International Math Bowl

Open Round Practice Test 1

Time limit: 60 minutes.
Instructions: In this round, you will compete in teams to answer as many of the 25 short answer questions as possible. All answers must be expressed as integers. **No calculators are allowed.**
Scoring: 1 point will be awarded for each correct answer; incorrect or blank answers will be given 0 points.

1. What is $\gcd(108, 144) \times \operatorname{lcm}(108, 144)$?

2. Let x be the answer to this question. Find $6 - 2x$.

3. Circle ω is centered at point O with radius 15. Chord AC and radius OB are perpendicular and intersect at M with $MB = 6$. What is the area of $\triangle AOC$?

4. If x and y are real numbers, the minimum value of $x^2 - 2xy + 6y^2 - 4y + 3$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

5. There are 14 pillars made of gold in a row that support a hallway in one of the temples in Olympus. Rob the robber has snuck into Olympus and wishes to steal some of these pillars and sell them back in the human world to get rich. However, they are immensely heavy, so he will only carry $1, 2, 3$, or 4 of them back to the human world. Also, the roof of the temple is made of solid silver, so Rob cannot steal two adjacent pillars or the roof will cave in. In how many ways can he safely steal the pillars and become rich?

6. A yard is equal to three feet. For positive integers a and b, the area of an a foot by b foot rectangle in square yards is numerically equal to the perimeter of the same a foot by b foot rectangle in feet. Find the number of possible ordered pairs (a, b).

7. If $\sin(\theta) - \cos(\theta) = \frac{2}{5}$ for $\pi \leq \theta \leq \frac{3\pi}{2}$, $\cos(2\theta)$ can be written in the form $\frac{a\sqrt{b}}{c}$ where a, b, and c are positive integers, a and c are relatively prime, and b is square-free. What is $a + b + c$?

8. An ant is on the coordinate plane at the point $(0, 0)$, and wishes to escape the region $|x| + |y| < 3$. Every second, the ant randomly chooses a number n between 0 and 1, and moves to the right n units and up $1 - n$ units. What is the expected number of seconds it will take for the ant to escape?

9. There is a positive integer n such that $2024n$ has 30 positive integer divisors (including 1 and $2024n$ itself) and $2024n + 1$ has 4 positive integer divisors (including 1 and $2024n + 1$ itself). What is n?

10. Sam and Ritwik are going to a math competition. They will each arrive at a uniform randomly selected time between 12 AM on October 12 and 12 AM on October 13. The probability that Sam and Ritwik arrive within one hour of one another can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

11. What is the remainder when $\sum_{i=1}^{100} \sum_{j=1}^{100} \sum_{k=1}^{100} (ijk)$ is divided by 7?

12. In $\triangle ABC$, $AB \cdot BC \cdot CA = 180$, and $\sin(\angle ABC) \cdot \sin(\angle ABC + \angle BCA) \cdot \sin(\angle BCA) = \frac{1}{150}$. Find the area of $\triangle ABC$.

13. If $f(x+y) = f(x) + f(y) + xy - 1$ for all real x and y, find

$$f(-20) + f(-19) + \cdots + f(19) + f(20).$$

14. Given that x and y are positive real numbers and $x + 2y = 14$, let N be the maximum value of x^3y^4. Find the last two digits of N^{100}.

15. Triangle ABC has side lengths $AB = 5$, $BC = 6$ and $AC = 7$. Points D, E, and F are the midpoints of sides BC, AC, and AB respectively. The quantity $AD^2 + BE^2 + CF^2$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. What is $m + n$?

16. Two points are chosen uniformly at random in rectangular prism $\mathcal{R}$ with side lengths $6, 8$, and 9. What is the expected value of the volume of rectangular prism with the least possible volume such that the rectangular prism has sides parallel to the sides of $\mathcal{R}$ and it must contain both points?

17. Fearless Fiona is wandering into the Forest of Functions. Fiona fumbles on the following functional equation for $f : \mathbb{R} \to \mathbb{R}$

$$f(xy + f(x) + f(y)) = f(x) + f(xy) + y$$

Find the final sum of all possible $f(1048575)$.

18. Let x, y, z be positive real numbers such that $xyz = 576$ and

$$\log_x(2y) = \log_y(3z) = \log_z(4x).$$

Then x can be represented in the form $2^a 3^b$ for rational numbers a and b. The quantity $a + b$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

19. How many ways can the letters in **INTERNATIONAL** be rearranged so that the letters **I** and **N** never touch?

20. Let $X > 999$ be a positive integer with its last three digits being $\underline{A}\ \underline{B}\ \underline{C}$ such that X^3 also has the same last three digits. Find the sum of all possible values of

$$(100A + 10B + C).$$

21. Let $\triangle ABC$ have side lengths $AB = 6$, $AC = 10$, $BC = 14$. Let X be a randomly chosen point on the incircle of $\triangle ABC$. Find the expected value of $AX^2+BX^2+CX^2$.

22. Polynomial $f(x) = x^3 + Ax^2 + Bx + C$ has roots r, s, t with nonzero integers A, B, C, and polynomial $g(x) = x^3 + Dx^2 + Ex + F$ has roots r^2, s^2, t^2. If $\frac{D}{A} = \frac{F}{C} = -5$ and $\frac{E}{B} = 3$. Find $f(2)$.

23. Two regular hexagons share a side and together make a concave decagon w after removing the shared side. Define a triangle partition of w as a set of non-degenerate triangles with fixed position which satisfy the following properties:

    (a) All the vertices of the triangles are vertices of w.

    (b) No two triangles overlap (shared sides don't count as overlap).

    (c) The union of the regions bounded by the triangles is the region bounded by w.

    Find the number of all such triangle partitions of w.

24. Let ABC be a triangle with $AB = 13$, $AC = 14$, $BC = 15$. Let D be on segment AC. If O, O_1, and O_2 are the circumcenters of $\triangle ABC$, $\triangle ABD$, and $\triangle DBC$ respectively, then the maximum possible area of triangle $\triangle OO_1O_2$ is $\frac{p}{q}$ for relatively prime positive integers p and q. Find $p + q$.

25. Let S_n be the set of all strings of length n whose only characters are **1** and $*$. For any element x in S_n for some n, let $f(x)$ be the result obtained after evaluating the string x as an expression by removing all unused or repeated $*$ terms. For example, the string $*\mathbf{1}**\mathbf{11}**\mathbf{111}**$ would be evaluated as $1 \cdot 11 \cdot 111 = 1221$, ignoring the repeated, beginning, and end multiplication characters. Let $a_n = \sum_{x \in S_n} f(x)$, as in the sum of $f(x)$ across all $x \in S_n$. Then $\sum_{n=1}^{\infty} \frac{a_n}{11^n} = \frac{p}{q}$ for relatively prime positive integers p and q. Find $p + q$ (Note that a string x with only $*$ characters is defined to have $f(x) = 1$).

ANSWERS:

1. 15552
2. 2
3. 108
4. 16
5. 642
6. 15
7. 73
8. 3
9. 22
10. 623
11. 6
12. 3
13. 2911
14. 76
15. 167
16. 16
17. 1048575
18. 65
19. 41368320
20. 7000
21. 125
22. 13
23. 422
24. 391
25. 231