Here you go—step-by-step and exam-ready.

1(A) Total Probability Theorem

Statement

Let $\{B_1, B_2, \dots, B_n\}$ be a **partition** of the sample space S:

- $B_i \cap B_j = \emptyset$ for $i \neq j$ (mutually exclusive),
- $\bigcup_{i=1}^{n} B_i = S$,
- $P(B_i) > 0$ for all i.

Then for any event A,

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) P(B_i).$$

(For a countable partition, the sum extends to ∞.)

Proof

1. Because the B_i's partition S, we can write

$$A = A \cap S = A \cap \left(\bigcup_{i=1}^{n} B_i\right) = \bigcup_{i=1}^{n} (A \cap B_i).$$

   The sets $A \cap B_i$ are **pairwise disjoint**.

2. By **countable additivity** of probability,

$$P(A)=\sum_{i=1}^{n} P(A\cap B_i).$$

3. Using the definition of conditional probability,

$$P(A\cap B_i)=P(A\mid B_i)P(B_i)\quad (\text{since } P(B_i)>0).$$

4. Substituting into the sum gives

$$P(A)=\sum_{i=1}^{n} P(A\mid B_i)P(B_i).$$

QED.

Special 2-event form

If S is partitioned by $\{B,B^c\}$,

$$P(A)=P(A\mid B)P(B)+P(A\mid B^c)P(B^c).$$

1(B) Cloud height is Gaussian: $X\sim \mathrm{N}(\mu=1800,\ \sigma^2=450)$.

Find $P(X>1850)$.

Step 1: Standardize.

$$\sigma=\sqrt{450}=\sqrt{9\times 50}=3\sqrt{50}\approx 3\times 7.071067\approx 21.213.$$

$$z=\frac{1850-\mu}{\sigma}=\frac{1850-1800}{21.213}=\frac{50}{21.213}\approx 2.357.$$

Step 2: Use the standard normal CDF $\Phi(z)$.

$$P(X>1850)=1-\Phi(2.357).$$

From standard normal tables (or calculation),

$$\Phi(2.357)\approx 0.99079 \text{ to } 0.99082.$$

Step 3: Tail probability.

$$P(X>1850)\approx 1-0.99081\approx 0.00919.$$

Answer:

$$\boxed{P(X>1850)\approx 0.0092 \text{ (about } 0.92\%).}$$

Sure! Let's do **Q.2 step-by-step**.

2(A) Define a PDF and prove its key properties

Definition

A random variable X is **continuous** if there exists a function $f_X(x)$ (the **probability density function**, PDF) such that for any real numbers $a < b$,

$$P(a < X \le b) = \int_a^b f_X(x)\,dx.$$

Its cumulative distribution function (CDF) is

$$F_X(x) = P(X \le x) = \int_{-\infty}^x f_X(t)\,dt.$$

Properties (with proofs)

1. **Non-negativity:** $f_X(x) \ge 0$ for all x.

   Reason: F_X is nondecreasing, so for small $h > 0$,

   $$\frac{F_X(x+h) - F_X(x)}{h} \ \ge\ 0.$$

   Taking $h \to 0^+$ gives $f_X(x) = F_X'(x) \ge 0$ (where the derivative exists). If f were negative on any interval, $\int f$ over that interval would be negative, contradicting that probabilities are ≥ 0.

2. **Unit area:** $\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$

   Reason: Using the CDF relation,

   $$\int_{-\infty}^{\infty} f_X(x)\,dx = \lim_{x\to\infty} F_X(x) - \lim_{x\to-\infty} F_X(x) = 1 - 0 = 1.$$

3. **Interval probability (area under the curve):**

   For any $a < b$,

$$P(a < X \le b) = F_X(b) - F_X(a) = \int_a^b f_X(x)\,dx.$$

4. **Point probability is zero (for continuous X):**

$$P(X = c) = \int_c^c f_X(x)\,dx = 0.$$

5. **Recovering the PDF from CDF:**
   If F_X is differentiable at x,

$$f_X(x) = \frac{d}{dx}F_X(x).$$

(You can also note the expectation formula: $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$ when it exists.)

2(B) Given $f_X(x) = k\,x$ for $0 \le x \le 1$ (and 0 elsewhere)

(i) Find k

Use Property 2 (unit area):

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 kx\,dx = k\left[\frac{x^2}{2}\right]_0^1 = \frac{k}{2} = 1 \;\Rightarrow\; k = 2.$$

So $f_X(x) = 2x,\ 0 \le x \le 1.$

(ii) Find $P(\frac{1}{4} < X < \frac{1}{2})$

$$P(\tfrac{1}{4} < X < \tfrac{1}{2}) = \int_{1/4}^{1/2} 2x\,dx = [x^2]_{1/4}^{1/2} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16} = 0.1875.$$

Final answers:

- $k = 2.$
- $P(\frac{1}{4} < X < \frac{1}{2}) = \boxed{\frac{3}{16}\ (\approx 0.1875)}.$

Great—let's do **Q.3 step by step**.

(Interpreting 3(A) as: *derive the CDFs from the given PDFs* for Exponential and Rayleigh.)

3(A) From PDF → CDF

Exponential($\lambda > 0$)

PDF:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

CDF definition: $F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$.

- If $x < 0$: $F_X(x) = 0$ (no mass left of 0).
- If $x \geq 0$:

$$F_X(x) = \int_0^x \lambda e^{-\lambda t}\,dt = \left[-e^{-\lambda t}\right]_0^x = 1 - e^{-\lambda x}.$$

Result:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\lambda x}, & x \geq 0. \end{cases}$$

Rayleigh($\sigma > 0$)

PDF:

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

CDF:

- If $x < 0$: $F_X(x) = 0$.
- If $x \geq 0$:

$$F_X(x) = \int_0^x \frac{t}{\sigma^2} e^{-\frac{t^2}{2\sigma^2}}\,dt.$$

Substitute $u = \dfrac{t^2}{2\sigma^2} \Rightarrow du = \dfrac{t}{\sigma^2}dt$:

$$F_X(x) = \int_0^{\frac{x^2}{2\sigma^2}} e^{-u}\,du = \left[-e^{-u}\right]_0^{\frac{x^2}{2\sigma^2}} = 1 - e^{-\frac{x^2}{2\sigma^2}}.$$

Result:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-\frac{x^2}{2\sigma^2}}, & x \geq 0. \end{cases}$$

3(B) Discrete RV: given PMF, find $F_X(x)$ and plot

PMF table

x	0	1	2	3
$P(X = x)$	$1/8$	$1/8$	$1/4$	$1/2$

(Check: $1/8 + 1/8 + 1/4 + 1/2 = 1$ ✅)

CDF: $F_X(x) = P(X \leq x)$. It is right-continuous and jumps at 0,1,2,3.

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{8}, & 0 \leq x < 1, \\ \frac{1}{4}, & 1 \leq x < 2, \\ \frac{1}{2}, & 2 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$$

Exact values at the jump points (since F is right-continuous):
$F_X(0) = 1/8,\ F_X(1) = 1/4,\ F_X(2) = 1/2,\ F_X(3) = 1.$

How to plot (exam sketch): Draw a step function:

- Start at 0 for $x < 0$.
- Closed dot at $(0, 1/8)$ and open circle just left of it.
- Next steps at heights $1/4$ (from $x = 1$ to < 2), $1/2$ (from $x = 2$ to < 3), and finally 1 for $x \geq 3$.

Nice — here's a clear, step-by-step solution for **Question 4 (both A and B)**.

4(A) Definitions + examples

i. Equally likely events

Definition: Two or more events are *equally likely* if each event has the same probability.
Example: When rolling a fair six-sided die, the events $E_i = \{\text{roll } i\}$ for $i = 1, \dots, 6$ are equally likely because $P(E_i) = \frac{1}{6}$ for every i.

ii. Exhaustive events

Definition: A collection of events $\{E_1, E_2, \dots, E_n\}$ is *exhaustive* if their union equals the sample space S: $\bigcup_i E_i = S$. That means at least one of the events must occur.
Example: For a die roll, the two events $A = \{\text{even}\} = \{2, 4, 6\}$ and $A^c = \{\text{odd}\} = \{1, 3, 5\}$ are exhaustive because every outcome is either even or odd, so $A \cup A^c = S$.

iii. Mutually exclusive events

Definition: Two events A and B are *mutually exclusive* if they cannot occur together: $A \cap B = \emptyset$. For more than two events, every pair in the collection has empty intersection.
Example: In a single card draw, the events $H = \{\text{heart}\}$ and $S = \{\text{spade}\}$ are mutually exclusive because one card cannot be both a heart and a spade.

Relation / remark:

- Events can be both mutually exclusive and exhaustive (e.g., the individual die-face events $\{1\}, \{2\}, \dots, \{6\}$ are pairwise mutually exclusive and their union is S).
- Mutually exclusive ≠ independent (mutual exclusivity often implies dependence unless one event has probability 0).

4(B) Exponential waiting time: $f_X(x) = 0.5\, e^{-0.5x}\, u(x)$

Here $u(x)$ is the unit step (so PDF is for $x \geq 0$). The rate parameter is $\lambda = 0.5$.

(i) Find the CDF $F_X(x)$.

By definition $F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$.

- For $x < 0$: $F_X(x) = 0$ (no mass left of 0).
- For $x \geq 0$:

$$F_X(x) = \int_0^x 0.5\, e^{-0.5t}\, dt.$$

Integrate:

$$\int 0.5\, e^{-0.5t}\, dt = -e^{-0.5t} + C.$$

So

$$F_X(x) = \left[-e^{-0.5t}\right]_{t=0}^{t=x} = 1 - e^{-0.5x}.$$

Therefore

$$\boxed{F_X(x) = \begin{cases} 0, & x < 0, \\ 1 - e^{-0.5x}, & x \geq 0. \end{cases}}$$

(ii) Compute $P(0.1 < X \leq 0.5)$.

Use the CDF:

$$P(0.1 < X \leq 0.5) = F_X(0.5) - F_X(0.1).$$

Substitute F:

$$= (1 - e^{-0.5\times0.5}) - (1 - e^{-0.5\times0.1}) = e^{-0.5\times0.1} - e^{-0.5\times0.5}.$$

Compute the exponents:

- $0.5 \times 0.1 = 0.05 \Rightarrow e^{-0.05} \approx 0.951229424500714.$
- $0.5 \times 0.5 = 0.25 \Rightarrow e^{-0.25} \approx 0.778800783071405.$

Difference:

$$P(0.1 < X \leq 0.5) \approx 0.951229424500714 - 0.778800783071405 \approx 0.172428641429309.$$

Rounded reasonably,

$$\boxed{P(0.1 < X \leq 0.5) \approx 0.1724\ (17.24\%).}$$

Sure — here's a clear, exam-style step-by-step solution for **Question 5 (A and B)**.

5(A) — Definition of a Random Variable and its types

Definition (precise):

A **random variable** X is a measurable function from a sample space S to the real numbers $\mathbb{R}$:

$$X : S \to \mathbb{R},$$

so every outcome $\omega \in S$ is assigned a real number $X(\omega)$. Probabilities of events about X are given by the probability measure on S via preimages (e.g. $P(X \in A) = P(\{\omega : X(\omega) \in A\})$).

Classification with one example each

1. **Discrete random variable**
   - *Definition:* Takes values from a countable set. Described by a PMF $p_X(x) = P(X = x)$.
   - *Example:* Number of heads in 3 fair coin tosses. Values $\{0, 1, 2, 3\}$ with probabilities given by the binomial distribution $P(X = k) = \binom{3}{k}(1/2)^3$.
2. **Continuous random variable**
   - *Definition:* Takes values in an interval (uncountable); probabilities are given by a PDF $f_X(x)$ with $P(a < X \le b) = \int_a^b f_X(x)\,dx$.
   - *Example:* Height of a randomly chosen adult (modeled as $X \sim \mathrm{N}(\mu, \sigma^2)$).
3. **Mixed random variable**
   - *Definition:* Has both a discrete part (point masses) and a continuous part (density). Its distribution has atoms plus an absolutely continuous component.
   - *Example:* Insurance claim amount where $X = 0$ with probability p (no claim) and, with probability $1 - p$, a positive continuous claim amount (say Uniform(0,1)). So $P(X = 0) = p$ and for $x > 0$ there's a density scaled by $1 - p$.

(You may also mention special cases like a degenerate/constant RV $X = c$, which is discrete with all mass at c.)

5(B) — Given PDF $f_X(x) = k(1 + x^2)$ for $0 \le x \le 1$ (and 0 elsewhere).

Find the constant k and the distribution function $F_X(x)$. Step-by-step:

Step 1 — Use normalisation $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.
Because $f_X(x) = 0$ outside $[0, 1]$,

$$\int_0^1 k(1 + x^2)\,dx = 1.$$

Step 2 — Evaluate the integral.
Compute $\int_0^1 (1 + x^2)\,dx = [x + \frac{x^3}{3}]_0^1 = 1 + \frac{1}{3} = \frac{4}{3}.$

So

$$k \cdot \frac{4}{3} = 1 \quad \Rightarrow \quad k = \frac{3}{4}.$$

Answer for k:

$$\boxed{k = \tfrac{3}{4}.}$$

Step 3 — Find the CDF $F_X(x) = P(X \le x)$.
We consider three regions.

- For $x < 0$: no mass yet, so $F_X(x) = 0$.
- For $0 \le x \le 1$:

  $$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{3}{4}(1 + t^2)\,dt = \frac{3}{4}\,[t + \frac{t^3}{3}]_0^x.$$

  Simplify:

  $$F_X(x) = \frac{3}{4}(x + \frac{x^3}{3}) = \frac{3x}{4} + \frac{x^3}{4} = \frac{3x + x^3}{4}.$$

- For $x > 1$: all probability mass accumulated, so $F_X(x) = 1$.

Compact form of the CDF:

$$F_X(x) = \begin{cases} 0, & x < 0, \\ \dfrac{3x + x^3}{4}, & 0 \le x \le 1, \\ 1, & x > 1. \end{cases}$$

Step 4 — Quick check (consistency).

- $F_X(0) = 0.$
- $F_X(1) = \dfrac{3(1) + 1^3}{4} = \dfrac{4}{4} = 1.$
- Differentiate $F_X(x)$ on $(0, 1)$:

$$\frac{d}{dx}\left(\frac{3x + x^3}{4}\right) = \frac{3 + 3x^2}{4} = \frac{3}{4}(1 + x^2) = f_X(x),$$

so the derivative returns the PDF — consistent.

Final boxed answers

- $\boxed{k = \frac{3}{4}}$.
- $\boxed{F_X(x) = 0 \ (x < 0),\ F_X(x) = \dfrac{3x + x^3}{4} \ (0 \le x \le 1),\ F_X(x) = 1 \ (x > 1).}$

6(A) State and prove properties of the conditional distribution function

1) Definition (discrete & continuous)

Given two random variables X and Y with joint distribution:

- **Discrete:** $p_{X|Y}(x \mid y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$ for $p_Y(y) > 0$.
  The conditional CDF is $F_{X|Y}(x \mid y) = P(X \le x \mid Y = y) = \sum_{t \le x} p_{X|Y}(t \mid y)$.
- **Continuous:** $f_{X|Y}(x \mid y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$ for $f_Y(y) > 0$.
  The conditional CDF is $F_{X|Y}(x \mid y) = P(X \le x \mid Y = y) = \int_{-\infty}^{x} f_{X|Y}(t \mid y)\, dt$.

Conditioning on the event $\{Y = y\}$ turns probabilities into a probability measure on the X-axis; all usual CDF properties hold for fixed y (when the conditional denominator > 0).

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2) Properties (state + proof sketches)

We fix a value y with $p_Y(y) > 0$ (or $f_Y(y) > 0$).

Property 1 — Bounded between 0 and 1

$$0 \le F_{X|Y}(x \mid y) \le 1 \quad \text{for all } x.$$

Proof: $F_{X|Y}(x \mid y) = P(X \le x \mid Y = y)$ is a conditional probability — probabilities are between 0 and 1. QED.

Property 2 — Non-decreasing in x

If $x_1 < x_2$ then $F_{X|Y}(x_1 \mid y) \le F_{X|Y}(x_2 \mid y)$.

Proof:

$$F_{X|Y}(x_2 \mid y) - F_{X|Y}(x_1 \mid y) = P(x_1 < X \le x_2 \mid Y = y) \ge 0,$$

because the probability of any event is nonnegative. So CDF is nondecreasing.

Property 3 — Right-continuity

$\lim_{h \downarrow 0} F_{X|Y}(x + h \mid y) = F_{X|Y}(x \mid y)$.

Proof (idea): For each $h > 0$ define the event $A_h = \{X \le x + h\}$. The sets A_h decrease to $A = \{X \le x\}$ as $h \downarrow 0$. By continuity from above of (unconditional) probability,

$$P(A_h \cap \{Y = y\}) \to P(A \cap \{Y = y\}).$$

Divide both sides by $P(Y = y) > 0$ to get the conditional probabilities converge, hence right-continuity.

(For discrete X the CDF is a right-continuous step function; same conclusion.)

Property 4 — Limits at $\pm\infty$

$$\lim_{x \to -\infty} F_{X|Y}(x \mid y) = 0, \qquad \lim_{x \to \infty} F_{X|Y}(x \mid y) = 1.$$

Proof: $F_{X|Y}(x \mid y) = P(X \le x \mid Y = y)$. As $x \to -\infty$ the event $\{X \le x\}$ becomes empty so probability $\to 0$. As $x \to \infty$ the event becomes the whole conditional sample space so probability $\to 1$.

Property 5 — Relation to conditional density / pmf

- If X is continuous conditional on $Y = y$ and $F_{X|Y}$ is differentiable at x,

$$f_{X|Y}(x \mid y) = \frac{d}{dx} F_{X|Y}(x \mid y).$$

  Also $\int_{-\infty}^{\infty} f_{X|Y}(x \mid y)\, dx = 1$.
- If X is discrete,

$$p_{X|Y}(x \mid y) = F_{X|Y}(x \mid y) - \lim_{t \uparrow x} F_{X|Y}(t \mid y),$$

  i.e. the jump size at x.

Proof sketch: These follow directly from the definitions (CDF as integral/sum of conditional density/pmf) and fundamental theorem of calculus / properties of sums.

Property 6 — Law of total probability (marginal from conditional)

- Continuous case:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X|Y}(x \mid y) f_Y(y)\, dy.$$

- Discrete case:

$$p_X(x) = \sum_y p_{X|Y}(x \mid y) p_Y(y).$$

Proof: Use $f_{X,Y}(x,y) = f_{X|Y}(x \mid y) f_Y(y)$. Integrate (or sum) both sides over y to obtain $f_X(x) = \int f_{X,Y}(x,y)\,dy$. Same for discrete. This is exactly the law of total probability expressed with conditional densities/pmfs.

Short summary for (A)

The conditional CDF $F_{X|Y}(x \mid y)$ (for any fixed admissible y) satisfies all the usual CDF properties: range $[0,1]$, nondecreasing, right-continuous, limits 0 and 1, and connects to conditional density/pmf by differentiation/jumps. Marginals are recovered by integrating/summing the conditional distributions weighted by Y's distribution.

6(B) Explain in detail: (i) Binomial distribution and (ii) Uniform distribution

(i) Binomial distribution — detailed

Definition / model:
Let X be the number of successes in n independent Bernoulli trials, each with success probability p. Then X has a **Binomial** distribution with parameters n and p, written $X \sim \mathrm{Bin}(n,p)$.

PMF (probability mass function):

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad k = 0, 1, \ldots, n.$$

Reason: choose which k trials succeeded ($\binom{n}{k}$ ways) and multiply success and failure probabilities.

Mean (expected value): $E[X] = np$.
Proof (indicator method, step-by-step):

- Let I_j be the indicator of success in trial j: $I_j = 1$ if trial j succeeds, else 0. Then

$$X = \sum_{j=1}^{n} I_j.$$

- For each j, $E[I_j] = 1 \cdot p + 0 \cdot (1-p) = p$.
- By linearity of expectation,

$$E[X] = \sum_{j=1}^{n} E[I_j] = \sum_{j=1}^{n} p = np.$$

Variance: $\mathrm{Var}(X) = np(1-p)$.

Proof (step-by-step):

- For Bernoulli I_j, $\mathrm{Var}(I_j) = p(1-p)$.
- For $i \neq j$, I_i and I_j are independent, so $\mathrm{Cov}(I_i, I_j) = 0$.
- For the sum,

$$\mathrm{Var}(X) = \sum_{j=1}^{n} \mathrm{Var}(I_j) + 2\sum_{i<j} \mathrm{Cov}(I_i, I_j) = \sum_{j=1}^{n} p(1-p) + 0 = np(1-p).$$

MGF (moment generating function):

$$M_X(t) = E[e^{tX}] = (1 - p + pe^t)^n.$$

Derivatives of M_X at $t = 0$ give moments (confirm $M_X'(0) = np$, $M_X''(0) - [M_X'(0)]^2 = np(1-p)$).

Examples / usage:

- Example: $n = 5, p = 0.3$. Probability of exactly $k = 2$ successes:

$$P(X = 2) = \binom{5}{2}(0.3)^2(0.7)^3.$$

  Compute numerically:
  $\binom{5}{2} = 10$. $0.3^2 = 0.09$. $0.7^3 = 0.343$.
  Multiply: $0.09 \times 0.343 = 0.03087$. Then $10 \times 0.03087 = 0.3087$.
  So $P(X = 2) = 0.3087$.

Important properties / approximations:

- Sum of independent Bernoulli trials — natural model for counts.
- If n large and p not too small, approximate by Normal with mean np, variance $np(1-p)$ (use continuity correction).
- If n large but p small with $\lambda = np$ fixed, Binomial $\approx$ Poisson(λ).

(ii) Uniform distribution — detailed

There are two common forms: **discrete uniform** and **continuous uniform**. I'll describe both, with emphasis on the continuous uniform $\mathrm{U}(a, b)$.

Discrete Uniform (brief)

If X takes each value in the finite set $\{x_1, \dots, x_m\}$ with equal probability $1/m$, then X is discrete uniform. Example: fair die $\{1, \dots, 6\}$ with $P(X = k) = 1/6$. Mean = average of the support.

Continuous Uniform $X \sim U(a, b)$ **(detailed)**

PDF:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

CDF (piecewise):

$$F_X(x) = \begin{cases} 0, & x < a, \\ \frac{x-a}{b-a}, & a \le x \le b, \\ 1, & x > b. \end{cases}$$

Mean $E[X]$ — compute step by step:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx.$$

Evaluate integral:

$$\int_a^b x\, dx = [\frac{x^2}{2}]_a^b = \frac{b^2 - a^2}{2}.$$

So

$$E[X] = \frac{1}{b-a} \cdot \frac{b^2 - a^2}{2} = \frac{b^2 - a^2}{2(b-a)}.$$

Factor numerator $b^2 - a^2 = (b-a)(b+a)$, cancel $(b-a)$:

$$E[X] = \frac{b+a}{2}.$$

Variance $\mathrm{Var}(X)$ — compute step by step:
First compute $E[X^2]$:

$$E[X^2] = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\, [\frac{x^3}{3}]_a^b = \frac{b^3 - a^3}{3(b-a)}.$$

Factor $b^3 - a^3 = (b-a)(b^2 + ab + a^2)$ so

$$E[X^2] = \frac{b^2 + ab + a^2}{3}.$$

Now

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2.$$

Compute $\left(\frac{a+b}{2}\right)^2 = \frac{a^2+2ab+b^2}{4}$. Put over common denominator 12:

$$E[X^2] = \frac{4(b^2 + ab + a^2)}{12}, \qquad (E[X])^2 = \frac{3(a^2 + 2ab + b^2)}{12}.$$

Subtract:

$$\mathrm{Var}(X) = \frac{4(b^2 + ab + a^2) - 3(a^2 + 2ab + b^2)}{12}.$$

Compute numerator:

$$\begin{aligned}4b^2 + 4ab + 4a^2 - 3a^2 - 6ab - 3b^2 &= (4b^2 - 3b^2) + (4a^2 - 3a^2) + (4ab - 6ab) \\ &= b^2 + a^2 - 2ab = (b-a)^2.\end{aligned}$$

So

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}.$$

Key properties / remarks:

- Symmetric about the midpoint $(a+b)/2$.
- Constant density on $[a, b]$, so no preference for any subinterval of equal length.
- Entropy (continuous) is maximal among distributions supported on $[a, b]$ — not required, but useful fact.

Example (numbers): $X \sim U(0, 1)$. Then $E[X] = (0+1)/2 = 0.5$. $\mathrm{Var}(X) = (1 - 0)^2/12 = 1/12 \approx 0.08333$.

Final quick recap

- **6(A):** Conditional CDF $F_{X|Y}(x \mid y)$ behaves exactly like an ordinary CDF for each fixed admissible y: it is between 0 and 1, nondecreasing, right-continuous, tends to 0 and 1 at $\pm\infty$; it differentiates to the conditional density and integrates (or sums) to 1.

Marginal laws are recovered by integrating/summing conditional laws against Y's law.

- **6(B)(i):** Binomial $X \sim \mathrm{Bin}(n,p)$: PMF $\binom{n}{k}p^k(1-p)^{n-k}$, $E[X] = np$, $\mathrm{Var}(X) = np(1-p)$, MGF $(1-p+pe^t)^n$.
- **6(B)(ii):** Continuous Uniform $U(a,b)$: PDF $1/(b-a)$ on $[a,b]$; CDF $(x-a)/(b-a)$ on $[a,b]$; $E[X] = (a+b)/2$, $\mathrm{Var}(X) = (b-a)^2/12$. Discrete uniform = equal mass on finitely many values.

If you want, I can: (1) give a one-page printable summary you can revise from, (2) show worked numerical examples for Binomial/Uniform (specific numbers), or (3) derive mean/variance of binomial using the MGF — tell me which and I'll produce it.

Nice — here's a clean, step-by-step derivation for the mean and variance of a continuous Uniform random variable X on the interval $[a, b]$.

Given: $X \sim \text{Uniform}(a, b)$ so the pdf is

$$f_X(x) = \frac{1}{b-a}, \qquad a \le x \le b.$$

---

1) Mean $E[X]$

By definition

$$E[X] = \int_a^b x f_X(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b x\,dx.$$

Compute the integral:

$$\int_a^b x\,dx = \left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2}.$$

So

$$E[X] = \frac{1}{b-a}\cdot\frac{b^2-a^2}{2} = \frac{b^2-a^2}{2(b-a)}.$$

Factor $b^2 - a^2 = (b-a)(b+a)$:

$$E[X] = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2}.$$

Result: $\boxed{E[X] = \frac{a+b}{2}}$ (the midpoint of the interval).

---

2) Second moment $E[X^2]$

$$E[X^2] = \int_a^b x^2 f_X(x)\,dx = \frac{1}{b-a}\int_a^b x^2\,dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{b^3-a^3}{3(b-a)}.$$

Factor $b^3 - a^3 = (b-a)(b^2+ab+a^2)$:

$$E[X^2] = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}.$$

3) Variance $\mathrm{Var}(X)$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{b^2 + ab + a^2}{3} - \left(\frac{a+b}{2}\right)^2.$$

Compute $\left(\frac{a+b}{2}\right)^2 = \frac{(a+b)^2}{4} = \frac{a^2 + 2ab + b^2}{4}$.

Put both terms over a common denominator 12:

$$\frac{b^2 + ab + a^2}{3} = \frac{4(b^2 + ab + a^2)}{12}, \qquad \frac{(a+b)^2}{4} = \frac{3(a^2 + 2ab + b^2)}{12}.$$

Subtract numerators:

$$4(b^2 + ab + a^2) - 3(a^2 + 2ab + b^2)$$

$$= 4b^2 + 4ab + 4a^2 \; - \; (3a^2 + 6ab + 3b^2)$$

$$= (4b^2 - 3b^2) + (4a^2 - 3a^2) + (4ab - 6ab) = b^2 + a^2 - 2ab = (a-b)^2.$$

So

$$\mathrm{Var}(X) = \frac{(a-b)^2}{12} = \frac{(b-a)^2}{12}.$$

Result: $\boxed{\mathrm{Var}(X) = \frac{(b-a)^2}{12}}$.

---

4) Quick special case

For $X \sim \mathrm{Uniform}(0, 1)$:
$E[X] = \frac{1}{2}, \ \mathrm{Var}(X) = \frac{1}{12}$.

Answer — Question 2 (step-by-step)

I'll answer both parts A and B step-by-step.

A) Moments about the mean (central moments) — definition, key examples and physical significance of variance

1. Definitions

- The **mean** (first raw moment) is

$$\mu = E[X] = \int \; x f_X(x)\, dx.$$

- The r**-th raw moment** (about the origin) is

$$m_r = E[X^r] = \int \; x^r f_X(x)\, dx.$$

- The r**-th central moment** (moment about the mean) is

$$\mu_r' = E[(X-\mu)^r] = \int \; (x-\mu)^r f_X(x)\, dx.$$

2. First few central moments (step by step)

- **First central moment** ($r = 1$):

$$\mu_1' = E[X-\mu] = E[X] - \mu = \mu - \mu = 0.$$

  So the first central moment is always zero.

- **Second central moment** ($r = 2$) — this is the **variance**:

$$\mu_2' = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2].$$

  Take expectations termwise:

$$\mu_2' = E[X^2] - 2\mu E[X] + \mu^2 = E[X^2] - 2\mu^2 + \mu^2 = E[X^2] - \mu^2.$$

  Therefore

$$\boxed{\mathrm{Var}(X) = \mu_2' = E[X^2] - (E[X])^2.}$$

- **Third central moment** ($r = 3$) — expansion and expression in raw moments:

$$(X-\mu)^3 = X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3,$$

  so

$$\mu_3' = E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3 = E[X^3] - 3\mu E[X^2] + 2\mu^3.$$

(The sign and magnitude of μ_3' measure asymmetry / skewness.)

3. Physical meaning of variance

- **Variance** $\mathrm{Var}(X) = E[(X-\mu)^2]$ is the *expected squared deviation* from the mean. It quantifies how spread out the values of X are around the mean.\
- Because deviations are squared, variance gives larger weight to larger deviations. Units of variance are the square of the units of X.
- The **standard deviation** $\sigma = \sqrt{\mathrm{Var}(X)}$ brings the measure back to the same units as X and is often easier to interpret.
- **Interpretation examples**
  - Small variance → most values lie close to the mean.
  - Large variance → values are widely dispersed.
- **Useful properties** (quick proofs):
  - $\mathrm{Var}(c) = 0$ for constant c.
  - $\mathrm{Var}(aX + b) = a^2\,\mathrm{Var}(X)$.
    Proof: $\mathrm{Var}(aX + b) = E[(aX + b - (a\mu + b))^2] = E[(a(X - \mu))^2] = a^2 E[(X - \mu)^2]$.

4. Higher central moments

- The 3rd central moment relates to **skewness** (direction of asymmetry).
- The 4th central moment relates to **kurtosis** (tailedness/peakedness).
- Often we use **standardized** versions: skewness $= \mu_3'/\sigma^3$, kurtosis $= \mu_4'/\sigma^4$.

B) If X is Uniform(a, b) and $Y = X$, find the density $f_Y(y)$

This is straightforward because Y is the identity transform of X.

Method 1 — CDF method

- $F_Y(y) = P(Y \le y) = P(X \le y) = F_X(y)$.
- Differentiate: $f_Y(y) = \dfrac{d}{dy}F_Y(y) = \dfrac{d}{dy}F_X(y) = f_X(y)$.

Method 2 — Change of variables formula (general)

- If $Y = g(X)$ is monotone and invertible, then

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|.$$

- Here $g(x) = x$, $g^{-1}(y) = y$, derivative $= 1$. So $f_Y(y) = f_X(y)$.

Since $X \sim \text{Uniform}(a, b)$, its pdf is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore

$$\boxed{f_Y(y) = f_X(y) = \begin{cases} \dfrac{1}{b-a}, & a \le y \le b, \\ 0, & \text{otherwise.} \end{cases}}$$

---

3 ANS

Question 3 — step-by-step answer

I'll answer both parts A and B in order.

A) State and prove Chebyshev's inequality

Statement.

Let X be any random variable with finite mean $\mu = E[X]$ and finite variance $\sigma^2 = \text{Var}(X)$. For any $k > 0$,

$$\boxed{P(|X - \mu| \ge k) \le \frac{\sigma^2}{k^2}.}$$

Equivalently,

$$P(|X - \mu| < k) \ge 1 - \frac{\sigma^2}{k^2}.$$

Proof (step-by-step using Markov's inequality).

1. Recall **Markov's inequality**: for any nonnegative random variable Y and any $a > 0$,

$$P(Y \geq a) \leq \frac{E[Y]}{a}.$$

2. Apply Markov's inequality to the nonnegative random variable

$$Y = (X-\mu)^2 \quad \text{(this is nonnegative).}$$

   Choose $a = k^2$ (note $k > 0 \Rightarrow k^2 > 0$). Then

$$P((X-\mu)^2 \geq k^2) \leq \frac{E[(X-\mu)^2]}{k^2}.$$

3. Observe that the event $(X-\mu)^2 \geq k^2$ is exactly the same as $|X-\mu| \geq k$. Also $E[(X-\mu)^2] = \text{Var}(X) = \sigma^2$. So

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2},$$

   which is Chebyshev's inequality. □

Interpretation: at least $1 - \sigma^2/k^2$ of the probability mass lies within distance k of the mean. Chebyshev is distribution-free (requires only finite variance), so it's conservative but very general.

B) Evaluate the characteristic function of a random variable with pdf

I'll interpret the pdf as the shifted exponential (scale $b > 0$, location a):

$$f_X(x) = \frac{1}{b}\, e^{-(x-a)/b}, \qquad x \geq a.$$

(This is the exponential distribution shifted by a.)

We want the characteristic function

$$\varphi_X(t) = E[e^{itX}] = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx.$$

Step-by-step computation.

1. Insert the pdf and change limits to $[a, \infty)$:

$$\varphi_X(t) = \int_a^{\infty} e^{itx}\, \frac{1}{b} e^{-(x-a)/b}\, dx.$$

2. Factor out the shift a. Let $y = x - a$ so $x = a + y$, $y \in [0, \infty)$ and $dx = dy$. Then

$$\varphi_X(t) = \frac{1}{b}\int_0^\infty e^{it(a+y)}e^{-y/b}\,dy = \frac{e^{ita}}{b}\int_0^\infty e^{y(it-1/b)}\,dy.$$

3. The integral is a standard exponential integral. For convergence we need $\Re(1/b) > 0$ (true since $b > 0$), so

$$\int_0^\infty e^{y(it-1/b)}\,dy = \int_0^\infty e^{-y(1/b-it)}\,dy = \frac{1}{1/b - it}.$$

(Because $\int_0^\infty e^{-cy}\,dy = 1/c$ for $\Re(c) > 0$.)

4. Put it together:

$$\varphi_X(t) = \frac{e^{ita}}{b}\cdot\frac{1}{1/b-it} = \frac{e^{ita}}{1-ibt}.$$

Answer (boxed):

$$\boxed{\varphi_X(t) = \frac{e^{iat}}{1-ibt}, \qquad b > 0.}$$

Checks and useful consequences.

- $\varphi_X(0) = 1$ (plug $t = 0$: right-hand side is $e^0/1 = 1$) — correct for any characteristic function.
- The MGF $M_X(s) = E[e^{sX}]$ (for $s < 1/b$) is $M_X(s) = \dfrac{e^{as}}{1-bs}$; the characteristic function is $M_X(it)$ as above.
- **Moments from derivatives (quick):**
  - $\varphi_X'(0) = iE[X]$. Differentiating $\varphi_X(t) = e^{ita}/(1-ibt)$ and evaluating at $t = 0$ gives $\varphi_X'(0) = i(a+b)$, so $E[X] = a + b$.
  - One can similarly compute $E[X^2]$ (or use M_X derivatives) to get $\mathrm{Var}(X) = b^2$. (So the shifted exponential has mean $a + b$ and variance b^2.)

---

4 ANS

Question 4 — step-by-step answer

I'll answer both parts A and B in order, with clear definitions and proofs.

A) What is meant by Expectation? — definition and key properties (with proofs)

Definition (intuitive / elementary):

- For a discrete random variable X taking values x_i with probabilities p_i,

$$E[X] = \sum_i x_i p_i,$$

  provided the sum of absolute values $\sum_i |x_i| p_i$ is finite.

- For a continuous random variable with pdf $f_X(x)$,

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx,$$

  provided $\int |x| f_X(x)\,dx < \infty$.

(General measure-theoretic definition: $E[X] = \int X\,dP$ when $\int |X|\,dP < \infty$.)

Important properties and proofs

Below E denotes expectation and all expectations are assumed finite where required.

1. Linearity (additivity & homogeneity)

Statement. For constants a, b and random variables X, Y with finite expectations,

$$\boxed{E[aX + bY] = aE[X] + bE[Y]}.$$

Proof (continuous case; discrete is analogous).

By definition,

$$E[aX + bY] = \iint (ax + by) f_{X,Y}(x, y)\,dx\,dy = a \iint x f_{X,Y}(x, y)\,dx\,dy + b \iint y f_{X,Y}(x, y)\,dx\,dy.$$

Those double integrals are $E[X]$ and $E[Y]$ respectively, so the identity follows. (Measure-theoretic: linearity of the Lebesgue integral gives the same result.)

---

2. Expectation of a constant

Statement. For constant c, $E[c] = c$.

Proof. Consider $X \equiv c$. Then $E[X] = \int c\, dP = c \int dP = c \cdot 1 = c$.

---

3. Nonnegativity for nonnegative r.v.

Statement. If $X \geq 0$ almost surely, then $E[X] \geq 0$.

Proof. Integral of a nonnegative function is nonnegative: $E[X] = \int X\, dP \geq 0$.

---

4. Monotonicity

Statement. If $X \leq Y$ almost surely, then $E[X] \leq E[Y]$.

Proof. $Y - X \geq 0$ a.s., so $E[Y - X] \geq 0$. Therefore $E[Y] - E[X] = E[Y - X] \geq 0 \Rightarrow E[X] \leq E[Y]$.

---

5. Absolute-value inequality (triangle inequality for expectation)

Statement. $|E[X]| \leq E[|X|]$.

Proof. $|E[X]| = \left|\int X\, dP\right| \leq \int |X|\, dP = E[|X|]$ by the triangle inequality for integrals.

---

6. Expectation of an indicator equals probability

Statement. For indicator I_A of event A, $E[I_A] = P(A)$.

Proof. $I_A(x)$ is 1 on A and 0 elsewhere, so $E[I_A] = \int I_A\, dP = P(A)$.

---

7. Product of independent random variables

Statement. If X and Y are independent and $E[|X|], E[|Y|] < \infty$, then

$$\boxed{E[XY] = E[X]\,E[Y].}$$

Proof (continuous case). Independence means joint pdf factors: $f_{X,Y}(x,y) = f_X(x)f_Y(y)$. Then

$$E[XY] = \iint xy\, f_X(x)f_Y(y)\, dx\, dy = (\int xf_X(x)\, dx)\,(\int yf_Y(y)\, dy) = E[X]E[Y].$$

These are the most commonly used expectation properties. (There are further results about monotone convergence, dominated convergence, Tonelli/Fubini for swapping sums/integrals under conditions — useful when dealing with infinite sums or limits — but above are the elementary properties you'll most often use.)

B) Evaluate the Moment Generating Function (MGF) for the pdf

$f_X(x) = \lambda e^{-\lambda x}u(x),\ \lambda > 0,$ where $u(x)$ is the unit step (so support $x \geq 0$)

Step 1 — definition of MGF.

The moment generating function is

$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx.$$

Step 2 — insert the pdf and reduce limits.

Since $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and 0 otherwise,

$$M_X(t) = \int_0^{\infty} e^{tx}\, \lambda e^{-\lambda x}\, dx = \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx.$$

Step 3 — determine convergence condition.

The integral $\int_0^{\infty} e^{-(\lambda-t)x}\, dx$ converges iff $\Re(\lambda - t) > 0$, i.e. for real t we require $\lambda - t > 0$ or

$$\boxed{t < \lambda.}$$

Step 4 — evaluate the integral.

For $t < \lambda$,

$$\int_0^{\infty} e^{-(\lambda-t)x}\, dx = \frac{1}{\lambda - t}.$$

Therefore

$$\boxed{M_X(t) = \frac{\lambda}{\lambda - t}, \qquad t < \lambda.}$$

Step 5 — checks and moments from the MGF.

- $M_X(0) = \lambda/(\lambda - 0) = 1$ ✔ (MGF at 0 is 1).
- First moment (mean): $E[X] = M_X'(0)$. Differentiate:

$$M_X(t) = \lambda(\lambda - t)^{-1} \quad \Rightarrow \quad M_X'(t) = \lambda(\lambda - t)^{-2}.$$

  So $M_X'(0) = \lambda\lambda^{-2} = 1/\lambda$. Hence

$$\boxed{E[X] = \frac{1}{\lambda}.}$$

- Second moment: $M_X''(t) = \lambda \cdot 2(\lambda - t)^{-3}$. Thus $M_X''(0) = 2/\lambda^2$. So

$$E[X^2] = M_X''(0) = \frac{2}{\lambda^2}.$$

  Variance:

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Summary (for the exponential $\lambda e^{-\lambda x}, x \geq 0$):

- MGF: $M_X(t) = \dfrac{\lambda}{\lambda - t}$ for $t < \lambda$.
- Mean: $E[X] = 1/\lambda$.
- Variance: $\mathrm{Var}(X) = 1/\lambda^2$.

5 ANS

Got it — I'll do **Q5 (A & B)** step-by-step.
(Note: from the image I read the pdf in part B as $f_X(x) = \dfrac{x}{6}$ *for* $2 \le x \le 4$ *and* 0 *otherwise — that choice normalizes to 1. If the interval or constant was different, tell me and I'll recompute.)*

A) Properties of the Moment Generating Function (MGF) — statement + proofs

Definition. The MGF of a random variable X is

$$M_X(t) = E[e^{tX}] = \int e^{tx}\, dF_X(x),$$

when the expectation exists for t in a neighborhood of 0.

Below "exists" means the integral is finite in a neighborhood of $t = 0$.

---

1. $M_X(0) = 1.$
Proof:

$$M_X(0) = E[e^{0\cdot X}] = E[1] = 1.$$

---

2. Derivatives give raw moments:

$$M_X^{(n)}(t) = \frac{d^n}{dt^n} M_X(t) = E[X^n e^{tX}], \qquad \text{so} \qquad M_X^{(n)}(0) = E[X^n].$$

Proof (first derivative, then generalize): differentiate under the integral (allowed if MGF exists in a neighborhood of 0):

$$M_X'(t) = \frac{d}{dt} E[e^{tX}] = E[Xe^{tX}].$$

Evaluate at $t = 0$: $M_X'(0) = E[X]$. Repeating gives the general result $M_X^{(n)}(0) = E[X^n]$.

---

3. Affine transform: If $Y = aX + b$ (constants a, b), then

$$M_{aX+b}(t) = E[e^{t(aX+b)}] = e^{bt} E[e^{(at)X}] = e^{bt} M_X(at).$$

Proof follows directly from algebra inside the expectation.

---

4. Sum of independent r.v.s: If X and Y are independent then

$$M_{X+Y}(t) = M_X(t)\, M_Y(t).$$

Proof:

$$M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}] = E[e^{tX}]\, E[e^{tY}] \quad \text{(independence)}.$$

More generally the MGF of a sum of independent variables is the product of their MGFs.

---

5. Uniqueness (sketch). If two distributions have MGFs that are equal for all t in an open interval around 0, then the distributions are identical. (This follows because an MGF, when it exists in a neighborhood of 0, is an analytic function whose power-series coefficients are the moments; equality of analytic functions on an interval forces equality of the coefficients and hence of all moments, which under the existence conditions implies equality of distributions.)

B) Calculations for $f_X(x) = \dfrac{x}{6},\ 2 \le x \le 4$ (0 otherwise)

Step B0 — check pdf normalizes to 1.

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_2^4 \frac{x}{6}\, dx = \frac{1}{6}\Big[\frac{x^2}{2}\Big]_2^4 = \frac{1}{12}(4^2 - 2^2) = \frac{1}{12}(16 - 4) = \frac{12}{12} = 1.$$

So it is a valid pdf.

(i) $E[X]$.

$$E[X] = \int_2^4 x \cdot f_X(x)\,dx = \int_2^4 x \cdot \frac{x}{6}\,dx = \frac{1}{6}\int_2^4 x^2\,dx = \frac{1}{6}\Big[\frac{x^3}{3}\Big]_2^4.$$

Compute the bracket: $4^3 = 64,\ 2^3 = 8 \Rightarrow 64 - 8 = 56$. So

$$E[X] = \frac{1}{6} \cdot \frac{56}{3} = \frac{56}{18} = \frac{28}{9}.$$

Numeric: $\frac{28}{9} \approx 3.111\overline{1}$.

(ii) $E[X^2]$.

$$E[X^2] = \int_2^4 x^2 \cdot f_X(x)\,dx = \int_2^4 x^2 \cdot \frac{x}{6}\,dx = \frac{1}{6}\int_2^4 x^3\,dx = \frac{1}{6}\Big[\frac{x^4}{4}\Big]_2^4.$$

Compute: $4^4 = 256,\ 2^4 = 16 \Rightarrow 256 - 16 = 240$. So

$$E[X^2] = \frac{1}{6} \cdot \frac{240}{4} = \frac{1}{6} \cdot 60 = 10.$$

(iii) $E[\,2X + 3\,]$. Use linearity:

$$E[2X + 3] = 2E[X] + 3 = 2 \cdot \frac{28}{9} + 3 = \frac{56}{9} + \frac{27}{9} = \frac{83}{9}.$$

Numeric: $\frac{83}{9} \approx 9.222\overline{2}$.

(You could also integrate $\int (2x + 3) f_X(x) dx$ and get the same result.)

(iv) Variance $\mathrm{Var}(X)$.

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2.$$

We have $E[X^2] = 10$ and $E[X] = \frac{28}{9}$. Compute $(E[X])^2$:

$$\Big(\frac{28}{9}\Big)^2 = \frac{28^2}{9^2} = \frac{784}{81}.$$

Write 10 as $\frac{810}{81}$ to subtract:

$$\mathrm{Var}(X) = \frac{810}{81} - \frac{784}{81} = \frac{26}{81}.$$

So

$$\boxed{\mathrm{Var}(X) = \frac{26}{81}.}$$

Standard deviation $\sigma_X = \sqrt{\mathrm{Var}(X)} = \frac{\sqrt{26}}{9} \approx 0.5666.$

Final answers (compact)

- (A) MGF properties proved: $M_X(0) = 1$; $M_X^{(n)}(0) = E[X^n]$; $M_{aX+b}(t) = e^{bt} M_X(at)$; for independent X, Y, $M_{X+Y}(t) = M_X(t) M_Y(t)$; uniqueness on a neighborhood of 0.
- (B) For $f_X(x) = \frac{x}{6},\ 2 \le x \le 4$:
  - $E[X] = \frac{28}{9}$,
  - $E[X^2] = 10$,
  - $E[2X+3] = \frac{83}{9}$,
  - $\mathrm{Var}(X) = \frac{26}{81}$ (so $\sigma_X = \frac{\sqrt{26}}{9}$).

Want me to (1) compute the actual MGF $M_X(t)$ for this distribution, or (2) show the same answers by using the MGF derivatives?

6 ANS

Sure — nere's Q6 worked **step-by-step**, parts A and B.

A) Monotonic vs non-monotonic transformations of a continuous r.v. (definitions, formulae & derivations)

Goal. Given a continuous random variable X with pdf $f_X(x)$ and $Y = g(X)$, find the pdf $f_Y(y)$.

1. Monotonic g (strictly one-to-one on the support)

Assume g is strictly **increasing** and differentiable, so an inverse g^{-1} exists.

- CDF method:

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le g^{-1}(y)) = F_X(g^{-1}(y)).$$

- Differentiate w.r.t. y:

$$f_Y(y) = \frac{d}{dy} F_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

If g is strictly **decreasing**, the same steps give an extra minus sign from differentiation of $1 - F_X$, so taking absolute value covers both cases:

Monotonic formula (unified):

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| .}$$

Example (linear). If $Y = aX + b$ with $a \neq 0$ then $g^{-1}(y) = (y - b)/a$ and $|d/dy\, g^{-1}(y)| = 1/|a|$. Thus

$$f_Y(y) = \frac{1}{|a|} f_X(\frac{y-b}{a}).$$

---

2. Non-monotonic g (many-to-one mapping)

If g is not one-to-one, for a given y the equation $g(x) = y$ may have several distinct real solutions $x_1, x_2, \dots, x_n$ (within the support of X). Partition the domain into pieces where g is monotone, or proceed with change-of-variables summing contributions of each root:

Non-monotonic formula:

$$\boxed{f_Y(y) = \sum_{i:\, g(x_i)=y} \frac{f_X(x_i)}{|g'(x_i)|},}$$

provided each root x_i is simple (i.e. $g'(x_i) \neq 0$) and g is differentiable at those points.

Derivation idea. On a small neighbourhood, each root contributes the probability from the corresponding small x-interval and change-of-variables gives the Jacobian factor $1/|g'(x_i)|$. Summing over roots yields the formula.

Example (square map). Let $Y = X^2$ and X uniform on $[-1, 1]$ with $f_X(x) = 1/2$ for $|x| \leq 1$. For $0 < y \leq 1$ the roots are $x = \pm\sqrt{y}$. So

$$f_Y(y) = \frac{f_X(\sqrt{y})}{|2\sqrt{y}|} + \frac{f_X(-\sqrt{y})}{|-2\sqrt{y}|} = \frac{1/2}{2\sqrt{y}} + \frac{1/2}{2\sqrt{y}} = \frac{1}{2\sqrt{y}}, \quad 0 < y \leq 1.$$

(Endpoints and points where derivative is zero need special care.)

Practical cautions & summary

- Always check the support: the formula yields $f_Y(y) > 0$ only for y in the image $g(\text{support}(X))$.
- If g' vanishes at a root, use the CDF method directly (may produce point masses or different behaviour).
- For discrete X use pmf transformation: $P(Y = y) = \sum_{x:g(x)=y} P(X = x)$ (no Jacobian).

B) Discrete problem — X takes values $-4, -1, 2, 3, 4$ each with probability $1/5$. Find pmf of $Y = 3X^3$, and compute mean & variance.

Step B1 — compute the mapping $x \mapsto y = 3x^3$ (do arithmetic carefully):

Make a table (compute x^3 then $y = 3x^3$):

- For $x = -4$: $x^3 = -64$. Then $y = 3(-64) = -192$.
- For $x = -1$: $x^3 = -1$. Then $y = 3(-1) = -3$.
- For $x = 2$: $x^3 = 8$. Then $y = 3 \cdot 8 = 24$.
- For $x = 3$: $x^3 = 27$. Then $y = 3 \cdot 27 = 81$.
- For $x = 4$: $x^3 = 64$. Then $y = 3 \cdot 64 = 192$.

All five y-values are distinct: $\{-192, -3, 24, 81, 192\}$.

Step B2 — pmf of Y. Since the map is one-to-one here,

$$P(Y = y) = \begin{cases} 1/5, & y \in \{-192, -3, 24, 81, 192\}, \\ 0, & \text{otherwise.} \end{cases}$$

Step B3 — mean $E[Y]$.

Compute sum of the y-values (showing arithmetic):

$$(-192) + (-3) + 24 + 81 + 192$$

Pair -192 and $+192$ cancel to 0, so remaining sum is

$$-3 + 24 + 81 = 21 + 81 = 102.$$

Therefore

$$E[Y] = \sum_y y\,P(Y = y) = \frac{1}{5} \cdot 102 = \frac{102}{5} = 20.4.$$

(As an exact fraction: $E[Y] = 102/5$.)

Step B4 — second moment $E[Y^2]$. Compute each y^2:

- $(-192)^2 = 192^2 = 36864$. (Check: $192^2 = (200 - 8)^2 = 40000 - 3200 + 64 = 36864$.)
- $(-3)^2 = 9$.
- $24^2 = 576$.
- $81^2 = 6561$.
- $192^2 = 36864$.

Sum them:

$$36864 + 9 + 576 + 6561 + 36864$$

First add the two big equal terms: $36864 + 36864 = 73728$. Then add $6561 \Rightarrow 80289$. Add $576 \Rightarrow 80865$. Add $9 \Rightarrow 80874$.

So $\sum y^2 = 80874$. Thus

$$E[Y^2] = \frac{1}{5} \cdot 80874 = \frac{80874}{5} = 16174.8.$$

(Exact fraction: $80874/5$.)

Step B5 — variance $\mathrm{Var}(Y)$.

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2.$$

We have $E[Y^2] = 80874/5$ and $E[Y] = 102/5$, so

$$(E[Y])^2 = \frac{102^2}{5^2} = \frac{10404}{25}.$$

Convert $E[Y^2]$ to denominator 25: $E[Y^2] = \frac{80874}{5} = \frac{404370}{25}$. Now subtract:

$$\mathrm{Var}(Y) = \frac{404370}{25} - \frac{10404}{25} = \frac{404370 - 10404}{25} = \frac{393966}{25}.$$

As a decimal:

$$\mathrm{Var}(Y) = \frac{393966}{25} = 15758.64.$$

You can also write it as

$$\boxed{E[Y] = \frac{102}{5} = 20.4, \qquad \mathrm{Var}(Y) = \frac{393966}{25} \approx 15758.64.}$$

(If you want the standard deviation: $\sigma_Y = \sqrt{15758.64} \approx 125.53$.)

Note: The answers and notes provided are only for study help. Sometimes they may be wrong. Please cross-check with your textbooks, class notes, or teacher before using.