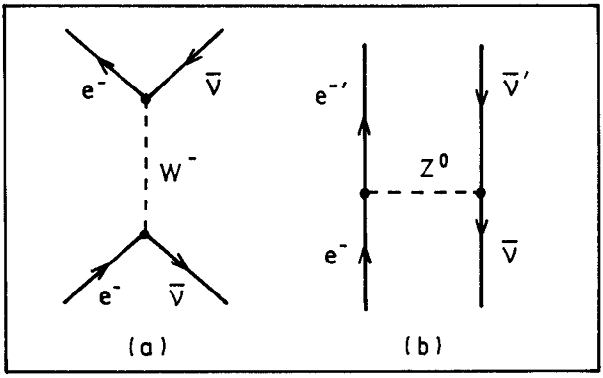

Fig. 3.16

$$L(e\bar{\nu}W) = \left(\sqrt{\frac{\pi\alpha}{2}}\frac{1}{\sin\theta}\right)^2 [\bar{\nu}\gamma^\mu(1-\gamma_5)e \cdot \frac{g^{\mu\nu}-(k^\mu k^\nu/M_W^2)}{M_W^2-k^2}\bar{e}\gamma_\mu(1-\gamma_5)\nu]$$

$$= \frac{\pi\alpha}{2\sin^2\theta}[\bar{\nu}\gamma^\mu(1-\gamma_5)e \cdot \frac{g^{\mu\nu}-(k^\mu k^\nu/M_W^2)}{M_W^2-k^2}\bar{e}\gamma_\mu(1-\gamma_5)\nu] .$$

At low energies, $M_W^2 \gg k^2$ and the above equation can be simplified to

$$L(e\bar{\nu}W) = \frac{\pi\alpha}{2\sin^2\theta M_W^2}[\bar{\nu}\gamma^\mu(1-\gamma_5)e][\bar{e}\gamma_\mu(1-\gamma_5)\nu] .$$

$\bar{\nu}$ and e being Dirac fields, we have

$$[\bar{\nu}\gamma^\mu(1-\gamma_5)e][\bar{e}\gamma_\mu(1-\gamma_5)\nu] = [\bar{\nu}\gamma^\mu(1-\gamma_5)\nu][\bar{e}\gamma_\mu(1-\gamma_5)e] ,$$

and the Lagrangian

$$L(e\bar{\nu}w) = \frac{G}{\sqrt{2}}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu][\bar{e}\gamma_\mu(1-\gamma_5)e] ,$$

as $G = \frac{\pi\alpha}{\sqrt{2}\sin^2\theta M_W^2}$.

For Fig. 3.16(b), the effective Lagrangian is

$$L(e\bar{\nu}Z^0) = \frac{\pi\alpha^2}{\sin^2\theta \cdot 4\cos^2\theta}$$
$$\times \left[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu \cdot \frac{g^{\mu\nu}-(k^\mu k^\nu/M_Z^2)}{M_Z^2-k^2} \cdot \bar{e}\gamma_\mu(g_V - g_A\gamma_5)e\right] ,$$

where $g_V = -1 + 4\sin^2\theta$, $g_A = -1$. If $M_Z^2 \gg k^2$ this can be simplified to a form for direct interaction of four Fermions:

$$\begin{aligned} L(e\bar{\nu}Z^0) &= \frac{\pi\alpha^2}{4\sin^2\theta\cos^2\theta M_Z^2}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu][\bar{e}\gamma_\mu(g_V - g_A\gamma_5)e] \\ &= \frac{G}{2\sqrt{2}}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu][\bar{e}\gamma_\mu(g_V - g_A\gamma_5)e]\,, \end{aligned}$$

as

$$M_Z^2 = \frac{M_W^2}{\cos^2\theta} = \frac{\pi\alpha^2}{\sqrt{2}G\sin^2\theta\cos^2\theta}\,.$$

The total effective Lagrangian is the sum of the two diagrams:

$$\begin{aligned} L_{\text{eff}} &= L(e\bar{\nu}W) + L(e\bar{\nu}Z^0) \\ &= \frac{G}{\sqrt{2}}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu][\bar{e}\gamma_\mu(1-\gamma_5)e] \\ &\quad + \frac{G}{2\sqrt{2}}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu][\bar{e}\gamma_\mu(g_V - g_A\gamma_5)e] \\ &= \frac{G}{\sqrt{2}}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu]\left[\bar{e}\gamma_\mu\left(1 + \frac{g_V}{2} - \gamma_5 - \frac{g_A}{2}\gamma_5\right)e\right] \\ &= \frac{G}{\sqrt{2}}[\bar{\nu}\gamma^\mu(1-\gamma_5)\nu][\bar{e}\gamma_\mu(A - B\gamma_5)e]\,, \end{aligned}$$

where $A = 1 + g_V/2$, $B = 1 + g_A/2$.

(c) Many experiments have been carried out to measure A and B, with the best results coming from neutrino scatterings such as $\nu_\mu e^-$, $\bar{\nu}_\mu e$ scatterings. Also experiments on the asymmetry of l-charge in $e^+e^- \to l^+l^-$ can give g_V and g_A, and hence A and B.

Note the $p\bar{p}$ colliding beams of CERN have been used to measure the masses of W and Z directly, yielding

$$\begin{aligned} M_W &= (80.8 \pm 2.7)\ \text{GeV}\,, \\ M_{Z^0} &= (92.9 \pm 1.6)\ \text{GeV}\,, \end{aligned}$$

and $\sin^2\theta = 0.224$.

3060

One of the important tests of the modern theory of weak interactions involves the elastic scattering of a μ-type neutrino off an electron:

$$\nu_\mu + e^- \to \nu_\mu + e^- .$$

For low energies this may be described by the effective interaction Hamiltonian density

$$H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \bar{\psi}_\nu \gamma^\alpha (1+\gamma_5) \psi_\nu \bar{\psi}_e \{ g_V \gamma_\alpha + g_A \gamma_\alpha \gamma_5 \} \psi_e ,$$

where G_F is the Fermi constant and g_V, g_A are dimensionless parameters. Let $\sigma(E)$ be the total cross section for this process, where E is the total center-of-mass energy, and take $E \gg m_e$. Suppose the target electron is unpolarized.

(a) On purely dimensional grounds, determine how $\sigma(E)$ depends on the energy E.

(b) Let $\frac{\partial \sigma}{\partial E}|_{0^0}$ be the differential cross section in the center-of-mass frame for forward scattering. Compute this in detail in terms of E, G_F, g_V, g_A.

(c) Discuss in a few words (and perhaps with a Feynman diagram) how this process is thought to arise from interaction of a vector boson with neutral "currents".

(*Princeton*)

Solution:

(a) Given $E \gg m_e$, we can take $m_e \approx 0$ and write the first order weak interaction cross section as $\sigma(E) \approx G_F^2 E^k$, where k is a constant to be determined. In our units, $\hbar = 1$, $c = 1$, $\hbar c = 1$. Then $[E] = M$. As $[\hbar c] = [ML] = 1$, $[\sigma] = [L^2] = M^{-2}$. Also, $[G_F] = \left[\frac{(\hbar c)^3}{(Mc^2)^2}\right] = M^{-2}$. Hence $k = -2 + 4 = 2$ and so

$$\sigma(E) \approx G_F^2 E^2 .$$

(b) The lowest order Faynman diagram for $\nu_\mu e \to \nu_\mu e$ is shown in Fig. 3.17. In the center-of-mass frame, taking $m_\nu = 0$, $m_e \approx 0$ we have

$$p_1 = p_3 = k = (p, \mathbf{p}) ,$$

$$p_2 = p_4 = p \approx (p, -\mathbf{p}) ,$$

$$\frac{d\sigma}{d\Omega} = \frac{|F|^2}{64\pi^2 S}$$

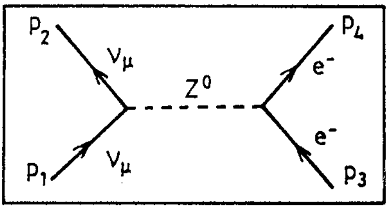

Fig. 3.17

with $S = E^2$. The square of the scattering amplitude based on H_{eff} is

$$|F|^2 = \frac{G_F^2}{2}\text{Tr}\,[\not{k}\gamma^\alpha(1+\gamma_5)\not{k}\gamma^\beta(1+\gamma_5)]$$
$$\times\frac{1}{2}\text{Tr}\,[\not{p}(g_V\gamma_\alpha + g_A\gamma_\alpha\gamma_5)\not{p}(g_V\gamma_\beta + g_A\gamma_\beta\gamma_5)]\,,$$

use having been made of the relation $\sum_{P_s}\bar{u}u = \not{p} + m$, where $\not{p} = \gamma_\mu p^\mu$. Note the factor $\frac{1}{2}$ arises from averaging over the spins of the interacting electrons, whereas the neutrinos are all left-handed and need not be averaged. Consider

$$\text{Tr}\,[\not{k}\gamma^\alpha(1+\gamma_5)\not{k}\gamma^\beta(1+\gamma_5)] = 2\text{Tr}\,[\not{k}\gamma^\alpha\not{k}\gamma^\beta(1+\gamma_5)]$$
$$= 8(k^\alpha k^\beta - k^2 g^{\alpha\beta} + k^\beta k^\alpha + i\varepsilon^{\alpha\beta\gamma\delta}k_\gamma k_\delta)\,.$$

The last term in the brackets is zero because its sign changes when the indices γ, δ are interchanged. Also for a neutrino, $k^2 = 0$. Hence the above expression can be simplified:

$$\text{Tr}\,(\not{k}\gamma^\alpha(1+\gamma_5)\not{k}\gamma^\beta(1+\gamma_5)] = 16k^\alpha k^\beta\,.$$

The second trace can be similarly simplified:

$$\frac{1}{2}\text{Tr}\,[\not{p}(g_V\gamma_\alpha + g_A\gamma_\alpha\gamma_5)\not{p}(g_V\gamma_\beta + g_A\gamma_\beta\gamma_5)]$$
$$= \frac{1}{2}\text{Tr}\,[g_V^2\not{p}\gamma_\alpha\not{p}\gamma_\beta + 2g_Vg_A\not{p}\gamma_\alpha\not{p}\gamma_\beta\gamma_5 + g_A^2\not{p}\gamma_\alpha\not{p}\gamma_\beta]$$
$$= 4(g_A^2 + g_V^2)^2 p_\alpha p_\beta\,.$$

Then as

$$k^\alpha p_\alpha^- k^\beta p_\beta = (p_1 \cdot p_2)(p_3 \cdot p_4) = (p^2 + \mathbf{p} \cdot \mathbf{p})^2 = \left[2\left(\frac{E}{2}\right)^2\right]^2 = \left(\frac{S}{2}\right)^2 .$$

we have

$$\begin{aligned} |F|^2 &= \frac{G_F^2}{2} \cdot 16 \times 4 \times \left(\frac{S}{2}\right)^2 (g_A^2 + g_V^2)^2 \\ &= 8G_F^2 S^2 (g_A^2 + g_V^2)^2 , \end{aligned}$$

and

$$\sigma = \int d\sigma = \int \frac{G_F^2}{8\pi^2} S(g_A^2 + g_V^2)^2 d\Omega = \frac{G_F^2 E^2}{2\pi}(g_A^2 + g_V^2)^2 .$$

Differentiating we have

$$\frac{d\sigma}{dE} = \frac{G_F^2 E}{\pi}(g_A^2 + g_V^2)^2 .$$

So the reaction cross section is isotropic in the center-of-mass frame, and the total cross section is proportional to E^2.

(c) The interaction is thought to take place by exchanging a neutral intermediate boson Z^0 as shown in Fig. 3.17 and is therefore called a neutral weak current interaction. Other such interactions are, for example,

$$\nu_\mu + N \to \nu_\mu + N , \qquad \nu_e + \mu \to \nu_e + \mu ,$$

where N is a nucleon.

3061

The Z-boson, mediator of the weak interaction, is eagerly anticipated and expected to weigh in at $M_Z \geq 80$ GeV.

(a) Given that the weak and electromagnetic interactions have roughly the same intrinsic strength (as in unified gauge theories) and that charged and neutral currents are of roughly comparable strength, show that this is a reasonable mass value (to a factor of 5).

(b) Estimate the width of Z^0 and its lifetime.

(c) Could you use Z^0 production in e^+e^- annihilation to experimentally determine the branching ratio of the Z^0 into neutrinos? If so, list explicitly what to measure and how to use it.

(*Princeton*)

Solution:

(a) The mediators of weak interactions are the massive intermediate vector bosons $W^\pm$ and Z^0. The weak coupling constant g_W can be related to the Fermi constant G_F in beta decays by

$$\frac{g_W^2}{8M_W^2} = \frac{G_F}{\sqrt{2}} .$$

In the Weinberg-Salam model, Z^0, which mediates neutrino and electron, has coupling constant g_Z related to the electromagnetic coupling constant g_e through

$$g_Z = \frac{g_e}{\sin\theta_W \cos\theta_W} ,$$

while g_W can be given as

$$g_W = \frac{g_e}{\sin\theta_W} ,$$

where θ_W is the weak mixing angle, called the Weinberg angle,

$$g_e = \sqrt{4\pi\alpha} ,$$

α being the fine structure constant. The model also gives

$$M_W = M_Z \cos\theta_W .$$

Thus

$$M_Z = \frac{M_W}{\cos\theta_W} = \frac{1}{\sin 2\theta_W}\left(\frac{4\pi\alpha}{\sqrt{2}G_F}\right)^{\frac{1}{2}} .$$

The Fermi constant G_F can be deduced from the observed muon mass and lifetime to be 1.166×10^{-5} GeV^{-2}. This gives

$$M_Z = \frac{74.6}{\sin 2\theta_W} \text{ GeV} .$$

For $M_Z \geq 80$ GeV, $\theta_W \leq 34.4^0$.

At the lower limit of 80 GeV, $\theta_W = 34.4°$ and the coupling constants are for electromagnetic interaction:

$$g_e = g_W \sin\theta_W = 0.6 g_W ,$$

for neutral current interaction:

$$g_Z = \frac{g_W}{\cos\theta_W} = 1.2 g_W ,$$

for charged current interaction:

$$g_W .$$

So the three interactions have strengths of the same order of magnitude if $M_Z \approx 80$ GeV.

(b) The coupling of Z^0 and a fermion can be written in the general form

$$L^Z_{\text{int}} = -\frac{g_W}{4\cos\theta_W} \bar{f}\gamma^\mu (g_V - g_A\gamma_5) f Z_\mu ,$$

where the values of g_V and g_A are for

$$\begin{aligned}
\nu_e, \nu_\mu, \cdots \quad & g_V = 1, \qquad g_A = 1 ; \\
e, \mu, \cdots \quad & g_V = -1 + 4\sin^2\theta_W , \qquad g_A = -1 ; \\
u, c, \cdots \quad & g_V = 1 - \frac{8}{3}\sin^2\theta_W , \qquad g_A = 1 ; \\
d, s, \cdots \quad & g_V = -1 + \frac{4}{3}\sin^2\theta_W , \qquad g_A = -1 .
\end{aligned}$$

Consider a general decay process

$$Z^0(P) \to f(p) + \bar{f}(q) .$$

The amplitude T is

$$T = -\frac{i g_W}{4\cos\theta_W} \varepsilon^\nu_\mu(p) \bar{u}_\sigma(p) \gamma^\mu (g_V - g_A\gamma_5) \nu_\rho(q) .$$

Summing over the fermion spins, quark decay channels and quark colors, and averageing over the three polarization directions of Z^0, we have

$$\begin{aligned}\sum |T|^2 &= \frac{4n}{3}\left(\frac{g_W}{4\cos\theta_W}\right)^2\left(-g_{\mu\nu}+\frac{p_\mu p_\nu}{M_Z^2}\right)\\ &\quad\times[(g_V^2+g_A^2)(p^\mu q^\nu+p^\nu q^\mu-g^{\mu\nu}p\cdot g)-(g_V^2-g_A^2)m^2g^{\mu\nu}]\\ &= \frac{4n}{3}\left(\frac{g_W}{4\cos\theta_W}\right)^2\left\{(g_V^2+g_A^2)\left[p\cdot q\right.\right.\\ &\quad\left.\left.+\frac{2}{M_Z^2}(P\cdot p)(P\cdot q)\right]+3(g_V^2-g_A^2)m^2\right\},\end{aligned}$$

where m is the fermion mass, n is the color number. In the rest system of Z^0, we have

$$\begin{aligned}&E=M_Z, \qquad \mathbf{p}=0\,,\\ &E_p=E_q=\frac{1}{2}M_Z\,, \qquad p=(E_p,\mathbf{p}), \quad q=(E_q,-\mathbf{p})\,,\\ &|\mathbf{p}|=|\mathbf{q}|=\frac{1}{2}(M_Z^2-4m^2)^{1/2}\,,\end{aligned}$$

and hence $p\cdot q=\left(\frac{M_Z}{2}\right)^2+\frac{1}{4}(M_Z^2-4m^2)=\frac{1}{2}M_Z^2-m^2$, $(P\cdot p)(P\cdot q)=\left(M_Z\cdot\frac{M_Z}{2}\right)^2=\frac{M_Z^4}{4}$.

Substitution gives

$$\sum |T|^2=\frac{4n}{3}\left(\frac{g_W}{4\cos\theta_W}\right)^2[(g_V^2+g_A^2)M_Z^2+2(g_V^2-2g_A^2)m^2]\,.$$

From the formula for the probability of two-body decay of a system at rest

$$d\Gamma=\frac{1}{32\pi^2}\frac{|\mathbf{p}|}{M_Z^2}\sum |T|^2d\Omega\,,$$

and neglecting the fermion mas m, we obtain

$$\Gamma(Z^0\to f\bar{f})=\frac{nG_FM_Z^3}{24\sqrt{2}\pi}(g_V^2+g_A^2)\,.$$

Note that in the above we have used

$$|\mathbf{p}| \approx \frac{M_Z}{2}, \int d\Omega = 4\pi, \left(\frac{g_W}{4\cos\theta_W}\right)^2 = \frac{G_F}{2\sqrt{2}} M_Z^2 .$$

Putting in the values of g_V, g_A, and n (contribution of color) we find

$$\Gamma(Z^0 \to \nu_e \bar{\nu}_e) = \Gamma(Z^0 \to \nu_\mu \bar{\nu}_\mu) = \frac{G_F M_Z^3}{12\sqrt{2}\pi},$$

$$\Gamma(Z^0 \to e^+ e^-) = \Gamma(Z^0 \to \mu^+ \mu^-) = \frac{G_F M_Z^3}{12\sqrt{2}\pi}(1 - 4\sin^2\theta_W + 8\sin^4\theta_W),$$

$$\Gamma(Z^0 \to u\bar{u}) = \Gamma(Z^0 \to c\bar{c}) = \frac{G_F M_Z^3}{4\sqrt{2}\pi}\left(1 - \frac{8}{3}\sin^2\theta_W + \frac{32}{9}\sin^4\theta_W\right),$$

$$\Gamma(Z^0 \to d\bar{d}) = \Gamma(Z^0 \to s\bar{s}) = \frac{G_F M_Z^3}{4\sqrt{2}\pi}\left(1 - \frac{4}{3}\sin^2\theta_W + \frac{8}{9}\sin^4\theta_W\right).$$

The sum of these branching widths gives the total width of Z^0:

$$\Gamma_Z = \frac{G_F M_Z^3}{12\sqrt{2}\pi} \cdot 8N\left(1 - 2\sin^2\theta_W + \frac{8}{3}\sin^4\theta_W\right),$$

where N is the number of generations of fermions, which is currently thought to be 3. The lifetime of Z^0 is $\tau = \Gamma_Z^{-1}$.

(c) Using the result of (b) and taking into accout the contribution of the quark colors, we have

$$\begin{aligned}\Gamma_{\nu\nu} : \Gamma_{\mu\mu} : \Gamma_{uu} : \Gamma_{dd} = \; & 1 : \left(1 - 4\sin^2\theta_W + \frac{8}{3}\sin^4\theta_W\right) \\ & : 3\left(1 - \frac{8}{3}\sin^2\theta_W + \frac{32}{9}\sin^4\theta_W\right) \\ & : 3\left(1 - \frac{4}{3}\sin^2\theta_W + \frac{8}{9}\sin^4\theta_W\right) \\ & \approx 1 : 0.5 : 1.8 : 2.3 ,\end{aligned}$$

employing the currently accepted value $\sin^2\theta_W = 0.2196$.

If we adopt the currently accepted 3 generations of leptons and quarks, then

$$B_{\mu\mu} = \frac{\Gamma_{\mu\mu}}{\Gamma_z} = \frac{1 - 4\sin^2\theta_W + \frac{8}{3}\sin^4\theta_W}{8 \times 3(1 - 2\sin^2\theta_W + \frac{8}{3}\sin^4\theta_W)} \approx 3\% .$$

Similarly, for

$$\Sigma\Gamma_{\nu\nu} = \Gamma_{\nu_e} + \Gamma_{\nu_\mu} + \Gamma_{\mu_\tau} = \frac{G_F M_Z^3}{4\sqrt{2}\pi}\,,$$

$$B_{\nu\nu} = \frac{\sum \Gamma_{\nu\nu}}{\Gamma_z} = 18\%\,.$$

According to the standard model, the numbers of generations of leptons and quarks correspond; so do those of ν_i and l_i. If we measure Γ_Z, we can deduce the number of generations N. Then by measuring $B_{\mu\mu}$ we can get $\Gamma_{\mu\mu}$.

Using the number of generations N and $\Gamma_{\mu\mu}$ we can obtain $\Gamma_{\nu\nu} \approx 2\Gamma_{\mu\mu}$, $\sum \Gamma_{\nu\nu} = 2N\Gamma_{\mu\mu}$.

In the production of Z^0 in e^+e^- annihilation, we can measure Γ_Z directly. Because the energy dispersion of the electron beam may be larger than Γ_Z, we should also measure $\Gamma_{\mu\mu}$ and Γ_h by measuring the numbers of muon pairs and hadrons in the resonance region, for as

$$A_h = \int_{\text{resonance region}} \sigma_h dE \approx \frac{6\pi^2}{M_Z^2}\frac{\Gamma_h\Gamma_{ee}}{\Gamma_Z} = \frac{6\pi^2}{M_Z^2}\frac{\Gamma_h\Gamma_{\mu\mu}}{\Gamma_Z}\,,$$

$$A_\mu = \int_{\text{resonance region}} \sigma_{\mu\mu} dE \approx \frac{6\pi^2}{M_Z^2}\frac{\Gamma_h\Gamma_{\mu\mu}}{\Gamma_Z}\,,$$

we have

$$A_\mu/A_h = \Gamma_{\mu\mu}/\Gamma_h\,.$$

Now for

$$N = 3, \qquad \Gamma_{\mu\mu} : \Gamma_h \approx 0.041\,;$$

$$N = 4, \qquad \Gamma_{\mu\mu} : \Gamma_h \approx 0.030\,;$$

$$N = 5, \qquad \Gamma_{\mu\mu} : \Gamma_h \approx 0.024\,.$$

From the observed A_μ and A_h we can get N, which then gives

$$B_{\nu\nu} = \sum \Gamma_{\nu\nu}/\Gamma_Z = 2N\Gamma_{\mu\mu}/\Gamma_Z\,, \quad \Gamma_Z = 3NT_{\mu\mu} + \Gamma_h\,.$$

3062

Experiments which scatter electrons off protons are used to investigate the charge structure of the proton on the assumption that the electromagnetic interaction of the electron is well understood. We consider an analogous process to study the charge structure of the neutral kaon, namely,

$K^0 + e \to K^0 + e$. (Call this amplitude A)

(a) Neglecting CP violation, express the amplitudes for the following processes in terms of A:

$K_L^0 + e \to K_L^0 + e$, (Scattering, call this A_s)

$K_L^0 + e \to K_S^0 + e$. (Regeneration, call this A_R)

(b) Consider the regeneration experiment

$K_L^0 + e \to K_S^0 + e$,

in which a kaon beam is incident on an electron target. At a very high energy E_K, what is the energy dependence of the differential cross section in the forward direction? (Forward means the scattering angle is zero, $\mathbf{p}_{K_L} = \mathbf{p}_{K_S}$). That is, how does $\left(\frac{d\sigma}{d\Omega}\right)_{0^0}$ vary with E_K? Define what you mean by very high energy.

(*Princeton*)

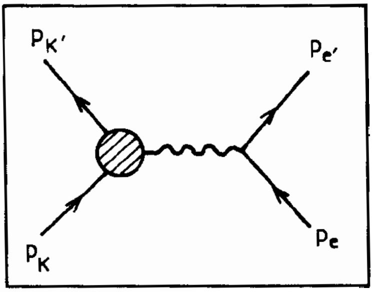

Fig. 3.18

Solution:

Consider the Feynman diagram Fig. 3.18, where p_K, $p_{K'}$, p_e, $p_{e'}$ are the initial and final momenta of K^0 and e with masses M and m, respectively. The S-matrix elements are:

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta(p_K + p_e - p_{K'} - p_{e'})\frac{t_{fi}}{(2\pi)^6}\sqrt{\frac{m^2}{4E_K E_{K'} E_e E_{e'}}}$$

where t_{fi} is the invariant amplitude

$$t_{fi} = ie^2(2\pi)^3\sqrt{4E_K E_{K'}}\bar{u}(p_{e'})\gamma^\mu u(p_e)\frac{1}{q^2}\langle K^0 p_{K'}|j_\mu(0)|K^0 p_K\rangle \approx A\,,$$

j_μ being the current operator.

(a) We have

$$|K_S^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle)\,, \quad |K_L^0\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle)\,.$$

If CP violation is neglected, K_L^0, K_S^0, and K^0 have the same mass. Then

$$\langle K_L^0 p_{K'}|j_\mu(0)|K_L^0 p_K\rangle = \frac{1}{2}\{\langle K^0 p_{K'}|j_\mu(0)|K^0 p_K\rangle + \langle \bar{K}^0 p_{K'}|j_\mu(0)|\bar{K}^0 p_K\rangle\}\,.$$

As

$$\begin{aligned}\langle \bar{K}^0 p_{K'}|j_\mu(0)|\bar{K}^0 p_K\rangle &= \langle \bar{K}^0 p_{K'}|C^{-1}Cj_\mu(0)C^{-1}C|\bar{K}^0 p_K\rangle \\ &= -\langle K^0 p_{K'}|j_\mu(0)|K^0 p_K\rangle\,,\end{aligned}$$

$A_S = 0$. Similarly we have $A_R = A$.

(b) Averaging over the spins of the initial electrons and summing over the final electrons we get the differential cross section

$$d\sigma = \frac{1}{2v_r}\frac{m^2}{4E_K E_{K'}E_e E_{e'}}(2\pi)^4\delta(p_e + p_K - p_{e'} - p_{K'})\sum_{\text{spin}}|t_{fi}|^2\frac{d\mathbf{p}_e d\mathbf{p}_{K'}}{(2\pi)^6}\,.$$

Integration over $\mathbf{p}_{e'}$ and $E_{K'}$ gives

$$\frac{d\sigma}{d\Omega'} = \frac{m}{32\pi^2}\frac{p_{K'}}{p_K}\frac{\sum_{\text{spin}}|t_{fi}|^2}{m + E_K - (p_K E_{K'}/p_{K'})\cos\theta'}\,,$$

where θ' is the angle $\mathbf{p}_{K'}$ makes with $\mathbf{p}_K$. Momentum conservations requires

$$p_{e'} + p_{K'} - p_e - p_K = 0\,,$$

giving

$$m + \sqrt{M_L^2 + \mathbf{p}_K^2} = \sqrt{m^2 + \mathbf{p}_e^2} + \sqrt{M_S^2 + \mathbf{p}_{K'}^2}\,,$$

where M_L, M_S are the masses of K_L^0 and K_S^0 respectively and m is the electron mass. Consider

$$E_L = \sqrt{M_L^2 + \mathbf{p}_K^2} = \sqrt{(M_S + \Delta M)^2 + \mathbf{p}_K^2}$$

with $\Delta M = M_L - M_S$. If $E_L^2 \gg M_S \Delta M$, or $E_L \gg \Delta M$, K_L^0 is said to have high energy. At this time the momentum equation becomes

$$m + \sqrt{M_S^2 + \mathbf{p}_K^2} = \sqrt{m^2 + \mathbf{p}_e^2} + \sqrt{M_S^2 + \mathbf{p}_{K'}^2}\,,$$

which represents an elastic scattering process.

For forward scattering, $p_K = p_{K'}$, $p_e = 0$, and

$$\left.\frac{d\sigma}{d\Omega}\right|_0 = \frac{1}{32\pi} \sum_{\text{spin}} |t_{fi}|^2\,.$$

Now

$$(2\pi)^3 \sqrt{4E_K E_{K'}} \langle K^0 p_{K'} | j_\mu(0) | K^0 p_K \rangle = (p_K + p_{K'})_\mu F_K (p_{K'} - p_K)^2\,,$$

where F_K is the electromagnetic form factor of K^0, $F_K(q^2) = q^2 g(q^2)$. Note $g(q^2)$ is not singular at $q^2 = 0$. Thus

$$\begin{aligned} t_{fi} &= ie^2 \bar{u}^+(p_{e'}) \gamma^\mu u(p_e) g[(p_{K'} - p_K)^2](p_{K'} + p_K)_\mu \\ &= ie^2 \bar{u}^+(\mathbf{p}_{e'} = 0) u(\mathbf{p}_e = 0) \cdot 2E_K g(0) \\ &= \begin{cases} ie^2 2E_K g(0) & \text{if the initial and final electrons have the same spin,} \\ 0 & \text{if the initial and final electrons have different spins.} \end{cases} \end{aligned}$$

Thus the forward scattering differential cross section has the energy dependence

$$\left.\frac{d\sigma}{d\Omega}\right|_0 \propto E_K^2\,.$$

3063

Inelastic neutrino scattering in the quark model. Consider the scattering of neutrinos on free, massless quarks. We will simplify things and discuss only strangeness-conserving reactions, i.e. transitions only between the u and d quarks.

(a) Write down all the possible charged-current elastic reactions for both ν and $\bar{\nu}$ incident on the u and d quarks as well as the $\bar{u}$ and $\bar{d}$ antiquarks. (There are four such reactions.)

(b) Calculate the cross section for one such process, e.g. $\frac{d\sigma}{d\Omega}(\nu d \to \mu^- u)$.

(c) Give helicity arguments to predict the angular distribution for each of the reactions.

(d) Assume that inelastic ν (or $\bar{\nu}$)-nucleon cross sections are given by the sum of the cross sections for the four processes that have been listed above. Derive the quark model prediction for the ratio of the total cross section for antineutrino-nucleon scattering compared with neutrino-nucleon scattering, $\sigma^{\bar{\nu}N}/\sigma^{\nu N}$.

(e) The experimental value is $\sigma^{\bar{\nu}N}/\sigma^{\nu N} = 0.37 \pm 0.02$. What does this value tell you about the quark/antiquark structure of the nucleon?

(*Princeton*)

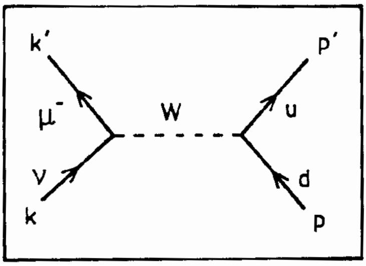

Fig. 3.19

Solution:

(a) The four charged-current interactions are (an example is shown in Fig. 3.19)

$$\nu_\mu d \to \mu^- u\,,$$

$$\bar{\nu}_\mu \bar{d} \to \mu^+ \bar{u}\,,$$

$$\nu_\mu \bar{u} \to \mu^- \bar{d}\,,$$

$$\bar{\nu}_\mu u \to \mu^+ d\,.$$

(b) For $\nu_\mu d \to \mu^- u$, ignoring m_μ, m_d, m_u and considering the reaction in the center-of-mass system, we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 S}|F|^2\,,$$

where the invariant mass squared is $S = -(k+p)^2 = -2kp$, and

$$|F|^2 = \frac{G_F^2}{2} \mathrm{Tr}\,[\not{k}'\gamma^\mu(1-\gamma_5)\not{k}\gamma^\nu(1-\gamma_5)] \times \frac{1}{2} \mathrm{Tr}\,[\not{p}'\gamma_\mu(1-\gamma_5)\not{p}\gamma_\nu(1-\gamma_5)] \cos^2\theta_c\,,$$

where θ_c is the Cabbibo mixing angle, and the factor $\frac{1}{2}$ arises from averaging over the spins of the initial muons. As

$$\begin{aligned} \mathrm{Tr}\,[\not{k}'\gamma^\mu(1-\gamma_5)\not{k}\gamma^\nu(1-\gamma_5)] &= \mathrm{Tr}\,[\not{k}'\gamma^\mu\not{k}\gamma^\nu(1-\gamma_5)^2] \\ &= 2\mathrm{Tr}\,[\not{k}'\gamma^\mu\not{k}\gamma^\nu] - 2\mathrm{Tr}\,[\not{k}'\gamma^\mu\not{k}\gamma^\nu\gamma_5] \\ &= 8\left(k'^\mu k^\nu + k'^\nu k^\mu + \frac{q^2}{2} g^{\mu\nu} - i\varepsilon^{\mu\nu\gamma\delta} k'_\gamma k_\delta\right), \end{aligned}$$

and similarly

$$\mathrm{Tr}\,[\not{p}'(\gamma_\mu(1-\gamma_5)\not{p}\gamma_\nu(1-\gamma_5)] = 8\left[p'_\mu p_\nu + p'_\nu p_\mu + \frac{q^2}{2} g_{\mu\nu} - i\varepsilon_{\mu\nu\alpha\beta} p'^\alpha p^\beta\right],$$

where $q^2 = -(k-k')^2 = -2kk'$ is the four-momentum transfer squared, we have

$$|F|^2 = 64 G_F^2 (k\cdot p)(k'\cdot p')\cos^2\theta_c = 16 G_F^2 S^2 \cos^2\theta_c\,,$$

and so

$$\frac{d\sigma}{d\Omega}(\nu d \to \mu^- u)_{\mathrm{cm}} = \frac{16 G_F^2 S^2 \cos^2\theta_c}{64\pi^2 S} = \frac{G_F^2 S}{4\pi^2}\cos^2\theta_c\,.$$

(c) In the weak interaction of hadrons, only the left-handed u, d quarks and e^-, μ^- and the right-handed quarks $\bar{u}$, $\bar{d}$ and e^+, μ^+ contribute. In the center-of-mass system, for the reactions $\nu d \to \mu^- u$ and $\bar{\nu}\bar{d} \to \mu^+\bar{u}$, the orbital angular momentum is zero and the angular distribution is isotropic as shown.

In the reactions $\nu\bar{u} \to \bar{d}\mu^-$ and $\bar{\nu}u \to \mu^+ d$ (Fig. 3.20), the total spins of the incoming and outgoing particles are both 1 and the angular distributions are

$$\frac{d\sigma}{d\Omega}(\nu\bar{u} \to \mu^-\bar{d})_{\mathrm{cm}} = \frac{G_F^2 S}{16\pi^2}\cos^2\theta_c(1-\cos\theta)^2\,,$$

$$\frac{d\sigma}{d\Omega}(\bar{\nu}u \to \mu^+ d)_{\mathrm{cm}} = \frac{G_F^2 S}{16\pi^2}\cos^2\theta_c(1-\cos\theta)^2\,.$$

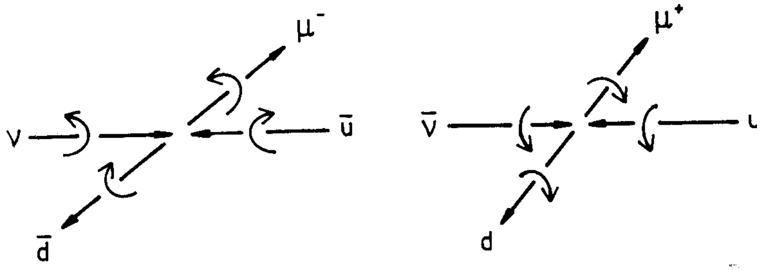

Fig. 3.20

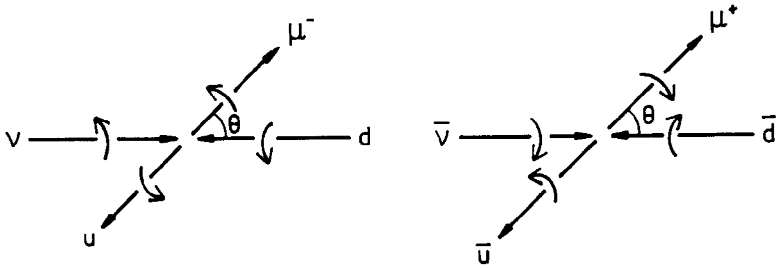

Fig. 3.21

(d) For the reactions $\nu d \to \mu^- u$ and $\bar{\nu}\bar{d} \to \mu^+ \bar{u}$ (Fig. 3.21) we have, similarly,

$$\frac{d\sigma}{d\Omega}(\nu d \to \mu^- u)_{\rm cm} = \frac{G_F^2 S}{4\pi^2} \cos^2\theta_c\,,$$

$$\frac{d\sigma}{d\Omega}(\bar{\nu}\bar{d} \to \mu^+ \bar{u})_{\rm cm} = \frac{G_F^2 S}{4\pi^2} \cos^2\theta_c\,.$$

Integrating over the solid angle Ω we have

$$\sigma_1 = \sigma(\nu d \to \mu^- u)_{cm} = \frac{G_F^2 S}{\pi} \cos^2\theta_c\,,$$

$$\sigma_2 = \sigma(\bar{\nu} u \to \mu^+ d)_{cm} = \frac{1}{3}\frac{G_F^2 S}{\pi} \cos^2\theta_c\,.$$

Neutron and proton contain quarks udd and uud respectively. Hence

$$\frac{\sigma(\nu n)}{\sigma(\bar{\nu} n)} = \frac{\sigma(\nu udd)}{\sigma(\bar{\nu} udd)} = \frac{2\sigma(\nu d)}{\sigma(\bar{\nu} u)} = \frac{2}{\left(\frac{1}{3}\right)} = 6 \,,$$

$$\frac{\sigma(\nu p)}{\sigma(\bar{\nu} p)} = \frac{\sigma(\nu uud)}{\sigma(\bar{\nu} uud)} = \frac{\sigma(\nu d)}{2\sigma(\bar{\nu} u)} = \frac{1}{2 \times \frac{1}{3}} = \frac{3}{2} \,.$$

If the target contains the same number of protons and neutrons,

$$\frac{\sigma(\nu N)}{\sigma(\bar{\nu} N)} = \frac{\sigma(\nu p) + \sigma(\nu n)}{\sigma(\bar{\nu} p) + \sigma(\bar{\nu} n)} = \frac{\frac{3}{2}\sigma(\bar{\nu} p) + \sigma(\nu n)}{\sigma(\bar{\nu} p) + \frac{1}{6}\sigma(\nu n)} = \frac{\frac{3}{3} + 3}{1 + \frac{3}{6}} = 3 \,,$$

where we have used $\sigma(\nu n) = 3\sigma(\bar{\nu} p)$.

(e) The experimental value $\sigma(\bar{\nu} N)/\sigma(\nu N) = 0.37 \pm 0.02$ is approximately the same as the theoretical value 1/3. This means that nucleons consist mainly of quarks, any antiquarks present would be very small in proportion. Let the ratio of antiquark to quark in a nucleon be α, then

$$\frac{\sigma(\bar{\nu} N)}{\sigma(\nu N)} = \frac{3\sigma(\bar{\nu} u) + 3\alpha\sigma(\bar{\nu} \bar{d})}{3\sigma(\nu d) + 3\alpha\sigma(\nu \bar{u})} = \frac{3 \times \frac{1}{3} + 3\alpha \times 1}{3 \times 1 + 3\alpha \times \frac{1}{3}} = \frac{1 + 3\alpha}{3 + \alpha} = 0.37 \,,$$

giving

$$\frac{1 + 3\alpha}{8} = \frac{0.37}{2.63} \,,$$

or

$$\alpha = 4 \times 10^{-2} \,.$$

3064

(a) According to the Weinberg-Salam model, the Higgs boson ϕ couples to every elementary fermion f (f may be a quark or lepton) in the form

$$\frac{em_f}{m_W} \phi \bar{f} f \,,$$

where m_f is the mass of the fermion f, e is the charge of the electron, and m_W is the mass of the W boson. Assuming that the Higgs boson decays primarily to the known quarks and leptons, calculate its lifetime in terms

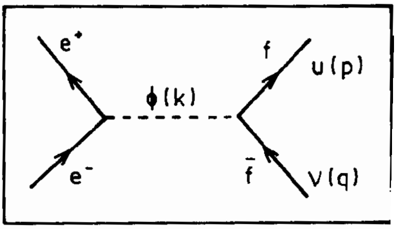

Fig. 3.22

of its mass m_H. You may assume that the Higgs boson is much heavier than the known quarks and leptons.

(b) Some theorists believe that the Higgs boson weighs approximately 10 GeV. If so do you believe it would be observed (in practice) as a resonance in e^+e^- annihilation (Fig. 3.22)? Roughly how large would the signal to background ratio be at resonance?

(*Princeton*)

Solution:

(a) Fermi's Golden Rule gives for decays into two fermions the transition probablity

$$\Gamma_f = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2p_0} \frac{d^3\mathbf{q}}{(2\pi)^3 2q_0} \cdot \frac{(2\pi)^4}{2k_0} \delta^4(k-p-q)|M|^2 ,$$

where

$$\begin{aligned} |M|^2 &= \mathrm{Tr} \sum_{s,t} \left[\left(\frac{em_f}{m_W} \right)^2 \bar{u}_s(p)\nu_t(q)\bar{\phi}\phi\bar{\nu}_t(q)u_s(p) \right] \\ &= \left(\frac{em_f}{m_W} \right)^2 \mathrm{Tr}\,[\not{p}\not{q} - m_f^2] \\ &= 4\left(\frac{em_f}{m_W} \right)^2 (p \cdot q - m_f^2) . \end{aligned}$$

As $p+q=k$, we have $p\cdot q = \frac{k^2-p^2-q^2}{2} = \frac{m_H^2-2m_f^2}{2}$,

$$|M|^2 = 4\left(\frac{em_f}{m_W} \right)^2 \left(\frac{m_H^2 - 4m_f^2}{2} \right) = 2\left(\frac{em_f}{m_W} \right)^2 m_H^2 \left(1 - \frac{4m_f^2}{m_H^2} \right)$$

in the rest system of the Higgs boson. Then

$$\begin{aligned}\Gamma_f &= \int \frac{d^3\mathbf{p}d^3\mathbf{q}}{(2\pi)^6 4p_0q_0}\frac{(2\pi)^4}{2m_H}\delta^4(k-p-q)|M|^2 \\ &= \frac{1}{(2\pi)^2}\int\frac{d^3\mathbf{p}}{4p_0q_0}\cdot\frac{1}{2m_H}\delta^4(m_H-p_0-q_0)|M|^2 \\ &= \frac{4\pi}{(2\pi)^2\cdot 4q_0^2\cdot 2m_H}\int q^2dq\cdot\delta(m_H-2q_0)|M|^2\,.\end{aligned}$$

With $qdq = q_0dq_0$, we have

$$\begin{aligned}\Gamma_f &= \frac{1}{8\pi m_H}\int\frac{q^2}{q_0^2}\frac{q_0}{q}dq_0\delta(m_H-2q_0)|M|^2 \\ &= \frac{1}{8\pi m_H}\frac{2}{m_H}\cdot\frac{1}{2}\left[\left(\frac{m_H}{2}\right)^2-m_f^2\right]^{1/2}\cdot 2\frac{e^2m_f^2m_H^2}{m_W^2}\left(1-\frac{4m_f^2}{m_H^2}\right) \\ &= \frac{e^2m_f^2m_H}{4\pi m_W^2}\left(1-\frac{4m_f^2}{m_H^2}\right)^{3/2}\cdot\frac{1}{2} \\ &\approx \frac{e^2m_f^2m_H}{8\pi m_W^2}\quad \text{if}\quad m_H\gg m_f\,.\end{aligned}$$

Then $\Gamma = \Sigma\Gamma_i = \frac{e^2m_H}{8\pi m_W^2}\sum a_fm_f^2$, with $a_f = 1$ for lepton and $a_f = 3$ for quark. Assuming $m_H \approx 10$ GeV, $m_W \approx 80$ GeV, and with $m_u = m_d = 0.35$ GeV, $m_s = 0.5$ GeV, $m_c = 1.5$ GeV, $m_b = 4.6$ GeV, $m_e = 0.5\times10^{-3}$ GeV, $m_\mu = 0.11$ GeV, $m_\tau = 1.8$ GeV, we have

$$\begin{aligned}\sum a_fm_f^2\left(1-\frac{4m_f^2}{m_H^2}\right)^{3/2} &\approx \sum_{f\neq b}a_fm_f^2+3m_b^2\left(1-\frac{4m_b^2}{m_H^2}\right)^{3/2} \\ &= 0.005^2+0.11^2+1.8^2+3\times(0.35^2+0.35^2+0.5^2+1.5^2) \\ &\quad+3\times4.6^2\left[1-4\times\left(\frac{4.6}{10}\right)^2\right]^{3/2} \\ &= 15.3\ \text{GeV}^2\,,\end{aligned}$$

and hence

$$\Gamma = \frac{1}{8\pi}\left(\frac{e^2}{\hbar c}\right)\hbar c \cdot \frac{m_H}{m_W^2}\sum a_f m_f^2 \left(1-\frac{4m_f^2}{m_H^2}\right)^{3/2}$$

$$\approx \frac{1}{8\pi \times 137} \times \frac{10}{80^2} \times 15.3 = 6.9 \times 10^{-6} \text{ GeV},$$

or

$$\tau = \Gamma^{-1} = 145 \text{ MeV}^{-1} = 6.58 \times 10^{-22} \times 145 \ s = 9.5 \times 10^{-20} \ s.$$

(b) The process $e^+e^- \to \bar{f}f$ consists of the following interactions:

$$e^+e^- \xrightarrow{\gamma, Z^0} \bar{f}f \quad \text{and} \quad e^+e^- \xrightarrow{H} \bar{f}f.$$

When $\sqrt{S} = 10$ GeV, Z^0 exchange can be ignored. Consider $e^+e^- \xrightarrow{\gamma} \bar{f}f$. The total cross section is given approximately by

$$\sigma_{\bar{f}f} \approx \frac{4\pi\alpha^2}{3S}Q_f^2,$$

where α is the fine structure constant and Q_f is the charge (in units of the electron charge) of the fermion. Thus

$$\sigma(^+e^- \xrightarrow{\gamma} \bar{f}f) = \frac{4\pi\alpha^2}{3S}\sum Q_f^2 \cdot a_f,$$

where $a_f = 1$ for lepton, $a_f = 3$ for quark. As $\Sigma Q_f^2 a_f = (\frac{1}{9} + \frac{4}{9} + \frac{1}{9} + \frac{4}{9} + \frac{1}{9}) \times 3 + 1 + 1 + 1 = \frac{20}{3}$, $S = m_H^2$, we have

$$\sigma(e^+e^- \xrightarrow{\gamma} \bar{f}f) = \frac{4\pi\alpha^2}{3S}\frac{20}{3} \approx 8\pi\alpha^2/S = \frac{8\pi\alpha^3}{m_H^2}.$$

For the $e^+e^- \xrightarrow{H} \bar{f}f$ process we have at resonance ($J_H = 0$)

$$\sigma(e^+e^- \xrightarrow{H} \bar{f}f) = \pi\lambda\!\!\!^{-2}\Gamma_{ee}/\Gamma \approx \pi p^{*-2}\Gamma_{ee}/\Gamma.$$

As a rough estimate, taking $\Gamma_{ee} \approx m^2$, i.e., $\Gamma_{ee}/\Gamma \approx (0.5 \times 10^{-3})^2/15.3 \approx 1.6 \times 10^{-8}$ and $p^{*2} = \frac{m_H^2}{4}$, we have

$$\sigma(e^+e^- \xrightarrow{H} \bar{f}f) : \sigma(e^+e^- \xrightarrow{\gamma} \bar{f}f) = \left(\frac{4\pi}{m_H^2}1.6 \times 10^{-8}\right)\left(\frac{8\pi\alpha^2}{m_H^2}\right)^{-1}$$

$$\approx 0.8 \times 10^{-8}/\alpha^2 \approx 1.5 \times 10^{-4}.$$

In e^+e^- annihilation in the 10 GeV region, the background, which is mainly due to the photon process, is almost 10^4 times as strong as the H_0 resonance process. The detection of the latter is all but impossible.

3065

Parity Violation. Recently the existence of a parity-violating neutral current coupled to electrons was demonstrated at SLAC. The experiment involved scattering of polarized electrons off (unpolarized) protons.

(a) Why are polarized electrons required? What is the signature for the parity violation?

(b) Estimate the magnitude of the effect.

(c) How would this parity violation manifest itself in the passage of light through matter?

(*Princeton*)

Solution:

(a) To observe the parity violation, we must measure the contribution of the pseudoscalar terms to the interaction, such as the electron and hadron spinor terms. Hence we must study the interaction between electrons of fixed helicity and an unpolarized target (or conversely electrons and a polarized target, or electrons and target both polarized). The signature for parity violation is a measureable quantity relating to electron helicity, such as the dependence of scattering cross section on helicity, etc.

(b) Electron-proton scattering involves two parts representing electromagnetic and weak interactions, or specifically scattering of the exchanged photons and exchanged Z^0 bosons. Let their amplitudes be A and B. Then

$$\sigma \approx A^2 + |A \cdot B| + B^2 \,.$$

In the energy range of the experiment, $A^2 \gg B^2$. As parity is conserved in electromagnetic interaction, parity violation arises from the interference term (considering only first order effect):

$$\frac{|A \cdot B|}{A^2 + B^2} \approx \frac{|A \cdot B|}{A^2} \approx \frac{|B|}{|A|} \approx \frac{G_f}{e^2/q^2} \,,$$

where G_F is the Fermi constant, e is the electron charge, q^2 is the square of the four-momentum transfer. We have

$$\frac{G_F}{e^2/q^2} \approx \frac{10^{-5}m_p^{-2}}{4\pi/137}q^2 \approx 10^{-4}q^2/m_p^2 \approx 10^{-4}q^2 \text{ GeV}^{-2}.$$

as $m_p \approx 1$ GeV. In the experiment at SLAC, $E_e \approx 20$ GeV, $q^2 \approx 10 \sim 20$ GeV^2, and the parity violation should be of order of magnitude 10^{-3}. The experiment specifically measured the scattering cross sections of electrons of different helicities, namely the asymmetry

$$A = \frac{\sigma(\lambda = 1/2) - \sigma(\lambda = -1/2)}{\sigma(\lambda = 1/2) + \sigma(\lambda = -1/2)} \approx q^2[a_1 + a_2 f(y)],$$

where a_1 and a_2 involve A_e, V_Q and A_Q, V_e respectively, being related to the quark composition of proton and the structure of the weak neutral current, $\sigma(\lambda = 1/2)$ is the scattering cross section of the incoming electrons of helicity 1/2, $y = (E-E')/E$, E and E' being the energies of the incoming and outgoing electrons respectively. From the experimental value of A, one can deduce the weak neutral current parameter.

(c) Parity violation in atomic range manifests itself as a slight discrepancy in the refractive indices of the left-handed and right-handed circularly polarized lights passing through a high-nuclear-charge material. For a linearly polarized light, the plane of polarization rotates as it passes through matter by an angle

$$\phi = \left(\frac{\omega L}{2c}\right) \mathrm{Re}(n_+ - n_-),$$

where L is the thickness of the material, ω is the angular frequency of the light, n_+ and n_- are the refractive indices of left-handed and right-handed circularly polarized lights

3066

There are now several experiments searching for proton decay. Theoretically, proton decay occurs when two of the quarks inside the proton exchange a heavy boson and become an antiquark and an antilepton. Suppose this boson has spin 1. Suppose, further, that its interactions conserve charge, color and the $SU(2)\times U(1)$ symmetry of the Weinberg–Salam model.

(a) It is expected that proton decay may be described by a fermion effective Lagrangian. Which of the following terms may appear in the effective

Lagrangian? For the ones which are not allowed, state what principle or facts forbid them, e.g., charge conservation.

(1) $u_R u_L d_R e_L^-$	(2) $u_R d_R d_L \nu_L$
(3) $u_R u_L d_L e_R^-$	(4) $u_L d_L d_L \nu_L$
(5) $u_R u_R d_R e_R^-$	(6) $u_L u_L d_R e_R^-$
(7) $u_L d_L d_R \nu_L$	(8) $u_L u_R d_R \nu_L$

All Fermions are incoming.

(b) Consider the decay $p \to e^+ H$, where H is any hadronic state with zero strangeness. Show that the average positron polarization defined by the ratio of the rates

$$P = \frac{\Gamma(p \to e_L^+ H) - \Gamma(p \to e_R^+ H)}{\Gamma(p \to e_L^+ H) + \Gamma(p \to e_R^+ H)}$$

is independent of the hadronic state H.

(c) If the spin-one boson has a mass of 5×10^{14} GeV and couples to fermions with electromagnetic strength (as predicted by grand unified theories), give a rough estimate of the proton lifetime (in years).

(*Princeton*)

Solution:

(a) (1), (2), (3), (4), (5) are allowed, (6), (7), (8) are forbidden. Note that (6) is forbidden because $u_L u_L$ is not an isospin singlet, (7) is forbidden because it does not contain ν_R (8) is forbidden because total charge is not zero.

(b) The decay process $p \to e^+ H$ can be described with the equivalent interaction Lagrangian

$$L_{\text{eff}} = [g_1(\bar{d}^c_{\alpha R}\mu_{\beta R})(\bar{\mu}^c_{\gamma L} e_L - \bar{d}^c_{\gamma L}\nu_L) + g_2(\bar{d}^c_{\alpha L}\mu_{\beta L})(\bar{\mu}^c_{\gamma R} e_R)]\varepsilon_{\alpha\beta\gamma}\,,$$

where g_1, g_2 are equavalent coupling coefficients, c denotes charge conjugation, α, β, γ are colors signatures, $\varepsilon_{\alpha\beta\gamma}$ is the antisymmetric matrix. Thus the matrix element of $p \to e_L^+ H$ is proportional to g_1, that of $e_R^+ H$ is proportional to g_2, both having the same structure. Hence

$$P = \frac{|g_1|^2 - |g_2|^2}{|g_1|^2 + |g_2|^2}$$

and is independent of the choice of the H state.

(c) An estimate of the lifetime of proton may be made, mainly on the basis of dimensional analysis, as follows. A massive spin-1 intermediate particle contributes a propagator $\sim m^{-2}$, where m is its mass. This gives rise to a transition matrix element of $\mathcal{M} \sim m^{-2}$. The decay rate of proton is thus

$$\Gamma_p \propto |\mathcal{M}|^2 \sim m^{-4}\,,$$

or

$$\Gamma_p \sim \frac{C\alpha^2}{m^4}\,,$$

where $\alpha = e^2/\hbar c$ is the dimensionless coupling constant for electromagnetic interaction (**Problem 3001**), and C is a constant. The lifetime of proton τ_p has dimension

$$[\tau_p] = M^{-1}\,,$$

since in our units $Et \sim \hbar = 1$ and so $[t] = [E]^{-1} = M^{-1}$. This means that

$$[C] = M^4 M^1 = M^5\,.$$

For a rough estimate we may take $C \sim m_p^5$, m_p being the proton mass. Hence, with $m \approx 5 \times 10^{14}$ GeV, $m_p \approx 1$ GeV,

$$\tau_p = \Gamma_p^{-1} \sim \frac{m^4}{\alpha^2 m_p^5} = 1.2 \times 10^{63}\ \text{GeV}^{-1}\,,$$

or, in usual units,

$$\tau_p \sim \frac{1.2 \times 10^{63}\hbar}{365 \times 24 \times 60 \times 60} = 3 \times 10^{31}\text{years}\,.$$

3067

It is generally recognized that there are at least three different kinds of neutrino. They can be distinguished by the reactions in which the neutrinos are created or absorbed. Let us call these three types of neutrino ν_e, ν_μ and ν_τ. It has been speculated that each of the neutrinos has a small but finite rest mass, possibly different for each type.

Let us suppose, for this question, that there is a small perturbing interaction between these neutrino types, in the absence of which all three types have the same nonzero rest mass M_0. Let the matrix element of this perturbation have the same real value $\hbar\omega_1$ between each pair of neutrino types. Let it have zero expectation value in each of the states ν_e, ν_μ and ν_τ.

(a) A neutrino of type ν_e is produced at rest at time zero. What is the probability, as a function of time, that the neutrino will be in each of the other states?

(b) (Can be answered independently of (a).) An experiment to detect these "neutrino oscillations" is being performed. The flight path of the neutrinos is 2000 meters. Their energy is 100 GeV. The sensitivity is such that the presence of 1% of neutrinos of one type different from that produced at the start of flight path can be measured with confidence. Take M_0 to be 20 electron volts. What is the smallest value of $\hbar\omega_1$ that can be detected? How does this depend on M_0?

(*UC, Berkeley*)

Solution:

(a) Let $|\psi\rangle = a_1(t)|\nu_e\rangle + a_2(t)|\nu_\mu\rangle + a_3(t)|\nu_\tau\rangle$. Initially the interaction Hamiltonian is zero. Use of the perturbation matrix

$$H' = \begin{pmatrix} 0 & \hbar\omega_1 & \hbar\omega_1 \\ \hbar\omega_1 & 0 & \hbar\omega_1 \\ \hbar\omega_1 & \hbar\omega_1 & 0 \end{pmatrix}$$

in the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \hbar\omega_1 \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

gives

$$\begin{cases} i\dot{a}_1 = \omega_1(a_2 + a_3)\,, \\ i\dot{a}_2 = \omega_1(a_1 + a_3)\,, \\ i\dot{a}_3 = \omega_1(a_1 + a_2)\,. \end{cases}$$

Eliminating a_1 from the last two equations gives

$$i(\dot{a}_3 - \dot{a}_2) = -\omega_1(a_3 - a_2)\,,$$

or

$$a_3(t) - a_2(t) = Ae^{i\omega_1 t}\,.$$

At time $t = 0$, $a_2(0) = a_3(0) = 0$, so $A = 0$, $a_2 = a_3$, with which the system of equations becomes

$$\begin{cases} i\dot{a}_1 = 2\omega_1 a_2\,, \\ i\dot{a}_2 = \omega_1(a_1 + a_2)\,. \end{cases}$$

Eliminating a_1 again, we have

$$\ddot{a}_2 + i\omega_1\dot{a}_2 + 2\omega_1^2 a_2 = 0\,,$$

whose solution is $a_2(t) = A_1e^{i\omega_1 t} + A_2e^{-i2\omega_1 t}$. At time $t = 0$, $a_2(0) = 0$, giving

$$A_1 + A_2 = 0, \quad \text{or} \quad a_2 = A_1(e^{i\omega_1 t} - e^{-i2\omega_1 t})\,.$$

Hence

$$\dot{a}_1 = -i2\omega_1 A_1(e^{i\omega_1 t} - e^{-i2\omega_1 t})\,,$$

or

$$a_1 = -2A_1e^{i\omega_1 t} - A_1e^{-i2\omega_1 t}\,.$$

Initially only $|\nu_e\rangle$ is present, so

$$a_1(0) = 1\,.$$

Thus $A_1 = -1/3$, and

$$a_2 = a_3 = \frac{1}{3}(e^{-i2\omega_1 t} - e^{i\omega_1 t})\,.$$

The probability that the neutrino is in $|\nu_\mu\rangle$ or $|\nu_\tau\rangle$ at time t is

$$\begin{aligned} P(|\nu_\mu\rangle) = \ & P(|\nu_\tau\rangle) = |a_2|^2 = \frac{1}{9}(e^{-i2\omega_1 t} - e^{i\omega_1 t})(e^{i2\omega_1 t} - e^{-i\omega_1 t}) \\ & = \frac{2}{9}[1 - \cos(3\omega_1 t)]\,. \end{aligned}$$

(b) For simplicity consider the oscillation between two types of neutrino only, and use a maximum mixing angle of $\theta = 45°$. From **Problem 3068** we have

$$P(\nu_1 \to \nu_2, t) = \sin^2 2\theta \sin^2\left(\frac{E_1 - E_2}{2}t\right) = \sin^2\left[1.27\left(\frac{l}{E}\Delta m^2\right)\right]\,,$$

where l is in m, E in MeV, and Δm^2 in eV^2. For detection of ν_2 we require $P \geq 0.01$, or $\sin\left[1.27\left(\frac{l}{E}\Delta m^2\right)\right] \geq 0.1$, giving

$$\Delta m^2 \geq \frac{100 \times 10^3}{1.27 \times 2000} \times \arcsin 0.1 = 3.944 \text{ eV}^2 .$$

As $\Delta m^2 = (M_0 + \hbar\omega_1)^2 - M_0^2 \approx 2M_0\hbar\omega_1$, we require

$$\hbar\omega_1 \geq \frac{3.944}{2 \times 20} = 9.86 \times 10^{-2} \text{ eV} \approx 0.1 \text{ eV}$$

Note that the minimum value of $\hbar\omega_1$ varies as M_0^{-1} if $M_0 \gg \hbar\omega_1$.

3068

Suppose that ν_e and ν_μ, the Dirac neutrinos coupled to the electron and the muon, are a mixture of two neutrinos ν_1 and ν_2 with masses m_1 and m_2:

$$\nu_e = \nu_1 \cos\theta + \nu_2 \sin\theta ,$$

$$\nu_\mu = -\nu_1 \sin\theta + \nu_2 \cos\theta ,$$

θ being the mixing angle.

The Hamiltonian has a mass term $H = m_1\bar{\nu}_1\nu_1 + m_2\bar{\nu}_2\nu_2$.

(a) Express the stationary-state masses m_1 and m_2, and the mixing angle θ in terms of the mass matrix elements of the Hamiltonian in the ν_e, ν_μ representation:

$$H = \bar{\nu}_l M_{ll'} \nu_{l'} \quad \text{with} \quad l, l' = e, \mu .$$

(b) Specify under what conditions there is maximal mixing or no mixing.

(c) Suppose that at $t = 0$ one has pure ν_e. What is the probability for finding a ν_μ at time t?

(d) Assuming that p (the neutrino momentum) is $\gg m_1$ and m_2, find the oscillation length.

(e) If neutrino oscillations were seen in a detector located at a reactor, what would be the order of magnitude of the oscillation parameter

$\Delta = |m_1^2 - m_2^2|$? (Estimate the particle energies and the distance between the source and the detector.)

(f) Answer (e) for the case of neutrino oscillations observed at a 100 GeV proton accelerator laboratory.

(*Princeton*)

Solution:

(a) In the ν_e, ν_μ representation the Hamiltonian is

$$H = \begin{pmatrix} M_{ee} & M_{e\mu} \\ M_{\mu e} & M_{\mu\mu} \end{pmatrix} .$$

For simplicity assume $M_{\mu e} = M_{e\mu}$. Then the eigenvalues are the solutions of

$$\begin{vmatrix} M_{ee} - m & M_{\mu e} \\ M_{\mu e} & M_{\mu\mu} - m \end{vmatrix} = 0 ,$$

i.e.

$$m^2 - (M_{ee} + M_{\mu\mu})m + (M_e M_\mu - M_{\mu e}^2) = 0 .$$

Solving the equation we have the eigenvalues

$$m_1 = \frac{1}{2}\left[(M_{ee} + M_{\mu\mu}) - \sqrt{(M_{ee} - M_{\mu\mu})^2 + 4M_{\mu e}^2}\right] ,$$

$$m_2 = \frac{1}{2}\left[(M_{ee} + M_{\mu\mu}) + \sqrt{(M_{ee} - M_{\mu\mu})^2 + 4M_{\mu e}^2}\right] .$$

In the ν_e, ν_μ representation let $\nu_2 = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. The operator equation

$$H\nu_2 = m_2\nu_2 ,$$

i.e.,

$$\begin{pmatrix} M_{ee} & M_{\mu e} \\ M_{\mu e} & M_{\mu\mu} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = m_2 \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} ,$$

gives, with the normalization condition $a_1^2 + a_2^2 = 1$,

$$a_1 = \frac{M_{\mu e}}{\sqrt{M_{\mu e}^2 + (m_2 - M_{ee})^2}} ,$$

$$a_2 = \frac{m_2 - M_{ee}}{\sqrt{M_{\mu e}^2 + (m_2 - M_{ee})^2}} .$$

The mixing equations

$$\nu_e = \nu_1 \cos\theta + \nu_2 \sin\theta\,,$$

$$\nu_\mu = -\nu_1 \sin\theta + \nu_2 \cos\theta$$

can be written as

$$\nu_1 = \nu_e \cos\theta - \nu_\mu \sin\theta\,,$$

$$\nu_2 = \nu_e \sin\theta + \nu_\mu \cos\theta\,.$$

However, as $\nu_2 = a_1\nu_e + a_2\nu_\mu$,

$$\begin{aligned}\tan\theta &= \frac{a_1}{a_2} = \frac{M_{\mu e}}{m_2 - M_{ee}} \\ &= \frac{2M_{\mu e}}{M_{\mu\mu} - M_{ee} + \sqrt{(M_{ee} - M_{\mu\mu})^2 + 4M_{\mu e}^2}}\,,\end{aligned}$$

or

$$\theta = a = \arctan\left(\frac{2M_{\mu e}}{M_{\mu\mu} - M_{ee} + \sqrt{(M_{ee} - M_{\mu\mu})^2 + 4M_{\mu e}^2}}\right)\,.$$

(b) When $M_{\mu\mu} = M_{ee}$, mixing is maximum and the mixing angles is $\theta = 45°$. In this case ν_1 and ν_2 are mixed in the ratio 1 : 1. When $M_{\mu e} = 0$, $\theta = 0$ and there is no mixing.

(c) At $t = 0$, the neutrinos are in a pure electron-neutrino state ν_e which is a mixture of states ν_1 and ν_2:

$$\nu_e = \nu_1 \cos\theta + \nu_2 \sin\theta\,.$$

The state ν_e changes with time. Denote it by $\psi_e(t)$. Then

$$\begin{aligned}\psi_e(t) &= \nu_1 e^{-iE_1 t}\cos\theta + \nu_2 e^{-iE_2 t}\sin\theta \\ &= (\nu_e\cos\theta - \nu_\mu\sin\theta)e^{-iE_1 t}\cos\theta + (\nu_e\sin\theta + \nu_\mu\cos\theta)e^{-iE_2 t}\sin\theta \\ &= (\cos^2\theta e^{-iE_1 t} + \sin^2\theta e^{-iE_2 t})\nu_e + \sin\theta\cos\theta(e^{-iE_1 t} + e^{-iE_2 t})\nu_\mu\,.\end{aligned}$$

So the probability of finding a ν_μ at time t is

$$\begin{aligned} P &= |\langle\nu_\mu|\psi_e(t)\rangle|^2 \\ &= \sin^2\theta\cos^2\theta| - e^{-iE_1t} + e^{-iE_2t}|^2 \\ &= \frac{1}{2}\sin^2(2\theta)\{1 - \cos[(E_1 - E_2)t]\} \\ &= \sin^2(2\theta)\sin^2\left(\frac{E_1 - E_2}{2}t\right), \end{aligned}$$

where E_1, E_2 are the eigenvalues of the states $|\nu_1\rangle$, $|\nu_2\rangle$ respectively

(d) As $E_1 - E_2 = \frac{E_1^2 - E_2^2}{E_1 + E_2} = \frac{1}{2E}[p_1^2 + m_1^2 - p_2^2 - m_2^2] \approx \frac{\Delta m^2}{2E}$ with $\Delta m^2 = m_1^2 - m_2^2$, $E = \frac{1}{2}(E_1 + E_2)$,

$$P = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E}t\right) = \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E}l\right),$$

since $l = t\beta \approx t$, the neutrino velocity being $\beta \approx 1$ as $p \gg m$. In ordinary units the second argument should be

$$\frac{\Delta m^2}{4E\hbar}\frac{l}{c} = \frac{10^{-12}}{4 \times 197 \times 10^{-13}}\frac{\Delta m^2}{E}l = \frac{1.27l\Delta m^2}{E}$$

with l in m, Δm^2 in eV^2, E in MeV.

Thus

$$P = \sin^2(2\theta)\sin^2(1.27l\Delta m^2/E),$$

and the oscillation period is $1.27l\Delta m^2/E \approx 2\pi$. Hence $\Delta m^2 l/E \ll 1$ gives the non-oscillation region, $\Delta m^2 l/E \approx 1$ gives the region of appreciable oscillation, and $\Delta m^2 l/E \gg 1$ gives the region of average effect.

(e) Neutrinos from a reactor have energy $E \approx 1$ MeV, and the distance between source and detector is several meters. As oscillations are observed,

$$\Delta m^2 = E/l \approx 0.1 \sim 1 \text{ eV}^2 .$$

(f) With protons of 100 GeV, the pions created have energy E_π of tens of GeV. Then the neutrino energy $E_\nu \geq 10$ GeV. With a distance of observation 100 m,

$$\Delta m^2 \approx E/l \approx 10^2 \sim 10^3 \text{ eV}^2 .$$

For example, for an experiment with $E_\nu \approx 10$ GeV, $l = 100\ m$,

$$\Delta m^2 = \frac{2\pi E}{1.27l} \approx 5 \times 10^2 \text{ eV}^2 \,.$$

3069

(a) Neutron n and antineutron $\bar{n}$ are also neutral particle and antiparticle just as K^0 and $\bar{K}^0$. Why is it not meaningful to introduce linear combinations of n_1 and n_2, similar to the K_1^0 and K_2^0? Explain this.

(b) How are the pions, muons and electrons distringuished in photographic emulsions and in bubble chambers? Discuss this briefly.

(*SUNY Buffalo*)

Solution:

(a) n and $\bar{n}$ are antiparticles with respect to each other with baryon numbers 1 and -1 respectively. As the baryon number B is conserved in any process, n and $\bar{n}$ are eigenstates of strong, electromagnetic and weak interactions. If they are considered linear combinations of n_1 and n_2 which are not eigenstates of strong, electromagnetic and weak interactions, as n and $\bar{n}$ have different B the linear combination is of no meaning. If some interaction should exist which does not conserve B, then the use of n_1 and n_2 could be meaningful. This is the reason for the absence of oscillations between neutron and antineutron.

(b) It is difficult to distinguish the charged particles e, μ, π over a general energy range merely by means of photographic emulsions or bubble chambers. At low energies ($E < 200 \sim 300$ MeV), they can be distinguished by the rate of ionization loss. The electron travels with the speed of light and causes minimum ionization. Muon and pion have different velocities for the same energy. As $-dE/dx \sim v^{-2}$, we can distinguish them in principle from the different ionization densities of the tracks in the photographic emulsion. However, it is difficult in practice because their masses are very similar.

At high energies, ($E > 1$ GeV), it is even more difficult to distinguish them as they all have velocity $v \approx c$. Pions may be distinguished by their interaction with the nuclei of the detecting medium. However the Z values of the materials in photographic emulsions and bubble chambers are rather low and the probability of nuclear reaction is not large. Muons and electrons do not cause nuclear reactions and cannot be destinguished this way. With bubble chambers, a transverse magnetic field is usually applied and the curvatures of the tracks can be used to distinguish the particles, provided the energy is not too high. For very low energies, muons and pions can be distinguished by their characteristic decays.

3070

Neutron-Antineutron Oscillations. If the baryon number is conserved, the transition $n \leftrightarrow \bar{n}$, know as "neutron oscillation" is forbidden. The experimental limit on the time scale of such oscillations in free space and zero magnetic field is $\tau_{n-\bar{n}} \geq 3 \times 10^6$ s. Since neutrons occur abundantly in stable nuclei, one would naively think it possible to obtain a much better limit on $\tau_{n-\bar{n}}$. The object of this problem is to understand why the limit is so poor.

Let H_0 be the Hamiltonian of the world in the absence of any interaction which mixes n and $\bar{n}$. Then

$$H_0|n\rangle = m_n c^2 |n\rangle, \quad H_0|\bar{n}\rangle = m_n c^2 |\bar{n}\rangle$$

for states at rest. Let H' be the interaction which turns n into $\bar{n}$ and vice versa:

$$H'|n\rangle = \varepsilon|\bar{n}\rangle, \quad H'|\bar{n}\rangle = \varepsilon|n\rangle\,,$$

where ε is real and H' does not flip spin.

(a) Start with a neutron at $t = 0$ and calculate the probability that it will be observed to be an antineutron at time t. When the probability is first equal to 50%, call that time $\tau_{n-\bar{n}}$. In this way convert the experimental limit on $\tau_{n-\bar{n}}$ into a limit on ε. Note $m_n c^2 = 940$ MeV.

(b) Now reconsider the problem in the presence of the earth's magnetic field $B_0 = 0.5$ Gs. The magnetic moment of the neutron is $\mu_n \approx -6 \times 10^{-18}$ MeV/Gs. The magnetic moment of the antineutron is opposite. Begin with a neutron at $t = 0$ and calculate the probability it will be observed to be an antineutron at time t. Ignore possible radioactive transitions. [Hint: work to lowest order in small quantities.]

(c) Nuclei with spin have non-vanishing magnetic fields. Explain briefly and qualitatively, in light of part (b), how neutrons in such nuclei can be so stable while $\tau_{n-\bar{n}}$ is only bounded by $\tau_{n-\bar{n}} \geq 3 \times 10^6$ sec.

(d) Nuclei with zero spin have vanishing average magnetic field. Explain briefly why neutron oscillation in such nuclei is also suppressed.

(*MIT*)

Solution:

(a) Consider the Hamiltonian $H = H_0 + H'$. As (using units where $c = 1$, $\hbar = 1$)

$$H(|n\rangle + |\bar{n}\rangle) = m_n(|n\rangle + |\bar{n}\rangle) + \varepsilon(|n\rangle + |\bar{n}\rangle) = (m_n + \varepsilon)(|n\rangle + |\bar{n}\rangle)\,,$$

$$H(|n\rangle - |\bar{n}\rangle) = m_n(|n\rangle - |\bar{n}\rangle) - \varepsilon(|n\rangle - |\bar{n}\rangle) = (m_n - \varepsilon)(|n\rangle - |\bar{n}\rangle)\,,$$

$|n\rangle \pm |\bar{n}\rangle$ are eigenstates of H. Denote these by $|n_\pm\rangle$.

Let Φ_0 be the wave function at $t = 0$. Then

$$\Phi_0|n\rangle = \frac{1}{2}(|n_+\rangle + |n_-\rangle)\,,$$

and the wave function at the time t is

$$\begin{aligned}\Phi &= \frac{1}{2}(|n_+\rangle e^{-i(m_n+\varepsilon)t} + |n_-\rangle e^{-i(m_n-\varepsilon)t}) \\ &= \frac{1}{2}e^{-im_nt}[(e^{-i\varepsilon t} + e^{i\varepsilon t})|n\rangle + (e^{-i\varepsilon t} - e^{i\varepsilon t})|\bar{n}\rangle \\ &= e^{-im_nt}(\cos\varepsilon t|n\rangle - i\sin\varepsilon t|\bar{n}\rangle)\,.\end{aligned}$$

The probability of observing an antineutron at time t is $P = \sin^2\varepsilon t$. As at $t = \tau_{n-\bar{n}}$, $\sin^2\varepsilon t|_{n-\bar{n}} = \sin^2\varepsilon\tau_{n-\bar{n}} = 1/2$,

$$\varepsilon\tau_{n-\bar{n}} = \pi/4\,.$$

Hence

$$\varepsilon \le \frac{\pi}{4}\cdot\frac{1}{3\times 10^6} = 2.62\times 10^{-7}\ s^{-1} = 2.62\times 10^{-7}\hbar = 1.73\times 10^{-28}\ \text{MeV}\,.$$

(b) The Hamiltonian is now $H = H_0 + H' - \boldsymbol{\mu}\cdot\mathbf{B}$. Then

$$H|n\rangle = m_n|n\rangle + \varepsilon|\bar{n}\rangle - \mu_n B|n\rangle = (m_n - \mu_n B)|n\rangle + \varepsilon|\bar{n}\rangle\,,$$

$$H|\bar{n}\rangle = m_n|\bar{n}\rangle + \varepsilon|n\rangle + \mu_n B|\bar{n}\rangle = (m_n + \mu_n B)|\bar{n}\rangle + \varepsilon|n\rangle\,.$$

Here we assume that $n, \bar{n}$ are polarized along z direction which is the direction of $\mathbf{B}$, i.e., $s_z(n) = 1/2$, $s_z(\bar{n}) = 1/2$. Note this assumption does not affect the generality of the result.

Let the eigenstate of H be $a|n\rangle + b|\bar{n}\rangle$. As

$$\begin{aligned} H(a|n\rangle + b|\bar{n}\rangle =& \ aH|n\rangle + bH|\bar{n}\rangle \\ =& \ [a(m_n - \mu_n B) + b\varepsilon)]|n\rangle + [b(m_n + \mu_n B) + a\varepsilon]|\bar{n}\rangle \,, \end{aligned}$$

we have

$$\frac{a(m_n - \mu_n B) + b\varepsilon}{a} = \frac{b(m_n + \mu_n B) + a\varepsilon}{b} \,,$$

or

$$b^2 - a^2 = \frac{2\mu_n B}{\varepsilon} ab = Aab \,,$$

where $A = \frac{2\mu_n B}{\varepsilon} \approx \frac{6\times 10^{-18}}{1.73\times 10^{-28}} = 3.47 \times 10^{10}$, and $b^2 + a^2 = 1$. Solving for a and b we have either

$$\begin{cases} a \approx 1, \\ b \approx -1/A, \end{cases} \quad \text{or} \quad \begin{cases} a \approx 1/A, \\ b \approx 1. \end{cases}$$

Hence the two eigenstates of H are

$$|n_+\rangle = \frac{1}{A}|n\rangle + |\bar{n}\rangle, \quad |n_-\rangle = |n\rangle - \frac{1}{A}|\bar{n}\rangle \,.$$

At $t = 0$, $\Phi_0 = |n\rangle = \frac{|n_+\rangle + A|n_-\rangle}{A + \frac{1}{A}} = \frac{A}{1+A^2}|n_+\rangle + \frac{A^2}{1+A^2}|n_-\rangle$.

At time t the wave function is

$$\Phi = \frac{A}{1+A^2}|n_+\rangle e^{-iE_+ t} + \frac{A^2}{1+A^2}|n_-\rangle e^{-iE_- t} \,,$$

where $E_+ = m_n - \mu_n B + A\varepsilon$, $E_- = m_n - \mu_n B - \varepsilon/A$. So

$$\begin{aligned} \Phi =& \ e^{-i(m_n - \mu_n B)t} \left(\frac{A}{1+A^2}|n_+\rangle e^{-iA\varepsilon t} + \frac{A^2}{1+A^2}|n_-\rangle e^{-i\frac{\varepsilon}{A}t} \right) \\ =& \ \frac{1}{1+A^2} e^{-i(m_n - \mu_n B)t} [(e^{-iA\varepsilon t} + A^2 e^{i\frac{\varepsilon}{A}t})|n\rangle + (Ae^{-iA\varepsilon t} - Ae^{i\frac{\varepsilon}{A}t})|\bar{n}\rangle] \,. \end{aligned}$$

The probability of observing an $\bar{n}$ at time t is

$$\begin{aligned} P &= \frac{A^2}{(1+A^2)^2}|e^{-iA\varepsilon t} - e^{i\frac{\varepsilon}{A}t}|^2 \\ &= \frac{A^2}{(1+A^2)^2}\left[2 - 2\cos\left(A\varepsilon - \frac{\varepsilon}{A}\right)t\right] \\ &= \frac{4A^2}{(1+A^2)^2}\sin^2\left(\frac{A^2-1}{2A}\varepsilon t\right) \\ &\approx \frac{4}{A^2}\sin^2\left(\frac{A}{2}\varepsilon t\right). \end{aligned}$$

(c) Nuclei with spin have non-vanishing magnetic fields and so the results of (b) are applicable. For $\tau_{n-\bar{n}} \geq 3 \times 10^6$ s, or $\varepsilon \leq 1.73 \times 10^{-28}$ MeV, $A = \frac{2\mu_n B}{\varepsilon}$ is quite a large number, so the probability of observing an $\bar{n}$ is almost zero ($\approx 1/A^2$). Thus there is hardly any oscillation between n and $\bar{n}$; the nuclei are very stable.

(d) While nuclei with zero spin have zero mean magnetic field $\langle B\rangle$, the mean square of B, $\langle B^2\rangle$, is not zero because the magnetic field is not zero everywhere in a nucleus. The probability of observing an $\bar{n}$, $P \approx 1/\langle A^2\rangle = \frac{\varepsilon^2}{4\mu_n^2\langle B^2\rangle}$, is still small and almost zero. Hence neutron oscillation in such nuclei is also suppressed.

3071

It has been conjectured that stable magnetic monopoles with magnetic charge $g = c\hbar/e$ and mass $\approx 10^4$ GeV might exist.

(a) Suppose you are supplied a beam of such particles. How would you establish that the beam was in fact made of monopoles? Be as realistic as you can.

(b) Monopoles might be pair-produced in cosmic ray collisions. What is the threshold for this reaction $(p + p \to M + \bar{M} + p + p)$?

(c) What is a practical method for recognizing a monopole in a cosmic ray event?

(*Princeton*)

Solution:

(a) The detection of magnetic monopoles makes use of its predicted characteristics as follows:

(1) Magnetic monopole has great ionizing power. Its specific ionization $-\frac{dE}{dx}$ is many times larger than that of a singly charged particle when it passes through matter, say a nuclear track detector like nuclear emulsion or cloud chamber.

(2) A charge does not suffer a force when moving parallel to a magnetic field, whereas a magnetic monopole is accelerated or decelerated (depending on the sign of its magnetic charge) when moving parallel to a magnetic field. A magnetic monopole can acquire an energy of 400 MeV when passing through a magnetic field of 10 kGs, whereas the energy of a charge does not change in the same process.

(3) When a magnetic monopole passes though a closed circuit, it would be equivalent to a large magnetic flux passing through the coil and a large current pulse would be induced in the circuit.

(4) When a charge and a magnetic monopole pass through a transverse magnetic field, they would suffer different deflections. The former is deflected transversely in the direction of $\mathbf{F} = \frac{1}{c}\mathbf{v} \times \mathbf{B}$, while the latter is deflected parallel or antiparallel to the magnetic field direction.

(b) Consider the process $p + p \to M + \bar{M} + p + p$, where one of the initial protons is assumed at rest, as is generally the case. As $E^2 - P^2$ is invariant and the particles are produced at rest in the center-of-mass system at threshold, we have

$$(E + m_p)^2 - P^2 = (2m_M + 2m_p)^2 \,,$$

where $E^2 - P^2 = m_p^2$, or

$$E = \frac{(2m_M + 2m_p)^2 - 2m_p^2}{2m_p} \,.$$

Taking $m_M = 10^4$ GeV, $m_p = 1$ GeV, we have $E \approx 2 \times 10^8$ GeV as the laboratory threshold energy.

If in the reaction the two initial protons have the same energy and collide head-on as in colliding beams, the minimum energy of each proton is given by

$$2E = 2m_M + 2m_p \,,$$

Hence $E \approx m_M = 10^4$ GeV.

(c) To detect magnetic monopoles in cosmic ray events, in principle, any one of the methods in (a) will do. A practical one is to employ a solid track detector telescope. When a particle makes a thick track in the system of detectors, several of the detectors together can distinguish the track due to a multiply charged particle from that due to a magnetic monopole, as in the former case the track thickness is a function of the particle velocity, but not in the latter case. Particle idetification is more reliable if a magnetic field is also used.

If magnetic monopoles are constantly created in cosmic-ray collisions above the earth they may be detected as follows. As a monopole loses energy rapidly by interacting with matter it eventually drops to the earth's surface. Based on their tendency of moving to the magnetic poles under the action of a magnetic field, we can collect them near the poles. To detect monopoles in a sample, we can place a coil and the sample between the poles of a strong magnet (Fig. 3.23). As a magnetic monopole moves from the sample to a pole a current pulse will be produced in the coil.

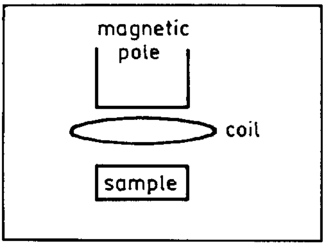

Fig. 3.23

3. STRUCTURE OF HADRONS AND THE QUARK MODEL (3072–3090)

3072

Describe the evidence (one example each) for the following conclusions:

(a) Existence of quarks (substructure or composite nature of mesons and baryons).

(b) Existence of the "color" quantum number.

(c) Existence of the "gluon".

(*Wisconsin*)

Solution:

(a) The main evidence supporting the quark theory is the non-uniform distribution of charge in proton and neutron as seen in the scattering of high energy electrons on nucleons, which shows that a nucleon has internal structure. Gell-Mann *et al.* discovered in 1961 the SU(3) symmetry of hadrons, which indicates the inner regularity of hadronic structure. Basing on these discoveries, Gell-Mann and Zweig separately proposed the quark theory. In it they assumed the existence of three types of quark, u, d, s and their antiparticles, which have fractional charges and certain quantum numbers, as constituents of hadrons: a baryon consists of three quarks; a meson, a quark and an antiquark. The quark theory was able to explain the structure, spin and parity of hadrons. It also predicted the existence of the Ω particle, whose discovery gave strong support to the quark theory. Later, three types of heavy quarks c, b, t were added to the list of quarks.

(b) The main purpose of postulating the color quantum number was to overcome the statistical difficulty that according to the quark theory Δ^{++}, a particle of spin 3/2, should consist of three u quarks with parallel spins, while the Pauli exclusion principle forbids three ferminions of parallel spins in the same ground state. To get over this Greenberg proposed in 1964 the color dimension for quarks. He suggested that each quark could have one of three colors. Although the three quarks of Δ^{++} have parallel spins, they have different colors, thus avoiding violation of the Pauli exclusion principle. The proposal of the 'color' freedom also explained the relative cross section R for producing hadrons in e^+e^- collisions. Quantum electrodynamics gives, for $E_{cm} < 3$ GeV, $R = \sum_i Q_i^2$, where Q_i is the charge of the ith quark, summing over all the quarks that can be produced at that energy. Without the color freedom, $R = 2/3$. Including the contribution of the color freedom, $R = 2$, in agreement with experiment.

(c) According to quantum chromodynamics, strong interaction takes place through exchange of gluons. The theory predicts the emission of hard gluons by quarks. In the electron-positron collider machine PETRA in DESY the "three-jet" phenomenon found in the hadronic final state provides strong evidence for the existence of gluons. The phenomenon is interpreted as an electron and a position colliding to produce a quark-antiquark pair, one of which then emits a gluon. The gluon and the two original quarks separately fragment into hadron jets, producing three jets in the final state. From the observed rate of three-jets events the coupling constant α_s for strong interaction can be deduced.

3073

Explain why each of the following particles cannot exist according to the quark model.

(a) A baryon of spin 1.
(b) An antibaryon of electric charge +2.
(c) A meson with charge +1 and strangeness −1.
(d) A meson with opposite signs of charm and strangeness.

(*Wisconsin*)

Solution:

(a) According to the quark model, a baryon consists of three quarks. Since the quark spin is 1/2, they cannot combine to form a baryon of spin 1.

(b) An antibaryon consists of three antiquarks. To combine three antiquarks to form an antribaryon of electric charge +2, we require antiquarks of electric charge +2/3. However, there is no such antiquark in the quark model.

(c) A meson consists of a quark and an antiquark. As only the s quark $(S=-1, Z=-\frac{1}{3})$ has nonzero strangeness, to form a meson of strangeness −1 and electric charge 1, we need an s quark and an antiquark of electric charge 4/3. There is, however, no such an antiquark.

(d) A meson with opposite signs of strangeness and charm must consist of a strange quark (antistrange quark) and anticharmed quark (charmed quark). Since the strangeness of strange quark and the charm of charm quark are opposite in sign, a meson will always have strangeness and charm of the same sign. Therefore there can be no meson with opposite signs of strangeness and charm.

3074

The Gell-Mann–Nishijima relationship which gives the charge of mesons and baryons in terms of certain quantum numbers is

$$q = e(I_3 + B/2 + S/2)\,.$$

(a) Identify the terms I_3, B and S, and briefly explain their usefulness in discussing particle reactions.

(b) Make a table of the values of these quantum numbers for the family: proton, antiproton, neutron, antineutron.

(*Wisconsin*)

Solution:

(a) I_3 is the third component of isospin and denotes the electric charge state of the isospin I. In strong and electromagnetic interactions I_3 is conserved, while in weak interaction it is not.

B is the baryon number. $B = 0$ for a meson and $B = 1$ for a baryon. $\Delta B = 0$ for any interaction. The conservation of baryon number means that proton is stable.

Table 3.7

Quantum number	p	$\bar{p}$	n	$\bar{n}$
I_3	1/2	$-1/2$	$-1/2$	1/2
B	1	-1	1	-1
S	0	0	0	0

S is the strangeness, introduced to account for the associated production of strange particles. S is conserved in strong and electromagnetic interactions, which implies that strange particles must be produced in pairs. S is not conserved in weak interaction, so a strange particle can decay through weak interaction to ordinary particles.

(b) The I_3, B, and S values of for nucleons are listed in Table 3.7.

3075

Give the quantum numbers and quark content of any 5 different hadrons.

(*Wisconsin*)

Solution:

The quantum numbers and quark content of five most common hadrons are listed in Table 3.8

Table 3.8

Hadron	Electric charge (Q)	Baryon number(B)	Spin(J)	Isospin(I)	I_3	*quark* content
n	0	1	1/2	1/2	$-1/2$	udd
p	1	1	1/2	1/2	1/2	uud
π^-	-1	0	0	1	-1	$d\bar{u}$
π^0	0	0	0	1	0	$\frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$
π^+	1	0	0	1	1	$u\bar{d}$

3076

Give a specific example of an SU(3) octet by naming all 8 particles. What is the value of the quantum numbers that are common to all the particles of the octet you have selected?

(*Wisconsin*)

Solution:

Eight nucleons and hyperons form an SU(3) octet, shown in Fig. 3.24. Their common quantum numbers are $J^{)} = \frac{1^+}{2}$, $B = 1$.

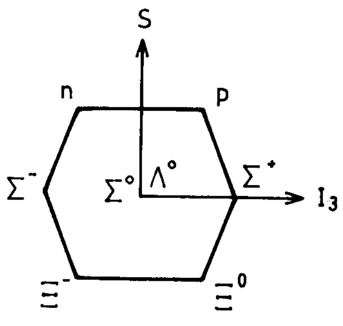

Fig. 3.24

3077

Calculate the ratio $R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$

(a) just below the threshold for "charm" production,
(b) above that threshold but below the b quark production threshold.

(*Wisconsin*)

Solution:

Quantum electrodynamics (QED) gives

$$\sigma(e^+e^- \to q_i\bar{q}_i \to \text{hadrons}) = \frac{4\pi\alpha^2}{S}Q_i^2\,,$$

where S is the square of the energy in the center-of-mass frame of e^+, e^-, α is the coupling constant, and Q_i is the electric charge (unit e) of the ith quark, and

$$\sigma(e^+e^- \to \mu^+\mu^-) = \frac{4\pi\alpha^2}{3S}\,.$$

Hence

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = \sum_i \frac{\sigma(e^+e^- \to q_i\bar{q}_i \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_i Q_i^2\,,$$

where $\sum_i$ sums over all the quarks which can be produced with the given energy.

(a) With such an energy the quarks which can be produced are u, d and s. Thus

$$R = 3\sum_i Q_i^2 = 3\times\left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}\right) = 2\,.$$

(b) The quarks that can be produced are now u, d, s and c. As the charge of c quark is $2/3$,

$$R = 3\sum_i Q_i^2 = 3\times\left(\frac{4}{9}+\frac{1}{9}+\frac{1}{9}+\frac{4}{9}\right) = \frac{10}{3}\,.$$

3078

(a) It is usually accepted that hadrons are bound states of elementary, strongly-interacting, spin-1/2 fermions called quarks. Briefly describe some evidence for this belief.

The lowest-lying mesons and baryons are taken to be bound states of the u, d, and s (or p, n, and λ in an alternative notation) quarks, which form an SU(3) triplet.

(b) Define what is meant by the approximate Gell-Mann–Neeman global SU(3) symmetry of strong interactions. How badly is this symmetry broken?

(c) Construct the lowest-lying meson and baryon SU(3) multiplets, giving the quark composition of each state and the corresponding quantum numbers J, P, I, Y, S, B and, where appropriate, G.

(d) What is the evidence for another quantum number "color", under which the strong interactions are exactly symmetric? How many colors are there believed to be? What data are used to determine this number?

(e) It is by now well established that there is a global SU(3) singlet quark c with charge 2/3 and a new quantum number C preserved by the strong interactions. Construct the lowest-lying $C = 1$ meson and baryon states, again giving J, P, I, Y, S and B.

(f) What are the main semileptonic decay modes (i.e., those decays that contain leptons and hadrons in the final state) of the $C = 1$ meson?

(g) Denoting the strange $J = 1$ and $J = 0$ charmed mesons by F^* and F respectively and assuming that $m_{F^*} > m_F + m_\pi$ (something not yet established experimentally), what rate do you expect for $F^* \to F\pi$. What might be the main decay mode of the F^*?

(*Princeton*)

Solution:

(a) The evidence supporting the quark model includes the following: (1) The deep inelastic scattering data of electrons on nucleons indicate that nucleon has substructure. (2) The SU(3) symmetry of hadrons can be explained by the quark model. (3) The quark model gives the correct cross-section relationship of hadronic reactions. (4) The quark model can explain the abnormal magnetic moments of nucleons.

(b) The approximate SU(3) symmetry of strong interactions means that isospin multiplets with the same spin and parity, i.e., same J^P, but different strangeness numbers can be transformed into each other. They are considered as the supermultiplet states of the same original particle U with different electric charges (I_3) and hypercharges (Y).

If SU(3) symmetry were perfect, particles of the same supermultiplet should have the same mass. In reality the difference of their masses can be quite large, which shows that such a supersymmetry is only approximate. The extent of the breaking of the symmetry can be seen from the difference between their masses, e.g., for the supermultiplet of 0^- mesons, $m_{\pi^0} = 135$ MeV, $m_{K^0} = 498$ MeV.

(c) The lowest-lying SU(3) multiplets of mesons and baryons formed by u, d and s quarks are as follows.

For mesons, the quarks can form octet and singlet of J^P equal to 0^- and 1^-. They are all ground states with $l = 0$, with quark contents and quantum numbers as listed in Table 3.9.

For baryons, which consist of three quarks each, the lowest-lying states are an octet of $J^P = \frac{1}{2}^+$ and a decuplet of $J^P = \frac{3}{2}^+$. They are ground states with $l = 0$ and other characteristics as given in Table 3.10.

(d) The purpose of introducing the color freedom is to overcome statistical difficulties. In the quark model, a quark has spin 1/2 and so must obey the Fermi statistics, which requires the wave function of a baryon to be antisymmetric for exchanging any two quarks. In reality, however, there are some baryons having quark contents sss or uuu, for which the wave functions are symmetric for quark exchange. To get over this contradiction, it is

Table 3.9 Quantum numbers and quark contents of meson supermultiplets of $J^P = 0^-, 1^-$

	0^-	1^-	quark content	I	I_3	Y	B	S	G
	π^+	ρ^+	$\bar{d}u$	1	+1	0	0	0	−1
	π^0	ρ^0	$(u\bar{u} - d\bar{d})/\sqrt{2}$	1	0	0	0	0	−1
	π^-	ρ^-	$\bar{u}d$	1	−1	0	0	0	−1
	K^+	K^{*+}	$\bar{s}u$	1/2	1/2	1	0	0	
octet	K^-	K^{*-}	$s\bar{u}$	1/2	−1/2	−1	0	0	
	K^0	K^{*0}	$\bar{s}d$	1/2	−1/2	1	0	0	
	$\bar{K}^0$	$\bar{K}^{*0}$	$s\bar{d}$	1/2	1/2	−1	0	0	
	$\eta(549)$		$\frac{(u\bar{u}+\bar{d}d-2\bar{s}s)}{\sqrt{6}}$	0	0	0	0	0	+1
		$\omega(783)$	$\frac{u\bar{u}+d\bar{d}}{\sqrt{2}}$	0	0	0	0	0	−1
singet	$\eta(958)$		$\frac{(u\bar{u}+\bar{d}d+\bar{s}s)}{\sqrt{3}}$	0	0	0	0	0	+1
		$\psi(1020)$	$\bar{s}s$	0	0	0	0	0	−1

Table 3.10 Characteristics of baryon octet ($\frac{1}{2}^+$) and decuplet ($\frac{3}{2}^+$)

J^P	particles	the quark content	I	I_3	Y	B	S
	p	uud	1/2	1/2	1	1	0
	n	udd	1/2	−1/2	1	1	0
	Σ^+	uus	1	1	0	1	−1
$\frac{1}{2}^+$	Σ^0	$s(ud+du)/\sqrt{2}$	1	0	0	1	−1
	Σ^-	dds	1	−1	0	1	−1
	Ξ^0	uss	1/2	1/2	−1	1	−2
	Ξ^-	dss	1/2	−1/2	−1	1	−2
	Λ^0	$s(du-ud)/\sqrt{2}$	0	0	0	1	−1
	Δ^-	ddd	3/2	−3/2	1	1	0
	Δ^0	ddu	3/2	−1/2	1	1	0
	Δ^+	duu	3/2	1/2	1	1	0
	Δ^{++}	uuu	3/2	3/2	1	1	0
$\frac{3}{2}^+$	Σ^{*-}	sdd	1	−1	0	1	−1
	Σ^{*0}	sdu	1	0	0	1	−1
	Σ^{*+}	suu	1	1	0	1	−1
	Ξ^{*-}	ssd	1/2	−1/2	−1	1	−2
	Ξ^{*0}	ssu	1/2	1/2	−1	1	−2
	Ω	sss	0	0	−2	1	−3

assumed that there is an additional quantum number called "color" which has three values. The hypothesis of color can be tested by the measurement of R in high-energy e^+e^- collisions, which is the ratio of the cross sections for producing hadrons and for producing a muonic pair

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} .$$

Suppose the energy of e^+e^- system is sufficient to produce all the three flavors of quarks. If the quarks are colorless,

$$R = \sum_i Q_i^2 = \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) = \frac{2}{3} ;$$

if each quark can have three colors,

$$R = 3\sum_i Q_i^2 = 3 \times \left(\frac{4}{9} + \frac{1}{9} + \frac{1}{9}\right) = 2 ,$$

The latter is in agreement with experiments.

(e) A $c(\bar{c})$ quark and an ordinary antiquark (quark) can combine into a charmed meson which can have J^P equal to 0^- or 1^-. The characteristics of charmed mesons are listed in Table 3.11. They can be regarded as the result of exchanging an $u(\bar{u})$ quark for a c $(\bar{c})$ quark in an ordinary meson. There are six meson states with $C = 1$, namely $D^+, D^0, F^+, D^{*+}, D^{*0}$ and F^{*+}. Also, a c quark and two ordinary quarks can combine into a charmed baryon of $J^P = \frac{1}{2}^+$ or $\frac{3}{2}^+$. Theoretically there should be 9 charmed baryons of $J^P = \frac{1}{2}^+$, whose characteristics are included in Table 3.12. Experimentally, the first evidence for charmed baryons Λ_c^+, Σ_c^{++} appeared in 1975, that for charmed mesons D^+, D^0, F^+ appeared in 1976–77.

Correspondingly, baryons with $C = 1$ and $J^P = \frac{3}{2}^+$ should exist. Theoretically there are six such baryons, with quark contents (ddc), (duc), (uuc), (cds), (css), (cus). Their expected quantum numbers, except for $J = 3/2$, have not been confirmed experimentally, but they should be the same as those of Σ_c^0, Σ_c^+, Σ_c^{++}, S^0, T^0 and S^+, respectively.

(f) The semileptonic decay of a meson with $C = 1$ actually arises from the semileptonic decay of its c quark:

$$c \to s\ l^+\ \nu_e, \quad \text{with amplitude} \sim \cos\theta_c\,,$$

$$c \to d\ l^+\ \nu_e, \quad \text{with amplitude} \sim \sin\theta_c\,,$$

Table 3.11 Characteristics of mesons with charmed quarks

J^P	particle	quark content	I	I_3	Y	S	C	B
	D^0	$\bar{u}c$	1/2	−1/2	1	0	1	0
	D^+	$\bar{d}c$	1/2	1/2	1	0	1	0
	$\bar{D}^0$	$\bar{c}u$	1/2	1/2	−1	0	−1	0
0^-	D^-	$\bar{c}d$	1/2	−1/2	−1	0	−1	0
	F^+	$\bar{s}c$	0	0	2	1	1	0
	F^-	$\bar{c}s$	0	0	−2	−1	−1	0
	η_0	$\bar{c}c$	0	0	0	0	0	0
	D^{*0}	$\bar{u}c$	1/2	−1/2	1	0	1	0
	D^{*+}	$\bar{d}c$	1/2	1/2	1	0	1	0
	$\bar{D}^{*0}$	$\bar{c}u$	1/2	1/2	−1	0	−1	0
1^-	D^{*-}	$\bar{c}d$	1/2	−1/2	−1	0	−1	0
	F^{*+}	$\bar{s}c$	0	0	2	1	1	0
	F^{*-}	$\bar{c}s$	0	0	−2	−1	−1	0
	J/ψ	$\bar{c}c$	0	0	0	0	0	0

Table 3.12 Characteristics of charmed baryons ($C = 1$) of $J^P = \frac{1}{2}^+$

Particle	Quark content	I	I_3	Y	S	C	B
Σ_c^{++}	cuu	1	1	2	0	1	1
Σ_c^+	$c(ud+du)/\sqrt{2}$	1	0	2	0	1	1
Σ_c^0	cdd	1	-1	2	0	1	1
S^+	$c(us+su)/\sqrt{2}$	1/2	1/2	1	-1	1	1
S^0	$c(ds+sd)/\sqrt{2}$	1/2	$-1/2$	1	-1	1	1
T^0	css	0	0	0	-2	1	1
Λ_c^+	$c(ud-du)/\sqrt{2}$	0	0	2	0	1	1
A^+	$c(us-su)/\sqrt{2}$	1/2	1/2	1	-1	1	1
A^0	$c(ds-sd)/\sqrt{2}$	1/2	$-1/2$	1	-1	1	1

where θ_c is the Cabibbo angle. For example, the reaction $D^0 \to K^- e^+ \nu_e$, is a Cabibbo-allowed decay, and $D^0 \to \pi^- e^+ \nu_e$, is a Cabibbo-forbidden decay.

(g) If F^* exists and $m_{F^*} > m_F + m_\pi$, then $F^* \to \pi^0 F$ is a strong decay and hence the main decay channel, as it obeys all the conservation laws. For example, F^* has $J^{PC} = 1^{--}$, F has $J^{PC} = 0^{-+}$, pion has $J^{PC} = 0^{-+}$. In the decay $F^* \to \pi^0 F$, the orbital angular momentum of the πF system is $l = 1$, the parity of the final state is $P(\pi^0)P(F)(-1)^l = -1$. Also, $C(\pi^0)C(F) = 1$. Thus the final state has $J^P = 1^-$, same as $J^P(F^*)$.

Another competiting decay channel is $F^* \to \gamma + F$, which is an electromagnetic decay with the relative amplitude determined by the interaction constant and the phase-space factor.

3079

Imagine that you have performed an experiment to measure the cross sections for the "inclusive" process

$$a + N \to \mu^+ + \mu^- + \text{anything}$$

where $a = p$, π^+ or π^-, and N is a target whose nuclei have equal numbers of protons and neutrons.

You have measured these three cross sections as a function of m, the invariant mass of the muon pair, and of s, the square of the energy in the center of mass.

The following questions are designed to test your understanding of the most common model used to describe these processes, the quark-antiquark annihilation model of Drell and Yan.

(a) In the simplest quark picture (baryons being composed of three quarks and mesons of a quark-antiquark pair), what is the predicted ratio

$$\frac{d\sigma_{pN}(s,m)}{dm} : \frac{d\sigma_{\pi^+N}(s,m)}{dm} : \frac{d\sigma_{\pi^-N}(s,m)}{dm} ?$$

(b) An accurate measurement shows each element of the ratio to be nonzero. How do you modify you answer to (a) to account for this? (A one or two sentence answer is sufficient.)

(c) Given this modification, how do you expect the ratio to behave with m (for fixed s)? (Again, a one or two sentence qualitative answer is sufficient.)

(d) How would the predicted values of the three cross sections change if the concept of color were introduced into the naive model?

(e) An important prediction of Drell and Yan is the concept of scaling. Illustrate this with a formula or with a sketch (labeling the ordinate and the abscissa).

(f) How would you determine the quark structure of the π^+ from your data?

(g) How would you estimate the antiquark content of the proton?

(*Princeton*)

Solution:

(a) According to the model of Drell and Yan, these reactions are processes of annihilation of a quark and an antiquark with emission of a leptonic pair. QED calculations show that if the square of the energy in the center of mass of the muons $s_{\mu u} \gg m_\mu^2, m_q^2$, the effect of m_μ, m_q can be neglected, yielding

$$\sigma(\mu^+\mu^- \to \gamma \to q_i\bar{q}_i) = \frac{4\pi}{3s_{\mu\mu}}\alpha^2 Q_i^2 \, ,$$

where Q_i is the charge number of the i quark, α is the fine structure constant. Making use of the principle of detailed balance, we find

$$\sigma(q_i\bar{q}_i \to \gamma \to \mu^+\mu^-) = \frac{4\pi}{3s}\alpha^2 Q_i^2 = Q_i^2\sigma_0 \, ,$$

where s is the square of total energy in the center-of-mass system of the two quarks, i.e., $s = s_{\mu\mu} = m^2$, m being the total energy in the center-of-mass system of the $\mu^+\mu^-$ (i.e. in the c.m.s. of $q_i\bar{q}_i$). Thus in the simplest quark picture,

$$\sigma(d\bar{d} \to \mu^+\mu^-) \approx \frac{1}{9}\sigma_0 \,,$$

$$\sigma(u\bar{u} \to \mu^+\mu^-) \approx \frac{4}{9}\sigma_0 \,,$$

For $pN \to \mu^+\mu^- + X$, as there is no antiquark in the proton and in the neutron,

$$\frac{d\sigma(s,m)}{dm} = 0 \,.$$

For the same s and m, recalling the quark contents of p, n, π^+ and π^- are uud, udd, $u\bar{d}$, $\bar{u}d$ respectively, we find

$$\sigma(\pi^+ N) = \sigma\left[(u\bar{d}) + \frac{1}{2}(uud + udd)\right] = \frac{1}{2}\sigma(d\bar{d})(1+2) \approx \frac{1}{6}\sigma_0 \,,$$

$$\sigma(\pi^- N) = \sigma\left[(\bar{u}d) + \frac{1}{2}(uud + udd)\right] = \frac{1}{2}\sigma(u\bar{u})(2+1) \approx \frac{2}{3}\sigma_0 \,,$$

and hence

$$\frac{d\sigma_{pN}(s,m)}{dm} : \frac{d\sigma_{\pi^+N}(s,m)}{dm} : \frac{d\sigma_{\pi^-N}(s,m)}{dm} = 0 : 1 : 4 \,.$$

(b) The result that $\frac{d\sigma_{pN}(s,m)}{dm}$ is not zero indicates that there are antiquarks in proton and neutron. Let the fraction of antiquarks in a proton or a neutron be α, where $\alpha \ll 1$. Then the fraction of quark is $(1-\alpha)$ and so

$$\begin{aligned}\sigma_{pN} = \sigma\Big\{ & 2\alpha\bar{u} + \frac{1}{2}[2(1-\alpha)u + (1-\alpha)u] + 2(1-\alpha)u \\ & + \frac{1}{2}(2\alpha\bar{u} + \alpha\bar{u}) + \alpha\bar{d} + \frac{1}{2}[(1-\alpha)d + 2(1-\alpha)d] + (1-\alpha)d \\ & + \frac{1}{2}(\alpha\bar{d} + 2\alpha\bar{d})]\Big\}\end{aligned}$$

$$= \sigma(u\bar{u})[3\alpha(1-\alpha)+3(1-\alpha)\alpha] + \sigma(d\bar{d})\left[\frac{3}{2}\alpha(1-\alpha)+\frac{3}{2}(1-\alpha)\alpha\right]$$

$$= 6\alpha(1-\alpha)\sigma(u\bar{u}) + 3\alpha(1-\alpha)\sigma(d\bar{d})$$

$$= 3\alpha(1-\alpha)[2\sigma(u\bar{u})+\sigma(d\bar{d})] \approx 3\alpha(1-\alpha)\sigma_0\,,$$

$$\sigma_{\pi^+N} = \sigma\left\{\bar{d}+\frac{1}{2}[(1-\alpha)d+2(1-\alpha)d]+u+\frac{1}{2}(2\alpha\bar{u}+\alpha\bar{u})\right\}$$

$$= \frac{3}{2}(1-\alpha)\sigma(d\bar{d}) + \frac{3}{2}\alpha\sigma(u\bar{u})$$

$$= \frac{3}{2}\sigma(d\bar{d}) + \frac{3}{2}\alpha[\sigma(u\bar{u})-\sigma(d\bar{d})] \approx \frac{1}{6}(1+3\alpha)\sigma_0\,,$$

$$\sigma_{\pi^-N} = \sigma\left\{\bar{u}+\frac{1}{2}[2(1-\alpha)u+(1-\alpha)u]+d+\frac{1}{2}(\alpha\bar{d}+2\alpha\bar{d})\right\}$$

$$= \frac{3}{2}(1-\alpha)\sigma(u\bar{u}) + \frac{3}{2}\alpha\sigma(d\bar{d})$$

$$= \frac{3}{2}\sigma(u\bar{u}) + \frac{3}{2}\alpha[\sigma(d\bar{d})-\sigma(u\bar{u})] \approx \frac{1}{6}(4-3\alpha)\sigma_0\,.$$

Hence

$$\frac{d\sigma_{pN}(s,m)}{dm} : \frac{d\sigma_{\pi^+N}(s,m)}{dm} : \frac{d\sigma_{\pi^-N}(s,m)}{dm} = 18\alpha(1-\alpha) : (1+3\alpha) : (4-3\alpha)\,.$$

For example if $\alpha = 0.01$, the cross sections are in the ratio $0.17 : 1 : 3.85$. Thus the cross sections, especially $\frac{d\sigma_{pN}(s,m)}{dm}$, is extremely sensitive to the fraction of antiquarks in the nucleon.

(c) An accurate derivation of the ratio is very complicated, as it would involve the structure functions of the particles (i.e., the distribution of quarks and their momenta in the nucleon and meson). If we assume that the momenta of the quarks in a nucleon are the same, then the cross section in the quark-antiquark center-of-mass system for a head-on collision is

$$\sigma(q_i\bar{q}_i \to \mu^+\mu^-) = \frac{4\pi}{3m^2}\alpha^2 Q_i^2\,,$$

or

$$\frac{d\sigma}{dm} \sim m^{-3}\alpha^2 Q_i^2\,.$$

Hence σ is proportional to m^{-2}, in agreement with experiments.

(d) The ratio would not be affected by the introduction of color.

(e) Scaling means that in a certain energy scale the effect on Drell-Yan process of smaller energies can be neglected. For instance, for second order electromagnetic processes, we have the general formula $d\sigma_{em} = \alpha^2 f(s, q^2, m_l)$, where s is the square of energy in the center-of-mass system, q^2 is the square of the transferred 4-momentum, and m_l is the mass of the charged particle. If s and $|q^2| \gg m_l^2$, it is a good approximation to set $m_l = 0$, yielding

$$d\sigma = \alpha^2 f(s, q^2)\,.$$

Thus, for example, in the process $q_i\bar{q}_i \to \mu^+\mu^-$, if $m \gg m_\mu$, m_q we can let $m_\mu \approx m_q \approx 0$ and obtain

$$\sigma(q_i\bar{q}_i \to \mu^+\mu^-) \propto Q_i^2/m^2\,.$$

(f) The good agreement between the calculated result

$$\frac{d\sigma_{\pi^+N}(s,m)}{dm} : \frac{d\sigma_{\pi^-N}(s,m)}{dm} = 1:4$$

and experiment supports the assumption of quark contents of $\pi^+(u\bar{d})$ and $\pi^-(\bar{u}d)$.

(g) By comparing the calculation in (b) with experiment we can determine the fraction α of antiquark in the quark content of proton.

3080

The bag model of hadron structure has colored quarks moving as independent spin-$\frac{1}{2}$ Dirac particles in a cavity of radius R. The confinement of the quarks to this cavity is achieved by having the quarks satisfy the free Dirac equation with a mass that depends on position: $m = 0$ for $r < R$ and $m = \infty$ for $r > R$. The energy operator for the quarks contains a term $\int d^3 r m(r)\bar{\psi}\psi$. In order for this term to give a finite contribution to the energy, the allowed Dirac wave functions must satisfy $\bar{\psi}\psi = 0$ where $m = \infty$ (i.e. for $r > R$), This is achieved by choosing a boundary condition at R on the solution of the Dirac equation.

(a) Show that the boundary conditions

(1) $\psi(|\mathbf{x}| = R) = 0$, (2) $i\hat{\mathbf{x}} \cdot \boldsymbol{\gamma}\psi(|\mathbf{x}| = R) = \psi(|\mathbf{x}| = R)$, where $\hat{\mathbf{x}}$ is the unit radial vector from the center of the cavity, both achieve the effect of setting $\bar{\psi}\psi = 0$ at $|\mathbf{x}| = R$. Which condition is physically acceptable?

(b) The general s-wave solution to the free massless Dirac equation can be written (using Bjorken–Drell conventions) as

$$\psi = N \begin{pmatrix} j_0(kR)x \\ i\boldsymbol{\sigma} \cdot \hat{\mathbf{x}} j_1(kR)x \end{pmatrix},$$

where x = 2-component spinor, j_l =spherical Bessel function, N = normalization constant. (Our convention is that $\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}$, $\boldsymbol{\sigma}$ =Pauli matrices). Use the boundary condition at $|\mathbf{x}| = R$ to obtain a condition that determines k (do not try to solve the equation).

(*Princeton*)

Solution:

(a) Clearly, the condition (1), $\psi(X = R) = 0$, satisfies the condition $\bar{\psi}\psi|_{X=R} = 0$. For condition (2), we have (at $X = R$)

$$\begin{aligned} \bar{\psi}\psi &= (i\hat{\mathbf{x}} \cdot \boldsymbol{\gamma}\psi)^+ \beta (i\hat{\mathbf{x}} \cdot \boldsymbol{\gamma}\psi) \\ &= (-i\psi^+ \hat{\mathbf{x}} \cdot \beta\boldsymbol{\gamma}\beta)\beta(i\hat{\mathbf{x}} \cdot \boldsymbol{\gamma}\psi) \\ &= \psi^+ \beta (\hat{\mathbf{x}} \cdot \boldsymbol{\gamma})(\hat{\mathbf{x}} \cdot \boldsymbol{\gamma})\psi \,. \end{aligned}$$

As

$$(\hat{\mathbf{x}} \cdot \boldsymbol{\gamma})(\hat{\mathbf{x}} \cdot \boldsymbol{\gamma}) = \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \\ -\boldsymbol{\sigma} \cdot \hat{\mathbf{x}} & 0 \end{pmatrix} \begin{pmatrix} 0 & \boldsymbol{\sigma} \cdot \hat{\mathbf{x}} \\ -\boldsymbol{\sigma} \cdot \hat{\mathbf{x}} & 0 \end{pmatrix} = -1 \,,$$

we have

$$\bar{\psi}\psi = -\psi^+ \beta\psi = -\bar{\psi}\psi \,,$$

and hence

$$\bar{\psi}\psi|_{X=R} = 0 \,.$$

The second condition is physically acceptable. The Dirac equation consists of four partial differential equations, each of which contains first partial differentials of the coordinates. Hence four boundary conditions are needed.

The requirement that wave functions should tend to zero at infinity places restriction on half of the solutions. This is equivalent to two boundary conditions, and we still need two more boundary conditions. $\Psi(X = R) = 0$ is equivalent to four boundary conditions, while the condition

$$i\hat{\mathbf{x}} \cdot \boldsymbol{\gamma}\psi(X = R) = \psi(X = R)\,,$$

i.e.,

$$i\begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot\hat{\mathbf{x}} \\ -\boldsymbol{\sigma}\cdot\hat{\mathbf{x}} & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix},$$

or

$$i(\boldsymbol{\sigma}\cdot\hat{\mathbf{x}})\beta = \alpha\,,$$

only has two equations which give the relationship between the major and minor components. Therefore, only the condition (2) is physically acceptable. We can see from the explicit expression of the solution in (b) that the major and minor components of the Dirac spinor contain Bessel functions of different orders and so cannot both be zero at $X = R$. Condition (1) is thus not appropriate.

(b) The condition $\alpha = i(\boldsymbol{\sigma}\cdot\hat{\mathbf{x}})\beta$ gives

$$j_0(kR)x = i \cdot i(\boldsymbol{\sigma}\cdot\hat{\mathbf{x}})(\boldsymbol{\sigma}\cdot\hat{\mathbf{x}})j_1(kR)x\,,$$

or

$$j_0(kR) = -j_1(kR)\,,$$

which determines k.

3081

The bag model of hadron structure has colored quarks moving as independent spin-half Dirac particles within a spherical cavity of radius R. To obtain wave functions for particular hadron states, the individual quark "orbitals" must be combined to produce states of zero total color and the appropriate values of the spin and flavor (isospin, charge, strangeness) quantum numbers.

In the very good approximation that the "up" and "down" quarks are massless one can easily obtain the lowest energy (s-wave) bag orbitals. These are given by the Dirac spinor

$$\psi = N \begin{pmatrix} j_0(kr)x \\ i\boldsymbol{\sigma} \cdot \hat{\mathbf{r}} j_1(kr)x \end{pmatrix} ,$$

where x is a 2-component spinor, $k = 2.04/R$, $j_l =$ spherical Bessel function.

(a) The lowest-lying baryons (proton and neutron) are obtained by putting three quarks in this orbital. How would you construct the wave function for the proton and for the neutron, i.e., which quarks would be combined and what is the structure of the spin wave function consistent with the quantum numbers of proton and neutron and Pauli's principle?

(b) The magnetic moment operator is defined as $\boldsymbol{\mu} = \int_{|\mathbf{x}|<R} d^3\mathbf{x}\frac{1}{2}\mathbf{r} \times \mathbf{J}_{EM}$, where $\mathbf{J}_{EM}$ is the usual Dirac electric current operator. Find an expression for this operator in terms of the spin operators of the constituent quarks. (You may leave integrals over Bessel functions undone.)

(c) Show that $\mu_n/\mu_p = -2/3$.

You may need the following Clebsch–Gordon coefficients:

$$\langle 1/2, 1/2|1, 1; 1/2, 1/2\rangle = \ (2/3)^{1/2} ,$$

$$\langle 1/2, 1/2|1, 0; 1/2, 1/2\rangle = \ -(1/3)^{1/2} .$$

(*Princeton*)

Solution:

(a) If we neglect "color" freedom, the lowest states of a baryon (p and n) are symmetric for quark exchange. Since the third component of the isospin of p is $I_3 = 1/2$, while u has $I_3 = \frac{1}{2}$, d has $I_3 = -\frac{1}{2}$, its quark content must be uud. As the system has isospin $\frac{1}{2}$ it cannot be completely symmetric for ud exchange, (i.e., the wave function cannot be in the form $uud+udu+duu$ as this would result in a decuplet with $I = 3/2$). Thus the wave function must have components of the form $uud - udu$. But, as mentioned above, the lowest-state baryon is perfectly symmetric for quark exchange. We have to multiply such forms with a spin wave function antisymmetric for exchanging the second and third quarks ($\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow$) to yield a wave function symmetric with respect to such an exchange:

$$u\uparrow(1)u\uparrow(2)d\downarrow(3)-u\uparrow(1)d\uparrow(2)u\downarrow(3)$$
$$-u\uparrow(1)u\downarrow(2)d\uparrow(3)+u\uparrow(1)d\downarrow(2)u\uparrow(3)\,.$$

Note this also satisfies the isospin conditions. Then use the following procedure to make the wave function symmetric for exchanging the first and second quarks, and the first and third quarks. Exchanging the first and second quarks gives

$$u\uparrow u\uparrow d\downarrow -d\uparrow u\uparrow u\downarrow -u\downarrow u\uparrow d\uparrow +d\downarrow u\uparrow u\uparrow\,,$$

and exchanging the first and third quarks gives

$$d\downarrow u\uparrow u\uparrow -u\downarrow d\uparrow u\uparrow -d\uparrow u\downarrow u\uparrow +u\uparrow d\downarrow u\uparrow\,.$$

Combining the above three wave functions and normalizing, we have

$$\frac{1}{\sqrt{18}}(2u\uparrow u\uparrow d\downarrow +2u\uparrow d\downarrow u\uparrow +2d\downarrow u\uparrow u\uparrow -u\uparrow u\downarrow d\uparrow -u\uparrow d\uparrow u\downarrow$$
$$-u\downarrow u\uparrow d\uparrow -u\downarrow d\uparrow u\uparrow -d\uparrow u\uparrow u\downarrow -d\uparrow u\downarrow u\uparrow)\,.$$

The color wave function antisymmetric for exchanging any two quarks takes the form

$$\frac{1}{\sqrt{6}}(RGB-RBG+GBR-GRB+BRG-BGR)\,.$$

Let

$$\psi_\uparrow=\begin{pmatrix} j_0(kr)x(\uparrow) \\ ij_1(kr)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}x(\uparrow)\end{pmatrix},\qquad x(\uparrow)=\begin{pmatrix}1\\0\end{pmatrix},$$

$$\psi_\downarrow=\begin{pmatrix} j_0(kr)x(\downarrow) \\ ij_1(kr)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}x(\downarrow)\end{pmatrix},\qquad x(\downarrow)=\begin{pmatrix}0\\1\end{pmatrix}.$$

To include in the orbital wave functions, we need only to change $\uparrow$ to $\psi\uparrow$, and $\downarrow$ to $\psi\downarrow$. Then the final result is

$$\frac{1}{6\sqrt{3}}(RGB-RBG+GBR-GRB+BRG-BGR)$$
$$\times(2u\psi\uparrow u\psi\uparrow d\psi\downarrow +2u\psi\uparrow d\psi\downarrow u\psi\uparrow +2d\psi\downarrow u\psi\uparrow u\psi\uparrow$$
$$-u\psi\uparrow u\psi\downarrow d\psi\uparrow -u\psi\uparrow d\psi\uparrow u\psi\downarrow -u\psi\downarrow u\psi\uparrow d\psi\uparrow$$
$$-u\psi\downarrow d\psi\uparrow u\psi\uparrow -d\psi\uparrow u\psi\uparrow u\psi\downarrow -d\psi\uparrow u\psi\downarrow u\psi\uparrow)\,.$$

The neutron wave function can be obtained by applying the isospin-flip operator on the proton wave function ($u \leftrightarrow d$), resulting in

$$\begin{aligned}
&\frac{1}{6\sqrt{3}}(RGB - RBG + GBR - GRB + BRG - BGR) \\
&\quad \times (2d\psi\uparrow d\psi\uparrow u\psi\downarrow + 2d\psi\uparrow u\psi\downarrow d\psi\uparrow + 2u\psi\downarrow d\psi\uparrow d\psi\uparrow \\
&\quad - d\psi\uparrow d\psi\downarrow u\psi\uparrow - d\psi\uparrow u\psi\uparrow d\psi\downarrow - d\psi\downarrow d\psi\uparrow u\psi\uparrow \\
&\quad - d\psi\downarrow u\psi\uparrow d\psi\uparrow - u\psi\uparrow d\psi\uparrow d\psi\downarrow - u\psi\uparrow d\psi\downarrow d\psi\uparrow)\,.
\end{aligned}$$

The above wave functions are valid only for spin-up proton and neutron. For spin-down nucleons, the wave functions can be obtained by changing $\uparrow$ into $\downarrow$, $\downarrow$ into $\uparrow$ in the spin-up wave function.

(b) The Dirac current operator is defined as

$$\mathbf{J} = Q\bar{\psi}^*\boldsymbol{\gamma}\psi = Q\bar{\psi}^*\beta\boldsymbol{\alpha}\psi = Q\psi_\downarrow^+\boldsymbol{\alpha}\psi = Q\psi_\downarrow^+\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\psi\,.$$

where $\boldsymbol{\sigma}$ is the Pauli matrix. Inserting the expression of ψ into the above, we have

$$\begin{aligned}
\mathbf{J}_{EM} =\ & QN^+N(j_0(kr)x_\downarrow^+, -ij_1(kr)x_\downarrow^+\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}) \\
& \times \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}\begin{pmatrix} j_0(kr)x \\ ij_1(kr)\boldsymbol{\sigma}\cdot\hat{\mathbf{r}}x \end{pmatrix} \\
=\ & iQ|N|^2 j_0(kr)j_1(kr)x_\downarrow^+[\boldsymbol{\sigma}, \boldsymbol{\sigma}\cdot\hat{\mathbf{r}}]x \\
=\ & iQ|N|^2 j_0(kr)j_1(kr)x_\downarrow^+(-2i\boldsymbol{\sigma}\times\hat{\mathbf{r}})x \\
=\ & 2Q|N|^2 j_0(kr)j_1(kr)x_\downarrow^+(\boldsymbol{\sigma}\times\hat{\mathbf{r}})x\,,
\end{aligned}$$

and hence

$$\begin{aligned}
\boldsymbol{\mu} &= \int_{|X|<R}\frac{1}{2}\mathbf{r}\times J_{EM}d^3X \\
&= \int_{|X|<R} Q|N|^2 j_0(kr)j_1(kr)x_\downarrow^+[r\boldsymbol{\sigma} - (\boldsymbol{\sigma}\cdot\mathbf{r})\mathbf{r}]xd^3X\,.
\end{aligned}$$

When we integrate this over the angles the second term in the brackets gives zero. Thus

$$\boldsymbol{\mu} = 4\pi Q|N|^2\left[\int_{r<R} r^3 j_0(kr)j_1(kr)\right]x_\downarrow^+\boldsymbol{\sigma} x dr\,.$$

(c) The expected value of the magnetic moment of a spin-up proton is

$$\begin{aligned}\langle p\uparrow|\boldsymbol{\mu}|p\uparrow\rangle = &\frac{Q}{18}\left[4\left(\frac{2}{3}+\frac{2}{3}+\frac{1}{3}\right)+4\left(\frac{2}{3}+\frac{2}{3}+\frac{1}{3}\right)\right.\\ &+4\left(\frac{2}{3}+\frac{2}{3}+\frac{1}{3}\right)+\left(\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right)\\ &+\left(\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right)+\left(\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right)\\ &\left.+\left(\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right)+\left(\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right)+\left(\frac{2}{3}-\frac{2}{3}-\frac{1}{3}\right)\right]\\ = &\ Q\,.\end{aligned}$$

Similarly,

$$\begin{aligned}\langle n\uparrow|\boldsymbol{\mu}|n\uparrow\rangle &= \frac{Q}{18}\left[3\times 4\left(-\frac{1}{3}-\frac{1}{3}-\frac{2}{3}\right)+6\left(-\frac{1}{3}+\frac{1}{3}-\frac{2}{3}\right)\right]\\ &= -\frac{2}{3}Q\,.\end{aligned}$$

Therefore

$$\frac{\mu_n}{\mu_p} = \frac{\langle n\uparrow|\boldsymbol{\mu}|n\uparrow\rangle}{\langle p\uparrow|\boldsymbol{\mu}|p\uparrow\rangle} = -\frac{2}{3}\,.$$

3082

Recent newspaper articles have touted the discovery of evidence for gluons, coming from colliding beam e^+e^- experiments. These articles are inevitably somewhat garbled and you are asked to do better.

(a) According to current theoretical ideas of quantum chromodynamics (based on gauge group SU(3)): What are gluons? How many different kinds are there? What are their electrical charge? What is spin of a gluon?

(b) One speaks of various quark types or 'flavors', e.g., 'up' quarks, 'down' quarks, etc. According to QCD how many types of quark are there for each flavor? What are their charges? Does QCD say anything about the number of different flavors? According to currently available evidence how many different flavors are in fact recently well established? Discuss the evidence. Discuss also what weak interaction ideas say about whether, given the present flavors, there is reason to expect more, and characterize the "morez". How do results on the inclusive cross section for $e^+ + e^- +$ hadrons, at various energies, bear on the number of flavors?

(c) At moderately high energies one finds that the hadrons coming from e^+e^- collisions form two 'jets (Fig. 3.25). This has made people happy. How does one account for this two-jet phenomenon on the quark-gluon picture? At still higher energies one occasionally sees three jets. This has also made people happy. Account for this three-jet phenomenon.

(*Princeton*)

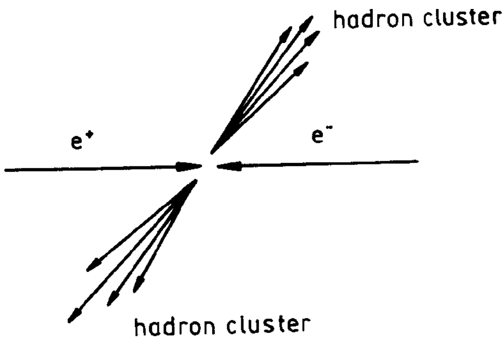

Fig. 3.25

Solution:

(a) According to QCD, hadrons consist of quarks and interactions between quarks are mediated by gluon field. Similar to the role of photons in electromagnetic interaction, gluons are propagators of strong interaction. There are eight kinds of gluons, all vector particles of electric charge zero, spin 1.

(b) In QCD theory, each kind of quark can have three colors, and quarks of the same flavor and different colors carry the same electric charge. An important characteristic of quarks is that they have fractional charges. QCD gives a weak limitation to the number of quarks, namely, if the number of quark flavors is larger than 16, asymototic freedom will be violated. The weak interaction does not restrict the number of quark flavors. However, cosmology requires the types of neutrino to be about 3 or 4 and the symmetry between leptons and quarks then restricts the number of flavors of quarks to be not more than 6 to 8. At various energies the relative total cross section for hadron production

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

has been found to agree with

$$R(E) = 3\sum_i Q_i^2 \,,$$

where the summation is over all quarks that can be produced at energy E, Q_i is the electric charge of the ith quark, and the factor 3 accounts for the three colors (**Problem 3078 (d)**).

(c) The two-jet phenomenon in e^+e^- collisions can be explained by the quark model. The colliding high energy e^+, e^- first produce a quark-antiquark pair of momenta $\mathbf{p}$ and $-\mathbf{p}$. When each quark fragments into hadrons, the sum of the hadron momenta in the direction of $\mathbf{p}$ is $\sum p_{||} = |\mathbf{p}|$, and in a transverse direction of $\mathbf{p}$ is $\sum p_\perp = 0$. In other words, the hadrons produced in the fragmentation of the quark and antiquark appear as two jets with axes in the directions of $\mathbf{p}$ and $-\mathbf{p}$. Measurements of the angular distribution of the jets about the electron beam direction have shown that quarks are ferminions of spin 1/2.

The three-jet phenomenon can be interperated as showing hard gluon emission in the QCD theory. At high energies, like electrons emitting photons, quarks can emit gluons. In e^+e^- collisions a gluon emitted with the quark pair can separately fragment into a hadron jet. From the rate of three-jet events it is possible to calculate α_s, the coupling constant of strong interaction.

3083

The observation of narrow long-lived states $(J/\psi, \psi')$ suggested the existence of a new quantum number (charm). Recently a new series of massive states has been observed through their decay into lepton pairs ($\Upsilon, \Upsilon', \ldots$ with masses ~ 10 GeV/c^2). Suppose the observation is taken to imply yet another quantum number (beauty).

(a) Estimate roughly the mass of the beauty quark.

(b) If this quark has an electric charge of $-1/3$ indicate how the Gell-Mann–Nishijima formula should be modified to incorporate the new quantum number.

(c) In the context of the conventional (colored) quark model, estimate the value of the ratio

$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

in the region well above the threshold for the production of the beauty.

(d) How would you expect the cross section for production of an $\Upsilon(b\bar{b}$ bound state) in colliding e^+e^- beams to change if the charge of b quark is $+2/3$ instead of $-1/3$? How would the branching ratio to lepton pairs change? What might be the change in its production cross section in hadronic collisions? Discuss this last answer briefly.

(*Princeton*)

Solution:

(a) The heavy meson Υ is composed of $b\bar{b}$. Neglecting the binding energy of b quarks, we have roughly

$$m_b \approx \frac{1}{2} M_\Upsilon \approx 5 \text{ GeV}/e^2 \,.$$

(b) For u, d, and s quarks, the Gell-Mann–Nishijima formula can be written as

$$Q = I_3 + \frac{1}{2}(B + S) \,.$$

Let the charm c of c quark be 1, the beauty b of b quark be -1. Then the Gell-Mann–Nishijima formula can be generalized as

$$Q = I_3 + \frac{1}{2}(B + S + c + b) \,,$$

which gives for c quark, $Q(c) = 0 + \frac{1}{2}(\frac{1}{3} + 0 + 1 + 0) = \frac{2}{3}$; for b quark $Q(b) = 0 + \frac{1}{2}(\frac{1}{3} + 0 + 0 - 1) = -\frac{1}{3}$

(c) If a certain quark q_i can be produced, its contribution to R is

$$R = \frac{\sigma(e^+e^- \to q_i\bar{q}_i)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3Q_i^2\,,$$

where Q_i is its charge, and the factor 3 accounts for the three colors. If the c.m.s energy is above the threshold for producing beauty, the five flavors of quarks u, d, s, c, and b can be produced. Hence

$$R(E) = 3\sum_i Q_i^2 = 3\left[3 \times \left(\frac{1}{3}\right)^2 + 2 \times \left(\frac{2}{3}\right)^2\right] = \frac{11}{3}\,.$$

(d) The cross section for the resonance state Υ is given by

$$\sigma = \frac{\pi(2J+1)}{m^2}\frac{\Gamma_{ee}\Gamma}{(E-m)^2 + \frac{\Gamma^2}{4}}\,,$$

where J and m are the spin and mass of Υ respectively, Γ is the total width of the resonance state, Γ_{ee} is the partial width of the e^+e^- channel. The partial width of $\Upsilon \to e^+e^-$ is

$$\Gamma_{ee}(\Upsilon \to e^+e^-) = 16\pi\frac{\alpha^2 Q_b^2}{m_b^2}|\psi(0)|^2\,,$$

where $\Psi(0)$ is the ground state wave function, Q_b and m_b are the charge and mass of b respectively, and α is the fine structure constant. At $E \approx m$,

$$\sigma = \frac{12\pi\Gamma_{ee}}{m^2\Gamma} \propto Q_b^2\,,$$

as Υ has spin $J = 1$.

When the charge of b quark changes from $-\frac{1}{3}$ to $\frac{2}{3}$, Q_p^2 changes from $\frac{1}{9}$ to $\frac{4}{9}$ and σ increases by 3 times. This means that both the total cross section and the partial width for the leptonic channel increase by 3 times.

There is no resonance in the production cross section in hadron collisions, because the hadron collision is a reaction process $h+\bar{h} \to \Upsilon + X$, but not a production process as $e^+ + e^- \to \Upsilon$. However, in the invariant mass spectrum of μ pairs (or e pairs) in hadron collisions we can see a small peak at the invariant mass $m(\mu\mu) = m_\Upsilon$. The height of this peak will increase by 3 times also.

3084

The recently discovered $\psi(M = 3.1\ \text{GeV}/c^2)$ and $\psi^*(M = 3.7\ \text{GeV}/c^2)$ particles are both believed to have the following quantum numbers:

$$
\begin{aligned}
J^P &= 1^-\,, \\
C &= -1(\text{charge conjugation})\,, \\
I &= 0(I\text{-spin})\,, \\
Q &= 0\,.
\end{aligned}
$$

Indicate which of the following decay modes are allowed by strong interaction, which by electromagnetic and which by weak interaction, and which are strictly forbidden. If strong decay is forbidden or if the decay is strictly forbidden, state the selection rule.

$$
\begin{aligned}
&\psi \to \mu^+\mu^- \\
&\psi \to \pi^0\pi^0 \\
&\psi^* \to \psi\pi^+\pi^- \\
&\psi^* \to \psi + \eta'(0.96\ \text{GeV}/c^2)
\end{aligned}
$$

(*Wisconsin*)

Solution:

The process $\psi \to \mu^+\mu^-$ is the result of electromagnetic interaction, and the decay $\psi^* \to \psi\pi^+\pi^-$ is a strong interaction process. The decay mode $\psi \to \pi^0\pi^0$ by strong interaction is forbidden since the C-parity of ψ is -1 and that of the two π^0 in the final state is $+1$, violating the conservation of C-parity in strong interaction. The decay mode $\psi^* \to \psi + \eta'(0.96\ \text{GeV}/c^2)$ is strictly forbidden as it violates the conservation of energy.

3085

At SPEAR (e^+e^- colliding-beam storage ring) several states called ψ, χ have been observed. The ψ's have quantum numbers of the photon ($J^P = 1^-$, $I^G = 0^-$) and have massses at 3.1 and 3.7 GeV/c^2. Suppose the following reaction was observed:

$$e^+e^- \to \psi(3.7) \to \gamma + \chi$$
$$\hookrightarrow \pi^+\pi^-$$

where $E^*_\gamma = 0.29$ GeV. What are the mass, spin, parity, isotopic spin, G-parity and charge conjugation possibilities for the χ? Assume an electric dipole $E1$ transition for the γ-ray emission and strong decay of the χ to 2π.

(*Wisconsin*)

Solution:

First we find the mass of χ. In the ψ rest frame

$$E_\chi + E_\gamma = m_\psi \, ,$$

or

$$E_\chi = 3.7 \text{ GeV} - 0.29 \text{ GeV} = 3.41 \text{ GeV} \, .$$

Momentum conservation gives

$$p_\chi = p_\gamma = 0.29 \text{ GeV}/c \, .$$

As

$$E_\chi^2 = p_\chi^2 + m_\chi^2 \, ,$$

we obtain

$$m_\chi = \sqrt{E_\chi^2 - p_\chi^2} = \sqrt{3.4^2 - 0.29^2} = 3.40 \text{ GeV}/c^2 \, .$$

Now to find the other quantum numbers of χ. As $\psi(3.7) \to \gamma\chi$ is an $E1$ transition, we see from its selection rules that the parities of ψ and χ are opposite and the change of spin is 0 or ± 1. Then the possible spin values of χ are $J = 0, 1, 2$ and its parity is positive, as ψ has $J^P = 1^-$.

Consider the strong decay $\chi \to \pi^+\pi^-$. As the parity of χ is $+1$, parity conservation requires $P(\pi^+)P(\pi^-)(-1)^l = (-1)^{2+l} = (-1)^l = +1$, giving $l = 0$ or 2. Thus the spin of χ can only be $J = 0$ or 2. Furthermore,

$$C(\chi) = (-1)^{l+s} = (-l)^l = +1 \,.$$

As π has positive G-parity, conservation of G-parity requires

$$G(\chi) = G(\pi^+)G(\pi^-) = +1 \,.$$

Now for mesons with C-parity, G-parity and C-parity are related through isospin I:

$$G(\chi) = (-1)^I C(\chi) \,.$$

As

$$G(\chi) = C(\chi) = 1 \,,$$

$(-1)^I = +1$, giving $I = 0$ or 2 for χ.

Up to now no meson with $I = 2$ has been discovered, so we can set $I = 0$. Hence the quantum numbers of χ can be set as

$$m_\chi = 3.40 \text{ GeV}/c^2, \quad I^G(J^P)C = 0^+(0^+)+ \quad \text{or} \quad 0^+(2^+)+ \ .$$

The angular distribution of γ emitted in ψ decay indicates that the spin of χ (3.40) is probably $J = 0$.

3086

It is well established that there are three $c\bar{c}$ states intermediate in mass between the $\psi(3095)$ and $\psi'(3684)$, namely,

$$\chi_0(3410): \qquad J^{PC} = 0^{++} \,,$$

$$\chi_1(3510): \qquad J^{PC} = 1^{++} \,,$$

$$\chi_2(3555): \qquad J^{PC} = 2^{++} \,.$$

The number in parentheses is the mass in MeV/c^2.

(a) What electric and magnetic multipoles are allowed for each of the three radioactive transitions:

$$\psi' \to \gamma + \chi_{0,1,2}?$$

(b) Suppose that the ψ' is produced in e^+e^- collisions at an electron-positron storage ring. What is the angular distribution of the photons relative to the beam direction for the decay $\psi' \to \gamma + \chi_0$?

(c) In the condition of part (b), could one use the angular distribution of the photons to decide the parity of the χ_0?

(d) For χ_0 and χ_1 states separately, which of the following decay modes are expected to be large, small, or forbidden?

$$\pi^0\pi^0,\ \gamma\gamma,\ p\bar{p},\ \pi^+\pi^-\pi^0,\ 4\pi^0,\ D^0\bar{K}^0,\ e^+e^-, \psi\eta^0\,.$$

$$(DATA : M_p = 938 \text{ MeV}/c^2; \quad M_{\pi^0} = 135 \text{ MeV}/c^2, \quad M_\eta = 549 \text{ MeV}/c^2)$$

(e) The strong decays of the χ states are pictured as proceeding through an intermediate state consisting of a small number of gluons which then interact to produce light quarks, which further interact and materialize as hadrons. If gluons are massless and have $J^P = 1^-$, what is the minimum number of gluons allowed in the pure gluon intermediate state of each of the $\chi_{0,1,2}$? What does this suggest about the relative hadronic decay widths for these three states?

(Princeton)

Solution:

(a) As γ and ψ both have $J^P = 1^-$, in the decay $|\Delta J| = 0, 1$ and parity changes. Hence it is an electric dipole transition.

(b) The partial width of the electric dipole transition is given by

$$\Gamma(2^3S_1 \to \gamma 2^3P_J) = \left(\frac{16}{243}\right) \alpha(2J+1)k^3|\langle 2P|\gamma|2S\rangle|^2\,,$$

where α is the fine structure constant. Thus

$$\Gamma(2^3S_1 \to \gamma_0 2^3P_0) : \Gamma(2^3S_1 \to \gamma_1 2^3P_1) : \Gamma(2^3S_1 \to \gamma_2 2^3P_2)$$

$$= k_0^3 : 3k_1^3 : 5k_2^3\,,$$

where k is the momentum of the emitted photon (setting $\hbar = 1$). The angular distributions of the photons are calculated to be

$$1 + \cos^2\theta \quad \text{for process } \psi' \to \gamma_0 + \chi_0\,,$$

$$1 - (1/3)\cos^2\theta \quad \text{for process } \psi' \to \gamma_1 + \chi_1\,,$$

$$1 + (1/13)\cos^2\theta \quad \text{for process } \psi' \to \gamma_2 + \chi_2\,.$$

(c) As the angular distributions of γ_1, γ_2, and γ_3 are different, they can be measured experimentally and used to determine the spin of χ_i. The other quantum numbers of χ_i may also be decided by the modes of their decay. For example, the χ_0 state decays to $\pi^+\pi^-$ or K^+K^-, and so $J^P = 0^+, 1^-, 2^+ \cdots$. Then from the angular distribution, we can set its $J^P = 0^+$. As $C(\pi^+\pi^-) = (-1)^l$, $J^{PC} = 0^{++}$. For the χ_1 state, $\pi^+\pi^-$ and K^+K^- are not among the final states so $J^P = 0^-, 1^+, 2^-$. The angular distribution then gives $J = 1$ and so $J^P = 1^+$. It is not possible to determine their J^P by angular distributions alone.

(d) $\chi_1 \to \pi^0\pi^0$ is forbidden. As π^0 has $J^P = 0^-$, $\pi^0\pi^0$ can only combine into states $0^+, 1^-, 2^+$. As χ_1 has $J^P = 1^+$, angular momentum and parity cannot both be conserved.

$\chi_0 \to \pi^0\pi^0$ satisfies all the conservation laws. However, it is difficult to detect. The whole process is $\psi' \to \gamma\chi_0 \to \gamma\pi^0\pi^0 \to \gamma\gamma\gamma\gamma\gamma$ and one would have to measure the five photons and try many combinations of invariant masses simultaneously to check if the above mode is satisfied. This mode has yet to be detected. Similarly we have the following:

$\chi_1 \to \gamma\gamma$ is forbidden. $\chi_0 \to \gamma\gamma$ is an allowed electromagnetic transition. However as χ_0 has another strong decay channel, the branching ratio of this decay mode is very small.

$\chi_0, \chi_1 \to p\bar{p}$ are allowed decays. However, their phase spaces are much smaller than that of $\chi_0 \to \pi^0\pi^0$, and so are their relative decay widths.

$\chi_0, \chi_1 \to \pi^+\pi^-\pi^0$ are forbidden as G-parity is not conserved;

$\chi_0, \chi_1 \to \pi^0\psi$ are forbidden as C-parity is not conserved;

$\chi_0, \chi_1 \to D^0\bar{K}^0$ are weak decays with very small branching ratios.

$\chi_0 \to e^+e^-$ is a high order electromagnetic decay with a very small branching ratio.

$\chi_1 \to e^+e^-$ is an electromagnetic decay. It is forbidden, however, by conservation of C-parity.

$\chi_0, \chi_1 \to \eta\psi$ are forbidden for violating energy conservation.

(e) As gluon has $J^P = 1^-$, it is a vector particle and the total wave function of a system of gluons must be symmetric. As a two-gluon system can only have states with 0^{++} or 2^{++}, a three-gluon system can only have states with 1^{++}, χ_0 and χ_2 have strong decays via a two-gluon intermediate state and χ_1 has strong decay via a three-gluons intermediate state. Then as decay probability is proportional to α_s^n, where α_s is the strong interaction constant ($\alpha_s \approx 0.2$ in the energy region of J/ψ) and n is the number of gluons in the intermediate state, the strong decay width of χ_1 is α_s times smaller than those of χ_0, χ_2. The result given by QCD is $\Gamma(\chi_0 \to hadrons) : \Gamma(\chi_2 \to hadrons) : \Gamma(\chi_1 \to hadrons) = 15 : 4 : 0.5$.

3087

Particles carrying a new quantum number called charm have recently been discovered. One such particle, D^+, was seen produced in e^+e^- annihilation, at center-of-mass energy $E = 4.03$ GeV, as a peak in the $K^-\pi^+\pi^+$ mass spectrum at $M_{K\pi\pi} = 1.87$ GeV. The Dalitz plot for the three-body decay shows nearly uniform population.

(a) Using the simplest quark model in which mesons are bound states of a quark and an antiquark, show that D^+ cannot be an ordinary strange particle resonance (e.g. K^{*+}).

(b) What are the spin and parity (J^P) of the $K\pi\pi$ final state?

(c) Another particle, D^0, was seen at nearly the same mass in the $K^-\pi^+$ mass spectrum from the same experiment. What are the allowed J^P assignments for the $K\pi$ state?

(d) Assume that these two particles are the same isospin multiplet, what can you infer about the type of interaction by which they decay?

(e) Suppose the $K_s \to 2\pi$ decay to be typical of strangeness-changing charm-conserving weak decays. Estimate the lifetime of D^0, assuming that the branching ratio $(D^0 \to K^-\pi^+)/(D^0 \to all) \approx 5\%$. The lifetime of K_s is $\sim 10^{-10}$ sec.

(*Princeton*)

Solution:

(a) According to the simplest quark model, K meson consists of an $\bar{s}$ quark and a u quark. All strange mesons are composed of an $\bar{s}$ and an ordinary quark, and only weak decays can change the quark flavor. If the s quark in a strange meson changes into a u or d quark, the strange meson will become an ordinary meson. On the other hand, strong and electromagnetic decays cannot change quark flavor. $D^+ \to K\pi\pi$ is a weak decay. So if there is an $\bar{s}$ quark in D^+, its decay product cannot include K meson, which also has an $\bar{s}$ quark. Hence there is no $\bar{s}$ but a quark of a new flavor in D^+, which changes into $\bar{s}$ in weak decay, resulting in a K meson in the final state.

(b) The Dalitz plot indicated $J = 0$ for a $K\pi\pi$ system. As the total angular momentum of the three particles is zero, the spin of D^+ is zero. Let the relative orbital angular momentum of the two π system be l, the orbital angular momentum of the K relative to the two π be l'. Since the spins of K, π are both zero, $\mathbf{J} = \mathbf{l} + \mathbf{l}' = 0$, i.e., $\mathbf{l} = -\mathbf{l}'$, or $|\mathbf{l}| = |\mathbf{l}'|$. Hence

$$P(K\pi\pi) = (-1)^{l+l'} P^2(\pi) P(K) = (-1)^2(-1) = (-1)^3 = -1 .$$

Thus the $K\pi\pi$ final state has $J^P = 0^-$.

(c) For the $K\pi$ state,

$$P(K\pi) = (-1)^l P(\pi) P(K) = (-1)^l , \qquad J = 0 + 0 + l .$$

Hence

$$J^P = 0^+, 1^-, 2^+ \cdots .$$

If $J(D) = 0$, then $l = 0$ and $J^P = 0^+$.

(d) If D^+, D^0 belong to an isospin multiplet, they must have the same J^P. As the above-mentioned $K\pi\pi$ and $K\pi$ systems have odd and even parities respectively, the decays must proceed through weak interaction in which parity is not conserved.

(e) Quark flavor changes in both the decays $D^0 \to K\pi$ and $K_s^0 \to \pi^+\pi^-$, which are both Cabibbo-allowed decays. If we can assume their matrix elements are roughly same, then the difference in lifetime is due to the difference in the phase-space factor. For the two-body weak decays, neglecting the difference in mass of the final states, we have

$$\Gamma(D_1^0 \to K^-\pi^+) = f_D^2 \cdot m_D \cdot m_K^2 \left(1 - \frac{m_K^2}{m_D^2}\right)^2 = f_D^2 \cdot \frac{m_K^2}{m_D^3}(m_D^2 - m_K^2)^2 ,$$

$$\Gamma(K_S \to 2\pi) = f_K^2 \cdot m_K \cdot m_\pi^2 \left(1 - \frac{m_\pi^2}{m_K^2}\right)^2 = f_K^2 \cdot \frac{m_\pi^2}{m_K^3}(m_K^2 - m_\pi^2)^2 ,$$

where f_D and f_K are coupling constants associated with the decays. Take $f_D = f_K$ and assume the branching ratio of $K_S^0 \to 2\pi$ is nearly 100%, we have

$$\frac{\tau_{D^0}}{\tau_K} = \frac{\Gamma(K \to 2\pi)}{\Gamma(D \to all)} = \frac{\Gamma(K \to 2\pi)}{20\Gamma(D \to K\pi)} = \frac{m_\pi^2 m_D^3 (m_K^2 - m_\pi^2)^2}{20 m_K^5 (m_D^2 - m_K^2)^2} ,$$

and hence

$$\tau_{D^0} = \frac{140^2 \times 1870^3}{20 \times 494^5} \left(\frac{494^2 - 140^2}{1870^2 - 494^2} \right)^2 \times 10^{-10} = 1.0 \times 10^{-13}\ s\,,$$

which may be compared with the experimental value

$$\tau_{D^0} = \left(4.4 \begin{matrix} +0.8 \\ -0.6 \end{matrix} \right) \times 10^{-13}\ s\,.$$

3088

A recent development in elementary particle physics is the discovery of charmed nonstrange mesons (called D^+, D^0, and their charge conjugates) with masses around 1870 MeV/c^2.

(a) Knowing the charge of charmed quark to be 2/3, give the quark contents of the D^+ and D^0 mesons.

(b) The D mesons decay weakly into ordinary mesons $(\pi, K, \cdots)$. Give estimates (with your reasoning) for the branching ratios of the following two-body decays:

$$\frac{BR(D^0 \to K^+K^-)}{BR(D^0 \to K^-\pi^+)}\,, \qquad \frac{BR(D^0 \to \pi^+\pi^-)}{BR(D^0 \to K^-\pi^+)}\,, \qquad \frac{BR(D^0 \to K^+\pi^-)}{BR(D^0 \to K^-\pi^+)}\,.$$

(c) How would you show that the decay of D mesons is by means of weak interaction?

(d) In a colliding beam at c.m. energy 4.03 GeV, a D^+ meson (mass = 1868.3 MeV/c^2) and a D^{*-} meson (mass = 2008.6 MeV/c^2) are produced. The D^{*-} decays into a $\bar{D}^0$ (mass = 1863.3 MeV/c^2) and a π^-. What is the maximum momentum in the laboratory of the D^{*-}? of the π?

(*Princeton*)

Solution:

(a) A D meson consists of a charmed quark c (charge $\frac{2}{3}$) and the antiparticle of a light quark u (charge $\frac{2}{3}$) or d (charge $-\frac{1}{3}$). To satisfy the charge requirements, the quark contents of D^+ and D^0 are $c\bar{d}$ and $c\bar{u}$ respectively.

(b) The essence of D meson decay is that one of its quarks changes flavor via weak interaction, the main decay modes arising from decay of the c quark as shown in Fig. 3.26.

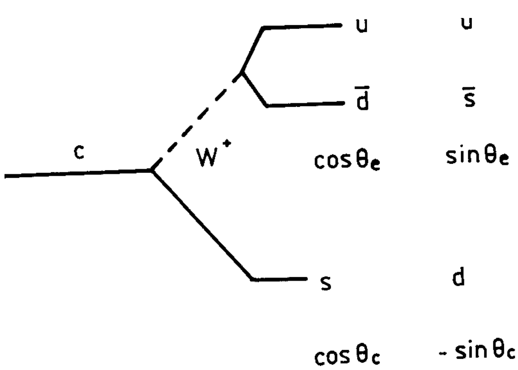

Fig. 3.26

Let θ_c be the Cabibbo mixing angle. We have

$c \to su\bar{d}$, amplitude $\sim \cos^2\theta_c$,
$c \to su\bar{s}$, amplitude $\sim \sin\theta_c \cos\theta_c$,
$c \to du\bar{d}$, amplitude $\sim -\sin\theta_c \cos\theta_c$,
$c \to du\bar{s}$, amplitude $\sim \sin^2\theta_c$,

and correspondingly

$$D^0 \to K^- + \pi^- \,, \quad \text{Cabibbo allowed}\,,$$

$$D^0 \to K^- + K^+ \,, \quad \text{first order Cabibbo forbidden}\,,$$

$$D^0 \to \pi^+ + \pi^- \,, \quad \text{first order Cabibbo forbidden}\,,$$

$$D^0 \to K^+ + \pi^- \,, \quad \text{second order Cabibbo forbidden}\,.$$

The value of θ_c has been obtained by experiment to be $\theta_c = 13.1^0$. Hence

$$\frac{BR(D^0 \to K^+K^-)}{BR(D^0 \to K^-\pi^+)} = \tan^2\theta_c \approx 0.05,$$

$$\frac{BR(D^0 \to \pi^+\pi^-)}{BR(D^0 \to K^-\pi^+)} = \tan^2\theta_c \approx 0.05,$$

$$\frac{BR(D^0 \to K^+\pi^-)}{BR(D^0 \to K^-\pi^+)} = \tan^4\theta_c \approx 2.5 \times 10^{-3}\,.$$

(c) In D^0 decay, the charm quantum number C changes. As only weak decays can change the flavor of a quark, the decays must all be weak decays.

(d) In the head-on collision of colliding beams the laboratory frame is the same as the center-of-mass frame. Let the masses, energies and momenta of D^{*-} and D^+ be m^*, m, E^*, E, p^*, p respectively and denote the total energy as E_0. Momentum and energy conservation gives

$$p^* = p, \qquad E^* + \sqrt{p^2 + m^2} = E_0 \,.$$

Thus

$$E^{*2} + E_0^2 - 2E^* E_0 = p^{*2} + m^2 \,.$$

With $E^{*2} = p^{*2} + m^{*2}$, we have

$$E^* = \frac{m^{*2} - m^2 + E_0^2}{2E_0} = \frac{2.0086^2 - 1.8683^2 + 4.03^2}{2 \times 4.03} = 2.08 \text{ GeV} \,,$$

$$p^* = \sqrt{2.08^2 - 2.008^2} = 0.54 \text{ GeV}/c \,,$$

giving

$$\beta = p^*/E^* = 0.26, \qquad \gamma = E^*/m^* = 1.04 \,.$$

In the D^{*-} rest frame, the decay takes place at rest and the total energy is equal to m^*. Using the above derivation we have

$$\bar{E}_\pi = \frac{m_\pi^2 - m_D^2 + m^{*2}}{2m^*} = 0.145 \text{ GeV} \,,$$

$$\bar{p}_\pi = \sqrt{E_\pi^2 - m_\pi^2} = 38 \text{ MeV}/c \,.$$

In the laboratory, the π meson will have the maximum momentum if it moves in the direction of D^{*-}. Let it be P_{max}. Then

$$\begin{aligned} p_{\text{max}} &= \gamma(\bar{p}_\pi + \beta \bar{E}_\pi) \\ &= 1.04(38 + 0.26 \times 145) = 79 \text{ MeV}/c \,. \end{aligned}$$

Hence the maximum momenta of D^{*-} and π^- are 540 MeV/c and 79 MeV/c respectively.

3089

In e^+e^- annihilation experiments, a narrow resonance (of width less than the intrinsic energy spread of the two beams) has been observed at

$E_{CM} = 9.5$ GeV for both

$$e^+e^- \to \mu^+\mu^-$$

and

$$e^+e^- \to \text{hadrons}\,.$$

The integrated cross sections for these reactions are measured to be

$$\int \sigma_{\mu\mu}(E)dE = 8.5 \times 10^{-33}\ \text{cm}^2 \cdot \text{MeV}\,,$$

$$\int \sigma_h(E)dE = 3.3 \times 10^{-31}\ \text{cm}^2 \cdot \text{MeV}\,.$$

Use the Breit–Wigner resonance formula to determine the partial widths $\Gamma_{\mu\mu}$ and Γ_h for the $\mu\mu$ and hadronic decays of the resonance.

Solution:

The Breit–Wigner formula can be written for the two cases as

$$\sigma_\mu(E) = \frac{\pi(2J+1)}{M^2}\frac{\Gamma_{ee}\Gamma_{\mu\mu}}{(E-M)^2+\frac{\Gamma^2}{4}}\,,$$

$$\sigma_h(E) = \frac{\pi(2J+1)}{M^2}\frac{\Gamma_{ee}\Gamma_h}{(E-M)^2+\frac{\Gamma^2}{4}}\,,$$

where M and J are the mass and spin of the resonance state, Γ, Γ_{ee}, Γ_h and $\Gamma_{\mu\mu}$ are the total width, and the partial widths for decaying into electrons, hadrons, and muons respectively. We have

$$\Gamma = \Gamma_{ee} + \Gamma_{\tau\tau} + \Gamma_{\mu\mu} + \Gamma_h\,,$$

where $\Gamma_{\tau\tau}$ is the partial width for decaying into τ particles. Because of the universality of lepton interactions, if we neglect the difference phase space factors, we have $\Gamma_{ee} = \Gamma_{\tau\tau} = \Gamma_{\mu\mu}$, and so

$$\Gamma = 3\Gamma_{\mu\mu} + \Gamma_h\,.$$

For the resonance at $M = 9.5$ GeV, $J = 1$. Therefore

$$\int \sigma_{\mu\mu}(E)dE = \frac{3\pi\Gamma^2_{\mu\mu}}{M^2}\int \frac{dE}{(E-M)^2+\frac{\Gamma^2}{4}}$$

$$= \frac{6\pi^2\Gamma^2_{\mu\mu}}{M^2\Gamma} = 8.5\times 10^{-33}\ \mathrm{cm}^2\cdot\mathrm{MeV}\,,$$

$$\int \sigma_h(E)dE = \frac{3\pi\Gamma_{\mu\mu}\Gamma_h}{M^2}\int \frac{dE}{(E-M)^2+\frac{\Gamma^2}{4}}$$

$$= \frac{6\pi^2\Gamma_{\mu\mu}\Gamma_h}{M^2\Gamma} = 3.3\times 10^{-31}\ \mathrm{cm}^2\cdot\mathrm{MeV}\,,$$

whose ratio gives

$$\Gamma_h = 38.8\Gamma_{\mu\mu}\,.$$

Hence

$$\Gamma = \Gamma_h + 3\Gamma_{\mu\mu} = 41.8\Gamma_{\mu\mu}\,,$$

and

$$\Gamma_{\mu\mu} = \frac{M^2}{6\pi^2}\frac{\Gamma}{\Gamma_h}\times 3.3\times 10^{-31} = 5.42\times 10^{-26}\ \mathrm{MeV}^3\mathrm{cm}^2\,.$$

To convert it to usual units, we note that

$$1 = \hbar c = 197\times 10^{-13}\ \mathrm{MeV}\cdot\mathrm{cm}\,,$$

or

$$1\ \mathrm{cm} = \frac{1}{197\times 10^{-13}}\ \mathrm{MeV}^{-1}\,.$$

Thus

$$\Gamma_{\mu\mu} = 1.40\times 10^{-3}\ \mathrm{MeV}\,,$$

and

$$\Gamma_h = 38.8\Gamma_{\mu\mu} = 5.42\times 10^{-2}\ \mathrm{MeV}\,,$$

$$\Gamma = 41.8\Gamma_{\mu\mu} = 5.84\times 10^{-2}\ \mathrm{MeV}\,.$$

3090

Suppose nature supplies us with massive charged spin-1 'quark' Q^+ and antiquark $\bar{Q}^-$. Using a model like the nonrelativistic charmonium model which successfully describes the J/ψ family, predict the spectrum of the neutral $Q\bar{Q}$ resonance. Make a diagram of the lowest few expected states, indicating the spin, charge conjugation parities, and allowed electromagnetic transitions, as well as the expected ordering of levels.

(*Princeton*)

Solution:

The current nonrelativistic model for dealing with heavy quarks employs a strong-interaction potential, approximated by a central potential. Then the angular part of the wave functions takes the form of spherical harmonic functions. To take account of quark confinement, a better potential is given by the Cornell model as $V(r) = -k/r + r/a^2$, which is a Coloumb potential superposed on a linear potential, with the former implying asymptotic freedom, and the latter quark confinement. By considering spin correlation the order of levels can be calculated numerically. For the quark-antiquark system,

spin: $\mathbf{J} = \mathbf{S} + \mathbf{l}$, where $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$, $s_1 = s_2 = 1$,

P-parity: $P(Q^+Q^-) = P(Q^+)P(Q^-)(-1)^l = (-1)^l$, as for a boson of spin 1, $P(\bar{Q}) = P(Q)$,

C-parity: $C(Q^+Q^-) = (-1)^{l+S}$.

Thus the system can have J^{PC} as follows:

$l = 0,$	$S = \lvert\mathbf{s}_1 + \mathbf{s}_2\rvert = 0\,,$	n^1S_0	$J^{PC} = 0^{++}$
	$S = \lvert\mathbf{s}_1 + \mathbf{s}_2\rvert = 1$	n^3S_1	1^{+-}
	$S = \lvert\mathbf{s}_1 + \mathbf{s}_2\rvert = 2$	n^5S_2	2^{++}
$l = 1,$	$S = \lvert\mathbf{s}_1 + \mathbf{s}_2\rvert = 0$	n^1P_1	1^{--}
	$S = \lvert\mathbf{s}_1 + \mathbf{s}_2\rvert = 1$	n^3P_0	0^{-+}
		n^3P_1	1^{-+}
		n^3P_2	2^{-+}
	$S = \lvert\mathbf{s}_1 + \mathbf{s}_2\rvert = 2$	n^5P_1	1^{--}
		n^5P_2	2^{--}
		n^5P_3	3^{--}

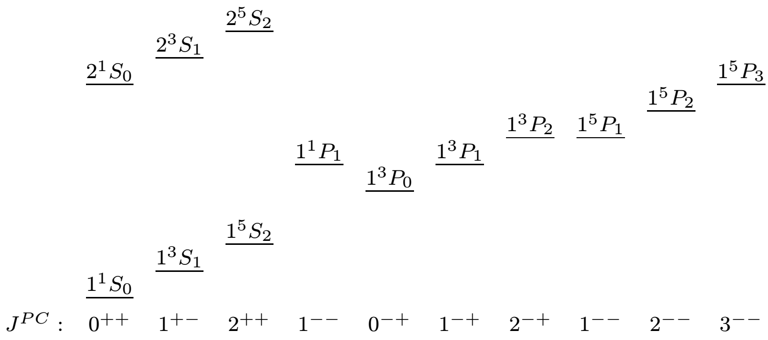

Fig. 3.27

In the above we have used spectroscopic symbols $n^{2S+1}S_J$, $n^{2S+1}P_J$ etc. to label states, with n denoting the principal quantum number, $2S+1$ the multiplicily, singlet, triplet or quintuplet, and J the total angular momentum. The order of the levels is shown in Fig. 3.27 (only S and P states are shown).

As the order of P states is related to the spin-correlation term, the order given here is only a possible one. The true order must be calculated using the assumed potential. Even the levels given here are seen more complicated than those for a spin-1/2 charm-anticharm system, with the addition of the $^5S_2$ and ^5P_J spectra. In accordance with the selection

Table 3.13. Possible γ transitions.

Transition	ΔJ	ΔP	ΔC	Type of transition
$2^3S_1 \to 1^3P_J$	$0, 1$	-1	-1	$E1$
$1^3P_J \to 1^3S_1$	$0, 1$	-1	-1	$E1$
$2^5S_2 \to 2^3S_1 \to 2^1S_0$	1	1	-1	$M1(E2)$
$1^5S_2 \to 1^3S_1 \to 1^1S_0$	1	1	-1	$M1(E2)$
$2^5S_2 \to 1^3S_1$	1	1	-1	$M1(E2)$
$2^3S_1 \to 1^1S_1$	1	1	-1	$M1(E2)$
$2^5S_2 \to 1^5P_J$	$0, 1$	-1	-1	$E1$
$1^5P_J \to 1^5S_2$	$0, 1$	-1	-1	$E1$
$2^1S_0 \to 1^1P_1, 1^5P_1$	1	-1	-1	$E1$
$1^1P_1, 1^5P_1 \to 1^1S_0$	1	-1	-1	$E1$
$2^5S_2 \to 1^1P_1$	1	-1	-1	$E1$
$1^1P_1 \to 1^5S_2$	1	-1	-1	$E1$

rules of electromagnetic transitions, the possible transitions are listed in Table 3.13.

Note that electromagnetic transitions between the P states are not included in the table because the level order cannot be ascertained. Higher order transitions ($M2, E3$, etc.) between $2^1S_0 \to 1^5P_{2,3}$ are also excluded. The transitions $^5S_2 \to {}^1S_0$, $^3S_1 \to {}^5P_2, {}^1P_1$, etc. are C-parity forbidden and so excluded.

This Page Intentionally Left Blank

PART IV

EXPERIMENTAL METHODS AND MISCELLANEOUS TOPICS

This Page Intentionally Left Blank

1. KINEMATICS OF HIGH-ENERGY PARTICLES (4001–4061)

4001

An accelerator under study at SLAC has as output bunches of electrons and positrons which are made to collide head-on. The particles have 50 GeV in the laboratory. Each bunch contains 10^{10} particles, and may be taken to be a cylinder of uniform charge density with a radius of 1 micron and a length of 2 mm as measured in the laboratory.

(a) To an observer traveling with a bunch, what are the radius and length of its bunch and also the one of opposite sign?

(b) How long will it take the two bunches to pass completely through each other as seen by an observer traveling with a bunch?

(c) Draw a sketch of the radial dependence of the magnetic field as measured in the laboratory when the two bunches overlap. What is the value of B in gauss at a radius of 1 micron?

(d) Estimate in the impulse approximation the angle in the laboratory by which an electron at the surface of the bunch will be deflected in passing through the other bunch. (Ignore particle-particle interaction.)

(*UC, Berkeley*)

Solution:

(a) Consider a particle P in the bunch traveling with the observer. Let Σ, Σ_0 be the reference frames attached to the laboratory and the observer respectively, taking the direction of motion of P as the x direction. The Lorentz factor of P, and hence of Σ_0, in Σ is

$$\gamma = \frac{E}{mc^2} = \frac{50 \times 10^9}{0.5 \times 10^6} = 1 \times 10^5 \,.$$

To an observer in Σ, the bunch is contracted in length:

$$L = \frac{1}{\gamma} L_0 \,,$$

where L_0 is its length in Σ_0. Thus

$$L_0 = \gamma L = 1 \times 10^5 \times 2 \times 10^{-3} = 200 \text{ m} \,.$$

The radius of the bunch is

$$r_0 = r = 1\ \mu m\,,$$

as there is no contraction in a transverse direction.

The bunch of opposite charge travels with velocity $-\beta c$ in Σ, where β is given by

$$\gamma^2 = \frac{1}{1-\beta^2}\,.$$

Its velocity in Σ_0 is obtained by the Lorentz transformation for velocity:

$$\beta' = \frac{-\beta-\beta}{1-\beta(-\beta)} = -\frac{2\beta}{1+\beta^2}\,.$$

Its length in Σ_0 is therefore

$$\begin{aligned} L' &= \frac{1}{\gamma'}L_0 = L_0\sqrt{1-\beta'^2} = L_0\sqrt{1-\left(\frac{2\beta}{1+\beta^2}\right)^2} \\ &= L_0\left(\frac{1-\beta^2}{1+\beta^2}\right) = \frac{L_0}{2\gamma^2-1} \\ &= \frac{200}{2\times 10^{10}-1} \approx 10^{-8} = 0.01\ \mu\mathrm{m}\,. \end{aligned}$$

(b) To an observer in Σ_0 the time taken for the two bunches to pass through each other completely is

$$t' = \frac{L_0 + L'}{\beta' c}\,.$$

As

$$\beta = \sqrt{1-\frac{1}{\gamma^2}} \approx 1$$

and so

$$\beta' = \frac{2\beta}{1+\beta^2} \approx 1\,,$$

$$\begin{aligned} t' &\approx \frac{200+10^{-8}}{c} \\ &= \frac{200}{3\times 10^8} = 6.67\times 10^{-7}\ \mathrm{s}\,. \end{aligned}$$

(c) Consider the bunch of positrons and let its length, radius, number of particles, and charge density be l, r_0, N and ρ respectively. Then

$$\rho = \frac{eN}{\pi r_0^2 l}\,.$$

The two bunches of positrons and electrons carry opposite charges and move in opposite directions, and so the total charge density is

$$J = 2\rho\beta c\,,$$

where βc is the speed of the particles given by

$$\gamma = \frac{E}{mc^2} = (1-\beta^2)^{-\frac{1}{2}}\,.$$

Applying Ampére's circuital law

$$\oint_c \mathbf{B}\cdot d\mathbf{l} = \mu_0 I\,,$$

we find for $r > r_0$,

$$2\pi r B = \mu_0 \cdot \frac{2eN}{\pi r_0^2 l}\beta c \cdot \pi r_0^2\,,$$

or

$$B = \frac{\mu_0 eN}{\pi l}\frac{\beta c}{r}\,;$$

for $r < r_0$,

$$2\pi r B = \mu_0 \cdot \frac{2eN}{\pi r_0^2 l}\beta c \pi r^2$$

or

$$B = \frac{\mu_0 eN}{\pi l}\frac{\beta c r}{r_0^2}\,.$$

Figure 4.1 shows the variation of B with r. At $r = r_0 = 1\ \mu$m,

$$B = \frac{4\pi\times 10^{-7}\times 1.6\times 10^{-19}\times 10^{10}}{\pi\times 2\times 10^{-3}\times 10^{-6}}\times 1\times 3\times 10^8 = 96\ \text{T}$$

$$= 9.6\times 10^5\ \text{Gs}\,.$$

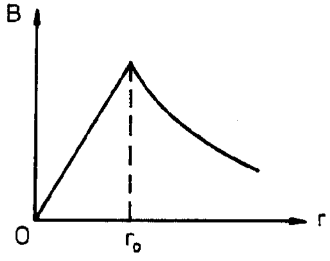

Fig. 4.1

(d) The magnetic field exerts a force vB perpendicular to the motion of an electron. If Δt is the duration of encounter with the opposite bunch, it will acquire a transverse momentum of

$$p_\perp = evB\Delta t\,.$$

Hence

$$\begin{aligned}\theta &\approx \frac{p_\perp}{p} = \frac{evB\Delta t}{m\gamma v} = \frac{eBl}{m\gamma v} = \frac{eBcl}{pc}\\ &= \frac{1.6\times10^{-19}\times 96\times 3\times10^{8}\times 2\times10^{-3}}{50\times10^{9}\times1.6\times10^{-19}} = 1.15\times10^{-3}\ \text{rad} = 39.6'\,.\end{aligned}$$

4002

A certain elementary process is observed to produce a relativistic meson whose trajectory in a magnetic field B is found to have a curvature given by $(\rho B)_1 = 2.7$ Tesla-meters.

After considerable energy loss by passage through a medium, the same meson is found to have $(\rho B)_2 = 0.34$ Telsa-meters while a time-of-flight measurement yields a speed of $v_2 = 1.8\times10^8$ m/sec for this 'slow' meson.

(a) Find the rest mass and the kinetic energies of the meson (in MeV) before and after slowing down (2-figure accuracy).

(b) If this 'slow' meson is seen to have a 50% probability of decaying in a distance of 4 meters, compute the intrinsic half life of this particle in its own rest frame, as well as the distance that 50% of the initial full-energy mesons would travel in the laboratory frame.

(*UC, Berkeley*)

Solution:

(a) As $evB = \frac{\gamma m v^2}{\rho}$, or $\rho B = \frac{\gamma\beta mc}{e}$, we have for the meson

$$\frac{(\rho B)_1}{(\rho B)_2} = \frac{\gamma_1\beta_1}{\gamma_2\beta_2}\,.$$

At $\beta_2 = \frac{v_2}{c} = \frac{1.8\times10^8}{3\times10^8} = 0.6$, or $\gamma_2\beta_2 = \frac{\beta_2}{\sqrt{1-\beta_2^2}} = 0.75$, we have

$$\begin{aligned} p_2c = \gamma_2\beta_2 mc^2 &= ec(\rho\beta)_2 \\ &= 1.6\times10^{-19}\times0.34c \text{ Joule} \\ &= 0.34\times3\times10^8 \text{ eV} \\ &= 0.102 \text{ GeV}\,. \end{aligned}$$

The rest mass of the meson is therefore

$$m = \frac{p_2c}{\gamma_2\beta_2c^2} = \frac{0.102}{0.75}\ \text{GeV}/c^2 = 0.14\ \text{GeV}/c^2\,.$$

Before slowing down, the meson has momentum

$$p_1c = ec(\rho B)_1 = 2.7\times0.3 = 0.81\ \text{GeV}\,,$$

and hence kinetic energy

$$T_1 = \sqrt{p_1^2c^2+m^2c^4} - mc^2 = \sqrt{0.81^2+0.14^2} - 0.14 = 0.68\ \text{GeV}\,.$$

After slowing down, the meson has kinetic energy

$$T_2 = \sqrt{p_2^2c^2+m^2c^4} - mc^2 = \sqrt{0.102^2+0.14^2} - 0.14 = 0.033\ \text{GeV}\,.$$

(b) The half life τ is defined by

$$\exp\left(-\frac{t}{\tau}\right) = \exp\left(-\frac{l}{\beta c\tau}\right) = \frac{1}{2}\,,$$

or

$$\tau = \frac{l}{\beta c\ln 2}\,.$$

In the rest frame of the meson, on account of time dilation, the half-life is

$$\tau_0 = \frac{\tau}{\gamma_2} = \frac{l_2}{\gamma_2\beta_2c\ln2} = \frac{4}{0.75\times3\times10^8\ \ln2} = 2.6\times10^{-8}\ \text{s}\,.$$

In the laboratory frame, the distance full-energy mesons travel before their number is reduced by 50% is given by

$$l_1 = \tau_1\beta_1c\ln2 = \tau_0\gamma_1\beta_1c\ln2\,.$$

As

$$\gamma_1\beta_1 = \frac{p_1 c}{mc^2} = \frac{0.81}{0.14} = 5.8\,,$$

$$l_1 = 2.6 \times 10^{-8} \times 5.8 \times 3 \times 10^8 \times \ln 2 = 31 \text{ m}\,.$$

4003

The Princeton synchrotron (PPA) has recently been used to accelerate highly charged nitrogen ions. If the PPA can produce protons of nominal total energy 3 GeV, what is the maximum kinetic energy of charge 6^+ ^{14}N ions?

(*Wisconsin*)

Solution:

After the ions enter the synchrotron, they are confined by magnetic field and accelerated by radio frequency accelerator. The maximum energy attainable is limited by the maximum value B_m of the magnetic field. The maximum momentum p_m is given by

$$p_m = |q|\rho B_m$$

where $|q|$ is the absolute charge of the ion and ρ the radius of its orbit. Considering protons and nitrogen ions we have

$$\frac{p_p}{p_N} = \frac{|q|_p}{|q|_N}\,, \qquad p_N = 6p_p\,.$$

As

$$\sqrt{p_p^2 + m_p^2} = \sqrt{p_p^2 + 0.938^2} = 3\,,$$

we have

$$p_p = 2.85 \text{ GeV}/c\,,$$

and

$$p_N = 17.1 \text{ GeV}/c\,.$$

Hence the maximum kinetic energy of the accelerated nitrogen ions is

$$T = \sqrt{17.1^2 + (0.938 \times 14)^2} - 0.938 \times 14 = 8.43 \text{ GeV}\,.$$

4004

(a) A muon at rest lives 10^{-6} sec and its mass is 100 MeV/c^2. How energetic must a muon be to reach the earth's surface if it is produced high in the atmosphere (say $\sim 10^4$ m up)?

(b) Suppose to a zeroth approximation that the earth has a 1-gauss magnetic field pointing in the direction of its axis, extending out to 10^4 m. How much, and in what direction, is a muon of energy E normally incident at the equator deflected by the field?

(c) Very high-energy protons in cosmic rays can lose energy through collision with 3-K radiation (cosmological background) in the process $p + \gamma \rightarrow p + \pi$. How energetic need a proton be to be above threshold for this reaction?

(*Princeton*)

Solution:

(a) Let the energy of the muons be $E \equiv \gamma m$, where m is their rest mass. In the laboratory frame the lifetime is $\tau = \tau_0 \gamma$, τ_0 being the lifetime in the muon rest frame. Then

$$l = \tau \beta c = \tau_0 \gamma \beta c \,,$$

giving

$$E = \frac{lm}{\beta \tau_0 c} \approx \frac{lm}{\tau_0 c} = \frac{10^4 \times 0.1}{10^{-6} \times 3 \times 10^8} = 3.3 \text{ GeV} \,.$$

(b) Consider a high energy μ^+ in the earth's magnetic field. The force exerted by the latter is balanced by the centripetal force:

$$evB = \frac{m\gamma v^2}{R} \,,$$

giving

$$R = \frac{pc}{ecB} \approx \frac{E}{ecB} \,,$$

where p and E are the momentum and total energy of the muon. With E in GeV and R in m,

$$\begin{aligned} R &\approx \frac{1.6 \times 10^{-10} \; E}{1.6 \times 10^{-19} \times 3 \times 10^8 \times 10^{-4}} \\ &= \frac{10^5}{3} \times E \,. \end{aligned}$$

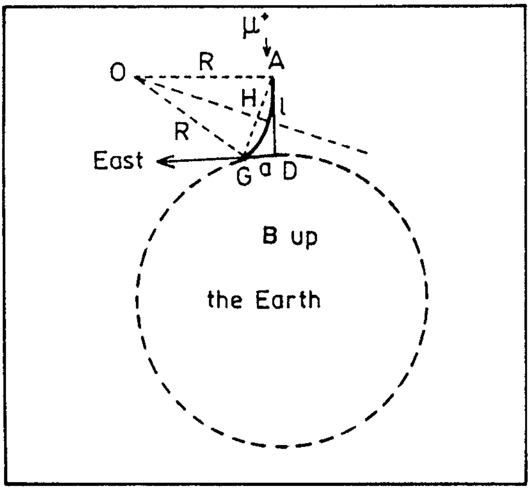

Fig. 4.2

A μ^+ incident vertically is deflected to the east and enters the earth's surface at a from the original path AD (Fig. 4.2). Let O be the center of curvature of the muon orbit and note that AD is tangential to the orbit. As $\angle OAD = \frac{\pi}{2}$, we have $\angle GAD = \angle AOH$. Hence ΔGAD and ΔAOH are similar and so

$$\frac{a}{\sqrt{l^2 + a^2}} = \frac{\sqrt{l^2 + a^2}}{2R},$$

or

$$a^2 - 2aR + l^2 = 0,$$

giving

$$a = \frac{2R \pm \sqrt{4R^2 - 4l^2}}{2} \approx \frac{l^2}{2R}$$

as $a \ll l \ll R$. Thus

$$a \approx \frac{3 \times 10^8}{2 \times 10^5 \times E} = \frac{1.5 \times 10^3}{E}.$$

For example, $a \approx 455$ m if $E = 3.3$ GeV; $a \approx 75$ m if $E = 20$ GeV.

As the earth's magnetic field points to the north, the magnetic force on a μ^+ going vertically down points to the east. It will be deflected to the east, while a μ^- will be deflected to the west.

(c) Radiation at $T = 3$ K consists of photons of energy $E = 3kT/2$, where $k = 8.6 \times 10^{-5}$ eV/K is the Boltzmann constant. Thus

$$E_\gamma = 8.6 \times 10^{-5} \times 3/2 \times 3 = 3.87 \times 10^{-4} \text{ eV}.$$

Consider the reaction $\gamma + p = p + \pi$. For head-on collision at threshold, taking $c = 1$ we have

$$(E_p + E_\gamma)^2 - (p_p - E_\gamma)^2 = (m_p + m_\pi)^2.$$

With $E_p^2 - p_p^2 = m_p^2$, and $p_p \approx E_p$ for very high energy protons, this becomes

$$E_p \approx \frac{m_\pi^2 + 2m_p m_\pi}{4E_\gamma}.$$

As $m_p = 0.938$ GeV, $m_\pi = 0.140$ GeV, $E_\gamma = 3.87 \times 10^{-13}$ GeV, the threshold energy is

$$E_p \approx \frac{0.14^2 + 2 \times 0.938 \times 0.14}{4 \times 3.87 \times 10^{-13}} = 1.82 \times 10^{11} \text{ GeV}.$$

4005

The mass of a muon is approximately 100 MeV/c^2 and its lifetime at rest is approximately two microseconds. How much energy would a muon need to circumnavigate the earth with a fair chance of completing the journey, assuming that the earth's magnetic field is strong enough to keep it in orbit? Is the earth's field actually strong enough?

(*Columbia*)

Solution:

To circumnavigate the earth, the life of a moving muon should be equal to or larger than the time required for the journey. Let the proper life of muon be τ_0. Then

$$\tau_0 \gamma \geq \frac{2\pi R}{\beta c},$$

where R is the earth's radius, βc is the muon's velocity and $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. The minimum momentum required by the muon is therefore

$$pc = m\gamma\beta c = \frac{2\pi Rmc}{\tau_0},$$

and the minimum energy required is

$$E = \sqrt{m^2c^4 + p^2c^2} = mc^2\sqrt{1 + \left(\frac{2\pi R}{\tau_0 c}\right)^2}$$

$$= 100 \times \sqrt{1 + \left(\frac{2\pi \times 6400 \times 10^3}{2 \times 10^{-6} \times 3 \times 10^8}\right)^2} = 6.7 \times 10^6 \text{ MeV}.$$

To keep the meson in orbit, we require

$$evB \geq \frac{m\gamma v^2}{R},$$

or

$$B \geq \frac{pc}{eRc} = \frac{6.7 \times 10^6 \times 1.6 \times 10^{-13}}{1.6 \times 10^{-19} \times 6400 \times 10^3 \times 3 \times 10^8}$$

$$= 3.49 \times 10^{-3}\ T \approx 35 \text{ Gs}.$$

As the average magnetic field on the earth's surface is about several tenths of one gauss, it is not possible to keep the muon in this orbit.

4006

(a) A neutron 5000 light-years from earth has rest mass 940 MeV and a half life of 13 minutes. How much energy must it have to reach the earth at the end of one half life?

(b) In the spontaneous decay of π^+ mesons at rest,

$$\pi^+ \to \mu^+ + \nu_\mu,$$

the μ^+ mesons are observed to have a kinetic energy of 4.0 MeV. The rest mass of the μ^+ is 106 MeV. The rest mass of neutrino is zero. What is the rest mass of π^+?

(*Wisconsin*)

Solution:

(a) Let the energy of the neutron be E, its velocity be βc, the half life in its rest frame be $\tau_{1/2}$. Then its half life in the earth's frame is $\tau_{1/2}\gamma$, where $\gamma = (1 - \beta^2)^{-\frac{1}{2}}$. For the neutron to reach the earth, we require

$$\gamma\beta c\tau_{\frac{1}{2}} = 5000 \times 365 \times 24 \times 60c,$$

or

$$\gamma\beta = 2.02 \times 10^8.$$

The energy of neutron is

$$E = \sqrt{m_0^2 + p^2} = m_0\sqrt{1+\gamma^2\beta^2} = 1.9 \times 10^{11} \text{ MeV}.$$

(b) Consider the decay $\pi^+ \to \mu^+ + \nu_\mu$ at rest. Conservation of momentum requires the momenta of μ and ν_μ be $\mathbf{p}$ and $-\mathbf{p}$ respectively. Then their energies are $E_\mu = \sqrt{m_\mu^2 + p^2}$, $E_\nu = p$ respectively. As $m_\mu = 106$ MeV, $E_\mu = 4 + 106 = 110$ MeV, we have

$$p = \sqrt{E_\mu^2 - m_\mu^2} = 29.4 \text{ MeV}.$$

Hence

$$m_\pi = E_\mu + E_\nu = 110 + 29.4 = 139.4 \text{ MeV}.$$

4007

A certain electron-positron pair produced cloud chamber tracks of radius of curvature 3 cm lying in a plane perpendicular to the applied magnetic field of magnitude 0.11 Tesla (Fig. 4.3). What was the energy of the γ-ray which produced the pair?

(*Wisconsin*)

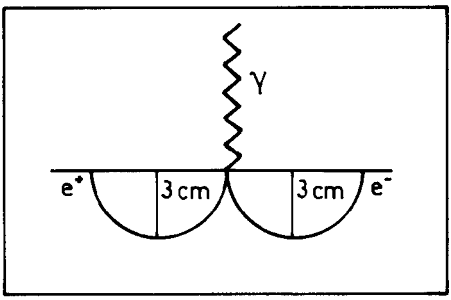

Fig. 4.3

Solution:

As

$$evB = \frac{m\gamma v^2}{\rho} = \frac{pv}{\rho},$$

we have

$$\begin{aligned} pc &= ecB\rho \\ &= \frac{1.6 \times 10^{-19} \times 3 \times 10^8}{1.6 \times 10^{-13}} B\rho \\ &= 300B\rho \end{aligned}$$

with B in Tesla, ρ in meter and p in MeV$/c$. Hence, on putting $c = 1$, the momentum of the e^+ or e^- is

$$p = 300B\rho = 300 \times 0.11 \times 0.03 = 0.99 \text{ MeV}/c\,,$$

and its energy is

$$E = \sqrt{p^2 + m_e^2} = \sqrt{0.99^2 + 0.51^2} = 1.1 \text{ MeV}\,.$$

Therefore the energy of the γ-ray that produced the e^+e^- pair is approximately

$$E_\gamma = 2E = 2.2 \text{ MeV}\,.$$

4008

Newly discovered D^0 mesons (mass = 1.86 GeV) decay by $D^0 \to K^+\pi^-$ in $\tau = 5 \times 10^{-13}$ sec. They are created with 18.6 GeV energy in a bubble chamber. What resolution is needed to observe more than 50% of the decays?

(a) 0.0011 mm.
(b) 0.44 mm.
(c) 2.2 mm.

(CCT)

Solution:

As

$$\begin{aligned} I &= I_0 e^{-t/\tau} \geq 0.5 I_0\,, \\ t &\leq \tau \ln 2\,. \end{aligned}$$

The mesons have $\gamma = \frac{18.6}{1.86} = 10$ and $\beta \approx 1$. Their proper lifetime is $\tau_0 = 5 \times 10^{-13}$ s, giving

$$\tau = \gamma\tau_0 = 5 \times 10^{-12} \text{ s}\,.$$

Thus the distance traveled by the mesons is

$$\begin{aligned} tc \leq \tau c \ln 2 &= 5 \times 10^{-12} \times 3 \times 10^{11} \times \ln 2 \\ &= 1 \text{ mm} \end{aligned}$$

Hence the resolution should be better than 1 mm, and the answer is (b).

4009

A collimated kaon beam emerges from an analyzing spectrometer with $E = 2$ GeV. At what distance is the flux reduced to 10% if the lifetime is 1.2×10^{-8} sec?

(a) 0.66 km.
(b) 33 m.
(c) 8.3 m.

(CCT)

Solution:

As $m_k = 0.494$ GeV, $\tau_0 = 1.2 \times 10^{-8}$ s, $E_k = 2$ GeV, we have

$$\gamma = \frac{2}{0.494} = 4.05\,, \qquad \beta = \sqrt{1 - \gamma^{-2}} = 0.97\,,$$

and the laboratory lifetime is

$$\tau = \gamma\tau_0 = 4.8 \times 10^{-8} \text{ s}\,.$$

The time t required to reduce the kaon flux from I_0 to $I_0/10$ is given by

$$I_0 e^{-t/\tau} = \frac{I_0}{10}\,,$$

or

$$t = \tau \ln 10 = 11.05 \times 10^{-8} \text{ s}\,.$$

The distance traveled by the beam during t is

$$t\beta c = 11.05 \times 10^{-8} \times 0.97 \times 3 \times 10^8 = 32 \text{ m}\,.$$

Hence the answer is (b).

4010

The Compton wavelength of a proton is approximately

(a) 10^{-6} cm.
(b) 10^{-13} cm.
(c) 10^{-24} cm.

(*CCT*)

Solution:

The Compton wavelength of proton is

$$\lambda = \frac{2\pi\hbar}{m_p c} = \frac{2\pi\hbar c}{m_p c^2} = \frac{2\pi \times 197 \times 10^{-13}}{938} = 1.32 \times 10^{-13} \text{ cm}\,.$$

Hence the answer is (b).

4011

In a two-body elastic collision:

(a) All the particle trajectories must lie in the same plane in the center of mass frame.
(b) The helicity of a participant cannot change.
(c) The angular distribution is always spherically symmetric.

(*CCT*)

Solution:

Conservation of momentum requires all the four particles involved to lie in the same plane. Hence the answer is (a).

4012

In a collision between a proton at rest and a moving proton, a particle of rest mass M is produced, in addition to the two protons. Find the minimum energy the moving proton must have in order for this process to take place. What would be the corresponding energy if the original proton were moving towards one another with equal velocity?

(*Columbia*)

Solution:

At the threshold of the reaction

$$p + p \to M + p + p\,,$$

the particles on the right-hand side are all produced at rest. Let the energy and momentum of the moving proton be E_p and p_p respectively. The invariant mass squared of the system at threshold is

$$S = (E_p + m_p)^2 - p_p^2 = (2m_p + M)^2\,.$$

As

$$E_p^2 = m_p^2 + p_p^2\,,$$

the above gives

$$\begin{aligned} E_p &= \frac{(2m_p + M)^2 - 2m_p^2}{2m_p} \\ &= m_p + 2M + \frac{M^2}{2m_p}\,. \end{aligned}$$

If the two protons move towards each other with equal velocity, the invariant mass squared at threshold is

$$S = (E_p + E_p)^2 - (p_p - p_p)^2 = (2m_p + M)^2\,,$$

giving

$$E_p = m_p + M/2\,.$$

4013

A relativistic particle of rest mass m_0 and kinetic energy $2m_0c^2$ strikes and sticks to a stationary particle of rest mass $2m_0$.

(a) Find the rest mass of the composite.
(b) Find its velocity.

(*SUNY, Buffalo*)

Solution:

(a) The moving particle has total energy $3m_0$ and momentum

$$p = \sqrt{(3m_0)^2 - m_0^2} = \sqrt{8}m_0 \,.$$

The invariant mass squared is then

$$S = (3m_0 + 2m_0)^2 - p^2 = 17m_0^2 \,.$$

Let the rest mass of the composite particle be M. Its momentum is also p on account of momentum conservation. Thus

$$S = (\sqrt{M^2 + p^2})^2 - p^2 = M^2 \,,$$

giving

$$M = \sqrt{S} = \sqrt{17}m_0 \,.$$

(b) For the composite,

$$\gamma\beta = \frac{p}{M} = \sqrt{\frac{8}{17}} \,,$$

$$\gamma = \sqrt{\gamma^2\beta^2 + 1} = \sqrt{\frac{8}{17} + 1} = \frac{5}{\sqrt{17}} \,.$$

Hence

$$\beta = \frac{\gamma\beta}{\gamma} = \frac{\sqrt{8}}{5}$$

and the velocity is

$$v = \beta c = 1.7 \times 10^{10} \text{ cm/s} \,.$$

4014

Find the threshold energy (kinetic energy) for a proton beam to produce the reaction

$$p + p \to \pi^0 + p + p$$

with a stationary proton target.

(*Wisconsin*)

Solution:

Problem 4012 gives

$$E_p = m_p + 2m_\pi + \frac{m_\pi^2}{2m_p} = 938 + 2 \times 135 + \frac{135^2}{2 \times 938} = 1218 \text{ MeV}\,.$$

Hence the threshold kinetic energy of the proton is $T_p = 1218 - 938 = 280$ MeV.

4015

In high energy proton-proton collisions, one or both protons may "diffractively dissociate" into a system of a proton and several charged pions. The reactions are

(1) $p + p \to p + (p + n\pi)$,
(2) $p + p \to (p + n\pi) + (p + m\pi)$,
where n and m count the number of produced pions.

In the laboratory frame, an incident proton (the projectile) of total energy E strikes a proton (the target) at rest. Find the incident proton energy E that is

(a) the minimum energy for reaction 1 to take place when the target dissociates into a proton and 4 pions,

(b) the minimum energy for reaction 1 to take place when the projectile dissociates into a proton and 4 pions,

(c) the minimum energy for reaction 2 to take place when both protons dissociate into a proton and 4 pions. ($m_\pi = 0.140$ GeV, $m_p = 0.938$ GeV)

(*Chicago*)

Solution:

Let p_p be the momentum of the incident proton, n_p and n_π be the numbers of protons and pions, respectively, in the final state. Then the invariant mass squared of the system is

$$S = (E + m_p)^2 - p_p^2 = (n_p m_p + n_\pi m_\pi)^2\,,$$

giving

$$E = \frac{(n_p m_p + n_\pi m_\pi)^2 - 2m_p^2}{2m_p}\,,$$

as

$$E^2 - p_p^2 = m_p^2 \, .$$

(a) For $p + p \to 2p + 4\pi$,

$$E = \frac{(2m_p + 4m_\pi)^2 - 2m_p^2}{2m_p} = 2.225 \; GeV \, .$$

(b) As the two protons are not distinguishable, the situation is identical with that of (a). Hence $E = 2.225$ GeV.

(c) For $p + p \to 2p + 8\pi$,

$$E = \frac{(2m_p + 8m_\pi)^2 - 2m_p^2}{2m_p} = 3.847 \text{ GeV} \, .$$

4016

Protons from an accelerator collide with hydrogen. What is the minimum energy to create antiprotons?

(a) 6.6 GeV.
(b) 3.3 GeV.
(c) 2 GeV.

(CCT)

Solution:

The reaction to produce antiprotons is

$$\mathbf{p} + \mathbf{p} \to \bar{\mathbf{p}} + \mathbf{p} + \mathbf{p} + \mathbf{p} \, .$$

The hydrogen can be considered to be at rest. Thus at threshold the invariant mass squared is

$$(E + m_p)^2 - (E^2 - m_p^2) = (4m_p)^2 \, ,$$

or

$$E = 7m_p \, .$$

Hence the threshold energy is

$$E = 7m_p = 6.6 \text{ GeV} \, ,$$

and the answer is (a).

4017

Determine the threshold energy for a gamma ray to create an electron-positron pair in an interaction with an electron at rest.

(*Wisconsin*)

Solution:

From the conservation of lepton number, the reaction is

$$\gamma + e^- \to e^+ + e^- + e^- .$$

At threshold the invariant mass squared is

$$S = (E_\gamma + m_e)^2 - p_\gamma^2 = (3m_e)^2 .$$

With $E_\gamma = p_\gamma$, the above becomes

$$E_\gamma = 4m_e = 2.044 \text{ MeV} .$$

4018

Consider a beam of pions impinging on a proton target. What is the threshold for K^- production?

(*Wisconsin*)

Solution:

Conservation of strangeness requires a K^+ be also produced. Then the conservation of I_z requires that the p be converted to n as π^- has $I_z = -1$. Hence the reaction is

$$\pi^- + p \to K^- + K^+ + n .$$

Let the threshold energy and momentum of π^- be E_π and p_π respectively. (Conservation of the invariant mass squared $S = (\Sigma E)^2 - (\Sigma \mathbf{P})^2$ requires

$$S = (E_\pi + m_p)^2 - p_\pi^2 = (2m_K + m_n)^2 .$$

With $E_\pi^2 - p_\pi^2 = m_\pi^2$, this gives

$$E_\pi = \frac{(2m_K + m_n)^2 - m_p^2 - m_\pi^2}{2m_p} = \frac{(2 \times 0.494 + 0.94)^2 - 0.938^2 - 0.14^2}{2 \times 0.938}$$

$$= 1.502 \text{ GeV}$$

4019

A particle of rest mass m whose kinetic energy is twice its rest energy collides with a particle of equal mass at rest. The two combine into a single new particle. Using only this information, calculate the rest mass such a new particle would have.

(*Wisconsin*)

Solution:

Let the mass of the new particle be M and that of the incident particle be m. The incident particle has total energy $E = m + T = 3m$. At threshold, M is produced at rest and the invariant mass squared is

$$S = (E+m)^2 - p^2 = M^2 .$$

With $E^2 - p^2 = m^2$, this gives

$$M^2 = 2Em + 2m^2 = 8m^2 ,$$

i.e.,

$$M = 2\sqrt{2}m .$$

4020

If a 1000 GeV proton hits a resting proton, what is the free energy to produce mass?

(a) 41.3 GeV.
(b) 1000 GeV.
(c) 500 GeV.

(*CCT*)

Solution:

Label the incident and target protons by 1 and 2 respectively. As the invariant mass squared

$$S = (\Sigma E_i)^2 - (\Sigma \mathbf{p})^2$$

is Lorentz-invariant,

$$(E_1 + m_p)^2 - p_1^2 = E^{*2} ,$$

where E^* is the total energy of the system in the center-of-mass frame. If the final state retains the two protons, the free energy for production of mass is

$$\begin{aligned} E^* - 2m_p &= \sqrt{2m_pE_1 + 2m_p^2} - 2m_p \\ &= \sqrt{2 \times 0.938 \times 1000 + 2 \times 0.938^2} - 2 \times 0.938 \\ &= 41.5 \text{ GeV} . \end{aligned}$$

As $E_1 \gg m_p$, a rough estimate is

$$\sqrt{2m_pE_1} \approx \sqrt{2000} = 45 \text{ GeV} .$$

Thus the answer is (a).

4021

In the CERN colliding-beam storage ring, protons of total energy 30 GeV collide head-on. What energy must a single proton have to give the same center-of-mass energy when colliding with a stationary proton?

(*Wisconsin*)

Solution:

Consider a proton of energy E and momentum P incident on a stationary proton in the laboratory. This is seen in the center-of-mass frame as two protons each of energy $\bar{E}$ colliding head-on. The invariant mass squared S is Lorentz-invariant. Hence

$$S = (2\bar{E})^2 = (E + m_p)^2 - P^2 = 2m_pE + 2m_p^2 ,$$

giving

$$E = \frac{4\bar{E}^2 - 2m_p^2}{2m_p} = \frac{4 \times 30^2 - 2 \times 0.938^2}{2 \times 0.938}$$
$$= 1.92 \times 10^3 \text{ GeV}\,.$$

4022

Calculate the fractional change in the kinetic energy of an α-particle when it is scattered through $180°$ by an O^{16} nucleus.

(*Wisconsin*)

Solution:

Let E be the kinetic energy of the incident α-particle, p be its momentum, m_α be its mass, and let E' and p' represent the kinetic energy and momentum of the scattered α-particle respectively. In the nonrelativistic approximation,

$$p = \sqrt{2m_\alpha E}\,, \qquad p' = \sqrt{2m_\alpha E'}\,.$$

Let the recoil momentum of ^{16}O be P_0, conservation of momentum and of energy require

$$P_0 = p + p' = \sqrt{2m_\alpha E} + \sqrt{2m_\alpha E'}\,,$$
$$E = E' + \frac{(\sqrt{2m_\alpha E} + \sqrt{2m_\alpha E'})^2}{2M}\,,$$

where M is the mass of ^{16}O nucleus. With $M \approx 4m_\alpha$ the last equation gives

$$E = E' + \frac{1}{4}(\sqrt{E} + \sqrt{E}\,')^2 = \frac{5}{4}E' + \frac{1}{2}\sqrt{EE'} + \frac{1}{4}E\,,$$

or

$$(5\sqrt{E'} - 3\sqrt{E})(\sqrt{E'} + \sqrt{E}) = 0\,.$$

Thus $5\sqrt{E'} - 3\sqrt{E} = 0$, yielding $E' = \frac{9}{25}E$.

Therefore the fractional change in the kinetic energy of α-particle is

$$\frac{E' - E}{E} = -\frac{16}{25}\,.$$

4023

A beam of π^+ mesons of kinetic energy T yields some μ^+ going backward. The μ^+'s are products of the reaction

$$\pi^+ \to \mu^+ + \nu\,.$$

With

$$m_\pi c^2 = 139.57 \text{ MeV}\,,$$

$$m_\mu c^2 = 105.66 \text{ MeV}\,,$$

$$m_\nu c^2 = 0.0 \text{ MeV}\,.$$

for what range of T is this possible?

(*Wisconsin*)

Solution:

μ^+ from π^+ decay can go backward in the laboratory frame if its velocity in the center-of-mass frame (c.m.s.), which is also the rest frame of π^+, is greater than the velocity of π^+ in the laboratory frame. Denoting quantities in c.m.s. by a bar, we have

$$m_\pi = \sqrt{\bar{\mathbf{p}}_\mu^2 + m_\mu^2} + \bar{\mathbf{p}}_\nu$$

since neutrino has zero rest mass. As $\bar{\mathbf{p}}_\mu = -\bar{\mathbf{p}}_\nu$, $\bar{p}_\mu = \bar{p}_\nu$ and the above gives

$$\bar{p}_\mu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}\,.$$

Hence

$$\bar{E}_\mu = \sqrt{\bar{p}_\mu^2 + m_\mu^2} = \frac{m_\pi^2 + m_\mu^2}{2m_\pi}\,,$$

and so

$$\bar{\beta}_\mu = \frac{\bar{p}_\mu}{\bar{E}_\mu} = \frac{m_\pi^2 - m_\mu^2}{m_\pi^2 + m_\mu^2}\,.$$

We require $\beta_\pi \leq \bar{\beta}_\mu$ for some μ^+ to go backward. Hence

$$E_\pi \leq \frac{m_\pi}{\sqrt{1 - \bar{\beta}_\mu{}^2}} = \frac{m_\pi^2 + m_\mu^2}{2m_\mu}\,,$$

or

$$T_\pi \le E_\pi - m_\pi = \frac{(m_\pi - m_\mu)^2}{2m_\mu} = 5.44 \text{ MeV}\,.$$

4024

State whether the following processes are possible or impossible and prove your statement:

(a) A single photon strikes a stationary electron and gives up all its energy to the electron.

(b) A single photon in empty space is transformed into an electron and a positron.

(c) A fast positron and a stationary electron annihilate, producing only one photon.

(*Wisconsin*)

Solution:

All the three reactions cannot take place because in each case energy and momentum cannot be both conserved.

(a) For the process

$$\gamma + e \to e'\,,$$

conservation of the invariant mass squared,

$$S = (E_\gamma + m_e)^2 - p_\gamma^2 = 2m_e E_\gamma + m_e^2 = E_{e'}^2 - p_{e'}^2 = m_e^2\,,$$

leads to $m_e E_\gamma = 0$, which contradicts the fact that neither E_γ nor m_e is zero.

(b) In the process $\gamma \to e^+ + e^-$, let the energies and momenta of the produced e^+ and e^- be E_1, E_2, $\mathbf{p}_1$, $\mathbf{p}_2$ respectively. The invariant mass squared of the initial state is

$$S(\gamma) = E_\gamma^2 - p_\gamma^2 = 0\,,$$

while for the final state it is

$$\begin{aligned} S(e^+e^-) &= (E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \\ &= 2m_e^2 + 2(E_1 E_2 - p_1 p_2 \cos\theta) \ge 2m_e^2\,, \end{aligned}$$

where θ is the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$. As $S(\gamma) \neq S(e^+e^-)$, its invariance is violated and the reaction cannot take place.

(c) The reaction is the inverse of that in (b). It similarly cannot take place.

4025

(a) Prove that an electron-positron pair cannot be created by a single isolated photon, i.e., pair production takes place only in the vicinity of a particle.

(b) Assuming that the particle is the nucleus of a lead atom, show numerically that we are justified in neglecting the kinetic energy of the recoil nucleus in estimating the threshold energy for pair production.

(*Columbia*)

Solution:

(a) This is not possible because energy and momentum cannot both be conserved, as shown in **Problem 4024**(b). However, if there is a particle in the vicinity to take away some momentum, it is still possible.

(b) Neglecting the kinetic energy of the recoiling nucleus, the threshold energy of the photon for e^+e^- pair production is

$$E_\gamma = 2m_e = 1.022 \text{ MeV}.$$

At most, the lead nucleus can take away all its momentum p_γ, i.e.,

$$p_{\text{Pb}} = p_\gamma = E_\gamma ,$$

and the recoil kinetic energy of the Pb nucleus is

$$T_{\text{Pb}} = {p_{\text{Pb}}}^2/(2m_{\text{Pb}}) = \left(\frac{E_\gamma}{2m_{\text{Pb}}}\right) E_\gamma .$$

As $m_{\text{Pb}} \approx 200 m_p = 1.88 \times 10^5$ MeV,

$$T_{\text{Pb}} \approx \frac{1.022}{2 \times 1.88 \times 10^5} \times E_\gamma = 2.7 \times 10^{-6} \times E_\gamma .$$

Hence it is reasonable to neglect the kinetic energy of the recoiling nucleus.

4026

(a) Write the reaction equation for the decay of a negative muon. Identify in words all the particles involved.

(b) A mu-minus decays at rest. Could a lepton from this decay convert a proton at rest into a neutron? If so, how; and in particular will there be enough energy?

(*Wisconsin*)

Solution:

(a) The decay reaction for μ^- is

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu ,$$

where e^- represents electron, $\bar{\nu}_e$ electron-antineutrino, ν_μ muon-neutrino.

(b) If the energy of the electron or the electron-antineutrino is equal to or larger than the respective threshold energy of the following reactions, a proton at rest can be converted into a neutron.

$$e^- + p \to n + \nu_e , \tag{1}$$

$$\bar{\nu}_e + p \to e^+ + n . \tag{2}$$

The threshold energy for reaction (1) is

$$E_1 \approx m_n - m_p - m_e \approx 0.8 \text{ MeV} .$$

The threshold energy for reaction (2) is

$$E_2 \approx m_n - m_p + m_e \approx 1.8 \text{ MeV} .$$

Mu-minus decay releases quite a large amount of energy, about 105 MeV. The maximum energy ν_μ can acquire is about $m_\mu/2 \approx 53$ MeV. Then the combined energy of $\bar{\nu}_e$ and e^- is at the least about 53 MeV. In the reactions, as the mass of proton is much larger than that of muon or neutrino, the threshold energy in the center-of-mass system is approximately equal to that in the laboratory system. Therefore, at least one of the two leptons,

$\bar{\nu}_e$ or e^-, from μ^- decay has energy larger than the threshold of the above reactions and so can convert a proton at rest into a neutron.

4027

Two accelerator facilities are under construction which will produce the neutral intermediate vector boson Z^0 via the process

$$e^+ + e^- \to Z^0 \,.$$

The mass of the Z^0 is $M_Z = 92$ GeV.

(a) Find the energy of the electron beam needed for the colliding beam facility under construction.

Assume that a fixed target facility is to be built, such that a beam of e^+ will strike a target of e^- at rest.

(b) What is the required e^+ beam energy for this case?

(c) What is the energy and velocity of the Z^0 (in the laboratory) after production?

(d) Find the maximum energy in the laboratory frame of muons from the subsequent decay $Z^0 \to \mu^+ + \mu^-$.

(*Columbia*)

Solution:

(a) For the colliding-beam machine, the center-of-mass and laboratory frames are identical, and so the threshold electron energy for Z^0 production is $E = M_Z/2 = 46$ GeV.

(b) For the fixed target facility, conservation of the invariant mass gives

$$(E_{e^+} + m_e)^2 - p_{e^+}^2 = M_Z^2 \,.$$

With $E_{e^+}^2 - p_{e^+}^2 = m_e^2$, we find the threshold energy

$$E_{e^+} = \frac{M_Z^2 - 2m_e^2}{2m_e} \approx \frac{M_Z^2}{2m_e} = 8.30 \times 10^6 \text{ GeV} \,.$$

(c) In the center-of-mass frame (c.m.s.), total momentum is zero, total energy is $2\bar{E}$, $\bar{E}$ being the energy of e^+ or e^-. Invariance of the invariant mass squared,

$$S = (E_{e^+} + m_e)^2 - p_{e^+}^2 = (2\bar{E})^2 \,,$$

gives

$$\bar{E} = \frac{\sqrt{2m_e E_{e^+} + 2m_e^2}}{2} \approx \sqrt{\frac{m_e E_{e^+}}{2}} = \frac{M_Z}{2}\,.$$

The Lorentz factor of c.m.s. is therefore

$$\gamma_0 = \frac{\bar{E}}{m_e} = \sqrt{\frac{E_{e^+}}{2m_e} + \frac{1}{2}} \approx \frac{M_Z}{2m_e}\,.$$

This is also the Lorentz factor of Z^0 as it is created at rest in c.m.s. Thus Z^0 has total energy $\gamma_0 M_z \approx \frac{M_Z^2}{2m_e} \approx E_{e^+}$ and velocity

$$\beta c = \left(1 - \frac{1}{\gamma_0^2}\right)^{\frac{1}{2}} c \approx \left[1 - \left(\frac{2m_e}{M_Z}\right)^2\right]^{\frac{1}{2}} c$$
$$\approx \left(1 - \frac{2m_e^2}{M_Z^2}\right) c\,.$$

(d) In the rest frame of Z^0 the angular distribution of the decay muons is isotropic. Those muons that travel in the direction of the incident e^+ have the maximum energy in the laboratory.

In c.m.s. Z^0 decays at rest into two muons, so that

$$\bar{E}_\mu = \frac{M_Z}{2}\,, \qquad \bar{\gamma}_\mu = \frac{\bar{E}_\mu}{m_\mu} = \frac{M_Z}{2m_\mu}\,.$$

For a muon moving in the direction of motion of e^+, inverse Lorentz transformation gives

$$\gamma_\mu = \gamma_0(\bar{\gamma}_\mu + \beta_0 \bar{\gamma}_\mu \bar{\beta}_\mu) \approx 2\gamma_0 \bar{\gamma}_\mu\,,$$

as $\beta_0 \approx \beta_\mu \approx 1$. Hence the maximum laboratory energy of the decay muons is

$$E_\mu = \gamma_\mu m_\mu \approx 2\gamma_0 \bar{\gamma}_\mu m_\mu = \frac{M_Z^2}{2m_e} \approx E_{e^+}\,.$$

This is to be expected physically as the velocity of the Z^0 is nearly equal to c. Compared to its kinetic energy, the rest mass of the muons produced in the reaction is very small. Thus the rest mass of the forward muon can be treated as zero, so that, like a photon, it takes all the momentum and energy of the Z^0.

4028

The following elementary-particle reaction may be carried out on a proton target at rest in the laboratory:

$$K^- + p \to \pi^0 + \Lambda^0 \,.$$

Find the special value of the incident K^- energy such that the Λ^0 can be produced at rest in the laboratory. Your answer should be expressed in terms of the rest masses m_{π^0}, m_{K^-}, m_p and m_{Λ^0}.

(MIT)

Solution:

The invariant mass squared $S = (\Sigma E)^2 - (\Sigma \mathbf{p})^2$ is conserved in a reaction. Thus

$$(E_K + m_p)^2 - p_K^2 = (E_\pi + m_\Lambda)^2 - p_\pi^2 \,.$$

As the Λ^0 is produced at rest, $p_\Lambda = 0$ and the initial momentum p_K is carried off by the π^0. Hence $p_\pi = p_K$ and the above becomes

$$E_K + m_p = E_\pi + m_\Lambda \,,$$

or

$$E_\pi^2 = p_\pi^2 + m_\pi^2 = p_K^2 + m_\pi^2 = E_K^2 + (m_\Lambda - m_p)^2 - 2E_K(m_\Lambda - m_p) \,,$$

or

$$2E_K(m_\Lambda - m_p) = m_K^2 - m_\pi^2 + (m_\Lambda - m_p)^2 \,,$$

giving

$$E_K = \frac{m_K^2 - m_\pi^2 + (m_\Lambda - m_p)^2}{2(m_\Lambda - m_p)} \,.$$

4029

K^+ mesons can be photoproduced in the reaction

$$\gamma + p \to K^+ + \Lambda^0 \,.$$

(a) Give the minimum γ-ray energy in the laboratory, where p is at rest, that can cause this reaction to take place.

(b) If the target proton is not free but is bound in a nucleus, then the motion of the proton in the nucleus (Fermi motion) allows the reaction of part (a) to proceed with a lower incident photon energy. Assume a reasonable value for the Fermi motion and compute the minimum photon energy.

(c) The Λ^0 decays in flight into a proton and a π^- meson. If the Λ^0 has a velocity of $0.8c$, what is (i) the maximum momentum that the π^- can have in the laboratory, and (ii) the maximum component of laboratory momentum perpendicular to the Λ^0 direction?

$$(m_{K^+} = 494 \text{ MeV}/c^2\,, m_{A^0} = 1116 \text{ MeV}/c^2\,, m_{\pi^-} = 140 \text{ Mev}/c^2)$$

(*CUSPEA*)

Solution:

(a) Let P denote 4-momentum. We have the invariant mass squared

$$S = -(P_\gamma + P_p)^2 = (m_p + E_\gamma)^2 - E_\gamma^2 = m_p^2 + 2E_\gamma m_p = (m_K + m_\Lambda)^2\,,$$

giving

$$E_\gamma = \frac{(m_K + m_\Lambda)^2 - m_p^2}{2m_p} = 913 \text{ MeV}\,.$$

as the minimum γ energy required for the reaction to take place.

(b) If we assume that the proton has Fermi momentum $p_p = 200$ MeV/ c then

$$S = -(P_\gamma + P_p)^2 = (E_\gamma + E_p)^2 - (\mathbf{p}_\gamma + \mathbf{p}_p)^2 = (m_K + m_\Lambda)^2\,.$$

With $E_\gamma = p_\gamma$, $E_p^2 - p_p^2 = m_p^2$, this gives

$$E_\gamma = \frac{(m_K + m_\Lambda)^2 - m_p^2 + 2\mathbf{p}_\gamma \cdot \mathbf{p}_p}{2E_p}\,.$$

The threshold energy E_γ is minimum when the proton moves opposite to the photon, in which case

$$\begin{aligned} E_\gamma &= \frac{(m_K + m_\Lambda)^2 - m_p^2}{2(E_p + p_p)} \\ &= \frac{(m_K + m_\Lambda)^2 - m_p^2}{2(\sqrt{p_p^2 + m_p^2} + p_p)} = 739 \text{ MeV}\,. \end{aligned}$$

(c) In the rest frame of Λ^0, conservation of energy and of momentum give

$$\bar{E}_\pi + \bar{E}_p = m_\Lambda \,, \qquad \bar{\mathbf{p}}_\pi + \bar{\mathbf{p}}_p = 0 \,.$$

Then

$$(m_\Lambda - \bar{E}_\pi)^2 = \bar{p}_p^2 + m_p^2 = \bar{p}_\pi^2 + m_p^2 \,,$$

or

$$\bar{E}_\pi = \frac{m_\Lambda^2 + m_\pi^2 - m_p^2}{2m_\Lambda} = 173 \text{ MeV} \,,$$

and so

$$\bar{p}_\pi = \sqrt{\bar{E}_\pi^2 - m_\pi^2} = 101 \text{ MeV}/c \,.$$

p_π is maximum in the laboratory if $\bar{p}_\pi$ is in the direction of motion of the Λ^0, which has $\beta_0 = 0.8$, $\gamma_0 = (1-\beta^2)^{-\frac{1}{2}} = \frac{5}{3}$ in the laboratory. Thus

$$p_\pi = \gamma_0(\bar{p}_\pi + \beta_0 \bar{E}_\pi) = 399 \text{ MeV}/c \,.$$

As $(p_\pi)_\perp = (\bar{p}_\pi)_\perp$, the maximum momentum in the transverse direction is given by the maximum $(\bar{p}_\pi)_\perp$, i.e., 101 MeV/c.

4030

The ρ^- meson is a meson resonance with mass 769 MeV and width 154 MeV. It can be produced experimentally by bombarding a hydrogen target with a π^--meson beam,

$$\pi^- + p \to \rho^0 + n \,.$$

(a) What is the lifetime and mean decay distance for a 5 GeV ρ^0?

(b) What is the π^- threshold energy for producing ρ^0 mesons?

(c) If the production cross section is 1 mb $\equiv 10^{-27}$ cm^2 and the liquid hydrogen target is 30 cm long, how many ρ^0 are produced on the average per incident π^-? (The density of liquid hydrogen is 0.07 g/c.c.)

(d) ρ^0 mesons decay almost instantaneously into $\pi^+ + \pi^-$. Given that the ρ^0 is produced in the forward direction in the laboratory frame with an energy of 5 GeV, what is the minimum opening angle between the outgoing π^+ and π^- in the laboratory frame?

(*Columbia*)

Solution:

(a) The ρ^0 has Lorentz factor

$$\gamma_0 = \frac{E_\rho}{m_\rho} = \frac{5}{0.769} = 6.50\,.$$

Its proper lifetime is

$$\tau_0 = \hbar/\Gamma = \frac{6.58 \times 10^{-22}}{154} = 4.27 \times 10^{-24}\ \mathrm{s}\,.$$

In laboratory frame the lifetime is

$$\tau = \gamma_0 \tau_0 = 2.78 \times 10^{-23}\ \mathrm{s}\,.$$

The mean decay distance for a 5 GeV ρ^0 is thus

$$\begin{aligned} d &= \tau\beta c = \tau_0\gamma_0\beta c = \tau_0 c\sqrt{\gamma_0^2 - 1} \\ &= 4.27 \times 10^{-24} \times 3 \times 10^{10} \times \sqrt{6.50^2 - 1} \\ &= 8.23 \times 10^{-13}\ \mathrm{cm}\,. \end{aligned}$$

(b) At threshold the invariant mass squared is

$$S = (E_\pi + m_p)^2 - p_\pi^2 = (m_\rho + m_n)^2\,.$$

With $E_\pi^2 = m_\pi^2 + p_\pi^2$ this gives the threshold pion energy

$$\begin{aligned} E_\pi &= \frac{(m_\rho + m_n)^2 - m_\pi^2 - m_p^2}{2m_p} \\ &= \frac{(769 + 940)^2 - 140^2 - 938^2}{2 \times 938} = 1077\ \mathrm{MeV}\,. \end{aligned}$$

(c) The average number of ρ^- events caused by an incident π is

$$\begin{aligned} N &= \rho l \sigma N_0/A = 0.07 \times 30 \times 10^{-27} \times 6.02 \times 10^{23} \\ &= 1.3 \times 10^{-3}\,, \end{aligned}$$

where $N_0 = 6.023 \times 10^{23}$ is the Avagadro number, $A = 1$ is the mass number of hydrogen, and ρ is the density of liquid hydrogen.

(d) In the rest frame $\bar{\Sigma}$ of the ρ^0, the pair of pions produced move in opposite directions with momenta $\bar{\mathbf{p}}_{\pi^+} = -\bar{\mathbf{p}}_{\pi^-}$ and energies $\bar{E}_{\pi^+} = \bar{E}_{\pi^-} = \frac{m_\rho}{2}$, corresponding to

$$\bar{\gamma}_\pi = \frac{\bar{E}_\pi}{m_\pi} = \frac{m_p}{2m_\pi}\,, \bar{\beta}_\pi = \sqrt{1 - \frac{1}{\bar{\gamma}_\pi^2}} = \frac{1}{m_\rho}\sqrt{m_\rho^2 - 4m_\pi^2} = 0.93\,.$$

$\bar{\Sigma}$ has Lorentz factor $\gamma_0 = 6.50$ in the laboratory, corresponding to

$$\beta_0 = \sqrt{1 - \frac{1}{6.50^2}} = 0.99\,.$$

Consider a pair of pions emitted in $\bar{\Sigma}$ parallel to the line of flight of ρ^0 in the laboratory. The forward-moving pion will move forward in the laboratory. As $\beta_0 > \bar{\beta}_\pi$, the backword-moving pion will also move forward in the laboratory. Hence the minimum opening angle between the pair is zero.

4031

(a) The Ω^- was discovered in the reaction $K^- + p \to \Omega^- + K^+ + K^0$. In terms of the masses of the various particles, what is the threshold kinetic energy for the reaction to occur if the proton is at rest?

(b) Suppose the K^0 travels at a speed of $0.8c$. It decays in flight into two neutral pions. Find the maximum angle (in the laboratory frame) that the pions can make with the K^0 line of flight. Express your answer in terms of the π and K masses.

(*Columbia*)

Solution:

(a) At threshold the invariant mass squared is

$$S = (E_K + m_p)^2 - p_K^2 = (m_\Omega + 2m_K)^2\,.$$

With $E_K^2 = p_K^2 + m_K^2$, this gives

$$E_K = \frac{(m_\Omega + 2m_K)^2 - m_p^2 - m_K^2}{2m_p}\,.$$

Hence the threshold kinetic energy is

$$T_K = E_K - m_K = \frac{(m_\Omega + 2m_K)^2 - (m_p + m_K)^2}{2m_p}\,.$$

(b) Denote the rest frame of K^0 by $\bar{\Sigma}$ and label the two π^0 produced by 1 and 2. In $\bar{\Sigma}$,

$$\bar{\mathbf{p}}_1 = -\bar{\mathbf{p}}_2, \qquad \bar{E}_1 + \bar{E}_2 = m_K\,,$$

and so

$$\bar{E}_1 = \bar{E}_2 = \frac{m_K}{2}\,,$$

$$\bar{p}_1 = \bar{p}_2 = \sqrt{\bar{E}^2 - m_\pi^2}$$

$$= \frac{1}{2}\sqrt{m_K^2 - 4m_\pi^2}\,.$$

Consider one of the pions, say pion 1. Lorentz transformation

$$p_1 \cos\theta_1 = \gamma_0(\bar{p}_1 \cos\bar{\theta}_1 + \beta_0 \bar{E}_1)\,,$$

$$p_1 \sin\theta_1 = \bar{p}_1 \sin\bar{\theta}_1\,,$$

gives

$$\tan\theta_1 = \frac{\sin\bar{\theta}_1}{\gamma_0\left(\cos\bar{\theta}_1 + \dfrac{\beta_0}{\bar{\beta}}\right)}\,,$$

where γ_0 and β_0 are the Lorentz factor and velocity of the K^0 in laboratory and $\bar{\beta} = \frac{\bar{p}_1}{\bar{E}_1}$ is the velocity of the pion in $\bar{\Sigma}$.

To find maximum θ_1, let $\frac{d\tan\theta_1}{d\bar{\theta}_1} = 0$, which gives

$$\cos\bar{\theta}_1 = -\frac{\bar{\beta}}{\beta_0}\,.$$

Note that under this condition $\frac{d^2\tan\theta_1}{d\bar{\theta}_1^2} < 0$. Also, we have $\beta_0 = 0.8$,

$$\bar{\beta} = \frac{\bar{p}_1}{\bar{E}_1} = \frac{1}{m_K}\sqrt{m_K^2 - 4m_\pi^2}$$

$$= \sqrt{494^2 - 135^2 \times 4}/494 = 0.84\,.$$

As $|\cos\bar{\theta}_1| \leq 1$, the condition cannot be satisfied. However, we see that as $\bar{\theta}_1 \to \pi$, $\cos\bar{\theta}_1 \to -1$, $\sin\bar{\theta}_1 \to 0$ and $\tan\theta_1 \to 0$, or $\theta_1 \to \pi$. Thus the maximum angle a pion can make with the line of flight of K^0 is π.

4032

The reaction

$$p + p \to \pi^+ + D\,, \tag{1}$$

in which energetic protons from an accelerator strike resting protons to produce positive pi-meson-deuteron pairs, was an important reaction in the "early days" of high-energy physics.

(a) Calculate the threshold kinetic energy T in the laboratory for the incident proton. That is, T is the minimum laboratory kinetic energy allowing the reaction to proceed. Express T in terms of the proton mass m_p, the pion mass m_π, and the deuteron mass m_D. Evaluate T, taking $m_p = 938$ MeV/c^2, $m_D = 1874$ MeV/c^2, $m_\pi = 140$ MeV/c^2.

(b) Assume that the reaction is isotropic in the center-of-mass system. That is, the probability of producing a π^+ in the solid angle element $d\Omega^* = d\phi^* d(\cos\theta^*)$ is constant, independent of angle. Find an expression for the normalized probability of the π^+ per unit solid angle in the laboratory, in terms of $\cos\theta_{\text{lab}}$, the velocity $\bar{\beta}c$ of the center of mass, the π^+ velocity βc in the laboratory, and the momentum p^* in the center of mass.

(c) In 2-body endothermic reactions such as (1) it can happen that the probability per unit solid angle in the laboratory for a reaction product can be singular at an angle $\theta \neq 0$. How does this relate to the result derived in (b)? Comment briefly but do not work out all of the relevant kinematics.

(*CUSPEA*)

Solution:

(a) At threshold the invariant mass squared is

$$(E + m_p)^2 - p^2 = (m_\pi + m_D)^2\,,$$

where E and p are the energy and momentum of the incident proton in the laboratory. With

$$E^2 = p^2 + m_p^2$$

this gives

$$E = \frac{(m_\pi + m_D)^2 - 2m_p^2}{2m_p},$$

or the threshold kinetic energy

$$T = E - m_p = \frac{(m_\pi + m_D)^2 - 4m_p^2}{2m_p} = 286.2 \text{ MeV}.$$

(b) Let the normalized probability for producing a π^+ per unit solid angle in the center-of-mass and laboratory frames be $\frac{dP}{d\Omega^*}$ and $\frac{dP}{d\Omega}$ respectively. Then

$$\frac{dP}{d\Omega^*} = \frac{1}{4\pi},$$

$$\frac{dP}{d\Omega} = \frac{dP}{d\Omega^*}\frac{d\Omega^*}{d\Omega} = \frac{1}{4\pi}\frac{d\cos\theta^*}{d\cos\theta},$$

where the star denotes quantities in the center-of-mass frame.

The Lorentz transformation for the produced π^+

$$p^* \sin\theta^* = p\sin\theta, \tag{1}$$

$$p^* \cos\theta^* = \bar{\gamma}(p\cos\theta - \bar{\beta}E), \tag{2}$$

$$E^* = \bar{\gamma}(E - \bar{\beta}p\cos\theta), \tag{3}$$

where $\bar{\gamma}$ and $\bar{\beta}$ are the Lorentz factor and velocity of the center of mass in the laboratory. Differentiating Eq. (2) with respect to $\cos\theta$, as p^* and E^* are independent of θ^* and hence of θ, we have

$$p^*\frac{d\cos\theta^*}{d\cos\theta} = \bar{\gamma}\left(p + \cos\theta\frac{dp}{d\cos\theta} - \bar{\beta}\frac{dE}{dp}\frac{dp}{d\cos\theta}\right).$$

As $E = (m^2 + p^2)^{1/2}$, $dE/dp = p/E = \beta$ and the above becomes

$$p^*\frac{d\cos\theta^*}{d\cos\theta} = \bar{\gamma}\left(p + \cos\theta\frac{dp}{d\cos\theta} - \bar{\beta}\beta\frac{dp}{d\cos\theta}\right). \tag{4}$$

Differentiate Eq. (3) with respect to $\cos\theta$, we find

$$\begin{aligned} 0 &= \bar{\gamma}\left(\frac{dE}{d\cos\theta} - \bar{\beta}p - \bar{\beta}\cos\theta\frac{dp}{d\cos\theta}\right) \\ &= \bar{\gamma}\left(\beta\frac{dp}{d\cos\theta} - \bar{\beta}p - \bar{\beta}\cos\theta\frac{dp}{d\cos\theta}\right), \end{aligned}$$

or

$$\frac{dp}{d\cos\theta} = \frac{p\bar{\beta}}{\beta - \bar{\beta}\cos\theta}\,.$$

Substituting this in Eq. (4) gives

$$\begin{aligned} p^* \frac{d\cos\theta^*}{d\cos\theta} &= \bar{\gamma}\left[p + \frac{(\cos\theta - \bar{\beta}\beta)\bar{\beta}p}{\beta - \bar{\beta}\cos\theta}\right] \\ &= \frac{(1-\bar{\beta}^2)\bar{\gamma}\beta p}{\beta - \bar{\beta}\cos\theta} = \frac{p}{\bar{\gamma}(1-\bar{\beta}\cos\theta/\beta)}\,. \end{aligned}$$

Hence the probability of producing a π^+ per unit solid angle in the laboratory is

$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{1}{4\pi}\frac{d\cos\theta^*}{d\cos\theta} \\ &= \frac{p}{4\pi\bar{\gamma}p^*(1-\bar{\beta}\cos\theta/\beta)} \\ &= \frac{m_\pi\beta\gamma}{4\pi\bar{\gamma}p^*(1-\bar{\beta}\cos\theta/\beta)}\,. \end{aligned}$$

(c) The result in (b) shows that $\frac{dP}{d\Omega}$ is singular if $1 - \frac{\bar{\beta}}{\beta}\cos\theta = 0$ which requires $\bar{\beta} > \beta$. When the π^+ goes backward in the center-of-mass frame, $\beta < \bar{\beta}$. Thus there will be an angle θ in the laboratory for which the condition is satisfied. Physically, this is the "turn around" angle, i.e., the maximum possible angle of π^+ emission in the laboratory.

4033

The Q-value (the energy released) of the $\mathrm{He}^3(n,p)$ reaction is reported to be 0.770 MeV. From this and the fact that the maximum kinetic energy of β-particles emitted by tritium (H^3) is 0.018 MeV, calculate the mass difference in amu between the neutron and a hydrogen atom ($^1\mathrm{H}$). (1 amu = 931 MeV)

(*SUNY, Buffalo*)

Solution:

The reaction

$$^3\mathrm{He} + n \to {}^3\mathrm{H} + p$$

has Q-value

$$Q = [M(^3\text{He}) + M(n) - M(^3\text{H}) - M(^1\text{H})] = 0.770 \text{ MeV},$$

whence

$$M(n) - M(^3\text{H}) = 0.770 + M(^1\text{H}) - M(^3\text{He}).$$

As in the decay $^3\text{H} \to ^3\text{He} + e^- + \bar{\nu}$ the electron has maximum energy

$$E_{\max} = [M(^3\text{H}) - M(^3\text{He})] = 0.018 \text{ MeV},$$

we find

$$M(n) - M(^1\text{H}) = 0.770 + 0.018 = 0.788 \text{ MeV}$$

$$= 8.46 \times 10^{-4} \text{ amu}.$$

4034

Suppose that a slowly moving antiproton is annihilated in a collision with a proton, leading to 2 negative pions and 2 positive pions. ($m_\pi c^2 =$ 140 MeV)

(a) What is the average kinetic energy per pion? (MeV)

(b) What is the magnitude of the momentum of a pion with such an energy? (MeV/c)

(c) What is the magnitude of the velocity? (In units of c)

(d) If the annihilation led instead to 2 photons, what would be the wavelength of each? (cm)

(*UC, Berkeley*)

Solution:

(a)

$$p + \bar{p} \to 2\pi^+ + 2\pi^-,$$

As the incident $\bar{p}$ is slowly moving, we can take $T_{\bar{p}} \approx 0$. Then each pion will have energy $E_\pi \approx \frac{2m_p}{4} = \frac{1}{2}m_p$, and so kinetic energy

$$\bar{T}_\pi \approx \frac{1}{2}m_p - m_\pi = \frac{1}{2}(938 - 2 \times 140) = 329 \text{ MeV}.$$

(b) The momentum of each pion is

$$p = \sqrt{E_\pi^2 - m_\pi^2} \approx \frac{1}{2}\sqrt{m_p^2 - 4m_\pi^2} = 448 \text{ MeV}/c\,.$$

(c) Its velocity is

$$\beta = \frac{p}{E} \approx \frac{2p}{m_p} = 0.955\,.$$

(d) If the annihilation had led to two photons, the energy of each photon would be

$$E_\gamma = \frac{2m_p}{2} = m_p = 938 \text{ MeV}\,.$$

The wavelength of each photon is

$$\lambda = \frac{c}{\nu} = \frac{2\pi\hbar c}{h\nu} = \frac{2\pi\hbar c}{E_\gamma} = \frac{2\pi \times 197 \times 10^{-13}}{938} = 1.32 \times 10^{-13} \text{ cm}\,.$$

4035

Consider the process of Compton scattering. A photon of wavelength λ is scattered off a free electron initially at rest. Let λ' be the wavelength of the photon scattered in a direction of θ.

(a) Compute λ' in terms of λ, θ and universal parameters.
(b) Compute the kinetic energy of the recoiled electron.

(*CUSPEA*)

Solution:

(a) Conservation of energy gives (Fig. 4.4)

$$pc + mc^2 = p'c + \sqrt{p_e^2c^2 + m^2c^4}\,,$$

or

$$(p - p' + mc)^2 = p_e^2 + m^2c^2\,, \tag{1}$$

where m is the mass of electron. Conservation of momentum requires

$$\mathbf{p} = \mathbf{p}' + \mathbf{p}_e$$

or

$$(\mathbf{p} - \mathbf{p}')^2 = \mathbf{p}_e^2\,. \tag{2}$$

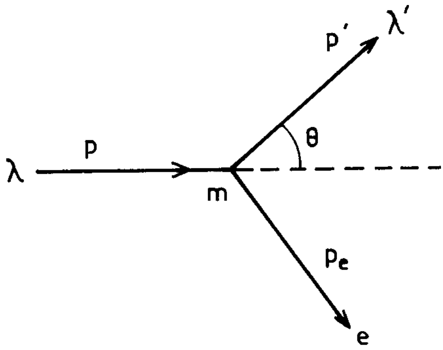

Fig. 4.4

The difference of Eqs. (1) and (2) gives

$$pp'(1-\cos\theta) = (p-p')mc\,,$$

i.e.,

$$\frac{1}{p'} - \frac{1}{p} = \frac{1}{mc}(1-\cos\theta)\,,$$

or

$$\frac{h}{p'} - \frac{h}{p} = \frac{h}{mc}(1-\cos\theta)\,.$$

Hence

$$\lambda' = \lambda + \frac{h}{mc}(1-\cos\theta)\,.$$

(b) The result of (a) gives

$$p'c = \frac{mc^2}{1-\cos\theta + \frac{mc}{p}}\,.$$

The kinetic energy of the recoiled electron is

$$T = \sqrt{p_e^2c^2 + m^2c^4} - mc^2 = pc - p'c$$

$$= \frac{pc(1-\cos\theta)}{1-\cos\theta + \frac{mc}{p}}$$

$$= \frac{(1-\cos\theta)\frac{hc}{\lambda}}{1-\cos\theta + \frac{mc\lambda}{h}}\,.$$

4036

An X-ray photon of initial frequency 3×10^{19} Hz collides with an electron at rest and is scattered through 90°. Find the new frequency of the X-ray. The electron Compton wavelength is 2.4×10^{-12} meters.

(*Wisconsin*)

Solution:

Suppose that the target electron is free. Then the wavelength of the scattered photon is given by (**Problem 4035**(a))

$$\lambda' = \lambda_0 + \frac{h}{mc}(1 - \cos\theta) \, ,$$

where λ_0 is the wavelength of the incident photon, $h/(mc)$ is the electron's Compton wavelength λ_c. At scattering angle 90° the wavelength of the scattered photon is

$$\lambda' = \lambda_0 + \lambda_c \, ,$$

and the new frequency is

$$\nu' = \frac{c}{\lambda'} = \frac{c}{\dfrac{c}{\nu_0} + \lambda_c} = 2.42 \times 10^{19} \text{ Hz} \, .$$

4037

Consider Compton scattering of photons colliding head-on with moving electrons. Find the energy of the back-scattered photons ($\theta = 180°$) if the incident photons have an energy $h\nu = 2$ eV and the electrons have a kinetic energy of 1 GeV.

(*Wisconsin, MIT, Columbia, Chicago, CCT*)

Solution:

Denote the energies and momenta of the electron and photon before and after collision by E_e, p_e, E_γ, p_γ, E'_e, p'_e, E'_γ, p'_γ respectively. Conservation of energy and of momentum give

$$E_\gamma + E_e = E'_\gamma + E'_e \, ,$$

or

$$p_\gamma + E_e = p'_\gamma + E'_e\,,$$

and

$$-p_\gamma + p_e = p'_\gamma + p'_e\,.$$

Addition and subtraction of the last two equations give

$$E'_e + p'_e = -2p'_\gamma + E_e + p_e\,,$$

$$E'_e - p'_e = 2p_\gamma + E_e - p_e\,,$$

which, after multiplying the respective sides together, give

$$E'^2_e - p'^2_e = E^2_e - p^2_e + 2p_\gamma(E_e + p_e) - 2p'_\gamma(E_e - p_e + 2p_\gamma)\,.$$

With $E'^2_e - p'^2_e = E^2_e - p^2_e = m^2_e$, this becomes

$$p'_\gamma = \frac{p_\gamma(E_e + p_e)}{E_e - p_e + 2p_\gamma} \approx \frac{2p_\gamma E_e}{\dfrac{m_e^2}{2E_e} + 2p_\gamma}$$

$$= \frac{2 \times 2 \times 10^{-6} \times 10^3}{\dfrac{0.511^2}{2 \times 10^3} + 2 \times 2 \times 10^{-6}} = 29.7 \text{ MeV}/c\,,$$

since $E_e - p_e = E_e - \sqrt{E_e^2 - m_e^2} \approx E_e - E_e(1 - \frac{m_e^2}{2E_e^2}) = \frac{m_e^2}{2E_e}$, $E_e + p_e \approx 2E_e$, $E_e \approx T_e$ as $m_e \ll E_e$. Hence the back-scattered photons have energy 29.7 MeV.

4038

(a) Two photons energy ε and E respectively collide head-on. Show that the velocity of the coordinate system in which the momentum is zero is given by

$$\beta = \frac{E - \varepsilon}{E + \varepsilon}$$

(b) If the colliding photons are to produce an electron-positron pair and ε is 1 eV, what must be the minimum value of the energy E?

(*Wisconsin*)

Solution:

(a) Let $\mathbf{P}$, $\mathbf{p}$ be the momenta of the photons, where $P = E$, $p = \varepsilon$. The total momentum of the system is $|\mathbf{P} + \mathbf{p}|$, and the total energy is $E + \varepsilon$. Hence the system as a whole has velocity

$$\beta = \frac{|\mathbf{P} + \mathbf{p}|}{E + \varepsilon} = \frac{E - \varepsilon}{E + \varepsilon}\,.$$

(b) At threshold the invariant mass squared of the system is

$$S = (E + \varepsilon)^2 - (\mathbf{P} + \mathbf{p})^2 = (2m_e)^2\,,$$

m_e being the electron mass.

As $(\mathbf{P} + \mathbf{p})^2 = (P - p)^2 = (E - \varepsilon)^2$, the above gives the minimum energy required:

$$E = \frac{m_e^2}{\varepsilon} = 261 \text{ GeV}\,.$$

4039

The universe is filled with black-body microwave radiation. The average photon energy is $E \sim 10^{-3}$ eV. The number density of the photons is ~ 300 cm^{-3}. Very high energy γ-rays make electron-positron-producing collisions with these photons. This pair-production cross section is $\sigma_T/3$, with σ_T being the nonrelativistic electron-photon scattering cross section $\sigma_T = (8\pi/3)r_e^2$, where $r_e = e^2/mc^2$ is the classical radius of electron.

(a) What energy γ-rays would have their lifetimes in the universe limited by this process?

(b) What is the average distance they would *travel* before being converted into e^+e^- pairs?

(c) How does this compare with the size of the universe?

(d) What physical process might limit lifetime of ultra-high-energy protons (energy $\geq 10^{20}$ eV) in this same microwave radiation? (Assume photon-proton scattering to be too small to be important.)

(*CUSPEA*)

Solution:

(a) Let the energies and momenta of the high energy photon and a microwave photon be E_1, $\mathbf{p}_1$, E_2, $\mathbf{p}_2$ respectively. For e^+e^- production we require

$$(E_1 + E_2)^2 - (\mathbf{p}_1 + \mathbf{p}_2)^2 \geq (2m)^2 ,$$

where m is the electron mass. As $E_1 = p_1$, $E_2 = p_2$, this becomes

$$2E_1E_2 - 2\mathbf{p}_2 \cdot \mathbf{p}_2 \geq (2m)^2 ,$$

or, if the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$ is θ,

$$E_1E_2(1 - \cos\theta) \geq 2m^2 .$$

Hence

$$E_1 \geq \frac{2m^2}{E_2(1 - \cos\theta)} .$$

E_1 is minimum when $\theta = \pi$, i.e., $\cos\theta = -1$. Thus the minimum energy for pair production is

$$E_{\min} = \frac{m^2}{E_2} = \frac{(0.51 \times 10^6)^2}{10^{-3}} = 2.6 \times 10^{14} \text{ eV} .$$

Photons of energies above this value would have lieftimes limited by the pair production process.

(b) The mean free path for pair production is

$$l = \frac{1}{\rho\sigma} \approx \frac{1}{\frac{\rho\sigma_T}{3}} = \frac{9}{8\pi\rho r_e^2}$$

$$= \frac{9}{8\pi \times 300 \times (2.8 \times 10^{-13})^2} = 1.5 \times 10^{22} \text{ cm} = 1.6 \times 10^4 \text{ light years} .$$

(c) The size of our universe is $R \approx 10^{10}$ light years. Thus

$$l \ll R .$$

(d) Suppose the proton collides head-on with a microwave photon. The total energy $\bar{E}$ in the center-of-mass frame is given by the invariant mass squared

$$(E_p + E_\gamma)^2 - (p_p - p_\gamma)^2 = \bar{E}^2 ,$$

or

$$2E_pE_\gamma + 2p_pp_\gamma + m_p^2 = \bar{E}^2 .$$

As $p_\gamma = E_\gamma$, $p_p \approx E_p$,

$$\begin{aligned}\bar{E} &= \sqrt{4E_pE_\gamma + m_p^2} \\ &= \sqrt{4 \times 10^{20} \times 10^{-3} + (10^9)^2} \\ &= 1.18 \times 10^9 \text{ eV}.\end{aligned}$$

Neglecting $\gamma p \to \gamma p$, we see that, as conservation of baryon number requires baryon number 1 in the products, the possible reactions are the following pion photoproduction

$$\gamma p \to \pi^0 p, \qquad \gamma p \to \pi^+ n.$$

4040

Consider the pion photoproduction reaction

$$\gamma + p \to \pi^0 + p,$$

where the rest energy is 938 MeV for the proton and 135 MeV for the neutral pion.

(a) If the initial proton is at rest in the laboratory find the laboratory threshold gamma-ray energy for this reaction to "go".

(b) The isotropic 3-K cosmic black-body radiation has average photon energy of about 0.001 eV. Consider a head-on collision between a proton and a photon of energy 0.001 eV. Find the minimum proton energy that will allow this pion photoproduction reaction to go.

(c) Speculate briefly on the implications of your result [to part (b)] for the energy spectrum of cosmic ray protons.

(*UC, Berkeley*)

Solution:

(a) The invariant mass squared of the reaction at threshold is

$$(E_\gamma + m_p)^2 - p_\gamma^2 = (m_p + m_\pi)^2.$$

With $E_\gamma = p_\gamma$, this gives

$$E_\gamma = \frac{(m_p + m_\pi)^2 - m_p^2}{2m_p} = m_\pi + \frac{m_\pi^2}{2m_p} = 145 \text{ MeV}.$$

(b) For head-on collision the invariant mass squared at threshold,

$$S = (E_\gamma + E_p)^2 - (p_\gamma - p_p)^2 = (m_\pi + m_p)^2,$$

gives

$$E_p - p_p = \frac{(m_p + m_\pi)^2 - m_p^2}{2E_\gamma} = 1.36 \times 10^{14} \text{ MeV}.$$

Writing $E_p - p_p = A$, we have

$$p_p^2 = (E_p - A)^2,$$

or

$$m_p^2 - 2AE_p + A^2 = 0,$$

giving the minimum proton energy for the reaction to go

$$E_p = \frac{1}{2A}(A^2 + m_p^2) \approx \frac{A}{2} = 6.8 \times 10^{13} \text{ MeV}.$$

(c) The photon density of 3-K black-body radiation is very large. Protons of energies $> E_p$ in cosmic radiation lose energy by constantly interacting with them. Hence the upper limit of the energy spectrum of cosmic-ray protons is E_p.

4041

The J/ψ particle has a mass of 3.097 GeV/c^2 and a width of 63 keV. A specific J/ψ is made with momentum 100 GeV/c and subsequently decays according to

$$J/\psi \to e^+ + e^-.$$

(a) Find the mean distance traveled by the J/ψ in the laboratory before decaying.

(b) For a symmetric decay (i.e., e^+ and e^- have the same laboratory momenta), find the energy of the decay electron in the laboratory.

(c) Find the laboratory angle of the electron with respect to the direction of the J/ψ.

(*Columbia*)

Solution:

(a) The total width Γ of J/ψ decay is 63 keV, so its proper lifetime is

$$\tau_0 = \hbar/\Gamma = \frac{6.58 \times 10^{-16}}{63 \times 10^3} = 1.045 \times 10^{-20} \text{ s} .$$

The laboratory lifetime is $\tau = \tau_0 \gamma$, where γ is its Lorentz factor. Hence the mean distance traveled by the J/ψ in the laboratory before decaying is

$$\begin{aligned} l &= \tau\beta c = \tau_0 \gamma \beta c = \frac{\tau_0 pc}{m} = 1.045 \times 10^{-20} \times \frac{100}{3.097} \times 3 \times 10^8 \\ &= 1.012 \times 10^{-10} \ m . \end{aligned}$$

(b) For symmetric decay, conservation of energy and of momentum give

$$E_J = 2E_e ,$$

$$p_J = 2p_e \cos\theta ,$$

where θ is the angle the electron makes with the direction of the J/ψ particle. Thus

$$E_e = \frac{1}{2} E_J = \frac{1}{2}\sqrt{p_J^2 + m_J^2} = \frac{1}{2}\sqrt{100^2 + 3.097^2} = 50.024 \text{ GeV} .$$

(c) The equations give

$$\left(\frac{E_J}{2}\right)^2 - \left(\frac{p_J}{2\cos\theta}\right)^2 = E_e^2 - p_e^2 = m_e^2 ,$$

or

$$\begin{aligned} \cos\theta &= \frac{p_J}{\sqrt{p_J^2 + m_J^2 - 4m_e^2}} \\ &= \frac{100}{\sqrt{100^2 + 3.097^2 - 4 \times (0.511 \times 10^{-3})^2}} = 0.9995 , \end{aligned}$$

i.e.

$$\theta = 1.77^\circ .$$

4042

A negative Ξ particle decays into a Λ^0 and a π^-:

$$\Xi^- \to \Lambda^0 + \pi^- .$$

The Ξ^- is moving in the laboratory in the positive x direction and has a momentum of 2 GeV/c. The decay occurs in such a way that in the Ξ^- center-of-mass system the Λ^0 goes at an angle of 30° from the initial Ξ^- direction.

Find the momenta and angles of the Λ^0 and the π^- in the laboratory after the decay.

Rest energies:

$$M_\Xi c^2 = 1.3 \text{ GeV} ,$$

$$M_\Lambda c^2 = 1.1 \text{ GeV} ,$$

$$M_\pi c^2 = 0.14 \text{ GeV} .$$

(*Columbia*)

Solution:

The kinematic parameters β, γ and energy E_Ξ for Ξ^- are as follows:

$$E_\Xi = \sqrt{p_\Xi^2 + m_\Xi^2} = 2.385 \text{ GeV} ,$$

$$\beta_\Xi = \frac{p_\Xi}{E_\Xi} = 0.839 ,$$

$$\gamma_\Xi = \frac{E_\Xi}{m_\Xi} = 1.835 .$$

Denote quantities in the Ξ^- rest frame by a bar. Conservation of momentum and of energy give

$$\bar{\mathbf{p}}_\pi + \bar{\mathbf{p}}_\Lambda = 0 ,$$

$$\bar{E}_\pi + \bar{E}_\Lambda = m_\Xi .$$

Then

$$\bar{p}_\Lambda = \bar{p}_\pi ,$$

and so

$$\bar{E}_\Lambda = \sqrt{\bar{p}_\pi^2 + m_\Lambda^2} = m_\Xi - \bar{E}_\pi \,.$$

Solving the last equation gives, with $\bar{E}_\pi^2 - \bar{p}_\pi^2 = m_\pi^2$,

$$\bar{E}_\pi = \frac{m_\Xi^2 + m_\pi^2 - m_\Lambda^2}{2m_\Xi} = 0.192 \text{ GeV}\,,$$

$$\bar{E}_\Lambda = m_\Xi - E_\pi = 1.108 \text{ GeV}\,,$$

$$\bar{p}_\Lambda = \bar{p}_\pi = \sqrt{\bar{E}_\pi^2 - m_\pi^2} = 0.132 \text{ GeV}/c\,.$$

The angle between $\bar{\mathbf{p}}_\Lambda$ and $\mathbf{p}_\Xi$ is $\bar{\theta}_\Lambda = 30°$, and the angle between $\bar{\mathbf{p}}_\pi$ and $\mathbf{p}_\Xi$ is $\bar{\theta}_\pi = 30° + 180° = 210°$.

Lorentz-transforming to the laboratory frame:

For π:

$$p_\pi \sin\theta_\pi = \bar{p}_\pi \sin\bar{\theta}_\pi = 0.132 \times \sin 210^0 = -0.064 \text{ GeV}/c\,,$$

$$p_\pi \cos\theta_\pi = \gamma(\bar{p}_\pi \cos\bar{\theta}_\pi + \beta\bar{E}_\pi) = 0.086 \text{ GeV}/c\,,$$

giving

$$\tan\theta_\pi = -0.767, \quad \text{or} \quad \theta_\pi = -37.5°\,,$$

$$p_\pi = \sqrt{0.086^2 + 0.064^2} = 0.11 \text{ GeV}/c\,.$$

For Λ:

$$p_\Lambda \sin\theta_\Lambda = \bar{p}_\Lambda \sin\bar{\theta}_\Lambda = 0.132 \times \sin 30° = 0.66 \text{ GeV}/c\,,$$

$$p_\Lambda \cos\theta_\Lambda = \gamma(\bar{p}_\Lambda \cos\bar{\theta}_\Lambda + \beta\bar{E}_\Lambda) = 1.92 \text{ GeV}/c\,,$$

$$\tan\theta_\Lambda = 0.034\,, \qquad \text{or} \qquad \theta_\Lambda = 1.9°\,,$$

$$p_\Lambda = \sqrt{1.92^2 + 0.066^2} = 1.92 \text{ GeV}/c\,.$$

The angle between directions of π and Λ in the laboratory is

$$\theta = \theta_\Lambda - \theta_\pi = 1.9 + 37.5 = 39.4°\,.$$

4043

A K-meson of rest energy 494 MeV decays into a μ of rest energy 106 MeV and a neutrino of zero rest energy. Find the kinetic energies of the μ and neutrino in a frame in which the K-meson decays at rest.

(*UC, Berkeley*)

Solution:

Consider the reaction

$$K \to \mu + \nu$$

in the rest frame of K. Conservation of momentum and of energy give

$$\mathbf{p}_\mu + \mathbf{p}_\nu = 0\,, \qquad \text{or} \qquad p_\mu = p_\nu\,,$$

and $E_\mu + E_\nu = m_K$.

We have

$$E_\mu^2 = (m_K - E_\nu)^2 = m_K^2 + E_\nu^2 - 2m_K E_\nu\,,$$

or, as $E_\nu = p_\nu = p_\mu$ and $E_\mu^2 = p_\mu^2 + m_\mu^2$,

$$p_\mu = \frac{m_K^2 - m_\mu^2}{2m_K} = \frac{494^2 - 106^2}{2 \times 494} = 236 \text{ MeV}/c\,.$$

The kinetic energies are

$$T_\nu = E_\nu = p_\nu c = p_\mu c = 236 \text{ MeV}\,,$$

$$T_\mu = \sqrt{p_\mu^2 + m_\mu^2} - m_\mu = 152 \text{ MeV}\,.$$

4044

Pions ($m = 140$ MeV) decay into muons and neutrinos. What is the maximum momentum of the emitted muon in the pion rest frame?

(a) 30 MeV/c.
(b) 70 MeV/c.
(c) 2.7 MeV/c.

(*CCT*)

Solution:

Denote total energy by E, momentum by $\mathbf{p}$, and consider the reaction $\pi \to \mu+\nu_\mu$ in the pion rest frame. Conservation of energy and of momentum give

$$E_\mu = m_\pi - E_\nu\,,$$

$$\mathbf{p}_\mu + \mathbf{p}_\nu = 0\,, \qquad \text{or} \qquad p_\mu = p_\nu\,.$$

As for neutrinos $E_\nu = p_\nu$, the first equation becomes, on squaring both sides,

$$p_\mu^2 + m_\mu^2 = (m_\pi - p_\mu)^2\,,$$

giving

$$p_\mu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi} = 29.9 \text{ MeV}/c\,.$$

Thus the answer is (a).

4045

The η' meson (let M denote its mass) can decay into a ρ^0 meson (mass m) and a photon (mass $= 0$): $\eta' \to \rho^0 + \gamma$. The decay is isotropic in the rest frame of the parent η' meson.

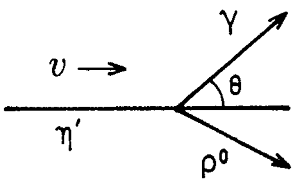

Fig. 4.5

Now suppose that a monoenergetic beam of η' mesons is traveling with speed v in the laboratory and let θ be the angle of the photon relative to the beam, as shown in Fig. 4.5. Let $P(\theta)d(\cos\theta)$ be the normalized probability that $\cos\theta$ lies in the interval $(\cos\theta, \cos\theta + d\cos\theta)$.

(a) Compute $P(\theta)$.

(b) Let $E(\theta)$ be the laboratory energy of the photon coming out at angle θ. Compute $E(\theta)$.

(*CUSPEA*)

Solution:

(a) Denote quantities in the rest frame of the η' particle by a bar and consider an emitted photon. Lorentz transformation for the photon,

$$\bar{p}\cos\bar{\theta} = \gamma(p\cos\theta - \beta E)\,,$$

$$\bar{E} = \gamma(E - \beta p\cos\theta)\,,$$

where γ, β are the Lorentz factor and velocity of the decaying η' in the laboratory frame, gives, as for the photon $\bar{p} = \bar{E}$, $p = E$,

$$\cos\bar{\theta} = \frac{\cos\theta - \beta}{1 - \beta\cos\theta}\,,$$

or

$$\frac{d\cos\bar{\theta}}{d\cos\theta} = \frac{1 - \beta^2}{(1 - \beta\cos\theta)^2}\,.$$

In the rest frame of the η', photon emission is isotropic, i.e., the probability of γ emission per unit solid angle is a constant. Thus

$$dP \propto d\bar{\Omega} = 2\pi\sin\bar{\theta}d\bar{\theta} = 2\pi d\cos\bar{\theta}\,,$$

or

$$\frac{dP}{1} = \frac{2\pi d\cos\bar{\theta}}{4\pi} = \frac{1}{2}d\cos\bar{\theta}\,.$$

Writing it as $dP = \bar{P}(\bar{\theta})d\cos\bar{\theta}$ we have $\bar{P}(\bar{\theta}) = \frac{1}{2}$. Transforming to the laboratory frame,

$$dP = \bar{P}(\bar{\theta})d\cos\bar{\theta} = P(\theta)d\cos\theta\,,$$

giving

$$P(\theta) = \frac{1}{2}\frac{d\cos\bar{\theta}}{d\cos\theta} = \frac{1 - \beta^2}{2(1 - \beta\cos\theta)^2}\,.$$

(b) In the rest frame of the η', conservation laws give

$$\bar{E}_p = M - \bar{E}\,, \qquad \bar{p}_p = \bar{p}\,,$$

or

$$\bar{E}_p^2 - \bar{p}_p^2 = m^2 = M^2 - 2M\bar{E}\,.$$

Thus

$$\bar{E} = \frac{M^2 - m^2}{2M}\,.$$

Lorentz transformation for energy

$$\bar{E} = \gamma E(1 - \beta\cos\theta)$$

gives

$$E = \frac{\bar{E}}{\gamma(1-\beta\cos\theta)} = \frac{M^2 - m^2}{2(E_\eta - p_\eta\cos\theta)}\,,$$

E_η, p_η being the energy and momentum of the η' in the laboratory.

4046

A K_L^0 meson ($Mc^2 = 498$ MeV) decays into $\pi^+\pi^-$ ($mc^2 = 140$ MeV) in flight. The ratio of the momentum of the K_L^0 to Mc is $p/Mc = 1$. Find the maximum transverse component of momentum that any decay pion can have in the laboratory. Find the maximum longitudinal momentum that a pion can have in the laboratory.

(*Wisconsin*)

Solution:

In the laboratory frame, K_L^0 has velocity

$$\beta_c = \frac{p}{E} = \frac{p}{\sqrt{p^2 + M^2}} = \frac{1}{\sqrt{2}}\,,$$

and hence $\gamma_c = \sqrt{2}$.

Let the energy and momentum of the pions in the rest frame of K_L^0 be $\bar{E}$ and $\bar{p}$ respectively. Energy conservation gives $2\bar{E} = M$, and hence

$$\bar{p} = \sqrt{\bar{E}^2 - m^2} = \frac{1}{2}\sqrt{M^2 - 4m^2} = \frac{1}{2}\sqrt{498^2 - 4\times 140^2} = 206\ \text{MeV}/c\,.$$

The transverse component of momentum is not changed by the Lorentz transformation. Hence its maximum value is the same as the maximum value in the rest frame, namely 206 MeV/c.

In the laboratory frame the longitudinal component of momentum of π is

$$p_l = \gamma_c(\bar{p}\cos\bar{\theta} + \beta_c \bar{E}) \, ,$$

and has the maximum value ($\cos\bar{\theta} = 1$)

$$\begin{aligned} p_{\text{lmax}} &= \gamma_c(\bar{p} + \beta_c \bar{E}) = \gamma_c \left(\bar{p} + \frac{\beta_c M}{2}\right) = \sqrt{2}\left(206 + \frac{498}{2\sqrt{2}}\right) \\ &= 540 \text{ MeV}/c \, . \end{aligned}$$

4047

(a) A D^0 charmed particle decays in the bubble chamber after traveling a distance of 3 mm. The total energy of the decay products is 20 GeV. The mass of D^0 is 1.86 GeV. What is the time that the particle lived in its own rest frame?

(b) If the decays of many D^0 particles are observed, compare the expected time distributions (in the D^0 rest frame) of the decays into decay mode of branching ratio 1% and the same for a decay mode of branching ratio 40%.

(*Wisconsin*)

Solution:

(a) The total energy of the D^0 before decay is 20 GeV. Hence the Lorentz factor γ of its rest frame is

$$\gamma = \frac{E}{m_0} = \frac{20}{1.86} = 10.75 \, .$$

The velocity of the D^0 (in units of c) is

$$\beta = \sqrt{\frac{\gamma^2 - 1}{\gamma^2}} = 0.996$$

The lifetime of the D^0 in the laboratory is

$$\tau = \frac{l}{\beta c} = \frac{3 \times 10^{-3}}{0.996 \times 3 \times 10^8} = 1.0 \times 10^{-11} \text{ s}$$

and its proper lifetime is

$$\tau_0 = \frac{\tau}{\gamma} = 9.3 \times 10^{-13} \text{ s} .$$

(b) The decay constant of D^0 is $\lambda = \frac{1}{\tau} = 1.07 \times 10^{12} \text{ s}^{-1}$.

In whatever decay mode, the expected time distribution of D^0 decays take the same form $f(t) \approx e^{-\lambda t} = \exp(-1.07 \times 10^{12} \times t)$. In other words, the decay modes of branching ratios 1% and 40% have the same expected time distribution.

4048

The charmed meson D^0 decays into $K^- \pi^+$. The masses of D, K, π = 1.8, 0.5, 0.15 GeV/c^2 respectively.

(a) What is the momentum of the K-meson in the rest frame of the D^0?

(b) Is the following statement true or false? Explain your answer.

"The production of single K^- mesons by neutrinos (ν_μ) is evidence for D^0 production"

(*Wisconsin*)

Solution:

In the rest frame of the D^0 meson, momentum conservation gives

$$\mathbf{p}_K + \mathbf{p}_\pi = 0 , \qquad \text{or} \qquad p_K = p_\pi .$$

Energy conservation gives

$$E_K + E_\pi = m_D .$$

i.e.,

$$\sqrt{p_K^2 + m_K^2} + \sqrt{p_K^2 + m_\pi^2} = m_D ,$$

leading to

$$p_K = \left[\left(\frac{m_D^2 + m_\pi^2 - m_K^2}{2m_D} \right)^2 - m_\pi^2 \right]^{\frac{1}{2}} = 0.82 \text{ GeV}/c .$$

(b) False. K^- has an s quark. Other particles such as Ξ^*, Ω^-, K^*, which can be produced in neutrino reactions, can also decay into single K^- mesons.

4049

The mean lifetime of a charged π-meson at rest is 2.6×10^{-8} sec. A monoenergetic beam of high-energy pions, produced by an accelerator, travels a distance of 10 meters, and in the process 10% of the pion decay. Find the momentum and kinetic energy of the pions.

(*Wisconsin*)

Solution:

Suppose the initial number of pions is N_0 and their velocity is β (in units of c). After traveling a distance of l the number becomes

$$N(l) = N_0 \exp\left(\frac{-\lambda l}{\beta c}\right),$$

where λ is the decay constant of pion in the laboratory. As

$$\lambda = \frac{1}{\tau} = \frac{1}{\gamma\tau_0},$$

where $\tau_0 = 2.6 \times 10^{-8}$ s is the proper lifetime of pion, $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, we have

$$\gamma\beta = \frac{l}{\tau_0 c \ln \dfrac{N_0}{N(l)}} = \frac{10}{2.6 \times 10^{-8} \times 3 \times 10^8 \times \ln \dfrac{1}{0.9}} = 12.2\,.$$

The momentum of the pions is

$$p = m\gamma\beta = 0.14 \times 12.2 = 1.71 \text{ GeV}/c\,,$$

and so the kinetic energy is

$$T = \sqrt{\mathbf{p}^2 + m^2} - m \approx 1.58 \text{ GeV}\,.$$

4050

Neutral mesons are produced by a proton beam striking a thin target. The mesons each decay into two γ-rays. The photons emitted in the forward direction with respect to the beam have an energy of 96 MeV, and the photons emitted in the backward direction have an energy of 48 MeV.

(a) Determine $\beta = v/c$ for the mesons.
(b) Determine the (approximate) rest energy of the mesons.

(*Wisconsin*)

Solution:

(a) In the decay of a π^0 in the laboratory, if one photon is emitted backward, the other must be emitted forward. Let their energies and momenta be E_2, p_2, E_1, p_1 respectively. Conservation of energy gives

$$E = E_1 + E_2 = 96 + 48 = 144 \text{ MeV}.$$

Conservation of momentum gives

$$p = p_1 - p_2 = 96 - 48 = 48 \text{ MeV}/c.$$

Hence the π^0 has velocity

$$\beta = \frac{p}{E} = \frac{48}{144} = \frac{1}{3}.$$

(b) The π^0 rest mass is

$$m = \frac{E}{\gamma} = E\sqrt{1-\beta^2} = \frac{144}{3} \times \sqrt{8} = 136 \text{ MeV}/c^2.$$

4051

A particle has mass $M = 3$ GeV$/c^2$ and momentum $p = 4$ GeV$/c$ along the x-axis. It decays into 2 photons with an angular distribution which is isotropic in its rest frame, i.e. $\frac{dP}{d\cos\theta^*} = \frac{1}{2}$. What are the maximum and minimum values of the component of photon momentum along the x-axis? Find the probability dP/dp_x of finding a photon with x component of momentum p_x, as a function of p_x.

(*Wisconsin*)

Solution:

In the rest frame of the particle, conservation of momentum and of energy require

$$\bar{E}_1 + \bar{E}_2 = M, \qquad \bar{\mathbf{p}}_1 + \bar{\mathbf{p}}_2 = 0.$$

Thus

$$\bar{p}_1 = \bar{p}_2 = \bar{p}\,, \qquad \bar{E}_1 = \bar{E}_2 = \bar{E} = \frac{M}{2}\,,$$

and the photons have energy

$$\bar{E} = \frac{3}{2} = 1.5 \text{ GeV}$$

and momentum

$$\bar{p} = \bar{E} = 1.5 \text{ GeV}/c\,.$$

The decaying particle has, in the laboratory,

$$\gamma\beta = \frac{p}{M} = \frac{4}{3}\,,$$

and so

$$\gamma = \sqrt{(\gamma\beta)^2 + 1} = \frac{5}{3}\,, \qquad \beta = \frac{\gamma\beta}{\gamma} = 0.8\,.$$

Lorentz transformation gives the x component of photon momentum in the laboratory as

$$p_x = \gamma(\bar{p}\cos\bar{\theta} + \beta\bar{E}) = \gamma\bar{p}(\cos\bar{\theta} + \beta)\,.$$

Hence, p_x is maximum when $\bar{\theta} = 0°$:

$$(p_x)_{\text{max}} = \frac{5}{3} \times 1.5(1 + 0.8) = 4.5 \text{ GeV}/c\,,$$

p_x is minimum when $\bar{\theta} = 180°$:

$$(p_x)_{\text{min}} = \frac{5}{3} \times 1.5(-1 + 0.8) = -0.5 \text{ GeV}/c\,.$$

Differentiating the transformation equation we have

$$dp_x = \gamma\bar{p}d\cos\bar{\theta}\,.$$

Hence

$$\frac{dP}{dp_x} = \frac{dP}{d\cos\bar{\theta}}\frac{d\cos\bar{\theta}}{dp_x} = \frac{1}{2}\cdot\frac{1}{\gamma\bar{p}} = 0.2\,.$$

4052

A neutral pion (π^0) decays into two γ rays. Suppose a π^0 is moving with a total energy E.

(a) What are the energies of the γ-rays if the decay process causes them to be emitted in opposite directions along the pion's original line of motion?

(b) What angle is formed between the two γ's if they are emitted at equal angles to the direction of the pion's motion?

(c) Taking $m_\pi = 135$ MeV and $E = 1$ GeV, give approximate numerical values for your above answers.

(*Columbia*)

Solution:

(a) Let the momenta and energies of the two γ's be p_{γ_1}, $p_{\gamma 2}$ and E_{γ_1}, E_{γ_2}, the momentum and energy of the π^0 be p_π, E, respectively. Conservation laws of energy and momentum require

$$E = E_{\gamma_1} + E_{\gamma_2} \, ,$$

$$p_\pi = p_{\gamma_1} - p_{\gamma_2} \, .$$

As

$$E^2 = p_\pi^2 + m_\pi^2 \, , \qquad E_{\gamma_1} = p_{\gamma_1} \, , \qquad E_{\gamma_2} = p_{\gamma_2} \, ,$$

the above equations give

$$m_\pi^2 = 4E_{\gamma_1} E_{\gamma_2} = 4E_{\gamma_1}(E - E_{\gamma_1}) \, .$$

The quadratic equation for E_{γ_1} has two solutions

$$E_{\gamma_1} = \frac{E + \sqrt{E^2 - m_\pi^2}}{2} \, ,$$

$$E_{\gamma_2} = \frac{E - \sqrt{E^2 - m_\pi^2}}{2} \, ,$$

which are the energies of the two photons.

(b) Let the angles the two photons make with the direction of the pion be θ and $-\theta$. Conservation laws give

$$E = 2E_\gamma \, ,$$

$$p_\pi = 2p_\gamma \cos\theta \, .$$

Note that, on account of symmetry, the two photons have the same energy and momentum E_γ, p_γ.

The two equations combine to give

$$m_\pi^2 = 4E_\gamma^2 - 4p_\gamma^2 \cos^2\theta = E^2(1-\cos^2\theta) = E^2\sin^2\theta\,,$$

or

$$\theta = \pm\arcsin\left(\frac{m_\pi}{E}\right)\,.$$

Thus the angle between the two photons is

$$\theta_{2\gamma} = 2\theta = 2\arcsin\left(\frac{m_\pi}{E}\right)\,.$$

(c) Numerically we have

$$E_{\gamma_1} = \frac{10^3 + \sqrt{10^6 - 135^2}}{2} = 995.4 \text{ MeV}\,,$$

$$E_{\gamma_2} = \frac{10^3 - \sqrt{10^6 - 135^2}}{2} = 4.6 \text{ MeV}\,,$$

$$\theta_{2\gamma} = 2\arcsin\left(\frac{135}{1000}\right) = 15.5^\circ\,.$$

4053

A π^0 meson decays isotropically into two photons in its rest system. Find the angular distribution of the photons in the laboratory as a function of the cosine of the polar angle in the laboratory for a π^0 with momentum $p = 280$ MeV/c. The rest energy of the pion is 140 MeV.

(UC, Berkeley)

Solution:

In the rest frame of the pion, the angular distribution of decay photons is isotropic and satisfies the normalization condition $\int W_0(\cos\theta^*, \phi^*)d\Omega^* = 1$. As a π^0 decays into two photons, $\int W(\cos\theta^*, \phi^*)d\Omega^* = 2$. Note that W is the probability of emitting a photon in the solid angle $d\Omega^*(\theta^*, \phi^*)$ in the decay of a π^0. As W is independent of θ^* and ϕ^*, the integral gives $W\int d\Omega^* = 4\pi W = 2$, or $W(\cos\theta^*, \phi^*) = \frac{1}{2\pi}$. Integrating over ϕ^*, we have

$$\int_0^{2\pi} W(\cos\theta^*) d\varphi^* = W\int_0^{2\pi} d\varphi^* = 1\,,$$

or

$$W(\cos\theta^*) = 1\,.$$

If θ^* corresponds to laboratory angle θ, then

$$W(\cos\theta) d\cos\theta = W(\cos\theta^*) d\cos\theta^*\,.$$

Let γ_0, β_0 be the Lorentz factor and velocity of the decaying π^0. The Lorentz transformation for a photon gives

$$p\cos\theta = \gamma_0(p^*\cos\theta^* + \beta_0 E^*) = \gamma_0 p^*(\cos\theta^* + \beta_0)\,,$$
$$E = p = \gamma_0(E^* + \beta_0 p^*\cos\theta^*) = \gamma_0 p^*(1 + \beta_0\cos\theta^*)\,.$$

Note E^*, p^* are constant since the angular distribution of the photons in the rest frame is isotropic. Differentiating the above equations with respect to $\cos\theta^*$, we have

$$\cos\theta\frac{dp}{d\cos\theta^*} + p\frac{d\cos\theta}{d\cos\theta^*} = \gamma_0 p^*\,,$$
$$\frac{dp}{d\cos\theta^*} = \gamma_0\beta_0 p^*\,,$$

which combine to give

$$\frac{d\cos\theta^*}{d\cos\theta} = \frac{p}{\gamma_0 p^*(1-\beta_0\cos\theta)} = \frac{1}{\gamma_0^2(1-\beta_0\cos\theta)^2}\,,$$

use having been made of the transformation equation

$$E^* = \gamma_0(E - \beta_0 p\cos\theta)\,,$$

or

$$p^* = \gamma_0 p(1-\beta_0\cos\theta)\,.$$

Hence

$$W(\cos\theta) = W(\cos\theta^*)\frac{d\cos\theta^*}{d\cos\theta} = \frac{1}{\gamma_0^2(1-\beta_0\cos\theta)^2}\,.$$

With π^0 of mass 140 MeV/c^2, momentum 280 MeV/c, we have

$$\gamma_0\beta_0 = \frac{280}{140} = 2\,,$$

$$\gamma_0 = \sqrt{(\gamma_0\beta_0)^2+1} = \sqrt{5}\,,$$

$$\beta_0 = \frac{\gamma_0\beta_0}{\gamma_0} = \frac{2}{\sqrt{5}}\,,$$

giving the laboratory angular distribution

$$W(\cos\theta) = \frac{1}{(\sqrt{5})^2\left(1-\frac{2}{\sqrt{5}}\cos\theta\right)^2} = \frac{1}{(\sqrt{5}-2\cos\theta)^2}\,.$$

4054

A neutral pion decays into two γ-rays, $\pi^0 \to \gamma + \gamma$, with a lifetime of about 10^{-16} sec. Neutral pions can be produced in the laboratory by stopping negative pions in hydrogen via the reaction

$$\pi^- + p \to \pi^0 + n\,.$$

The values of the rest masses of these particles are:

$$m(\pi^-) = 140 \text{ MeV}\,, \qquad m(\pi^0) = 135 \text{ MeV}\,, \qquad m(p) = 938 \text{ MeV}\,,$$

$$m(n) = 940 \text{ MeV}\,.$$

(a) What is the velocity of the π^0 emerging from this reaction? Assume that both the π^- and the proton are at rest before the reaction.

(b) What is the kinetic energy of the emerging neutron?

(c) How far does the π^0 travel in the laboratory if it lives for a time of 10^{-16} seconds measured in its own rest frame?

(d) What is the maximum energy in the laboratory frame of the γ-rays from the π^0 decay?

(*Columbia*)

Solution:

(a) Momentum conservation requires

$$\mathbf{p}_{\pi^0} + \mathbf{p}_n = 0\,, \qquad \text{or} \qquad p_{\pi^0} = p_n\,.$$

Energy conservation requires

$$E_n = m_{\pi^-} + m_p - E_{\pi^0}\,.$$

With $E^2 - p^2 = m^2$, these equations give

$$\begin{aligned} E_{\pi^0} &= \frac{(m_\pi + m_p)^2 + m_{\pi^0}^2 - m_n^2}{2(m_{\pi^-} + m_p)} \\ &= 137.62 \text{ MeV}\,. \end{aligned}$$

Hence

$$\gamma = \frac{E_{\pi^0}}{m_{\pi^0}} = 1.019\,,$$

and

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.194\,.$$

Thus the π^0 has velocity 5.8×10^7 m/s.

(b) The neutron has kinetic energy

$$\begin{aligned} T_n &= m_{\pi^-} + m_p - E_{\pi^0} - m_n \\ &= 0.38 \text{ MeV}\,. \end{aligned}$$

(c) The lifetime of π^0 in the laboratory is

$$\tau = \tau_0\gamma = 1.019 \times 10^{-16} \text{ s}\,.$$

Hence the distance it travels before decaying is

$$l = \tau\beta c = 1.019 \times 10^{-16} \times 5.8 \times 10^7 = 5.9 \times 10^{-9} \text{ m}\,.$$

(d) The π^0 has $\gamma = 1.019$, $\beta = 0.194$ in the laboratory. In its rest frame, each decay photon has energy

$$E_\gamma^* = \frac{1}{2} m_{\pi^0} = 67.5 \text{ MeV}\,.$$

Transforming to the laboratory gives

$$E_\gamma = \gamma(E_\gamma^* + \beta p_\gamma^* \cos\theta^*)\,.$$

Maximum E_γ corresponds to $\theta^* = 0$:

$$\begin{aligned}(E_\gamma)_{\max} &= \gamma E_\gamma^*(1+\beta) = 1.019 \times 67.5 \times (1 + 0.194) \\ &= 82.1 \text{ MeV}\,.\end{aligned}$$

4055

High energy neutrino beams at Fermilab are made by first forming a monoenergetic π^+ (or K^+) beam and then allowing the pions to decay by

$$\pi^+ \to \mu^+ + \nu\,.$$

Recall that the mass of the pion is 140 MeV/c^2 and the mass of the muon is 106 MeV/c^2.

(a) Find the energy of the decay neutrino in the rest frame of the π^+.

In the laboratory frame, the energy of the decay neutrino depends on the decay angle θ (see Fig. 4.6). Suppose the π^+ beam has an energy 200 GeV/c^2.

(b) Find the energy of a neutrino produced in the forward direction ($\theta = 0$).

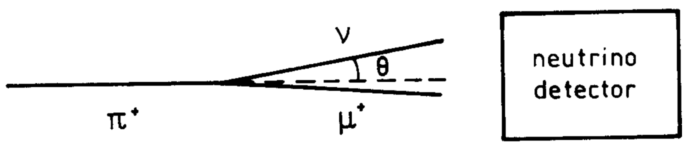

Fig. 4.6

(c) Find the angle θ at which the neutrino's energy has fallen to half of its maximum energy.

(*Chicago*)

Solution:

(a) In the π^+ rest frame conservation laws of energy and momentum require

$$E_\nu + E_\mu = m_\pi \,,$$

$$\mathbf{p}_\nu + \mathbf{p}_\mu = 0 \,, \qquad \text{or} \qquad p_\nu = p_\mu \,.$$

These equations combine to give

$$m_\mu^2 + p_\nu^2 = E_\nu^2 + m_\pi^2 - 2m_\pi E_\nu \,.$$

Assume that neutrino has zero mass. Then $E_\nu = p_\nu$ and the above gives

$$E_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_{\pi^+}} = \frac{140^2 - 106^2}{2 \times 140} = 30 \text{ MeV} \,.$$

(b) For π^+ of energy 200 GeV, $\gamma = \frac{E}{m} = \frac{200}{0.140} = 1429$, $\beta \approx 1$. Lorentz transformation for neutrino

$$E_\nu = \gamma(E_\nu^* + \beta p_\nu^* \cos\theta^*) = \gamma E_\nu^*(1 + \beta\cos\theta^*)$$

gives for $\theta^* = 0$

$$E_\nu = \gamma E_\nu^*(1 + \beta) \approx 1429 \times 30 \times (1 + 1) = 85.7 \text{ GeV} \,.$$

Note $\theta^* = 0$ corresponds to $\theta = 0$ in the laboratory as $p_\nu \sin\theta = p_\nu^* \sin\theta^*$.

(c) The laboratory energy of the neutrino is maximum when $\theta = \theta^* = 0$. Thus

$$(E_\nu)_{\max} = \gamma E_\nu^*(1 + \beta) \,.$$

For $E_\nu = \frac{1}{2}(E_\nu)_{\max}$, we have

$$\gamma E_\nu^*(1 + \beta\cos\theta^*) = \frac{1}{2}\gamma E_\gamma^*(1 + \beta) \,,$$

giving

$$\cos\theta^* = \frac{\beta - 1}{2\beta} \,,$$

which corresponds to

$$\sin\theta^* = \sqrt{1-\cos^2\theta^*} = \frac{1}{2\beta}\sqrt{3\beta^2+2\beta-1}\,.$$

Lorentz transformation equations for neutrino

$$p_\nu \sin\theta = p_\nu^* \sin\theta^*\,,$$

$$p_\nu \cos\theta = \gamma(p_\nu^*\cos\theta^* + \beta E_\nu^*) = \gamma p_\nu^*(\cos\theta^* + \beta)\,,$$

give

$$\tan\theta = \frac{\sin\theta^*}{\gamma(\cos\theta^*+\beta)}\,.$$

For $E_\nu = \frac{1}{2}(E_\nu)_{\max}$,

$$\tan\theta_{\frac{1}{2}} = \frac{\sqrt{3\beta^2+2\beta-1}}{\gamma(\beta-1+2\beta^2)} = \frac{1}{\gamma}\cdot\frac{1}{(2\beta-1)}\sqrt{\frac{3\beta-1}{\beta+1}} \approx \frac{1}{\gamma}$$

as $\beta \approx 1$. Hence at half the maximum angle,

$$\theta_{\frac{1}{2}} \approx \frac{1}{\gamma}\,.$$

4056

One particular interest in particle physics at present is the weak interactions at high energies. These can be investigated by studying high-energy neutrino interactions. One can produce neutrino beams by letting pi and K mesons decay in flight. Suppose a 200-GeV/c pi-meson beam is used to produce neutrinos via the decay $\pi^+ \to \mu^+ + \nu$. The lifetime of pi-mesons is $\tau_{\pi^\pm} = 2.60\times10^{-8}$ sec (in the rest frame of the pion), and its rest energy is 139.6 MeV. The rest energy of the muon is 105.7 MeV, and the neutrino is massless.

(a) Calculate the mean distance traveled by the pions before they decay.

(b) Calculate the maximum angle of the muon (relative to the pion direction) in the laboratory.

(c) Calculate the minimum and maximum momenta the neutrinos can have.

(*UC, Berkeley*)

Solution:

(a) The pions have Lorentz factor

$$\gamma = \frac{E}{m} \approx \frac{p}{m} = \frac{200000}{139.6} = 1433 \, .$$

The lifetime of the pions in the laboratory frame is then $\tau = \gamma\tau_0 = 2.6 \times 10^{-8} \times 1433 = 3.72 \times 10^{-5}$ s.

The speed of the pions is very close to that of light. Thus before decaying the distance traveled is on the average

$$l = c\tau = 3 \times 10^8 \times 3.72 \times 10^{-5} = 1.12 \times 10^4 \text{ m} \, .$$

(b) Figure 4.7 shows the decay in the laboratory frame Σ and the rest frame Σ^* of the pion.

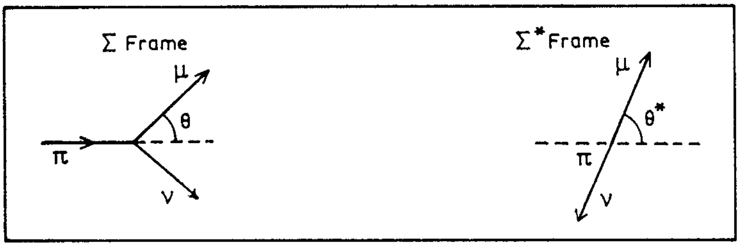

Fig. 4.7

In Σ^*, conservation laws of energy and momentum require

$$E_\nu^* + E_\mu^* = m_\pi \, ,$$

$$\mathbf{p}_\nu^* + \mathbf{p}_\mu^* = 0 \, , \qquad \text{or} \qquad p_\nu^* = p_\mu^* \, .$$

The above equations combine to give

$$E_\mu^* = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} = 109.8 \text{ MeV} \, .$$

Lorentz transformation for the muon gives

$$p_\mu \sin\theta = p_\mu^* \sin\theta^* \,,$$

$$p_\mu \cos\theta = \gamma(p_\mu^* \cos\theta^* + \beta E_\mu^*) \,,$$

where $\gamma = 1433$ is the Lorentz factor of Σ^*, $\beta \approx 1$. Thus

$$\tan\theta = \frac{\sin\theta^*}{\gamma\left(\cos\theta^* + \frac{E_\mu^*}{p_\mu^*}\right)} = \frac{\sin\theta^*}{\gamma\left(\cos\theta^* + \frac{1}{\beta_\mu^*}\right)} \,,$$

where $\beta_\mu^* = \frac{p_\mu^*}{E_\mu^*}$. To find maximum θ, let

$$\frac{d\tan\theta}{d\theta^*} = 0 \,.$$

This gives $\cos\theta^* = -\beta_\mu^*$, $\sin\theta^* = \sqrt{1-\beta_\mu^{*2}} = \frac{1}{\gamma_\mu^*}$. Hence

$$(\tan\theta)_{\max} = \frac{1}{\gamma\gamma_\mu^*\left(-\beta_\mu^* + \frac{1}{\beta_\mu^*}\right)} = \frac{\beta_\mu^*}{\gamma\gamma_\mu^*(\beta_\mu^{*2}-1)} = \frac{\gamma_\mu^*\beta_\mu^*}{\gamma} = \frac{\sqrt{\gamma_\mu^{*2}-1}}{\gamma} \,.$$

As $\gamma_\mu^* = \frac{E_\mu^*}{m_\mu} = \frac{109.8}{105.7} = 1.039$, $\gamma = 1433$, we have

$$\theta_{\max} = \arctan(\tan\theta)_{\max} \approx \frac{\sqrt{\gamma_\mu^{*2}-1}}{\gamma} = 1.97\times 10^{-4}\ \text{rad} = 0.011^0 \,.$$

(c) In the rest frame Σ^*, the neutrino has energy

$$E_\nu^* = m_\pi - E_\mu^* = \frac{m_\pi^2 - m_\mu^2}{2m\pi} = 29.8\ \text{MeV} \,,$$

and hence momentum 29.8 MeV/c. Lorentz transformation gives for the neutrino,

$$p_\nu = E_\nu = \gamma(E_\nu^* + \beta p_\nu^* \cos\theta^*) = \gamma p_\nu^*(1 + \beta\cos\theta^*) \,.$$

Hence

$$
\begin{aligned}
(p_\nu)_{\max} &= \gamma p_\nu^*(1+\beta) \\
&= 1433 \times 29.8(1+1) = 85.4 \text{ GeV}/c\,, \\
(p_\nu)_{\min} &= \gamma p_\nu^*(1-\beta) \\
&= [\sqrt{(\gamma\beta)^2+1} - \gamma\beta] p_\nu^* \\
&\approx \frac{p_\nu^*}{2\gamma\beta} = \frac{m_\pi p_\nu^*}{2p_\pi} = \frac{139.6 \times 29.4}{2 \times 200 \times 10^3} \\
&= 1.04 \times 10^{-2} \text{ MeV}/c\,.
\end{aligned}
$$

4057

A beam of pions of energy E_0 is incident along the z-axis. Some of these decay to a muon and a neutrino, with the neutrino emerging at an angle θ_ν relative to the z-axis. Assume that the neutrino is massless.

(a) Determine the neutrino energy as a function of θ_ν. Show that if $E_0 \gg m_\pi$ and $\theta_\nu \ll 1$,

$$
E_\nu \approx E_0 \frac{1 - \left(\dfrac{m_\mu}{m_\pi}\right)^2}{1 + \left(\dfrac{E_0}{m_\pi}\right)^2 \theta_\nu^2}\,.
$$

(b) The decay is isotropic in the center-of-mass frame. Determine the angle θ_m such that half the neutrinos will have $\theta_\nu < \theta_m$.

(*Columbia*)

Solution:

(a) Let the emission angle of the muon relative to the z-axis be θ. Conservation of energy and of momentum give

$$
E_0 = E_\mu + E_\nu = \sqrt{p_\mu^2 + m_\mu^2} + E_\nu\,,
$$

$$
\sqrt{E_0^2 - m_\pi^2} = p_\mu \cos\theta + p_\nu \cos\theta_\nu\,,
$$

$$
0 = p_\mu \sin\theta + p_\nu \sin\theta_\nu\,,
$$

As neutrino is assumed massless, $p_\nu = E_\nu$. The momentum equations combine to give

$$p_\mu^2 = E_0^2 - m_\pi^2 + p_\nu^2 - 2\sqrt{E_0^2 - m_\pi^2}E_\nu \cos\theta_\nu \,,$$

while the energy equation gives

$$p_\mu^2 = E_0^2 - m_\mu^2 + p_\nu^2 - 2E_0 E_\nu \,.$$

The difference of the last two equations then gives

$$\begin{aligned} E_\nu &= \frac{m_\pi^2 - m_\mu^2}{2(E_0 - \sqrt{E_0^2 - m_\pi^2}\cos\theta_\nu)} \\ &= \frac{m_\pi^2}{2E_0}\frac{\left[1 - \left(\frac{m_\mu}{m_\pi}\right)^2\right]}{\left[1 - \sqrt{1 - \left(\frac{m_\pi}{E_0}\right)^2}\cos\theta_\nu\right]} \,. \end{aligned}$$

If $E_0 \gg m_\pi$, $\theta_\nu \ll 1$, then

$$\sqrt{1 - \left(\frac{m_\pi}{E_0}\right)^2}\cos\theta_\nu \approx \left[1 - \frac{1}{2}\left(\frac{m_\pi}{E_0}\right)^2\right]\left(1 - \frac{\theta_\nu^2}{2}\right) \approx 1 - \frac{1}{2}\left(\frac{m_\pi}{E_0}\right)^2 - \frac{\theta_\nu^2}{2} \,,$$

and hence

$$E_\nu \approx \frac{m_\pi^2}{E_0} \times \frac{1 - \left(\frac{m_\mu}{m_\pi}\right)^2}{\left(\frac{m_\pi}{E_0}\right)^2 + \theta_\nu^2} = E_0 \frac{1 - \left(\frac{m_\mu}{m_\pi}\right)^2}{1 + \left(\frac{E_0}{m_\pi}\right)^2 \theta_\nu^2} \,.$$

(b) The center-of-mass frame (i.e. rest frame of π) has Lorentz factor and velocity

$$\gamma = \frac{E_0}{m_\pi} \,, \qquad \beta = \sqrt{1 - \frac{1}{\gamma^2}} \,.$$

Denote quantities in the rest frame by a bar. Lorentz transformation for the neutrino

$$\begin{aligned} p_\nu \sin\theta_\nu &= \bar{p}_\nu \sin\bar{\theta}_\nu \,, \\ p_\nu \cos\theta_\nu &= \gamma(\bar{p}_\nu \cos\bar{\theta}_\nu + \beta\bar{E}_\nu) = \gamma\bar{p}_\nu(\cos\bar{\theta}_\nu + \beta) \end{aligned}$$

gives

$$\tan\theta_\nu = \frac{\sin\bar{\theta}_\nu}{\gamma(\beta + \cos\bar{\theta}_\nu)}.$$

As the angular distribution of the neutrinos in the rest frame is isotropic, $\bar{\theta}_m = 90°$. Then

$$\tan\theta_m = \frac{\sin 90°}{\gamma(\beta + \cos 90°)} = \frac{1}{\gamma\beta} = \frac{1}{\sqrt{\gamma^2 - 1}}$$

$$= \frac{1}{\sqrt{\left(\frac{E_0}{m_\pi}\right)^2 - 1}} = \frac{m_\pi}{\sqrt{E_0^2 - m_\pi^2}},$$

or

$$\theta_m = \arctan\left(\frac{m_\pi}{\sqrt{E_0^2 - m_\pi^2}}\right).$$

Note that as

$$\frac{d\theta_\nu}{d\bar{\theta}_\nu} = \frac{\cos^2\theta_\nu}{\gamma}\frac{(1 + \beta\cos\bar{\theta}_\nu)}{(\beta + \cos\bar{\theta}_\nu)^2} \geq 0$$

θ_ν increases monotonically as $\bar{\theta}_\nu$ increases. This means that if $\bar{\theta}_\nu \leq \bar{\theta}_m$ contains half the number of the neutrinos emitted, $\theta_\nu \leq \theta_m$ also contains half the neutrinos.

4058

(a) Calculate the momentum of pions that have the same velocity as protons having momentum 400 GeV/c. This is the most probable momentum that produced-pions have when 400-GeV/c protons strike the target at Fermilab. The pion rest mass is 0.14 GeV/c^2. The proton rest mass is 0.94 GeV/c^2.

(b) These pions then travel down a decay pipe of 400 meter length where some of them decay to produce the neutrino beam for the neutrino detector located more than 1 kilometer away. What fraction of the pions decay in the 400 meters? the pions' proper mean lifetime is 2.6×10^{-8} sec.

(c) What is the length of the decay pipe as measured by observers in the pion rest frame?

(d) The pion decays into a muon and a neutrino ($\pi \to \mu + \nu_\mu$, the neutrino has zero rest-mass.) Using the relationship between total relativistic energy and momentum show that the magnitude of the decay fragments' momentum in the pion rest frame is given by $\frac{p}{c} = \frac{M^2 - m^2}{2M}$, where M is the rest mass of pion and m is the rest mass of muon.

(e) The neutrino detectors are, on the average, approximately 1.2 km from the point where the pions decay. How large should the transverse dimension (radius) of the detector be in order to have a chance of detecting all neutrinos that are produced in the forward hemisphere in the pion rest frame?

(UC, Berkeley)

Solution:

(a) The pions and the protons, having the same velocity, have the same γ and hence the same $\gamma\beta$. As

$$p_\pi = m_\pi \gamma\beta\,, \qquad p_p = m_p \gamma\beta\,,$$

$$p_\pi = \frac{m_\pi}{m_p} p_p = \frac{0.14}{0.94} \times 400 = 59.6 \text{ Gev}/c\,.$$

(b) The pions have

$$\gamma\beta = \frac{59.6}{0.14} = 426\,,$$

and hence $\gamma = \sqrt{(\gamma\beta)^2 + 1} \approx \gamma\beta = 426$.

The pions have proper mean lifetime $\tau_0 = 2.6 \times 10^{-8}$ s and hence mean lifetime $\tau = \gamma\tau_0 = 1.1 \times 10^{-5}$ s in the laboratory. Hence

$$\frac{N}{N_0} = (1 - e^{-\frac{l}{\tau c}}) = (1 - e^{-0.12}) = 0.114\,.$$

(c) In the pion rest frame, on account of Fitzgerald contraction the observed length of the decay pipe is

$$\bar{l} = \frac{l}{\gamma} = \frac{400}{426} = 0.94\ m\,.$$

(d) In the pion rest frame, energy and momentum conservation laws require

$$E_\mu + E_\nu = m_\pi\,,$$

$$\mathbf{p}_\mu + \mathbf{p}_\nu = 0\,, \qquad \text{or} \qquad p_\mu = p_\nu\,.$$

For a particle, total energy and momentum are related by (taking $c = 1$)

$$E^2 = p^2 + m^2 .$$

For neutrino, as $m = 0$ we have $E_\nu = p_\nu$. The energy equation thus becomes

$$p_\mu^2 + m_\mu^2 = m_\pi^2 - 2p_\nu m_\pi + p_\nu^2 ,$$

or

$$p_\nu = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}$$

i.e.,

$$p = \frac{M^2 - m^2}{2M} .$$

(e) The decay $\pi \to \mu\nu$ is isotropic in the rest frame of the pion. **Problem 4057**(b) gives the neutrinos' 'half-angle' as

$$\theta_{1/2} = \arctan\left(\frac{m_\pi}{\sqrt{E_0^2 - m_\pi^2}}\right) = \arctan\frac{1}{\sqrt{\gamma^2 - 1}} \approx \frac{1}{\gamma} .$$

Thus the diameter of the detector should be larger than

$$L = 2d \tan\theta_{\frac{1}{2}} \approx \frac{2d}{\gamma} = \frac{2 \times 1200}{426} = 5.63\ m .$$

4059

Consider the decay $K^0 \to \pi^+ + \pi^-$.

Assuming the following transition matrix element

$$T_{\text{if}} = \frac{G}{\sqrt{8E_K E_+ E_-}} \frac{P_K(P_+ + P_-)}{m_K} .$$

show that the lifetime of the K^0 meson as measured in its rest system is

$$\tau = \left[\frac{G^2}{8\pi\hbar^4 c}\sqrt{\frac{m_K^2}{4} - \mu^2}\right]^{-1} .$$

(E_K, E_+ and E_- are the relativistic energies of K^0, π^+ and π^- respectively, and P_K, P_+ and P_- are the corresponding 4-momenta. M_K is the K-meson mass and G is the coupling constant. μ is the π-meson mass).

(*SUNY, Buffalo*)

Solution:

The transition probability per unit time is given by

$$W = \frac{2\pi}{\hbar}|T_{\text{if}}|^2\rho(E)\,.$$

In the rest frame of K^0 meson,

$$E_K = m_K c^2\,, \qquad E_+ = E_- = \frac{1}{2}m_K c^2\,,$$

$$P_K^2 = \frac{E_K^2}{c^2} = m_K^2 c^2\,,$$

$$(P_+ + P_-)^2 = -(\mathbf{p}_+ + \mathbf{p}_-)^2 + \frac{(E_+ + E_-)^2}{c^2} = m_K^2 c^2\,.$$

Hence

$$\begin{aligned}|T_{\text{if}}|^2 &= \frac{G^2}{8E_K E_+ E_-}\frac{[P_K(P_+ + P_-)]^2}{m_K^2} \\ &= \frac{G^2}{8m_K c^2 \frac{m_K^2}{4} c^4}\frac{m_K^4 c^4}{m_K^2} = \frac{G^2}{2m_K c^2}\,.\end{aligned}$$

For a two-body decay, in the rest frame of the decaying particle,

$$\rho(E) = \frac{1}{(2\pi\hbar)^3}\frac{d}{dE}\int p_1^2 dp_1 d\Omega = \frac{4\pi}{(2\pi\hbar)^3}\frac{d}{dE}\left(\frac{1}{3}p_1^3\right)\,,$$

assuming the decay to be isotropic.

Noting $\mathbf{p}_1+\mathbf{p}_2 = 0$, or $p_1^2 = p_2^2$, i.e., $p_1dp_1 = p_2dp_2$, and $dE = dE_1+dE_2$, we find

$$\begin{aligned}\rho(E) &= \frac{4\pi}{(2\pi\hbar)^3}\frac{E_1E_2p_1}{E_1+E_2} = \frac{1}{(2\pi\hbar)^3c^2}\frac{m_Kc^2}{4}\left(\sqrt{\frac{m_K^2}{4}-\mu^2}\right)4\pi c \\ &= \frac{m_K c}{8\pi^2\hbar^3}\sqrt{\frac{m_K^2}{4}-\mu^2}\,,\end{aligned}$$

where we have used

$$\frac{d}{dt}\left(\frac{1}{3}p_1^3\right) = \frac{p_1^2 dp_1}{dE_1 + dE_2} = \frac{p_1}{\dfrac{dE_1}{p_1 dp_1} + \dfrac{dE_2}{p_2 dp_2}} = \frac{E_1 E_2 p_1}{E_1 + E_2}\,,$$

for as $E_1^2 = p_1^2 + m_1^2$

$$\frac{dE_1}{p_1 dp_1} = \frac{1}{E_1}\,, \qquad \text{etc.}$$

Therefore

$$\begin{aligned} W &= \frac{2\pi}{\hbar}\frac{G^2}{2m_K c^2}\frac{m_K c}{8\pi^2\hbar^3}\sqrt{\frac{m_K^2}{4} - \mu^2} \\ &= \frac{G^2}{8\pi\hbar^4 c}\sqrt{\frac{m_K^2}{4} - \mu^2}\,, \end{aligned}$$

and the lifetime of K^0 is

$$\tau = \left[\frac{G^2}{8\pi\hbar^4 c}\sqrt{\frac{m_K^2}{4} - \mu^2}\right]^{-1}\,.$$

4060

The possible radioactive decay of the proton is a topic of much current interest. A typical experiment to detect proton decay is to construct a very large reservoir of water and put into it devices to detect Čerenkov radiation produced by the products of proton decay.

(a) Suppose that you have built a reservoir with 10,000 metric tons (1 ton = 1000 kg) of water. If the proton mean life τ_p is 10^{32} years, how many decays would you expect to observe in one year? Assume that your detector is 100% efficient and that protons bound in nuclei and free protons decay at the same rate.

(b) A possible proton decay is $p \to \pi^0 + e^+$. The neutral pion π^0 immediately (in 10^{-16} sec) decays to two photons, $\pi^0 \to \gamma + \gamma$. Calculate the maximum and minimum photon energies to be expected from a proton decaying at rest. The masses: proton $m_p = 938$ MeV, positron $m_{e^+} = 0.51$ MeV, neutral pion $m_{\pi^0} = 135$ MeV.

(*CUSPEA*)

Solution:

(a) Each H_2O molecule has 10 protons and 8 neutrons and a molecular weight of 18. The number of protons in 10^4 tons of water is then

$$N = \frac{10}{18} \times 10^7 \times 10^3 \times 6.02 \times 10^{23} = 3.34 \times 10^{33} \, ,$$

using Avagadro's number $N_0 = 6.02 \times 10^{23}$ mole^{-1}. The number of expected decays per year is therefore

$$\Delta N = \frac{3 \cdot 34}{\tau_p} \times 10^{33} = \frac{3.34 \times 10^{33}}{10^{32}} = 33.4/\text{year} \, .$$

(b) In the rest frame of the proton, conservation laws of energy and momentum require

$$M_p = E_{\pi^0} + E_{e^+} \, ,$$

$$p_{\pi^0} = p_{e^+} \, .$$

With $E^2 = M^2 + p^2$, these give

$$E_\pi = \frac{M_p^2 + M_\pi^2 - M_e^2}{2M_p}$$

$$= \frac{938^2 + 135^2 - 0.5^2}{2 \times 938} = 479 \text{ MeV} \, .$$

In the rest frame of the π^0 the energy and momentum of each γ are

$$E' = p' = \frac{M_\pi}{2} \, .$$

The π^0 has Lorentz factor and velocity

$$\gamma_\pi = \frac{479}{135} = 3.548 \, ,$$

$$\beta_\pi = \sqrt{1 - \frac{1}{\gamma_\pi^2}} = 0.9595 \, .$$

Lorentz transformation between the π^0 rest frame and the laboratory frame for the photons

$$E_\gamma = \gamma_\pi (E' + \beta_\pi p' \cos\theta') = \frac{M_\pi}{2}\gamma_\pi (1 + \beta_\pi \cos\theta') = \frac{E_\pi}{2}(1 + \beta_\pi \cos\theta')$$

shows that the photons will have in the laboratory maximum energy ($\theta' = 0$)

$$(E_\gamma)_{\max} = \frac{E_\pi}{2}(1 + \beta_\pi) = \frac{479}{2}(1 + 0.9595) = 469.3 \text{ MeV},$$

and minimum energy ($\theta' = 180°$)

$$(E_\gamma)_{\min} = \frac{E_\pi}{2}(1 - \beta_\pi) = \frac{479}{2}(1 - 0.9595) = 9.7 \text{ MeV}.$$

4061

Consider the decay in flight of a pion of laboratory energy E_π by the mode $\pi \to \mu + \nu_\mu$. In the pion center-of-mass system, the muon has a helicity $h = \frac{\mathbf{s}\cdot\boldsymbol{\beta}}{s\beta}$ of 1, where $\mathbf{s}$ is the muon spin. For a given E_π there is a unique laboratory muon energy $E_\mu^{(0)}$ for which the muon has zero average helicity in the laboratory frame.

(a) Find the relation between E_π and $E_\mu^{(0)}$.

(b) In the nonrelativistic limit, find the minimum value of E_π for which it is possible to have zero-helicity muons in the laboratory.

(*Columbia*)

Solution:

(a) Consider the spin 4-vector of the muon emitted in the decay $\pi \to \mu + \nu$. In the rest frame of the muon, it is

$$S_\alpha = (\mathbf{S}, iS_0),$$

where $\mathbf{S}$ is the muon spin and $S_0 = 0$.

Now consider the spin 4-vector in the rest frame of the pion, Σ_π. The muon has parameters γ_μ, β_μ in this frame and

$$S'_\alpha = (\mathbf{S}', iS'_0)$$

with

$$\mathbf{S}' = \mathbf{S} + (\gamma_\mu - 1)\mathbf{S} \cdot \hat{\boldsymbol{\beta}}_\mu \hat{\boldsymbol{\beta}}_\mu \,,$$

$$S_0' = \gamma_\mu (S_0 + \mathbf{S} \cdot \boldsymbol{\beta}_\mu) = \gamma_\mu \mathbf{S} \cdot \boldsymbol{\beta}_\mu = \gamma_\mu S \beta_\mu h_\mu \,.$$

In the Σ_π frame,

$$h_\mu = \frac{\mathbf{S} \cdot \boldsymbol{\beta}_\mu}{S\beta_\mu} = 1 \,,$$

and so $\mathbf{S} \cdot \boldsymbol{\beta}_\mu = S\beta_\mu$, i.e. $\mathbf{S} // \boldsymbol{\beta}_\mu$. It follows that

$$\mathbf{S}' = \mathbf{S} + (\gamma_\mu - 1) S \beta_\mu^{-1} \boldsymbol{\beta}_\mu \,,$$

$$S_0' = \gamma_\mu S \beta_\mu \,.$$

Next transform from Σ_π to the laboratory frame Σ, in which the pion has parameters γ_π, β_π, the muon has parameters γ, β. Then

$$S_\alpha^{\text{Lab}} = (\mathbf{S}'', iS_0'') \,,$$

where

$$\begin{aligned} S_0'' &= \gamma_\pi (\mathbf{S}_0' + \boldsymbol{\beta}_\pi \cdot \mathbf{S}') \\ &= \gamma_\pi [\gamma_\mu \beta_\mu S + \boldsymbol{\beta}_\pi \cdot \mathbf{S} + (\gamma_\mu - 1)(\boldsymbol{\beta}_\pi \cdot \boldsymbol{\beta}_\mu) S \beta_\mu^{-1}] \,. \end{aligned}$$

As $\mathbf{S} // \boldsymbol{\beta}_\mu$,

$$(\boldsymbol{\beta}_\pi \cdot \boldsymbol{\beta}_\mu) S \beta_\mu^{-1} = (\boldsymbol{\beta}_\pi \cdot \mathbf{S}) \beta_\mu \beta_\mu^{-1} = \boldsymbol{\beta}_\pi \cdot \mathbf{S} \,,$$

and

$$S_0'' = \gamma_\pi \gamma_\mu S (\beta_\mu^2 + \boldsymbol{\beta}_\pi \cdot \boldsymbol{\beta}_\mu) \beta_\mu^{-1} = \gamma \beta S h \,,$$

with

$$h = \gamma_\pi \gamma_\mu \gamma^{-1} \beta^{-1} (\beta_\mu^2 + \boldsymbol{\beta}_\pi \cdot \boldsymbol{\beta}_\mu) \beta_\mu^{-1} \,.$$

At muon energy $E_\mu^{(0)}$, $h = 0$, or

$$\beta_\mu^2 = -\boldsymbol{\beta}_\pi \cdot \boldsymbol{\beta}_\mu \,.$$

Lorentz transformation then gives

$$\begin{aligned} \gamma &= \gamma_\pi \gamma_\mu (1 + \boldsymbol{\beta}_\pi \cdot \boldsymbol{\beta}_\mu) \\ &= \gamma_\pi \gamma_\mu (1 - \beta_\mu^2) = \frac{\gamma_\pi}{\gamma_\mu} \,. \end{aligned}$$

Hence

$$E_\mu^{(0)} = m_\mu \gamma = \frac{m_\mu}{m_\pi} \frac{E_\pi}{\gamma_\mu} .$$

Consider the decay in the rest frame of π. Conservation of momentum and of energy require

$$\mathbf{p}_\nu + \mathbf{p}_\mu = 0 , \qquad \text{or} \qquad p_\nu = p_\mu ,$$

$$E_\nu = m_\pi - E_\pi .$$

These combine to give

$$E_\mu = \frac{m_\pi^2 + m_\mu^2}{2m_\pi} ,$$

or

$$\gamma_\mu = \frac{E_\mu}{m_\mu} = \frac{m_\pi^2 + m_\mu^2}{2m_\pi m_\mu} .$$

Hence

$$E_\mu^{(0)} = \frac{m_\mu}{m_\pi} \cdot \frac{2m_\pi m_\mu}{m_\pi^2 + m_\mu^2} E_\pi = \frac{2m_\mu^2}{m_\pi^2 + m_\mu^2} E_\pi .$$

(b) For the average muon helicity $h = 0$ in the laboratory frame, we require

$$\boldsymbol{\beta}_\pi \cdot \boldsymbol{\beta}_\mu = -\beta_\mu^2 ,$$

or

$$\beta_\pi \cos\theta = -\beta_\mu .$$

This means that

$$\beta_\pi \geq \beta_\mu , \qquad \text{or} \qquad \gamma_\pi \geq \gamma_\mu .$$

Hence the minimum pion energy required is

$$(E_\pi)_{\min} = \gamma_\mu m_\pi = \frac{m_\pi^2 + m_\mu^2}{2m_\mu} .$$

2. INTERACTIONS BETWEEN RADIATION AND MATTER (4062–4085)

4062

The energy loss of an energetic muon in matter is due mainly to collisions with

(a) nucleons.
(b) nuclei.
(c) electrons.

(CCT)

Solution:

A muon loses energy in matter mainly due to collisions with electrons, transferring part of its kinetic energy to the latter, which can either jump to higher energy levels or to be separated from the atoms resulting in their ionization.

So the answer is (c).

4063

A beam of negative muons can be stopped in matter because a muon may be

(a) transformed into an electron by emitting a photon.
(b) absorbed by a proton, which goes into an excited state.
(c) captured by an atom into a bound orbit about the nucleus.

(CCT)

Solution:

A μ^- can be captured into a bound orbit by a nucleus to form a μ-atom. It can also decay into an electron and two neutrinos (γ_μ, $\bar{\nu}_e$) but not an electron and a photon. So the answer is (c).

4064

After traversing one radiation length, an electron of 1 GeV has lost:

(a) 0.368 GeV
(b) none
(c) 0.632 GeV

of its original energy.

(*CCT*)

Solution:

By definition $E = E_0 e^{-x/\lambda}$, where λ is the radiation length. Thus when $x = \lambda$, $E = E_0 e^{-1} = 0.368$ GeV. The loss of energy is $\Delta E = 1 - 0.368 = 0.632$ GeV, and the answer is (c).

4065

A relativistic proton loses 1.8 MeV when penetrating a 1-cm thick scintillator. What is the most likely mechanism?

(a) Ionization, excitation.
(b) Compton effect.
(c) Pair production.

(*CCT*)

Solution:

When a relativistic proton passes through a medium, energy loss by ionization and excitation comes to $-dE/dx \approx$ 1–2 MeV/g cm^{-2}. The density of the scintilator is $\rho \approx 1$ g cm^{-3}, so $dx = 1$ g cm^{-2}. The energy loss rate $-\frac{dE}{dx} = 1.8$ MeV/g cm^{-2} agrees with ionization loss rate. So the answer is (a).

4066

The mean energy loss of a relativistic charged particle in matter per g/cm^2 is about

(a) 500 eV.
(b) 10 KeV.
(c) 2 MeV.

(*CCT*)

Solution:

As $dE/dx \approx (1 \sim 2)$ MeV/g cm $^{-2}$, the answer is (c).

4067

The critical energy of an electron is the energy at which

(a) the radiation loss equals the ionization loss.
(b) the electron ionizes an atom.
(c) the threshold of nuclear reaction is reached.

(*CCT*)

Solution:

The critical energy is defined as the energy at which the radiation loss is equal to the ionization loss. The answer is (a).

4068

The straggling of heavy ions at low energy is mostly a consequence of

(a) finite momentum.
(b) fluctuating state of ionization.
(c) multiple scattering.

(*CCT*)

Solution:

Multiple scattering changes an ion's direction of motion, thus making them straggle. The answer is (c).

4069

The so-called "Fermi plateau" is due to

(a) a density effect.
(b) Lorentz contraction.
(c) relativistic mass increase.

(*CCT*)

Solution:

At Lorentz factor $\gamma \approx 3$, rate of ionization loss $dE/dx \approx (dE/dx)_{\min}$. At $\gamma > 3$, because of its logarithmic relationship with energy, dE/dx increases only slowly with increasing γ. Finally, $\frac{dE}{dz} \approx$ constant when $\gamma > 10$ for a dense medium (solid or liquid), and when $\gamma > 100$ for a dilute medium (gas), because of the effect of electron density. The plateau in the $\frac{dE}{dx}$ vs E curve is known as "Fermi plateau". Thus the answer is (a).

4070

The probability for an energy loss E' in the interval dE' of a charged particle with energy E and velocity v in a single collision is proportional to

(a) $\frac{E'}{E} dE'$.
(b) EdE'.
(c) $(\frac{1}{vE'})^2 dE'$.

(CCT)

Solution:

Take collisions with electrons as example. For a single collision, the energy loss of a particle of charge Ze depends only on its velocity v and the impact parameter b : $E' = \frac{2Z^2e^4}{m_0 v^2 b^2}$, where m_0 is the electron mass. Thus $dE' = -\frac{4Z^2e^4}{m_0 v^2 b^3}\, db = -A\frac{db}{v^2 b^3}$, where $A = \frac{4Z^2e^4}{m_0}$ is a constant.

Suppose the electrons are distributed uniformly in the medium. Then the probability of colliding with an electron with impact parameter in the interval between b and $b + db$ is

$$d\sigma = 2\pi b|db| = \frac{2\pi v^2 b^4}{A} dE' = \frac{\pi A dE'}{2(vE')^2} \propto \frac{dE'}{(vE')^2}\,.$$

Hence the answer is (c).

4071

The scattering of an energetic charged particle in matter is due mostly to interactions with (the)

(a) electrons.
(b) nuclei.

(c) quarks.

(CCT)

Solution:

In traversing a medium, a charged particle suffers Coulomb interactions with both electrons and nuclei. However, though collisions with the former are numerous, the momentum transfer in each is very small. Only collisions with the latter will result in appreciable scattering of the traversing particle.

Hence the answer is (b).

4072

The mean scattering angle of a charged particle in matter of a thickness x increases with

(a) x^2.
(b) $x^{1/2}$.
(c) x.

(CCT)

Solution:

The mean scattering angle of a particle of charge Ze in traversing matter of thickness x is $|\bar{\theta}| = \frac{KZ\sqrt{x}}{pv} \propto x^{1/2}$, where K is a constant. Hence the answer is (b).

4073

Consider a 2-cm thick plastic scintillator directly coupled to the surface of a photomultiplier with a gain of 10^6. A 10-GeV particle beam is incident on the scintillator as shown in Fig. 4.8(a).

(a) If the beam particle is a muon, estimate the charge collected at the anode of the photomultiplier.

(b) Suppose one could detect a signal on the anode of as little as 10^{-12} coulomb. If the beam particle is a neutron, estimate what is the smallest laboratory angle that it could scatter elastically from a proton in the scintillator and still be detected?

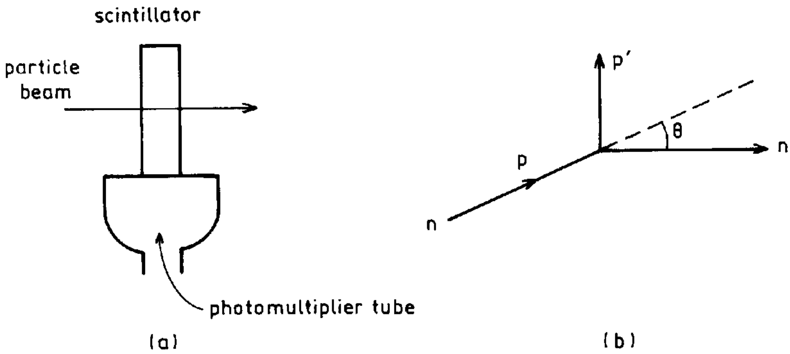

Fig. 4.8

(c) Same as Part (b), but it scatters elastically from a carbon nucleus.

(*Chicago*)

Solution:

(a) From its ionization loss curve, we see that a muon of energy 10 GeV will lose 4 MeV in a plastic scintillator of length 2 cm. Roughly, in a plastic scintillator, producing one photon requires 100 eV of energy. This amount of energy will produce $N_{\text{ph}} \approx 4 \times 10^4$ photons in the scintillator. Suppose about 50% of the photons make it to the photomultiplier tube and about 10% of these produce photoelectrons off the cathode. Then the number of photoelectrons emitted is $N_{\text{pe}} = 2 \times 10^3$. With a gain of 10^6, the charge collected at the anode of the photomultiplier is $Q = 2\times10^9 e = 3.2\times10^{-10}$ C.

(b) Figure 4.8(b) shows a neutron scatters by a small angle θ in the laboratory frame. Its momentum is changed by an amount $p\theta$ normal to the direction of motion. This is the momentum of the recoiling nucleus. Then the kinetic energy acquired by it is

$$\frac{p^2\theta^2}{2m}\,,$$

where m is the mass of the recoiling nucleus. As an energy loss of 4 MeV corresponds to 3.2×10^{-10}C of anode charge, the detection threshold of 10^{-12} C implies that recoil energy as little as 12.5 keV can be detected. Hence the smallest laboratory scattering angle $\theta_{\min}$ that can be detected is given by

$$\theta^2_{\min} = \frac{2m_p}{p_n^2} \times 12.5 \times 10^3 = \frac{2 \times 10^9}{(10^{10})^2} \times 12.5 \times 10^3 = 2.5 \times 10^{-7}\ \text{rad}^2\,,$$

i.e.

$$\theta_{\min} = 5.0 \times 10^{-4}\ \text{rad}\,,$$

assuming the recoiling nucleus is a proton.

(a) If the recoiling particle is a carbon nucleus, then

$$\theta^2_{\min} = \frac{2m_c}{p_n^2} \times 12.5 \times 10^3 = \frac{2 \times 12 \times 10^9}{(10^{10})^2} \times 12.5 \times 10^3 = 3.0 \times 10^{-6}\ \text{rad}^2\,,$$

i.e.,

$$\theta_{\min} = 1.73 \times 10^{-3}\ \text{rad}\,.$$

4074

How many visible photons ($\sim$ 5000 Å) does a 100-W bulb with 3% efficiency emit per second?

(a) 10^{19}.
(b) 10^9.
(c) 10^{33}.

(CCT)

Solution:

Each photon of $\lambda = 5000$ Å has energy

$$E = h\nu = hc/\lambda = \frac{2\pi \times 197 \times 10^{-7}}{5000 \times 10^{-8}} = 2.5\ \text{eV}\,.$$

So the number of photons is

$$N = \frac{W}{E} = \frac{100 \times 0.03}{2.5 \times 1.6 \times 10^{-19}} = 0.75 \times 10^{19} \approx 10^{19}\,.$$

Hence the answer is (a).

4075

Estimate the attenuation (absorption/scattering) of a beam of 50-keV X-rays in passage through a layer of human tissue (no bones!) one centimeter thick.

(Columbia)

Solution:

As the human body is mostly water, we can roughly take its density as that of water, $\rho \approx 1$ g/cm^3. Generally, the absorption coefficient of 50 keV X-rays is about 0.221 cm^2/g. Then the attenuation resulting from the passage through one centimeter of tissue (thickness = 1 cm $\times$ 1 g cm^{-3} = 1 g cm^{-2}) is

$$1 - \exp(-0.221 \times 1) = 0.20 = 20\% .$$

4076

Photons of energy 0.3 eV, 3 eV, 3 keV, and 3 MeV strike matter. What interactions would you expect to be important? Match one or more interactions with each energy.

0.3 eV	(a) Pair production	(e) Atomic Ionization
3 eV	(b) Photoelectric effect	(f) Raman Scattering (rotational
3 keV	(c) Compton Scattering	and vibrational excitation)
3 MeV	(d) Rayleigh Scattering	

(*Wisconsin*)

Solution:

Raman scattering is important in the region of 0.3 eV. Atomic ionization, Rayleigh scattering and Raman scattering are important around 3 eV. Photoelectric effect is important in the region of 3 keV. In the region of 3 MeV, Compton scattering and pair production are dominant.

4077

Discuss the interaction of gamma radiation with matter for photon energies less than 10 MeV. List the types of interaction that are important in this energy range; describe the physics of each interaction and sketch the relative contribution of each type of interaction to the total cross section as a function of energy.

(*Columbia*)

Solution:

Photons of energies less than 10 MeV interact with matter mainly through photoelectric effect, Compton scattering, and pair production.

(1) *Photoelectric effect*: A single photon gives all its energy to a bound electron in an atom, detaching it completely and giving it a kinetic energy $E_e = E_\gamma - E_b$, where E_γ is the energy of the photon and E_b is the binding energy of the electron. However, conservation of momentum and of energy prevent a free electron from becoming a photoelectron by absorbing all the energy of the photon. In photoelectric effect, conservation of momentum must be satisfied by the recoiling of the nucleus to which the electron was attached. The process generally takes place with the inner electrons of an atom (mostly K- and L-shell electrons). The cross section $\sigma_{p-e} \propto Z^5$, where Z is the nuclear charge of the medium. If $\varepsilon_K < E_\gamma < 0.5$ MeV, $\sigma_{p-e} \propto E_\gamma^{-\frac{7}{2}}$, where ε_K is the binding energy of K-electron. If $E_\gamma >$ 0.5 MeV, $\sigma_{p-e} \propto E_\gamma^{-1}$. Thus photoelectric effect is dominant in the low-energy region and in high-Z materials.

(2) *Compton scattering*: A photon is scattered by an electron at rest, the energies of the electron and the scattered photon being determined by conservation of momentum and energy to be respectively

$$E_e = E_\gamma \left[1 + \frac{mc^2}{E_\gamma(1-\cos\theta)}\right]^{-1},$$

$$E_{\gamma'} = E_\gamma \left[1 + \frac{E_\gamma}{mc^2}(1-\cos\theta)\right]^{-1},$$

where m is the electron mass, E_γ is the energy of the incident photon, and θ is the angle the scattered photon makes with the incident direction. The cross section is $\sigma_c \propto ZE_\gamma^{-1}\ln E_\gamma$ (if $E_\gamma > 0.5$ MeV).

(3) *Pair production*: If $E_\gamma > 2m_ec^2$, a photon can produce a positron-electron pair in the field of a nucleus. The kinetic energy of the positron-electron pair is given by $E_{e^+} + E_{e^-} = E_\gamma - 2m_ec^2$. In low-energy region $\sigma_{e^+e^-}$ increases with increasing E_γ, while in high-energy region, it is approximately constant. Figure 4.9 shows the relative cross sections of lead for absorption of γ-rays as a function of E_γ. It is seen that for $E_\gamma \gtrsim 4$ MeV, pair production dominates, while for low energies, photoelectric and Compton effects are important. Compton effect predominates in the energy region from several hundred keV to several MeV.

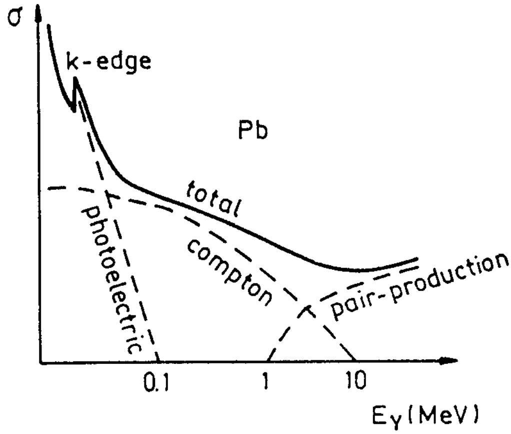

Fig. 4.9

4078

Fast neutrons can be detected by observing scintillations caused by recoil protons in certain (optically transparent) hydrocarbons. Assume that you have a 5 cm thick slab of scintillator containing the same number-density of C and H, namely 4×10^{22} atoms/cm^3 of each kind.

(a) What fraction of ~ 5 MeV neutrons incident normal to the slab will pass through the slab without interacting with either C or H nuclei?

(b) What fraction of the incident neutrons will produce a recoil proton? [Assume $\sigma_H = 1.5$ barns, $\sigma_C = 1.0$ barn. Note 1 barn $= 10^{-24}$ cm^2.]

(*Wisconsin*)

Solution:

(a) Denote the number of neutrons by N. The number decreases by ΔN after traveling a distance Δx in the scintillator, given by

$$\Delta N = -N(\sigma_H n_H + \sigma_C n_C)\Delta x\,,$$

where n is the number density of the nuclei of the scintillator. After passing through a distance d, the number of neutrons that have not undergone any interaction is then

$$N = N_0 \exp[-(\sigma_H n_H + \sigma_C n_C)d]\,,$$

giving

$$\eta = N/N_0 = \exp[-(1.5 + 1.0) \times 10^{-24} \times 4 \times 10^{22} \times 5]$$
$$= e^{-0.5} = 60.5\%\,.$$

(b) The fraction of incident neutrons undergoing at least one interaction is

$$\eta' = 1 - \eta = 39.5\%\,.$$

Of these only those interacting with protons can produce recoil protons. Thus the fraction of neutrons that produce recoil protons is

$$\eta'' = \eta' \cdot \frac{1.5}{1.5 + 1.0} = \frac{3}{5}\eta' = 23.7\%\,.$$

4079

The mean free path of fast neutrons in lead is about 5 cm. Find the total neutron cross section of lead (atomic mass number $\sim$ 200, density $\sim$ 10 g/cm^3).

(*Wisconsin*)

Solution:

The number of Pb atoms per unit volume is

$$n = \frac{\rho}{A} \times N_0 = \frac{10}{200} \times 6.022 \times 10^{23} = 3.01 \times 10^{22}\ \text{cm}^{-3}\,.$$

The mean free path of neutron in lead is $l = 1/(n\sigma)$, where σ is the interaction cross section between neutron and lead. Hence

$$\sigma = \frac{1}{nl} = \frac{1}{3.01 \times 10^{22} \times 5} = 6.64 \times 10^{-24}\ \text{cm}^2 = 6.64\ b\,.$$

4080

It is desired to reduce the intensity of a beam of slow neutrons to 5% of its original value by placing into the beam a sheet of Cd (atomic weight 112, density 8.7×10^3 kg/m^3). The absorption cross section of Cd is 2500 barns. Find the required thickness of Cd.

(*Wisconsin*)

Solution:

The intensity of a neutron beam after passing through a Cd foil of thickness t is given by $I(t) = I_0 e^{-n\sigma t}$, where I_0 is the initial intensity, n is the number density of Cd, and σ is the capture cross section. As

$$n = \frac{\rho N_0}{A} = \frac{8.7}{112} \times 6.022 \times 10^{23} = 4.7 \times 10^{22} \text{ cm}^{-3},$$

the required thickness of Cd foil is

$$t = \frac{1}{n\sigma} \ln \frac{I_0}{I(t)} = \frac{1}{4.7 \times 10^{22} \times 2500 \times 10^{-24}} \ln \frac{1}{0.05} = 0.025 \text{ cm}.$$

4081

A beam of neutrons passes through a hydrogen target (density 4×10^{22} atom/cm^3) and is detected in a counter C as shown in Fig. 4.10. For equal incident beam flux, 5.0×10^5 counts are recorded in C with the target empty, and 4.6×10^5 with the target full of hydrogen. Estimate the total n-p scattering cross section, and its statistical error.

(*Wisconsin*)

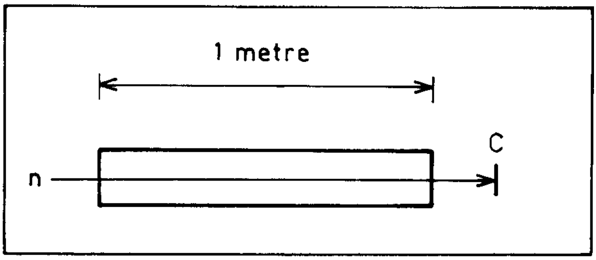

Fig. 4.10

Solution:

Let the total cross section of n-p interaction be σ. After passing through the hydrogen target, the number of neutrons decreases from N_0 to $N_0 e^{-n\sigma t}$, where $n = 4\times 10^{22}$ cm^{-3} is the atomic concentration of the target. Suppose the numbers of neutrons detected without and with the hydrogen target are N', N'' respectively and η is the neutron-detecting efficiency of C. Then

$$N' = \eta N_0 \,, \qquad N'' = \eta N_0 e^{-n\sigma t} = N' e^{-n\sigma t}\,,$$

and thus

$$N''/N' = e^{-n\sigma t}\,,$$

giving the n-p scattering cross section as

$$\sigma = \frac{1}{nt}\ln\frac{N'}{N''} = \frac{1}{4\times 10^{22}\times 100}\ln\frac{5\times 10^5}{4.6\times 10^5} = 2.08\times 10^{-26}\ \text{cm}^2\,.$$

To estimate the statistical error of σ we note

$$\begin{aligned}\Delta\sigma &= \frac{\partial\sigma}{\partial N'}(\Delta N') + \frac{\partial\sigma}{\partial N'}(\Delta N')\,,\\ \frac{\partial\sigma}{\partial N'} &= \frac{1}{ntN'}\,,\\ \frac{\partial\sigma}{\partial N''} &= -\frac{1}{ntN''}\,,\\ \Delta N' &= \sqrt{N'}\,, \qquad \Delta N'' = \sqrt{N''}\,.\end{aligned}$$

Hence

$$(\Delta\sigma)^2 = \left(\frac{\partial\sigma}{\partial N'}\right)^2(\Delta N')^2 + \left(\frac{\partial\sigma}{\partial N''}\right)^2(\Delta N'')^2 = \frac{1}{(nt)^2}\left(\frac{1}{N'}+\frac{1}{N''}\right)\,,$$

or

$$\begin{aligned}\Delta\sigma &= \frac{1}{(nt)}\sqrt{\frac{1}{N'}+\frac{1}{N''}} = \frac{1}{4\times 10^{22}\times 100}\sqrt{\frac{1}{4.6\times 10^5}+\frac{1}{5\times 10^5}}\\ &\approx 5\times 10^{-28}\ \text{cm}^2\,.\end{aligned}$$

Therefore

$$\sigma = (2.08\pm 0.05)\times 10^{-26}\ \text{cm}^2 = (20.8\pm 0.5)\ \text{mb}\,.$$

4082

A beam of energetic neutrons with a broad energy spectrum is incident down the axis of a very long rod of crystalline graphite as shown in Fig. 4.11. It is found that the faster neutrons emerge from the sides of the rod, but only slow neutrons emerge from the end. Explain this very briefly and estimate numerically the maximum velocity of the neutrons which emerge from the end of the rod. Introduce no symbols.

(*Columbia*)

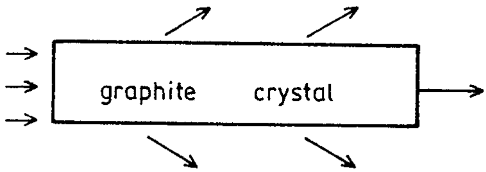

Fig. 4.11

Solution:

Crystalline graphite is a cold neutron filter. High-energy neutrons change directions on elastic scattering with the nuclei in the crystalline graphite and finally go out of the rod. Because of their wave property, if the wavelengths of the neutrons are comparable with the lattice size, interference occurs with the diffraction angle θ satisfying Bragg's law

$$m\lambda = 2d \sin\theta\,, \qquad \text{with } m = 1, 2, 3 \dots .$$

In particular for $\lambda > 2d$, there is no coherent scattering except for $\theta = 0$. At $\theta = 0$, the neutrons can go through the crystal without deflection. Furthermore, as the neutron absorption cross section of graphite is very small, attenuation is small for the neutrons of $\lambda > 2d$. Graphite is polycrystalline with irregular lattice orientation. The high-energy neutrons change directions by elastic scattering and the hot neutrons change directions by Bragg scattering from microcrystals of different orientations. Finally both leave the rod through the sides. Only the cold neutrons with wavelength $\lambda > 2d$ can go through the rod without hindrance. For graphite, $\lambda > 2d = 6.69$ Å. The maximum velocity of such cold neutrons is

$$v_{\max} = \frac{p}{m} = \frac{h}{m\lambda} = \frac{2\pi\hbar c^2}{\lambda mc^2} = \frac{2\pi \times 197 \times 10^{-13} \times 3 \times 10^{10}}{6.69 \times 10^{-8} \times 940}$$

$$= 0.59 \times 10^5 \text{ cm}/s = 590\ m/s\,.$$

4083

Mean free path for 3-MeV electron-neutrinos in matter is

$$10, 10^7, 10^{17}, 10^{27}\ g/\text{cm}^2\,.$$

(*Columbia*)

Solution:

The interaction cross section between neutrino and matter is $\sigma \approx 10^{-41}$ cm^2, and typically the atomic number-density of matter $n \approx 10^{23}$ cm^{-3}, density of matter $\rho \approx 1$ g/cm^3. Hence the mean free path of neutrino in matter is $1 = \rho/n\sigma \approx 10^{18}$ g/cm^2. The third answer is correct.

4084

Čerenkov radiation is emitted by a high-energy charged particle which moves through a medium with a velocity greater than the velocity of electromagnetic-wave propagation in the medium.

(a) Derive the relationship among the particle velocity $v = \beta c$, the index of refraction n of the medium, and the angle θ at which the Čerenkov radiation is emitted relative to the line of flight of the particle.

(b) Hydrogen gas at one atmosphere and 20°C has an index of refraction $n = 1 + 1.35 \times 10^{-4}$. What is the minimum kinetic energy in MeV which an electron (of mass 0.5 MeV/c^2) would need in order to emit Čerenkov radiation in traversing a medium of hydrogen gas at 20°C and one atmosphere?

(c) A Čerenkov-radiation particle detector is made by fitting a long pipe of one atmosphere, 20°C hydrogen gas with an optical system capable of detecting the emitted light and of measuring the angle of emission θ to an accuracy of $\delta\theta = 10^{-3}$ radian. A beam of charged particles with momentum

100 GeV/c are passed through the counter. Since the momentum is known, measurement of the Čerenkov angle is, in effect, a measurement of the rest mass m_0. For a particle with m_0 near 1 GeV/c^2, and to first order in small quantities, what is the fractional error (i.e., $\delta m_0/m_0$) in the determination of m_0 with the Čerenkov counter?

(*CUSPEA*)

Solution:

(a) Figure 4.12 shows the cross section of a typical Čerenkov wavefront. Suppose the particle travels from O to A in t seconds. The radiation sent out while it is at O forms a spherical surface with center at O and radius $R = ct/n$. The Čerenkov radiation wavefront which is tangent to all such spherical surfaces is a conic surface. In the triangle AOB, OB = R = ct/n, OA = $vt = \beta ct$, and so $\cos\theta$ = OB/OA = $1/(n\beta)$.

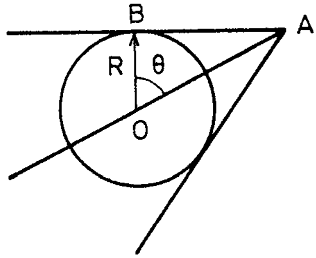

Fig. 4.12

(b) As $\cos\theta = \frac{1}{n\beta}$, we require

$$\beta \geq \frac{1}{n}\,.$$

Thus

$$\beta_{\min} = \frac{1}{n} = \frac{1}{1 + 1.35 \times 10^{-4}} \approx 1 - 1.35 \times 10^{-4}\,,$$

and so

$$\gamma_{\min} = \frac{1}{\sqrt{(1+\beta)(1-\beta)}} \approx \frac{1}{\sqrt{2 \times 1.35 \times 10^{-4}}} = 60.86\,.$$

The minimum kinetic energy required by an electron is therefore

$$T = (\gamma - 1)mc^2 = 59.86 \times 0.5 = 29.9 \text{ MeV}\,.$$

(c) The rest mass m_0c^2 is calculated from (taking $c = 1$)

$$m_0^2 = \frac{p^2}{(\gamma\beta)^2} = \frac{p^2(1-\beta^2)}{\beta^2} = \frac{p^2}{\beta^2} - p^2$$
$$= p^2n^2\cos^2\theta - p^2\,.$$

Differentiating with respect to θ gives

$$2m_0dm_0 = -2p^2n^2\cos\theta\sin\theta d\theta\,.$$

Hence

$$\delta m_0 = \frac{p^2n^2}{2m_0}\sin 2\theta\delta\theta\,.$$

With $m_0 \approx 1$ GeV/c^2, $p = 100$ GeV/c,

$$\gamma = \frac{\sqrt{p^2+m_0^2}}{m_0} = \sqrt{10^4+1}\,.$$

Thus

$$\cos\theta = \frac{1}{n\beta} = \frac{\gamma}{n\sqrt{\gamma^2-1}}$$
$$= \frac{\sqrt{10^4+1}}{(1+1.35\times 10^{-4})\times 10^2}$$
$$\approx \frac{1+0.5\times 10^{-4}}{1+1.35\times 10^{-4}}$$
$$\approx 1 - 0.85\times 10^{-4}$$
$$\approx 1 - \frac{\theta^2}{2}\,,$$

and hence

$$\theta^2 \approx 1.7 \times 10^{-4}$$

or

$$\theta \approx 1.3 \times 10^{-2} \text{ rad}\,.$$

As θ is small, $\sin 2\theta \approx 2\theta$, and

$$\frac{\delta m_0}{m_0} = \frac{p^2 n^2 \theta}{m_0^2} \delta\theta$$

$$\approx 10^4 \times 1.3 \times 10^{-2} \times 10^{-3} = 0.13\,.$$

4085

A proton with a momentum of 1.0 GeV/c is passing through a gas at high pressure. The index of refraction of the gas can be changed by changing the pressure.

(a) What is the minimum index of refraction at which the proton will emit Čerenkov radiation?

(b) At what angle will the Čerenkov radiation be emitted when the index of refraction of the gas is 1.6? (Take rest mass of proton as 0.94 GeV/c^2.)

(*Columbia*)

Solution:

(a) The proton has Lorentz factor

$$\gamma = \frac{\sqrt{p^2 + m^2}}{m} = \frac{\sqrt{1 + 0.94^2}}{0.94} = 1.46$$

and hence

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.729\,.$$

For the proton to emit Čerenkov radiation we require

$$\frac{1}{n\beta} \leq 1\,,$$

or

$$n \geq \frac{1}{\beta} = \frac{1}{0.729} = 1.37\,.$$

(b)

$$\cos\theta = \frac{1}{n\beta} = \frac{1}{1.6 \times 0.729} = 0.86\,,$$

giving

$$\theta = 31^\circ\,.$$

3. DETECTION TECHNIQUES AND EXPERIMENTAL METHODS (4086–4105)

4086

The mean energy for production of a free ion pair in gases by radiation is

(a) equal to the ionization potential.
(b) between 20 $\sim$ 40 eV.
(c) in good approximation 11.5Z.

(CCT)

Solution:

The average energy needed to produce a pair of free ions is larger than the ionization potential, as part of the energy goes to provide for the kinetic energy of the ions. The answer is (b).

4087

At low E/p the drift velocity of electrons in gases, v_{Dr}, follows precisely the relation $v_{Dr} \propto E/p$. This can be explained by the fact that

(a) the electrons each gains an energy $\varepsilon = eE \int ds$.
(b) the electrons thermalize completely in inelastic encounters with the gas molecules.
(c) the cross section is independent of electron velocity.

(CCT)

Solution:

The electrons acquire an average velocity $v_{Dr} = \frac{p}{2m_e} = \frac{eE\tau}{2m_e}$, in the electric field E, where τ is the average time-interval between two consecutive collisions. As $\tau = \frac{l}{v_{Dr}} \propto \frac{1}{\sigma v_{Dr}}$, where l is the mean free path of the electrons in the gas and σ is the interaction cross section, we have

$$v_{Dr} \propto \frac{E}{\sigma p} \propto \frac{E}{p}$$

if σ is independent of velocity. If σ is dependent on velocity, the relationship would be much more complicated. Hence the answer is (c).

4088

The mean ionization potential is a mean over energies of different

(a) atomic excitation levels.
(b) molecular binding energies.
(c) electronic shell energies.

(CCT)

Solution:

The mean ionization potential is defined as the average energy needed to produce a pair of positive and negative ions, which is the average of the molecular binding energies. The answer is (b).

4089

The efficiency of a proportional counter for charged particles is ultimately limited by

(a) signal-to-noise ratio.
(b) total ionization.
(c) primary ionization.

(CCT)

Solution:

If the mean primary ionization of a charged particle is very small, there is a finite probability that the charged particle may not produce sufficient primary ionization for its observation because of statistical fluctuation. Hence the answer is (c).

4090

Spectra of monoenergetic X-rays often show two peaks in proportional counters. This is due to

(a) escape of fluorescent radiation.
(b) Auger effect.
(c) Compton scattering.

(CCT)

Solution:

The escape of fluorescent radiation causes the spectrum to have two peaks. The larger peak is the total energy peak of the X-rays, while the smaller one is due to the fluorescent X-rays escaping from the detector. The answer is (a).

4091

A Geiger counter consists of a 10 mm diameter grounded tube with a wire of 50 μm diameter at +2000 V in the center. What is the electrical field at the wire?

(a) 200^2 V/cm.
(b) 150 kV/cm.
(c) 1.5×10^9 V/cm.

(CCT)

Solution:

With $R_0 = 0.5\times10^{-2}$ m, $R_i = 75\times10^{-6}$ m, $V = 200$ V, $\gamma = 25\times10^{-6}$ m,

$$E(r) = \frac{V}{r\ln\dfrac{R_0}{R_i}} = 1.51 \times 10^7 \text{ V}/m$$
$$= 151 \text{ kV/cm}\,.$$

Hence the answer is (b).

4092

For Question 4091, the electrical field at the tube wall is

(a) 0 V/cm.
(b) 377 V/cm.
(c) 754 V/cm.

(CCT)

Solution:

Same as for **Problem 4091** but with $r = 0.5 \times 10^{-2}$ m:

$$E(r) = 7.55 \times 10^4 \text{ V/m} = 755 \text{ V/cm}\,.$$

The answer is (c).

4093

What limits the time resolution of a proportional counter?

(a) Signal-to-noise ratio of the amplifier.
(b) Slow signal formation at the anode (slow rise time).
(c) Random location of the ionization and therefore variable drift time.

(CCT)

Solution:

Randomness of the location of the primary ionization causes the time it takes for the initial ionization electrons to reach the anode to vary. The anode signals are produced mainly by the avalanche of the electrons which reach the anode first. Thus large fluctuation results, making the resolution poor. The answer is (c).

4094

What is the mechanism of discharge propagation in a self-quenched Geiger counter?

(a) Emission of secondary electrons from the cathode by UV quanta.
(b) Ionization of the gas near the anode by UV quanta.
(c) Production of metastable states and subsequent de-excitation.

(CCT)

Solution:

The answer is (b).

4095

Does very pure NaI work as a good scintillator?

(a) No.
(b) Only at low temperatures.
(c) Yes.

(CCT)

Solution:

The answer is (b).

4096

What is the advantage of binary scintillators?

(a) They are faster.
(b) They give more amplitude in the photodetector.
(c) They are cheaper.

(CCT)

Solution:

The advantage of binary scintillators is their ability to restrain the Compton and escape peaks, and so to increase the total energy-peak amplitudes in the photodetector. The answer is (b).

4097

A charged particle crosses a NaI(TI)-scintillator and suffers an energy loss per track length dE/dx. The light output dL/dx

(a) is proportional to dE/dx.
(b) shows saturation at high dE/dx.
(c) shows saturation at high dE/dx and deficiency at low dE/dx.

(CCT)

Solution:

NaI(T1) is not a strictly linear detector. Its photon output depends on both the type of the traversing particle and its energy loss. When the energy loss is very small the departure from nonlinearity of the photon output is large, while when dE/dX is very large it becomes saturated. The answer is (c).

4098

Monoenergetic γ-rays are detected in a NaI detector. The events between the Compton edge and the photopeak occur

(a) predominantly in thin detectors.
(b) predominantly in thick detectors.
(c) never.

(CCT)

Solution:

In general, the number of events in the region between the Compton edge and the photopeak is smaller than in other regions. In the spectrum, such events appear as a valley. In neither a thin detector or a thick detector can they become dominant. The answer is (c).

4099

The light emission in organic scintillators is caused by transitions between

(a) levels of delocalized σ electrons.
(b) vibrational levels.
(c) rotational levels.

(CCT)

Solution:

Actually the fast component of the emitted light from an organic scintillator is produced in the transition between the ${}^0S_1$ level and the delocalized ${}^1S_0$ level. The answer is (a).

4100

A proton with total energy 1.4 GeV transverses two scintillation counters 10 m apart. What is the time of flight?

(a) 300 ns.
(b) 48 ns.
(c) 33 ns.

(CCT)

Solution:

The proton has rest mass $m_p = 0.938$ GeV and hence

$$\gamma = \frac{E}{m_p} = \frac{1.4}{0.938} = 1.49\,,$$

$$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.74\,.$$

The time of flight is therefore

$$t = \frac{10}{0.74 \times 3 \times 10^8} = 4.5 \times 10^{-8}\ s = 45\ ns\,.$$

The answer is (b).

4101

What is the time of flight if the particle in Question 4100 is an electron?

(a) 330 ns.
(b) 66 ns.
(c) 33 ns.

(CCT)

Solution:

An electron with energy 1.4 GeV $\gg m_e c^2 = 0.51$ MeV has $\beta \approx 1$. Hence the time of flight is

$$t \approx \frac{10}{3 \times 10^8} = 3.3 \times 10^{-8}\ s = 33\ ns\,.$$

Thus the answer is (c).

4102

How would you detect 500 MeV γ-rays? With

(a) hydrogen bubble chamber.
(b) shower counter (BGO).
(c) Geiger counter.

(CCT)

Solution:

As 500 MeV γ-rays will cause cascade showers in a medium, we need a total-absorption electromagnetic shower counter for their detection. The BGO shower counter makes a good choice because of its short radiation length and high efficiency. Hence the answer is (b).

4103

How would one measure the mean lifetime of the following particles?

(1) $U^{238} : \tau = 4.5 \times 10^9$ years,
(2) Λ^0 hyperon : $\tau = 2.5 \times 10^{-10}$ sec,
(3) ρ^0 meson : $\tau \approx 10^{-22}$ sec.

(Wisconsin)

Solution:

(1) The lifetime of ^{238}U can be deduced from its radioactivity $-dN/dt = \lambda N$, where the decay rate is determined directly by measuring the counting rate. Given the number of the nuclei, λ can be worked out and $\tau = 1/\lambda$ calculated.

(2) The lifetime of Λ^0 hyperon can be deduced from the length of its trajectory before decaying according to $\Lambda^0 \to p^+\pi^-$ in a strong magnetic field in a bubble chamber. From the opening angle and curvatures of the tracks of p and π^-, we can determine the momentum of the Λ^0, which is the sum of the momenta of p and π^-. Given the rest mass of Λ^0, its mean lifetime can be calculated from the path length of Λ^0 (**Problem 3033**).

(3) The lifetime of ρ^0 meson can be estimated from the invariant mass spectrum. From the natural width ΔE of its mass in the spectrum, its lifetime can be estimated using the uncertainty principle $\Delta E \Delta \tau \approx \hbar$.

4104

The "charmed" particles observed in e^+e^- storage rings have not yet been seen in hadron-hadron interactions. One possible means for detecting such particles is the observation of muons resulting from their leptonic decays. For example, consider a charmed particle c with decay mode

$$c \to \mu\nu\,.$$

Unfortunately, the experimental situation is complicated by the presence of muons from π decays.

Consider an experiment at Fermilab in which 400 GeV protons strike a thick iron target (beam dump) as depicted in Fig. 4.13.

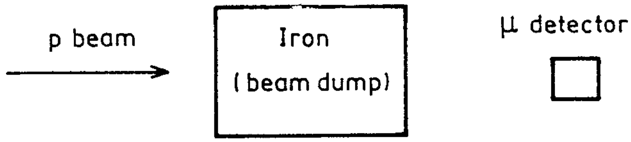

Fig. 4.13

Some of the muons entering the detector will be from π decays and some from c decays (ignore other processes). Calculate the ratio of muons from c decays to those from π decays under the following assumptions:

(a) the pions that have suffered interaction in the dump completely disappear from the beam,

(b) the energy spectra of both π and c are flat from minimum up to the maximum possible energy,

(c) the mass of the c is 2 GeV/c^2 and its lifetime is $\ll 10^{-10}$ sec,

(d) one can ignore muon energy loss in the iron,

(e) one can ignore any complications due to the geometry of the muon detector,

(f) the p–p inelastic cross section is 30 mb, and the mean charged pion multiplicity in inelastic interactions is 8.

Be specific. State any additional assumptions. Give a numerical value for the ratio at $E_\mu = 100$ GeV assuming the total production cross section for c to be 10 μb per Fe nucleus and that it decays to $\mu\nu$ 10% of the time.

(*Princeton*)

Solution:

In addition to the assumptions listed in the question, we also assume the charge independence of nucleon interactions so that $\sigma_{pp} = \sigma_{pn}$ and the mean charged-pion multiplicities are the same for pp and pn collisions.

For ^{56}Fe, the number densities of protons and neutrons are the same, being

$$N_p = N_n = \frac{28}{56} \times 7.8 \times 6.02 \times 10^{23} = 2.35 \times 10^{24} \text{ cm}^{-3} .$$

Denote the flux of protons in the incident beam by $\phi(x)$, where x is the target thickness from the surface of incidence. As

$$\frac{d\phi}{dx} = -(\sigma_{pp} N_p + \sigma_{pn} N_n)\phi = -2\sigma_{pp} N_p \phi ,$$

$$\phi = \phi_0 e^{-2\sigma_{pp} N_p x} .$$

If the target is sufficiently thick, say $x = 10$ m $= 10^3$ cm, then

$$\phi = \phi_0 \exp(-2 \times 30 \times 10^{-27} \times 2.35 \times 10^{24} \times 10^3) = 5.8 \times 10^{-62} \phi_0 ,$$

at the exit surface, showing that the beam of protons is completely dissipated in the target. This will be assumed in the following.

Consider first the c quarks produced in p–Fe interactions in the target. From the given data $\sigma_{p\text{Fe}}(c) = 10$ μb, $\sigma_{pp} = 30$ mb, we find the number of c quarks so produced as

$$N_c = \int N_{\text{Fe}} \sigma(c) d\phi \approx N_{\text{Fe}} \sigma(c) \phi_0 \int_0^\infty e^{-2\sigma_{pp} N_p x} dx$$

$$= \frac{N_{\text{Fe}}}{2N_{pp}} \frac{\sigma(c)}{\sigma_{pp}} \phi_0 = \frac{1}{56} \times \frac{10^{-5}}{30 \times 10^{-3}} \phi_0 = 5.95 \times 10^{-6} \phi_0 .$$

As the c quarks have lifetime $\ll 10^{-10}$ s, all those produced in p–Fe interactions will decay in the target, giving rise to muons 10% of the times. Thus

$$N_{\mu c} = 0.1 N_c = 5.95 \times 10^{-7} \phi_0 .$$

Next consider the muons arising from the decay of charged pions produced in p-nucleon interactions. After emission the pions may interact with the nucleons of the target and disappear from the beam, as assumed, or decay in flight giving rise to muons. For the former case we assume $\sigma_{\pi p} = \sigma_{\pi n} \approx \frac{2}{3}\sigma_{pp} = 20\ mb$ at high energies. For the latter case the lifetime of the charged pions in the laboratory is γ_π/λ, where λ is the decay constant and $\gamma_\pi = (1-\beta_\pi^2)^{-\frac{1}{2}}$, $\beta_\pi c$ being the mean velocity of the pions. Then the change of N_π per unit interval of x is

$$\begin{aligned}\frac{dN_\pi}{dx} &= 8(\sigma_{pp}N_p + \sigma_{pn}N_n)\phi(x) - \left(\frac{\lambda}{\gamma_\pi\beta_\pi c} + \sigma_{\pi p}N_p + \sigma_{\pi n}N_n\right)N_\pi \\ &= 16\sigma_{pp}N_p\varphi_0 e^{-2\sigma_{pp}N_p x} - \left(\frac{\lambda}{\gamma_\pi\beta_\pi c} + 2\sigma_{\pi p}N_p\right)N_\pi \\ &= 8B\phi_0 e^{-Bx} - B'N_\pi\,,\end{aligned}$$

where $B = 2\sigma_{pp}N_p$, $B' = 2\sigma_{\pi p}N_p + \lambda'$, $\lambda' = \frac{\lambda}{\gamma_\pi\beta_\pi c}$. The solution of the differential equation is

$$N_\pi = \frac{8B}{B'-B}(e^{-Bx} - e^{-B'x})\phi_0\,.$$

Hence the number of charged pions which decay in the target per unit interval of x is

$$\frac{dN_\pi(\lambda)}{dx} = \frac{\lambda}{\gamma_\pi\beta_\pi c}N_\pi(\lambda) = \frac{8B\lambda'}{B'-B}(e^{-Bx} - e^{-B'x})\phi_0\,.$$

Integration from $x=0$ to $x=\infty$ gives

$$N_\pi(\lambda) = \frac{8B\lambda'}{B'-B}\left(\frac{1}{B} - \frac{1}{B'}\right)\phi_0 = \frac{8\lambda'\phi_0}{B'}\,.$$

The branching ratio for $\pi \to \mu\nu \approx 100\%$, so that $N_{\mu\pi} \approx N_\pi(\lambda)$. This means that the energy spectrum of muons is also flat (though in actual fact high-energy muons are more likely than low-energy ones), making the comparison with $N_{\mu c}$ much simpler.

Take for example $E_\mu \sim 100$ GeV. Then $E_\pi \gtrsim 100$ GeV, $\beta_\pi \approx 1$, $\gamma_\pi \gtrsim 714$, and so

$$\lambda' = \frac{\lambda}{\gamma_\pi\beta_\pi c} = \frac{1}{2.6\times10^{-8}\times714\times3\times10^{10}} = 1.8\times10^{-6}\ \text{cm}^{-1}\,.$$

As

$$\sigma_{\pi p} N_p = 20 \times 10^{-27} \times 2.34 \times 10^{24} = 4.7 \times 10^{-2}\ \mathrm{cm}^{-1} \gg \lambda' .$$

$$N_\pi(\lambda) \simeq \frac{8\lambda'\phi_0}{2\sigma_{\pi p} N_p} = \frac{8 \times 1.8 \times 10^{-6}\phi_0}{2 \times 4.7 \times 10^{-2}} = 1.5 \times 10^{-4}\phi_0 .$$

Hence

$$\frac{N_{\mu c}}{N_{\mu\pi}} = \frac{5.95 \times 10^{-7}}{1.5 \times 10^{-4}} = 4 \times 10^{-3} .$$

4105

An experiment has been proposed to study narrow hadronic states that might be produced in $p\bar{p}$ annihilation. Antiprotons stored inside a ring would collide with a gas jet of hydrogen injected into the ring perpendicular to the beam. By adjusting the momentum of the beam in the storage ring the dependence of the $p\bar{p}$ cross section on the center-of-mass energy can be studied. A resonance would show up as a peak in the cross section to some final state.

Assume that there exists a hadron that can be produced in this channel with a mass of 3 GeV and a total width of 100 keV.

(a) What beam momentum should be used to produce this state?

(b) One of the motivations for this experiment is to search for charmonium states (bound states of a charmed quark-antiquark pair) that cannot be seen directly as resonance in e^+e^- annihilation. Which spin-parity states of charmonium would you expect to be visible as resonance in this experiment but not in e^+e^- annihilation?

Rough answers are O.K. for the remaining questions.

(c) Assume that the beam momentum spread is 1%. If the state shows up as a peak in the total cross section vs. center-of-mass energy plot, how wide would it appear to be?

(d) How wide would the state appear to be if oxygen were used in the gas jet instead of hydrogen?

(e) Assume that the jet is of thickness 1 mm and of density 10^{-9} gram/cm^3, and that there are 10^{11} circulating antiprotons in a ring of diameter 100 m. How many events per second occur per cm^2 of cross section? (In other words, what is the luminosity?) How many $p\bar{p}$ annihilations would occur per second?

(f) If the state (whose total width is 100 keV) has a branching ratio of 10% to $p\bar{p}$, what is the value of the total cross section expected at the peak (assuming the target jet is hydrogen)?

(*Princeton*)

Solution:

(a) In the laboratory frame, the velocity of the gas jet is very small and the target protons can be considered as approximately at rest. At threshold, the invariant mass squared is

$$S = (E_p + m_p)^2 - p_p^2 = M^2 \,.$$

With $E_p^2 = m_p^2 + p_p^2$, $M = 3$ GeV, this gives

$$E_p = \frac{M^2 - 2m_p^2}{2m_p} = \frac{3^2 - 2 \times 0.938^2}{2 \times 0.938} = 3.86 \text{ GeV}\,,$$

and hence the threshold momentum

$$p_p = \sqrt{E_p^2 - m_p^2} = 3.74 \text{ GeV}/c\,.$$

(b) In e^+e^- collisions, as e^+e^- annihilation gives rise to a virtual photon whose J^P is 1^-, only the resonance state of $J^P = 1^-$ can be produced. But for $p\bar{p}$ reaction, many states can be created, e.g.,

$$\begin{aligned} \text{for } S = 0,\ l = 0,\ & J^p = 0^+\,; \\ S = 1,\ l = 0,\ & J^P = 1^-\,; \\ S = 1,\ l = 1,\ & J^P = 0^-,\ 1^-,\ 2^-\,; \\ l = 2,\ & J^P = 1^+,\ 2^+,\ 3^+\,. \end{aligned}$$

Therefore, besides the state $J^P = 1^-$, other resonance states with $J^P = 0^-, 0^+, 1^+, 2^-, 2^+, 3^+ \cdots$ can also be produced in $p\bar{p}$ annihilation.

(c) At threshold

$$p^2 = E^2 - m^2 = \frac{M^4}{4m_p^2} - M^2\,.$$

Differentiating we have

$$2p\Delta p = M^3 \frac{\Delta M}{m_p^2} - 2M\Delta M \,,$$

or

$$\Delta M = \frac{2m_p^2 p^2 \dfrac{\Delta p}{p}}{M^3 - 2m_p^2 M} \,.$$

With $\frac{\Delta p}{p} = 0.01$, this gives

$$\begin{aligned}\Delta M &= \frac{2 \times 0.938^2 \times 3.74^2 \times 0.01}{3^3 - 2 \times 0.938^2 \times 3} \\ &= 1.13 \times 10^{-2} \text{ GeV} \,.\end{aligned}$$

Since $\Delta M \gg \Gamma$, the observed linewidth is due mainly to Δp.

(d) If oxygen was used instead of hydrogen, the proton that interacts with the incident antiproton is inside the oxygen nucleus and has a certain kinetic energy known as the Fermi energy. The Fermi motion can be in any direction, thus broadening the resonance peak. For a proton in an oxygen nucleus, the maximum Fermi momentum is

$$\begin{aligned}p_F &\approx \frac{\hbar}{R_0}\left(\frac{9\pi Z}{4A}\right)^{1/3} = \frac{\hbar c}{R_0 c}\left(\frac{9\pi}{8}\right)^{1/3} \\ &= \frac{197 \times 10^{-13}}{1.4 \times 10^{-13} c}\left(\frac{9\pi}{8}\right)^{1/3} = 210 \text{ MeV}/c \,,\end{aligned}$$

where the nuclear radius is taken to be $R = R_0 A^{1/3}$, which is much larger than the spread of momentum ($\Delta p = 3.47$ MeV/c). This would make the resonance peak much too wide for observation. Hence it is not practicable to use oxygen instead of hydrogen in the experiment.

(e) The antiprotons have velocity βc, where

$$\beta = \frac{p_p}{E_p} = \frac{3.74}{3.86} = 0.97 \,.$$

The number of times they circulate the ring per second is

$$\frac{\beta c}{100\pi}$$

and so the number of encounters of $p\bar{p}$ per second per cm^2 of cross section is

$$B = 10^{11} \times \frac{0.97 \times 3 \times 10^{10}}{100 \times 10^2 \times \pi} \times 0.1 \times 10^{-9} \times 6.023 \times 10^{23}$$

$$= 5.6 \times 10^{30} \text{ cm}^{-2} \text{ s}^{-1} .$$

Suppose $\sigma_{p\bar{p}} \approx 30$ mb. The number of $p\bar{p}$ annihilation expected per second is

$$\sigma_{p\bar{p}} B = 30 \times 10^{-27} \times 5.6 \times 10^{30}$$

$$= 1.68 \times 10^5 \text{ s}^{-1} .$$

(f) The cross section at the resonance peak is given by

$$\sigma = \frac{(2J+1)}{(2J_p+1)(2J_{\bar{p}}+1)} \frac{\pi \lambda^2 \Gamma_{p\bar{p}} \Gamma}{(E-M)^2 + \frac{\Gamma^2}{4}} .$$

At resonance $E = M$. Suppose the spin of the resonance state is zero. Then as $J_p = J_{\bar{p}} = \frac{1}{2}$,

$$\sigma(J=0) = \pi \lambda^2 \frac{\Gamma_{p\bar{p}}}{\Gamma} .$$

With $\lambda = \frac{\hbar}{p_p}$, $\frac{\Gamma_{p\bar{p}}}{\Gamma} = 0.1$, we have

$$\sigma = \pi \times \left(\frac{\hbar c}{p_p c}\right)^2 \times 0.1 = \pi \times \left(\frac{197 \times 10^{-13}}{3740}\right)^2 \times 0.1$$

$$= 8.7 \times 10^{-30} \text{ cm}^2$$

$$= 8.7\ \mu b .$$

4. ERROR ESTIMATION AND STATISTICS (4106–4118)

4106

Number of significant figures to which α is known: 4, 8, 12, 20

(*Columbia*)

Solution:

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.03604(11)},$$

the answer is 8.

4107

If the average number of counts in a second from a radioactive source is 4, what is the probability of recording 8 counts in one second?

(*Columbia*)

Solution:

The count rate follows Poisson distribution. Hence

$$P(8) = 4^8 e^{-4}/8! = 0.03\,.$$

4108

Suppose it is intended to measure the uniformity of the thickness of an aluminium filter placed perpendicular to an X-ray beam. Using an X-ray detector and source, equal-exposure transmission measurements are taken at various points on the filter. The number of counts, N, obtained in 6 trials were 1.00×10^4, 1.02×10^4, 1.04×10^4, 1.06×10^4, 1.08×10^4, 1.1×10^4.

(a) Calculate the standard deviation associated with these measurements.

(b) What do the measurements tell you about the uniformity of the filter?

(c) Given that $N = N_0 e^{-\mu t}$, how is a fractional uncertainty in N related to a fractional uncertainty in t?

(d) For a given number of counts at the detector, would the fractional error in t be larger for small t or large t?

(*Wisconsin*)

Solution:

(a) The mean of the counts is $\bar{N} = \frac{1}{n}\sum_1^n N_i = 1.05 \times 10^4$. The standard deviation of a reading is

$$\sigma = \sqrt{\frac{1}{n-1}\sum_i^n (N_i - \bar{N})^2} = 0.037 \times 10^4 \,.$$

(b) If the Al foil is uniform, the counts taken at various locations should follow the Poisson distribution with a standard deviation

$$\Delta N = \sqrt{N} \approx \sqrt{1.05 \times 10^4} = 0.01 \times 10^4 \,.$$

Since the standard deviation of the readings (0.037×10^4) is more than three times ΔN, the foil cannot be considered uniform.

(c) Write $N = N_0 e^{-\mu t}$ as $\ln N = \ln N_0 - \mu t$. As $\frac{dN}{N} = -\mu dt$, we have

$$\frac{\Delta N}{N} = \mu \Delta t \,,$$

or

$$\frac{\Delta N}{N} = \mu t \left(\frac{\Delta t}{t}\right) \,.$$

(d) As

$$\frac{\Delta t}{t} = \frac{1}{\mu t}\frac{\Delta N}{N} \,,$$

for a given set of data, the smaller t is, the larger is the fractional error of t.

4109

You have measured 25 events $J \to e^+e^-$ by reconstructing the mass of the e^+e^- pairs. The apparatus measures with $\Delta m/m = 1\%$ accuracy. The average mass is 3.100 GeV. What is the error?

(a) 6.2 MeV
(b) 1.6 MeV
(c) 44 MeV.

(CCT)

Solution:

As Δm is the error in a single measurement, the standard deviation is

$$\sigma = \sqrt{\frac{1}{25-1}\sum (\Delta m)^2} = \sqrt{\frac{25}{24}}\Delta m \approx \Delta m = 31 \text{ MeV} \,.$$

Hence the standard deviation of the mean, or the standard error, is

$$e = \frac{\sigma}{\sqrt{25}} = 6.2 \text{ MeV}\,.$$

Thus the answer is (a).

4110

In a cloud chamber filled with air at atmospheric pressure, 5 MeV alpha particles make tracks about 4 cm long. Approximately how many such tracks must one observe to have a good chance of finding one with a distinct sharp bend resulting from a nuclear encounter?

(*Columbia*)

Solution:

As the nuclear radius is $R = r_0 A^{1/3}$, where $r_0 = 1.2$ fm and $A = 14.7$ for the average air nucleus, the nuclear cross section σ is

$$\sigma \approx \pi R^2 = \pi \times (1.2 \times 10^{-13} \times 14.7^{1/3})^2 = 2.7 \times 10^{-25} \text{ cm}^2\,.$$

The number density of nuclei in the cloud chamber is

$$n = \frac{\rho N_A}{A} = \frac{0.001293 \times 6.023 \times 10^{23}}{14.7} = 5.3 \times 10^{19} \text{ cm}^{-3}\,.$$

Hence the mean free path is $\lambda = \frac{1}{n\sigma} = 7.0 \times 10^4$ cm.

Therefore, for a good chance of finding a large-angle scattering one should observe about $7 \times 10^4/4 \approx 20000$ events.

4111

The positive muon (μ^+) decays into a positron and two neutrinos,

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu\,,$$

with a mean lifetime of about 2 microseconds. Consider muons at rest polarized along the z-axis of a coordinate system with a degree of polarization

P, and confine our observations to the highest-energy positrons from muon decays. These positrons are emitted with an angular distribution

$$I(\cos\theta)d\Omega = (1 + P\cos\theta)\frac{d\Omega}{4\pi}\,,$$

where θ is the angle between the positron direction and z-axis, and $d\Omega$ is the solid angle element into which the positron is emitted.

(a) Assume $P = +1$. What is the probability that for the first six positrons observed, three are in the forward hemisphere ($\cos\theta > 0$) and three are in the backward hemisphere ($\cos\theta < 0$)?

(b) Assume that P is in the neighborhood of 1, but not accurately known. You wish to determine P by comparing the numbers of observed forward (N_f) and backward (N_b) decay positrons. How many muon decays, N ($N = N_f + N_b$), must you observe to determine P to an accuracy of $\pm 1\%$?

(*CUSPEA*)

Solution:

(a) As $d\Omega = 2\pi d\cos\theta$, the probability of a forward decay is

$$P_f = 2\pi\int_0^1 \frac{(1 + P\cos\theta)d\cos\theta}{4\pi} = \frac{1}{2}\left(1 + \frac{P}{2}\right)\,,$$

and the probability of a backward decay is

$$P_b = 2\pi\int_{-1}^0 \frac{(1 + P\cos\theta)d\cos\theta}{4\pi} = \frac{1}{2}\left(1 - \frac{P}{2}\right)\,.$$

If we observe N positrons, the probability of finding N_f positrons in the forward and N_b positrons in the backward hemisphere, where $N = N_f + N_b$, is according to binomial distribution

$$W = \frac{N!}{N_f!N_b!}(P_f)^{N_f}(P_b)^{N_b}\,.$$

For $P = 1$, the above give $P_f = 3/4$, $P_b = 1/4$. With $N = 6$, $N_f = N_b = 3$, the probability is

$$W = \frac{6!}{3!3!}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^3 = 0.132\,.$$

(b) P can be determined from

$$P_f - P_b = \frac{P}{2},$$

i.e.,

$$P = 2(P_f - P_b) = 2(2P_f - 1),$$

where $P_f = \frac{N_f}{N}$, $P_b = \frac{N_b}{N}$ are to be obtained from experimental observations. With N events observed, the standard deviation of N_f is

$$\Delta N_f = \sqrt{NP_f(1 - P_f)}.$$

So

$$\Delta P_f = \frac{\Delta N_f}{N} = \sqrt{\frac{P_f(1 - P_f)}{N}}.$$

Hence

$$\Delta P = 4\Delta P_f = 4\sqrt{\frac{P_f(1 - P_f)}{N}},$$

or

$$N = \frac{16P_f(1 - P_f)}{(\Delta P)^2}.$$

With $P \approx 1$, $\Delta P \approx 0.01P = 0.01$, $P_f \approx \frac{3}{4}$, N must be at least

$$N_{\min} = \frac{16 \cdot \frac{3}{4} \cdot \frac{1}{4}}{(10^{-2})^2} = 30000.$$

4112

Carbon dioxide in the atmosphere contains a nearly steady-state concentration of radioactive ^{14}C which is continually produced by secondary cosmic rays interacting with atmosphere nitrogen. When a living organism dies, its carbon contains ^{14}C at the atmospheric concentration, but as time passes the fraction of ^{14}C decreases due to radioactive decay. This is the basis for the technique of radiocarbon dating.

In the following you may assume that the atmospheric value for the ratio $^{14}C/^{12}C$ is 10^{-12} and that the half life for the ^{14}C β-decay is 5730 years.

(a) It is desired to use radiocarbon dating to determine the age of a carbon sample. How many grams of a sample are needed to measure the age to a precision of ± 50 years (standard deviation of 50 years)? Assume that the sample is actually 5000 years old, that the radioactivity is counted for one hour with a 100% efficient detector, and that there is no background.

(b) Repeat part (a), but now assume that there is a background counting rate in the detector (due to radioactivity in the detector itself, cosmic rays, etc.) whose average value is accurately known to be 4000 counts per hour.

(*CUSPEA*)

Solution:

(a) ^{14}C decays according to

$$N = N_0 e^{-\lambda t} .$$

Its counting rate is thus

$$A = -dN/dt = \lambda N_0 e^{-\lambda t} = \lambda N .$$

Differentiating we have

$$\frac{dA}{dt} = -\lambda^2 N_0 e^{-\lambda t} = -\lambda A ,$$

and hence

$$\Delta A/A = \lambda \Delta t .$$

The decay constant is $\lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5730} = 1.21 \times 10^{-4}\ \mathrm{yr}^{-1}$. As the counting rate per hour A follows the Poisson distribution,

$$\frac{\Delta A}{A} = \frac{\sqrt{A}}{A} = \frac{1}{\sqrt{A}} = 50\lambda ,$$

giving

$$A = \left(\frac{1}{50 \times 1.21 \times 10^{-4}} \right)^2 = 2.73 \times 10^4\ h^{-1} .$$

Let the mass of carbon required be x grams. Then

$$A = \frac{\lambda x N_A}{12} \times 10^{-12} \times \exp(-5000\lambda)\,,$$

giving

$$\begin{aligned} x &= \frac{12A \times 10^{12} \times e^{5000\lambda}}{N_A \lambda} \\ &= \frac{12 \times 2.73 \times 10^4 \times 365 \times 24}{6.023 \times 10^{23} \times 1.21 \times 10^{-4}} \times 10^{12} \times e^{5000 \times 1.21 \times 10^{-4}} \\ &= 72.1\ g\,. \end{aligned}$$

(b) With a background counting rate of A_B, the total rate is $A + A_B \pm \sqrt{A + A_B}$. As A_B is known precisely, $\Delta A_B = 0$. Hence

$$\Delta(A + A_B) = \Delta A = \sqrt{A + A_B}\,,$$

or

$$\frac{\Delta A}{A} = \sqrt{\frac{1}{A} + \frac{A_B}{A^2}}\,.$$

With $\frac{\Delta A}{A} = \lambda \Delta t = C$, say, the above becomes

$$C^2 A^2 - A - A_B = 0\,.$$

Hence

$$\begin{aligned} A &= \frac{1}{2C^2}(1 + \sqrt{1 + 4C^2 A_B}) \\ &= \frac{1}{2 \times (1.21 \times 10^{-4} \times 50)^2}\left[1 + \sqrt{1 + 4 \times (1.21 \times 10^{-4} \times 50)^2 \times 4000}\right] \\ &= 3.09 \times 10^4\ h^{-1}\,, \end{aligned}$$

and the mass of sample required is

$$m = \frac{3.09 \times 10^4}{2.73 \times 10^4} \times 72.1 = 81.6\ g\,.$$

4113

A Čerenkov counter produces 20 photons/particle. The cathode of the photomultiplier converts photons with 10% efficiency into photoelectrons. One photoelectron in the multiplier will produce a signal. Of 1000 particles, how many passes unobserved?

(a) none
(b) 3
(c) 130

(CCT)

Solution:

Consider the passage of a particle. It produces 20 photons, each of which has a probability $P = 0.1$ of producing a photoelectron and so being detected. The particle will not be observed if none of the 20 photons produces photoelectrons. The probability of this happening is

$$\begin{aligned} P(0) &= \frac{20!}{0!20!}(0.1)^0(0.9)^{20} \\ &= 0.122\,. \end{aligned}$$

Hence of the 1000 incident particles, it is expected that 122 will not be observed. Thus the answer is (c).

4114

A radioactive source is emitting two types of radiation A and B, and is observed by means of a counter that can distinguish between the two. In a given interval, 1000 counts of type A and 2000 of type B are observed. Assuming the processes producing A and B are independent, what is the statistical error on the measured ratio $r = \frac{N_A}{N_B}$?

(Wisconsin)

Solution:

Writing the equation as

$$\ln r = \ln N_a - \ln N_B$$

and differentiating both sides, we have

$$\frac{dr}{r} = \frac{dN_A}{N_A} - \frac{dN_B}{N_B}\,.$$

As N_A and N_B are independent of each other,

$$\left(\frac{\Delta r}{r}\right)^2 = \left(\frac{\Delta N_A}{N_A}\right)^2 + \left(\frac{\Delta N_B}{N_B}\right)^2\,.$$

Now N_A and N_B follow Poisson's distribution. So $\Delta N_A = \sqrt{N_A}$, $\Delta N_B = \sqrt{N_B}$, and hence

$$\frac{\Delta r}{r} = \sqrt{\frac{1}{N_A} + \frac{1}{N_B}} = \sqrt{\frac{1}{1000} + \frac{1}{2000}} = 3.9\%\,,$$

or

$$\Delta r = \frac{1000}{2000} \times 0.039 = 0.020\,,$$

which is the standard error of the ratio r.

4115

A sample of β-radioactive isotope is studied with the aid of a scintillation counter which is able to detect the decay electrons and accurately determine the individual decay times.

(a) Let τ denote the mean decay lifetime. The sample contains a large number N of atoms, and the detection probability per decay is ε. Calculate the average counting rate in the scintillator. You may assume τ to be much longer than any period of time over which measurements are made. In a measurement of τ, 10,000 counts are collected over a period of precisely one hour. The detection efficiency of the scintillator is independently determined to be 0.4 and N is determined to be 10^{23}. What is the measured value of τ? What is the statistical error in this determination of τ (standard deviation)?

(b) Let $P(t)dt$ be the probability that two successive counts in the scintillator are at t and $t + dt$. Compute $P(t)$ in terms of t, ε, N, τ.

(*CUSPEA*)

Solution:

(a) As $\tau \gg$ time of measurement, N can be considered constant and the average counting rate is

$$R = \frac{\varepsilon N}{\tau}\,.$$

Hence

$$\tau = \frac{\varepsilon N}{R} = \frac{0.4 \times 10^{23}}{10^4} = 0.4 \times 10^{19}\ h = 4.6 \times 10^{14}\ \text{yr}\,.$$

The statistical error of R is $\sqrt{R}$ as counting rates follow Poisson's distribution. Then

$$\frac{\Delta\tau}{\tau} = \frac{\Delta R}{R} = \frac{1}{\sqrt{R}} = \frac{1}{\sqrt{10^4}} = 0.01\,,$$

or

$$\Delta\tau = 4.6 \times 10^{12}\ \text{yr}\,.$$

(b) The first count occurs at time t. This means that no count occurs in the time interval 0 to t. As the expected mean number of counts for the interval is $m = Rt$, the probability of this happening is

$$\frac{e^{-m} m^0}{0!} = e^{-m} = e^{-Rt}\,.$$

The second count can be taken to occur in the time dt. As $m' = Rdt$, the probability is

$$\frac{e^{-m'} m'}{1!} = e^{-Rdt} Rdt \approx Rdt\,.$$

Hence

$$P(t)dt = Re^{-Rt}dt$$

or

$$P(t) = \frac{\varepsilon N}{\tau} \exp\left(-\frac{\varepsilon N t}{\tau}\right)\,.$$

4116

A minimum-ionizing charged particle traverses about 1 mg/cm^2 of gas. The energy loss shows fluctuations. The full width at half maximum (fwhm) divided by the most probable energy loss (the relative fwhm) is about

(a) 100%.

(b) 10%.
(c) 1%.

(CCT)

Solution:

The energy loss of a minimum-ionizing charged particle when it transverses about 1 mg/cm^2 of gas is about 2 keV. The average ionization energy for a gas molecule is about 30 eV. The relative fwhm is then about

$$\eta = 2.354 \left(\frac{\varepsilon F}{E_0}\right)^{1/2} = 2.354 \left(\frac{30F}{2000}\right)^{1/2} = 29(F)^{1/2}\%\,,$$

where $F < 1$ is the Fanor factor. The answer is (b).

4117

An X-ray of energy ε is absorbed in a proportional counter and produces in the mean $\bar{n}$ ion pairs. The rms fluctuation σ of this number is given by

(a) $\sqrt{\bar{n}}$.
(b) $\sqrt{F\bar{n}}$, with $F < 1$.
(c) $\pi \ln \bar{n}$.

(CCT)

Solution:

The answer is (b).

4118

A 1 cm thick scintillator produces 1 visible photon/100 eV of energy loss. It is connected by a light guide with 10% transmission to a photomultiplier (10% efficient) converting the light into photoelectrons. What is the variation σ in pulse height for the proton in Problem 4065?

(a) 21.2%
(b) 7.7%
(c) 2.8%

(CCT)

Solution:

The energy loss of the proton in the scintillator is $\Delta E = 1.8$ MeV $= 1.8 \times 10^6$ eV per cm path length. Then the mean number of photons produced in the scintillator is

$$\bar{n} = \frac{1.8 \times 10^6}{100} = 1.8 \times 10^4 \,.$$

With a transmission efficiency of 10% and a conversion efficiency of 10%, the number of observed photoelectrons is $\bar{N} = 1.8 \times 10^4 \times 0.1 \times 0.1 = 180$. The percentage standard deviation is therefore

$$\sigma = \frac{\sqrt{\bar{N}}}{\bar{N}} = \frac{1}{\sqrt{\bar{N}}} = \frac{1}{\sqrt{180}} = 7.5\% \,.$$

Hence the answer is (b).

5. PARTICLE BEAMS AND ACCELERATORS (4119–4131)

4119

(a) Discuss the basic principles of operation of cyclotrons, synchrocyclotrons and synchrotrons. What are the essential differences among them? What limits the maximum energy obtainable from each?

(b) Discuss the basic principles of operation of linear accelerators such as the one at SLAC. What are the advantages and disadvantages of linear accelerators as compared to circular types?

(c) For what reason have colliding-beam accelerators ("intersecting storage ring") been constructed in recent years? What are their advantages and disadvantages as compared to conventional fixed-target accelerators?

(*Columbia*)

Solution:

(a) The cyclotron basically consists of two hollow, semicircular metal boxes — the dees — separated along their straight edges by a small gap. An ion source at the center of the gap injects particles of charge Ze into one of the dees. A uniform and constant magnetic field is applied perpendicular to the dees, causing the particles to orbit in circular paths of radius r given by

$$\frac{mv^2}{r} = ZevB\,.$$

The particles are accelerated each time it crosses the gap by a radio-frequency electric field applied across the gap of angular frequency $\omega_r = \frac{ZeB}{m} = w_p$, the angular frequency of revolution of the particles. As w_p is independent of the orbit radius r, the particles always take the same time to cover the distance between two successive crossings, arriving at the gap each time at the proper phase to be accelerated.

An upper limit in the energy attainable in the cyclotron is imposed by the relativistic increase of mass accompanying increase of energy, which causes them to reach the accelerating gap progressively later, to finally fall out of resonance with the rf field and be no longer accelerated.

In the synchrocyclotron, this basic limitation on the maximum energy attainable is overcome by varying the frequency of the rf field, reducing it step by step in keeping with the decrease of w_p due to relativistic mass change. While in principle there is no limit to the attainable energy in the synchrocyclotron, the magnet required to provide the magnetic field, which covers the entire area of the orbits, has a weight proportional to the third power of the maximum energy. The weight and cost of the magnet in practice limit the maximum attainable energy.

In the synchrotron the particles are kept in an almost circular orbit of a fixed radius between the poles of a magnet annular in shape, which provides a magnetic field increasing in step with the momentum of the particles. Accelerating fields are provided by one or more rf stations at points on the magnetic ring, the rf frequency increasing in step with the increasing velocity of the particles. The highest energy attainable is limited by the radiation loss of the particles, which on account of the centripetal acceleration radiate electromagnetic radiation at a rate proportional to the fourth power of energy.

Comparing the three types of accelerators, we note that for the cyclotron both the magnitude of the magnetic field and the frequency of the rf field are constant. For the synchrocyclotron, the magnitude of the magnetic field is constant while the frequency of the rf field changes synchronously with the particle energy, and the orbit of a particle is still a spiral. For the synchrotron, both the magnitude of the magnetic field and the frequency of the rf field are to be tuned to keep the particles in a fixed orbit.

(b) In a linear accelerator such as SLAC, charged particles travel in a straight line along the axis of a cylindrical pipe that acts as a waveguide, which has a rf electromagnetic field pattern with an axial electric field component to provide the accelerating force. Compared to ring-shaped accelerators, the linear accelerator has many advantages. As the particles move along a straight line they are easily injected and do not need extraction. In addition, as there is no centripetal acceleration radiation loss is neglectable. It is especially suited for acclerating electrons to very high energies. Another advantage is its flexibility in construction. It can be lengthened in steps. Its downside is its great length and high cost as compared to a ring accelerator of equal energy.

(c) In the collision of a particle of mass m and energy E with a stationary particle of equal mass the effective energy for interaction is $\sqrt{2mE}$, while for a head-on collision between colliding beams of energy E the effective energy is $2E$. It is clear then that the higher the energy E, the smaller will be the fraction of the total energy available for interaction in the former case. As it is difficult and costly to increase the energy attainable by an accelerated particle, many colliding-beam machines have been constructed in recent years. However, because of their lower beam intensity and particle density, the luminosity of colliding-beam machines is much lower than that of stationary-target machines.

4120

(a) Briefly describe the cyclotron and the synchrotron, contrasting them. Tell why one does not use:

(b) cyclotrons to accelerate protons to 2 GeV?

(c) synchrotrons to accelerate electrons to 30 GeV?

(*Columbia*)

Solution:

(a) In the cyclotron, a charged particle is kept in nearly circular orbits by a uniform magnetic field and accelerated by a radio frequency electric field which reverses phase each time the particle crosses the gap between the two D-shape electrodes. However, as its mass increases accompanying the increase of energy, the cyclotron radius of the particle $r = \frac{mv}{eB}$ increases, and the cyclotron frequency $w = \frac{eB}{m}$ decreases. Hence the relative phase of particle revolution relative to the rf field changes constantly. In the synchrotron the bending magnetic field is not constant, but changes with the energy of the particle, causing it to move in a fixed orbit. Particles are accelerated by resonant high frequency field at one or several points on the orbit, continually increasing the energy (cf. **Problem 4119**(a)).

(b) In the cyclotron, as the energy of the particle increases, the radius of its orbit also increases and the accelerating phase of the particle changes constantly. When the kinetic energy of the particle is near to its rest energy, the accumulated phase difference can be quite large, and finally the particle will fall in the decelerating range of the radio frequency field when it reaches the gap between the D-shaped electrorodes. Then the energy of the particle cannot be further increased. The rest mass of the proton is ~ 1 GeV. To accelerate it to 2 GeV with a cyclotron, we have to accomplish this before it falls in the decelerating range. The voltage required is too high in practice.

(c) In the synchrotron the phase-shift probem does not arise, so the particle can be accelerated to a much higher energy. However at high energies, on account of the large centripetal acceleration the particle will radiate electromagnetic radiation, the synchrotron radiation, and lose energy, making the increase in energy per cycle negative. The higher the energy and the smaller the rest mass of a particle, the more intense is the synchrotron radiation. Obviously, when the loss of energy by synchrotron radiation is equal to the energy acquired from the accelerating field in the same interval of time, further acceleration is not possible. As the rest mass of electron is only 0.511 MeV, to accelerate an electron to 30 GeV, we must increase the radius of the accelerator, or the accelerating voltage, or both to very large values, which are difficult and costly in practice. For example, a 45 GeV e^+e^- colliding-beams facility available at CERN has a circumference of 27 km.

4121

Radius of 500 GeV accelerator at Batavia is 10^2, 10^3, 10^4, 10^5 m.

(*Columbia*)

Solution:

In a magnetic field of induction B, the radius of the orbit of a proton is

$$R = \frac{m\gamma\beta c}{eB} = \frac{m\gamma\beta c^2}{eBc} .$$

For a proton of energy 500 GeV, $\beta \approx 1$, $m\gamma c^2 = 500$ GeV. Hence, if $B \sim 1\,T$ as is generally the case,

$$R = \frac{500 \times 10^9 \times 1.6 \times 10^{-19}}{1.6 \times 10^{-19} \times 1 \times 3 \times 10^8} = \frac{5}{3} \times 10^3 \text{ m} .$$

Thus the answer is 10^3 m.

4122

In a modern proton synchrotron (particle accelerator) the stability of the protons near the equilibrium orbit is provided by the fact that the magnetic field B required to keep the particles in the equilibrium orbit (of radius R) is nonuniform, independent of θ, and can often be parametrized as

$$B_z = B_0 \left(\frac{R}{r}\right)^n ,$$

where z is the coordinate perpendicular to the plane of the equilibrium orbit (i.e., the vertical direction) with $z = 0$ at the equilibrium orbit, B_0 is a constant field required to keep the particles in the equilibrium orbit of radius R, r is the actual radial position of the particle (i.e. $\rho = r - R$ is the horizontal displacement of the particle from the equilibrium orbit), and n is some constant. Derive the frequencies of the vertical and horizontal betatron oscillations for a particular value of n. For what range of values of n will the particles undergo stable oscillations in both the vertical and horizontal directions around the equilibrium orbit?

(*Columbia*)

Solution:

Using the cylindrical coordinates (r, θ, z), we can write the equation of motion of the particle

$$\frac{d}{dt}(m\mathbf{v}) = e\mathbf{E} + e\mathbf{v} \times \mathbf{B}$$

as

$$\frac{d}{dt}\left(m\frac{dr}{dt}\right) - mr\left(\frac{d\theta}{dt}\right)^2 = eE_r + eB_z r\frac{d\theta}{dt} - eB_\theta\frac{dz}{dt}\,,$$

$$\frac{1}{r}\frac{d}{dt}\left(mr^2\frac{d\theta}{dt}\right) = eE_\theta + eB_r\frac{dz}{dt} - eB_z\frac{dr}{dt}\,,$$

$$\frac{d}{dt}\left(m\frac{dz}{dt}\right) = eE_z + eB_\theta\frac{dz}{dt} - eB_r r\frac{d\theta}{dt}\,.$$

On the orbit of the particle the electric field is zero and the magnetic field is independent of θ, i.e.,

$$E_\theta = E_r = E_z = B_\theta = 0\,.$$

The first and third of the above equations reduce to

$$\frac{d}{dt}\left(m\frac{dr}{dt}\right) - mr\left(\frac{d\theta}{dt}\right)^2 = eB_z r\frac{d\theta}{dt}\,, \tag{1}$$

$$\frac{d}{dt}\left(m\frac{dz}{dt}\right) = -eB_r r\frac{d\theta}{dt}\,. \tag{2}$$

On the equilibrium orbit, $r = R$ and Eq. (1) becomes

$$mR\left(\frac{d\theta}{dt}\right)^2 = -eB_0 R\left(\frac{d\theta}{dt}\right)\,,$$

or

$$\frac{d\theta}{dt} = -\frac{eB_0}{m} = -\omega_0\,, \qquad \text{say}\,.$$

ω_0 is the angular velocity of revolution of the particle, i.e., its angular frequency.

The actual orbit fluctuates about the equilibrium orbit. Writing $r = R + \rho$, where ρ is a first order small quantity, same as z, and retaining only first order small quantities we have, near the equilibrium orbit,

$$B_z(r,z) \approx B_0\left(\frac{B}{r}\right)^n \approx B_0\left(1+\frac{\rho}{R}\right)^{-n} \approx B_0\left(1-\frac{n\rho}{R}\right)\,.$$

As

$$\nabla \times \mathbf{B} = 0\,,$$

considering the θ component of the curl we have

$$\frac{\partial B_r}{\partial z} = \frac{\partial B_z}{\partial r}\,,$$

from which follows

$$B_r(\rho, z) \approx B_r(\rho, 0) + \left(\frac{\partial B_r}{\partial z}\right)_{z=0} z = 0 + \left(\frac{\partial B_z}{\partial r}\right)_{z=0} z$$

$$= -\left(\frac{nB_z}{r}\right)_{z=0} z = -\frac{nB_0}{R} z\,,$$

since $B = B_z = B_0$ for $\rho = 0$.

To consider oscillations about R, let $r = R + \rho$. On using the approximate expressions for B_z and B_r and keeping only first order small quantities, Eqs. (1) and (2) reduce to

$$\frac{d^2\rho}{dt^2} = -\omega_0^2(1-n)\rho\,,$$

$$\frac{d^2 z}{dt^2} = -\omega_0^2 n z\,.$$

Hence if $n < 1$, there will be stable oscillations in the radial direction with frequency

$$\omega_\rho = \sqrt{1-n}\,\omega_0 = \frac{\sqrt{1-n}\,eB_0}{m}\,.$$

If $n > 0$, there will be stable oscillations in the vertical direction with frequency

$$\omega_z = \sqrt{n}\,\omega_0 = \frac{\sqrt{n}\,eB_0}{m}\,.$$

Thus only when the condition $0 < n < 1$ is satisfied can the particle undergo stable oscillations about the equilibrium orbit in both the horizontal and vertical directions.

4123

A modern accelerator produces two counter-rotating proton beams which collide head-on. Each beam has 30 GeV protons.

(a) What is the total energy of collision in the center-of-mass system?

(b) What would be the required energy of a conventional proton accelerator in which protons strike a stationary hydrogen target to give the same center-of-mass energy?

(c) If the proton-proton collision rate in this new machine is 10^4/sec, estimate the required vacuum in the system such that the collision rate of protons with residual gas be of this same order of magnitude in 5 m of pipe. Take 1000 m as the accelerator circumference, $\sigma_{p-\text{air}} = 10^{-25}$ cm^2, and the area of the beam as 1 mm^2.

(*Columbia*)

Solution:

(a) The center-of-mass system is defined as the frame in which the total momentum of the colliding particles is zero. Thus for the colliding beams, the center-of-mass system (c.m.s.) is identical with the laboratory system. It follows that the total energy of collision in c.m.s. is $2E_p = 2 \times 30 = 60$ GeV.

(b) If a conventional accelerator and a stationary target are used, the invariant mass squared is

$$\begin{aligned} S &= (E_p + m_p)^2 - p_p^2 \\ &= E_p^2 - p_p^2 + 2E_p m_p + m_p^2 \\ &= 2E_p m_p + 2m_p^2 \,. \end{aligned}$$

In c.m.s.

$$S = (60)^2 = 3600 \text{ GeV}^2 \,.$$

As S is invariant under Lorentz transformation we have

$$2E_p m_p + 2m_p^2 = 3600 \,,$$

or

$$E_p = \frac{1800 - 0.938^2}{0.938} = 1918 \text{ GeV} \,,$$

as the required incident proton energy.

(c) Let n, s be the number density of protons and cross sectional area of each colliding beam, L be the circumference of the beam orbit, l be the length of the pipe of residual air with density ρ. The number of collisions per unit time in the colliding beam machine is

$$r = \frac{N}{\Delta t} = \frac{N_p N_p \sigma_{pp}}{\left(\frac{L}{c}\right)} = \frac{(nsL)^2 c\sigma_{pp}}{L} = n^2 s^2 Lc\sigma_{pp} \,.$$

The number of collisions per unit time in the air pipe is

$$r' = \frac{N'}{\Delta t'} = \frac{N_p N_a \sigma_{pa}}{\left(\frac{L+l}{c}\right)} \approx (nsL)\left(\frac{\rho s_l N_A}{A}\right)\frac{c\sigma_{pa}}{L} \,,$$

where A is the molecular weight of air and N_A is Avodagro's number.

If $r' = r$, the above give

$$\rho = \frac{A}{N_A}\frac{L}{l}\frac{\sigma_{pp}}{\sigma_{pa}} n \,.$$

As $r = 10^4$ s^{-1}, we have

$$n = \left(\frac{10^4}{s^2 Lc\sigma_{pp}}\right)^{\frac{1}{2}} = \left(\frac{10^4}{10^{-4} \times 10^5 \times 3 \times 10^{10} \times 3 \times 10^{-26}}\right)^{\frac{1}{2}}$$
$$= 1.8 \times 10^9 \text{ cm}^{-3} \,,$$

taking $\sigma_{pp} = 30$ mb$= 3 \times 10^{-26}$ cm^2. Hence

$$\rho = \frac{29}{6.02 \times 10^{23}}\left(\frac{1000}{5}\right)\left(\frac{3 \times 10^{-26}}{10^{-25}}\right) \times 1.8 \times 10^9$$
$$= 5.3 \times 10^{-12} \text{ g cm}^{-3} \,,$$

The pressure P of the residual air is given by

$$\frac{5.3 \times 10^{-12}}{1.3 \times 10^{-3}} = \frac{P}{1} \,,$$

i.e., $P = 4 \times 10^{-9}$ atm.

4124

Suppose you are able to produce a beam of protons of energy E in the laboratory (where $E \gg m_p c^2$) and that you have your choice of making a single-beam machine in which this beam strikes a stationary target, or dividing the beam into two parts (each of energy E) to make a colliding-beam machine.

(a) Discuss the relative merits of these two alternatives from the following points of view:

(1) the threshold energy for particle production,

(2) the event rate,

(3) the angular distribution of particles produced and its consequences for detector design.

(b) Consider the production of the Z^0 particle ($Mc^2 \approx 90$ GeV) at threshold in a $p+p$ collision. What is the energy E required for each type of machine?

(c) At beam energy E, what is the maximum energy of a π meson produced in each machine?

(*CUSPEA*)

Solution:

(a) (i) The invariant mass squared is the same before and after reaction:

$$S = -(p_1 + p_2)^2 = -(p_1' + p_2' + p)^2 \,,$$

where p_1, p_2 are the initial 4-momenta of the two protons, p_1', p_2' are their final 4-momenta, respectively, and p is the 4-momentum of the new particle of rest mass M.

Then for one proton being stationary initially, $p_1 = (\mathbf{p}_1, E_p)$, $p_2 = (0, m_p)$ and so

$$\begin{aligned} S &= (E_1 + m_p)^2 - \mathbf{p}_1^2 \\ &= (E_1^2 - \mathbf{p}_1^2) + m_p^2 + 2E_1 m_p \\ &= 2m_p^2 + 2E_1 m_p \,. \end{aligned}$$

At threshold the final state has

$$p_1' = p_2' = (0, m_p), \qquad p = (0, M) \qquad \text{and so}$$

$$S' = (2m_p + M)^2 \,.$$

For the reaction to proceed we require

$$S \geq S' ,$$

or

$$E_1 \geq m_p + 2M + \frac{M^2}{2m_p} .$$

For colliding beams, we have $p_1 = (\mathbf{p}_c, E_c)$, $p_2 = (-\mathbf{p}_c, E_c)$ and the invariant mass squared

$$S'' = (2E_c)^2 - (\mathbf{p}_c - \mathbf{p}_c)^2 = 4E_c^2 .$$

The requirement $S'' \geq S'$ then gives

$$E_c \geq m_p + \frac{M}{2} .$$

Note that $E_1 \gg E_c$ if $M \gg m_p$. Hence colliding-beam machine is able to produce the same new particle with particles of much lower energies.

(ii) Since a fixed target provides an abundance of target protons which exist in its nuclei, the event rate is much higher for a stationary-target machine.

(iii) With a stationary-target machine, most of the final particles are collimated in the forward direction of the beam in the laboratory. Detection of new particles must deal with this highly directional geometry of particle distribution and may have difficulty in separating them from the background of beam particles.

With a colliding-beam machine the produced particles will be more uniformly distributed in the laboratory since the total momentum of the colliding system is zero. In this case the detectors must cover most of the 4π solid angle.

(b) Using the formulas in (a) (i) we find, with $m_p = 0.94$ GeV, $M = 90$ GeV, the threshold energies for a fixed-target machine,

$$E_1 = m_p + 2M + \frac{M^2}{2m_p} = 0.94 + 2 \times 90 + \frac{90^2}{2 \times 0.94} = 4489 \text{ GeV} ,$$

and for a colliding-beam machine,

$$E_c = m_P + \frac{M}{2} = 0.94 + \frac{90}{2} = 45.94 \text{ GeV} .$$

(c) *Colliding-beam machine*

Let p_1, p_2 be the momenta of the protons in the final state, and p_π be the momentum of the pion produced. Conservation of energy requires

$$2E = \sqrt{m_p^2 + p_1^2} + \sqrt{m_p^2 + p_2^2} + \sqrt{m_\pi^2 + p_\pi^2} \, .$$

Conservation of momentum requires

$$\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_\pi = 0 \, ,$$

or

$$p_\pi^2 = p_1^2 + p_2^2 + 2p_1 p_2 \cos\alpha \, .$$

This means that for p_π to have the maximum value, the angle α between $\mathbf{p}_1$, $\mathbf{p}_2$ must be zero, since $\frac{\partial p_\pi}{\partial \alpha} = \frac{p_1 p_2}{p_\pi} \sin\alpha$. Thus at maximum p_π, the three final particles must move in the same line. Write

$$\mathbf{p}_2 = -(\mathbf{p}_1 + \mathbf{p}_\pi) \, .$$

The energy equation becomes

$$2E = \sqrt{m_p^2 + (p_\pi + p_1)^2} + \sqrt{m_\pi^2 + p_\pi^2} + \sqrt{m_p^2 + p_1^2} \, .$$

Differentiating we have

$$0 = \frac{(p_\pi + p_1)d(p_\pi + p_1)}{\sqrt{m_p^2 + (p_\pi + p_1)^2}} + \frac{p_\pi dp_\pi}{\sqrt{m_\pi^2 + p_\pi^2}} + \frac{p_1 dp_1}{\sqrt{m_p^2 + p_1^2}} \, .$$

Letting $dp_\pi / dp_1 = 0$, we find

$$-\frac{p_1}{\sqrt{m_p^2 + p_1^2}} = \frac{(p_\pi + p_1)}{\sqrt{m_p^2 + (p_\pi + p_1)^2}} \, .$$

Hence

$$p_\pi = -2p_1 \, , \quad p_2 = p_1 \, .$$

Thus at maximum E_π,

$$2E = 2\sqrt{m_p^2 + p_1^2} + \sqrt{m_\pi^2 + (2p_1)^2} \, ,$$

or

$$4E^2 - 4EE_{\pi\,\max} + m_\pi^2 + 4p_1^2 = 4m_p^2 + 4p_1^2 \,,$$

giving the maximum pion energy

$$E_{\pi\,\max} = \frac{4E^2 + m_\pi^2 - 4m_p^2}{4E} \approx E \,,$$

as $E \gg m_p$.

Stationary-target machine: When E_π is maximum, the two final-state protons are stationary and the pion takes away the momentum of the incident proton. Thus

$$E_\pi + 2m_p = E + m_p \,,$$

or

$$E_\pi = E - m_p \approx E \qquad \text{as} \qquad E \gg m_p \,.$$

4125

An electron (mass m, charge e) moves in a plane perpendicular to a uniform magnetic field. If energy loss by radiation is neglected the orbit is a circle of some radius R. Let E be the total electron energy, allowing for relativistic kinematics so that $E \gg mc^2$.

(a) Explain the needed field strength B analytically in terms of the above parameters. Compute B numerically, in gauss, for the case where $R = 30$ meters, $E = 2.5 \times 10^9$ electron volts. For this part of the problem you will have to recall some universal constants.

(b) Actually, the electron radiates electromagnetic energy because it is being accelerated by the B field. However, suppose that the energy loss per revolution ΔE is small compared to E. Explain the ratio $\Delta E/E$ analytically in terms of the parameters. Then evaluate this ratio numerically for the particular value of R given above.

(*CUSPEA*)

Solution:

(a) Let $\mathbf{v}$ be the velocity of the electron. Its momentum is $\mathbf{p} = m\gamma\mathbf{v}$, where $\gamma = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}}$. Newton's second law of motion gives

$$\frac{d\mathbf{p}}{dt} = m\gamma\frac{d\mathbf{v}}{dt} = e\mathbf{v} \times \mathbf{B} \,,$$

as $|\mathbf{v}|$ and hence γ are constant since $\mathbf{v} \perp \mathbf{B}$, or

$$\left|\frac{d\mathbf{v}}{dt}\right| = \frac{evB}{m\gamma}\,.$$

As

$$\left|\frac{d\mathbf{v}}{dt}\right| = \frac{v^2}{R}\,,$$

where R is the radius of curvature of the electron orbit,

$$B = \frac{m\gamma v}{eR}\,,$$

or

$$\begin{aligned} B &= \frac{pc}{eRc} = \frac{\sqrt{E^2 - m^2c^4}}{eRc} \approx \frac{E}{eRc} \\ &= \frac{2.5 \times 10^9 \times 1.6 \times 10^{-19}}{1.6 \times 10^{-19} \times 30 \times 3 \times 10^8} = 2.8 \times 10^{-1}\ \mathrm{T} \\ &= 2.8 \times 10^3\ \mathrm{Gs}\,. \end{aligned}$$

(b) The power radiated by the electron is

$$\begin{aligned} P &= \frac{e^2}{6\pi\varepsilon_0 c^3}\gamma^6\left[\dot{v}^2 - \left(\frac{\mathbf{v} \times \dot{\mathbf{v}}}{c}\right)^2\right] \\ &= \frac{e^2\dot{v}^2}{6\pi\varepsilon_0 c^3}\gamma^4 \\ &= \frac{e^2 v^4}{6\pi\varepsilon_0 c^3}\frac{\gamma^4}{R^2}\,, \end{aligned}$$

as $\dot{\mathbf{v}} \perp \mathbf{v}$. The energy loss per revolution is then

$$\begin{aligned} \Delta E &= \frac{2\pi RP}{v} = \frac{4\pi}{3}\left(\frac{e^2}{4\pi\varepsilon_0 mc^2 R}\right)(\gamma\beta)^3\gamma mc^2 \\ &= \frac{4\pi}{3}\left(\frac{r_0}{R}\right)(\gamma\beta)^3 E = \frac{4\pi}{3}\left(\frac{r_0}{R}\right)(\gamma^2 - 1)^{\frac{3}{2}}E\,, \end{aligned}$$

where $r_0 = 2.8 \times 10^{-15}$ m is the classical radius of electron and $\beta = \frac{v}{c}$. With $\gamma = \frac{2.5\times10^9}{0.51\times10^6} = 4.9 \times 10^3$,

$$\begin{aligned} \frac{\Delta E}{E} &\approx \frac{4\pi}{3} \times \frac{2.8 \times 10^{-15}}{30} \times (4.9 \times 10^3)^3 \\ &= 4.6 \times 10^{-5}\,. \end{aligned}$$

The results can also be obtained using the relevant formulas as follows.

(a)

$$p(\mathrm{GeV}/c) = 0.3B(\mathrm{T})R(\mathrm{m})$$

giving

$$B = \frac{p}{0.3R} = \frac{2.5}{0.3 \times 30} \approx 0.28 \text{ T}.$$

(b)

$$\Delta E(\mathrm{keV}) \approx 88E(\mathrm{GeV})^4/R(\mathrm{m})$$

giving

$$\begin{aligned}\frac{\Delta E}{E} &= 88E^3 \times 10^{-6}/R \\ &= 88 \times 2.5^3 \times 10^{-6}/30 \\ &= 4.6 \times 10^{-5}.\end{aligned}$$

4126

Draw a simple, functional cyclotron magnet in cross section, showing pole pieces of 1 m diameter, yoke and windings. Estimate the number of ampere-turns required for the coils if the spacing between the pole pieces is 10 cm and the required field is 2 $T(= 20$ kgauss). $\mu_0 = 4\pi \times 10^{-7}$ J/A$^2\cdot$ m.

(*Columbia*)

Solution:

Figure 4.14 shows the cross section of a cyclotron magnet. The magnetic flux ϕ crossing the gap betwen the pole pieces is

$$\phi = \frac{NI}{R},$$

where

$$R = \frac{d}{\mu_0 S},$$

d being the gap spacing and S the area of each pole piece, is the reluctance. By definition the magnetic induction is $B = \frac{\phi}{S}$. Thus

$$NI = \phi R = \frac{Bd}{\mu_0} = \frac{2 \times 10 \times 10^{-2}}{4\pi \times 10^{-7}} = 1.59 \times 10^5 \text{ A-turns}.$$

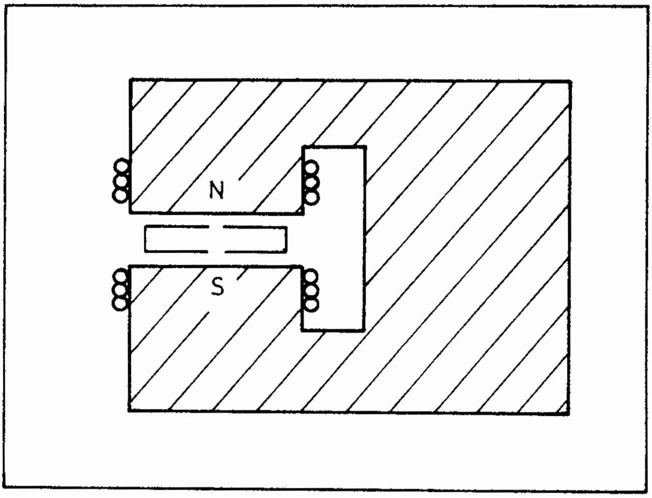

Fig. 4.14

4127

In general, when one produces a beam of ions or electrons, the space charge within the beam causes a potential difference between the axis and the surface of the beam. A 10-mA beam of 50-keV protons ($v = 3 \times 10^6$ m/sec) travels along the axis of an evacuated beam pipe. The beam has a circular cross section of 1-cm diameter. Calculate the potential difference between the axis and the surface of the beam, assuming that the current density is uniform over the beam diameter.

(*Wisconsin*)

Solution:

The beam carries a current

$$I = \int \mathbf{j} \cdot d\mathbf{S} = \int_0^R j 2\pi r dr = \pi R^2 j = \pi R^2 \rho v \,,$$

where j and ρ are the current and charge densities respectively. Thus

$$\rho = \frac{I}{\pi R^2 v} \,.$$

At a distance r from the axis, Gauss' flux theorem

$$2\pi rlE = \pi r^2 l\rho/\varepsilon_0$$

gives the electric field intensity as

$$E = \frac{r\rho}{2\varepsilon_0} = \frac{r}{2\pi\varepsilon_0}\frac{I}{vR^2}\,.$$

As $E = -\frac{dV}{dr}$, the potential difference is

$$\Delta V = \int_0^R E(r)dr = \frac{I}{2\pi\varepsilon_0 vR^2}\int rdr = \frac{I}{4\pi\varepsilon_0 v}$$

$$= \frac{9\times 10^9\times 10\times 10^{-3}}{3\times 10^6} = 30\ \text{V}\,.$$

4128

Cosmic ray flux at ground level is 1/year, 1/min, 1/ms, 1/μs, cm^{-2} sterad^{-1}.

(*Columbia*)

Solution:

The answer is $1/(\text{min}\cdot\text{cm}^2\cdot\text{sterad})$. At ground level, the total cosmic ray flux is $1.1\times 10^2/(\text{m}^2\cdot\text{s}\cdot\text{sterad})$, which consists of a hard component of $0.8\times 10^2/(\text{m}^2\cdot\text{s}\cdot\text{sterad})$ and a soft component of $0.3\times 10^2/(\text{m}^2\cdot\text{s}\cdot\text{sterad})$.

4129

Particle flux in a giant accelerator is 10^4, 10^8, 10^{13}, 10^{18} per pulse.

(*Columbia*)

Solution:

A typical particle flux in a proton accelerator is 10^{13}/pulse.

4130

Which particle emits the most synchrotron radiation light when bent in a magnetic field?

(a) Proton.
(b) Muon.
(c) Electron.

(CCT)

Solution:

The synchrotron radiation is emitted when the trajectory of a charged particle is bent by a magnetic field. **Problem 4125** gives the energy loss per revolution as

$$\Delta E = \left(\frac{4\pi}{3}\right)\left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{1}{R}\beta^3\gamma^4 \,,$$

where R, the radius of curvature of the trajectory, is given by

$$R = \frac{m\gamma\beta c}{eB} \,.$$

Thus for particles of the same charge and γ, $\Delta E \propto m^{-1}$. Hence the answer is (c).

4131

The magnetic bending radius of a 400 GeV particle in 15 kgauss is:

(a) 8.8 km.
(b) 97 m.
(c) 880 m.

(CCT)

Solution:

The formula

$$p(\mathrm{GeV}/c) = 0.3B(T)R(m)$$

gives

$$R = \frac{p}{0.3B} = \frac{400}{0.3 \times 1.5} = 880 \text{ m} \,.$$

Or, from first principles one can obtain

$$R = \frac{m\gamma\beta c}{eB} \approx \frac{m\gamma c^2}{eBc} = \frac{400 \times 10^9 \times 1.6 \times 10^{-19}}{1.6 \times 10^{-19} \times 1.5 \times 3 \times 10^8} = 880 \text{ m} \,,$$

as $B = 15$ kGs $= 1.5$ T.

Hence the answer is (c).

This Page Intentionally Left Blank

INDEX TO PROBLEMS