Chapter 4

Discrete-time systems

Discrete-Time Systems

- **A discrete-time system processes a given input sequence x[n] to generates an output sequence y[n] with more desirable properties**
- **In most applications, the discrete-time system is a single-input, single-output system:**

$$x[n] \longrightarrow \boxed{\text{Discrete-Time System}} \longrightarrow y[n]$$

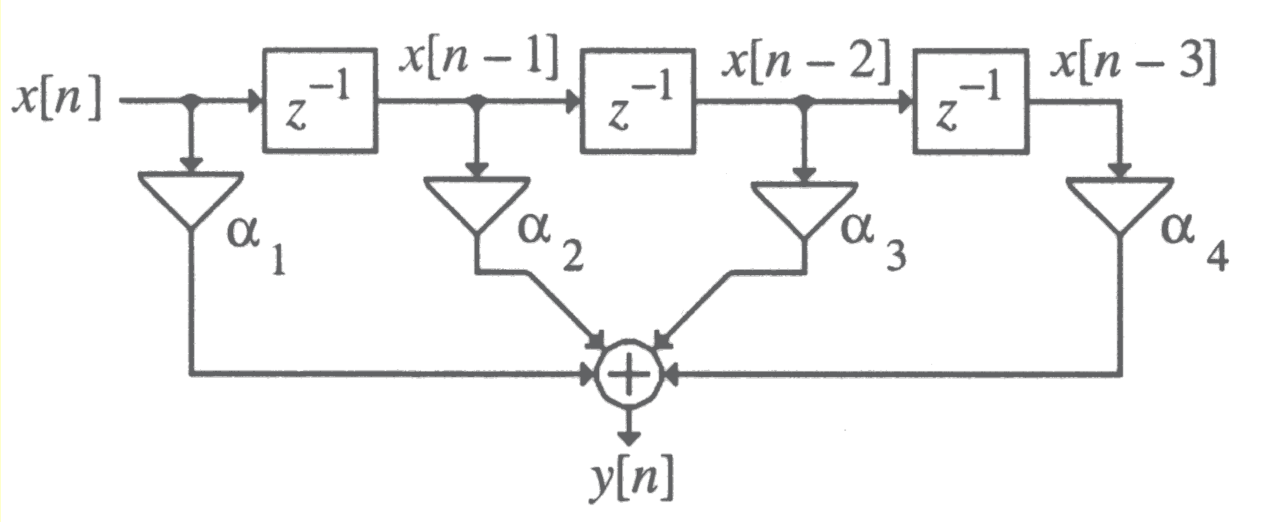
Input sequence **Output sequence**

Discrete-Time Systems

- Mathematically, the discrete-time system is characterized by an operator $\mathcal{H}(\cdot)$ that transforms the input sequence $x[n]$ into another sequence $y[n]$ at the output
- The discrete-time system may also have more than one input and/or more than one output

Discrete-Time Systems Examples

- **2-input, 1-output discrete-time systems - Modulator, adder**
- **1-input, 1-output discrete-time systems - Multiplier, unit delay, unit advance**



Discrete-Time Systems Examples

- **Accumulator** : $y[n] = \sum_{=-\infty}^{n} x[\ ]$

  累加器

$$= \sum_{=-\infty}^{n-1} x[\ ] + x[n] = y[n-1] + x[n]$$

- **The output y[n] at time instant n is the sum of the input sample x[n] at time instant n and the previous output y[n-1] at time instant n-1 which is the sum of all previous input sample values from -∞ to n-1**

- **The system accumulatively adds, i.e., it accumulates all input sample values**

Discrete-Time Systems Examples

- **Accumulator - Input-output relation can also be written in the form**

$$y[n] = \sum_{=-\infty}^{-1} x[\ \ ] + \sum_{=0}^{n} x[\ \ ]$$

$$= y[-1] + \sum_{=0}^{n} x[\ \ ], \quad n \geq 0$$

- **The second form is used for a causal input sequence, in which case y[-1] is called the initial condition**

初始条件

Discrete-Time Systems Examples

- Moving-Average Filter

  滑动平均滤波器

- Linear Interpolator

  线性内插器

- Median Filter

  中值滤波器

Discrete-Time Systems:Examples

- **M-point moving-average system** -

$$y[n] = \frac{1}{M} \sum_{k=0}^{M-1} x[n-k]$$

- Used in smoothing random variations in data
- In most applications, the data $x[n]$ is a bounded sequence
- M-point average $y[n]$ is also a bounded sequence

Discrete-Time Systems:Examples

- If there is no bias in the measurements, an improved estimate of the noisy data is obtained by simply increasing M
- A direct implementation of the M-point moving average system requires $M-1$ additions, 1 division, and storage of $M-1$ past input data samples
- A more efficient implementation is developed next

Discrete-Time Systems:Examples

$$y[n] = \frac{1}{M}\left(\sum_{\ell=0}^{M-1} x[n-\ell] + x[n-M] - x[n-M]\right)$$

$$= \frac{1}{M}\left(\sum_{\ell=1}^{M} x[n-\ell] + x[n] - x[n-M]\right)$$

$$= \frac{1}{M}\left(\sum_{\ell=0}^{M-1} x[n-1-\ell] + x[n] - x[n-M]\right)$$

Hence

$$y[n] = y[n-1] + \frac{1}{M}\left(x[n] - x[n-M]\right)$$

Discrete-Time Systems:Examples

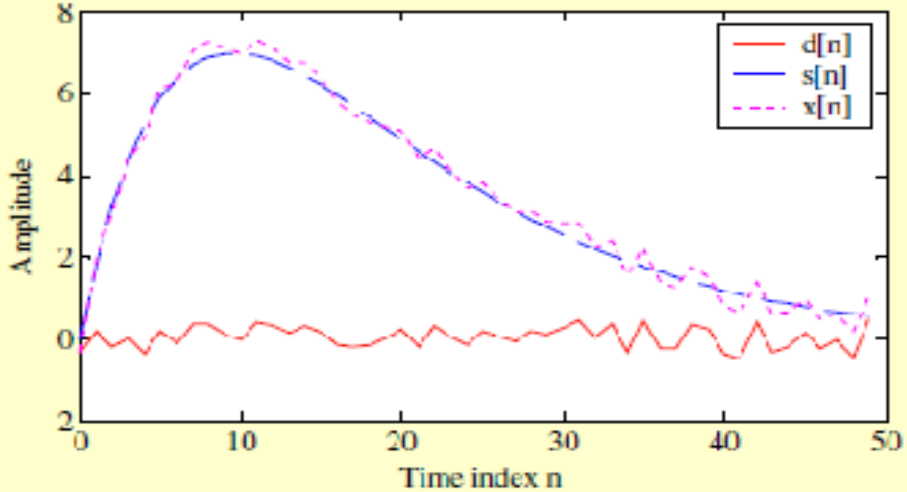
- Computation of the modified M-point moving average system using the recursive equation now requires 2 additions and 1 division
- An application: Consider

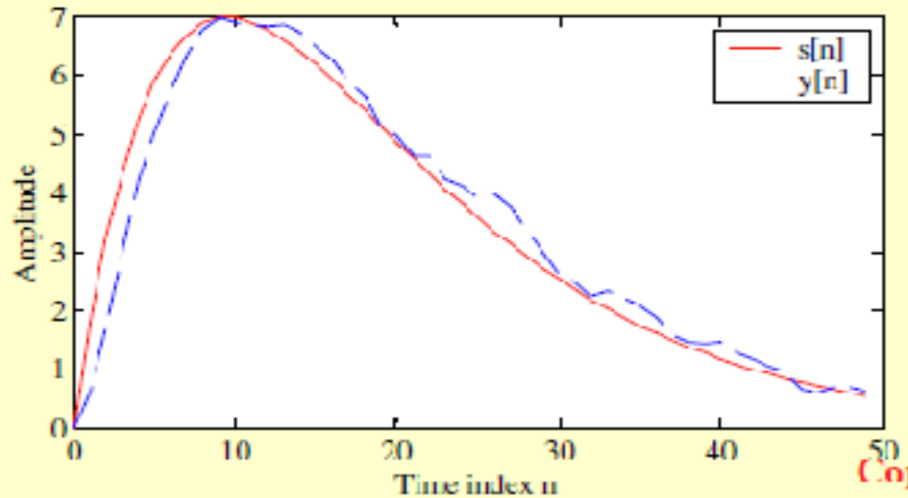
$$x[n] = s[n] + d[n],$$

where $s[n]$ is the signal corrupted by a noise $d[n]$

Discrete-Time Systems:Examples

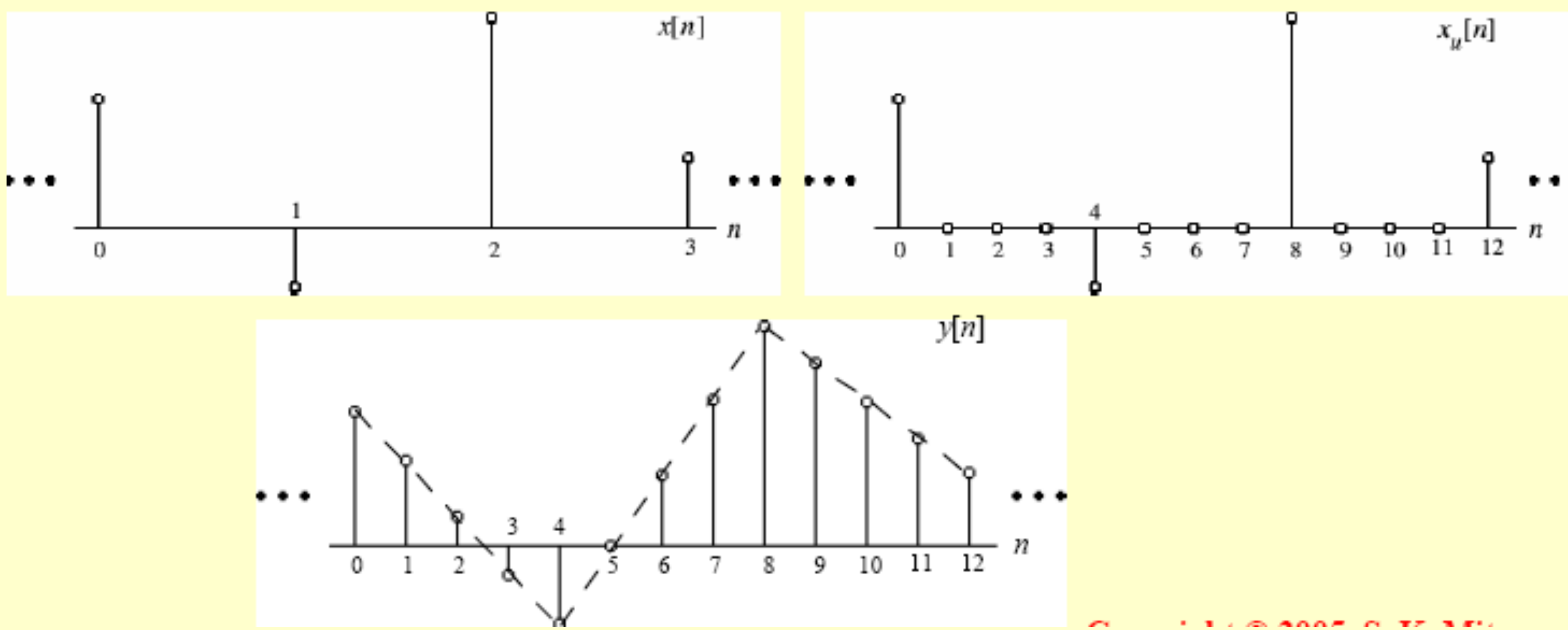
$$s[n] = 2[n(0.9)^n],\ d[n]$$ - random signal





Discrete-Time Systems:Examples

- **Linear interpolation** - Employed to estimate sample values between pairs of adjacent sample values of a discrete-time sequence
- **Factor-of-4 interpolation**



Discrete-Time Systems: Examples

- **Factor-of-2 interpolator** -

$$y[n] = x_u[n] + \frac{1}{2}\left(x_u[n-1] + x_u[n+1]\right)$$

- **Factor-of-3 interpolator** -

$$y[n] = x_u[n] + \frac{1}{3}\left(x_u[n-1] + x_u[n+2]\right)$$

$$+ \frac{2}{3}\left(x_u[n-2] + x_u[n+1]\right)$$

Discrete-Time Systems: Examples

- **Factor-of-2 interpolator** -

Original (512×512)

Down–sampled (256×256)

Interpolated (512 × 512)

Discrete-Time Systems: Examples

Median Filter –

- The median of a set of $(2K+1)$ numbers is the number such that K numbers from the set have values greater than this number and the other K numbers have values smaller
- Median can be determined by rank-ordering the numbers in the set by their values and choosing the number at the middle

Discrete-Time Systems: Examples

Median Filter –

- **Example**: Consider the set of numbers

$$\{2, \quad -3, \quad 10, \quad 5, \quad -1\}$$

- Rank-order set is given by

$$\{-3, \quad -1, \quad 2, \quad 5, \quad 10\}$$

- Hence,

$$\text{med}\{2, \quad -3, \quad 10, \quad 5, \quad -1\} = 2$$

Discrete-Time Systems: Examples

Median Filter –

- Implemented by sliding a window of odd length over the input sequence $\{x[n]\}$ one sample at a time
- Output $y[n]$ at instant n is the median value of the samples inside the window centered at n

- To show this, consider a median filter with a window of length 3
- Output of the filter for an input

$$\{x_1[n]\} = \{3, \quad 4, \quad 5\}, 0 \le n \le 2$$

is

$$\{y_1[n]\} = \{3, \quad 4, \quad 4\}, 0 \le n \le 2$$

- Output for an input

$$\{x_2[n]\} = \{2, \quad -1, \quad -1\}, 0 \le n \le 2$$

is

$$\{y_2[n]\} = \{0, \quad -1, \quad -1\}, 0 \le n \le 2$$

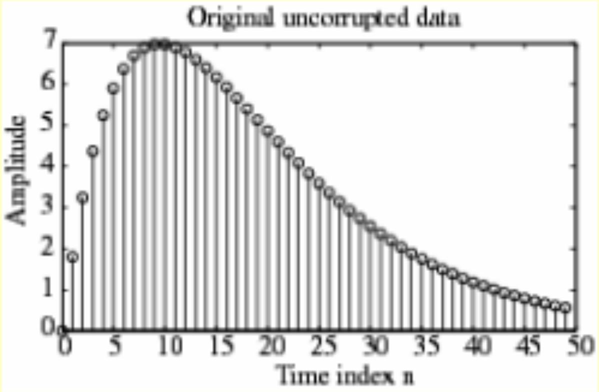
Discrete-Time Systems: Examples

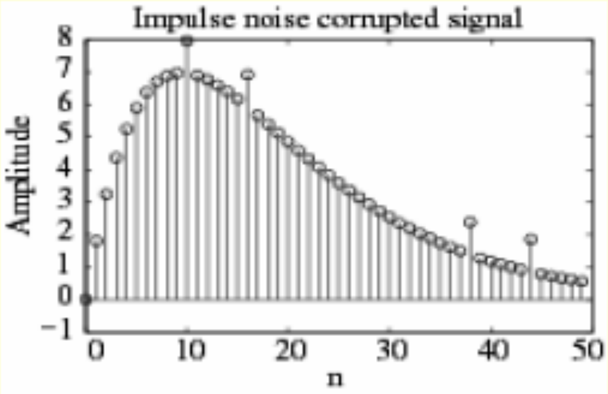
Median Filter –

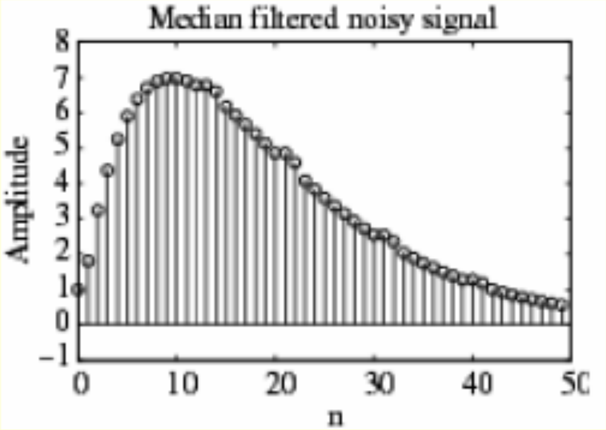
- Finds applications in removing additive random noise, which shows up as sudden large errors in the corrupted signal
- Usually used for the smoothing of signals corrupted by impulse noise

Discrete-Time Systems: Examples

Median Filtering Example –







Classification of Discrete-Time Systems

- **Linear System**
- **Shift-Invariant System**
- **Causal System**
- **Stable System**
- **Passive and Lossless Systems**

  被动的　　　　无源和无损系统

Linear Systems

- **Definition** - If $y_1[n]$ is the output due to an input $x_1[n]$ and $y_2[n]$ is the output due to an input $x_2[n]$ then for an input

  $$x[n]=ax_1[n]+bx_2[n]$$

  the output is given by

  $$y[n]=ay_1[n]+by_2[n]$$

- Above property must hold for any arbitrary constants a and b and for all possible inputs $x_1[n]$ and $x_2[n]$
- Hence, the above system is **linear**

Linear Discrete-Time Systems

- **Accumulator** - $y_1[n] = \sum_{\ell=-\infty}^{n} x_1[\ell], \quad y_2[n] = \sum_{\ell=-\infty}^{n} x_2[\ell]$

  For an input

$$x[n] = \alpha\, x_1[n] + \beta\, x_2[n]$$

  the output is

$$y[n] = \sum_{\ell=-\infty}^{n} \left(\alpha\, x_1[\ell] + \beta\, x_2[\ell]\right)$$

$$= \alpha \sum_{\ell=-\infty}^{n} x_1[\ell] + \beta \sum_{\ell=-\infty}^{n} x_2[\ell] = \alpha\, y_1[n] + \beta\, y_2[n]$$

- Hence, the above system is **linear**

Linear Discrete-Time Systems

- The outputs $y_1[n]$ and $y_2[n]$ for inputs $x_1[n]$ and $x_2[n]$ are given by

$$y_1[n] = y_1[-1] + \sum_{\ell=0}^{n} x_1[\ell]$$

$$y_2[n] = y_2[-1] + \sum_{\ell=0}^{n} x_2[\ell]$$

- The output $y[n]$ for an input $\alpha x_1[n] + \beta x_2[n]$ is given by

$$y[n] = y[-1] + \sum_{\ell=0}^{n} (\alpha x_1[\ell] + \beta x_2[\ell])$$

Linear Discrete-Time Systems

- Now $\alpha\, y_1[n] + \beta\, y_2[n]$

$$= \alpha\big(y_1[-1] + \sum_{\ell=0}^{n} x_1[\ell]\big) + \beta\big(y_2[-1] + \sum_{\ell=0}^{n} x_2[\ell]\big)$$

$$= (\alpha\, y_1[-1] + \beta\, y_2[-1]) + (\alpha \sum_{\ell=0}^{n} x_1[\ell] + \beta \sum_{\ell=0}^{n} x_2[\ell])$$

- Thus $y[n] = \alpha\, y_1[n] + \beta\, y_2[n]$ if

$$y[-1] = \alpha\, y_1[-1] + \beta\, y_2[-1]$$

Linear Discrete-Time System

- For the causal accumulator to be **linear** the condition $y[-1] = \alpha\, y_1[-1] + \beta\, y_2[-1]$ must hold for all initial conditions $y[-1]$, $y_1[-1]$, $y_2[-1]$, and all constants α and β
- This condition cannot be satisfied unless the accumulator is initially at rest with zero initial condition
- For nonzero initial condition, the system is **nonlinear**

- The median filter described earlier is a nonlinear discrete-time system
- To show this, consider a median filter with a window of length 3
- Output of the filter for an input

$$\{x_1[n]\} = \{3, \quad 4, \quad 5\}, 0 \le n \le 2$$

is

$$\{y_1[n]\} = \{3, \quad 4, \quad 4\}, 0 \le n \le 2$$

- Output for an input

$$\{x_2[n]\} = \{2, \quad -1, \quad -1\}, 0 \le n \le 2$$

is

$$\{y_2[n]\} = \{0, \quad -1, \quad -1\}, 0 \le n \le 2$$

- However, the output for an input

$$\{x[n]\} = \{x_1[n] + x_2[n]\}$$

is

$$\{y[n]\} = \{3, \quad 4, \quad 3\}$$

- Note

$$\{y_1[n]+y_2[n]\}=\{3, \quad 3, \quad 3\}\neq\{y[n]\}$$

- Hence, the median filter is a nonlinear discrete-time system

Shift-Invariant System

- For a shift-invariant system, if $y_1[n]$ is the response to an input $x_1[n]$, then the response to an input

$$x[n]=x_1[n-n_0]$$

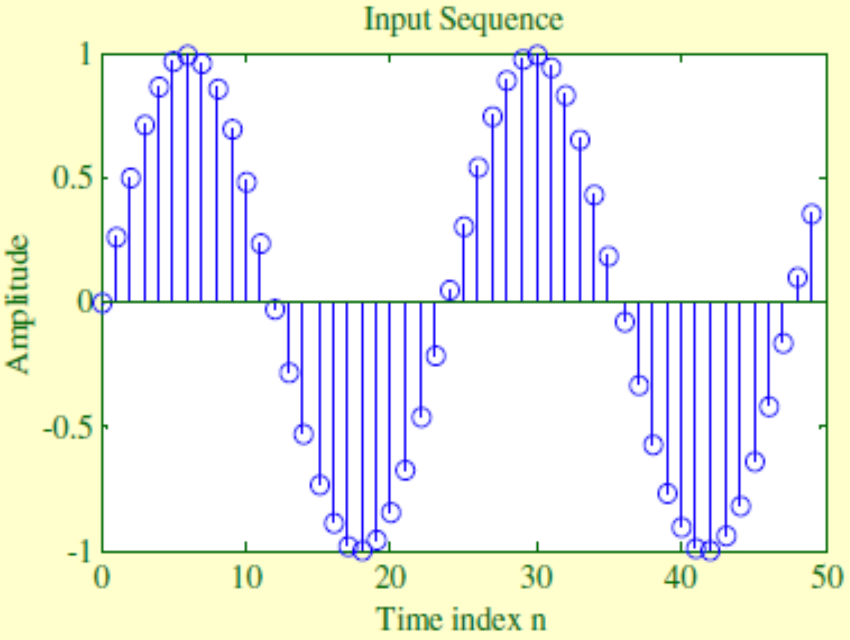
  is simply

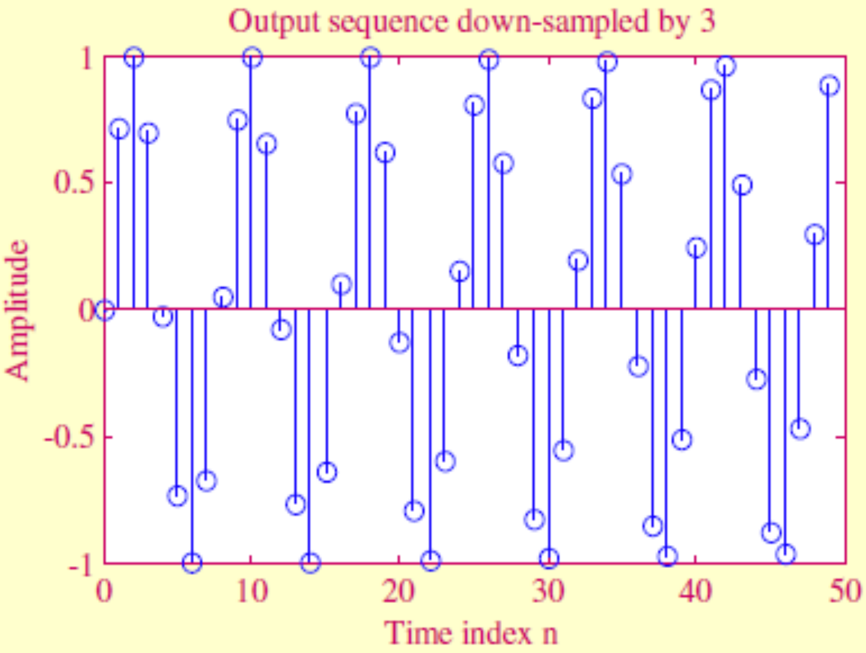
$$y[n]=y_1[n-n_0]$$

  where n_0 is any positive or negative integer
- The above relation must hold for any arbitrary input and its corresponding output
- The above property is called time-invariance property, or shift-invariant property

Sampling Rate Alteration

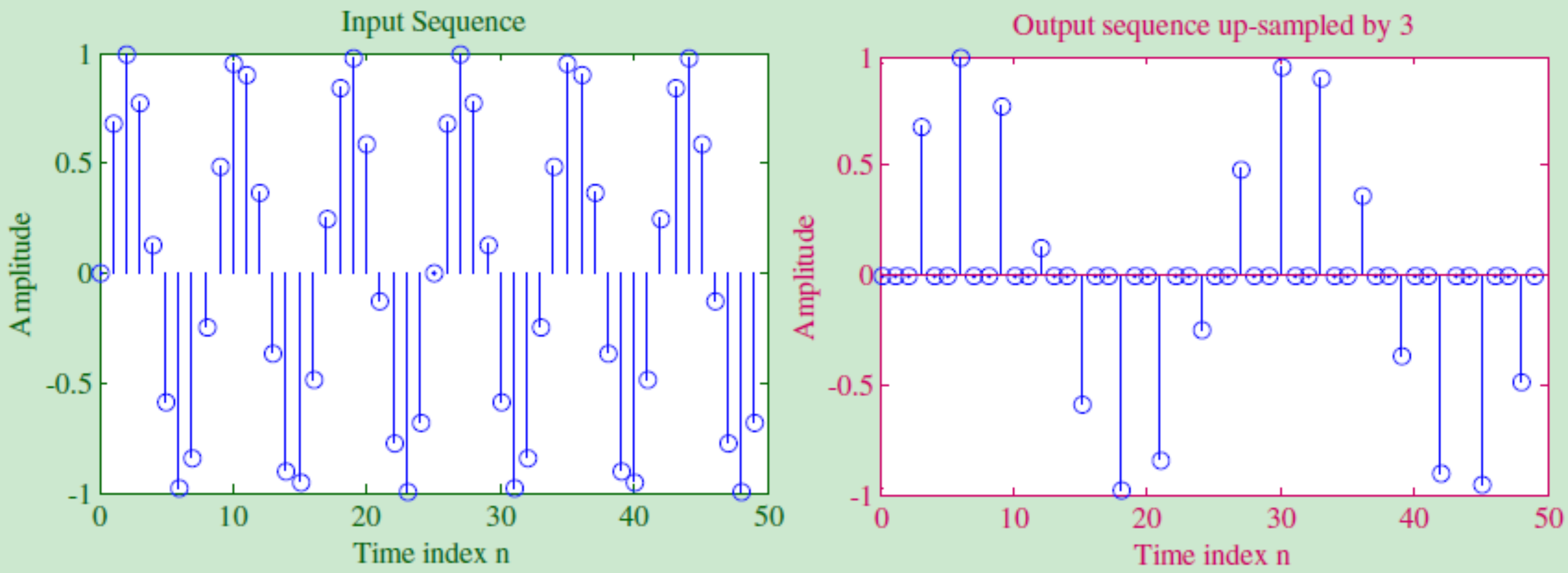
- An example of the down-sampling operation





Sampling Rate Alteration

- An example of the up-sampling operation



Shift-Invariant System

- Example - Consider the up-sampler with an input-output relation given by

$$x_u[n] = \begin{cases} x[n/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

- For an input $x_1[n] = x[n - n_o]$ the output $x_{1,u}[n]$ is given by

$$x_{1,u}[n] = \begin{cases} x_1[n/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} x[(n - Ln_o)/L], & n = 0, \pm L, \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$

Shift-Invariant System

- However from the definition of the up-sampler

$$x_u[n-n_o]$$
$$=\begin{cases} x[(n-n_o)/L], & n=n_o, n_o \pm L, n_o \pm 2L, \ldots.. \\ 0, & \text{otherwise} \end{cases}$$
$$\neq x_{1,u}[n]$$

- Hence, the up-sampler is a time-varying system

Linear Time-Invariant system

- **Linear Time-Invariant (LTI) System -**
  A system satisfying both the linearity and the time-invariance property
- **LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design**
- **Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades**

Causal System

- In a **causal system**, the n_o-th output sample $y[n_o]$ depends only on input samples $x[n]$ for $n \le n_o$ and does not depend on input samples for $n > n_o$
- Let $y_1[n]$ and $y_2[n]$ be the responses of a causal discrete-time system to the inputs $x_1[n]$ and $x_2[n]$, respectively

Causal System

- Then

$$x_1[n] = x_2[n] \text{ for } n < N$$

implies also that

$$y_1[n] = y_2[n] \text{ for } n < N$$

- For a causal system, changes in output samples do not precede changes in the input samples

Causal System

- Examples of causal systems:

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

$$y[n] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + a_1 y[n-1] + a_2 y[n-2]$$

$$y[n] = y[n-1] + x[n]$$

- Examples of noncausal systems:

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

$$y[n] = x_u[n] + \frac{1}{3}(x_u[n-1] + x_u[n+2]) + \frac{2}{3}(x_u[n-2] + x_u[n+1])$$

Discrete-Time Systems: Examples

- **Factor-of-2 interpolator** -

$$y[n] = x_u[n] + \frac{1}{2}\left(x_u[n-1] + x_u[n+1]\right)$$

- **Factor-of-3 interpolator** -

$$y[n] = x_u[n] + \frac{1}{3}\left(x_u[n-1] + x_u[n+2]\right)$$

$$+ \frac{2}{3}\left(x_u[n-2] + x_u[n+1]\right)$$

Causal System

- A noncausal system can be implemented as a causal system by delaying the output by an appropriate number of samples
- For example a causal implementation of the factor-of-2 interpolator is given by

$$y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n])$$

Stable system

- A discrete-time system to be **stable** if and only if for every bounded input, the output is also bounded. i.e.

  $|x[n]|<B_x$ for all values of n, then

  $|y[n]|<B_y$

Stable System

- There are various definitions of stability
- We consider here the **bounded-input, bounded-output (BIBO) stability**
- If $y[n]$ is the response to an input $x[n]$ and if

$$|x[n]| \leq B_x \quad \text{for all values of } n$$

then

$$|y[n]| \leq B_y \quad \text{for all values of } n$$

Stable System

- Example - The M-point moving average filter is BIBO stable:

$$y[n] = \frac{1}{M}\sum_{k=0}^{M-1} x[n-k]$$

- For a bounded input $|x[n]| \le B_x$ we have

$$|y[n]| = \left|\frac{1}{M}\sum_{k=0}^{M-1} x[n-k]\right| \le \frac{1}{M}\sum_{k=0}^{M-1} |x[n-k]|$$

$$\le \frac{1}{M}(MB_x) \le B_x$$

Passive and Lossless Systems

- **A discrete-time system is defined to be passive if, for every finite-energy input x[n], the output y[n] has, at most, the same energy, i.e.**

$$\sum_{n=-\infty}^{\infty} \left|y[n]\right|^2 \leq \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 < \infty$$

- **For a lossless system, the above inequality is satisfied with an equal sign for every input**

Passive and Lossless Systems

- **<u>Example</u> - Consider the discrete-time system defined by y[n]=αx[n-N] with N a positive integer**
- **Its output energy is given by**

$$\sum_{n=-\infty}^{\infty} \left|y[n]\right|^2 = \left|\alpha\right|^2 \sum_{n=-\infty}^{\infty} \left|x[n]\right|^2$$

- **Hence, it is a passive system if $|\alpha| \leq 1$ and is a lossless system if $|\alpha| = 1$**

Impulse and Step Responses

- The response of a discrete-time system to a unit sample sequence $\{\delta[n]\}$ is called the **unit impulse response** or simply, the **impulse response**, and is denoted by $\{h[n]\}$
- The response of a discrete-time system to a unit step sequence $\{\mu[n]\}$ is called the **unit step response** or simply, the **step response**, and is denoted by $\{s[n]\}$

Impulse and Step Responses

- **<u>Example</u> - The impulse response of the system**

  $$y[n]=a_1x[n]+a_2x[n-1]+a_3x[n-2]+a_4x[n-3]$$

  is obtained by setting $x[n] = \delta[n]$ resulting in

  $$h[n]=a_1\delta[n]+a_2\delta[n-1]+a_3\delta[n-2]+a_4\delta[n-3]$$

- **The impulse response is thus a finite-length sequence of length 4 given by**

  $$\{h[n]=\{a_1, a_2, a_3, a_4\}$$

Impulse and Step Responses

- **Example - The impulse response of the discrete-time accumulator**

$$y[n] = \sum_{=-\infty}^{n} x[\quad]$$

is obtained by setting $x[n] = \delta[n]$ resulting in

$$h[n] = \sum_{=-\infty}^{n} \delta[\quad] = \mu[n]$$

Impulse and Step Responses

- **<u>Example</u> - The impulse response {h[n]} of the factor-of-2 interpolator**

$$y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$$

is obtained by setting $x_u[n]=\delta[n]$ and is given by

$$h[n] = \delta[n] + \frac{1}{2}(\delta[n-1] + \delta[n+1])$$

- **The impulse response is thus a finite-length sequence of length 3:**

$$\{h[n]\} = \{0.5, \quad \underset{\uparrow}{1} \quad 0.5\}$$

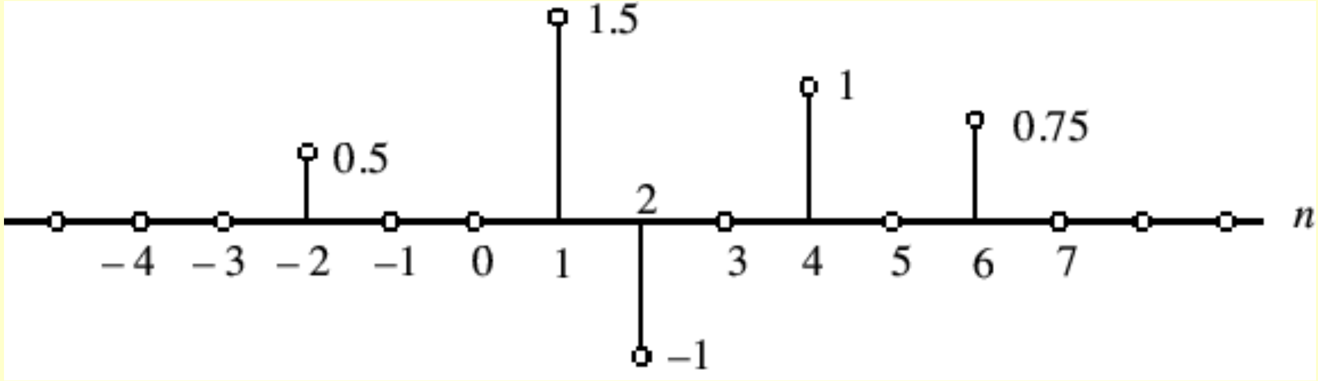
时域特性

Time-Domain Characterization of LTI Discrete-Time System

- **Input-Output Relationship :**
  A consequence of the linear, time-invariance property is that an LTI discrete-time system is completely characterized by its impulse response
- **Knowing the impulse response one can compute the output of the system for any arbitrary input**

Representation of an Arbitrary Sequence

- **An arbitrary sequence can be represented in the time-domain as a 加权和 weighted sum of some basic sequence and its delayed (advanced) versions**



$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + \delta[n-4] + 0.75\delta[n-6]$$

Time-Domain Characterization of LTI Discrete-Time System

- Let **h[n]** denote the impulse response of a LTI discrete-time system
- Compute its output y[n] for the input:

$$x[n] = 0.5\delta[n+2] + 1.5\delta[n-1] - \delta[n-2] + 0.75\delta[n-5]$$

- As the system is linear, we can compute its outputs for each member of the input separately and add the individual outputs to determine y[n]

Time-Domain Characterization of LTI Discrete-Time System

- **Likewise, as the system is linear**

Input **output**

$$0.5\delta[n+2] \rightarrow 0.5h[n+2]$$

$$1.5\delta[n-1] \rightarrow 1.5h[n-1]$$

$$-\delta[n-2] \rightarrow -h[n-2]$$

$$0.75\delta[n-5] \rightarrow 0.75h[n-5]$$

Hence because of the linearity property we get

$$y[n] = 0.5h[n+2] + 1.5h[n-1] - h[n-2] + 0.75h[n-5]$$

Time-Domain Characterization of LTI Discrete-Time System

- **Now, any arbitrary input sequence $x[n]$ can be expressed as a linear combination of delayed and advanced unit sample sequences in the form**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

- **The response of the LTI system to an input x[k]δ[n-k] will be x[k]h[n-k]**

Time-Domain Characterization of LTI Discrete-Time System

- **Hence, the response $y[n]$ to an input**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

will be $$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

which can be alternately written as

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k]h[k]$$

Convolution Sum

- **The summation**

$$\boxed{y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=\sum_{k=-\infty}^{\infty}x[n-k]h[n]}$$

卷积和

is called the convolution sum of the sequences $x[n]$ and $h[n]$ and represented compactly as **简记为**

$$y[n]=x[n]\circledast h[n]$$

Convolution Sum

Properties -

- **Commutative property: 交换律**

$$x[n] \circledast h[n] = h[n] \circledast x[n]$$

- **Associative property : 结合律**

$$(x[n] \circledast h[n]) \circledast y[n] = x[n] \circledast (h[n] \circledast y[n])$$

- **Distributive property : 分配律**

$$x[n] \circledast (h[n] + y[n]) = x[n] \circledast h[n] + x[n] \circledast y[n]$$

Convolution Sum

解释

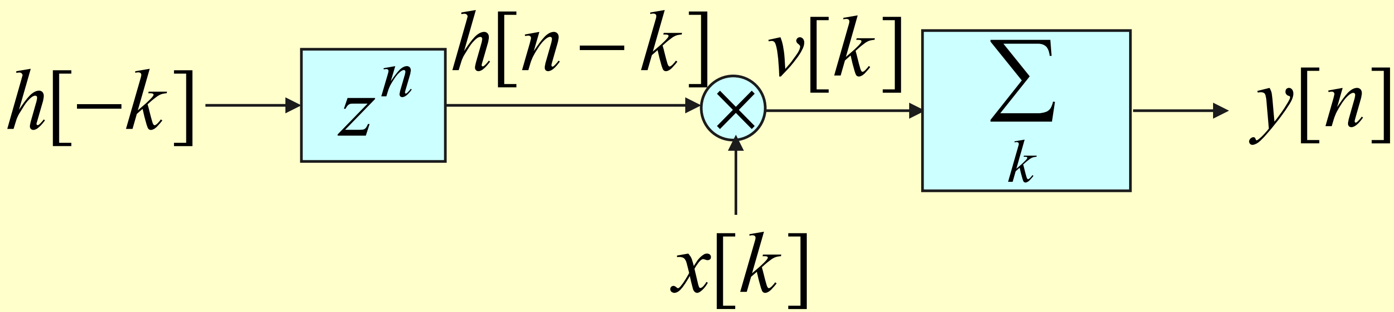
Interpretation -

- **1) Time-reverse h[k] to form h[-k]**
- **2) Shift h[-k] to the right by n sampling periods if $n > 0$ or shift to the left by n sampling periods if $n < 0$ to form h[n-k]**
- **3) Form the product v[k]=x[k]h[n-k]**
- **4) Sum all samples of v[k] to develop the n-th sample of y[n] of the convolution sum**

Convolution Sum

示意图

- **Schematic Representation** -



- **The computation of an output sample using the convolution sum is simply a sum of products**
- **Involves fairly simple operations such as additions, multiplications, and delays**

Convolution Sum

- In practice, if either the input or the impulse response is of finite length, the convolution sum can be used to compute the output sample as it involves a finite sum of products
- If both the input sequence and the impulse response sequence are of finite length, the output sequence is also of finite length

Convolution Sum

- If both the input sequence and the impulse response sequence are of **infinite length**, convolution sum cannot be used to compute the output
- For systems characterized by an infinite impulse response sequence, an alternate time-domain description involving a finite sum of products will be considered

Tabular Method of Convolution Sum Computation

- Can be used to convolve two finite-length sequences
- Consider the convolution of $\{g[n]\}, 0 \le n \le 3$, with $\{h[n]\}, 0 \le n \le 2$, generating the sequence $y[n] = g[n] \circledast h[n]$
- Samples of $\{g[n]\}$ and $\{h[n]\}$ are then multiplied using the conventional multiplication method without any carry operation

Tabular Method of Convolution Sum Computation

n:	0	1	2	3	4	5
$g[n]$:	$g[0]$	$g[1]$	$g[2]$	$g[3]$		
$h[n]$:	$h[0]$	$h[1]$	$h[2]$			
	$g[0]h[0]$	$g[1]h[0]$	$g[2]h[0]$	$g[3]h[0]$		
		$g[0]h[1]$	$g[1]h[1]$	$g[2]h[1]$	$g[3]h[1]$	
			$g[0]h[2]$	$g[1]h[2]$	$g[2]h[2]$	$g[3]h[2]$
$y[n]$:	$y[0]$	$y[1]$	$y[2]$	$y[3]$	$y[4]$	$y[5]$

- The samples $y[n]$ generated by the convolution sum are obtained by adding the entries in the column above each sample

Tabular Method of Convolution Sum Computation

- The samples of $\{y[n]\}$ are given by

$$y[0] = g[0]h[0]$$

$$y[1] = g[1]h[0] + g[0]h[1]$$

$$y[2] = g[2]h[0] + g[1]h[1] + g[0]h[2]$$

$$y[3] = g[3]h[0] + g[2]h[1] + g[1]h[2]$$

$$y[4] = g[3]h[1] + g[2]h[2]$$

$$y[5] = g[3]h[2]$$

Convolution Sum

- The M-file **conv** implements the convolution sum of two finite-length sequences
- If a=[-2 0 1 -1 3]

  b=[1 2 0 -1]

  then **conv(a,b)** yields

  [-2 -4 1 3 1 5 1 -3]

Convolution Using MATLAB

- The M-file `conv` implements the convolution sum of two finite-length sequences
- If $a = [-2 \quad 0 \quad 1 \quad -1 \quad 3]$

  $b = [1 \quad 2 \quad 0 \quad -1]$

  then `conv(a,b)` yields

  $[-2 \quad -4 \quad 1 \quad 3 \quad 1 \quad 5 \quad 1 \quad -3]$

Stability condition in terms of the Impulse Response

- **BIBO Stability Condition- A discrete-time is BIBO stable if the output sequence {y[n]} remains bounded for all bounded input sequence {x[n]}**
- **An LTI discrete-time system is stable if and only if its impulse response sequence {h(n)} is absolutely summable, that is,**

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

Stability Condition of an LTI Discrete-Time System

- Proof: Assume $h[n]$ is a real sequence
- Since the input sequence $x[n]$ is bounded we have

$$|x[n]| \le B_x < \infty$$

- Therefore

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k] x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|$$

$$\le B_x \sum_{k=-\infty}^{\infty} |h[k]| = B_x S$$

Stability Condition of an LTI Discrete-Time System

- Thus, $S < \infty$ implies $|y[n]| \leq B_y < \infty$ indicating that $y[n]$ is also bounded
- To prove the converse, assume $y[n]$ is bounded, i.e., $|y[n]| \leq B_y$
- Consider the input given by

$$x[n] = \begin{cases} \operatorname{sgn}(h[-n]), & \text{if } h[-n] \neq 0 \\ K, & \text{if } h[-n] = 0 \end{cases}$$

Stability Condition of an LTI Discrete-Time System

where $\text{sgn}(c) = +1$ if $c > 0$ and $\text{sgn}(c) = -1$ if $c < 0$ and $|K| \leq 1$

- Note: Since $|x[n]| \leq 1$, $\{x[n]\}$ is obviously bounded
- For this input, $y[n]$ at $n = 0$ is

$$y[0] = \sum_{k=-\infty}^{\infty} \text{sgn}(h[k])h[k] = S \leq B_y < \infty$$

- Therefore, $|y[n]| \leq B_y$ implies $S < \infty$

Stability Condition of an LTI Discrete-Time System

- Example - Consider a causal LTI discrete-time system with an impulse response

$$h[n] = (\alpha)^n \mu[n]$$

- For this system

$$S = \sum_{n=-\infty}^{\infty} \left|\alpha^n \mu[n]\right| = \sum_{n=0}^{\infty} |\alpha|^n = \frac{1}{1-|\alpha|} \quad \text{if } |\alpha| < 1$$

- Therefore $S < \infty$ if $|\alpha| < 1$ for which the system is BIBO stable
- If $|\alpha| = 1$, the system is not BIBO stable

Causaliity condition in terms of the Impulse Response

- **An LTI discrete-time system is causal if and only if its impulse response sequence {h(n)} is a causal sequence satisfying the condition:**

$$H[k]=0 \qquad\qquad for \quad k<0$$

Causality Condition of an LTI Discrete-Time System

- Let $x_1[n]$ and $x_2[n]$ be two input sequences with

$$x_1[n] = x_2[n] \text{ for } n \le n_o$$

- The corresponding output samples at $n = n_o$ of an LTI system with an impulse response $\{h[n]\}$ are then given by

Causality Condition of an LTI Discrete-Time System

$$y_1[n_o] = \sum_{k=-\infty}^{\infty} h[k]x_1[n_o - k] = \sum_{k=0}^{\infty} h[k]x_1[n_o - k]$$

$$+ \sum_{k=-\infty}^{-1} h[k]x_1[n_o - k]$$

$$y_2[n_o] = \sum_{k=-\infty}^{\infty} h[k]x_2[n_o - k] = \sum_{k=0}^{\infty} h[k]x_2[n_o - k]$$

$$+ \sum_{k=-\infty}^{-1} h[k]x_2[n_o - k]$$

Causality Condition of an LTI Discrete-Time System

- If the LTI system is also causal, then

$$y_1[n_o] = y_2[n_o]$$

- As $x_1[n] = x_2[n]$ for $n \le n_o$

$$\sum_{k=0}^{\infty} h[k] x_1[n_o - k] = \sum_{k=0}^{\infty} h[k] x_2[n_o - k]$$

- This implies

$$\sum_{k=-\infty}^{-1} h[k] x_1[n_o - k] = \sum_{k=-\infty}^{-1} h[k] x_2[n_o - k]$$

Causality Condition of an LTI Discrete-Time System

- As $x_1[n] \neq x_2[n]$ for $n > n_o$ the only way the condition

$$\sum_{k=-\infty}^{-1} h[k] x_1[n_o - k] = \sum_{k=-\infty}^{-1} h[k] x_2[n_o - k]$$

will hold if both sums are equal to zero, which is satisfied if

$$h[k] = 0 \quad \text{for } k < 0$$

Causality Condition of an LTI Discrete-Time System

- An LTI discrete-time system is **causal** if and only if its impulse response $\{h[n]\}$ is a causal sequence
- <u>Example</u> - The discrete-time system defined by

$$y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]$$

is a causal system as it has a causal impulse response $\{h[n]\} = \{\underset{\uparrow}{\alpha_1} \quad \alpha_2 \quad \alpha_3 \quad \alpha_4\}$

Causality Condition of an LTI Discrete-Time System

- Example - The discrete-time accumulator defined by

$$y[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

is a causal system as it has a causal impulse response given by

$$h[n] = \sum_{\ell=-\infty}^{n} \delta[\ell] = \mu[n]$$

Causality Condition of an LTI Discrete-Time System

- Example - The factor-of-2 interpolator defined by

$$y[n] = x_u[n] + \frac{1}{2}\left(x_u[n-1] + x_u[n+1]\right)$$

is noncausal as it has a noncausal impulse response given by

$$\{h[n]\} = \{0.5 \quad \underset{\uparrow}{1} \quad 0.5\}$$

Causality Condition of an LTI Discrete-Time System

- Note: A noncausal LTI discrete-time system with a finite-length impulse response can often be realized as a causal system by inserting an appropriate amount of delay
- For example, a causal version of the factor-of-2 interpolator is obtained by delaying the input by one sample period:

$$y[n] = x_u[n-1] + \tfrac{1}{2}\left(x_u[n-2] + x_u[n]\right)$$

Simple Interconnection Schemes

- **Cascade Connection**

$$\longrightarrow \boxed{h_1[n]} \longrightarrow \boxed{h_2[n]} \longrightarrow \;\equiv\; \longrightarrow \boxed{h_2[n]} \longrightarrow \boxed{h_1[n]} \longrightarrow$$

$$\equiv\; \longrightarrow \boxed{h_1[n] \circledast h_2[n]} \longrightarrow$$

Impulse response h[n] of the cascade of two LTI discrete-time systems with impulse responses $\mathbf{h_1[n]}$ and $\mathbf{h_2[n]}$ is given by

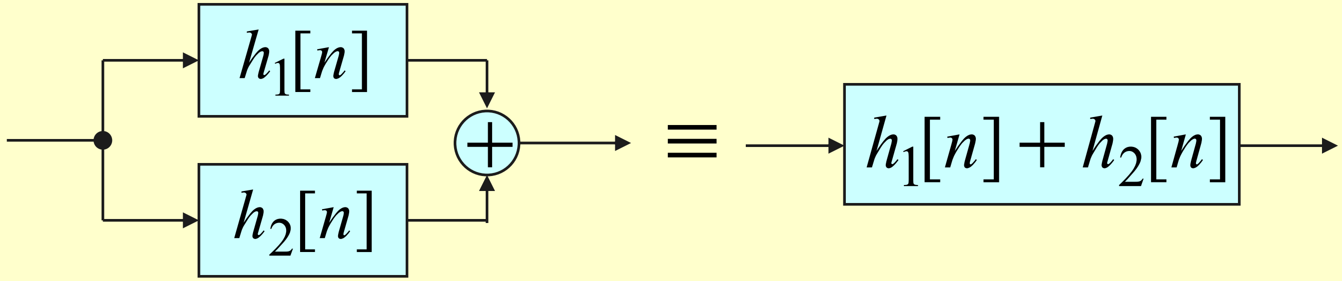
$$h[n] = h_1[n] \circledast h_2[n]$$

Simple Interconnection Schemes

- **Note: The ordering of the systems in the cascade has no effect on the overall impulse response because of the commutative property of convolution**
- **A cascade connection of two stable systems is stable**
- **A cascade connection of two passive (lossless) systems is passive (lossless)**

Simple Interconnection Schemes

- **Parallel Connection**



•Impulse response h[n] of the parallel connection of two LTI discrete-time systems with impulse responses $h_1[n]$ and $h_2[n]$ is given by

$$h[n] = h_2[n] + h_1[n]$$

Simple Interconnection Schemes

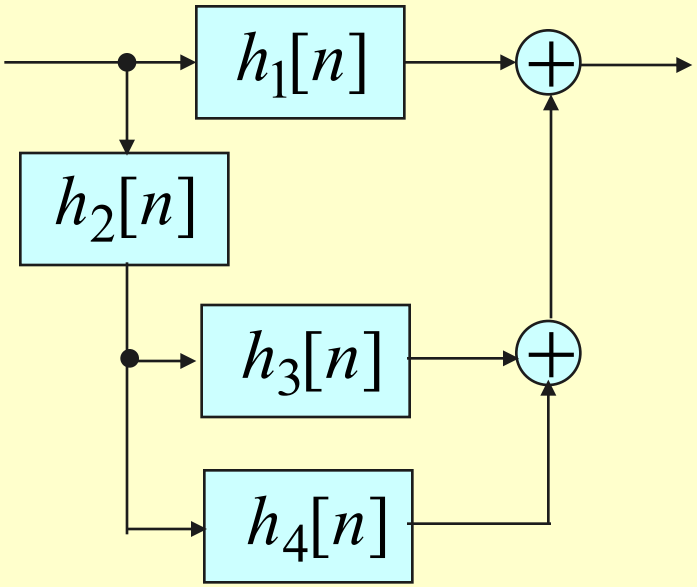
Consider the discrete-time system where

$$h_1[n] = \delta[n] + 0.5\delta[n-1]$$

$$h_2[n] = 0.5\delta[n] + 0.25\delta[n-1]$$

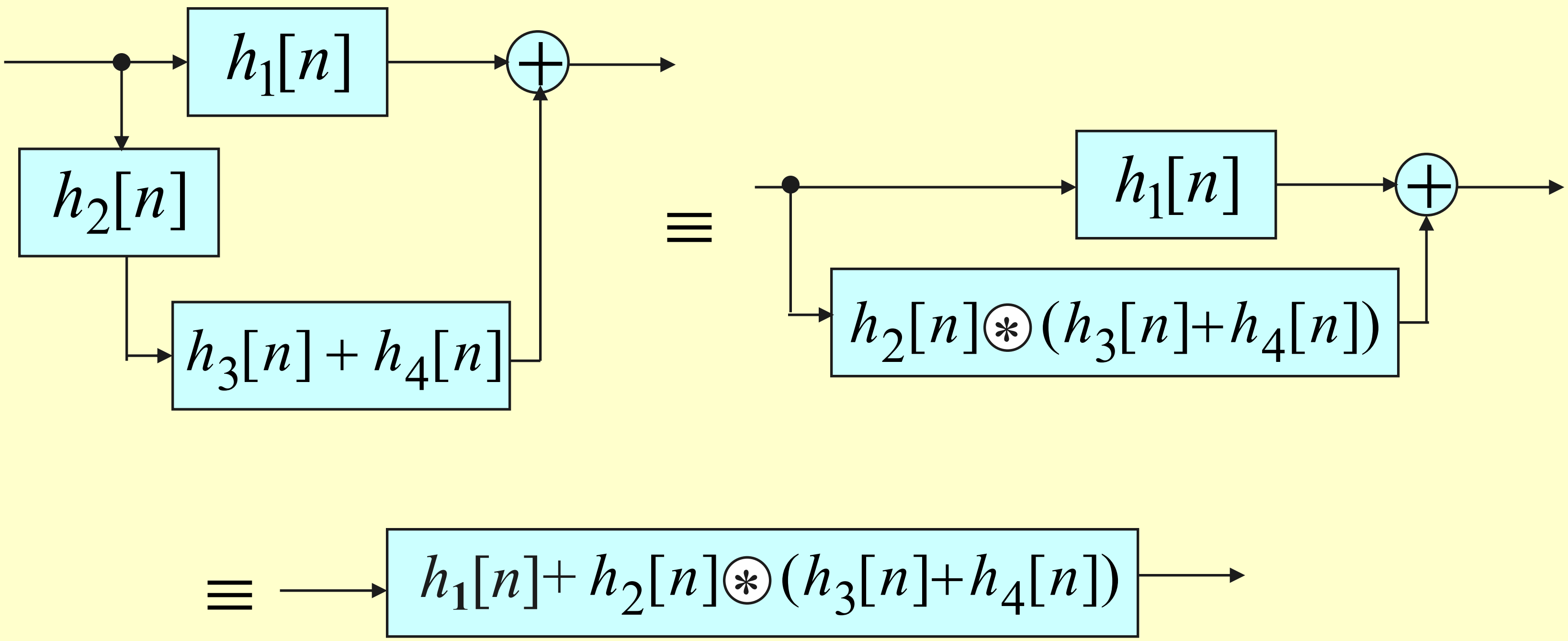
$$h_3[n] = 2\delta[n]$$

$$h_4[n] = 2(0.5)^n\mu[n]$$



Simple Interconnection Schemes

- **Simplifying the block-diagram we obtain**



Simple Interconnection Schemes

- **Overall impulse response $h[n]$ is given by**

$$h[n] = h_1[n] + h_2[n] \circledast (h_3[n] + h_4[n])$$
$$= h_1[n] + h_2[n] \circledast h_3[n] + h_2[n] \circledast h_4[n]$$

Now,

$$h_2[n] \circledast h_3[n] = (\tfrac{1}{2}\delta[n] - \tfrac{1}{4}\delta[n-1]) \circledast 2\delta[n]$$
$$= \delta[n] - \tfrac{1}{2}\delta[n-1]$$

Simple Interconnection Schemes

$$h_2[n] \circledast h_4[n] = (\tfrac{1}{2}\delta[n] - \tfrac{1}{4}\delta[n-1]) \circledast \left(-2(\tfrac{1}{2})^n \mu[n]\right)$$

$$= -(\tfrac{1}{2})^n \mu[n] + \tfrac{1}{2}(\tfrac{1}{2})^{n-1} \mu[n-1]$$

$$= -(\tfrac{1}{2})^n \mu[n] + (\tfrac{1}{2})^n \mu[n-1]$$

$$= -(\tfrac{1}{2})^n \delta[n] = -\delta[n]$$

Therefore

$$h[n] = \delta[n] + \tfrac{1}{2}\delta[n-1] + \delta[n] - \tfrac{1}{2}\delta[n-1] - \delta[n] = \delta[n]$$

Finite-Dimensional LTI Discrete-Time Systems

- An important subclass of LTI discrete-time systems is characterized by a linear constant coefficient difference equation of the form

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- $x[n]$ and $y[n]$ are, respectively, the input and the output of the system
- $\{d_k\}$ and $\{p_k\}$ are constants characterizing the system

Finite-Dimensional LTI Discrete-Time Systems

- The **order** of the system is given by $\max(N,M)$, which is the order of the difference equation
- It is possible to implement an LTI system characterized by a constant coefficient difference equation as here the computation involves two finite sums of products

Finite-Dimensional LTI Discrete-Time Systems

- If we assume the system to be causal, then the output $y[n]$ can be recursively computed using

$$y[n] = -\sum_{k=1}^{N} \frac{d_k}{d_0} y[n-k] + \sum_{k=0}^{M} \frac{p_k}{d_0} x[n-k]$$

provided $d_0 \neq 0$

- $y[n]$ can be computed for all $n \geq n_o$, knowing $x[n]$ and the initial conditions

$$y[n_o - 1], y[n_o - 2], \ldots, y[n_o - N]$$

Classification of LTI Discrete-Time Systems

Based on Impulse Response Length -

- **If the impulse response h[n] is of finite length, i.e.,**

  $$h[n]=0 \quad \text{for } n<N_1 \text{ and } n>N_2 \text{ with } N_1<N_2$$

  then it is known as a finite impulse response (FIR) discrete-time system 有限冲击响应

- **The convolution sum description here is**

$$y[n]=\sum_{k=N_1}^{N_2} h[k]x[n-k]$$

Classification of LTI Discrete-Time Systems

- The output y[n] of an FIR LTI discrete-time system can be computed directly from the convolution sum as it is a finite sum of products
- Examples of FIR LTI discrete-time systems are the moving-average system and the linear interpolators

Classification of LTI Discrete-Time Systems

- If the impulse response is of infinite length, then it is known as an **infinite impulse response (IIR)** discrete-time system 无限冲击响应
- The class of IIR systems we are concerned with in this course are characterized by **linear constant coefficient difference equations**

Classification of LTI Discrete-Time Systems

- **Example** - The discrete-time accumulator defined by

$$y[n]=y[n-1]+x[n]$$

is seen to be an IIR system

Classification of LTI Discrete-Time Systems

- **<u>Example</u> - The familiar numerical integration formulas that are used to numerically solve integrals of the form**

$$y(t) = \int_{0}^{t} x(\tau)d\tau$$

can be shown to be characterized by linear constant coefficient difference equations, and hence, are examples of IIR systems

Classification of LTI Discrete-Time Systems

- **If we divide the interval of integration into *n* equal parts of length T, then the previous integral can be rewritten as**

$$y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau)d\tau$$

where we have set t = nT and used the notation

$$y(nT) = \int_{0}^{nT} x(\tau)d\tau$$

Classification of LTI Discrete-Time Systems

- **Using the trapezoidal method we can write**

$$\int_{(n-1)T}^{nT} x(\tau)\,d\tau = \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

Hence, a numerical representation of the definite integral is given by

$$y(nT) = y((n-1)T) + \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

Classification of LTI Discrete-Time Systems

- **Let y[n] = y(nT) and x[n] = x(nT)**
- **Then**

$$y(nT) = y((n-1)T) + \frac{T}{2}\{x((n-1)T) + x(nT)\}$$

reduces to

$$y[n] = y[n-1] + \frac{T}{2}\{x[n] + x[n-1]\}$$

which is recognized as the difference equation representation of a first-order IIR discrete-time system

LTI Discrete-Time Systems in the Transform Domain

- Such transform-domain representations provide additional insight into the behavior of such systems
- It is easier to design and implement these systems in the transform-domain for certain applications
- We consider now the use of the DTFT in developing the transform-domain representations of an LTI system

LTI Discrete-Time Systems in the Transform Domain

- **In this course we shall be concerned with LTI discrete-time systems characterized by linear constant coefficient difference equations of the form:**

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

LTI Discrete-Time Systems in the Transform Domain

- **Applying the DTFT to the difference equation and making use of the linearity and the time-invariance properties we arrive at the input-output relation in the transform-domain as**

$$\sum_{k=0}^{N} d_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} p_k e^{-j\omega k} X(e^{j\omega})$$

where $Y(e^{j\omega})$ and $X(e^{j\omega})$ are the DTFTs of y[n] and x[n], respectively

LTI Discrete-Time Systems in the Transform Domain

- **In developing the transform-domain representation of the difference equation, it has been tacitly assumed that $X(e^{j\omega})$ and $Y(e^{j\omega})$ exist**
- **The previous equation can be alternately written as**

$$\left(\sum_{k=0}^{N} d_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} p_k e^{-j\omega k}\right) X(e^{j\omega})$$

Definition

- Consider the LTI discrete-time system with an impulse response {h[n]} shown below

- Its input-output relationship in the time-domain is given by the convolution sum

$$y[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Frequency-domain characterization of LTI system

If $Y(e^{j\omega})$ and $X(e^{j\omega})$ denote the DTFT of the output sequence y[n] and input sequence x[n], respectively, then applying the convolution theorem to the following equation:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k),$$

We arrive at

$$\boxed{Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})}$$

The above equation relates the input and the output of an LTI system in the frequency domain.

Frequency-domain characterization of LTI system

where $H(e^{j\omega})$ is the frequency response of the LTI system, and can be written as

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Thus, the frequency response of an LTI discrete-time system is given by the ratio of the Fourier transform $Y(e^{j\omega})$ of the output sequence y[n] to the Fourier transform $X(e^{j\omega})$ of the input sequence x[n].

The Concept of Filtering

- **One application of an LTI discrete-time system is to pass certain frequency components in an input sequence without any distortion (if possible) and to block other frequency components**
- **Such systems are called digital filters and one of the main subjects of discussion in this course**

The Concept of Filtering

- **The key to the filtering process is**

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

  - **It expresses an arbitrary input as a linear weighted sum of an infinite number of exponential sequences, or equivalently, as a linear weighted sum of sinusoidal sequences**

The Concept of Filtering

- **Thus, by appropriately choosing the values of the magnitude function $|H(e^{j\omega})|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others**

The Concept of Filtering

- **To understand the mechanism behind the design of frequency-selective filters, consider a real-coefficient LTI discrete-time system characterized by a magnitude function**

$$\left|H(e^{j\omega})\right| \cong \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The Concept of Filtering

- **We apply an input**

  $x[n]=A\cos\omega_1 n+B\cos\omega_2 n, \quad 0<\omega_1<\omega_c<\omega_2<\pi$

  to this system

- **Because of linearity, the output of this system is of the form**

$$y[n]=A\left|H(e^{j\omega_1})\right|\cos\left(\omega_1 n+\theta(\omega_1)\right)$$

$$+B\left|H(e^{j\omega_2})\right|\cos\left(\omega_2 n+\theta(\omega_2)\right)$$

The Concept of Filtering

- **As**

$$\left|H(e^{j\omega_1})\right| \cong 1, \quad \left|H(e^{j\omega_2})\right| \cong 0$$

the output reduces to

$$y[n] \cong A\left|H(e^{j\omega_1})\right|\cos\left(\omega_1 n + \theta(\omega_1)\right)$$

- **Thus, the system acts like a lowpass filter**
- **In the following example, we consider the design of a very simple digital filter**

The Concept of Filtering

- **<u>Example</u> - The input consists of a sum of two sinusoidal sequences of angular frequencies 0.1 rad/sample and 0.4 rad/sample**
- **We need to design a highpass filter that will pass the high-frequency component of the input but block the low-frequency component**
- **For simplicity, assume the filter to be an FIR filter of length 3 with an impulse response:**

$$h[0] = h[2] = \alpha, \qquad h[1] = \beta$$

The Concept of Filtering

- **The convolution sum description of this filter is then given by**

  $$\begin{aligned} y[n] &= h[0]x[n]+h[1]x[n-1]+h[2]x[n-2] \\ &= \alpha x[n] + \beta x[n-1] + \alpha x[n-2] \end{aligned}$$

- **y[n] and x[n] are, respectively, the output and the input sequences**
- **Design Objective: Choose suitable values of α and β so that the output is a sinusoidal sequence with a frequency 0.4 rad/sample**

The Concept of Filtering

- **Now, the frequency response of the FIR filter is given by**

$$H(e^{j\omega}) = h[0] + h[1]e^{-j\omega} + h[2]e^{-j2\omega}$$

$$= \alpha(1 + e^{-j2\omega}) + \beta e^{-j\omega}$$

$$= 2\alpha\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right)e^{-j\omega} + \beta e^{-j\omega}$$

$$= (2\alpha\cos\omega + \beta)e^{-j\omega}$$

The Concept of Filtering

- The magnitude and phase functions are

$$|H(e^{j\omega})| = 2\alpha\cos\omega + \beta$$

$$\theta(\omega) = -\omega$$

- In order to block the low-frequency component, the magnitude function at $\omega = 0.1$ should be equal to zero
- Likewise, to pass the high-frequency component, the magnitude function at $\omega = 0.4$ should be equal to one

The Concept of Filtering

- Thus, the two conditions that must be satisfied are

$$|H(e^{j0.1})| = 2\alpha\cos(0.1) + \beta = 0$$

$$|H(e^{j0.4})| = 2\alpha\cos(0.4) + \beta = 1$$

- Solving the above two equations we get

$$\alpha = -6.76195$$

$$\beta = 13.456335$$

The Concept of Filtering

- **Thus the output-input relation of the FIR filter is given by**

  $$y[n] = -\,6.76195(x[n]+x[n-2])+13.456335x[n-2]$$

  where the input is

  $$x[n] = \{\cos(0.1n) + \cos(0.4n)\}\mu[n]$$

The Concept of Filtering

- The first seven samples of the output are shown below

n	$\cos(0.1n)$	$\cos(0.4n)$	$x[n]$	$y[n]$
0	1.0	1.0	2.0	−13.52390
1	0.9950041	0.9210609	1.9160652	13.956333
2	0.9800665	0.6967067	1.6767733	0.9210616
3	0.9553364	0.3623577	1.3176942	0.6967064
4	0.9210609	−0.0291995	0.8918614	0.3623572
5	0.8775825	−0.4161468	0.4614357	−0.0292002
6	0.8253356	−0.7373937	0.0879419	−0.4161467

The Concept of Filtering

- From this table, it can be seen that, neglecting the least significant digit,

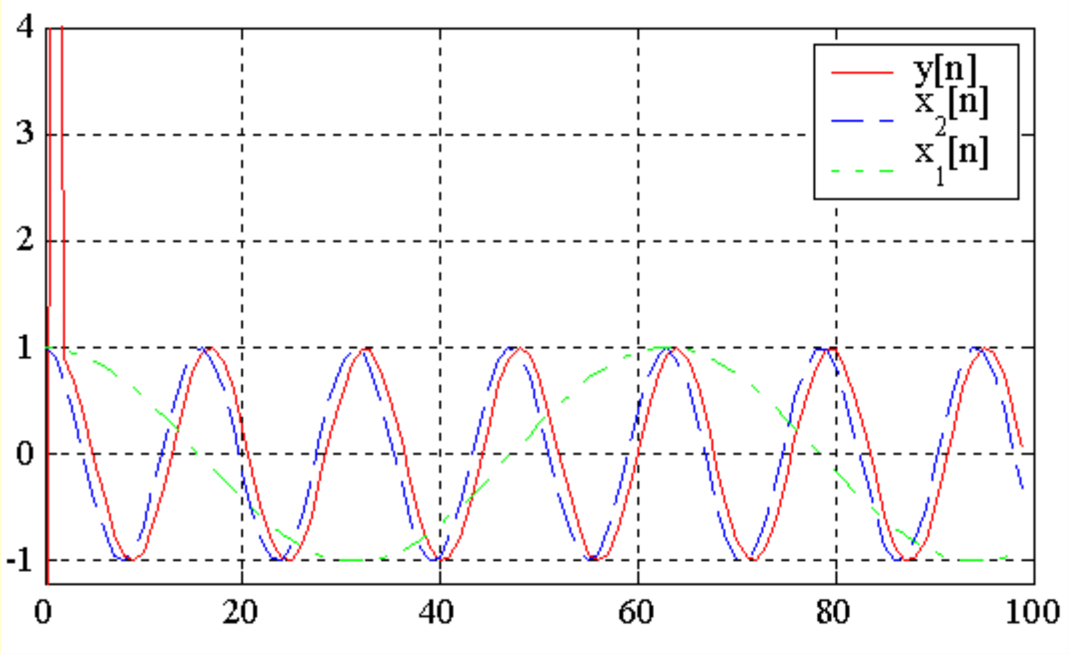
  $y[n] = \cos(0.4(n-1)) \qquad\qquad \text{for } n \geq 2$
- Computation of the present value of the output requires the knowledge of the present and two previous input samples
- Hence, the first two output samples, $y[0]$ and $y[1]$, are the result of assumed zero input sample values at $n = -1$ and $n = -2$

The Concept of Filtering

- **Note also that the output is delayed version of the high-frequency component cos(0.4n) of the input, and the delay is one sample period**

The Concept of Filtering

- **Figure below shows the plots generated by running this program**



Exercise 4.3

- For each of the following discrete-time systems, where y[n] and x[n] are, respectively, the output and the input sequences, determine whether or not the system is (1) linear, (2) causal, (3) stable, and (4) shift-invariant;

(a) $y[n] = x[n+3]$

(c) $y[n] = \ln(1 - |x[n]|)$.

Exercise 4.69

- An FIR filter of length 3 is defined by a symmetric impulse response, i.e., h[0]=h[2]. Let the input to this filter be a sum of two cosine sequences of angular frequencies 0.2 rad/samples and 0.5 rad/samples, respectively. Determine the impulse response coefficients so that the filter passes only the high-frequency component of the input.