



# ADITYA COLLEGE OF ENGINEERING & TECHNOLOGY

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## III SEMESTER (2025-26), SESSIONAL 1 EXAMINATION QUESTION BANK ELECTRONICS AND COMMUNICATION ENGINEERING PROBABILITY THEORY AND STOCHASTIC PROCESS (231BS3T04)

Q.No	Question	Marks	CO No.	Taxonomy										
	<b>UNIT-1</b>													
1	A) State and Prove Total Probability Theorem  B) Consider the height of clouds is a Gaussian random variable, X, with mean 1800 and variance 450. Find the probability that the height of clouds is greater than 1850 meter.	5	CO1	Apply										
2	A) Define probability density function with relevant expressions and prove the properties of pdf  B) A random variable X has pdf shown below. i) Find the value of k. ii) Find $P(1/4 < X < 1/2)$ . $f_X(x) = \begin{cases} kx, & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$	5	CO1	Understand										
3	A) Deduce the expressions of PDF from the pdf expressions of the Exponential and Rayleigh random variables.  B) Consider a random variable X, with the PMF as tabulated below: Find $F_X(x)$ and plot <table border="1"><tr><td>X=x</td><td>0</td><td>1</td><td>2</td><td>3</td></tr><tr><td>P(X=x)</td><td>1/8</td><td>1/8</td><td>1/4</td><td>1/2</td></tr></table>	X=x	0	1	2	3	P(X=x)	1/8	1/8	1/4	1/2	5	CO1	Evaluate
X=x	0	1	2	3										
P(X=x)	1/8	1/8	1/4	1/2										
4	A) Define and explain the following with an example: i. Equally likely events ii. Exhaustive events iii. Mutually exclusive events  B) Suppose the waiting time of data packets in a computer network is an exponential random variable with Pdf $f_X(x) = 0.5 \exp(-0.5x)u(x)$ . (i) Find $F_X(x)$ (ii) Find the $P(0.1 < X \leq 0.5)$ .	5	CO1	Remember										
5	A) Define Random Variable. Classify its types each with one example  B) A random variable X has pdf $f_X(x) = k(1+x^2)$ , for $0 \leq x \leq 1$ . Find the constant k and Distribution function of random variable.	5	CO1	Remember										

<b>6</b>	A)	State and prove Conditional Distribution Function Properties	5	CO1	Apply
	B)	Explain in detail (i) Binomial distribution (ii) Uniform Distribution	5	CO1	Understand
<b>UNIT-2</b>			<b>Marks</b>	<b>CO No.</b>	<b>Taxonomy</b>
<b>1</b>	A)	Evaluate mean and Variance of Uniform random variable.	5	CO2	Evaluate
	B)	Estimate the nth moment about origin using Moment Generating Function	5	CO2	Analyse
<b>2</b>	A)	Explain about the Moments about the Mean. Write the physical significance of Variance	5	CO2	Understand
	B)	X is a uniformly distributed random variable in the interval (a, b). If $Y = \sqrt{X}$ , Obtain the density of Y.	5	CO2	Apply
<b>3</b>	A)	State and prove Chebyshev's inequality	5	CO2	Apply
	B)	Evaluate the characteristic function of a random variable with Pdf $f_X(x) = \frac{1}{b} e^{-(\frac{x-a}{b})}$ , $x>a$	5	CO2	Evaluate
<b>4</b>	A)	What is meant by Expectation? State and prove its properties.	5	CO2	Remember
	B)	Evaluate the Moment generating function of a random variable with Pdf $f_X(x) = \lambda e^{-\lambda x} u(x)$ , $\lambda > 0$	5	CO2	Evaluate
<b>5</b>	A)	Prove the properties of Moment Generating Function	5	CO2	Apply
	B)	Consider the random variable with probability density function $f_X(x) = \begin{cases} \left(\frac{1}{6}\right)x, & 2 \leq x \leq 4 \\ 0, & \text{otherwise} \end{cases}$ Find (i) $E[X]$ (ii) $E[X^2]$ (iii) $E[2X+3]$ and (iii) $\sigma_X^2$	5	CO2	Understand
<b>6</b>	A)	Explain about Monotonic and Non-Monotonic transformation of a continuous random variable	5	CO2	Understand
	B)	A random variable X has the values -4,-1,2,3 and 4 , each with probability 1/5 . Find the density function, mean and variance of the random variable $Y = 3X^3$	5	CO2	Understand

Here you go—step-by-step and exam-ready.

## 1(A) Total Probability Theorem

### Statement

Let  $\{B_1, B_2, \dots, B_n\}$  be a **partition** of the sample space  $S$ :

- $B_i \cap B_j = \emptyset$  for  $i \neq j$  (mutually exclusive),
- $\bigcup_{i=1}^n B_i = S$ ,
- $P(B_i) > 0$  for all  $i$ .

Then for any event  $A$ ,

$$P(A) = \sum_{i=1}^n P(A | B_i)P(B_i).$$

(For a countable partition, the sum extends to  $\infty$ .)

### Proof

1. Because the  $B_i$ 's partition  $S$ , we can write

$$A = A \cap S = A \cap \left( \bigcup_{i=1}^n B_i \right) = \bigcup_{i=1}^n (A \cap B_i).$$

The sets  $A \cap B_i$  are **pairwise disjoint**.

2. By **countable additivity** of probability,

$$P(A) = \sum_{i=1}^n P(A \cap B_i).$$

3. Using the definition of conditional probability,

$$P(A \cap B_i) = P(A | B_i)P(B_i) \quad (\text{since } P(B_i) > 0).$$

4. Substituting into the sum gives

$$P(A) = \sum_{i=1}^n P(A | B_i)P(B_i).$$

QED.

### Special 2-event form

If  $S$  is partitioned by  $\{B, B^c\}$ ,

$$P(A) = P(A | B)P(B) + P(A | B^c)P(B^c).$$


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**1(B) Cloud height is Gaussian:**  $X \sim N(\mu = 1800, \sigma^2 = 450)$ .

Find  $P(X > 1850)$ .

**Step 1: Standardize.**

$$\sigma = \sqrt{450} = \sqrt{9 \times 50} = 3\sqrt{50} \approx 3 \times 7.071067 \approx 21.213.$$

$$z = \frac{1850 - \mu}{\sigma} = \frac{1850 - 1800}{21.213} = \frac{50}{21.213} \approx 2.357.$$

**Step 2: Use the standard normal CDF  $\Phi(z)$ .**

$$P(X > 1850) = 1 - \Phi(2.357).$$

From standard normal tables (or calculation),

$$\Phi(2.357) \approx 0.99079 \text{ to } 0.99082.$$

**Step 3: Tail probability.**

$$P(X > 1850) \approx 1 - 0.99081 \approx 0.00919.$$

**Answer:**

$P(X > 1850) \approx 0.0092 \text{ (about 0.92%).}$

Sure! Let's do Q.2 step-by-step.

## 2(A) Define a PDF and prove its key properties

### Definition

A random variable  $X$  is **continuous** if there exists a function  $f_X(x)$  (the **probability density function**, PDF) such that for any real numbers  $a < b$ ,

$$P(a < X \leq b) = \int_a^b f_X(x) dx.$$

Its cumulative distribution function (CDF) is

$$F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt.$$

### Properties (with proofs)

1. **Non-negativity:**  $f_X(x) \geq 0$  for all  $x$ .

*Reason:*  $F_X$  is nondecreasing, so for small  $h > 0$ ,

$$\frac{F_X(x+h) - F_X(x)}{h} \geq 0.$$

Taking  $h \rightarrow 0^+$  gives  $f'_X(x) = F'_X(x) \geq 0$  (where the derivative exists). If  $f$  were negative on any interval,  $\int f$  over that interval would be negative, contradicting that probabilities are  $\geq 0$ .

2. **Unit area:**  $\int_{-\infty}^{\infty} f_X(x) dx = 1$ .

*Reason:* Using the CDF relation,

$$\int_{-\infty}^{\infty} f_X(x) dx = \lim_{x \rightarrow \infty} F_X(x) - \lim_{x \rightarrow -\infty} F_X(x) = 1 - 0 = 1.$$

3. **Interval probability (area under the curve):**

For any  $a < b$ ,

$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx.$$

4. Point probability is zero (for continuous  $X$ ):

$$P(X = c) = \int_c^c f_X(x) dx = 0.$$

5. Recovering the PDF from CDF:

If  $F_X$  is differentiable at  $x$ ,

$$f_X(x) = \frac{d}{dx} F_X(x).$$

(You can also note the expectation formula:  $E[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx$  when it exists.)

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**2(B) Given  $f_X(x) = kx$  for  $0 \leq x \leq 1$  (and 0 elsewhere)**

(i) Find  $k$

Use Property 2 (unit area):

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_0^1 kx dx = k \left[ \frac{x^2}{2} \right]_0^1 = \frac{k}{2} = 1 \Rightarrow k = 2.$$

So  $f_X(x) = 2x$ ,  $0 \leq x \leq 1$ .

(ii) Find  $P(\frac{1}{4} < X < \frac{1}{2})$

$$P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{1/4}^{1/2} 2x dx = [x^2]_{1/4}^{1/2} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16} = 0.1875.$$

Final answers:

- $k = 2$ .
- $P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \boxed{\frac{3}{16} (\approx 0.1875)}$ .

Great—let's do Q.3 step by step.

(Interpreting 3(A) as: *derive the CDFs from the given PDFs for Exponential and Rayleigh.*)

### 3(A) From PDF → CDF

#### Exponential( $\lambda > 0$ )

PDF:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

CDF definition:  $F_X(x) = P(X \leq x) = \int_{-\infty}^x f_X(t) dt.$

- If  $x < 0$ :  $F_X(x) = 0$  (no mass left of 0).
- If  $x \geq 0$ :

$$F_X(x) = \int_0^x \lambda e^{-\lambda t} dt = [-e^{-\lambda t}]_0^x = 1 - e^{-\lambda x}.$$

Result:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$


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#### Rayleigh( $\sigma > 0$ )

PDF:

$$f_X(x) = \begin{cases} \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

CDF:

- If  $x < 0$ :  $F_X(x) = 0$ .
- If  $x \geq 0$ :

$$F_X(x) = \int_0^x \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}} dt.$$

$$\text{Substitute } u = \frac{t^2}{2\sigma^2} \Rightarrow du = \frac{t}{\sigma^2} dt;$$

$$F_X(x) = \int_0^{\frac{x^2}{2\sigma^2}} e^{-u} du = [-e^{-u}]_0^{\frac{x^2}{2\sigma^2}} = 1 - e^{-\frac{x^2}{2\sigma^2}}.$$

**Result:**

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0. \end{cases}$$

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### 3(B) Discrete RV: given PMF, find $F_X(x)$ and plot

**PMF table**

$x$	0	1	2	3
$P(X = x)$	1/8	1/8	1/4	1/2

(Check:  $1/8 + 1/8 + 1/4 + 1/2 = 1$

**CDF:**  $F_X(x) = P(X \leq x)$ . It is right-continuous and jumps at 0,1,2,3.

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{8}, & 0 \leq x < 1, \\ \frac{1}{4}, & 1 \leq x < 2, \\ \frac{1}{2}, & 2 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$$

**Exact values at the jump points (since  $F$  is right-continuous):**

$$F_X(0) = 1/8, F_X(1) = 1/4, F_X(2) = 1/2, F_X(3) = 1.$$

**How to plot (exam sketch):** Draw a step function:

- Start at 0 for  $x < 0$ .
- Closed dot at  $(0, 1/8)$  and open circle just left of it.
- Next steps at heights  $1/4$  (from  $x = 1$  to  $< 2$ ),  $1/2$  (from  $x = 2$  to  $< 3$ ), and finally 1 for  $x \geq 3$ .

Nice — here's a clear, step-by-step solution for **Question 4** (both A and B).

## 4(A) Definitions + examples

### i. Equally likely events

Definition: Two or more events are *equally likely* if each event has the same probability.

Example: When rolling a fair six-sided die, the events  $E_i = \{\text{roll } i\}$  for  $i = 1, \dots, 6$  are equally likely because  $P(E_i) = \frac{1}{6}$  for every  $i$ .

### ii. Exhaustive events

Definition: A collection of events  $\{E_1, E_2, \dots, E_n\}$  is *exhaustive* if their union equals the sample space  $S$ :  $\bigcup_i E_i = S$ . That means at least one of the events must occur.

Example: For a die roll, the two events  $A = \{\text{even}\} = \{2, 4, 6\}$  and  $A^c = \{\text{odd}\} = \{1, 3, 5\}$  are exhaustive because every outcome is either even or odd, so  $A \cup A^c = S$ .

### iii. Mutually exclusive events

Definition: Two events  $A$  and  $B$  are *mutually exclusive* if they cannot occur together:  $A \cap B = \emptyset$ . For more than two events, every pair in the collection has empty intersection.

Example: In a single card draw, the events  $H = \{\text{heart}\}$  and  $S = \{\text{spade}\}$  are mutually exclusive because one card cannot be both a heart and a spade.

**Relation / remark:**

- Events can be both mutually exclusive and exhaustive (e.g., the individual die-face events  $\{1\}, \{2\}, \dots, \{6\}$  are pairwise mutually exclusive and their union is  $S$ ).
- Mutually exclusive  $\neq$  independent (mutual exclusivity often implies dependence unless one event has probability 0).

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## 4(B) Exponential waiting time: $f_X(x) = 0.5 e^{-0.5x} u(x)$

Here  $u(x)$  is the unit step (so PDF is for  $x \geq 0$ ). The rate parameter is  $\lambda = 0.5$ .

(i) Find the CDF  $F_X(x)$ .

By definition  $F_X(x) = \int_{-\infty}^x f_X(t) dt$ .

- For  $x < 0$ :  $F_X(x) = 0$  (no mass left of 0).
- For  $x \geq 0$ :

$$F_X(x) = \int_0^x 0.5 e^{-0.5t} dt.$$

Integrate:

$$\int 0.5 e^{-0.5t} dt = -e^{-0.5t} + C.$$

So

$$F_X(x) = [-e^{-0.5t}]_{t=0}^{t=x} = 1 - e^{-0.5x}.$$

Therefore

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-0.5x}, & x \geq 0. \end{cases}$$

(ii) Compute  $P(0.1 < X \leq 0.5)$ .

Use the CDF:

$$P(0.1 < X \leq 0.5) = F_X(0.5) - F_X(0.1).$$

Substitute  $F$ :

$$= (1 - e^{-0.5 \times 0.5}) - (1 - e^{-0.5 \times 0.1}) = e^{-0.5 \times 0.1} - e^{-0.5 \times 0.5}.$$

Compute the exponents:

- $0.5 \times 0.1 = 0.05 \Rightarrow e^{-0.05} \approx 0.951229424500714$ .
- $0.5 \times 0.5 = 0.25 \Rightarrow e^{-0.25} \approx 0.778800783071405$ .

Difference:

$$P(0.1 < X \leq 0.5) \approx 0.951229424500714 - 0.778800783071405 \approx 0.172428641429309.$$

Rounded reasonably,

$$P(0.1 < X \leq 0.5) \approx 0.1724 (17.24\%).$$

Sure — here's a clear, exam-style step-by-step solution for Question 5 (A and B).

## 5(A) — Definition of a Random Variable and its types

**Definition (precise):**

A random variable  $X$  is a measurable function from a sample space  $S$  to the real numbers  $\mathbb{R}$ :

$$X : S \rightarrow \mathbb{R},$$

so every outcome  $\omega \in S$  is assigned a real number  $X(\omega)$ . Probabilities of events about  $X$  are given by the probability measure on  $S$  via preimages (e.g.  $P(X \in A) = P(\{\omega : X(\omega) \in A\})$ ).

**Classification with one example each**

### 1. Discrete random variable

- *Definition:* Takes values from a countable set. Described by a PMF  $p_X(x) = P(X = x)$ .
- *Example:* Number of heads in 3 fair coin tosses. Values  $\{0, 1, 2, 3\}$  with probabilities given by the binomial distribution  $P(X = k) = \binom{3}{k}(1/2)^3$ .

### 2. Continuous random variable

- *Definition:* Takes values in an interval (uncountable); probabilities are given by a PDF  $f_X(x)$  with  $P(a < X \leq b) = \int_a^b f_X(x) dx$ .
- *Example:* Height of a randomly chosen adult (modeled as  $X \sim N(\mu, \sigma^2)$ ).

### 3. Mixed random variable

- *Definition:* Has both a discrete part (point masses) and a continuous part (density). Its distribution has atoms plus an absolutely continuous component.
- *Example:* Insurance claim amount where  $X = 0$  with probability  $p$  (no claim) and, with probability  $1 - p$ , a positive continuous claim amount (say Uniform(0,1)).  
So  $P(X = 0) = p$  and for  $x > 0$  there's a density scaled by  $1 - p$ .

(You may also mention special cases like a degenerate/constant RV  $X = c$ , which is discrete with all mass at  $c$ .)

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**5(B) — Given PDF  $f_X(x) = k(1 + x^2)$  for  $0 \leq x \leq 1$  (and 0 elsewhere).**

Find the constant  $k$  and the distribution function  $F_X(x)$ . Step-by-step:

**Step 1 — Use normalisation  $\int_{-\infty}^{\infty} f_X(x) dx = 1$ .**

Because  $f_X(x) = 0$  outside  $[0, 1]$ ,

$$\int_0^1 k(1 + x^2) dx = 1.$$

**Step 2 — Evaluate the integral.**

$$\text{Compute } \int_0^1 (1 + x^2) dx = \left[ x + \frac{x^3}{3} \right]_0^1 = 1 + \frac{1}{3} = \frac{4}{3}.$$

So

$$k \cdot \frac{4}{3} = 1 \quad \Rightarrow \quad k = \frac{3}{4}.$$

**Answer for  $k$ :**

$$k = \frac{3}{4}.$$

**Step 3 — Find the CDF  $F_X(x) = P(X \leq x)$ .**

We consider three regions.

- For  $x < 0$  : no mass yet, so  $F_X(x) = 0$ .
- For  $0 \leq x \leq 1$  :

$$F_X(x) = \int_{-\infty}^x f_X(t) dt = \int_0^x \frac{3}{4}(1 + t^2) dt = \frac{3}{4} \left[ t + \frac{t^3}{3} \right]_0^x.$$

Simplify:

$$F_X(x) = \frac{3}{4} \left( x + \frac{x^3}{3} \right) = \frac{3x}{4} + \frac{x^3}{4} = \frac{3x + x^3}{4}.$$

- For  $x > 1$  : all probability mass accumulated, so  $F_X(x) = 1$ .

**Compact form of the CDF:**

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{3x + x^3}{4}, & 0 \leq x \leq 1, \\ 1, & x > 1. \end{cases}$$

**Step 4 — Quick check (consistency).**

- $F_X(0) = 0.$
- $F_X(1) = \frac{3(1) + 1^3}{4} = \frac{4}{4} = 1.$
- Differentiate  $F_X(x)$  on  $(0, 1)$ :

$$\frac{d}{dx} \left( \frac{3x + x^3}{4} \right) = \frac{3 + 3x^2}{4} = \frac{3}{4}(1 + x^2) = f_X(x),$$

so the derivative returns the PDF — consistent.

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### Final boxed answers

- $k = \frac{3}{4}.$
  - $F_X(x) = 0 (x < 0), F_X(x) = \frac{3x + x^3}{4} (0 \leq x \leq 1), F_X(x) = 1 (x > 1).$
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### 6(A) State and prove properties of the conditional distribution function

## 1) Definition (discrete & continuous)

Given two random variables  $X$  and  $Y$  with joint distribution:

- **Discrete:**  $p_{X|Y}(x | y) = \frac{p_{X,Y}(x, y)}{p_Y(y)}$  for  $p_Y(y) > 0$ .

The conditional CDF is  $F_{X|Y}(x | y) = P(X \leq x | Y = y) = \sum_{t \leq x} p_{X|Y}(t | y)$ .

- **Continuous:**  $f_{X|Y}(x | y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$  for  $f_Y(y) > 0$ .

The conditional CDF is  $F_{X|Y}(x | y) = P(X \leq x | Y = y) = \int_{-\infty}^x f_{X|Y}(t | y) dt$ .

Conditioning on the event  $\{Y = y\}$  turns probabilities into a probability measure on the  $X$ -axis; all usual CDF properties hold for fixed  $y$  (when the conditional denominator  $> 0$ ).

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## 2) Properties (state + proof sketches)

We fix a value  $y$  with  $p_Y(y) > 0$  (or  $f_Y(y) > 0$ ).

### Property 1 — Bounded between 0 and 1

$$0 \leq F_{X|Y}(x | y) \leq 1 \quad \text{for all } x.$$

**Proof:**  $F_{X|Y}(x | y) = P(X \leq x | Y = y)$  is a conditional probability — probabilities are between 0 and 1. QED.

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### Property 2 — Non-decreasing in $x$

If  $x_1 < x_2$  then  $F_{X|Y}(x_1 | y) \leq F_{X|Y}(x_2 | y)$ .

**Proof:**

$$F_{X|Y}(x_2 | y) - F_{X|Y}(x_1 | y) = P(x_1 < X \leq x_2 | Y = y) \geq 0,$$

because the probability of any event is nonnegative. So CDF is nondecreasing.

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### Property 3 — Right-continuity

$$\lim_{h \downarrow 0} F_{X|Y}(x + h | y) = F_{X|Y}(x | y).$$

**Proof (idea):** For each  $h > 0$  define the event  $A_h = \{X \leq x + h\}$ . The sets  $A_h$  decrease to  $A = \{X \leq x\}$  as  $h \downarrow 0$ . By continuity from above of (unconditional) probability,

$$P(A_h \cap \{Y = y\}) \rightarrow P(A \cap \{Y = y\}).$$

Divide both sides by  $P(Y = y) > 0$  to get the conditional probabilities converge, hence right-continuity.

(For discrete  $X$  the CDF is a right-continuous step function; same conclusion.)

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#### Property 4 — Limits at $\pm\infty$

$$\lim_{x \rightarrow -\infty} F_{X|Y}(x | y) = 0, \quad \lim_{x \rightarrow \infty} F_{X|Y}(x | y) = 1.$$

**Proof:**  $F_{X|Y}(x | y) = P(X \leq x | Y = y)$ . As  $x \rightarrow -\infty$  the event  $\{X \leq x\}$  becomes empty so probability  $\rightarrow 0$ . As  $x \rightarrow \infty$  the event becomes the whole conditional sample space so probability  $\rightarrow 1$ .

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#### Property 5 — Relation to conditional density / pmf

- If  $X$  is continuous conditional on  $Y = y$  and  $F_{X|Y}$  is differentiable at  $x$ ,

$$f_{X|Y}(x | y) = \frac{d}{dx} F_{X|Y}(x | y).$$

Also  $\int_{-\infty}^{\infty} f_{X|Y}(x | y) dx = 1$ .

- If  $X$  is discrete,

$$p_{X|Y}(x | y) = F_{X|Y}(x | y) - \lim_{t \uparrow x} F_{X|Y}(t | y),$$

i.e. the jump size at  $x$ .

**Proof sketch:** These follow directly from the definitions (CDF as integral/sum of conditional density/pmf) and fundamental theorem of calculus / properties of sums.

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#### Property 6 — Law of total probability (marginal from conditional)

- Continuous case:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x | y) f_Y(y) dy.$$

- Discrete case:

$$p_X(x) = \sum_y p_{X|Y}(x | y) p_Y(y).$$

**Proof:** Use  $f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y)$ . Integrate (or sum) both sides over  $y$  to obtain  $f_X(x) = \int f_{X,Y}(x,y) dy$ . Same for discrete. This is exactly the law of total probability expressed with conditional densities/pmf's.

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### Short summary for (A)

The conditional CDF  $F_{X|Y}(x|y)$  (for any fixed admissible  $y$ ) satisfies all the usual CDF properties: range  $[0, 1]$ , nondecreasing, right-continuous, limits 0 and 1, and connects to conditional density/pmf by differentiation/jumps. Marginals are recovered by integrating/summing the conditional distributions weighted by  $Y$ 's distribution.

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## 6(B) Explain in detail: (i) Binomial distribution and (ii) Uniform distribution

### (i) Binomial distribution — detailed

**Definition / model:**

Let  $X$  be the number of successes in  $n$  independent Bernoulli trials, each with success probability  $p$ . Then  $X$  has a **Binomial** distribution with parameters  $n$  and  $p$ , written  $X \sim \text{Bin}(n, p)$ .

**PMF (probability mass function):**

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n.$$

**Reason:** choose which  $k$  trials succeeded ( $\binom{n}{k}$  ways) and multiply success and failure probabilities.

**Mean (expected value):**  $E[X] = np$ .

**Proof (indicator method, step-by-step):**

- Let  $I_j$  be the indicator of success in trial  $j$ :  $I_j = 1$  if trial  $j$  succeeds, else 0. Then

$$X = \sum_{j=1}^n I_j.$$

- For each  $j$ ,  $E[I_j] = 1 \cdot p + 0 \cdot (1-p) = p$ .
- By linearity of expectation,

$$E[X] = \sum_{j=1}^n E[I_j] = \sum_{j=1}^n p = np.$$

**Variance:**  $\text{Var}(X) = np(1 - p)$ .

**Proof (step-by-step):**

- For Bernoulli  $I_j$ ,  $\text{Var}(I_j) = p(1 - p)$ .
- For  $i \neq j$ ,  $I_i$  and  $I_j$  are independent, so  $\text{Cov}(I_i, I_j) = 0$ .
- For the sum,

$$\text{Var}(X) = \sum_{j=1}^n \text{Var}(I_j) + 2 \sum_{i < j} \text{Cov}(I_i, I_j) = \sum_{j=1}^n p(1 - p) + 0 = np(1 - p).$$

**MGF (moment generating function):**

$$M_X(t) = E[e^{tX}] = (1 - p + pe^t)^n.$$

Derivatives of  $M_X$  at  $t = 0$  give moments (confirm  $M'_X(0) = np$ ,  $M''_X(0) - [M'_X(0)]^2 = np(1 - p)$ ).

**Examples / usage:**

- Example:  $n = 5, p = 0.3$ . Probability of exactly  $k = 2$  successes:

$$P(X = 2) = \binom{5}{2}(0.3)^2(0.7)^3.$$

Compute numerically:

$$\binom{5}{2} = 10, 0.3^2 = 0.09, 0.7^3 = 0.343.$$

Multiply:  $0.09 \times 0.343 = 0.03087$ . Then  $10 \times 0.03087 = 0.3087$ .

So  $P(X = 2) = 0.3087$ .

**Important properties / approximations:**

- Sum of independent Bernoulli trials — natural model for counts.
- If  $n$  large and  $p$  not too small, approximate by Normal with mean  $np$ , variance  $np(1 - p)$  (use continuity correction).
- If  $n$  large but  $p$  small with  $\lambda = np$  fixed, Binomial  $\approx$  Poisson( $\lambda$ ).

## (ii) Uniform distribution — detailed

There are two common forms: **discrete uniform** and **continuous uniform**. I'll describe both, with emphasis on the continuous uniform  $U(a, b)$ .

### Discrete Uniform (brief)

If  $X$  takes each value in the finite set  $\{x_1, \dots, x_m\}$  with equal probability  $1/m$ , then  $X$  is discrete uniform. Example: fair die  $\{1, \dots, 6\}$  with  $P(X = k) = 1/6$ . Mean = average of the support.

### Continuous Uniform $X \sim U(a, b)$ (detailed)

PDF:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

CDF (piecewise):

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \leq x \leq b, \\ 1, & x > b. \end{cases}$$

Mean  $E[X]$  — compute step by step:

$$E[X] = \int_{-\infty}^{\infty} xf_X(x) dx = \int_a^b x \cdot \frac{1}{b-a} dx.$$

Evaluate integral:

$$\int_a^b x dx = \left[ \frac{x^2}{2} \right]_a^b = \frac{b^2 - a^2}{2}.$$

So

$$E[X] = \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{b^2 - a^2}{2(b-a)}.$$

Factor numerator  $b^2 - a^2 = (b-a)(b+a)$ , cancel  $(b-a)$ :

$$E[X] = \frac{b+a}{2}.$$

Variance  $\text{Var}(X)$  — compute step by step:

First compute  $E[X^2]$ :

$$E[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \left[ \frac{x^3}{3} \right]_a^b = \frac{b^3 - a^3}{3(b-a)}.$$

Factor  $b^3 - a^3 = (b - a)(b^2 + ab + a^2)$  so

$$E[X^2] = \frac{b^2 + ab + a^2}{3}.$$

Now

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2.$$

Compute  $\left(\frac{a+b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4}$ . Put over common denominator 12:

$$E[X^2] = \frac{4(b^2 + ab + a^2)}{12}, \quad (E[X])^2 = \frac{3(a^2 + 2ab + b^2)}{12}.$$

Subtract:

$$\text{Var}(X) = \frac{4(b^2 + ab + a^2) - 3(a^2 + 2ab + b^2)}{12}.$$

Compute numerator:

$$\begin{aligned} 4b^2 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2 &= (4b^2 - 3b^2) + (4a^2 - 3a^2) + (4ab - 6ab) \\ &= b^2 + a^2 - 2ab = (b - a)^2. \end{aligned}$$

So

$$\text{Var}(X) = \frac{(b - a)^2}{12}.$$

**Key properties / remarks:**

- Symmetric about the midpoint  $(a + b)/2$ .
- Constant density on  $[a, b]$ , so no preference for any subinterval of equal length.
- Entropy (continuous) is maximal among distributions supported on  $[a, b]$  — not required, but useful fact.

**Example (numbers):**  $X \sim U(0, 1)$ . Then  $E[X] = (0 + 1)/2 = 0.5$ .  $\text{Var}(X) = (1 - 0)^2/12 = 1/12 \approx 0.08333$ .

---

## Final quick recap

- 6(A): Conditional CDF  $F_{X|Y}(x | y)$  behaves exactly like an ordinary CDF for each fixed admissible  $y$ : it is between 0 and 1, nondecreasing, right-continuous, tends to 0 and 1 at  $\pm\infty$ ; it differentiates to the conditional density and integrates (or sums) to 1.

Marginal laws are recovered by integrating/summing conditional laws against  $Y$ 's law.

- **6(B)(i):** Binomial  $X \sim \text{Bin}(n, p)$ : PMF  $\binom{n}{k} p^k (1-p)^{n-k}$ ,  $E[X] = np$ ,  $\text{Var}(X) = np(1-p)$ , MGF  $(1-p+pe^t)^n$ .
  - **6(B)(ii):** Continuous Uniform  $U(a, b)$ : PDF  $1/(b-a)$  on  $[a, b]$ ; CDF  $(x-a)/(b-a)$  on  $[a, b]$ ;  $E[X] = (a+b)/2$ ,  $\text{Var}(X) = (b-a)^2/12$ . Discrete uniform = equal mass on finitely many values.
- 

If you want, I can: (1) give a one-page printable summary you can revise from, (2) show worked numerical examples for Binomial/Uniform (specific numbers), or (3) derive mean/variance of binomial using the MGF — tell me which and I'll produce it.

Nice — here's a clean, step-by-step derivation for the mean and variance of a continuous Uniform random variable  $X$  on the interval  $[a, b]$ .

**Given:**  $X \sim \text{Uniform}(a, b)$  so the pdf is

$$f_X(x) = \frac{1}{b-a}, \quad a \leq x \leq b.$$


---

### 1) Mean $E[X]$

By definition

$$E[X] = \int_a^b x f_X(x) dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x dx.$$

Compute the integral:

$$\int_a^b x dx = \left[ \frac{x^2}{2} \right]_a^b = \frac{b^2 - a^2}{2}.$$

So

$$E[X] = \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{b^2 - a^2}{2(b-a)}.$$

Factor  $b^2 - a^2 = (b-a)(b+a)$ :

$$E[X] = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}.$$

**Result:**  $E[X] = \frac{a+b}{2}$  (the midpoint of the interval).

---

### 2) Second moment $E[X^2]$

$$E[X^2] = \int_a^b x^2 f_X(x) dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \left[ \frac{x^3}{3} \right]_a^b = \frac{b^3 - a^3}{3(b-a)}.$$

Factor  $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$ :

$$E[X^2] = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}.$$

---

### 3) Variance $\text{Var}(X)$

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2.$$

Compute  $\left(\frac{a+b}{2}\right)^2 = \frac{(a+b)^2}{4} = \frac{a^2 + 2ab + b^2}{4}$ .

Put both terms over a common denominator 12:

$$\frac{b^2 + ab + a^2}{3} = \frac{4(b^2 + ab + a^2)}{12}, \quad \frac{(a+b)^2}{4} = \frac{3(a^2 + 2ab + b^2)}{12}.$$

Subtract numerators:

$$\begin{aligned} & 4(b^2 + ab + a^2) - 3(a^2 + 2ab + b^2) \\ &= 4b^2 + 4ab + 4a^2 - (3a^2 + 6ab + 3b^2) \\ &= (4b^2 - 3b^2) + (4a^2 - 3a^2) + (4ab - 6ab) = b^2 + a^2 - 2ab = (a - b)^2. \end{aligned}$$

So

$$\text{Var}(X) = \frac{(a - b)^2}{12} = \frac{(b - a)^2}{12}.$$

Result:  $\boxed{\text{Var}(X) = \frac{(b - a)^2}{12}}.$

---

### 4) Quick special case

For  $X \sim \text{Uniform}(0, 1)$ :

$$E[X] = \frac{1}{2}, \quad \text{Var}(X) = \frac{1}{12}.$$

---

## Answer — Question 2 (step-by-step)

I'll answer both parts A and B step-by-step.

---

### A) Moments about the mean (central moments) — definition, key examples and physical significance of variance

#### 1. Definitions

- The **mean** (first raw moment) is

$$\mu = E[X] = \int x f_X(x) dx.$$

- The  **$r$ -th raw moment** (about the origin) is

$$m_r = E[X^r] = \int x^r f_X(x) dx.$$

- The  **$r$ -th central moment** (moment about the mean) is

$$\mu'_r = E[(X - \mu)^r] = \int (x - \mu)^r f_X(x) dx.$$

#### 2. First few central moments (step by step)

- First central moment ( $r = 1$ ):**

$$\mu'_1 = E[X - \mu] = E[X] - \mu = \mu - \mu = 0.$$

So the first central moment is always zero.

- Second central moment ( $r = 2$ )** — this is the **variance**:

$$\mu'_2 = E[(X - \mu)^2] = E[X^2 - 2\mu X + \mu^2].$$

Take expectations termwise:

$$\mu'_2 = E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2.$$

Therefore

$\text{Var}(X) = \mu'_2 = E[X^2] - (E[X])^2.$

- Third central moment ( $r = 3$ )** — expansion and expression in raw moments:

$$(X - \mu)^3 = X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3,$$

so

$$\mu'_3 = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3 = E[X^3] - 3\mu E[X^2] + 2\mu^3.$$

(The sign and magnitude of  $\mu'_3$  measure asymmetry / skewness.)

### 3. Physical meaning of variance

- **Variance**  $\text{Var}(X) = E[(X - \mu)^2]$  is the *expected squared deviation* from the mean. It quantifies how spread out the values of  $X$  are around the mean.
- Because deviations are squared, variance gives larger weight to larger deviations. Units of variance are the square of the units of  $X$ .
- The **standard deviation**  $\sigma = \sqrt{\text{Var}(X)}$  brings the measure back to the same units as  $X$  and is often easier to interpret.
- **Interpretation examples**
  - Small variance  $\rightarrow$  most values lie close to the mean.
  - Large variance  $\rightarrow$  values are widely dispersed.
- **Useful properties** (quick proofs):
  - $\text{Var}(c) = 0$  for constant  $c$ .
  - $\text{Var}(aX + b) = a^2 \text{Var}(X)$ .  
Proof:  $\text{Var}(aX + b) = E[(aX + b - (a\mu + b))^2] = E[(a(X - \mu))^2] = a^2 E[(X - \mu)^2]$ .

### 4. Higher central moments

- The 3rd central moment relates to **skewness** (direction of asymmetry).
  - The 4th central moment relates to **kurtosis** (tailedness/peakedness).
  - Often we use **standardized** versions: skewness =  $\mu'_3/\sigma^3$ , kurtosis =  $\mu'_4/\sigma^4$ .
- 

## B) If $X$ is Uniform( $a, b$ ) and $Y = X$ , find the density $f_Y(y)$

This is straightforward because  $Y$  is the identity transform of  $X$ .

### Method 1 — CDF method

- $F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y)$ .
- Differentiate:  $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(y) = f_X(y)$ .

### Method 2 — Change of variables formula (general)

- If  $Y = g(X)$  is monotone and invertible, then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}g^{-1}(y) \right|.$$

- Here  $g(x) = x$ ,  $g^{-1}(y) = y$ , derivative = 1. So  $f_Y(y) = f_X(y)$ .

Since  $X \sim \text{Uniform}(a, b)$ , its pdf is

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$f_Y(y) = f_X(y) = \begin{cases} \frac{1}{b-a}, & a \leq y \leq b, \\ 0, & \text{otherwise.} \end{cases}$$

---

3 ANS

### Question 3 — step-by-step answer

I'll answer both parts A and B in order.

---

#### A) State and prove Chebyshev's inequality

Statement.

Let  $X$  be any random variable with finite mean  $\mu = E[X]$  and finite variance  $\sigma^2 = \text{Var}(X)$ . For any  $k > 0$ ,

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2}.$$

Equivalently,

$$P(|X - \mu| < k) \geq 1 - \frac{\sigma^2}{k^2}.$$

**Proof (step-by-step using Markov's inequality).**

1. Recall **Markov's inequality**: for any nonnegative random variable  $Y$  and any  $a > 0$ ,

$$P(Y \geq a) \leq \frac{E[Y]}{a}.$$

2. Apply Markov's inequality to the nonnegative random variable

$$Y = (X - \mu)^2 \quad (\text{this is nonnegative}).$$

Choose  $a = k^2$  (note  $k > 0 \Rightarrow k^2 > 0$ ). Then

$$P((X - \mu)^2 \geq k^2) \leq \frac{E[(X - \mu)^2]}{k^2}.$$

3. Observe that the event  $(X - \mu)^2 \geq k^2$  is exactly the same as  $|X - \mu| \geq k$ . Also

$$E[(X - \mu)^2] = \text{Var}(X) = \sigma^2. \text{ So}$$

$$P(|X - \mu| \geq k) \leq \frac{\sigma^2}{k^2},$$

which is Chebyshev's inequality.  $\square$

**Interpretation:** at least  $1 - \sigma^2/k^2$  of the probability mass lies within distance  $k$  of the mean. Chebyshev is distribution-free (requires only finite variance), so it's conservative but very general.

---

## B) Evaluate the characteristic function of a random variable with pdf

I'll interpret the pdf as the shifted exponential (scale  $b > 0$ , location  $a$ ):

$$f_X(x) = \frac{1}{b} e^{-(x-a)/b}, \quad x \geq a.$$

(This is the exponential distribution shifted by  $a$ .)

We want the characteristic function

$$\varphi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f_X(x) dx.$$

### Step-by-step computation.

1. Insert the pdf and change limits to  $[a, \infty)$ :

$$\varphi_X(t) = \int_a^{\infty} e^{itx} \frac{1}{b} e^{-(x-a)/b} dx.$$

2. Factor out the shift  $a$ . Let  $y = x - a$  so  $x = a + y, y \in [0, \infty)$  and  $dx = dy$ . Then

$$\varphi_X(t) = \frac{1}{b} \int_0^\infty e^{it(a+y)} e^{-y/b} dy = \frac{e^{ita}}{b} \int_0^\infty e^{y(it-1/b)} dy.$$

3. The integral is a standard exponential integral. For convergence we need  $\Re(1/b) > 0$  (true since  $b > 0$ ), so

$$\int_0^\infty e^{y(it-1/b)} dy = \int_0^\infty e^{-y(1/b-it)} dy = \frac{1}{1/b-it}.$$

(Because  $\int_0^\infty e^{-cy} dy = 1/c$  for  $\Re(c) > 0$ .)

4. Put it together:

$$\varphi_X(t) = \frac{e^{ita}}{b} \cdot \frac{1}{1/b-it} = \frac{e^{ita}}{1-ibt}.$$

**Answer (boxed):**

$$\varphi_X(t) = \frac{e^{iat}}{1-ibt}, \quad b > 0.$$

**Checks and useful consequences.**

- $\varphi_X(0) = 1$  (plug  $t = 0$ : right-hand side is  $e^0/1 = 1$ ) — correct for any characteristic function.
  - The MGF  $M_X(s) = E[e^{sX}]$  (for  $s < 1/b$ ) is  $M_X(s) = \frac{e^{as}}{1-bs}$ ; the characteristic function is  $M_X(it)$  as above.
  - **Moments from derivatives (quick):**
    - $\varphi'_X(0) = iE[X]$ . Differentiating  $\varphi_X(t) = e^{ita}/(1-ibt)$  and evaluating at  $t = 0$  gives  $\varphi'_X(0) = i(a+b)$ , so  $E[X] = a+b$ .
    - One can similarly compute  $E[X^2]$  (or use  $M_X$  derivatives) to get  $\text{Var}(X) = b^2$ . (So the shifted exponential has mean  $a+b$  and variance  $b^2$ .)
-

#### 4 ANS

## Question 4 — step-by-step answer

I'll answer both parts A and B in order, with clear definitions and proofs.

---

### A) What is meant by Expectation? — definition and key properties (with proofs)

Definition (intuitive / elementary):

- For a discrete random variable  $X$  taking values  $x_i$  with probabilities  $p_i$ ,

$$E[X] = \sum_i x_i p_i,$$

provided the sum of absolute values  $\sum_i |x_i| p_i$  is finite.

- For a continuous random variable with pdf  $f_X(x)$ ,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx,$$

provided  $\int |x| f_X(x) dx < \infty$ .

(General measure-theoretic definition:  $E[X] = \int X dP$  when  $\int |X| dP < \infty$ .)

---

### Important properties and proofs

Below  $E$  denotes expectation and all expectations are assumed finite where required.

#### 1. Linearity (additivity & homogeneity)

**Statement.** For constants  $a, b$  and random variables  $X, Y$  with finite expectations,

$$E[aX + bY] = aE[X] + bE[Y].$$

**Proof (continuous case; discrete is analogous).**

By definition,

$$E[aX + bY] = \iint (ax + by) f_{X,Y}(x, y) dx dy = a \iint xf_{X,Y}(x, y) dx dy + b \iint yf_{X,Y}(x, y) dx dy.$$

Those double integrals are  $E[X]$  and  $E[Y]$  respectively, so the identity follows.

(Measure-theoretic: linearity of the Lebesgue integral gives the same result.)

---

## 2. Expectation of a constant

**Statement.** For constant  $c$ ,  $E[c] = c$ .

**Proof.** Consider  $X \equiv c$ . Then  $E[X] = \int c dP = c \int dP = c \cdot 1 = c$ .

---

## 3. Nonnegativity for nonnegative r.v.

**Statement.** If  $X \geq 0$  almost surely, then  $E[X] \geq 0$ .

**Proof.** Integral of a nonnegative function is nonnegative:  $E[X] = \int X dP \geq 0$ .

---

## 4. Monotonicity

**Statement.** If  $X \leq Y$  almost surely, then  $E[X] \leq E[Y]$ .

**Proof.**  $Y - X \geq 0$  a.s., so  $E[Y - X] \geq 0$ . Therefore  $E[Y] - E[X] = E[Y - X] \geq 0 \Rightarrow E[X] \leq E[Y]$ .

---

## 5. Absolute-value inequality (triangle inequality for expectation)

**Statement.**  $|E[X]| \leq E[|X|]$ .

**Proof.**  $|E[X]| = \left| \int X dP \right| \leq \int |X| dP = E[|X|]$  by the triangle inequality for integrals.

---

## 6. Expectation of an indicator equals probability

**Statement.** For indicator  $I_A$  of event  $A$ ,  $E[I_A] = P(A)$ .

**Proof.**  $I_A(x)$  is 1 on  $A$  and 0 elsewhere, so  $E[I_A] = \int I_A dP = P(A)$ .

---

## 7. Product of independent random variables

**Statement.** If  $X$  and  $Y$  are independent and  $E[|X|], E[|Y|] < \infty$ , then

$$E[XY] = E[X]E[Y].$$

**Proof (continuous case).** Independence means joint pdf factors:  $f_{X,Y}(x,y) = f_X(x)f_Y(y)$ .

Then

$$E[XY] = \iint xyf_X(x)f_Y(y) dx dy = (\int xf_X(x) dx)(\int yf_Y(y) dy) = E[X]E[Y].$$

---

These are the most commonly used expectation properties. (There are further results about monotone convergence, dominated convergence, Tonelli/Fubini for swapping sums/integrals under conditions — useful when dealing with infinite sums or limits — but above are the elementary properties you'll most often use.)

---

## B) Evaluate the Moment Generating Function (MGF) for the pdf

$f_X(x) = \lambda e^{-\lambda x} u(x)$ ,  $\lambda > 0$ , where  $u(x)$  is the unit step (so support  $x \geq 0$ )

**Step 1 — definition of MGF.**

The moment generating function is

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x) dx.$$

**Step 2 — insert the pdf and reduce limits.**

Since  $f_X(x) = \lambda e^{-\lambda x}$  for  $x \geq 0$  and 0 otherwise,

$$M_X(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx.$$

**Step 3 — determine convergence condition.**

The integral  $\int_0^{\infty} e^{-(\lambda-t)x} dx$  converges iff  $\Re(\lambda - t) > 0$ , i.e. for real  $t$  we require  $\lambda - t > 0$  or

$$t < \lambda.$$

**Step 4 — evaluate the integral.**

For  $t < \lambda$ ,

$$\int_0^{\infty} e^{-(\lambda-t)x} dx = \frac{1}{\lambda - t}.$$

Therefore

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad t < \lambda.$$

**Step 5 — checks and moments from the MGF.**

- $M_X(0) = \lambda/(\lambda - 0) = 1$  ✓ (MGF at 0 is 1).
- First moment (mean):  $E[X] = M'_X(0)$ . Differentiate:

$$M_X(t) = \lambda(\lambda - t)^{-1} \Rightarrow M'_X(t) = \lambda(\lambda - t)^{-2}.$$

So  $M'_X(0) = \lambda\lambda^{-2} = 1/\lambda$ . Hence

$$E[X] = \frac{1}{\lambda}.$$

- Second moment:  $M''_X(t) = \lambda \cdot 2(\lambda - t)^{-3}$ . Thus  $M''_X(0) = 2/\lambda^2$ . So

$$E[X^2] = M''_X(0) = \frac{2}{\lambda^2}.$$

Variance:

$$\text{Var}(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

**Summary (for the exponential  $\lambda e^{-\lambda x}, x \geq 0$ ):**

- MGF:  $M_X(t) = \frac{\lambda}{\lambda - t}$  for  $t < \lambda$ .
  - Mean:  $E[X] = 1/\lambda$ .
  - Variance:  $\text{Var}(X) = 1/\lambda^2$ .
-

## 5 ANS

Got it — I'll do Q5 (A & B) step-by-step.

(Note: from the image I read the pdf in part B as  $f_X(x) = \frac{x}{6}$  for  $2 \leq x \leq 4$  and 0 otherwise — that choice normalizes to 1. If the interval or constant was different, tell me and I'll recompute.)

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## A) Properties of the Moment Generating Function (MGF) — statement + proofs

**Definition.** The MGF of a random variable  $X$  is

$$M_X(t) = E[e^{tX}] = \int e^{tx} dF_X(x),$$

when the expectation exists for  $t$  in a neighborhood of 0.

Below "exists" means the integral is finite in a neighborhood of  $t = 0$ .

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1.  $M_X(0) = 1$ .

Proof:

$$M_X(0) = E[e^{0 \cdot X}] = E[1] = 1.$$

---

2. Derivatives give raw moments:

$$M_X^{(n)}(t) = \frac{d^n}{dt^n} M_X(t) = E[X^n e^{tX}], \quad \text{so} \quad M_X^{(n)}(0) = E[X^n].$$

Proof (first derivative, then generalize): differentiate under the integral (allowed if MGF exists in a neighborhood of 0):

$$M'_X(t) = \frac{d}{dt} E[e^{tX}] = E[Xe^{tX}].$$

Evaluate at  $t = 0$ :  $M'_X(0) = E[X]$ . Repeating gives the general result  $M_X^{(n)}(0) = E[X^n]$ .

---

**3. Affine transform:** If  $Y = aX + b$  (constants  $a, b$ ), then

$$M_{aX+b}(t) = E[e^{t(aX+b)}] = e^{bt} E[e^{atX}] = e^{bt} M_X(at).$$

Proof follows directly from algebra inside the expectation.

---

**4. Sum of independent r.v.s:** If  $X$  and  $Y$  are independent then

$$M_{X+Y}(t) = M_X(t) M_Y(t).$$

Proof:

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}] = E[e^{tX}] E[e^{tY}] \quad (\text{independence}).$$

More generally the MGF of a sum of independent variables is the product of their MGFs.

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**5. Uniqueness (sketch).** If two distributions have MGFs that are equal for all  $t$  in an open interval around 0, then the distributions are identical. (This follows because an MGF, when it exists in a neighborhood of 0, is an analytic function whose power-series coefficients are the moments; equality of analytic functions on an interval forces equality of the coefficients and hence of all moments, which under the existence conditions implies equality of distributions.)

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**B) Calculations for  $f_X(x) = \frac{x}{6}$ ,  $2 \leq x \leq 4$  (0 otherwise)**

Step B0 — check pdf normalizes to 1.

$$\int_{-\infty}^{\infty} f_X(x) dx = \int_2^4 \frac{x}{6} dx = \frac{1}{6} \left[ \frac{x^2}{2} \right]_2^4 = \frac{1}{12} (4^2 - 2^2) = \frac{1}{12} (16 - 4) = \frac{12}{12} = 1.$$

So it is a valid pdf.

---

(i)  $E[X]$ .

$$E[X] = \int_2^4 x \cdot f_X(x) dx = \int_2^4 x \cdot \frac{x}{6} dx = \frac{1}{6} \int_2^4 x^2 dx = \frac{1}{6} \left[ \frac{x^3}{3} \right]_2^4.$$

Compute the bracket:  $4^3 = 64$ ,  $2^3 = 8 \Rightarrow 64 - 8 = 56$ . So

$$E[X] = \frac{1}{6} \cdot \frac{56}{3} = \frac{56}{18} = \frac{28}{9}.$$

Numeric:  $\frac{28}{9} \approx 3.111\overline{1}$ .

---

(ii)  $E[X^2]$ .

$$E[X^2] = \int_2^4 x^2 \cdot f_X(x) dx = \int_2^4 x^2 \cdot \frac{x}{6} dx = \frac{1}{6} \int_2^4 x^3 dx = \frac{1}{6} \left[ \frac{x^4}{4} \right]_2^4.$$

Compute:  $4^4 = 256$ ,  $2^4 = 16 \Rightarrow 256 - 16 = 240$ . So

$$E[X^2] = \frac{1}{6} \cdot \frac{240}{4} = \frac{1}{6} \cdot 60 = 10.$$

---

(iii)  $E[2X + 3]$ . Use linearity:

$$E[2X + 3] = 2E[X] + 3 = 2 \cdot \frac{28}{9} + 3 = \frac{56}{9} + \frac{27}{9} = \frac{83}{9}.$$

Numeric:  $\frac{83}{9} \approx 9.222\overline{2}$ .

(You could also integrate  $\int (2x + 3)f_X(x)dx$  and get the same result.)

---

(iv) Variance  $\text{Var}(X)$ .

$$\text{Var}(X) = E[X^2] - (E[X])^2.$$

We have  $E[X^2] = 10$  and  $E[X] = \frac{28}{9}$ . Compute  $(E[X])^2$ :

$$\left(\frac{28}{9}\right)^2 = \frac{28^2}{9^2} = \frac{784}{81}.$$

Write 10 as  $\frac{810}{81}$  to subtract:

$$\text{Var}(X) = \frac{810}{81} - \frac{784}{81} = \frac{26}{81}.$$

So

$$\boxed{\text{Var}(X) = \frac{26}{81}.}$$

$$\text{Standard deviation } \sigma_X = \sqrt{\text{Var}(X)} = \frac{\sqrt{26}}{9} \approx 0.5666.$$


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### Final answers (compact)

- (A) MGF properties proved:  $M_X(0) = 1$ ;  $M_X^{(n)}(0) = E[X^n]$ ;  $M_{aX+b}(t) = e^{bt}M_X(at)$ ; for independent  $X, Y$ ,  $M_{X+Y}(t) = M_X(t)M_Y(t)$ ; uniqueness on a neighborhood of 0.
- (B) For  $f_X(x) = \frac{x}{6}$ ,  $2 \leq x \leq 4$ :
  - $E[X] = \frac{28}{9}$ ,
  - $E[X^2] = 10$ ,
  - $E[2X + 3] = \frac{83}{9}$ ,
  - $\text{Var}(X) = \frac{26}{81}$  (so  $\sigma_X = \frac{\sqrt{26}}{9}$ ).

Want me to (1) compute the actual MGF  $M_X(t)$  for this distribution, or (2) show the same answers by using the MGF derivatives?

### 6 ANS

Sure — here's Q6 worked **step-by-step**, parts A and B.

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## A) Monotonic vs non-monotonic transformations of a continuous r.v. (definitions, formulae & derivations)

**Goal.** Given a continuous random variable  $X$  with pdf  $f_X(x)$  and  $Y = g(X)$ , find the pdf  $f_Y(y)$ .

## 1. Monotonic $g$ (strictly one-to-one on the support)

Assume  $g$  is strictly **increasing** and differentiable, so an inverse  $g^{-1}$  exists.

- CDF method:

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)).$$

- Differentiate w.r.t.  $y$ :

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

If  $g$  is strictly **decreasing**, the same steps give an extra minus sign from differentiation of  $1 - F_X$ , so taking absolute value covers both cases:

**Monotonic formula (unified):**

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|.$$

**Example (linear).** If  $Y = aX + b$  with  $a \neq 0$  then  $g^{-1}(y) = (y - b)/a$  and  $|d/dy g^{-1}(y)| = 1/|a|$ . Thus

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y - b}{a}\right).$$


---

## 2. Non-monotonic $g$ (many-to-one mapping)

If  $g$  is not one-to-one, for a given  $y$  the equation  $g(x) = y$  may have several distinct real solutions  $x_1, x_2, \dots, x_n$  (within the support of  $X$ ). Partition the domain into pieces where  $g$  is monotone, or proceed with change-of-variables summing contributions of each root:

**Non-monotonic formula:**

$$f_Y(y) = \sum_{i: g(x_i)=y} \frac{f_X(x_i)}{|g'(x_i)|},$$

provided each root  $x_i$  is simple (i.e.  $g'(x_i) \neq 0$ ) and  $g$  is differentiable at those points.

**Derivation idea.** On a small neighbourhood, each root contributes the probability from the corresponding small  $x$ -interval and change-of-variables gives the Jacobian factor  $1/|g'(x_i)|$ . Summing over roots yields the formula.

**Example (square map).** Let  $Y = X^2$  and  $X$  uniform on  $[-1, 1]$  with  $f_X(x) = 1/2$  for  $|x| \leq 1$ . For  $0 < y \leq 1$  the roots are  $x = \pm\sqrt{y}$ . So

$$f_Y(y) = \frac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|2\sqrt{y}|} = \frac{1/2}{2\sqrt{y}} + \frac{1/2}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}, \quad 0 < y \leq 1.$$

(Endpoints and points where derivative is zero need special care.)

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### Practical cautions & summary

- Always check the support: the formula yields  $f_Y(y) > 0$  only for  $y$  in the image  $g(\text{support}(X))$ .
  - If  $g'$  vanishes at a root, use the CDF method directly (may produce point masses or different behaviour).
  - For discrete  $X$  use pmf transformation:  $P(Y = y) = \sum_{x:g(x)=y} P(X = x)$  (no Jacobian).
- 

**B) Discrete problem —  $X$  takes values  $-4, -1, 2, 3, 4$  each with probability  $1/5$ . Find pmf of  $Y = 3X^3$ , and compute mean & variance.**

**Step B1 — compute the mapping  $x \mapsto y = 3x^3$  (do arithmetic carefully):**

Make a table (compute  $x^3$  then  $y = 3x^3$ ):

- For  $x = -4$ :  $x^3 = -64$ . Then  $y = 3(-64) = -192$ .
- For  $x = -1$ :  $x^3 = -1$ . Then  $y = 3(-1) = -3$ .
- For  $x = 2$ :  $x^3 = 8$ . Then  $y = 3 \cdot 8 = 24$ .
- For  $x = 3$ :  $x^3 = 27$ . Then  $y = 3 \cdot 27 = 81$ .
- For  $x = 4$ :  $x^3 = 64$ . Then  $y = 3 \cdot 64 = 192$ .

All five  $y$ -values are distinct:  $\{-192, -3, 24, 81, 192\}$ .

**Step B2 — pmf of  $Y$ .** Since the map is one-to-one here,

$$P(Y = y) = \begin{cases} 1/5, & y \in \{-192, -3, 24, 81, 192\}, \\ 0, & \text{otherwise.} \end{cases}$$

**Step B3 — mean  $E[Y]$ .**

Compute sum of the  $y$ -values (showing arithmetic):

$$(-192) + (-3) + 24 + 81 + 192$$

Pair  $-192$  and  $+192$  cancel to 0, so remaining sum is

$$-3 + 24 + 81 = 21 + 81 = 102.$$

Therefore

$$E[Y] = \sum_y y P(Y = y) = \frac{1}{5} \cdot 102 = \frac{102}{5} = 20.4.$$

(As an exact fraction:  $E[Y] = 102/5$ .)

**Step B4 — second moment  $E[Y^2]$ .** Compute each  $y^2$ :

- $(-192)^2 = 192^2 = 36864$ . (Check:  $192^2 = (200 - 8)^2 = 40000 - 3200 + 64 = 36864$ .)
- $(-3)^2 = 9$ .
- $24^2 = 576$ .
- $81^2 = 6561$ .
- $192^2 = 36864$ .

Sum them:

$$36864 + 9 + 576 + 6561 + 36864$$

First add the two big equal terms:  $36864 + 36864 = 73728$ . Then add  $6561 \Rightarrow 80289$ .

Add  $576 \Rightarrow 80865$ . Add  $9 \Rightarrow 80874$ .

So  $\sum y^2 = 80874$ . Thus

$$E[Y^2] = \frac{1}{5} \cdot 80874 = \frac{80874}{5} = 16174.8.$$

(Exact fraction:  $80874/5$ .)

**Step B5 — variance  $\text{Var}(Y)$ .**

$$\text{Var}(Y) = E[Y^2] - (E[Y])^2.$$

We have  $E[Y^2] = 80874/5$  and  $E[Y] = 102/5$ , so

$$(E[Y])^2 = \frac{102^2}{5^2} = \frac{10404}{25}.$$

Convert  $E[Y^2]$  to denominator 25:  $E[Y^2] = \frac{80874}{5} = \frac{404370}{25}$ . Now subtract:

$$\text{Var}(Y) = \frac{404370}{25} - \frac{10404}{25} = \frac{404370 - 10404}{25} = \frac{393966}{25}.$$

As a decimal:

$$\text{Var}(Y) = \frac{393966}{25} = 15758.64.$$

You can also write it as

$$E[Y] = \frac{102}{5} = 20.4, \quad \text{Var}(Y) = \frac{393966}{25} \approx 15758.64.$$

(If you want the standard deviation:  $\sigma_Y = \sqrt{15758.64} \approx 125.53$ .)

Note: The answers and notes provided are only for study help. Sometimes they may be wrong. Please cross-check with your textbooks, class notes, or teacher before using.