Unit – I

Mathematical Logic

INTRODUCTION

Proposition: A **proposition** or **statement** is a declarative sentence which is either true or false but not both. The truth or falsity of a proposition is called its **truth-value.**

These two values 'true' and 'false' are denoted by the symbols T and F respectively. Sometimes these are also denoted by the symbols 1 and 0 respectively.

Example 1: Consider the following sentences:

1. Delhi is the capital of India.
2. Kolkata is a country.
3. 5 is a prime number.
4. 2 + 3 = 4.

These are propositions (or statements) because they are either true of false.

Next consider the following sentences:

5. How beautiful are you?
6. Wish you a happy new year
7. $x + y = z$
8. Take one book.

These are not propositions as they are not declarative in nature, that is, they do not declare a definite truth value T or F.

Propositional Calculus is also known as **statement calculus.** It is the branch of mathematics that is used to describe a logical system or structure. A logical system consists of (1) a universe of propositions, (2) truth tables (as axioms) for the logical operators and (3) definitions that explain equivalence and implication of propositions.

Connectives

The words or phrases or symbols which are used to make a proposition by two or more propositions are called **logical connectives** or **simply connectives.** There are five basic connectives called negation, conjunction, disjunction, conditional and biconditional.

Negation

The **negation** of a statement is generally formed by writing the word 'not' at a proper place in the statement (proposition) or by prefixing the statement with the phrase 'It is not the case that'. If p denotes a statement then the negation of p is written as p and read as 'not p'. If the truth value of p is T then the truth value of p is F. Also if the truth value of p is F then the truth value of p is T.

Table 1. Truth table for negation

p	¬p
T	F
F	T

Example 2: Consider the statement p: Kolkata is a city. Then $\neg p$: Kolkata is not a city.

Although the two statements 'Kolkata is not a city' and 'It is not the case that Kolkata is a city' are not identical, we have translated both of them by p. The reason is that both these statements have the same meaning.

Conjunction

The **conjunction** of two statements (or propositions) p and q is the statement $p \wedge q$ which is read as 'p and q'. The statement $p \wedge q$ has the truth value T whenever both p and q have the truth value T. Otherwise it has truth value F.

Table 2. Truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3: Consider the following statements p : It is raining today.

q : There are 10 chairs in the room.

Then $p \wedge q$: It is raining today and there are 10 chairs in the room.

Note: Usually, in our everyday language the conjunction 'and' is used between two statements which have some kind of relation. Thus a statement 'It is raining today and 1 + 1 = 2' sounds odd, but in logic it is a perfectly acceptable statement formed from the statements 'It is raining today' and '1 + 1 = 2'.

Example 4: Translate the following statement:

'Jack and Jill went up the hill' into symbolic form using conjunction.

Solution: Let p : Jack went up the hill, q : Jill went up the hill.

Then the given statement can be written in symbolic form as $p \wedge q$.

Disjunction

The **disjunction** of two statements p and q is the statement $p \vee q$ which is read as 'p or q'. The statement $p \vee q$ has the truth value F only when both p and q have the truth value F. Otherwise it has truth value T.

Table 3: Truth table for disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 5: Consider the following statements p : I shall go to the game.

q : I shall watch the game on television.

Then $p \vee q$: I shall go to the game or watch the game on television.

Conditional proposition

If p and q are any two statements (or propositions) then the statement $p \rightarrow q$ which is read as, 'If p, then q' is called a **conditional statement** (or **proposition**) or **implication** and the connective is the **conditional connective.**

The conditional is defined by the following table:

Table 4. Truth table for conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

In this conditional statement, p is called the **hypothesis** or **premise** or **antecedent** and q is called the **consequence** or **conclusion.**

To understand better, this connective can be looked as a conditional promise. If the promise is violated (broken), the conditional (implication) is false. Otherwise it is true. For this reason, the only circumstances under which the conditional $p \rightarrow q$ is false is when p is true and q is false.

Example 6: *Translate the following statement:*

'The crop will be destroyed if there is a flood' into symbolic form using conditional connective.

Solution: Let c : the crop will be destroyed; f : there is a flood.
Let us rewrite the given statement as
'If there is a flood, then the crop will be destroyed'. So, the symbolic form of the given statement is $f \rightarrow c$.

Example 7: Let p and q denote the statements:
p : You drive over 70 km per hour.
q : You get a speeding ticket.

Write the following statements into symbolic forms.

(i) You will get a speeding ticket if you drive over 70 km per hour.

(ii) Driving over 70 km per hour is sufficient for getting a speeding ticket.

(iii) If you do not drive over 70 km per hour then you will not get a speeding ticket.
(iv) Whenever you get a speeding ticket, you drive over 70 km per hour.
Solution: (i) $p \rightarrow q$ (ii) $p \rightarrow q$ (iii) $p \rightarrow q$ (iv) $q \rightarrow p$.

Notes: 1. In ordinary language, it is customary to assume some kind of relationship between the antecedent and the consequent in using the conditional. But in logic, the antecedent and the consequent in a conditional statement are not required to refer to the same subject matter. For example, the statement 'If I get sufficient money then I shall purchase a high-speed computer' sounds reasonable. On the other hand, a statement such as 'If I purchase a computer then this pen is red' does not make sense in our conventional language. But according to the definition of conditional, this proposition is perfectly acceptable and has a truth-value which depends on the truth-values of the component statements.

2. Some of the alternative terminologies used to express $p \to q$ (if p, then q) are the following: (*i*) p implies q

(*ii*) p only if q ('If p, then q' formulation emphasizes the antecedent, whereas 'p only if q' formulation emphasizes the consequent. The difference is only stylistic.)

(*iii*) q if p, or q when p.

(*iv*) q follows from p, or q whenever p.

(*v*) p is sufficient for q, or a sufficient condition for q is p. (*vi*) q is necessary for p, or a necessary condition for p is q. (*vii*) q is consequence of p.

Converse, Inverse and Contrapositive

If $P \to Q$ is a conditional statement, then

(1). $Q \to P$ is called its *converse*

(2). $\neg P \to \neg Q$ is called its *inverse*

(3). $\neg Q \to \neg P$ is called its *contrapositive*.

Truth table for $Q \to P$ (converse of $P \to Q$)

P	Q	$Q \to P$
T	T	T
T	F	T
F	T	F
F	F	T

Truth table for $\neg P \to \neg Q$ (inverse of $P \to Q$)

P	Q	$\neg P$	$\neg Q$	$\neg P \to \neg Q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Truth table for $\neg Q \to \neg P$ (contrapositive of $P \to Q$)

P	Q	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
T	T	F	F	T
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T

Example: Consider the statement

P : It rains.

Q: The crop will grow.

The implication $P \rightarrow Q$ states that

R: If it rains then the crop will grow.

The converse of the implication $P \rightarrow Q$, namely $Q \rightarrow P$ sates that S: If the crop will grow then there has been rain.

The inverse of the implication $P \rightarrow Q$, namely $\neg P \rightarrow \neg Q$ sates that

U: If it does not rain then the crop will not grow.

The contraposition of the implication $P \rightarrow Q$, namely $\neg Q \rightarrow \neg P$ states that T : If the crop do not grow then there has been no rain.

Example 9: *Construct the truth table for* $(p \rightarrow q) \wedge (q \rightarrow p)$

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$
T	*T*	*T*	*T*	*T*
T	*F*	*F*	*T*	*F*
F	*T*	*T*	*F*	*F*
F	*F*	*T*	*T*	*T*

Biconditional proposition

If p and q are any two statements (propositions), then the statement $p\leftrightarrow q$ which is read as 'p if and only if q' and abbreviated as 'p iff q' is called a **biconditional statement** and the connective is the **biconditional connective.**

The truth table of p↔q is given by the following table:

Table 6. Truth table for biconditional

p	q	p↔q
T	*T*	*T*
T	*F*	*F*
F	*T*	*F*
F	*F*	*T*

It may be noted that $p\ q$ is true only when both p and q are true or when both p and q are false. Observe that $p\ q$ is true when both the conditionals $p \rightarrow q$ and $q \rightarrow p$ are true, *i.e.*, the truth-values of $(p \rightarrow q) \wedge (q \rightarrow p)$, given in Ex. 9, are identical to the truth-values of $p\ q$ defined here.

Note: The notation $p \leftrightarrow q$ is also used instead of p↔q.

TAUTOLOGY AND CONTRADICTION

Tautology: A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a **universally valid formula** or a **logical truth** or a **tautology.**

Contradiction: A statement formula which is false regardless of the truth values of the statements which replace the variables in it is said to be a **contradiction.**

Contingency: A statement formula which is neither a tautology nor a contradiction is known as a **contingency.**

Substitution Instance

A formula A is called a substitution instance of another formula B if A can be obtained form B by substituting formulas for some variables of B, with the condition that the same formula is substituted for the same variable each time it occurs.

Example: Let $B : P \rightarrow (J \wedge P)$.

Substitute $R \leftrightarrow S$ for P in B, we get

$$(i): (R \leftrightarrow S) \rightarrow (J \wedge (R \leftrightarrow S))$$

Then A is a substitution instance of B.

Note that $(R \leftrightarrow S) \rightarrow (J \wedge P)$ is not a substitution instance of B because the variables

P in $J \wedge P$ was not replaced by $R \leftrightarrow S$.

Equivalence of Formulas

Two formulas A and B are said to equivalent to each other if and only if $A \leftrightarrow B$ is a tautology.

If $A \leftrightarrow B$ is a tautology, we write $A \Leftrightarrow B$ which is read as A is equivalent to B.

Note : 1. $\Leftrightarrow$ is only symbol, but not connective.

2. $A \leftrightarrow B$ is a tautology if and only if truth tables of A and B are the same.
3. Equivalence relation is symmetric and transitive.

Method I. Truth Table Method: One method to determine whether any two statement formulas are equivalent is to construct their truth tables.

Example: Prove $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

Solution:

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$(P \vee Q) \Leftrightarrow \neg(\neg P \wedge \neg Q)$
T	T	T	F	F	F	T	T
T	F	T	F	T	F	T	T
F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T

As $P \vee Q \quad \neg(\neg P \wedge \neg Q)$ is a tautology, then $P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$.

Example: Prove $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.

Solution:

P	Q	$P \rightarrow Q$	$\neg P$	$\neg P \vee Q$	$(P \rightarrow Q) \quad (\neg P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(P \rightarrow Q) \quad (\neg P \vee Q)$ is a tautology then $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$.

Equivalence Formulas:

1. Idempotent laws:

   (a) $P \vee P \Leftrightarrow P$ (b) $P \wedge P \Leftrightarrow P$

2. Associative laws:

   (a) $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$ (b) $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3. Commutative laws:

   (a) $P \vee Q \Leftrightarrow Q \vee P$ (b) $P \wedge Q \Leftrightarrow Q \wedge P$

4. Distributive laws:

   $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

5. Identity laws:

   (a) (i) $P \vee F \Leftrightarrow P$ (ii) $P \vee T \Leftrightarrow T$

   (b) (i) $P \wedge T \Leftrightarrow P$ (ii) $P \wedge F \Leftrightarrow F$

6. Component laws:

   (a) (i) $P \vee \neg P \Leftrightarrow T$ (ii) $P \wedge \neg P \Leftrightarrow F$

   (b) (i) $\neg\neg P \Leftrightarrow P$ (ii) $\neg T \Leftrightarrow F$, $\neg F \Leftrightarrow T$

7. Absorption laws:

   (a) $P \vee (P \wedge Q) \Leftrightarrow P$ (b) $P \wedge (P \vee Q) \Leftrightarrow P$

8. Demorgan's laws:

   (a) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$ (b) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Method II. Replacement Process: Consider a formula $A : P \to (Q \to R)$. The formula $Q \to R$ is a part of the formula A. If we replace $Q \to R$ by an equivalent formula $\neg Q \vee R$ in A, we get another formula $B : P \to (\neg Q \vee R)$. One can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A as the replacement process.

Example: Prove that $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \to R$.(May. 2010)

Solution: $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R)$ $[\because Q \to R \Leftrightarrow \neg Q \vee R]$

$\Leftrightarrow \neg P \vee (\neg Q \vee R)$ $[\because P \to Q \Leftrightarrow \neg P \vee Q]$

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ [by Associative laws]

$\Leftrightarrow \neg(P \wedge Q) \vee R$ [by De Morgan's laws]

$\Leftrightarrow (P \wedge Q) \to R [\because P \to Q \Leftrightarrow \neg P \vee Q]$.

Example: Prove that $(P \to Q) \wedge (R \to Q) \Leftrightarrow (P \vee R) \to Q$.

Solution: $(P \to Q) \wedge (R \to Q) \Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$

$\Leftrightarrow (\neg P \wedge \neg R) \vee Q \Leftrightarrow$

$\neg(P \vee R) \vee Q \Leftrightarrow P \vee$

$R \to Q$

Example: Prove that $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$.

Solution: $P \to (Q \to P) \Leftrightarrow \neg P \vee (Q \to P)$

$$\Leftrightarrow \neg P \vee (\neg Q \vee P)$$

$$\Leftrightarrow (\neg P \vee P) \vee \neg Q$$

$$\Leftrightarrow T \vee \neg Q$$

$$\Leftrightarrow T$$

and

$$\neg P \to (P \to Q) \Leftrightarrow \neg(\neg P) \vee (P \to Q)$$

$$\Leftrightarrow P \vee (\neg P \vee Q) \Leftrightarrow$$

$$(P \vee \neg P) \vee Q \Leftrightarrow T$$

$$\vee Q$$

$$\Leftrightarrow T$$

So, $P \to (Q \to P) \Leftrightarrow \neg P \to (P \to Q)$.

***Example: Prove that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow R$. (Nov. 2009)

Solution:

$$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ [Associative and Distributive laws]

$\Leftrightarrow (\neg(P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R)$ [De Morgan's laws]

$\Leftrightarrow (\neg(P \vee Q) \vee (P \vee Q)) \wedge R$ [Distributive laws]

$\Leftrightarrow T \wedge R$ [$\because \neg P \vee P \Leftrightarrow T$]

$\Leftrightarrow R$

**Example: Show $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is tautology.

Solution: By De Morgan's laws, we have

$$\neg P \wedge \neg Q \Leftrightarrow \neg(P \vee Q)$$

$$\neg P \vee \neg R \Leftrightarrow \neg(P \wedge R)$$

Therefore

$$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Leftrightarrow \neg(P \vee Q) \vee \neg(P \wedge R)$$

$$\Leftrightarrow \neg((P \vee Q) \wedge (P \vee R))$$

Also

$$\neg(\neg P \wedge (\neg Q \vee \neg R)) \Leftrightarrow \neg(\neg P \wedge \neg(Q \wedge R))$$

$$\Leftrightarrow P \vee (Q \wedge R)$$

$$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Hence $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \Leftrightarrow (P \vee Q) \wedge (P \vee Q) \wedge (P \vee R)$

$$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Thus $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

$\Leftrightarrow [(P \vee Q) \wedge (P \vee R)] \vee \neg[(P \vee Q) \wedge (P \vee R)]$

$\Leftrightarrow T$

Hence the given formula is a tautology.

Example: Show that $(P \wedge Q) \rightarrow (P \vee Q)$ is a tautology. (Nov. 2009)

Solution: $(P \wedge Q) \rightarrow (P \vee Q) \Leftrightarrow \neg(P \wedge Q) \vee (P \vee Q)$ [$\because P \rightarrow Q \Leftrightarrow \neg P \vee Q$]

$\Leftrightarrow (\neg P \vee \neg Q) \vee (P \vee Q)$ [by De Morgan's laws]

$\Leftrightarrow (\neg P \vee P) \vee (\neg Q \vee Q)$ [by Associative laws and commutative laws]

$\Leftrightarrow (T \vee T)$[by negation laws]

$\Leftrightarrow T$

Hence, the result.

Example: Write the negation of the following statements.

(a). Jan will take a job in industry or go to graduate school.
(b). James will bicycle or run tomorrow.
(c). If the processor is fast then the printer is slow.

Solution: (a). Let P : Jan will take a job in industry.
Q: Jan will go to graduate school.

The given statement can be written in the symbolic as $P \vee Q$.

The negation of $P \vee Q$ is given by $\neg(P \vee Q)$.

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

$\neg P \wedge \neg Q$: Jan will not take a job in industry and he will not go to graduate school.

(b). Let P : James will bicycle.
Q: James will run tomorrow.

The given statement can be written in the symbolic as $P \vee Q$.

The negation of $P \vee Q$ is given by $\neg(P \vee Q)$.

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

$\neg P \wedge \neg Q$: James will not bicycle and he will not run tomorrow.

(c). Let P : The processor is fast.
Q: The printer is slow.

The given statement can be written in the symbolic as $P \rightarrow Q$.

The negation of $P \rightarrow Q$ is given by $\neg(P \rightarrow Q)$.

$$\neg(P \rightarrow Q) \Leftrightarrow \neg(\neg P \vee Q) \Leftrightarrow P \wedge \neg Q.$$

$P \wedge \neg Q$: The processor is fast and the printer is fast.

Example: Use Demorgans laws to write the negation of each statement.

(a). I want a car and worth a cycle.
(b). My cat stays outside or it makes a mess.
(c). I've fallen and I can't get up.
(d). You study or you don't get a good grade.

Solution: (a). I don't want a car or not worth a cycle.
(b). My cat not stays outside and it does not make a mess.

(c). I have not fallen or I can get up.

(d). You can not study and you get a good grade.

Exercises: 1. Write the negation of the following statements.

(a). If it is raining, then the game is canceled.

(b). If he studies then he will pass the examination.

Are $(p \to q) \to r$ and $p \to (q \to r)$ logically equivalent? Justify your answer by using the rules of logic to simply both expressions and also by using truth tables. Solution: $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent because

Method I: Consider

$$(p \to q) \to r \Leftrightarrow (\neg p \lor q) \to r$$

$$\Leftrightarrow \neg(\neg p \lor q) \lor r \Leftrightarrow$$

$$(p \land \neg q) \lor r$$

$$\Leftrightarrow (p \land r) \lor (\neg q \land r)$$

and

$$p \to (q \to r) \Leftrightarrow p \to (\neg q \lor r)$$

$$\Leftrightarrow \neg p \lor (\neg q \lor r) \Leftrightarrow$$

$$\neg p \lor \neg q \lor r.$$

Method II: (Truth Table Method)

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

Here the truth values (columns) of $(p \to q) \to r$ and $p \to (q \to r)$ are not identical.

Consider the statement: "If you study hard, then you will excel". Write its converse, contra positive and logical negation in logic.

Duality Law

Two formulas A and A^* are said to be *duals* of each other if either one can be obtained from the other by replacing $\land$ by $\lor$ and $\lor$ by $\land$. The connectives $\lor$ and $\land$ are called *duals* of each other. If the formula A contains the special variable T or F, then A^*, its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Example: Write the dual of the following formulas:

(i). $(P \vee Q) \wedge R$ (ii). $(P \wedge Q) \vee T$ (iii). $(P \wedge Q) \vee (P \vee \neg(Q \wedge \neg S))$

Solution: The duals of the formulas may be written as

(i). $(P \wedge Q) \vee R$ (ii). $(P \vee Q) \wedge F$ (iii). $(P \vee Q) \wedge (P \wedge \neg(Q \vee \neg S))$

Result 1: The negation of the formula is equivalent to its dual in which every variable is replaced by its negation.
We can prove

$$\neg A(P_1, P_2, ..., P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, ..., \neg P_n)$$

Example: Prove that (a). $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$

(b). $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$

Solution: (a). $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q))$ $[\because P \to Q \Leftrightarrow \neg P \vee Q]$

$$
\begin{aligned}
&\Leftrightarrow (P \wedge Q) \vee (\neg P \vee Q) \\
&\Leftrightarrow (P \wedge Q) \vee \neg P \vee Q \\
&\Leftrightarrow ((P \wedge Q) \vee \neg P)) \vee Q \\
&\Leftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q \\
&\Leftrightarrow (T \wedge (Q \vee \neg P)) \vee Q \\
&\Leftrightarrow (Q \vee \neg P) \vee Q \\
&\Leftrightarrow Q \vee \neg P \\
&\Leftrightarrow \neg P \vee Q
\end{aligned}
$$

(b). From (a)

$$(P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \Leftrightarrow \neg P \vee Q$$

Writing the dual

$$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$$

Tautological Implications

A statement formula A is said to *tautologically imply* a statement B if and only if $A \to B$ is a tautology.

In this case we write $A \Rightarrow B$, which is read as 'A implies B'.

Note: $\Rightarrow$ is not a connective, $A \Rightarrow B$ is not a statement formula.

$A \Rightarrow B$ states that $A \to B$ is tautology.

Clearly $A \Rightarrow B$ guarantees that B has a truth value T whenever A has the truth value T.

One can determine whether $A \Rightarrow B$ by constructing the truth tables of A and B in the same manner as was done in the determination of $A \Leftrightarrow B$. Example: Prove that $(P \to Q) \Rightarrow (\neg Q \to \neg P)$.

Solution:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg Q \to \neg P$	$(P \to Q) \to (\neg Q \to \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since all the entries in the last column are true, $(P \to Q) \to (\neg Q \to \neg P)$ is a tautology.

Hence $(P \to Q) \Rightarrow (\neg Q \to \neg P)$.

In order to show any of the given implications, it is sufficient to show that an assignment of the truth value T to the antecedent of the corresponding condi-

tional leads to the truth value T for the consequent. This procedure guarantees that the conditional becomes tautology, thereby proving the implication.

Example: Prove that $\neg Q \wedge (P \to Q) \Rightarrow \neg P$.

Solution: Assume that the antecedent $\neg Q \wedge (P \to Q)$ has the truth value T, then both $\neg Q$ and $P \to Q$ have the truth value T, which means that Q has the truth value F, $P \to Q$ has the truth value T. Hence P must have the truth value F.

Therefore the consequent $\neg P$ must have the truth value T.

$$\neg Q \wedge (P \to Q) \Rightarrow \neg P.$$

Another method to show $A \Rightarrow B$ is to assume that the consequent B has the truth value F and then show that this assumption leads to A having the truth value F. Then $A \to B$ must have the truth value T.

Example: Show that $\neg(P \to Q) \Rightarrow P$.

Solution: Assume that P has the truth value F. When P has F, $P \to Q$ has T, then $\neg(P \to Q)$ has F. Hence $\neg(P \to Q) \to P$ has T.

$$\neg(P \to Q) \Rightarrow P$$

Other Connectives

We introduce the connectives NAND, NOR which have useful applications in the design of computers.

NAND: The word NAND is a combination of 'NOT' and 'AND' where 'NOT' stands for negation and 'AND' for the conjunction. It is denoted by the symbol ↑.

If P and Q are two formulas then

$$P \uparrow Q \Leftrightarrow \neg(P \wedge Q)$$

The connective ↑ has the following equivalence:

$$P \uparrow P \Leftrightarrow \neg(P \wedge P) \Leftrightarrow \neg P \vee \neg P \Leftrightarrow \neg P.$$

$(P \uparrow Q) \uparrow (P \uparrow Q) \Leftrightarrow \neg(P \uparrow Q) \Leftrightarrow \neg(\neg(P \wedge Q)) \Leftrightarrow P \wedge Q.$

$(P \uparrow P) \uparrow (Q \uparrow Q) \Leftrightarrow \neg P \uparrow \neg Q \Leftrightarrow \neg(\neg P \wedge \neg Q) \Leftrightarrow P \vee Q.$

NAND is Commutative: Let P and Q be any two statement formulas.

$$(P \uparrow Q) \Leftrightarrow \neg(P \wedge Q)$$

$$\Leftrightarrow \neg(Q \wedge P) \Leftrightarrow$$

$$(Q \uparrow P)$$

$\therefore$ NAND is commutative.

NAND is not Associative: Let P, Q and R be any three statement formulas.

Consider $\uparrow (Q \uparrow R) \Leftrightarrow \neg(P \wedge (Q \uparrow R)) \Leftrightarrow \neg(P \wedge (\neg(Q \wedge R)))$

$$\Leftrightarrow \neg P \vee (Q \wedge R))$$

$$(P \uparrow Q) \uparrow R \Leftrightarrow \neg(P \wedge Q) \uparrow R$$

$$\Leftrightarrow \neg(\neg(P \wedge Q) \wedge R) \Leftrightarrow$$

$$(P \wedge Q) \vee \neg R$$

Therefore the connective $\uparrow$ is not associative.

NOR: The word NOR is a combination of 'NOT' and 'OR' where 'NOT' stands for negation and 'OR' for the disjunction. It is denoted by the symbol $\downarrow$.

If P and Q are two formulas then

$$P \downarrow Q \Leftrightarrow \neg(P \vee Q)$$

The connective $\downarrow$ has the following equivalence:

$P \downarrow P \Leftrightarrow \neg(P \vee P) \Leftrightarrow \neg P \wedge \neg P \Leftrightarrow \neg P.$

$(P \downarrow Q) \downarrow (P \downarrow Q) \Leftrightarrow \neg(P \downarrow Q) \Leftrightarrow \neg(\neg(P \vee Q)) \Leftrightarrow P \vee Q.$

$(P \downarrow P) \downarrow (Q \downarrow Q) \Leftrightarrow \neg P \downarrow \neg Q \Leftrightarrow \neg(\neg P \vee \neg Q) \Leftrightarrow P \wedge Q.$

NOR is Commutative: Let P and Q be any two statement formulas.

$$(P \downarrow Q) \Leftrightarrow \neg(P \vee Q)$$

$$\Leftrightarrow \neg(Q \vee P) \Leftrightarrow$$

$$(Q \downarrow P)$$

$\therefore$ NOR is commutative.

NOR is not Associative: Let P, Q and R be any three statement formulas. Consider

$$P \downarrow (Q \downarrow R) \Leftrightarrow \neg(P \vee (Q \downarrow R))$$

$$\Leftrightarrow \neg(P \vee (\neg(Q \vee R)))$$

$$\Leftrightarrow \neg P \wedge (Q \vee R)$$

$$(P \downarrow Q) \downarrow R \Leftrightarrow \neg(P \vee Q) \downarrow R$$

$$\Leftrightarrow \neg(\neg(P \vee Q) \vee R) \Leftrightarrow$$

$$(P \vee Q) \wedge \neg R$$

Therefore the connective $\downarrow$ is not associative.

Evidently, $P \uparrow Q$ and $P \downarrow Q$ are duals of each other.

Since

$$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$$

$$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q.$$

Example: Express $P \downarrow Q$ interms of $\uparrow$ only.
Solution:

$$\begin{aligned} \downarrow Q &\Leftrightarrow \neg(P \vee Q) \\ &\Leftrightarrow (P \vee Q) \uparrow (P \vee Q) \\ &\Leftrightarrow [(P \uparrow P) \uparrow (Q \uparrow Q)] \uparrow [(P \uparrow P) \uparrow (Q \uparrow Q)] \end{aligned}$$

Example: Express $P \uparrow Q$ interms of $\downarrow$ only. (May-2012)

Solution:

$$\begin{aligned} \uparrow Q &\Leftrightarrow \neg(P \wedge Q) \\ &\Leftrightarrow (P \wedge Q) \downarrow (P \wedge Q) \\ &\Leftrightarrow [(P \downarrow P) \downarrow (Q \downarrow Q)] \downarrow [(P \downarrow P) \downarrow (Q \downarrow Q)] \end{aligned}$$

Truth Tables

Example: Show that $(A \oplus B) \vee (A \downarrow B) \Leftrightarrow (A \uparrow B)$. (May-2012)

Solution: We prove this by constructing truth table.

A	B	$A \oplus B$	$A \downarrow B$	$(A \oplus B) \vee (A \downarrow B)$	$A \uparrow B$
T	T	F	F	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	T	T	T

As columns $(A \oplus B) \vee (A \downarrow B)$ and $(A \uparrow B)$ are identical.

$$\therefore (A \oplus B) \vee (A \downarrow B) \Leftrightarrow (A \uparrow B).$$

Normal Forms

If a given statement formula $A(p_1, p_2, ...p_n)$ involves n atomic variables, we have 2^n possible combinations of truth values of statements replacing the variables.

The formula A is a tautology if A has the truth value T for all possible assignments of the truth values to the variables $p_1, p_2, ...p_n$ and A is called a contradiction if A has the truth value F for all possible assignments of the truth values of the n variables. A is said to be *satis able* if A has the truth value T for atleast one combination of truth values assigned to $p_1, p_2, ...p_n$.

The problem of determining whether a given statement formula is a Tautology, or a Contradiction is called a decision problem.

The construction of truth table involves a finite number of steps, but the construc-tion may not be practical. We therefore reduce the given statement formula to normal form and find whether a given statement formula is a Tautology or Contradiction or atleast satisfiable.

It will be convenient to use the word "product" in place of "conjunction" and "sum" in place of "disjunction" in our current discussion.

A product of the variables and their negations in a formula is called an *elementary product*. Similarly, a sum of the variables and their negations in a formula is called an *elementary sum*.

Let P and Q be any atomic variables. Then P , $\neg P \wedge Q$, $\neg Q \wedge P \;\; \neg P$, $P \;\; \neg P$, and $Q \wedge \neg P$ are some examples of elementary products. On the other hand, P , $\neg P \vee Q$, $\neg Q \vee P \vee \neg P$, $P \vee \neg P$, and $Q \vee \neg P$ are some examples of elementary sums.

Any part of an elementary sum or product which is itself an elementary sum or product is called a *factor* of the original elementary sum or product. Thus $\neg Q, \wedge \neg P$, and $\neg Q \wedge P$ are some of the factors of $\neg Q \wedge P \wedge \neg P$.

Disjunctive Normal Form (DNF)

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a *disjunctive normal form* of the given formula.

Example: Obtain disjunctive normal forms of

(a) $P \wedge (P \rightarrow Q)$; (b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$.

Solution: (a) We have

$$P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$$
$$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$$

(b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q))$ [using

$R \leftrightarrow S \Leftrightarrow (R \wedge S) \vee (\neg R \wedge \neg S)$

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q))$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q)$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$

which is the required disjunctive normal form.

Note: The DNF of a given formula is not unique.

Conjunctive Normal Form (CNF)

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a *conjunctive normal form* of the given formula.

The method for obtaining conjunctive normal form of a given formula is similar to the one given for disjunctive normal form. Again, the conjunctive normal form is not unique.

Example: Obtain conjunctive normal forms of

(a) $P \wedge (P \rightarrow Q)$; (b) $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$.

Solution: (a). $P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$

(b). $\neg(P \vee Q) \leftrightarrow (P \wedge Q)$

$$\Leftrightarrow (\neg(P \vee Q) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q))$$

$$\Leftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))$$

$$\Leftrightarrow [(P \vee Q \vee P) \wedge (P \vee Q \vee Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)]$$

$$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$$

Note: A given formula is tautology if every elementary sum in CNF is tautology.

Example: Show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.

Solution: First we obtain a CNF of the given formula.

$$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \Leftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q)$$

$$\Leftrightarrow (Q \vee (P \vee \neg P)) \wedge (Q \vee \neg Q)$$

$$\Leftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)$$

Since each of the elementary sum is a tautology, hence the given formula is tautology.

Principal Disjunctive Normal Form

In this section, we will discuss the concept of principal disjunctive normal form (PDNF).

Minterm: For a given number of variables, the minterm consists of conjunctions in which each statement variable or its negation, but not both, appears only once.

Let P and Q be the two statement variables. Then there are 2^2 minterms given by $P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, and $\neg P \wedge \neg Q$.

Minterms for three variables P, Q and R are $P \wedge Q \wedge R$, $P \wedge Q \wedge \neg R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$, $\neg P \wedge Q \wedge R$, $\neg P \wedge Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge R$ and $\neg P \wedge \neg Q \wedge \neg R$. From the truth tables of these minterms of P and Q, it is clear that

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

(i). no two minterms are equivalent

(ii). Each minterm has the truth value T for exactly one combination of the truth values of the variables P and Q.

Definition: For a given formula, an equivalent formula consisting of disjunctions of minterms only is called the Principal disjunctive normal form of the formula.

The principle disjunctive normal formula is also called the sum-of-products canonical form.

Methods to obtain PDNF of a given formula

(a). By Truth table:

(i). Construct a truth table of the given formula.

(ii). For every truth value T in the truth table of the given formula, select the minterm which also has the value T for the same combination of the truth values of P and Q.

(iii). The disjunction of these minterms will then be equivalent to the given formula.

Example: Obtain the PDNF of $P \rightarrow Q$.
Solution: From the truth table of $P \rightarrow Q$

P	Q	$P \rightarrow Q$	Minterm
T	T	T	$P \wedge Q$
T	F	F	$P \wedge \neg Q$
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

The PDNF of $P \rightarrow Q$ is $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$.

$$\therefore P \rightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q).$$

Example: Obtain the PDNF for $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.
Solution:

P	Q	R	Minterm	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$
T	T	T	$P \wedge Q \wedge R$	T	F	T	T
T	T	F	$P \wedge Q \wedge \neg R$	T	F	F	T
T	F	T	$P \wedge \neg Q \wedge R$	F	F	F	F
T	F	F	$P \wedge \neg Q \wedge \neg R$	F	F	F	F
F	T	T	$\neg P \wedge Q \wedge R$	F	T	T	T
F	T	F	$\neg P \wedge Q \wedge \neg R$	F	F	F	F
F	F	T	$\neg P \wedge \neg Q \wedge R$	F	T	F	T
F	F	F	$\neg P \wedge \neg Q \wedge \neg R$	F	F	F	F

The PDNF of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ is

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R).$$

(b). Without constructing the truth table:

In order to obtain the principal disjunctive normal form of a given formula is constructed as follows:

(1). First replace →, by their equivalent formula containing only $\wedge$, $\vee$ and ¬.
(2). Next, negations are applied to the variables by De Morgan's laws followed by the application of distributive laws.
(3). Any elementarily product which is a contradiction is dropped. Minterms are ob-tained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

Example: Obtain the principal disjunctive normal form of

$(a)\ \neg P \vee Q$; $(b)\ (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.

Solution:

(a) $\neg P \vee Q \Leftrightarrow (\neg P \wedge T) \vee (Q \wedge T) \quad [\because A \wedge T \Leftrightarrow A]$

$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P)) \ [\because P \vee \neg P \Leftrightarrow T]$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

$[\because P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \quad [\because P \vee P \Leftrightarrow P]$

$(b)\ (P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge T) \vee (\neg P \wedge R \wedge T) \vee (Q \wedge R \wedge T)$

$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge R \wedge (Q \vee \neg Q)) \vee (Q \wedge R \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q)(\neg P \wedge R \wedge \neg Q)$

$\vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

$P \vee (P \wedge Q) \Leftrightarrow P$

$P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q$

Solution: We write the principal disjunctive normal form of each formula and com-pare these normal forms.

$(a)\ P \vee (P \wedge Q) \Leftrightarrow (P \wedge T) \vee (P \wedge Q) \quad [\because P \wedge Q \Leftrightarrow P]$

$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (P \wedge Q) \quad [\because P \vee \neg P \Leftrightarrow T]$

$\Leftrightarrow ((P \wedge Q) \vee (P \wedge \neg Q)) \vee (P \wedge Q)$ [by distributive laws]

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)\ [\because P \vee P \Leftrightarrow P]$

which is the required PDNF.

Now, $\Leftrightarrow P \wedge T$

$\Leftrightarrow P \wedge (Q \vee \neg Q)$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$

which is the required PDNF.

Hence, $P \vee (P \wedge Q) \Leftrightarrow P$.

$(b)\ P \vee (\neg P \wedge Q) \Leftrightarrow (P \wedge T) \vee (\neg P \wedge Q)$

$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (\neg P \wedge Q)$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

which is the required PDNF.

Now,

$$P \vee Q \Leftrightarrow (P \wedge T) \vee (Q \wedge T)$$

$$\Leftrightarrow (P \wedge (Q \vee \neg Q)) \vee (Q \wedge (P \vee \neg P))$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$$

$$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

which is the required PDNF.

Hence, $\quad P \vee (\neg P \wedge Q) \Leftrightarrow P \vee Q.$

Example: Obtain the principal disjunctive normal form of

$$P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P)). \qquad \text{(Nov. 2011)}$$

Solution: Using $P \to Q \Leftrightarrow \neg P \vee Q$ and De Morgan's law, we obtain

$$\to ((P \to Q) \wedge \neg(\neg Q \vee \neg P)) \Leftrightarrow \neg P$$
$$\vee ((\neg P \vee Q) \wedge (Q \wedge P))$$

$$\Leftrightarrow \neg P \vee ((\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P)) \Leftrightarrow$$
$$\neg P \vee F \vee (P \wedge Q)$$

$$\Leftrightarrow \neg P \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge T) \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge (Q \vee \neg Q)) \vee (P \wedge Q)$$

$$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

Hence $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ is the required PDNF.

Principal Conjunctive Normal Form

The dual of a minterm is called a Maxterm. For a given number of variables, the *maxterm* consists of disjunctions in which each variable or its negation, but not both, appears only once. Each of the maxterm has the truth value F for exactly one com-bination of the truth values of the variables. Now we define the principal conjunctive normal form.

For a given formula, an equivalent formula consisting of conjunctions of the max-terms only is known as its *principle conjunctive normal form*. This normal form is also called the *product-of-sums canonical form.*The method for obtaining the PCNF for a given formula is similar to the one described previously for PDNF.

Example: Obtain the principal conjunctive normal form of the formula (¬P→R)∧(Q↔P)

Solution:

$$(\neg P \to R) \wedge (Q \leftrightarrow P)$$

$$\Leftrightarrow [\neg(\neg P) \vee R] \wedge [(Q \to P) \wedge (P \to Q)]$$

$$\Leftrightarrow (P \vee R) \wedge [(\neg Q \vee P) \wedge (\neg P \vee Q)]$$

$$\Leftrightarrow (P \vee R \vee F) \wedge [(\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)]$$

$$\Leftrightarrow [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [\neg Q \vee P) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$$

$$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R)$$

$$\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

$$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

which is required principal conjunctive normal form.

Note: If the principal disjunctive (conjunctive) normal form of a given formula A containing n variables is known, then the principal disjunctive (conjunctive) normal form of $\neg A$ will consist of the disjunction (conjunction) of the remaining minterms (maxterms) which do not appear in the principal disjunctive (conjunctive) normal form of A. From $A \Leftrightarrow \neg\neg A$ one can obtain the principal conjunctive (disjunctive) normal form of A by repeated applications of De Morgan's laws to the principal disjunctive (conjunctive) normal form of $\neg A$.

Example: Find the PDNF form PCNF of $S : P \vee (\neg P \to (Q \vee (\neg Q \to R)))$.

Solution:

$$\Leftrightarrow P \vee (\neg P \to (Q \vee (\neg Q \to R)))$$

$$\Leftrightarrow P \vee (\neg(\neg P) \vee (Q \vee (\neg(\neg Q) \vee R))$$

$$\Leftrightarrow P \vee (P \vee Q \vee (Q \vee R)))$$

$$\Leftrightarrow P \vee (P \vee Q \vee R)$$

$$\Leftrightarrow P \vee Q \vee R$$

which is the PCNF.

Now PCNF of $\neg S$ is the conjunction of remaining maxterms, so

$$\text{PCNF of } \neg S : (P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$$

$$\wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$$

Hence the PDNF of S is

$$\neg(\text{PCNF of } \neg S) : (\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$$

$$\vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$$

Theory of Inference for Statement Calculus

Definition: The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as inference theory .

Definition: If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a deduction or a formal proof and the argument is called a *valid argument* or conclusion is called a *valid conclusion*.

Note: Premises means set of assumptions, axioms, hypothesis.

Definition: Let A and B be two statement formulas. We say that "*B logically follows from A*" or "B is a *valid conclusion (consequence)* of the premise A" iff $A \rightarrow B$ is a tautology, that is $A \Rightarrow B$.
We say that from a set of premises $\{H_1, H_2, \cdots, H_m\}$, a conclusion C follows logically iff

$$H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow C$$

(1)

Note: To determine whether the conclusion logically follows from the given premises, we use the following methods:

- Truth table method
- Without constructing truth table method.

Validity Using Truth Tables

Given a set of premises and a conclusion, it is possible to determine whether the conclusion logically follows from the given premises by constructing truth tables as follows.

Let $P_1, P_2, \cdots, P_n$ be all the atomic variables appearing in the premises $H_1, H_2, \cdots, H_m$ and in the conclusion C. If all possible combinations of truth values are assigned to $P_1, P_2, \cdots, P_n$ and if the truth values of $H_1, H_2, ..., H_m$ and C are entered in a table. We look for the rows in which all $H_1, H_2, \cdots, H_m$ have the value T. If, for every such row, C also has the value T, then (1) holds. That is, the conclusion follows logically.

Alternatively, we look for the rows on which C has the value F. If, in every such row, at least one of the values of $H_1, H_2, \cdots, H_m$ is F, then (1) also holds. We call such a method a 'truth table technique' for the determination of the validity of a conclusion.

Example: Determine whether the conclusion C follows logically from the premises

H_1 and H_2.

$(a)\ H_1 : P \to Q \qquad H_2 : P \quad C : Q$

$(b)\ H_1 : P \to Q \qquad H_2 : \neg P \quad C : Q$

$(c)\ H_1 : P \to Q \qquad H_2 : \neg(P \wedge Q) \quad C : \neg P$

$(d)\ H_1 : \neg P \qquad H_2 : P \quad Q \quad C : \neg(P \wedge Q)$

$(e)\ H_1 : P \to Q \qquad H_2 : Q \quad C : P$

Solution: We first construct the appropriate truth table, as shown in table.

P	Q	$P \to Q$	$\neg P$	$\neg(P \wedge Q)$	$P \quad Q$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	T	F
F	F	T	T	T	T

(a) We observe that the first row is the only row in which both the premises have the value T . The conclusion also has the value T in that row. Hence it is valid.

In (b) the third and fourth rows, the conclusion Q is true only in the third row, but not in the fourth, and hence the conclusion is not valid.
Similarly, we can show that the conclusions are valid in (c) and (d) but not in (e).

Rules of Inference

The following are two important rules of inferences.

Rule P: A premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is tautologically implied by one or more of the preceding formulas in the derivation.

Implication Formulas

$I_1 : P \wedge Q \Rightarrow P$ (simplification)

$I_2 : P \wedge Q \Rightarrow Q$

$I_3 : P \Rightarrow P \vee Q$

$I_4 : Q \Rightarrow P \vee Q$

$I_5 : \neg P \Rightarrow P \rightarrow Q$

$I_6 : Q \Rightarrow P \rightarrow Q$

$I_7 : \neg(P \rightarrow Q) \Rightarrow P$

$I_8 : \neg(P \rightarrow Q) \Rightarrow \neg Q$

$I_9 : P, Q \Rightarrow P \wedge Q$

$I_{10} : \neg P, P \vee Q \Rightarrow Q$ (disjunctive syllogism)

$I_{11} : P, P \rightarrow Q \Rightarrow Q$ (modus ponens)

$I_{12} : \neg Q, P \rightarrow Q \Rightarrow \neg P$ (modus tollens)

$I_{13} : P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (hypothetical syllogism)

$I_{14} : P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (dilemma)

Example: Demonstrate that R is a valid inference from the premises $P \rightarrow Q$, $Q \rightarrow R$, and P .
Solution:

{1}	(1)	$P \rightarrow Q$	Rule P
{2}	(2)	P	Rule P,
{1, 2}	(3)	Q	Rule T, (1), (2), and I_{13}
{4}	(4)	$Q \rightarrow R$	Rule P
{1, 2, 4}	(5)	R	Rule T, (3), (4), and I_{13}

Hence the result.

Example: Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$, and $(A \wedge \neg B) \rightarrow (R \vee S)$.

Solution:

{1}	(1)	$(C \vee D) \rightarrow \neg H$	Rule P
{2}	(2)	$\neg H \rightarrow (A \wedge \neg B)$	Rule P
{1, 2}	(3)	$(C \vee D) \rightarrow (A \wedge \neg B)$	Rule T, (1), (2), and I_{13}
{4}	(4)	$(A \wedge \neg B) \rightarrow (R \vee S)$	Rule P
{1, 2, 4}	(5)	$(C \vee D) \rightarrow (R \vee S)$	Rule T, (3), (4), and I_{13}
{6}	(6)	$C \vee D$	Rule P
{1, 2, 4, 6}	(7)	$R \vee S$	Rule T, (5), (6), and I_{11}

Hence the result.

Example: Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

Solution:

{1}	(1)	$P \vee Q$	Rule P
{1}	(2)	$\neg P \rightarrow Q$	Rule T, (1) $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
{3}	(3)	$Q \rightarrow S$	Rule P
{1, 3}	(4)	$\neg P \rightarrow S$	Rule T, (2), (3), and I_{13}
{1, 3}	(5)	$\neg S \rightarrow P$	Rule T, (4), $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
{6}	(6)	$P \rightarrow R$	Rule P
{1, 3, 6}	(7)	$\neg S \rightarrow R$	Rule T, (5), (6), and I_{13}
{1, 3, 6}	(8)	$S \vee R$	Rule T, (7) and $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

Hence the result.

Example: Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$,

$Q \rightarrow R$, $P \rightarrow M$, and $\neg M$.

Solution:

{1}	(1)	$P \rightarrow M$	Rule P
{2}	(2)	$\neg M$	Rule P
{1, 2}	(3)	$\neg P$	Rule T, (1), (2), and I_{12}
{4}	(4)	$P \vee Q$	Rule P
{1, 2, 4}	(5)	Q	Rule T, (3), (4), and I_{10}
{6}	(6)	$Q \rightarrow R$	Rule P

{1, 2, 4, 6}	(7)	R	Rule T, (5), (6), and I_{11}
{1, 2, 4, 6}	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7) and I_9

Hence the result.

Example: Show I_{12} : $\neg Q, P \rightarrow Q \Rightarrow \neg P$.
Solution:

{1}	(1)	$P \rightarrow Q$	Rule P
{1}	(2)	$\neg Q \rightarrow \neg P$	Rule T, (1), and $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$
{3}	(3)	$\neg Q$	Rule P
{1, 3}	(4)	$\neg P$	Rule T, (2), (3), and I_{11}

Hence the result.

Example: Test the validity of the following argument:

”If you work hard, you will pass the exam. You did not pass. Therefore, you did not work hard”.

Example: Test the validity of the following statements:

”If Sachin hits a century, then he gets a free car. Sachin does not get a free car.

Therefore, Sachin has not hit a century”.

Rules of Conditional Proof or Deduction Theorem

We shall now introduce a third inference rule, known as CP or rule of conditional proof.
Rule CP: If we can derive S from R and a set of premises, then we can derive $R \rightarrow S$ from the set of premises alone.
Rule CP is not new for our purpose her because it follows from the equivalence

$$(P \wedge R) \rightarrow S \Leftrightarrow P \rightarrow (R \rightarrow S)$$

Let P denote the conjunction of the set of premises and let R be any formula. The above equivalence states that if R is included as an additional premise and S is derived from $P \wedge R$, then $R \rightarrow S$ can be derived from the premises P alone.

Rule CP is also called the *deduction theorem* and is generally used if the conclu-sion of the form $R \rightarrow S$. In such cases, R is taken as an additional premise and S is derived from the given premises and R.

Example: Show that $R \rightarrow S$ can be derived from the premises $P \rightarrow (Q \rightarrow S)$, $\neg R \vee P$, and Q.
(Nov. 2011)

Solution: Instead of deriving $R \rightarrow S$, we shall include R as an additional premise and show S first.

$\{1\}$	(1) $\neg R \vee P$	Rule P
$\{2\}$	(2) R	Rule P (assumed premise)
$\{1, 2\}$	(3) P	Rule T, (1), (2), and I_{10}
$\{4\}$	(4) $P \rightarrow (Q \rightarrow S)$	Rule P
$\{1, 2, 4\}$	(5) $Q \rightarrow S$	Rule T, (3), (4), and I_{11}
$\{6\}$	(6) Q	Rule P
$\{1, 2, 4, 6\}$	(7) S	Rule T, (5), (6), and I_{11}
$\{1, 2, 4, 6\}$	(8) $R \rightarrow S$	Rule CP

Example: Show that $P \rightarrow S$ can be derived from the premises $\neg P \vee Q$, $\neg Q \vee R$, and $R \rightarrow S$.
Solution: We include P as an additional premise and derive S.

$\{1\}$	(1) $\neg P \vee Q$	Rule P
$\{2\}$	(2) P	Rule P (assumed premise)
$\{1, 2\}$	(3) Q	Rule T, (1), (2), and I_{10}
$\{4\}$	(4) $\neg Q \vee R$	Rule P
$\{1, 2, 4\}$	(5) R	Rule T, (3), (4), and I_{10}
$\{6\}$	(6) $R \rightarrow S$	Rule P
$\{1, 2, 4, 6\}$	(7) S	Rule T, (5), (6), and I_{11}
$\{1, 2, 4, 6\}$	(8) $P \rightarrow S$	Rule CP

Example: 'If there was a ball game, then traveling was difficult. If they arrived on time, then traveling was not difficult. They arrived on time. Therefore, there was no ball game'. Show that these statements constitute a valid argument. Solution: Let us indicate the statements as follows:

P : There was a ball game.
Q: Traveling was difficult.
R: They arrived on time.

Hence, the given premises are $P \to Q$, $R \to \neg Q$, and R. The conclusion is $\neg P$.

{1}	(1) $R \to \neg Q$	Rule P
{2}	(2) R	Rule P
{1, 2}	(3) $\neg Q$	Rule T, (1), (2), and I_{11}
{4}	(4) $P \to Q$	Rule P
{4}	(5) $\neg Q \to \neg P$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
{1, 2, 4}	(6) $\neg P$	Rule T, (3), (5), and I_{11}

Example: By using the method of derivation, show that following statements con-stitute a valid argument: "If A works hard, then either B or C will enjoy. If B enjoys, then A will not work hard. If D enjoys, then C will not. Therefore, if A works hard, D will not enjoy.

Solution: Let us indicate statements as follows:

Given premises are $P \to (Q \vee R)$, $Q \to \neg P$, and $S \to \neg R$. The conclusion is $P \to \neg S$.
We include P as an additional premise and derive $\neg S$.

{1}	(1) P	Rule P (additional premise)
{2}	(2) $P \to (Q \vee R)$	Rule P
{1, 2}	(3) $Q \vee R$	Rule T, (1), (2), and I_{11}
{1, 2}	(4) $\neg Q \to R$	Rule T, (3) and $P \to Q \Leftrightarrow P \vee Q$
{1, 2}	(5) $\neg R \to Q$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
{6}	(6) $Q \to \neg P$	Rule P
{1, 2, 6}	(7) $\neg R \to \neg P$	Rule T, (5), (6), and I_{13}
{1, 2, 6}	(8) $P \to R$	Rule T, (7) and $P \to Q \Leftrightarrow \neg Q \to \neg P$
{9}	(9) $S \to \neg R$	Rule P
{9}	(10) $R \to \neg S$	Rule T, (9) and $P \to Q \Leftrightarrow \neg Q \to \neg P$
{1, 2, 6, 9}	(11) $P \to \neg S$	Rule T, (8), (10) and I_{13}
{1, 2, 6, 9}	(12) $\neg S$	Rule T, (1), (11) and I_{11}

Example: Determine the validity of the following arguments using propositional logic: "Smoking is healthy. If smoking is healthy, then cigarettes are prescribed by physicians. Therefore, cigarettes are prescribed by physicians". (May-2012)

Solution: Let us indicate the statements as follows:

P : Smoking is healthy.

Q: Cigarettes are prescribed by physicians.

Hence, the given premises are P , $P \to Q$. The conclusion is Q.

{1}	(1) $P \to Q$	Rule P
{2}	(2) P	Rule P

{1, 2}	(3) Q	Rule T, (1), (2), and I_{11}

Hence, the given statements constitute a valid argument.

Consistency of Premises

A set of formulas $H_1, H_2, \cdots, H_m$ is said to be *consistent* if their conjunction has the truth value T for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, \cdots, H_m$.

If, for every assignment of the truth values to the atomic variables, at least one of the formulas $H_1, H_2, \cdots, H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \cdots, H_m$ are called *inconsistent*.

Alternatively, a set of formulas $H_1, H_2, \cdots, H_m$ is inconsistent if their conjunction implies a contradiction, that is,

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow R \wedge \neg R$$

where R is any formula.

Example: Show that the following premises are inconsistent:

(1). If Jack misses many classes through illness, then he fails high school.
(2). If Jack fails high school, then he is uneducated.
(3). If Jack reads a lot of books, then he is not uneducated.
(4). Jack misses many classes through illness and reads a lot of books.

Solution: Let us indicate the statements as follows:

E: Jack misses many classes through illness.
S: Jack fails high school.
A: Jack reads a lot of books.
H: Jack is uneducated.

The premises are $E \to S$, $S \to H$, $A \to \neg H$, and $E \wedge A$.

{1}	(1) $E \to S$	Rule P
{2}	(2) $S \to H$	Rule P
{1, 2}	(3) $E \to H$	Rule T, (1), (2), and I_{13}
{4}	(4) $A \to \neg H$	Rule P
{4}	(5) $H \to \neg A$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
{1, 2, 4}	(6) $E \to \neg A$	Rule T, (3), (5), and I_{13}
{1, 2, 4}	(7) $\neg E \vee \neg A$	Rule T, (6) and $P \to Q \Leftrightarrow \neg P \vee Q$
{1, 2, 4}	(8) $\neg(E \wedge A)$	Rule T, (7), and $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
{9}	(9) $E \wedge A$	Rule P
{1, 2, 4, 9}	(10) $\neg(E \wedge A) \wedge (E \wedge A)$	Rule T, (8), (9) and I_9

Thus, the given set of premises leads to a contradiction and hence it is inconsistent.

Example: Show that the following set of premises is inconsistent: "If the contract is valid, then John is liable for penalty. If John is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid, and the bank will loan him money."

Solution: Let us indicate the statements as follows:

V : The contract is valid.

L: John is liable for penalty.

M: Bank will loan him money.

B: John will go bankrupt.

$\{1\}$	(1)	$V \to L$	Rule P
$\{2\}$	(2)	$L \to B$	Rule P
$\{1, 2\}$	(3)	$V \to B$	Rule T, (1), (2), and I_{13}
$\{4\}$	(4)	$M \to \neg B$	Rule P
$\{4\}$	(5)	$M \to \neg M$	Rule T, (4), and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{1, 2, 4\}$	(6)	$V \to \neg M$	Rule T, (3), (5), and I_{13}
$\{1, 2, 4\}$	(7)	$\neg V \vee \neg M$	Rule T, (6) and $P \to Q \Leftrightarrow \neg P \vee Q$
$\{1, 2, 4\}$	(8)	$\neg(V \wedge M)$	Rule T, (7), and $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$
$\{9\}$	(9)	$V \wedge M$	Rule P
$\{1, 2, 4, 9\}$	(10)	$\neg(V \wedge M) \wedge (V \wedge M)$	Rule T, (8), (9) and I_9

Thus, the given set of premises leads to a contradiction and hence it is inconsistent.

Indirect Method of Proof

The method of using the rule of conditional proof and the notion of an inconsistent set of premises is called the *indirect method of proof* or *proof by contradiction*.

In order to show that a conclusion C follows logically from the premises $H_1, H_2, \cdots,$ H_m, we assume that C is false and consider $\neg C$ as an additional premise. If the new set of premises is inconsistent, so that they imply a contradiction. Therefore, the assump-tion that $\neg C$ is true does not hold.

Hence, C is true whenever $H_1, H_2, \cdots, H_m$ are true. Thus, C follows logically from the premises $H_1, H_2, \cdots, H_m$.

Example: Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Solution: We introduce $\neg\neg(P \wedge Q)$ as additional premise and show that this additional premise leads to a contradiction.

{1}	(1) $\neg\neg(P \wedge Q)$	Rule P (assumed)
{1}	(2) $P \wedge Q$	Rule T, (1), and $\neg\neg P \Leftrightarrow P$
{1}	(3) P	Rule T, (2), and I_1
{4}	(4) $\neg P \wedge \neg Q$	Rule P
{4}	(5) $\neg P$	Rule T, (4), and I_1
{1, 4}	(6) $P \wedge \neg P$	Rule T, (3), (5), and I_9

Hence, our assumption is wrong.

Thus, $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Example: Using the indirect method of proof, show that

$$P \rightarrow Q,\ Q \rightarrow R,\ \neg(P \wedge R),\ P \vee R \Rightarrow R.$$

Solution: We include $\neg R$ as an additional premise. Then we show that this leads to a contradiction.

{1}	(1) $P \rightarrow Q$	Rule P
{2}	(2) $Q \rightarrow R$	Rule P
{1, 2}	(3) $P \rightarrow R$	Rule T, (1), (2), and I_{13}
{4}	(4) $\neg R$	Rule P (assumed)
{1, 2, 4}	(5) $\neg P$	Rule T, (4), and I_{12}
{6}	(6) $P \vee R$	Rule P
{1, 2, 4, 6}	(7) R	Rule T, (5), (6) and I_{10}
{1, 2, 4, 6}	(8) $R \wedge \neg R$	Rule T, (4), (7), and I_9

Hence, our assumption is wrong.

Example: Show that the following set of premises are inconsistent, using proof by contradiction

$$P \rightarrow (Q \vee R),\ Q \rightarrow \neg P,\ S \rightarrow \neg R,\ P \Rightarrow P \rightarrow \neg S.$$

Solution: We include $\neg(P \rightarrow \neg S)$ as an additional premise. Then we show that this leads to a contradiction.

$$\therefore \neg(P \rightarrow \neg S) \Leftrightarrow \neg(\neg P \vee \neg S) \Leftrightarrow P \wedge S.$$

{1}	(1) $P \rightarrow (Q \vee R)$	Rule P
{2}	(2) P	Rule P
{1, 2}	(3) $Q \vee R$	Rule T, (1), (2), and Modus Ponens
{4}	(4) $P \wedge S$	Rule P (assumed)
{1, 2, 4}	(5) S	Rule T, (4), and $P \wedge Q \Rightarrow P$

{6}	(6) $S \to \neg R$	Rule P
{1, 2, 4, 6}	(7) $\neg R$	Rule T, (5), (6) and Modus Ponens
{1, 2, 4, 6}	(8) Q	Rule T, (3), (7), and $P \wedge Q, \neg Q \Rightarrow P$
{9}	(9) $Q \to \neg P$	Rule P
{1, 2, 4, 6}	(10) $\neg P$	Rule T, (8), (9), and $P \wedge Q, \neg Q \Rightarrow P$
{1, 2, 4, 6}	(11) $P \wedge \neg P$	Rule T, (2), (10), and $P, Q \Rightarrow P \wedge Q$
{1, 2, 4, 6}	(12) F	Rule T, (11), and $P \wedge \neg P \Leftrightarrow F$

Hence, it is proved that the given premises are inconsistent.

The Predicate Calculus

Predicate

A part of a declarative sentence describing the properties of an object is called a predicate. The logic based upon the analysis of predicate in any statement is called predicate logic.

Consider two statements:

John is a bachelor

Smith is a bachelor.

In each statement "is a bachelor" is a predicate. Both John and Smith have the same property of being a bachelor. In the statement logic, we require two different symbols to express them and these symbols do not reveal the common property of these statements. In predicate calculus these statements can be replaced by a single statement "x is a bachelor". A predicate is symbolized by a capital letters which is followed by the list of variables. The list of variables is enclosed in parenthesis. If P stands for the predicate "is a bachelor", then $P(x)$ stands for "x is a bachelor",where x is a predicate variable.

`The domain for $P(x)$: x is a bachelor, can be taken as the set of all human names. Note that $P(x)$ is not a statement, but just an expression. Once a value is assigned to x, $P(x)$ becomes a statement and has the truth value. If x is Ram, then $P(x)$ is a statement and its truth value is true.

Quantifiers

Quantifiers: Quantifiers are words that are refer to quantities such as 'some' or 'all'.

Universal Quantifier: The phrase 'forall' (denoted by $\forall$) is called the universal quantifier.

For example, consider the sentence "All human beings are mortal".

Let $P(x)$ denote 'x is a mortal'.

Then, the above sentence can be written as

$$(\forall x \in S)P(x) \text{ or } \forall x P(x)$$

where S denote the set of all human beings.

$\forall x$ represents each of the following phrases, since they have essentially the same for all x

For every x

For each x.

Existential Quantifier: The phrase 'there exists' (denoted by $\exists$) is called the exis-tential quantifier.

For example, consider the sentence

"There exists x such that $x^2 = 5$.

This sentence can be written as

$$(\exists x \in R)P(x) \text{ or } (\exists x)P(x),$$

where $P(x) : x^2 = 5$.

$\exists x$ represents each of the following phrases

There exists an x
There is an x
For some x
There is at least one x.

Example: Write the following statements in symbolic form:

(i). Something is good
(ii). Everything is good
(iii). Nothing is good
(iv). Something is not good.

Solution: Statement (i) means "There is atleast one x such that, x is good".
Statement (ii) means "Forall x, x is good".
Statement (iii) means, "Forall x, x is not good".
Statement (iv) means, "There is atleast one x such that, x is not good.
Thus, if $G(x) : x$ is good, then

statement (i) can be denoted by $(\exists x)G(x)$

statement (ii) can be denoted by $(\forall x)G(x)$

statement (iii) can be denoted by $(\forall x)\neg G(x)$

statement (iv) can be denoted by $(\exists x)\neg G(x)$.

Example: Let $K(x) : x$ is a man
$L(x) : x$ is mortal
$M(x) : x$ is an integer
$N(x) : x$ either positive or negative

Express the following using quantifiers:

- All men are mortal
- Any integer is either positive or negative.

Solution: (a) The given statement can be written as
for all x, if x is a man, then x is mortal and this can be expressed as
$(x)(K(x) \to L(x))$.

(b) The given statement can be written as
for all x, if x is an integer, then x is either positive or negative and this can be expressed as $(x)(M(x) \to N(x))$.

Free and Bound Variables

Given a formula containing a part of the form $(x)P(x)$ or $(\exists x)P(x)$, such a part is called an x-bound part of the formula. Any occurrence of x in an x-bound part of the formula is called a bound occurrence of x, while any occurrence of x or of any variable that is not a bound occurrence is called a free occurrence. The smallest formula immediately following $(\forall x)$ or $(\exists x)$ is called the scope of the quantifier.

Consider the following formulas:

- $(x)P(x, y)$
- $(x)(P(x) \rightarrow Q(x))$
- $(x)(P(x) \rightarrow (\exists y)R(x, y))$
- $(x)(P(x) \rightarrow R(x)) \vee (x)(R(x) \rightarrow Q(x))$
- $(\exists x)(P(x) \wedge Q(x))$
- $(\exists x)P(x) \wedge Q(x)$.

In (1), $P(x, y)$ is the scope of the quantifier, and occurrence of x is bound occurrence, while the occurrence of y is free occurrence.

In (2), the scope of the universal quantifier is $P(x) \rightarrow Q(x)$, and all concrescences of x are bound.

In (3), the scope of (x) is $P(x) \rightarrow (\exists y)R(x, y)$, while the scope of $(\exists y)$ is $R(x, y)$. All occurrences of both x and y are bound occurrences.

In (4), the scope of the first quantifier is $P(x) \rightarrow R(x)$ and the scope of the second is $R(x) \rightarrow Q(x)$. All occurrences of x are bound occurrences.

In (5), the scope $(\exists x)$ is $P(x) \wedge Q(x)$.

In (6), the scope of $(\exists x)$ is $P(x)$ and the last of occurrence of x in $Q(x)$ is free.

Negations of Quantified Statements

(i). $\neg(x)P(x) \Leftrightarrow (\exists x)\neg P(x)$

(ii). $\neg(\exists x)P(x) \Leftrightarrow (x)(\neg P(x))$.

Example: Let $P(x)$ denote the statement ”x is a professional athlete” and let $Q(x)$ denote the statement ”x plays soccer”. The domain is the set of all people.

(a). Write each of the following proposition in English.

- $(x)(P(x) \rightarrow Q(x)$
- $(\exists x)(P(x) \wedge Q(x))$
- $(x)(P(x) \vee Q(x))$

(b). Write the negation of each of the above propositions, both in symbols and in words.

Solution:

(a). (i). For all x, if x is an professional athlete then x plays soccer.

”All professional athletes plays soccer” or ”Every professional athlete plays soccer”.

(ii). There exists an x such that x is a professional athlete and x plays soccer.

”Some professional athletes paly soccer”.

(iii). For all x, x is a professional athlete or x plays soccer.

”Every person is either professional athlete or plays soccer”.

(b). (i). In symbol: We know that

$$\neg(x)(P(x) \rightarrow Q(x)) \Leftrightarrow (\exists x)\neg(P(x) \rightarrow Q(x)) \Leftrightarrow (\exists x)\neg(\neg(P(x)) \vee Q(x))$$

$$\Leftrightarrow (\exists x)(P(x) \wedge \neg Q(x))$$

There exists an x such that, x is a professional athlete and x does not paly soccer.
In words: ”Some professional athlete do not play soccer”.

(ii). $\neg(\exists x)(P(x) \wedge Q(x)) \Leftrightarrow (x)(\neg P(x) \vee \neg Q(x))$

In words: ”Every people is neither a professional athlete nor plays soccer” or All people either not a professional athlete or do not play soccer”.

(iii). $\neg(x)(P(x) \vee Q(x)) \Leftrightarrow (\exists x)(\neg P(x) \wedge \neg Q(x))$.

In words: ”Some people are not professional athlete or do not paly soccer”.

Inference Theory of the Predicate Calculus

To understand the inference theory of predicate calculus, it is important to be famil-iar with the following rules:

Rule US: Universal specification or instaniation

$$(x)A(x) \Rightarrow A(y)$$

From $(x)A(x)$, one can conclude $A(y)$.

Rule ES: Existential specification

$$(\exists x)A(x) \Rightarrow A(y)$$

From $(\exists x)A(x)$, one can conclude $A(y)$.

Rule EG: Existential generalization

$$A(x) \Rightarrow (\exists y)A(y)$$

From $A(x)$, one can conclude $(\exists y)A(y)$.

Rule UG: Universal generalization

$$A(x) \Rightarrow (y)A(y)$$

From $A(x)$, one can conclude $(y)A(y)$.

Equivalence formulas:

$E_{31} : (\exists x)[A(x) \vee B(x)] \Leftrightarrow (\exists x)A(x) \vee (\exists x)B(x)$

$E_{32} : (x)[A(x) \wedge B(x)] \Leftrightarrow (x)A(x) \wedge (x)B(x)$

$E_{33} : \neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$

$E_{34} : \neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$

$E_{35} : (x)(A \vee B(x)) \Leftrightarrow A \vee (x)B(x)$

$E_{36} : (\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x)B(x)$

$E_{37} : (x)A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$

$E_{38} : (\exists x)A(x) \rightarrow B \Leftrightarrow (x)(A(x) \rightarrow B)$

$E_{39} : A \rightarrow (x)B(x) \Leftrightarrow (x)(A \rightarrow B(x))$

$E_{40} : A \rightarrow (\exists x)B(x) \Leftrightarrow (\exists x)(A \rightarrow B(x))$

$E_{41} : (\exists x)(A(x) \rightarrow B(x)) \Leftrightarrow (x)A(x) \rightarrow (\exists x)B(x)$

$E_{42} : (\exists x)A(x) \rightarrow (x)B(X) \Leftrightarrow (x)(A(x) \rightarrow B(X))$.

Example: Verify the validity of the following arguments:

"All men are mortal. Socrates is a man. Therefore, Socrates is mortal".

or

Show that $(x)[H(x) \rightarrow M(x)] \wedge H(s) \Rightarrow M(s)$.

Solution: Let us represent the statements as follows:

$H(x)$: x is a man

$M(x)$: x is a mortal

s : Socrates

Thus, we have to show that $(x)[H(x) \rightarrow M(x)] \wedge H(s) \Rightarrow M(s)$.

{1}	(1)	$(x)[H(x) \rightarrow M(x)]$	Rule P
{1}	(2)	$H(s) \rightarrow M(s)$	Rule US, (1)
{3}	(3)	$H(s)$	Rule P
{1, 3}	(4)	$M(s)$	Rule T, (2), (3), and I_{11}

Example: Establish the validity of the following argument:"All integers are ratio-nal numbers. Some integers are powers of 2. Therefore, some rational numbers are powers of 2".

Solution: Let $P(x)$: x is an integer

$R(x)$: x is rational number

$S(x)$: x is a power of 2

Hence, the given statements becomes

$$(x)(P(x) \rightarrow R(x)), (\exists x)(P(x) \wedge S(x)) \Rightarrow (\exists x)(R(x) \wedge S(x))$$

Solution:

{1}	(1)	$(\exists x)(P(x) \wedge S(x))$	Rule P
{1}	(2)	$P(y) \wedge S(y)$	Rule ES, (1)
{1}	(3)	$P(y)$	Rule T, (2) and $P \wedge Q \Rightarrow P$
{1}	(4)	$S(y)$	Rule T, (2) and $P \wedge Q \Rightarrow Q$
{5}	(5)	$(x)(P(x) \rightarrow R(x))$	Rule P
{5}	(6)	$P(y) \rightarrow R(y)$	Rule US, (5)
{1, 5}	(7)	$R(y)$	Rule T, (3), (6) and $P, P \rightarrow Q \Rightarrow Q$
{1, 5}	(8)	$R(y) \wedge S(y)$	Rule T, (4), (7) and $P, Q \Rightarrow P \wedge Q$
{1, 5}	(9)	$(\exists x)(R(x) \wedge S(x))$	Rule EG, (8)

Hence, the given statement is valid.

Example: Show that $(x)(P(x) \to Q(x)) \wedge (x)(Q(x) \to R(x)) \Rightarrow (x)(P(x) \to R(x))$.
Solution:

$\{1\}$	(1) $(x)(P(x) \to Q(x))$	Rule P
$\{1\}$	(2) $P(y) \to Q(y)$	Rule US, (1)
$\{3\}$	(3) $(x)(Q(x) \to R(x))$	Rule P
$\{3\}$	(4) $Q(y) \to R(y)$	Rule US, (3)
$\{1, 3\}$	(5) $P(y) \to R(y)$	Rule T, (2), (4), and I_{13}
$\{1, 3\}$	(6) $(x)(P(x) \to R(x))$	Rule UG, (5)

Example: Show that $(\exists x)M(x)$ follows logically from the premises

$(x)(H(x) \to M(x))$ and $(\exists x)H(x)$.

Solution:

$\{1\}$	(1) $(\exists x)H(x)$	Rule P
$\{1\}$	(2) $H(y)$	Rule ES, (1)
$\{3\}$	(3) $(x)(H(x) \to M(x))$	Rule P
$\{3\}$	(4) $H(y) \to M(y)$	Rule US, (3)
$\{1, 3\}$	(5) $M(y)$	Rule T, (2), (4), and I_{11}
$\{1, 3\}$	(6) $(\exists x)M(x)$	Rule EG, (5)

Hence, the result.

Example: Show that $(\exists x)[P(x) \wedge Q(x)] \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$.
Solution:

$\{1\}$	(1) $(\exists x)(P(x) \wedge Q(x))$	Rule P
$\{1\}$	(2) $P(y) \wedge Q(y)$	Rule ES, (1)
$\{1\}$	(3) $P(y)$	Rule T, (2), and I_1
$\{1\}$	(4) $(\exists x)P(x)$	Rule EG, (3)
$\{1\}$	(5) $Q(y)$	Rule T, (2), and I_2
$\{1\}$	(6) $(\exists x)Q(x)$	Rule EG, (5)
$\{1\}$	(7) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule T, (4), (5) and I_9

Hence, the result.

Note: Is the converse true?

$\{1\}$	(1) $(\exists x)P(x) \wedge (\exists x)Q(x)$	Rule P
$\{1\}$	(2) $(\exists x)P(x)$	Rule T, (1) and I_1
$\{1\}$	(3) $(\exists x)Q(x)$	Rule T, (1), and I_1
$\{1\}$	(4) $P(y)$	Rule ES, (2)
$\{1\}$	(5) $Q(s)$	Rule ES, (3)

Here in step (4), y is fixed, and it is not possible to use that variable again in step (5). Hence, the *converse is not true.*

Example: Show that from $(\exists x)[F(x) \wedge S(x)] \rightarrow (y)[M(y) \rightarrow W(y)]$ and $(\exists y)[M(y) \wedge \neg W(y)]$ the conclusion $(x)[F(x) \rightarrow \neg S(x)]$ follows.

{1}	(1)	$(\exists y)[M(y) \wedge \neg W(y)]$	Rule P
{1}	(2)	$[M(z) \wedge \neg W(z)]$	Rule ES, (1)
{1}	(3)	$\neg[M(z) \rightarrow W(z)]$	Rule T, (2), and $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$
{1}	(4)	$(\exists y)\neg[M(y) \rightarrow W(y)]$	Rule EG, (3)
{1}	(5)	$\neg(y)[M(y) \rightarrow W(y)]$	Rule T, (4), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1}	(6)	$(\exists x)[F(x) \wedge S(x)] \rightarrow (y)[M(y) \rightarrow W(y)]$	Rule P
{1, 6}	(7)	$\neg(\exists x)[F(x) \wedge S(x)]$	Rule T, (5), (6) and I_{12}
{1, 6}	(8)	$(x)\neg[F(x) \wedge S(x)]$	Rule T, (7), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1, 6}	(9)	$\neg[F(z) \wedge S(z)]$	Rule US, (8)
{1, 6}	(10)	$\neg F(z) \vee \neg S(z)$	Rule T, (9), and De Morgan's laws
{1, 6}	(11)	$F(z) \rightarrow \neg S(z)$	Rule T, (10), and $P \rightarrow Q \Leftrightarrow \neg P \vee Q$
{1, 6}	(12)	$(x)(F(x) \rightarrow \neg S(x))$	Rule UG, (11)

Hence, the result.

Example: Show that $(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$. (May. 2012)

Solution: We shall use the indirect method of proof by assuming $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional premise.

{1}	(1)	$\neg((x)P(x) \vee (\exists x)Q(x))$	Rule P (assumed)
{1}	(2)	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	Rule T, (1) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
{1}	(3)	$\neg(x)P(x)$	Rule T, (2), and I_1
{1}	(4)	$(\exists x)\neg P(x)$	Rule T, (3), and $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
{1}	(5)	$\neg(\exists x)Q(x)$	Rule T, (2), and I_2
{1}	(6)	$(x)\neg Q(x)$	Rule T, (5), and $\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$
{1}	(7)	$\neg P(y)$	Rule ES, (5), (6) and I_{12}
{1}	(8)	$\neg Q(y)$	Rule US, (6)
{1}	(9)	$\neg P(y) \wedge \neg Q(y)$	Rule T, (7), (8)and I_9
{1}	(10)	$\neg(P(y) \vee Q(y))$	Rule T, (9), and $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
{11}	(11)	$(x)(P(x) \vee Q(x))$	Rule P
{11}	(12)	$(P(y) \vee Q(y))$	Rule US
{1, 11}	(13)	$\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	Rule T, (10), (11), and I_9
{1, 11}	(14)	F	Rule T, and (13)

which is a contradiction.Hence, the statement is valid.

Example: Using predicate logic, prove the validity of the following argument: ”Every husband argues with his wife. x is a husband. Therefore, x argues with his wife”.

Solution: Let $P(x)$: x is a husband.

$Q(x)$: x argues with his wife.

Thus, we have to show that $(x)[P(x) \rightarrow Q(x)] \wedge P(x) \Rightarrow Q(y)$.

{1}	(1)	$(x)(P(x) \rightarrow Q(x))$	Rule P
{1}	(2)	$P(y) \rightarrow Q(y)$	Rule US, (1)
{1}	(3)	$P(y)$	Rule P
{1}	(4)	$Q(y)$	Rule T, (2), (3), and I_{11}

Example: Prove using rules of inference

Duke is a Labrador retriever.
All Labrador retriever like to swim.
Therefore Duke likes to swim.

Solution: We denote

$L(x)$: x is a Labrador retriever.

$S(x)$: x likes to swim.

d: Duke.

We need to show that $L(d) \wedge (x)(L(x) \rightarrow S(x)) \Rightarrow S(d)$.

{1}	(1)	$(x)(L(x) \rightarrow S(x))$	Rule P
{1}	(2)	$L(d) \rightarrow S(d)$	Rule US, (1)
{2}	(3)	$L(d)$	Rule P
{1, 2}	(4)	$S(d)$	Rule T, (2), (3), and I_{11}.

JNTUK Previous questions

1. Test the Validity of the Following argument: “All dogs are barking. Some animals are dogs. Therefore, some animals are barking”.
2. Test the Validity of the Following argument:
   “Some cats are animals. Some dogs are animals. Therefore, some cats are dogs”.
3. Symbolizes and prove the validity of the following arguments :
   (i) Himalaya is large. Therefore every thing is large.
   (ii) Not every thing is edible. Therefore nothing is edible.
4. a) Find the PCNF of $(\sim p \leftrightarrow r) \wedge (q \leftrightarrow p)$?
   b) Explain in brief about duality Law?

   c) Construct the Truth table for $\sim(\sim p \wedge \sim q)$?
   d) Find the disjunctive Normal form of $\sim(p \rightarrow (q \wedge r))$?
5. Define Well Formed Formula? Explain about Tautology with example?
6. Explain in detail about the Logical Connectives with Examples?

7. Obtain the principal conjunctive normal form of the formula $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$
8. Prove that $(\exists x)P(x) \wedge Q(x) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$. Does the converse hold?
9. Show that from i) $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$

   ii) $(\exists y)\ (M(y) \wedge \neg W(y))$ the conclusion $(x)(F(x) \rightarrow \neg S(x))$ follows.

10. Obtain the principal disjunctive and conjunctive normal forms of $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$. Is this formula a tautology?
11. Prove that the following argument is valid: No Mathematicians are fools. No one who is not a fool is an administrator. Sitha is a mathematician. Therefore Sitha is not an administrator.
12. Test the Validity of the Following argument: If you work hard, you will pass the exam. You did not pass. Therefore you did not work hard.
13. Without constructing the Truth Table prove that $(p \rightarrow q) \rightarrow q = p v q$?
14. Using normal forms, show that the formula $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$ is a tautology.
15. Show that $(x)\ (P(x) \vee Q(x)) \rightarrow (x)P(x) \vee (\exists x)Q(x$
16. Show that $\neg (P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Leftrightarrow (\neg P \vee Q)$

    $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$
17. Prove that $(\exists x)\ (P(x) \wedge Q(x)) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$
18. Example: Prove or disprove the validity of the following arguments using the rules of inference. (i) All men are fallible (ii) All kings are men (iii) Therefore, all kings are fallible.
19. Test the Validity of the Following argument:

    "Lions are dangerous animals, there are lions, and therefore there are dangerous animals."

MULTIPLE CHOICE QUESTIONS

1: Which of the following propositions is tautology?
A.(p v q)→q B. p v (q→p) C.p v (p→q) D.Both (b) & (c)
Option: C

2: Which of the proposition is p^ (~ p v q) is
A.A tautology B.A contradiction C.Logically equivalent to p ^ q D.All of above
Option: C

3: Which of the following is/are tautology?
A.a v b → b ^ c B.a ^ b → b v c C.a v b → (b → c) D.None of these
Option: B

4: Logical expression (A^ B) → (C' ^ A) → (A ≡ 1) is
A.ContradictionB.Valid C.Well-formed formula D.None of these
Option: D

5: Identify the valid conclusion from the premises Pv Q, Q → R, P → M, ˥M
A.P ^ (R v R) B.P ^ (P ^ R) C.R ^ (P v Q) D.Q ^ (P v R)
Option: D

6: Let a, b, c, d be propositions. Assume that the equivalence a ↔ (b v ˥b) and b ↔ c hold. Then truth value of the formula (a ^ b) → ((a ^ c) v d) is always
A.True B.False C.Same as the truth value of a D.Same as the truth value of b
Option: A

7: Which of the following is a declarative statement?
A. It's right B. He says C.Two may not be an even integer D.I love you
Option: B

8: P → (Q → R) is equivalent to
A. (P ^ Q) → R B.(P v Q) → R C.(P v Q) → ˥R D.None of these
Option: A

9: Which of the following are tautologies?
A.((P v Q) ^ Q) ↔ Q B.((P v Q) ^ ˥P) → Q C.((P v Q) ^ P) → P D.Both (a) & (b)
Option: D

10: If F1, F2 and F3 are propositional formulae such that F1 ^ F2 → F3 and F1 ^ F2→F3 are both tautologies, then which of the following is TRUE?
A.Both F1 and F2 are tautologies B.The conjuction F1 ^ F2 is not satisfiable
C.Neither is tautologies D.None of these

Option: B

11. Consider two well-formed formulas in propositional logic
    F1 : P →˥P F2 : (P →˥P) v (˥P →) Which of the following statement is correct?
    A.F1 is satisfiable, F2 is unsatisfiable B.F1 is unsatisfiable, F2 is satisfiable
    C.F1 is unsatisfiable, F2 is valid D.F1 & F2 are both satisfiable
    Option: C

12: What can we correctly say about proposition P1 : (p v ˥q) ^ (q →r) v (r v p)
    A.P1 is tautology B.P1 is satisfiable
    C.If p is true and q is false and r is false, the P1 is true
    D.If p as true and q is true and r is false, then P1 is true
    Option: C

13: (P v Q) ^ (P → R)^ (Q →S) is equivalent to
    A.S ^ R B.S → R C.S v R D.All of above
    Option: C

14: The functionally complete set is
    A.{ ˥, ^, v } B.{↓, ^ }C.{↑} D.None of these
    Option: C

15: (P v Q) ^ (P→R) ^ (Q → R) is equivalent to
    A.P B.Q C.R D.True = T
    Option: C

16: ˥(P → Q) is equivalent to
    A.P ^ ˥Q B.P ^ QC.˥P v Q D.None of these
    Option: A

17: In propositional logic , which of the following is equivalent to p → q?
    A.~p → q B.~p v q C.~p v~ q D.p →q
    Option: B

18: Which of the following is FALSE? Read ^ as And, v as OR, ~as NOT, →as one way implication and ↔ as two way implication?
    A.((x → y)^ x) →y B.((~x →y)^ (~x ^ ~y))→y C.(x → (x v y)) D.((x v y) ↔(~x v ~y))
    Option: D

19: Which of the following well-formed formula(s) are valid?
    A.((P → Q)^(Q → R))→ (P → R) B.(P → Q) →(˥P → ˥Q)
    C.(P v (˥P v ˥Q)) →P D.((P → R) v (Q → R)) → (P v Q}→R)
    Option: A

20: Let p and q be propositions. Using only the truth table decide whether p ↔ q does not imply p → ˥q is
    A.True B.False C.None D.Both A and B
    Option: A

UNIT-1

MATHEMATICAL LOGIC

Propositional calculus

Statements and Notations:

A number of words making a complete gra-mmatical structure having a sense and meaning and also meant an assertion in logic or mathematics is called a sentence. This assertion may be of two types declarative and non-declarative.

Definition: A proposition (or) statement is a declarative sentence that is either true (or) false, but not both. The truth or falsity of a statement is called its truth value.

Examples:

(1) $8+3=11$

(2) Paris is in England

(3) where are you going ?

(4) $4-x=8$

(5) close the door

(6) what a hot day

The sentences (1) and (2) are statements, the first is true and the second is false.
(3) is a question, not a declarative sentence, hence it is not a statement.

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(4) is a declarative sentence, but not a statement, since it is true or false depends on the value of x.

(5) is not a statement, it is a command.

(6) is not a statement, it is exclamation.

<u>Note:</u> Statements are denoted by P. Q, R, ... (or) p. q, r, ... which is also known as proposition variables. propositional variables can assume only two values, true (or) false. There are also two propositi-onal constants, T and F, that represent true or false respectively. If p denotes the proposition " The capital of U.P is Agra" then instead of saying the proposition " The capital of U.P is Agra" is false, we can simply say that the value of p is F.

Connectives

A proposition consisting of only a single propositional variable (or) a single propositional constant is called an atomic (primary) proposition. A proposition obtained from the combination of two (or) more propositions by means of Logical operators (or) connectives of two (or) more propositions (or) by negating a single proposition is called a molecular (or) compound (or) composite proposition.

The words and Symbols used to form Compound proposition are called connectives. The following Symbols are used to represent connectives.

S.No	Symbol used	Connective word	Name of the compound statement formed by the connective	Symbolic form
1	$\neg$	not	Negation	$\sim p$
2	$\wedge$	and	Conjunction	$p \wedge q$
3	$\vee$	or	Disjunction	$p \vee q$
4	$\Rightarrow, \rightarrow$	if .... then	Implication (or) Conditional	$p \Rightarrow q$
5	$\Leftrightarrow, \leftrightarrow$	if and only if	Equivalence (or) Bi-Conditional	$p \Leftrightarrow q$

If p and q are any two statements then $\neg p$, $p \wedge q$, $p \vee q$, $p \rightarrow q$ and $p \rightleftarrows q$ are also statements.

<u>Negation</u> : If p is any proposition, the negation of p, denoted by $\neg p$ (or) $\sim p$ and read as not p, is a proposition which is false when p is true and true when p is false.

For example, consider the statement p: Paris is in France. Then the negation of p is the statement $\sim p$: Paris is not in France.

The negation of the proposition q: No student is intelligent is $\sim q$: Some students are intelligent.

The truth table of $\sim p$ is

p	$\sim p$
T	F
F	T

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Find the negation of the following propositions

(i) It is cold

(ii) Ravi is rich.

(iii) Today is Thursday.

(iv) There are 12 months in a year.

(v) Vijayawada is the capital of A.P.

Conjunction

If p and q are two statements, then Conjunction of p and q is the compound statement denoted by $p \wedge q$ and read as "p and q". The compound statement $p \wedge q$ is true when both p and q are true, otherwise it is false. The truth values of p and q ($p \wedge q$) are given in the following truth table.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

Note: $p \wedge q$, $q \wedge p$ have same truth values.
Conjunction is symmetric.

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Example: (1) The Conjunction of

p : It is raining today.

q : There are 20 tables in this room is

$p \wedge q$: It is raining today and there are 20 tables in this room.

(2) If p : Jack went up the hill

q : Jill went up the hill then

$p \wedge q$: Jack and Jill went up the hill.

Disjunction

If p and q are two statements, then the disjunction of p and q is the compound statem-ent denoted by $p \vee q$ and read as "p or q". The compound statement $p \vee q$ is true if at least one of p or q is true. It is false when both p and q are false. The truth values of $p \vee q$ are given the following truth table.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Note: $p \vee q$, $q \vee p$ have same truth values. Disjunction is Symmetric.

Example: 1) The Disjunction of

p : I shall watch the game on T.V

q : go to the stadium is $p \vee q$: I shall watch the game on T.V (or) go to the stadium.

2) There is something wrong with the bulb or wiring.

Conditional Proposition

If p and q are propositions the compound proposition "if p then q" denoted by $p \Rightarrow q$ is called the Conditional proposition (or) Implication and the connective is the Conditional connective. The proposition 'p' is called antecedent (or) hypothesis and the proposition q is called the consequence (or) conclusion.

For example, Consider if it rains then I will carry an umbrella.

Here p : It rains, is antecedent

q : I will carry an umbrella, is Consequence.

The proposition $p \rightarrow q$ has a truth value F when p has the truth value T and q has the truth value F. otherwise its is truth value T.

The truth table is

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Example: 1) If I get the book then I begin to read.   Here p : I get the book

q : I begin to read

Symbolic form is $p \to q$.

2) Express in English the statement $p \to q$ where

p : The sun rises in the East.

Q : $4+3=7$.

Sol: If the sun rises in the East then $4+3=7$.

3) Write the following statement in Symbolic form.

statement: If either John prefers tea or Jim prefers coffee, then Rita prefers milk.

Sol: - p : John prefers tea

q : Jim prefers coeffee

r : Rita prefers milk.

Symbolic form is $(p \vee q) \to r$.

Bi-Conditional statement

The Compound statement formed by using the connective 'if and only if' is called Bi-conditional statement. If p and q are any two statements then the statement $p \rightleftarrows q$, which is read as p if and only if q is called a Bi-Conditional statement. The truth value of the statement $p \rightleftarrows q$ is T whenever both p, q have identical truth values. The truth table is

p	q	$p \rightleftarrows q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: The statement "p if and only if q" may also be expressed as

p is necessary and sufficient for q

if p then q and conversely.

Exercises

1. Find the truth values of each of the following statements.

(i) Tirupati is in A.P if and only if $1+3=4$

(ii) Hyderabad is in karnataka if and only if $1+3=4$

(iii) Chennai is in Tamilnadu iff $4 \div 2 = 2$

(iv) New Delhi is the capital of SriLanka iff $5-2=3$.

Sol: (i) and (iii) are true

and (ii) and (iv) are false.

(2) Determine the truth value of each of the following statements:

(i) Paris is in France and $2+2=4$

(ii) Paris is in France and $2+2=5$

(iii) Paris is in England and $2+2=4$

(iv) Paris is in England and $2+2=5$.

Sol: (i) T $(T \wedge T = T)$ (ii) F $(T \wedge F = F)$

(iii) F $(F \wedge T = F)$ (iv) F $(F \wedge F = F)$

(3) Determine the truth value of each of the following statements:

(i) $1+1=5$ (or) $2+2=4$

(ii) $7+1=5$ or $3+7=8$

(iii) $1+1=5$ or $3+3=4$

(iv) $2+5=7$ or $1+7=8$

Sol: (i) T $(F \vee T = T)$

(ii) F $(F \vee F = F)$

(iii) F $(F \vee F = F)$

(iv) T $(T \vee T = T)$

4. Let P: Ravi is rich, Q: Ravi is happy

Write each of the following in symbolic form

(i) Ravi is poor but happy

(ii) Ravi is neither rich nor happy.

(iii) Ravi is rich and unhappy.

Sol:- (i) $\neg P \wedge Q$ (ii) $\neg P \wedge \neg Q$ (iii) $P \wedge \neg Q$

5. Write the symbolic statement of

"If Rita and Sita go to I.T Camp and Jim and John go to P.C Camp then the College gets the good name".

Sol: Consider p: Rita go to I.T Camp

q: Sita go to I.T camp

r: Jim go to P.C Camp

s: John go to P.C Camp

t: college gets good name.

Symbolic form of given statement is

$(p \wedge q) \wedge (r \wedge s) \rightarrow t$.

6. Write the following statements in symbolic form.

(i) If the sun is shining today then $2+3>4$.

(ii) The crop will be destroyed if there is a flood.

Sol: (i) Take p: The sun is shining today.

$q : 2+3 > 4$

Given statement can be written as in Symbolic form as $p \to q$.

(ii) Consider p : The crop will be destroyed

q : There is a flood.

given statement can be written as in Symbolic form as $q \to p$.

Well Formed Formulas

Statement formulas: A statement formula is an expression denoted by a string consisting of variables, parentheses and connective symbols.

Well-Formed Formulas:

A well-formed formula can be generated by the following Rules:

(i) A statement variable standing alone is a well-formed formula.

(ii) If A is a well formed formula, then $\neg A$ is a well formed formula.

(iii) If A and B are well-formed formulas then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$ and $(A \leftrightarrow B)$ are well-formed formulas.

(iv) A string of symbols containing the statement variables, connectives and parentheses is a well-formed formula if and only if it can be obtained by finite applications of rules (1), (2) and (3).

<u>Example</u>: From the formulas given below select those which are well-formed formula according to the definition.

(1) $(p \rightarrow (p \vee q))$ ✓

(2) $((p \rightarrow (\sim p)) \rightarrow (\sim p))$ ✓

(3) $((p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)))$

It is not well-formed formula, because one of the parentheses in the beginning is missing.

(4) $((\sim p \rightarrow q) \rightarrow (q \rightarrow p)))$.

It is not well-formed formula, because one more parentheses in the end.

(5) $((p \wedge q) \rightleftarrows p)$ ✓

Truth tables

The truth value of a proposition is either true (denoted by T) or false (denoted by F). A truth table is a table that shows the truth value of a compound proposition for all possible cases.

1. Construct the truth table for $\sim p \wedge q$.

Sol:-

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	F

2. Construct the table for $(p \vee q) \vee \sim p$.

p	q	$p \vee q$	$\sim p$	$(p \vee q) \vee \sim p$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

3. Construct the truth tables for the following formulas.

(i) $\sim(\sim p \vee \sim q)$ (ii) $\sim(\sim p \wedge \sim q)$

4. Construct the truth tables for the following formulas:

(i) $(p \wedge q) \vee (q \wedge r) \vee (r \wedge p)$

(ii) $(p \vee q) \wedge (r \vee s)$.

5. Construct a truth table for each of the following compound statements

(i) $(p \to q) \wedge (\sim p \to q)$ (ii) $p \to (\sim q \vee r)$

Sol: 4 (ii)

p	q	r	s	$p \vee q$	$\sim r \vee s$	$(p \vee q) \wedge (\sim r \vee s)$
T	T	T	T	T	T	T
T	T	T	F	T	T	T
T	T	F	T	T	T	T
T	T	F	F	T	F	F
T	F	T	T	T	T	T
T	F	T	F	T	T	T
T	F	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	T	T	T	T
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	T	F	F	T	F	F
F	F	T	T	F	T	F
F	F	T	F	F	T	F
F	F	F	T	F	T	F
F	F	F	F	F	F	F

5 (ii)

p	q	r	$\sim q$	$\sim q \vee r$	$p \to (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

6. Construct the truth table for the following :

$[(p \vee q) \wedge (\sim r)] \leftrightarrow (q \rightarrow r)$.

Tautologies

Definition : A compound proposition that is always true for all possible truth values of its variables or, in otherwords contain only T in the last column of its truth table is called a tautology. A compound proposition that is always false for all possible truth values of its variables or in otherwords contain only F in the last column of its truth table is called a contradiction. A proposition that is neither a tautology nor a contradiction is called a contingen-cy.

propositions like

(i) The professor is either a woman or a man.

(ii) People either like watching TVs or they don't.

are always true and are called tautologies.

Example: $p \vee \sim p$ is a tautology since it always has truth value T.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Propositions like

(i) x is prime and x is an even integer greater than 8.

(ii) All men are good and all men are bad are always false and are called contradictions.

Example: $p \wedge \sim p$ is a contradiction.

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Exercise

Indicate which of the following formulas are tautologies or contradictions.

(i) $(p \rightarrow (p \vee q))$

(ii) $((p \vee q) \rightarrow p)$

(iii) $((p \rightarrow \sim p) \rightarrow \sim p)$

(iv) $((\sim q \wedge p) \wedge q)$

(v) $[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$

(vi) $(\sim p \rightarrow q) \rightarrow (q \rightarrow p)$

(vii) $((p \wedge q) \leftrightarrow p)$

Sol:- (i) The truth table for $(p \rightarrow (p \vee q))$ is

p	q	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Since all the entries in the last column of the truth table of $(p \to (p \vee q))$ are true.

Hence $(p \to (p \vee q))$ is a tautology.

(ii) The truth table for $((p \vee q) \to p)$

p	q	$p \vee q$	$p \vee q \to p$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Since all the entries in the last column of the truth table of $((p \vee q) \to p)$ are true and false.

Hence $((p \vee q) \to p)$ is a contingency.

(iii) The truth table for $((\sim q \wedge p) \wedge q)$

p	q	$\sim q$	$\sim q \wedge p$	$(\sim q \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

Since all the entries in the last column of the truth table of $((\sim q \wedge p) \wedge q)$ are false.

Hence $((\sim q \wedge p) \wedge q)$ is a contradiction.

Equivalence formulas

The propositions p and q are said to be logically equivalent if $p \leftrightarrow q$ is a tautology and is denoted by $p \Leftrightarrow q$ (or) $p \equiv q$

Example: $\sim(\sim p)$ is equivalent to p.

$p \wedge p$ is equivalent to p.

p	$\sim p$	$\sim(\sim p)$	$p \leftrightarrow \sim(\sim p)$
T	F	T	T
F	T	F	T

$p \leftrightarrow \sim(\sim p)$ is a tautology.

Hence $\sim(\sim p) \equiv p$.

p	p	$p \wedge p$	$p \leftrightarrow p \wedge p$
T	T	T	T
F	F	F	T

$p \leftrightarrow p \wedge p$ is a tautology.

Hence $p \wedge p \equiv p$.

Note: $\Leftrightarrow$ is only symbol, but not connective symbol.

1. Show that the propositions are $p \rightarrow q$ and $\sim p \vee q$ are logically equivalent.

(OR)

prove that $p \rightarrow q \equiv \sim p \vee q$.

Sol: The truth table for $(p \to q) \leftrightarrow (\sim p \vee q)$

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

$(p \to q) \leftrightarrow (\sim p \vee q)$ is a tautology.

Hence, $(p \to q) \equiv (\sim p \vee q)$.

2. prove that $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$.

3. Show the following equivalences

(i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(ii) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

(iii) $\sim (p \to q) \equiv p \wedge \sim q$

(iv) $\sim (p \rightleftarrows q) \equiv (p \wedge \sim q) \vee (\sim p \wedge q)$.

Sol:- (iv) Let $A = (p \wedge \sim q) \vee (\sim p \wedge q)$, $B = \sim (p \rightleftarrows q)$.

To show A and B are logically equivalent.

i.e $A \leftrightarrow B$ is a tautology.

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	A	$p \rightleftarrows q$	B	$A \leftrightarrow B$
T	T	F	F	F	F	F	T	F	T
T	F	F	T	T	F	T	F	T	T
F	T	T	F	F	T	T	F	T	T
F	F	T	T	F	F	F	T	F	T

All the truth values of $(A \leftrightarrow B)$ are true, hence $A \leftrightarrow B$ is a tautology.

$\therefore A \equiv B$.

4. Show that p is equivalent to the following formulas $\sim(\sim p)$, $p \wedge p$, $p \vee p$, $p \vee (p \wedge q)$, $p \wedge (p \vee q)$, $(p \wedge q) \vee (p \wedge \sim q)$, $(p \vee q) \wedge (p \vee \sim q)$.

Sol:

p	$\sim p$	$\sim(\sim p)$	$p \wedge p$	$p \vee p$
T	F	T	T	T
F	T	F	F	F

$\therefore p \Leftrightarrow \sim(\sim p) \Leftrightarrow p \wedge p \Leftrightarrow p \vee p$.

Consider A : $(p \vee (p \wedge q))$

B : ~~$p \vee$~~ $p \wedge (p \vee q)$

C : $(p \wedge q) \vee (p \wedge \sim q)$

D : $(p \vee q) \wedge (p \vee \sim q)$.

p	q	$p \wedge q$	$p \vee q$	$\sim q$	$p \wedge \sim q$	$p \vee \sim q$	A	B	C	D
T	T	T	T	F	F	T	T	T	T	T
T	F	F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F	F	F	F
F	F	F	F	T	F	T	F	F	F	F

$\therefore p \Leftrightarrow A \Leftrightarrow B \Leftrightarrow C \Leftrightarrow D$

Hence, all the formulas are equivalent.

Equivalent formulas

Let p, q and r be any three statements. Then all possible formulas may be written as

1. $p \vee p \Leftrightarrow p$, $p \wedge p \Leftrightarrow p$ (Idempotent laws)

2. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
   $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ (Associative laws)

3. $p \vee q \Leftrightarrow q \vee p$
   $p \wedge q \Leftrightarrow q \wedge p$ (Commutative laws)

4. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
   $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ (Distributive laws)

5. $p \wedge T \Leftrightarrow p$, $p \vee F \Leftrightarrow p$ (Identity laws)

6. $p \vee T \Leftrightarrow T$, $p \wedge F \Leftrightarrow F$ (Domination laws)

7. $p \vee \sim p \Leftrightarrow T$, $p \wedge \sim p \Leftrightarrow F$ (Negation laws)

8. $\sim(\sim p) \Leftrightarrow p$ (Double Negation law)

9. $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$
   $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$ (Demorgan's laws)

10. $p \vee (p \wedge q) \Leftrightarrow p$
    $p \wedge (p \vee q) \Leftrightarrow p$ (Absorption laws).

* Use truth tables to prove the distributive law,

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

Sol:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Since the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are identical.

$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$ is a tautology.

Therefore $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

i.e $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

Replacement process:

Consider the formula $A : p \rightarrow (q \rightarrow r)$.

The formula $q \rightarrow r$ is a part of the formula A. If we replace $q \rightarrow r$ by an equivalent formula $\sim q \vee r$ in A, we get another formula $B : p \rightarrow (\sim q \vee r)$. Here, we can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A is known as the Replacement process.

1. Show that $p \to (q \to r) \Leftrightarrow p \to (\sim q \vee r)$
$\Leftrightarrow (p \wedge q) \to r$

Sol: Consider $p \to (q \to r)$

$\Leftrightarrow p \to (\sim q \vee r)$ $\quad (\because q \to r \Leftrightarrow \sim q \vee r)$

$\Leftrightarrow \sim p \vee (\sim q \vee r)$

$\Leftrightarrow (\sim p \vee \sim q) \vee r$ $\quad$ (Associative Law)

$\Leftrightarrow \sim (p \wedge q) \vee r$ $\quad$ (DeMorgan's Law)

$\Leftrightarrow p \wedge q \to r.$

Hence, $p \to (q \to r) \Leftrightarrow (p \to (\sim q \vee r)) \Leftrightarrow (p \wedge q) \to r$

2. Show that the following equivalences

(i) $p \to (q \to p) \Leftrightarrow \sim p \to (p \to q)$

(ii) $p \to (q \vee r) \Leftrightarrow (p \to q) \vee (p \to r)$

Sol:- (i) Consider $p \to (q \to p)$

$\Leftrightarrow p \to (\sim q \vee p)$

$\Leftrightarrow \sim p \vee (\sim q \vee p)$

$\Leftrightarrow (\sim p \vee \sim q) \vee p$ $\quad$ (Associative law)

$\Leftrightarrow p \vee (\sim p \vee \sim q)$ $\quad$ (Commutative law)

$\Leftrightarrow (p \vee \sim p) \vee \sim q$ $\quad$ ($\because$ Associative law)

$\Leftrightarrow T \vee \sim q$ $\quad$ (Negation Law)

$\Leftrightarrow T$ $\quad$ (Domination law)

Also, $\sim p \to (p \to q)$

$\Leftrightarrow \sim p \to (\sim p \vee q)$ $\quad (\because p \to q \Leftrightarrow \sim p \vee q)$

$\Leftrightarrow \sim (\sim p) \vee (\sim p \vee q)$

$\Leftrightarrow p \vee (\sim p \vee q)$ $\quad$ (Double Negation law).

$\Leftrightarrow (p \vee \sim p) \vee q$ $\quad$ (Associative law)

$\Leftrightarrow T \vee q \Leftrightarrow T$ ($\because$ Domination law)

Hence, $p \rightarrow (q \rightarrow p) \Leftrightarrow \neg p \rightarrow (p \rightarrow q)$.

(ii) Consider $p \rightarrow (q \vee r)$

$\Leftrightarrow \neg p \vee (q \vee r)$

$\Leftrightarrow (\neg p \vee \neg p) \vee (q \vee r)$

$\Leftrightarrow (\neg p \vee q) \vee (\neg p \vee r)$

$\Leftrightarrow (p \rightarrow q) \vee (p \rightarrow r)$

Hence, $p \rightarrow (q \vee r) \Leftrightarrow (p \rightarrow q) \vee (p \rightarrow r)$.

(3) Show that $(\neg p \wedge (\neg q \wedge r)) \vee ((q \wedge r) \vee (p \wedge r))$ $\Leftrightarrow r$ by using Replacement process.

Sol: Consider, $(\neg p \wedge (\neg q \wedge r)) \vee ((q \wedge r) \vee (p \wedge r))$

$\Leftrightarrow ((\neg p \wedge \neg q) \wedge r) \vee ((q \wedge r) \vee (p \wedge r))$ ($\because$ Associative)

$\Leftrightarrow (\neg(p \vee q) \wedge r) \vee ((q \vee p) \wedge r)$ ($\because$ Demorgans Law Distributive)

$\Leftrightarrow (\neg(p \vee q) \wedge r) \vee ((p \vee q) \wedge r)$ ($\because$ Commutative law)

$\Leftrightarrow (\neg(p \vee q) \vee (p \vee q)) \wedge r$ (Distributive law).

$\Leftrightarrow T \wedge r$ (by negation Law)

$\Leftrightarrow r$ (Identity law).

Hence, $(\neg p \wedge (\neg q \wedge r)) \vee ((q \wedge r) \vee (p \wedge r)) \Leftrightarrow r$.

(4) Show that $\neg(p \rightleftarrows q) \Leftrightarrow (p \vee q) \wedge \neg(p \wedge q)$.

Sol: Consider, $\sim(p \rightleftarrows q)$

$\Leftrightarrow \sim((p \to q) \wedge (q \to p))$

$\Leftrightarrow \sim((\sim p \vee q) \wedge (\sim q \vee p))$

$\Leftrightarrow \sim(((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p)$

($\because$ Distributive Law)

$\Leftrightarrow \sim((\sim p \wedge \sim q) \vee (q \wedge \sim q)) \vee ((\sim p \wedge p) \vee (q \wedge p))$

($\because$ Distributive law).

$\Leftrightarrow \sim(((\sim p \wedge \sim q) \vee F) \vee (F \vee (q \wedge p))$ ($\because p \wedge \sim p \Leftrightarrow F$)

$\Leftrightarrow \sim(\sim(p \vee q) \vee F \vee (p \wedge q))$ (De morgan Law) Commutative

$\Leftrightarrow \sim(\sim(p \vee q) \vee (p \wedge q))$ ($F \vee P \equiv P$)

$\Leftrightarrow (p \vee q) \wedge \sim(p \wedge q)$.

Hence, $\sim(p \rightleftarrows q) \Leftrightarrow (p \vee q) \wedge \sim(p \wedge q)$.

5. Are $(p \to q) \to r$ and $p \to (q \to r)$ logically
* equivalent ? Justify your answer by using the rules of Logic to simply both expressions and also by using truth tables.

Sol: $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent because

consider $(p \to q) \to r$

$\Leftrightarrow (\sim p \vee q) \to r$ ($p \to q \Leftrightarrow \sim p \vee q$)

$\Leftrightarrow \sim(\sim p \vee q) \vee r$.

$\Leftrightarrow (\sim(\sim p) \wedge \sim q) \vee r$.

$\Leftrightarrow (p \wedge \sim q) \vee r$

$\Leftrightarrow (p \wedge r) \vee (\sim q \wedge r)$

Also, $p \to (q \to r) \Leftrightarrow p \to (\sim q \vee r)$

$\Leftrightarrow \sim p \vee (\sim q \vee r)$.

$\Leftrightarrow \sim p \vee \sim q \vee r$.

Truth table Method:

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	T	T	T
T	F	F	F	T	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	F	T	T

Hence the truth values of $(p \to q) \to r$ and $p \to (q \to r)$ are not identical.

Duality Law

Two formulas A and A^* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of each other. If the formulae A contains the special variables T or F, then

A^* its dual is obtained by replacing T by F and F by T in addition to the above mentioned interchanges.

Ex: 1. write the duals of (i) $(p \vee q) \wedge r$
(ii) $(p \wedge q) \wedge T$

Sol: Duals are (i) $(p \wedge q) \vee r$
(ii) $(p \vee q) \vee F$.

2. If the formula A is given by $A: \sim(p \vee q) \wedge (p \vee \sim(q \wedge \sim r))$, then find A^*.

Sol: Dual of A is $A^*: \sim(p \wedge q) \vee (p \wedge \sim(q \vee \sim r))$.

Tautological Implications

- In $p \to q$, the statement p is called antecedent and q is called consequent.

For any statement formula $p \to q$, the statement formula $q \to p$ is called its converse.

$\sim p \to \sim q$ is called the inverse.

and $\sim q \to \sim p$ is called the contra-positive.

Ex:- write the converse, inverse and contra-positive of the following proposition.

* If ΔABC is a right triangle then $|AB|^2 + |BC|^2 = |AC|^2$.

Converse: If $|AB|^2 + |BC|^2 = |AC|^2$ then ΔABC is a right triangle.

Inverse: If ΔABC is not a right triangle, then $|AB|^2 + |BC|^2 \neq |AC|^2$

Contra-positive: If $|AB|^2 + |BC|^2 \neq |AC|^2$ then ΔABC is not a right triangle.

Definition: A statement A is said to tautologically imply a statement B if and only if $A \to B$ is a tautology. We shall denote it by $A \Rightarrow B$, which is read as "A implies B".

Note: $\Rightarrow$ is not a connective, $A \Rightarrow B$ is not a statement formula.

* Prove that $(p \to q) \Rightarrow (\sim q \to \sim p)$.

Sol:

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$(p \to q) \to (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since all the entries in the last column are true.

$(p \to q) \to (\sim q \to \sim p)$ is a tautology.

Hence $(p \to q) \Rightarrow (\sim q \to \sim p)$.

Normal Forms

In this section, we will denote the word "Product" in place of "conjunction" and "sum" in place of "disjunction".

Definition: A product of the variables and their negations in a formula is called an elementary product.

Example: Let p, q be any two atomic variables. Then p, $\sim p$, $\sim p \wedge q$, $\sim q \wedge p$, $\sim q \wedge p \wedge \sim p$, $p \wedge \sim p$, $\sim p \wedge \sim q$ are some examples of elementary products.

Definition: A sum of the variables and their negations in a formula is called an elementary sum.

Example: Let p, q be any two atomic variables. Then p, $\sim p$, $\sim p \vee q$, $\sim q \vee p$, $p \vee \sim p$, $\sim q \vee p \vee \sim p$ are some examples of elementary sums.

Definition: Any part of an elementary sum (product) which is itself an elementary sum (product) is called a factor of the original sum (product).

Example: q, $q \vee \sim p$, $\sim p \vee \sim q$ are some of the factors of $q \vee \sim p \vee \sim q$ and $\sim q$, $p \wedge \sim p$, $\sim q \wedge p$

are some of the factors of $\sim q \wedge p \wedge \sim p$.

Disjunctive Normal Form (d.n.f)

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

Procedure to obtain a disjunctive normal form of a given Logical expression:

Three steps are required to obtain a disjunctive normal form through algebraic manipulations.

1. Remove all $\Rightarrow$ and $\Leftrightarrow$ by an equivalent expression containing the connectives $\wedge, \vee, \sim$ only.
2. Eliminate $\sim$ before sums and products by using the double negation or by using De Morgans Laws.
3. Apply the distributive law until a sum of elementary product is obtained.

Note: 1) The d.n.f of a given formula is not unique because different d.n.f's can be obtained for given formula if the distributive laws are applied in different ways.

2) Extended distributive law

$(p\wedge q)\vee(r\wedge s)=(p\wedge r)\vee(p\wedge s)\vee(q\wedge r)\vee(q\wedge s)$ is is very useful to obtain the disjunctive normal form:

*1. Obtain the disjunctive normal forms of the following.

(i) $p \wedge (p \rightarrow q)$

(ii) $\sim (p \rightarrow (q \wedge r))$

(iii) $\sim (p \vee q) \rightleftarrows (p \wedge q)$.

Sol:- (i) $p \wedge (p \rightarrow q) \Leftrightarrow p \wedge (\sim p \vee q)$

$\Leftrightarrow (p \wedge \sim p) \vee (p \wedge q)$ ($\because$ distributive)

which is the required disjunctive normal form.

(ii) $\sim (p \rightarrow (q \wedge r))$

$\Leftrightarrow \sim (\sim p \vee (q \wedge r))$ ($\because p \rightarrow q \Leftrightarrow \sim p \vee q$)

$\Leftrightarrow \sim(\sim p) \wedge (\sim q \vee \sim r)$

$\Leftrightarrow p \wedge (\sim q \vee \sim r)$

$\Leftrightarrow (p \wedge \sim q) \vee (p \wedge \sim r)$ ($\because$ distributive).

Which is the required disjunctive normal form.

(iii) $\sim (p \vee q) \rightleftarrows (p \wedge q)$

$\Leftrightarrow (\sim (p \vee q) \wedge (p \wedge q)) \vee (\sim(\sim(p \vee q)) \wedge \sim (p \wedge q))$

($\because r \rightleftarrows s \Leftrightarrow (r \wedge s) \vee (\sim r \wedge \sim s)$).

$\Leftrightarrow ((\sim p \wedge \sim q) \wedge (p \wedge q)) \vee ((p \vee q) \wedge (\sim p \vee \sim q))$

($\because$ by De morgans Law)

$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee ((p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q))$

($\because$ by distributive Law)

$\Leftrightarrow (\sim p \wedge \sim q \wedge p \wedge q) \vee (p \wedge \sim p) \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee (q \wedge \sim q)$

which is the required d.n.f of the given formula.

2. obtain disjunctive normal form of

$p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

Sol:-

$p \to ((p \to q) \wedge \sim(\sim q \vee \sim p))$

$\Leftrightarrow \sim p \vee ((p \to q) \wedge \sim(\sim q \vee \sim p))$ $(\because p \to q \Leftrightarrow \sim p \vee q)$

$\Leftrightarrow \sim p \vee ((\sim p \vee q) \wedge (q \wedge p))$ ($\because$ double negation law)

$\Leftrightarrow \sim p \vee ((\sim p \wedge q \wedge p) \vee (q \wedge q \wedge p))$ ($\because$ distributive law)

$\Leftrightarrow \sim p \vee (\sim p \wedge p \wedge q) \vee (q \wedge p)$ $(\because F \wedge q \Leftrightarrow F$, $p \vee F \Leftrightarrow p)$

$\Leftrightarrow \sim p \vee (F \wedge q) \vee (p \wedge q) \Leftrightarrow \sim p \vee (p \wedge q)$.

which the required d.n.f of the given formula.

Conjunctive Normal Form (c.n.f)

A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form' of the given formula.

The procedure of obtaining a c.n.f of a given formula is similar to the one given for d.n.f.

Example: $p \wedge (p \to q) \Leftrightarrow p \wedge (\sim p \vee q)$

which is the required c.n.f

Note: The conjunctive normal form is not unique.

1. obtain c.n.f of the following formulas:

(i) $((p \to q) \wedge \sim q) \to \sim p$ (ii) $((p \to q) \wedge \sim p) \to \sim q$

Sol: (i) $((p \to q) \wedge \sim q) \to \sim p$

$\Leftrightarrow ((\sim p \vee q) \wedge \sim q) \to \sim p$

$\Leftrightarrow \sim((\sim p \vee q) \wedge \sim q) \vee \sim p$

$\Leftrightarrow (\sim(\sim p \vee q) \vee \sim(\sim q)) \vee \sim p$

$\Leftrightarrow ((p \wedge \sim q) \vee q) \vee \sim p$

$\Leftrightarrow ((p \vee q) \wedge (\sim q \vee q)) \vee \sim p$

$\Leftrightarrow (p \vee q \vee \sim p) \wedge (\sim q \vee q \vee \sim p)$

which is the required c.n.f.

(ii) $((p \to q) \wedge \sim p) \to \sim q$

$\Leftrightarrow ((\sim p \vee q) \wedge \sim p) \to \sim q$

$\Leftrightarrow \sim((\sim p \vee q) \wedge \sim p) \vee \sim q$

$\Leftrightarrow \sim(\sim p \vee q) \vee \sim(\sim p)) \vee \sim q$

$\Leftrightarrow ((p \wedge \sim q) \vee p) \vee \sim q$

$\Leftrightarrow ((p \vee p) \wedge (\sim q \vee p)) \vee \sim q$

$\Leftrightarrow (p \vee p \vee \sim q) \wedge (\sim q \vee p \cancel{\vee \sim q})$

$\Leftrightarrow (p \vee \sim q) \wedge (\sim q \vee p)$

which is the required c.n.f.

2. obtain c.n.f. of the following

$\sim(p \vee q) \rightleftarrows (p \wedge q)$

Principal Disjunctive normal form (P.d.n.f)

Let p and q be two statement variables, then $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$ and $\sim p \wedge \sim q$ are called minterms of p and q. It may be noted that none of the minterms should contain both a variable and its negation. For given two variables, there are 2^2 minterms. The number of minterms in n variables are 2^n. For example, minterms for the three variables p, q and r are

$p \wedge q \wedge r$, $p \wedge q \wedge \sim r$, $p \wedge \sim q \wedge r$, $p \wedge \sim q \wedge \sim r$
$\sim p \wedge q \wedge r$, $\sim p \wedge q \wedge \sim r$, $\sim p \wedge \sim q \wedge r$, $\sim p \wedge \sim q \wedge \sim r$

Definition: Principal disjunctive normal form of a given formula can be defined as an equivalent formula consisting of disjunctives of minterms only. This is also called the Sum of products canonical form.

Methods to obtain P.d.n.f of a given formula:

1. **Truth table method:** For every truth value T of the given formula, select the minterm which is also has the value T for the same combination of the

truth values of the variables.

* obtain p.d.n.f of each of the following

(i) $p \to q$ (ii) $p \vee q$ (iii) $\neg(p \wedge q)$.

Sol:- The truth table is

p	q	$p \to q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	T	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	F	T

Hence $p \to q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$.

$p \vee q \Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$.

$\neg(p \wedge q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

which are p.d.n.forms of given formulas.

* obtain the p.d.n.f of the following formula.

$p \vee (\neg p \to (q \vee (\neg q \to r)))$.

Sol: Consider $A: \neg p \to (q \vee (\neg q \to r))$

$B: p \vee (\neg p \to (q \vee (\neg q \to r)))$.

p	q	r	$\neg p$	$\neg q$	$\neg q \to r$	$q \vee (\neg q \to r)$	A	B	Necessary minterms
T	T	T	F	F	T	T	T	T	$p \wedge q \wedge r$
T	T	F	F	F	T	T	T	T	$p \wedge q \wedge \neg r$
T	F	T	F	T	T	T	T	T	$p \wedge \neg q \wedge r$
T	F	F	F	T	F	F	T	T	$p \wedge \neg q \wedge \neg r$
F	T	T	T	F	T	T	T	T	$\neg p \wedge q \wedge r$
F	T	F	T	F	T	T	T	T	$\neg p \wedge q \wedge \neg r$
F	F	T	T	T	T	T	T	T	$\neg p \wedge \neg q \wedge r$
F	F	F	T	T	F	F	F	F	—

Therefore, p.d.n.f of $p \vee (\sim p \rightarrow (q \vee (\sim q \rightarrow r)))$ is
$(p \wedge q \wedge r) \vee (p \wedge q \wedge \sim r) \vee (p \wedge \sim q \wedge r) \vee (p \wedge \sim q \wedge \sim r)$
$\vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge q \wedge r)$.

2. Replacement Method (without constructing truth table): To obtain the p.d.n.f of a given formula without using its truth table, we use the following steps.

1. First replace the conditionals and Biconditionals by their equivalent formulas containing only $\wedge$, $\vee$ and $\sim$.
2. The negations are applied to the variables by using De morgan's Laws followed by the application of distributive laws.
3. Any elementary product which is a contradiction is droped.
4. Min terms are obtained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

truth values of the variables.

* obtain p.d.n.f of each of the following

(i) $p \to q$ (ii) $p \vee q$ (iii) $\neg(p \wedge q)$

Sol:- The truth table is

p	q	$p \to q$	$p \vee q$	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	T	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	F	T

Hence $p \to q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$.

$p \vee q \Leftrightarrow (p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge q)$.

$\neg(p \wedge q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

which are p.d.n.forms of given formulas.

* obtain the p.d.n.f of the following formula.

$p \vee (\neg p \to (q \vee (\neg q \to r)))$.

Sol: Consider $A : \neg p \to (q \vee (\neg q \to r))$

$B : p \vee (\neg p \to (q \vee (\neg q \to r)))$.

p	q	r	$\neg p$	$\neg q$	$\neg q \to r$	$q \vee (\neg q \to r)$	A	B	Necessary minterms
T	T	T	F	F	T	T	T	T	$p \wedge q \wedge r$
T	T	F	F	F	T	T	T	T	$p \wedge q \wedge \neg r$
T	F	T	F	T	T	T	T	T	$p \wedge \neg q \wedge r$
T	F	F	F	T	F	F	T	T	$p \wedge \neg q \wedge \neg r$
F	T	T	T	F	T	T	T	T	$\neg p \wedge q \wedge r$
F	T	F	T	F	T	T	T	T	$\neg p \wedge q \wedge \neg r$
F	F	T	T	T	T	T	T	T	$\neg p \wedge \neg q \wedge r$
F	F	F	T	T	F	F	F	F	—

1. obtain the p.d.n.f. for the following formulas, without using truth tables.

(i) $p \rightleftarrows q$ (ii) $\neg p \vee q$ (iii) $(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$.

Sol: (i) $p \rightleftarrows q$

$\Leftrightarrow (\neg p \vee q) \wedge (\neg q \vee p)$

$\Leftrightarrow ((\neg p \vee q) \wedge \neg q) \vee ((\neg p \vee q) \wedge p)$

$\Leftrightarrow ((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee (\neg p \wedge p) \vee (q \wedge p))$.

$\Leftrightarrow (\neg p \wedge \neg q) \vee F \vee F \vee (p \wedge q)$

$\Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

which is the required p.d.n.f.

(ii) $\neg p \vee q \Leftrightarrow (\neg p \wedge T) \vee (T \wedge q)$ $(\because \neg p \Leftrightarrow \neg p \wedge T)$

$\Leftrightarrow (\neg p \wedge (q \vee \neg q)) \vee ((p \vee \neg p) \wedge q)$

$\Leftrightarrow ((\neg p \wedge q) \vee (\neg p \wedge \neg q)) \vee ((p \wedge q) \vee (\neg p \wedge q))$

$\Leftrightarrow (p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$

which is the required p.d.n.f.

(iii) $(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r)$

$\Leftrightarrow ((p \wedge q) \wedge T) \vee ((\neg p \wedge r) \wedge T) \vee ((q \wedge r) \wedge T)$

$\Leftrightarrow ((p \wedge q) \wedge (r \vee \neg r)) \vee ((\neg p \wedge r) \wedge (q \vee \neg q)) \vee$
$((q \wedge r) \wedge (p \vee \neg p))$.

$\Leftrightarrow ((p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r)) \vee ((\neg p \wedge r \wedge q) \vee$
$(\neg p \wedge r \wedge \neg q)) \vee ((q \wedge r \wedge p) \vee (q \wedge r \wedge \neg p))$

$\Leftrightarrow (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (\neg p \wedge q \wedge r) \vee$
$(\neg p \wedge \neg q \wedge r)$.

which is the required p.d.n.f.

Principal Conjunctive Normal Form (P.C.N.F)

The dual of a minterm is called a maxterm. For a given number of variables the maxterm consists of disjunctions in which each variable or its negation, but not both, appears only once.

For two variables p and q, there are 2^2 maxterms given by $p \vee q$, $p \vee \sim q$, $\sim p \vee q$, $\sim p \vee \sim q$.

Maxterms for three variables p, q and r are

$p \vee q \vee r$, $p \vee q \vee \sim r$, $p \vee \sim q \vee r$, $p \vee \sim q \vee \sim r$

$\sim p \vee q \vee r$, $\sim p \vee q \vee \sim r$, $\sim p \vee \sim q \vee r$, $\sim p \vee \sim q \vee \sim r$.

Each of the maxterms has the truth value F for exactly one combination of the truth values of the variables. Different maxterms have the truth value F for different combinations of the truth values of the variables.

Definition: Principal conjunctive normal form of a given formula can be defined as an equivalent formula consists of conjunctive of maxterms only. This is also called the Product of Sums Canonical form. The process for obtaining P.C.N.F is similar to the one followed for P.D.N.F. For obtaining P.C.N.F of α, one can also construct the P.D.N.F of $\sim\alpha$ and apply negation.

1. obtain the principal conjunctive normal form of the following

(a) $p \wedge q$ using truth table.

(b) $(\sim p \Rightarrow r) \wedge (q \Leftrightarrow p)$ without using truth table.

Sol:- a) The truth table of $p \wedge q$ is

p	q	$p \wedge q$	Necessary Maxterms
T	T	T	—
T	F	F ✓	$p \vee \sim q$
F	T	F ✓	$\sim p \vee q$
F	F	F ✓	$\sim p \vee \sim q$

$\therefore p \wedge q \Leftrightarrow (p \vee \sim q) \wedge (\sim p \vee q) \wedge (\sim p \vee \sim q)$

which is the required p.c.n.f.

(b) $(\sim p \Rightarrow r) \wedge (q \Leftrightarrow p)$

$\Leftrightarrow (\sim(\sim p) \vee r) \wedge ((q \rightarrow p) \wedge (p \rightarrow q))$

$\Leftrightarrow (p \vee r) \wedge ((\sim q \vee p) \wedge (\sim p \vee q))$

$\Leftrightarrow [(p \vee r) \vee F] \wedge [(\sim q \vee p) \vee F] \wedge [(\sim p \vee q) \vee F]$ $\quad [\because p \vee F \equiv p]$

$\Leftrightarrow [(p \vee r) \vee (q \wedge \sim q)] \wedge [(\sim q \vee p) \vee (r \wedge \sim r)] \wedge [(\sim p \vee q) \vee (r \wedge \sim r)]$.

$\Leftrightarrow [(p \vee r \vee q) \wedge (p \vee r \vee \sim q)] \wedge [(\sim q \vee p \vee r) \wedge (\sim q \vee p \vee \sim r)] \wedge [(\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)]$.

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \sim q \vee r) \wedge (p \vee \sim q \vee \sim r) \wedge (\sim p \vee q \vee r) \wedge (\sim p \vee q \vee \sim r)$

which is the required p.c.n.f.

2. obtain the PCNF of the following formula $(\neg p \to r) \wedge (q \to p)$ by (i) using Truth table (ii) without using Truth table.

Sol: The truth table is

p	q	r	$\neg p$	A $\neg p \to r$	B $q \to p$	$A \wedge B$	Necessary maxterms
T	T	T	F	T	T	T	—
T	T	F	F	T	T	T	—
T	F	T	F	T	T	T	—
T	F	F	F	T	T	T	—
F	T	F	T	F	F	F ✓	$\neg p \vee q \vee \neg r$
F	T	T	T	T	F	F ✓	$\neg p \vee q \vee r$
F	F	T	T	T	T	T	—
F	F	F	T	F	T	F ✓	$\neg p \vee \neg q \vee \neg r$

$$(\neg p \to r) \wedge (q \to p) \Leftrightarrow (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$$
$$\wedge (\neg p \vee \neg q \vee \neg r)$$

which the required p.c.n.f.

(ii) $(\neg p \to r) \wedge (q \to p)$

$\Leftrightarrow (\neg(\neg p) \vee r) \wedge (\neg q \vee p)$

$\Leftrightarrow ((p \vee r) \vee T) \wedge ((\neg q \vee p) \vee T)$

$\Leftrightarrow ((p \vee r) \vee (q \wedge \neg q)) \wedge ((\neg q \vee p) \vee (r \wedge \neg r))$

$\Leftrightarrow ((p \vee r \vee q) \wedge (p \vee r \vee \neg q) \wedge (\neg q \vee p \vee r) \wedge$
$(\neg q \vee p \vee \neg r)$

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee \neg q \vee r) \wedge (p \vee \neg q \vee \neg r)$

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Theory of Inference for Statement Calculus

Consider a set of propositions $p_1, p_2, \ldots\ldots p_n$ and a proposition q. Then a compound proposition of the form $(p_1 \wedge p_2 \wedge p_3 \wedge \cdots\cdots p_n) \rightarrow q$ is called an argument. Here $p_1, p_2, \cdots p_n$ are called the Premises of the argument and q is called a conclusion of the argument.

It is a practice to write the above argument in the following tabular form :

$$\begin{array}{c} p_1 \\ p_2 \\ \vdots \\ p_n \\ \hline \therefore q \end{array}$$

Here, the three-dot symbol stands for " Therefore".

An argument with Premises $p_1, p_2, \cdots\cdots, p_n$ and Conclusion q is said to Valid if whenever each of Premises $p_1, p_2, \cdots\cdots, p_n$ is true, then the conclusion q is also true. otherwise the argument is invalid.

checking the validity of an argument form:

step1: construct truth table for the premises and the conclusion.

2: find the rows in which all the premises are true (critical rows)

3: check conclusion of all critical rows

a) If in each critical row the conclusion is true then the argument form is valid.

b) If there is a row in which conclusion is false then the argument form is invalid.

1. using truth table, show that the conclusion $c: \sim p$ follows from the premises $H_1: \sim q$, $H_2: p \to q$.

Sol:-

$$\left.\begin{array}{l} H_1: \sim q \\ H_2: p \to q \end{array}\right\} \text{premises}$$

$\therefore c: \sim p$ } conclusion

we construct a truth table for the premises and conclusion.

p	q	H_1 $\sim q$	H_2 $p \to q$	C $\sim p$
T	T	F	T	F
T	F	T	F	F
F	T	F	T	T
F	F	T	T	T ✓

We see in the above table that there is only one case in which both premises are true, namely, the last case and in this case the conclusion is true, hence the argument is valid.

2. Determine whether the following statement form is valid (or) invalid

$$\left.\begin{array}{l} p \to q \vee \sim r \\ q \to p \wedge r \end{array}\right\} \text{premises}$$

$$\therefore\ p \to r \quad \} \text{ conclusion}$$

Sol:- Construct truth table for the premises and the conclusion

p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \to q \wedge \sim r$	$q \to p \wedge r$	$p \to r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	F
T	F	T	F	F	~~F~~ T	F	T	T
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	T
F	T	F	T	T	F	T	F	T
F	F	T	F	F	F	T	T	T
F	F	F	T	F	F	T	T	T

The above table, in fourth critical row the conclusion is false.

Hence the given argument is invalid.

3. Show that the conclusion C follows from the premises $H_1, H_2, \ldots\ldots$ in the following cases.

(a) $H_1 : p \to q$ $\quad C : p \to p \wedge q$

(b) $H_1 : \sim p$, $H_2 : p \vee q$ $\quad C : q$

(c) $H_1 : \sim p \vee q$, $H_2 : \sim(q \wedge \sim r)$ $\quad H_3 : \sim r$ $\quad C : \sim p$

Sol: a) Given $H_1 : p \to q$

$C : p \to p \wedge q$.

we draw a truth table for the premises and conclusion.

		H_1		C
p	q	$p \to q$	$p \wedge q$	$p \to p \wedge q$
T	T	T ✓	T	T ✓
T	F	F	F	F
F	T	T ✓	F	T ✓
F	F	T ✓	F	T ✓

Therefore, C follows from the premises H_1 because in first, third, fourth rows H_1 value is T and C value T.

b) Given $H_1 : \sim p$, $H_2 : p \vee q$ } premises

$\therefore$ $C : q \to$ conclusion.

we draw a truth table for the premises and conclusion

p	q	$\sim p$	$p \vee q$
T	T	F	T
T	F	F	T
F	T ✓	T	T
F	F	T	F

Therefore, C follows from the premises H_1 and H_2 because in third row H_1, H_2 values are T and C value is T.

c) Given $H_1 : \sim p \vee q$
$H_2 : \sim(q \wedge \sim r)$
$H_3 : \sim r$
} premises

$\therefore C : \sim p$ Conclusion

We draw a truth table for the premises and conclusion

p	q	C $\sim p$	H_1 $\sim p \vee q$	r	H_3 $\sim r$	$q \wedge \sim r$	H_2 $\sim(q \wedge \sim r)$
T	T	F	T	T	F	F	T
T	T	F	T	F	T	T	F
T	F	F	F	T	F	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	F	F	T
F	F	T	T	T	F	F	T
F	T	T	T	F	T	T	F
F	F	T ✓	T ✓	F	T ✓	F	T ✓

Therefore, c follows from the premises H_1, H_2, H_3 because in last row H_1, H_2, H_3 have value T and c value T.

Rules of Inference

(i) Method (Without using truth table):

The truth table technique becomes tedious when the numbers of atomic variables present in all the formulae representing the premises and the conclusion is large. To overcome this disadvantage, we need to investigate other possible methods, without using the truth table. We now describe the process of derivation by which one demonstrates that a particular formula is a valid consequence of a given set of premises. Before we do this, we give two rules of inference which are called rules P and T.

Rule P: A Premise may be introduced at any point in the derivation.

Rule T: A formula S may be introduced in a derivation if S is a tautologically implied by one or more of the preceding formulas in the derivation.

Before we Proceed with the actual process of derivation, we list some important implications and equivalences that will be referred to frequently.

Implication formulas :

$I_1 : p \wedge q \Rightarrow p$
$I_2 : p \wedge q \Rightarrow q$ } (Simplification)

$I_3 : p \Rightarrow p \vee q$
$I_4 : q \Rightarrow p \vee q$ } (addition)

$I_5 : \sim p \Rightarrow p \rightarrow q$

$I_6 : q \Rightarrow p \rightarrow q$

$I_7 : \sim(p \rightarrow q) \Rightarrow p$

$I_8 : \sim(p \rightarrow q) \Rightarrow \sim q$

$I_9 : p, q \Rightarrow p \wedge q$

$I_{10} : \sim p, p \vee q \Rightarrow q$ (disjunctive Syllogism)

$I_{11} : p, p \rightarrow q \Rightarrow q$ (modus ponens)

$I_{12} : \sim q, p \rightarrow q \Rightarrow \sim p$ (modus tollens)

$I_{13} : p \rightarrow q, q \rightarrow r \Rightarrow p \rightarrow r$ (hypothetical Syllogism)

$I_{14} : p \vee q, p \rightarrow q, q \rightarrow r \Rightarrow r$ (dilemma)

Equivalent formulas

$E_1 : \sim(\sim p) \Leftrightarrow p$ (double negation law)

$E_2 : p \wedge q \Leftrightarrow q \wedge p$
$E_3 : p \vee q \Leftrightarrow q \vee p$ } (Commutative laws)

$E_4 : (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
$E_5 : (p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$ } (associative laws)

$E_6 : p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$
$E_7 : p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$ } (distributive laws)

$E_8 : \sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

$E_9 : \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

(De Morgan's laws)

$E_{10} : p \vee p \Leftrightarrow p$

$E_{11} : p \wedge p \Leftrightarrow p$

(idempotent laws)

$E_{12} : r \vee (p \wedge \sim p) \Leftrightarrow r$

$E_{13} : r \wedge (p \vee \sim p) \Leftrightarrow r$

$E_{14} : r \vee (p \vee \sim p) \Leftrightarrow T$

$E_{15} : r \wedge (p \wedge \sim p) \Leftrightarrow F$

$E_{16} : p \rightarrow q \Leftrightarrow \sim p \vee q$

$E_{17} : \sim(p \rightarrow q) \Leftrightarrow p \wedge \sim q$

$E_{18} : p \rightarrow q \Leftrightarrow \sim q \rightarrow \sim p$

$E_{19} : p \rightarrow (q \rightarrow r) \Leftrightarrow (p \wedge q) \rightarrow r$

$E_{20} : \sim(p \leftrightarrow q) \Leftrightarrow (p \leftrightarrow \sim q)$

$E_{21} : (p \leftrightarrow q) \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

$E_{22} : (p \leftrightarrow q) \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q)$

$E_{23} : (p \wedge T) \Leftrightarrow p$

$E_{24} : (p \vee F) \Leftrightarrow p$

(identity laws)

$E_{25} : (p \vee T) \Leftrightarrow T$

$E_{26} : (p \wedge F) \Leftrightarrow F$

(domination laws)

$E_{27} : (p \vee \sim p) \Leftrightarrow T$

$E_{28} : (p \wedge \sim p) \Leftrightarrow F$

(negation laws)

$E_{29} : p \vee (p \wedge q) \Leftrightarrow p$

$E_{30} : p \wedge (p \vee q) \Leftrightarrow p$

(absorption laws)

1. Demonstrate that r is a valid inference from the premises $p \to q$, $q \to r$ and p.

Sol:

$\{1\}$	(1)	$p \to q$	rule P
$\{2\}$	(2)	p	rule P
$\{1,2\}$	(3)	q	rule T, (1), (2) and $p, p \to q \Rightarrow q$.
$\{4\}$	(4)	$q \to r$	rule P
$\{1,2,4\}$	(5)	r	rule T (3), (4) and $p, p \to q \Rightarrow q$.

Hence the result.

2. Show that t is a valid conclusion from the premises $p \Rightarrow q$, $q \Rightarrow r$, $r \Rightarrow s$, $\sim s$ and $p \vee t$.

Sol:

$\{1\}$	(1)	$p \Rightarrow q$	rule P
$\{2\}$	(2)	$q \Rightarrow r$	rule P
$\{1,2\}$	(3)	$p \Rightarrow r$	rule T, (1),(2) and I_{13}.
$\{4\}$	(4)	$r \Rightarrow s$	rule P
$\{1,2,4\}$	(5)	$p \Rightarrow s$	rule T, (4),(5) and I_{13}.
$\{6\}$	(6)	$\sim s$	rule P
$\{1,2,4,6\}$	(7)	$\sim p$	rule T, (5),(6) and I_{12}.
$\{8\}$	(8)	$p \vee t$	rule P
$\{1,2,4,6,8\}$	(9)	t	rule T, (7), (8) and I_{10}.

Hence the result.

* 3. Show that $R \vee S$ follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$.

<u>Sol</u>:

$\{1\}$	1.	$C \vee D$	rule P
$\{2\}$	2.	$(C \vee D) \rightarrow \neg H$	rule P
$\{1,2\}$	3.	$\neg H$	rule T, (1), (2) and I_{11}
$\{4\}$	4.	$\neg H \rightarrow (A \wedge \neg B)$	rule P
$\{1,2,4\}$	5.	$(A \wedge \neg B)$	rule T, (3), (4) and I_{11}
$\{6\}$	6.	$(A \wedge \neg B) \rightarrow (R \vee S)$	rule P
$\{1,2,4,6\}$	7.	$(R \vee S)$	rule T, (5), (6) and I_{11}

Hence the result.

(OR)

$\{1\}$	1.	$(C \vee D) \rightarrow \neg H$,	rule P.
$\{2\}$	2	$\neg H \rightarrow (A \wedge \neg B)$	rule P
$\{1,2\}$	3.	$(C \vee D) \rightarrow (A \wedge \neg B)$	rule T, (1), (2) and I_{13}
$\{4\}$	4.	$(A \wedge \neg B) \rightarrow (R \vee S)$	rule P
$\{1,2,4\}$	5.	$(C \vee D) \rightarrow (R \vee S)$	rule T, (3), (4) and I_{13}
$\{6\}$	6.	$(C \vee D)$	rule P
$\{1,2,4,6\}$	7.	$(R \vee S)$	rule T, (5), (6) and I_{11}

Hence the result.

(4) Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

Sol:

{1}	1.	$P \vee Q$	Rule P
{1}	2.	$\neg P \rightarrow Q$	Rule T, E_1 and E_{16}
{3}	3.	$Q \rightarrow S$	Rule P
{1, 3}	4.	$\neg P \rightarrow S$	Rule T, (2), (3) and I_{13}
{1, 3}	5.	$\neg S \rightarrow P$	Rule T., (4) and E_{18}, E_1
{6}	6.	$P \rightarrow R$	Rule P
{1,3,6}	7.	$\neg S \rightarrow R$	Rule T, (5), (6) and I_{13}
{1,3,6}	8.	$S \vee R$	Rule T, (7) and E_{16}, E_1

Hence the result.

(5) Show that r logically follows from premises $p \rightarrow (q \rightarrow r)$, $p \wedge q$

Sol:

{1}	1.	$p \wedge q$	Rule P
{1}	2.	p	Rule T and I_1
{3}	3.	$p \rightarrow (q \rightarrow r)$	Rule P
{1, 3}	4.	$q \rightarrow r$	Rule T, (2), (3) and I_{11}
{1}	5.	q	Rule T and I_1
{1, 3}	6.	r	Rule T and I_{11}

Hence, r logically follows from given premises.

(6) Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

Sol:

$\{1\}$	1.	$P \to M$	Rule P
$\{2\}$	2.	$\neg M$	Rule P
$\{1,2\}$	3.	$\neg P$	Rule T, (1), (2) and I_{12}
$\{4\}$	4.	$P \vee Q$	Rule P
$\{1,2,4\}$	5.	Q	Rule T, (3), (4) and I_{10}
$\{6\}$	6.	$Q \to R$	Rule P
$\{1,2,4,6\}$	7.	R	Rule T, (5), (6) and I_{11}
$\{1,2,4,6\}$	8.	$R \wedge (P \vee Q)$	Rule T, (4), (7) and I_9

Hence the result.

(7) Show that $\neg Q, P \to Q \Rightarrow \neg P$.

Sol:

$\{1\}$	1.	$P \to Q$	Rule P
$\{1\}$	2.	$\neg Q \to \neg P$	Rule T, (1) and $P \to Q \Leftrightarrow \neg Q \to \neg P$
$\{3\}$	3.	$\neg Q$	Rule P
$\{1,3\}$	4.	$\neg P$	Rule T, (2), (3) and I_{11}

Hence the result.

8. Test whether the following is a valid argument.
"If Sachin hits a century, then we gets a free car. Sachin hits a century. Therefore, Sachin gets a free car".

Sol: Let us indicate the statements as follows:

p: Sachin hits a century.

q: Sachin gets a free car.

Hence, the given argument is of the form

$p \to q,\ p \Rightarrow q$.

$\{1\}$	1. $p \to q$		Rule P
$\{2\}$	2. p		Rule P
$\{1,2\}$	3. q		Rule T, (1), (2) and I_{11}.

Hence the result.

9. Test the validity of the following argument:

If a man is a bachelor, he is worried.
If a man is worried, he dies young.
Therefore, Bachelors die young.

Sol: Let us indicate the statements as follows:

p: a man is a bachelor

q: he is worried

r: he dies young

Hence, the given premises are $p \to q$, $q \to r$.
The conclusion is $p \to r$.
Thus, we need to prove that $p \to q, q \to r \Rightarrow p \to r$.

I^{st} Method:

{1}	1.	$p \to q$,	Rule P
{2}	2.	$q \to r$,	Rule P
{1, 2}	3.	$p \to r$.	Rule T, (1), (2) and I_{13}.

Hence, the given statements constitute a valid argument.

II^{nd} Method: $(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \wedge (q \to r)$	$(p \to q) \wedge (q \to r) \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Since all the truth values of $(p \to q) \wedge (q \to r) \to (p \to r)$ are true,

$(p \to q) \wedge (q \to r) \to (p \to r)$ is a tautology.

Hence, the given statements constitute a valid argument.

10. "If there was a ball game, then travelling was difficult. If they arrived on time, then travelling was not difficult. They arrived on time. Therefore, there was no ball game". Show that these statements constitute a valid argument.

Sol:- Let us indicate the statements as follows:

p: There was a ball game

q: Travelling was difficult

r: They arrived on time.

Hence, the given premises are $p \to q$, $r \to \sim q$ and r.

The conclusion is $\sim p$.

Thus, we need to prove that $p \to q, r \to \sim q, r \Rightarrow \sim p$.

$\{1\}$	1. $r \to \sim q$	Rule P	
$\{2\}$	2. r	Rule P	
$\{1,2\}$	3. $\sim q$	Rule T,	(1), (2) and I_{11}
$\{4\}$	4. $p \to q$	Rule P	
$\{4\}$	5. $\sim q \to \sim p$	Rule T,	(4) and E_{18}
$\{1,2,4\}$	6. $\sim p$	Rule T,	(3), (5) and I_{11}

Hence, the given statements constitute a valid argument.

11. By using the method of derivation, show that the following statements constitute a valid argument:

" If A works hard, then either B or C will enjoy. If B enjoys, then A will not work hard. If D enjoys, then C will not. Therefore, if A works hard, D will not enjoy".

Sol: Let us indicate the statements as follows:

p: A works hard

q: B will enjoy

r: C will enjoy

s: D will enjoy

Given premises are $p \to (q \vee r)$, $q \to \sim p$, $s \to \sim r$.

The conclusion is $p \to \sim s$.

Thus, we need to prove that $p \to (q \vee r)$, $q \to \sim p$, $s \to \sim r \Rightarrow p \to \sim s$.

{1}	1.	p	rule P (additional premise)
{2}	2.	$p \to (q \vee r)$	rule P
{1,2}	3.	$q \vee r$	rule T, (1),(2) and I_{11}.
{1,2}	4.	$\sim q \to r$	rule T, (3) and '$\sim p \to q \Leftrightarrow p \vee q$'
{1,2}	5.	$\sim r \to q$	rule T, (4) and '$p \to q \Leftrightarrow \sim q \to \sim p$'
{6}	6.	$q \to \sim p$,	rule P
{1,2,6}	7.	$\sim r \to \sim p$,	rule T, (5), (6) and I_{13}.
{1,2,6}	8.	$p \to r$,	rule T, (7) and '$p \to q \Leftrightarrow \sim q \to \sim p$'
{9}	9.	$s \to \sim r$,	rule P
{9}	10.	$r \to \sim s$	rule T, (9) and $p \to q \Leftrightarrow \sim q \to \sim p$
{1,2,6,9}	11.	$p \to \sim s$	rule T, (8), (10) and I_{13}

Hence the statements constitute a valid argument

Rules of Conditional Proof (or Deduction Theorem)

Another important rule used in Logic is the rule of conditional proof (or Rule CP). This rule can be defined as follows.

<u>Rule CP</u>: If we can derive S from R and a set of Premises, then we can derive $R \to S$ from a set of Premises alone.

<u>Note</u>: Rule CP is generally used if the Conclusion is of the form $R \to S$. In such Cases, R is taken as an additional premise and S is derived from the given premises and R.

12. Show that $r \to s$ can be derived from the premises $p \to (q \to s)$, $\neg r \vee p$ and q.

<u>Sol</u>: To prove $r \to s$, we can ~~to~~ include r as an additional premise.

$\{1\}$	1.	r	Rule P (assumed premise)
$\{2\}$	2.	$\neg r \vee p$	Rule P
$\{3\}$	3.	$r \to p$	Rule T, (2) and $p \to q \Leftrightarrow \neg p \vee q$.
$\{1,3\}$	4.	p	Rule T, (1),(3) and $p, p \to q \Rightarrow q$
$\{5\}$	5.	$p \to (q \to s)$	Rule P
$\{1,3,5\}$	6.	$q \to s$	Rule T, (4),(5) and $p, p \to q \Rightarrow q$.
$\{7\}$	7.	q	Rule P
$\{1,3,5,7\}$	8.	s	Rule T, (6),(7) and $p, p \to q \Rightarrow q$
$\{1,3,5,7\}$	9.	$r \to s$	Rule CP, (1) and (8).

Hence the result.

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Consistency of Premises

A set of formulas $H_1, H_2, \dots H_m$ is said to be consistent if their conjunction has truth value T for some assignment of truth values to the atomic variables appearing in $H_1, H_2, \dots, H_m$.

If for every assignment of the truth values to the atomic variables, at least one of the formulas $H_1, H_2, \dots, H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \dots H_m$ are called inconsistent if their conjunction implies a contradiction.

1. Show that the premises $a \to (b \to c)$, $d \to (b \land \neg c)$, $a \land d$ are inconsistent.

Sol:

{1}	1.	$a \land d$	Rule P
{1}	2.	a	Rule T, (1) and $p \land q \Rightarrow p$
{1}	3.	d	Rule T, (1) and $p \land q \Rightarrow q$
{4}	4.	$a \to (b \to c)$	Rule P.
{1, 4}	5.	$b \to c$	Rule T, (1), (4) and "$p, p \to q \Rightarrow q$"
{1, 4}	6.	$\neg b \lor c$	Rule T, (5) and $p \to q \Leftrightarrow \neg p \lor q$
{7}	7.	$d \to (b \land \neg c)$	Rule P
{7}	8.	$\neg(b \land \neg c) \to \neg d$	Rule T, (7) and $p \to q \Leftrightarrow \neg q \to \neg p$
{7}	9.	$\neg b \lor c \to \neg d$	Rule T, (8) and by DeMorgan Law
{1, 4, 7}	10.	$\neg d$	Rule T, (6), (9) and $p, p \to q \Rightarrow q$
{1, 4, 7}	11.	$d \land \neg d$	Rule T, (3), (10) and $p, q \Rightarrow p \land q$

But, $d \wedge \sim d \Leftrightarrow F$

Therefore our assumption is wrong.

Hence, the given premises are inconsistent.

Indirect Method of Proof:-

Proof by Contradiction is often referred to as indirect proof; in contrast with the direct proofs which have appeared earlier. A contradiction is typically in the form $R \wedge \sim R$, where R is any statement.

1. Using indirect method of proof, show that if $p \Rightarrow q \wedge r$, $q \vee s \Rightarrow t$ and $p \vee s$, then t.

Sol:-

{1}	1.	$p \Rightarrow q \wedge r$	Rule P
{2}	2.	$q \vee s \Rightarrow t$	Rule P
{3}	3.	$p \vee s$	Rule P
{4}	4.	$\sim t$	Rule P (additional premise)
{2, 4}	5.	$\sim(q \vee s)$	Rule T, (2), (4) and "$\sim q, p \rightarrow q \Rightarrow \sim p$"
{2, 4}	6.	$\sim q \wedge \sim s$	Rule T, (5) and De Morgan's law
{2, 4}	7.	$\sim q$	Rule T, (6) and "$p \wedge q \Rightarrow p$"
{2, 4}	8.	$\sim s \wedge \sim q$	Rule T, (6) Commutative
{2, 4}	9.	$\sim s$	Rule T, (6) and "$p \wedge q \Rightarrow q$"
{3}	10.	$s \vee p$	Rule T, (3) and Commutative
{2, 3, 4}	11.	p	Rule T, (9), (10) and $\sim p, p \vee q \Rightarrow q$
{1, 2, 3, 4}	12.	$q \wedge r$	Rule T, (11), (1) and I_{11}
{1, 2, 3, 4}	13.	q	Rule T, (12) and $p \wedge q \Rightarrow p$
{1, 2, 3, 4}	14.	$q \wedge \sim q$	Rule T, (7), (13)

Contradiction.

2. Prove by indirect method

$\neg q$, $p \Rightarrow q$ and $p \vee t$, then t.

Sol:

{1}	1. $\neg q$		Rule P
{2}	2. $p \Rightarrow q$		Rule P
{3}	3. $p \vee t$		Rule P
{4}	4. $\neg t$		Rule P. Additional premise.
{3}	5. $t \vee p$		Rule T, (3) and Commutative.
{4, 3}	6. p		Rule T (4), (5) and I_{10}.
{2,4,3}	7. q		Rule T, (2), (6) and I_{11}
{1,2,4,3}	8. $q \wedge \neg q$		Rule T, (1), (7)

($\because q \wedge \neg q \Leftrightarrow F$).

Contradiction.

3. Show that the following set of premises ~~is inconsistent~~ inconsistent:

"If the contract is valid, then John is liable for penalty. If John is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid and the bank will loan him money."

Sol:- Let us indicate the statements as follows:

p : The contract is valid

q : John is liable for penalty

r : John will go bankrupt

s : Bank will loan him money.

Then, the given premises are $p \to q$, $q \to r$, $s \to \sim r$, $p \wedge s$.

$\{1\}$	1.	$p \to q$	Rule P
$\{2\}$	2.	$q \to r$	Rule P
$\{1,2\}$	3.	$p \to r$	Rule T (1),(2) and I_{13}
$\{4\}$	4.	$s \to \sim r$	Rule P
$\{4\}$	5.	$r \to \sim s$	Rule T, (4) and $p \to q \Leftrightarrow \sim q \to \sim p$
$\{1,2,4\}$	6.	$p \to \sim s$	Rule T, (3),(5) and I_{13}
$\{1,2,4\}$	7.	$\sim p \vee \sim s$	Rule T, (6) and "$p \to q \Rightarrow \sim p \vee q$"
$\{1,2,4\}$	8.	$\sim(p \wedge s)$	Rule T, (7) and De morgan Laws
$\{9\}$	9.	$(p \wedge s)$	Rule P
$\{1,2,4,9\}$	10.	$(p \wedge s) \wedge \sim(p \wedge s)$	Rule T, (8),(9) and "$p, q \Rightarrow p \wedge q$"

But, $(p \wedge s) \wedge \sim(p \wedge s) \Leftrightarrow F$.

Thus, our assumption is wrong.

Hence, the given premises are inconsistent.

4. Show that the following premises are inconsistent.

1. If Jack misses many classes because of illness, then he fails high school.
2. If Jack fails high school, then he is uneducated
3. If Jack reads a lot of books, then he is not uneducated.
4. Jack misses many classes because of illness and reads a lot of books.

Sol: Let us indicate the statements as follows.

p : Jack misses many classes because of illness

q : Jack fails high school

r : Jack is uneducated

s : Jack reads a lot of books.

Then, the given premises are

$p \to q$, $q \to r$, $s \to \sim r$, $p \wedge s$.

$\{1\}$	1.	$p \to q$	Rule P
$\{2\}$	2.	$q \to r$	Rule P
$\{1,2\}$	3.	$p \to r$	Rule T, (1), (2) and I_{13}
$\{4\}$	4.	$s \to \sim r$	Rule P
$\{4\}$	5.	$r \to \sim s$	Rule T, (4) and "$p \to q \Leftrightarrow \sim q \to \sim p$"
$\{1,2,4\}$	6.	$p \to \sim s$	Rule T, (3), (5) and I_{13}
$\{1,2,4\}$	7.	$\sim p \vee \sim s$	Rule T, (6) and "$p \to q \Leftrightarrow \sim p \vee q$"
$\{1,2,4\}$	8.	$\sim (p \wedge s)$	Rule T, (7) and De morgan's law

{9} 9. $p \wedge s$ Rule P

{1,2,4,9} 10. $\sim(p \wedge s) \wedge (p \wedge s)$ Rule T, (8), (9) and $p, q \Rightarrow p \wedge q$

But, $\sim(p \wedge s) \wedge (p \wedge s) \Leftrightarrow F$; which is a contradiction

Thus, our assumption is wrong.

Hence, the given premises are inconsistent.

Predicate Calculus

A part of a declarative sentence describing the properties of an object (or) relation among objects is called a predicate.

The Logic based upon the analysis of predicate in any statement is called Predicate logic.

Consider the following two statements

1. Radha is a girl.
2. Seeta is a girl.

Here, the first part Radha (or) Seeta is the subject of the statement. The second part "is a girl", which refers to a property that the subject can have, is called the predicate. We symbolize a predicate by a capital letter and the names of individuals or objects in general by lower case letters. Here, the predicate "is a girl" is denoted by G, Radha is r and Seeta by s.

Hence, the statement "Radha is a girl" is denoted by $G(r)$, while "Seeta is a girl" is denoted by $G(s)$.

In general, any statement of the type p is Q, where Q is a predicate and p is the subject, can be denoted by $Q(p)$.

A predicate requiring m $(m>0)$ names is called an m-place predicate.

<u>Examples:</u> Consider the examples

(i) Indu is a student.

The Predicate S: is a student is a 1-place predicate because it is related to one object (or) name, say Indu (i).

The statement can be translated as $S(i)$.

(ii) phaneendhar is taller than Mohan.

The predicate "is taller than" is a 2-place predicate. This statement can be translated as $T(p, m)$ where T symbolizes "taller than", p denotes "phaneendhar" and m denotes "Mohan".

Examples of 3-place predicates and 4-place predicates are

(i) Suman sits between Rahul and Nithin.

(ii) Venkat and Karthik played bridge against Rohith and Raj.

* A Statement is called a 0-place predicate when no names are associated with the statement.

Statement Function, variables and Quantifiers

Consider the following statements:

1. Somu is mortal
2. India is mortal
3. A table is mortal

Let H be the Predicate 'is mortal', s be the name 'Somu', i be the name 'India' and t be the table. Then $H(s)$, $H(i)$ and $H(t)$ denote the above statements. If we write $H(x)$ for 'x is mortal', then $H(s)$, $H(i)$ and $H(t)$ can be obtained from $H(x)$ by replacing x by an appropriate name.

Note that $H(x)$ is not a statement, but when x is replaced by the name of an object, it becomes a statement.

* A simple statement function of one variable

is defined to be an expression consisting of a Predicate Symbol and an individual Variable. Such a statement function becomes a statement when the variable is replaced by the name of any object. we can form compound statement functions by combining one or more simple statement functions and the logical connectives.

Note:- 1. $M(x) \wedge H(x)$, $M(x) \rightarrow H(x)$, $\neg H(x)$, $M(x) \vee \neg H(x)$, etc, are all examples of compound statement functions, where $M(x)$ represents "x is a man" and $H(x)$ represents "x is mortal".

2. Consider the statement function of two variables

$G(x, y)$: x is taller than y.

If both x and y are replaced by the names of objects, we get a statement.

If m represents Mr. Madhu and f represents Mr. Gopi, then we have $G(m, f)$: Mr. Madhu is taller than Mr. Gopi; and

$G(f, m)$: Mr. Gopi is taller than Mr. Madhu.

* Form a compound statement from the following

(i) $M(x)$: x is a man

(ii) $H(y)$: y is mortal.

Given $M(x)$: x is a man

and $H(y)$: y is a mortal. Then we write

$M(x) \wedge H(y)$: x is a man and y is mortal.

2. Let $P(x)$ denote the statement '$x > 3$'. What are the truth values of $P(4)$ and $P(2)$?

Sol: Given $P(x)$ denote the statement $x > 3$.

$P(4)$ is the statement $4 > 3$, which is true

$P(2)$ is the statement $2 > 3$, which is false.

Hence, the truth values of $P(4)$ and $P(2)$ are T and F, respectively.

3. Let $Q(x, y)$ denote the statement '$x = y + 3$'. What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Sol:

$Q(x, y)$: $x = y + 3$.

$Q(1, 2)$: $1 = 2 + 3$, which is False

$Q(3, 0)$: $3 = 0 + 3$, which is True.

Hence, the truth values of $Q(1, 2)$ and $Q(3, 0)$ are F and T respectively.

Quantifiers

There are two types of quantifiers. They are the following:

1. Universal quantifiers
2. Existential quantifiers

Universal quantifiers:

Consider the following statements:

1. All men are mortal
2. Every apple is red.
3. An integer is either positive or negative.

These statements can be written as

for all x, if x is a man, then x is mortal.

for all x, if x is an apple, then x is red.

for all x, if x is an integer, then x is either positive or negative.

These statements can be symbolized as

$(x)\ (M(x) \rightarrow H(x))$

$(x)\ (A(x) \rightarrow R(x))$

$(x)\ (Z(x) \rightarrow (P(x) \vee N(x)))$, where

$M(x)$: x is a man ; $H(x)$: x is mortal

$A(x)$: x is an apple ; $R(x)$: x is red

$Z(x)$: x is an integer ; $P(x)$: x is positive

$N(x)$: x is negative.

We symbolize 'for all x' by the symbol $(\forall x)$ (or) (x) and it is called universal quantifier.

The universal quantification of $P(x)$ is the proposition $P(x)$ that is true for all values of x in the universe of discourse.

The notation '$(\forall x)\ P(x)$' denotes the universal quantification of $P(x)$.

The proposition $\forall x\ P(x)$ is also expressed as 'for all x $P(x)$' (or) 'for every x $P(x)$' (or) 'for each x $P(x)$'

Existential quantifiers

Consider the following statements:

1. There exists a man
2. Some men are clever.
3. Some real numbers are rational.

These statements can be written as

* There exists an x such that x is a man
* There exists an x such that x is a man and x is clever.
* There exists an x such that x is a real number and x is a rational number.

The above statements can be symbolized as

$(\exists x)\ M(x)$; $(\exists x)\ (M(x) \wedge C(x))$

$(\exists x)\ (R_1(x) \wedge R_2(x))$, respectively.

Where $M(x)$: x is a man

$C(x)$: x is clever

$R_1(x)$: x is a real number

$R_2(x)$: x is a rational number.

Existential quantifier is denoted by the symbol $(\exists x)$ and stands for 'there is at least one x such that' (or) 'there exists an x such that' (or) "for some x".

The existential quantification of $P(x)$ is the proposition "there exists an element 'x' in the universe of discourse such that $P(x)$ is true."

We use the notation $\exists x\, P(x)$ for the existential quantification; here, '$\exists$' is called the existential quantifier.

<u>Ex</u>: What is the truth value of the following quantifications?

(i) $(\forall x)\, Q(x)$, where $Q(x)$ is the statement '$x < 2$' and the universe of discourse consists of all real numbers?

(ii) $(\forall x)\, P(x)$, where $P(x)$ is the statement '$x^2 < 10$' and the universe of discourse consists of the positive integers not exceeding 4?

<u>Sol</u>: (i) Given, $(\forall x)\, Q(x) : x < 2$

$Q(x)$ is not true for every real number x, since $Q(3)$ is false.

Thus, $(\forall x)\, Q(x)$ is false

(ii) Given $(\forall x)\ P(x) : x^2 < 10$ is $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$.

Since the universe of discourse is $\{1, 2, 3, 4\}$.

$P(4) : 4^2 < 10$, which is not true i.e false

Hence, $(\forall x)\ P(x)$ is false.

(2) What is the truth value of the following quantifications?

(i) $\exists x\ P(x)$, where $P(x)$ denotes the statement '$x > 3$' and the universe of discourse consists of all real numbers?

(ii) $\exists x\ Q(x)$, where $Q(x)$ denotes the statement '$x = x+1$' and the universe of discourse is the set of all real numbers?

Sol: (i) Given, $P(x) : x > 3$.

when $x = 4$, $4 > 3$ which is true.

Hence $x = 4$, $\exists x\ P(x)$ is true.

(ii) Given $Q(x) : x = x + 1$.

Since $Q(x)$ is false for every real number x,

i.e $\exists x\ Q(x)$ is false.

(3) Write the following statements in Symbolic form:

(i) Something is good

(ii) Everything is good

(iii) Nothing is good

(iv) Something is not good.

Sol: Statement (i) means "There exists at least one x such that, x is good"

Statement (ii) means, "for all x, x is good."

Statement (iii) means, "for all x, x is not good."

Statement (iv) means, "There is at least one x such that, x is not good."

Thus if $G(x)$: x is good, then

Statement (i) Can be denoted by $(\exists x) G(x)$

Statement (ii) Can be denoted by $(\forall x) G(x)$

Statement (iii) Can be denoted by $(\forall x) \sim G(x)$.

Statement (iv) Can be denoted by $(\exists x) \sim G(x)$.

(4). Let $K(x)$: x is a man

$L(x)$: x is mortal

$M(x)$: x is an integer

$N(x)$: x is either positive (or) negative.

Express the following using quantifiers

a) All men are mortal

b) Any integer is either positive (or) Negative.

Sol: a) The given statement Can be written as for all x, if x is a man then x is mortal and this can be expressed as $(x)(K(x) \to L(x))$

(b) The given statement can be written as for all x, if x is an integer then x is either positive or negative and this can be expressed as $(x)(M(x) \to N(x))$.