(1) z_H $\sqrt[4]{z}$ $\quad z = -72(1-\sqrt{3}i)$ $\quad |z| = \sqrt{a^2+b^2}$

$z = -72 + 72\sqrt{3}i \qquad a = -72 \qquad b = 72\sqrt{3}$

$r = 144 \qquad |z| = \sqrt{(-72)^2 + (72\sqrt{3})^2} = 72\sqrt{1+3} = 72 \cdot 2 = 144$

$\begin{cases} \cos\varphi = \frac{a}{r} \\ \sin\varphi = \frac{b}{r} \end{cases} \qquad \begin{cases} \cos\varphi = \frac{-72}{144} = -\frac{1}{2} \\ \sin\varphi = \frac{72\sqrt{3}}{144} = \frac{\sqrt{3}}{2} \end{cases} \Rightarrow \varphi\ \frac{2\pi}{3}$

$w_k = \sqrt[n]{r}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi+2\pi k}{n}\right), k = 0, 1, 2 \ldots$

$w_k = \sqrt[4]{144}\left(\cos\frac{\frac{2\pi}{3} + 2\pi k}{4} + i\sin\frac{\frac{2\pi}{3} + 2\pi k}{4}\right)$

$12 \cdot 12$
$4 \cdot 3 \; 4 \cdot 3$

$w_0 = 2\sqrt{3}\left(\cos\frac{2\pi}{3}\cdot\frac{1}{4} + i\sin\frac{2\pi}{3}\cdot\frac{1}{4}\right) =$

$= 2\sqrt{3}\left(\cos\frac{\pi}{6} + i\cdot\sin\frac{\pi}{6}\right) = 2\sqrt{3}\cdot\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right) =$

$= \cancel{2}\sqrt{3}\,\frac{\sqrt{3}}{\cancel{2}} + \cancel{2}\sqrt{3}\cdot\frac{1}{\cancel{2}}i = 3 + \sqrt{3}i$

$w_1 = 2\sqrt{3}\left(\cos\frac{\frac{2\pi}{\cancel{3}} + \overset{3}{\cancel{2\pi}}}{4} + i\sin\frac{\frac{2\pi}{3} + \overset{3}{\cancel{2\pi}}}{4}\right) =$

$= 2\sqrt{3}\left(\cos\frac{8\pi}{3}\cdot\frac{1}{4} + i\sin\frac{8\pi}{3}\cdot\frac{1}{4}\right) =$

$= 2\sqrt{3}\left(\cos\frac{2\pi}{3} + i\cdot\sin\frac{2\pi}{3}\right) = \cancel{2}\sqrt{3}\left(-\frac{1}{\cancel{2}}\right) + i\cdot\cancel{2}\sqrt{3}\,\frac{\sqrt{3}}{\cancel{2}}$

$= -\sqrt{3} + 3i$

(2) $\left(\frac{\sqrt{3}+i}{1-i}\right)^{30} \qquad z_1 = \sqrt{3} + i \qquad z_2 = 1 - i$

$a = \sqrt{3} \quad b = 1 \qquad\qquad a = 1 \quad b = -1$

$|z_1| = \sqrt{a^2+b^2} = \sqrt{3+1} = 2 \qquad r_1 = 2$

$\begin{cases} \cos\varphi = \frac{\sqrt{3}}{2} \\ \sin\varphi = \frac{1}{2} \end{cases} \qquad \varphi = \frac{\pi}{6}$

$$|z_2| = \sqrt{1^2 + (-1)^2} = \sqrt{2} \qquad r_2 = \sqrt{2}$$

$$\begin{cases} \cos \varphi = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{7\pi}{4} \\ \sin \varphi = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2} \quad \varnothing \end{cases}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \left((\cos \varphi_1 - \varphi_2) + (i \sin \varphi_1 - \varphi_2) \right)$$

$$\frac{z_1}{z_2} = \frac{2}{\sqrt{2}} \left(\cos\left(\frac{\pi}{6} - \frac{7\pi}{4}\right) + i \cdot \sin\left(\frac{\pi}{6} - \frac{7\pi}{4}\right) \right) =$$

$$= \sqrt{2} \left(\cos\left(-\frac{19\pi}{12}\right) + i \sin\left(-\frac{19\pi}{12}\right) \right) = \qquad 2\pi = \frac{24\pi}{12}$$

$$= \sqrt{2} \left(\cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right)$$

$$\left(\frac{\sqrt{3} + i}{1 - i} \right)^{30} = (\sqrt{2})^{30} \left(\cos\left(30 \cdot \frac{5\pi}{12}\right) + i \sin\left(30 \cdot \frac{5\pi}{12}\right) \right)$$

$$= 2^{15} \left(\cos \frac{25\pi}{2} + i \cdot \sin \frac{25\pi}{2} \right) =$$

$$= 2^{15} \left(\cos \frac{\pi}{2} + i \cdot \sin \frac{\pi}{2} \right) = 2^{15} (0 + 1i) = 2^{15} i$$

(3) $f(x) = x^2 + 4x - 8$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & -1 \\ -3 & 5 & -2 \end{pmatrix} \qquad A^2 + 4A - 8E$$

$$A^2 = A \cdot A$$

$$A^2 = \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & -1 \\ -3 & 5 & -2 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & -1 \\ -3 & 5 & -2 \end{pmatrix} = \begin{pmatrix} -5 & 25 & -5 \\ 3 & 11 & -2 \\ 3 & 1 & -7 \end{pmatrix}$$

$$4A = 4 \begin{pmatrix} 1 & 3 & 2 \\ 0 & 4 & -1 \\ -3 & 5 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 12 & 8 \\ 0 & 16 & -4 \\ -12 & 20 & -8 \end{pmatrix}$$

$$8E = 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

$$f(A) = A^2 + 4A - 8E$$

$$f(A) = \begin{pmatrix} -5 & 25 & -5 \\ 3 & 11 & -2 \\ 3 & 1 & -7 \end{pmatrix} + \begin{pmatrix} 4 & 12 & 8 \\ 0 & 16 & -4 \\ -12 & 20 & -8 \end{pmatrix} - \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} =$$

$$= \begin{pmatrix} -9 & 37 & 3 \\ 3 & 19 & -6 \\ -9 & 21 & -7 \end{pmatrix}$$

(4) $\begin{cases} 2x_1 + 3x_2 - 5x_3 = 12 \\ 4x_1 + 5x_2 + 3x_3 = 10 \\ x_1 - 4x_2 + 4x_3 = -6 \end{cases}$ Метод Гаусса

$$\left(\begin{array}{ccc|c} 2 & 3 & -5 & 12 \\ 4 & 5 & 3 & 10 \\ 1 & -4 & 4 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 4 & -6 \\ 4 & 5 & 3 & 10 \\ 2 & 3 & -5 & 12 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 4 & -6 \\ 0 & -21 & 13 & -34 \\ 0 & 11 & -13 & 24 \end{array}\right) \sim$$

$1p(-4) + 2p$

$1p(-2) + 3p$

$2p(11) + 3p(21)$

$$\sim \left(\begin{array}{ccc|c} 1 & -4 & 4 & -6 \\ 0 & -21 & 13 & -34 \\ 0 & 0 & -130 & 130 \end{array}\right)$$

$$\begin{cases} x_1 - 4x_2 + 4x_3 = -6 \\ -x_2 + 13x_3 = -14 \\ 130x_3 = -130 \end{cases} \Rightarrow \underline{x_3 = -1}$$

$\underline{x_1 = 2}$

$-x_2 + (-13) = -14 \quad \underline{x_2 = 1}$

$x_1 - 4\cdot 1 + 4(-1) = -6$

$x_1 - 4 - 4 = -6$

$x_1 = -6 + 4 + 4$

$\underline{x_1 = 2}$

Метод Крамера

$$\Delta=\begin{vmatrix} 2 & 3 & -5 \\ 4 & 5 & 3 \\ 1 & -4 & 4 \end{vmatrix} = 40+9+80+25-48+24=130$$

$$\Delta_1=\begin{vmatrix} 12 & 3 & -5 \\ 10 & 5 & 3 \\ -6 & -4 & 4 \end{vmatrix} = 240-54+200-150-120+144=260$$

$$\Delta_2=\begin{vmatrix} 2 & 12 & -5 \\ 4 & 10 & 3 \\ 1 & -6 & 4 \end{vmatrix} = 80+120+36+50-192+36=130$$

$$\Delta_3=\begin{vmatrix} 2 & 3 & 12 \\ 4 & 5 & 10 \\ 1 & -4 & -6 \end{vmatrix} = -60+30-192-60+72+80=-130$$

$x_1=\frac{\Delta_1}{\Delta}$ $\quad x_1=\frac{260}{130}=2$ $\quad x_2=\frac{130}{130}=1$ $\quad x_3=-\frac{130}{130}=-1$

Матричний спосіб

$$\begin{vmatrix} 2 & 3 & -5 \\ 4 & 5 & 3 \\ 1 & -4 & 4 \end{vmatrix} = 130 \qquad B=\begin{pmatrix} 12 \\ 10 \\ -6 \end{pmatrix} \qquad X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$X=A^{-1}B$$

$$A_{11}=\begin{vmatrix} 5 & 3 \\ -4 & 4 \end{vmatrix} = 20+12=32$$

$$A_{12}=(-1)^3\begin{vmatrix} 4 & 3 \\ 1 & 4 \end{vmatrix} = -(16-3)=-13$$

$$A_{13}=\begin{vmatrix} 4 & 5 \\ 1 & -4 \end{vmatrix} = -16-5=-21$$

$$A_{21}=(-1)^3\begin{vmatrix} 3 & -5 \\ -4 & 4 \end{vmatrix} = -(12-20)=-(-8)=8$$

$$A_{22}=\begin{vmatrix} 2 & -5 \\ 1 & 4 \end{vmatrix} = 8+5=13$$

$$A_{23}=(-1)^5\begin{vmatrix} 2 & 3 \\ 1 & -4 \end{vmatrix} = -(-8-3)=11$$

$$A_{31} = \begin{vmatrix} 3 & -5 \\ 5 & 3 \end{vmatrix} = 9 + 25 = 34$$

$$A_{32} = (-1)^5 \begin{vmatrix} 2 & -5 \\ 4 & 3 \end{vmatrix} = -(6+20) = -26$$

$$A_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2$$

$$A^{-1} = \begin{pmatrix} 32 & -13 & -21 \\ 8 & 13 & 11 \\ 34 & -26 & -2 \end{pmatrix} \cdot \frac{1}{130}$$

$$A^{T} = \begin{pmatrix} 32 & 8 & 34 \\ -13 & 13 & -26 \\ -21 & 11 & -2 \end{pmatrix}$$

$$A^{-1} = \frac{1}{130} \cdot \begin{pmatrix} 32 & 8 & 34 \\ -13 & 13 & -26 \\ -21 & 11 & -2 \end{pmatrix} = \begin{pmatrix} \frac{16}{65} & \frac{4}{65} & \frac{17}{65} \\ -1/10 & 1/10 & -1/5 \\ -21/130 & 11/130 & -1/65 \end{pmatrix}$$

$$\begin{pmatrix} 16/65 & 4/65 & 17/65 \\ -1/10 & 1/10 & -1/5 \\ -21/130 & 11/130 & -1/65 \end{pmatrix} \begin{pmatrix} 12 \\ 10 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

5. $a_1 = (1,0,2,2) \quad a_2 = (2,1,0,2) \quad a_3 = (2,2,1,0)$

$b_1 = a_1 = (1,0,2,2)$

$$b_2 = a_2 - \frac{(a_2 b_1)}{b_1 b_1} b_1 = (2,1,0,2) - \frac{6}{9}(1,0,2,2) =$$

$$= \frac{1}{3}(4,3,-4,2)$$

$b_1 b_1 = (1,0,2,2)(1,0,2,2) =$
$= 1 + 0 + 4 + 4 = 9$

$$b_3 = a_3 - \frac{(a_3 b_2)}{(b_2 b_2)} b_2 = (2,2,1,0) - \frac{4}{9}(1,0,2,2) -$$

$$- \frac{10/3}{15/3} \cdot \frac{1}{3}(4,3,-4,2) = \frac{1}{3}(2,4,3,-4)$$

$(a_3 b_2) = (2,2,1,0) \cdot (1,0,2,2) = 2 \cdot 1 + 2 \cdot 0 + 1 \cdot 2 + 0 \cdot 2 =$
$= 2 + 2 = 4$

⑥ $\varphi(\vec{x}) = (x_1 + 2x_2,\ x_1 + x_2 - x_3,\ 3x_3 - x_1)$

$\varphi(\vec{x} + \vec{y}) = \varphi(x_1 + y_1,\ x_2 + y_2,\ x_3 + y_3) =$

$= ((x_1 + y_1) + 2(x_2 + y_2),\ (x_1 + y_1) + (x_2 + y_2) -$

$- (x_3 + y_3),\ 3(x_3 + y_3) - (x_1 + y_1)) =$

$= (x_1 + y_1 + 2x_2 + 2y_2,\ x_1 + y_1 + x_2 + y_2 - x_3 - y_3,$

$3x_3 + 3y_3 - x_1 - y_1) =$

$= (x_1 + 2x_2,\ x_1 + x_2 - x_3,\ 3x_3 - x_1) +$

$+ (y_1 + 2y_2,\ y_1 + y_2 - y_3,\ 3y_3 - y_1) =$

$= \varphi(x) + \varphi(y) = (x_1 + 2x_2 + y_1 + 2y_2,$

$x_1 + x_2 - x_3 + y_1 + y_2 - y_3,\ 3x_3 - x_1 + 3y_3 - y_1)$

$\Rightarrow \varphi(\vec{x} + \vec{y}) = \varphi(\vec{x}) + \varphi(\vec{y})$

Нехай λ - довільне, тоді

$\varphi(\lambda \vec{x}) = \varphi(\lambda x_1, \lambda x_2, \lambda x_3) = (\lambda x_1 + 2\lambda x_2,$

$\lambda x_1 + \lambda x_2 - \lambda x_3,\ 3\lambda x_3 - \lambda x_1) =$

$= \lambda(x_1 + 2x_2,\ x_1 + x_2 - x_3,\ 3x_3 - x_1) =$

$= \lambda \varphi(\vec{x}) \Rightarrow \varphi$ - лін. перетв.

Образи: $\varphi(\vec{e}_1), \varphi(\vec{e}_2), \varphi(\vec{e}_3)$

$\varphi(\vec{e}_1) = \varphi(1,0,0) = (1 + 2\cdot 0,\ 1 + 0 - 0,\ 3\cdot 0 - 1) =$
$= (1, 1, -1)$

$\varphi(\vec{e}_2) = \varphi(0,1,0) = (0 + 2\cdot 1,\ 0 + 1 - 0,\ 3\cdot 0 - 0) =$
$= (2, 1, 0)$

$\varphi(\vec{e}_3) = \varphi(0,0,1) = (0 + 2\cdot 0,\ 0 + 0 - 1,$
$3\cdot 0 - 0) = (0, -1, 0)$

(7)

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

$$\lambda(A) = |A - \lambda I| = \begin{vmatrix} 3-\lambda & 2 & 0 \\ 2 & 2-\lambda & 2 \\ 0 & 2 & 1-\lambda \end{vmatrix} =$$

$= (3-\lambda)(2-\lambda)(1-\lambda) + 2\cdot 2\cdot 0 + 2\cdot 2\cdot 0 - (2-\lambda)\cdot 0\cdot 0 -$
$- 2\cdot 2(1-\lambda) - 2\cdot 2(3-\lambda) = (6 - 3\lambda - 2\lambda + \lambda^2)(1-\lambda) -$
$- 4 + 4\lambda - 12 + 4\lambda = 6 - 6\lambda - 5\lambda + 5\lambda^2 + \lambda^2 - \lambda^3 -$
$- 4 + 4\lambda - 12 + 4\lambda = -3\lambda + 6\lambda^2 - \lambda^3 - 10 =$
$= -\lambda^3 + 6\lambda^2 - 3\lambda - 10$

$$-\lambda^3 + 6\lambda^2 - 3\lambda - 10 = 0$$

$$\lambda_1 = -1 \quad \lambda_2 = 2 \quad \lambda_3 = 5$$

Для $\lambda_1 = -1$

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 2 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 0 \\ 0 & -4 & -4 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 4 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$1p + 2p(-2)$

$2p + 3p(-2)$

$$\begin{cases} 4x_1 + 2y_1 = 0 \\ y_1 + z_1 = 0 \end{cases}$$