Lecture 3
MIMO&OFDM

- Refer to <<Fundamentals of Wireless Communication>> Chapter 7.1 & 8 & 3.4.4

Part 1: MIMO

- Refer to <<Fundamentals of Wireless Communication>> Chapter 7.1 & 8

Capacity of *SISO* AWGN Channel

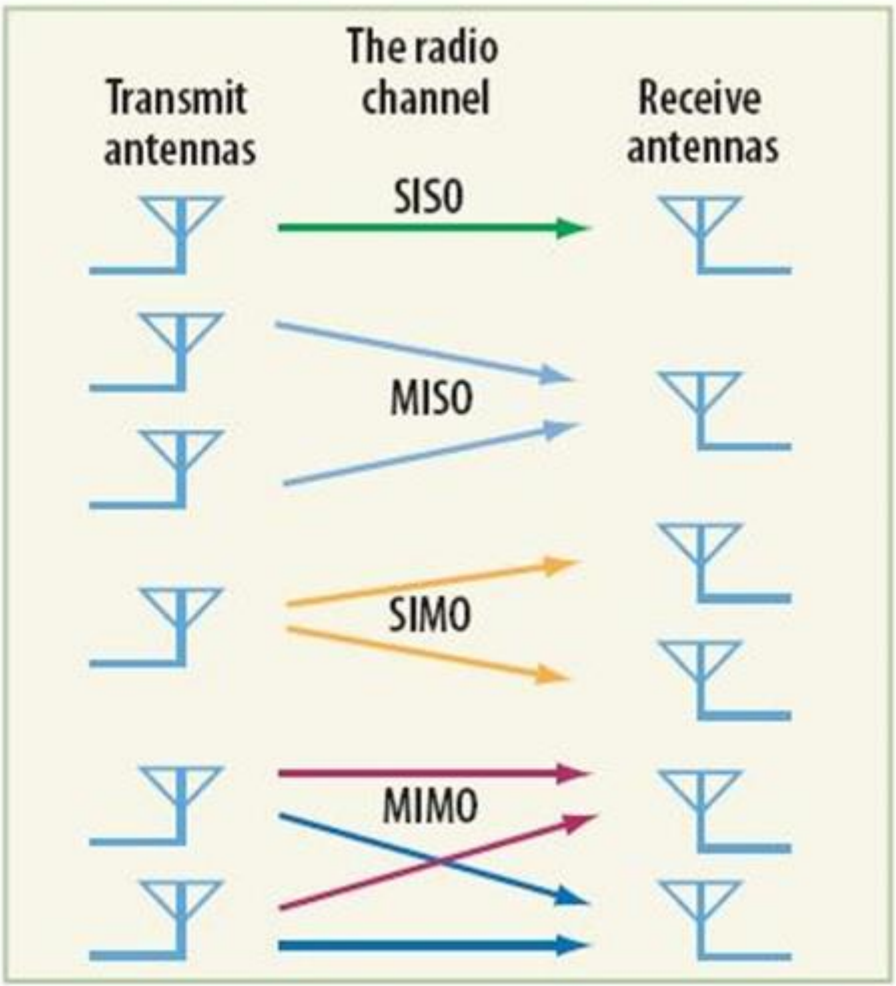
- Capacity of AWGN channel

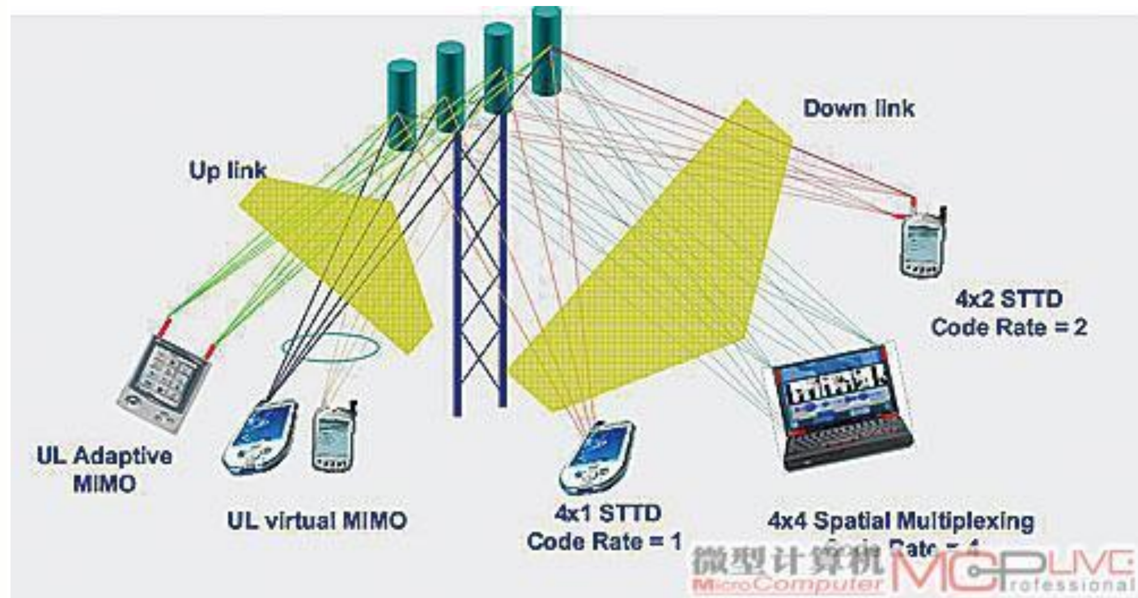
$$
\begin{aligned}
C_{\mathsf{awgn}} &= \log(1+\mathsf{SNR}) \quad \mathsf{bits/s/Hz} \\
&= W\log(1+\mathsf{SNR}) \quad \mathsf{bits/s}
\end{aligned}
$$

- If average transmit power constraint is $\bar{P}$ watts and noise psd is N_0 watts/Hz,

$$
C_{\mathsf{awgn}} = W\log\left(1+\frac{\bar{P}}{N_0 W}\right) \quad \mathsf{bits/s.}
$$

Multiple-Antenna System



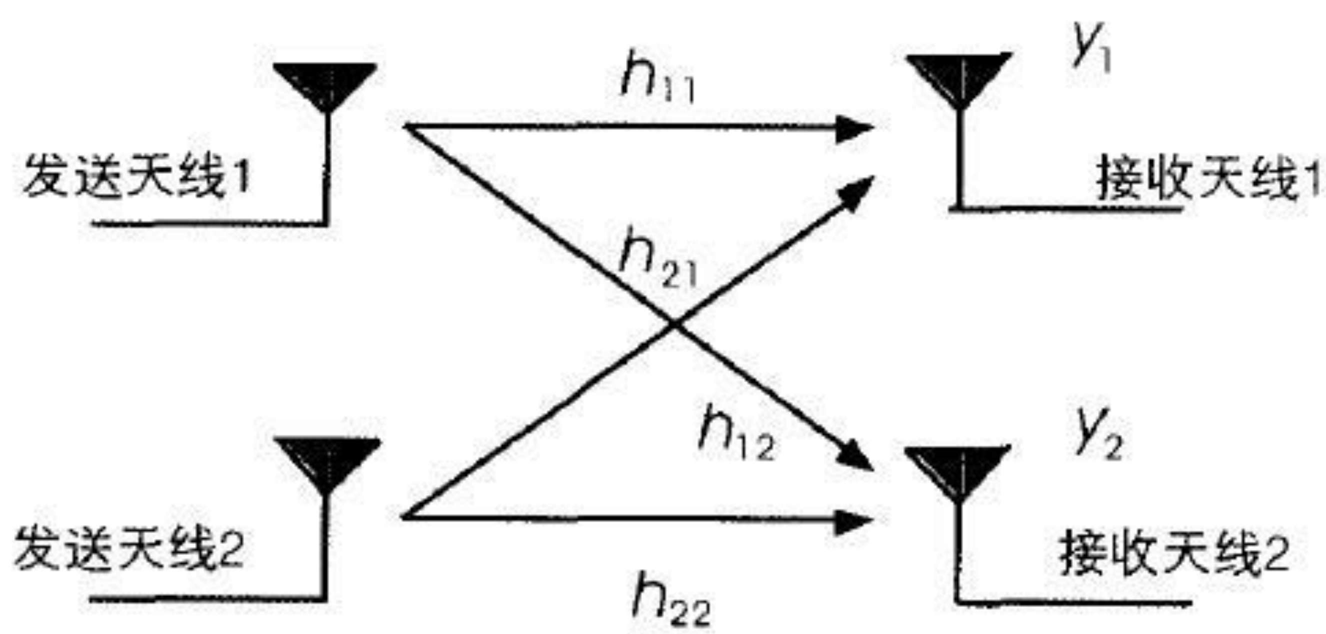


MIMO Channel

Right

MIMO Channel

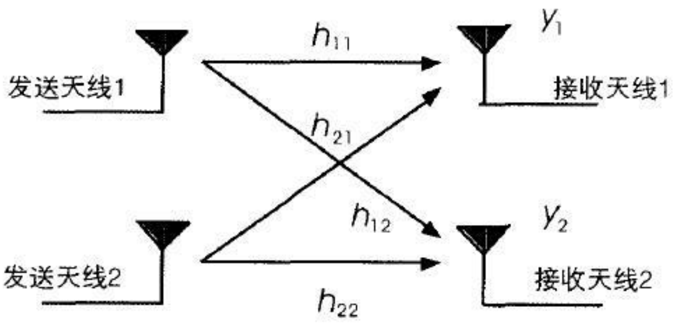
2x2 Example



MIMO Capacity via SVD

Narrowband MIMO channel:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$$



$\mathbf{H}$ is n_r by n_t, fixed channel matrix.

Singular value decomposition (SVD):

$$\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^*$$

$\mathbf{U}$, $\mathbf{V}$ are complex orthogonal matrices and $\mathbf{\Lambda}$ real diagonal (singular values).

Singular Value Decomposition: SVD

Suppose $\mathbf{M}$ is a $m \times n$ matrix whose entries come from the field K, which is either the field of real numbers or the field of complex numbers. Then there exists a factorization, called a 'singular value decomposition' of $\mathbf{M}$, of the form

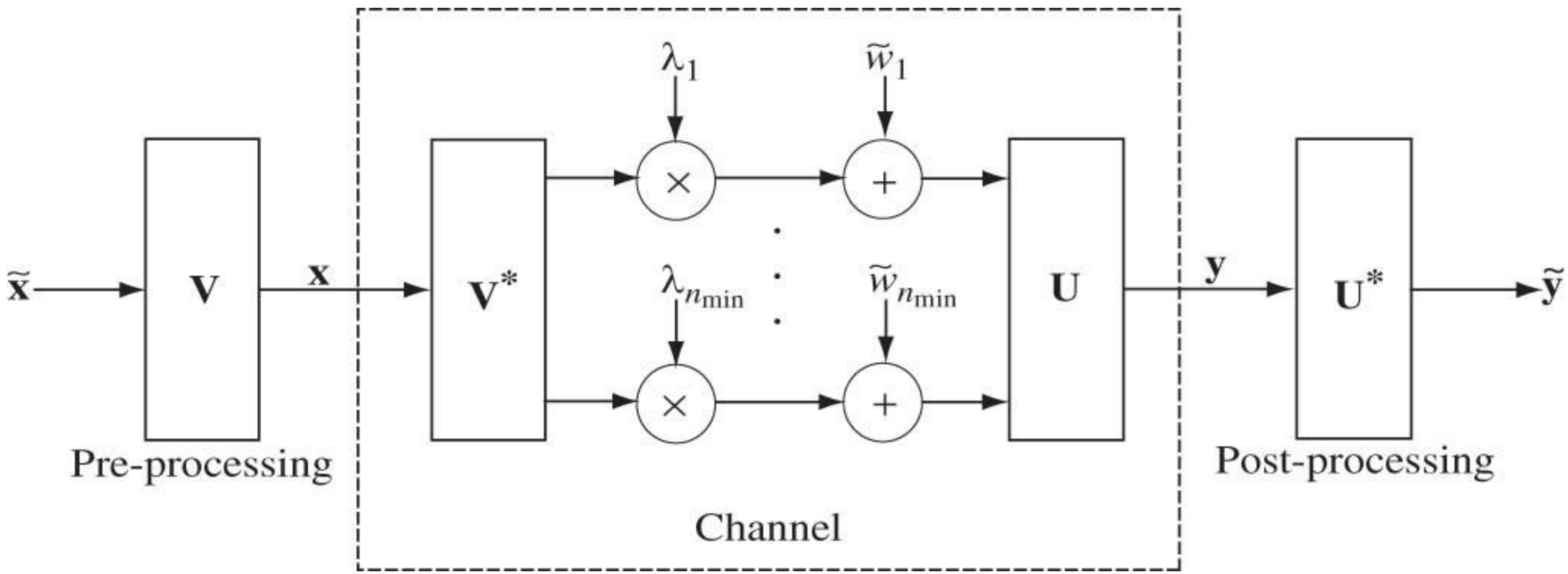
$$\mathbf{M} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$$

where

- $\mathbf{U}$ is an $m \times m$ unitary matrix over K (if $K = \mathbb{R}$, unitary matrices are orthogonal matrices),
- $\boldsymbol{\Sigma}$ is a diagonal $m \times n$ matrix with non-negative real numbers on the diagonal,
- $\mathbf{V}$ is an $n \times n$ unitary matrix over K, and $\mathbf{V}^*$ is the conjugate transpose of $\mathbf{V}$.

The diagonal entries σ_i of $\boldsymbol{\Sigma}$ are known as the **singular values** of $\mathbf{M}$. A common convention is to list the singular values in descending order. In this case, the diagonal matrix, $\boldsymbol{\Sigma}$, is uniquely determined by $\mathbf{M}$ (though not the matrices $\mathbf{U}$ and $\mathbf{V}$ if $\mathbf{M}$ is not square, see below).

Spatial Parallel Channel



$$\tilde{\mathbf{y}} = \Lambda \tilde{\mathbf{x}} + \tilde{\mathbf{w}} \qquad \tilde{y}_i = \lambda_i \tilde{x}_i + \tilde{w}_i, \qquad i = 1, 2, \ldots, n_{\min}$$

- Capacity is achieved by waterfilling over the eigenmodes of $\mathbf{H}$. (Analogy to frequency-selective channels.)

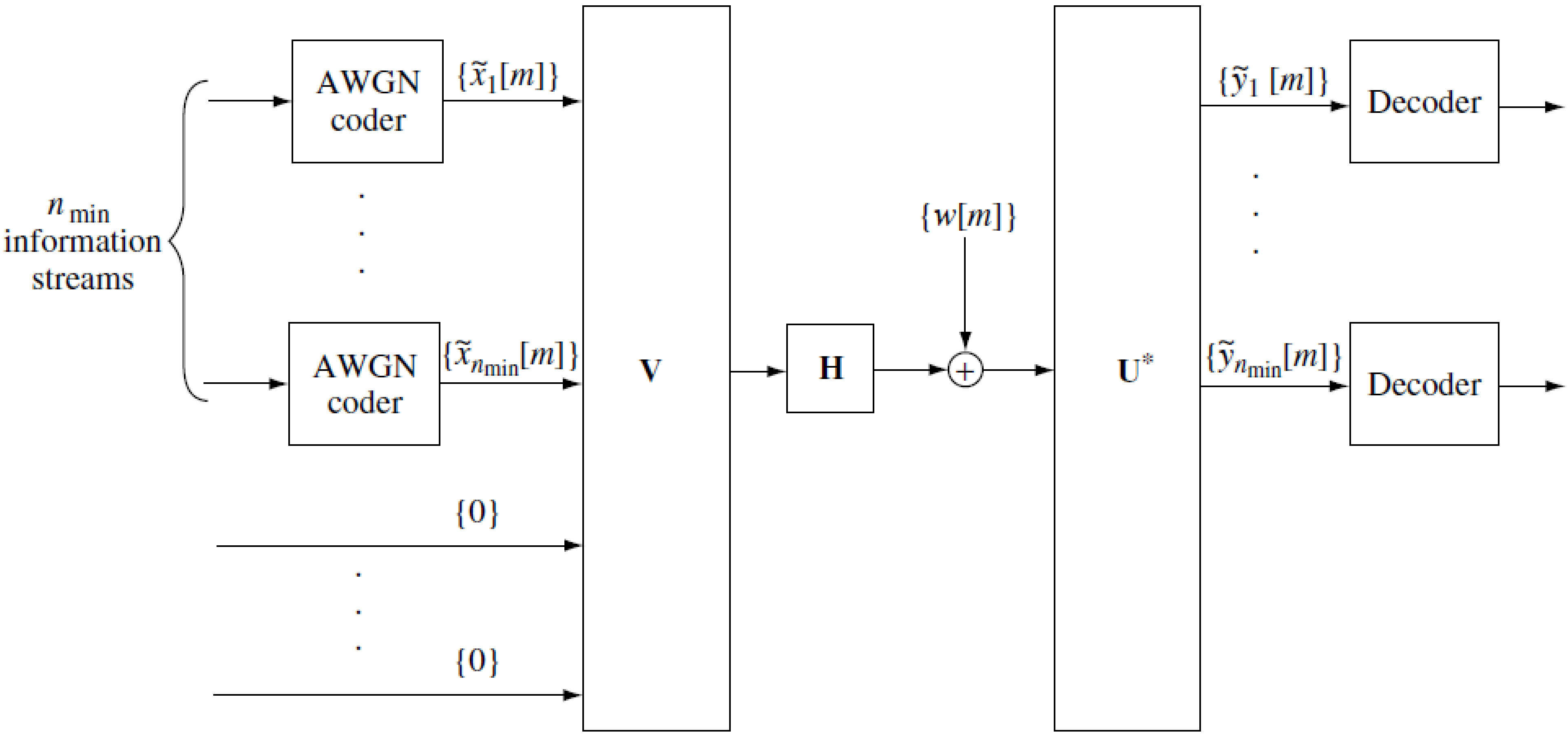
MIMO Capacity by waterfilling

$$\boxed{C = \sum_{i=1}^{n_{\min}} \log\left(1 + \frac{P_i^* \lambda_i^2}{N_0}\right) \text{ bits/s/Hz},}$$

where $P_1^*, \ldots, P_{n_{\min}}^*$ are the waterfilling power allocations:

$$\boxed{P_i^* = \left(\mu - \frac{N_0}{\lambda_i^2}\right)^+,}$$

The SVD architecture for MIMO communication



Rank and Condition Number (1)

At high SNR, equal power allocation is near optimal:

$$C \approx \sum_{i=1}^{k} \log\left(1 + \frac{P\lambda_i^2}{kN_0}\right) \approx k \log \mathsf{SNR} + \sum_{i=1}^{k} \log\left(\frac{\lambda_i^2}{k}\right)$$

where k is the number of nonzero λ_i^2 's, i.e. the rank of $\mathbf{H}$.

- *The parameter k is the number of spatial degrees of freedom per second per hertz.*

Rank and Condition Number (2)

By Jensen's inequality,

$$\frac{1}{k}\sum_{i=1}^{k}\log\left(1+\frac{P}{kN_0}\lambda_i^2\right) \le \log\left(1+\frac{P}{kN_0}\left(\frac{1}{k}\sum_{i=1}^{k}\lambda_i^2\right)\right)$$

Note

$$\sum_{i=1}^{k}\lambda_i^2 = \mathrm{Tr}[\mathbf{H}\mathbf{H}^*] = \sum_{i,j}|h_{ij}|^2.$$

The closer the condition number:

$$\frac{\max_i \lambda_i}{\min_i \lambda_i}$$

to 1, the higher the capacity.

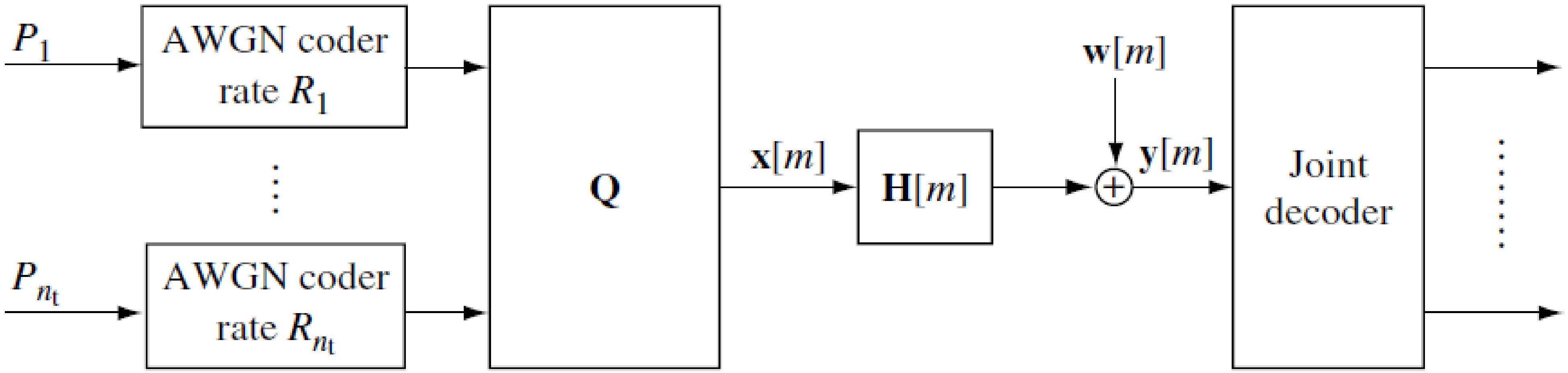
Rank and Condition Number (3)

At low SNR, the near optimal policy is to allocate power only to the strongest Eigenmode.Theresulting capacity is,

$$C \approx \frac{P}{N_0}\left(\max_i \lambda_i^2\right)\log_2 \mathrm{e}\ \text{bits/s/Hz}.$$

- In this regime, the rank or condition number of the channel matrix is less relevant.
- What matters is how much energy gets transferred from the transmitter to the receiver.

The V-BLAST architecture



- If Q = V and the powers are given by the waterfilling allocations.
- If Q = I_{nr} , then independent data streams are sent on the different transmit antennas

Capacity with precoding $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$

The rate achieved in a given channel state **H** is

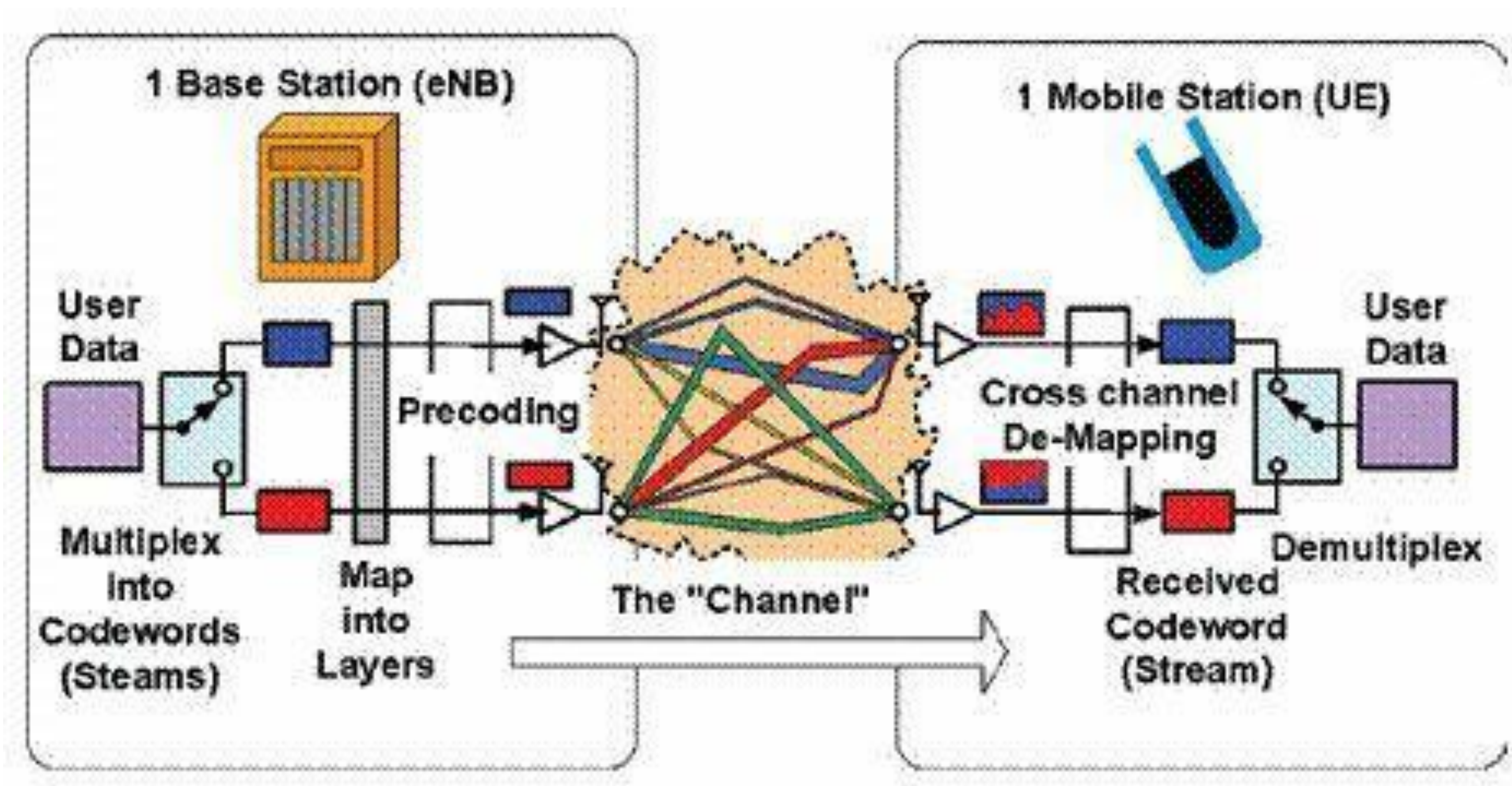
$$\log\det\left(\mathbf{I}_{n_r} + \frac{1}{N_0}\mathbf{H}\mathbf{K}_x\mathbf{H}^*\right)$$

K_x = E{x*x'} is the covariance matrix of the transmitted signal x:

$$K_x = PP^*$$

We call the matrix P as precoder.

Precoding


1 Base Station (eNB)
1 Mobile Station (UE)
User
Data
Precoding
Cross channel
De-Mapping
User
Data
Multiplex
into
Codewords
(Steams)
Map
into
Layers
The "Channel"
Demultiplex
Received
Codeword
(Stream)

Capacity with non-AWGN

If w is non-AWGN, and its covariance matrix is R=E{w*w'}

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$$

The rate achieved in a given channel state **H** is

$$C = \log\det\left(\mathrm{I} + \mathrm{H}\mathrm{K}_x\mathrm{H}^*\mathrm{R}^{-1}\right)$$

Proof:

Fast fading MIMO channel with Receiver CSI Only

$$\mathbf{y}[m] = \mathbf{H}[m]\mathbf{x}[m] + \mathbf{w}[m], \qquad m = 1, 2, \ldots,$$

where {H[m]} is a random fading process.

A long-term rate of reliable communication equal to

$$\mathbb{E}_{\mathbf{H}}\left[\log\det\left(\mathbf{I}_{n_r} + \frac{1}{N_0}\mathbf{H}\mathbf{K}_x\mathbf{H}^*\right)\right]$$

We can now choose the covariance Kx as a function of the channel *statistics* to achieve a reliable communication rate of

$$C = \max_{\mathbf{K}_x:\mathrm{Tr}[\mathbf{K}_x]\le P} \mathbb{E}\left[\log\det\left(\mathbf{I}_{n_r} + \frac{1}{N_0}\mathbf{H}\mathbf{K}_x\mathbf{H}^*\right)\right]$$

i.i.d. Rayleigh fading model

For i.i.d. Rayleigh fading model, equal powers are the optimal:

$$C = \mathbb{E}\left[\log\det\left(\mathbf{I}_{n_r} + \frac{\mathsf{SNR}}{n_t}\mathbf{H}\mathbf{H}^*\right)\right]$$

where SNR = P/N0 is the common SNR at each receive antenna.

$$\begin{aligned} C &= \mathbb{E}\left[\sum_{i=1}^{n_{\min}} \log\left(1 + \frac{\mathsf{SNR}}{n_t}\lambda_i^2\right)\right] \\ &= \sum_{i=1}^{n_{\min}} \mathbb{E}\left[\log\left(1 + \frac{\mathsf{SNR}}{n_t}\lambda_i^2\right)\right]. \end{aligned}$$

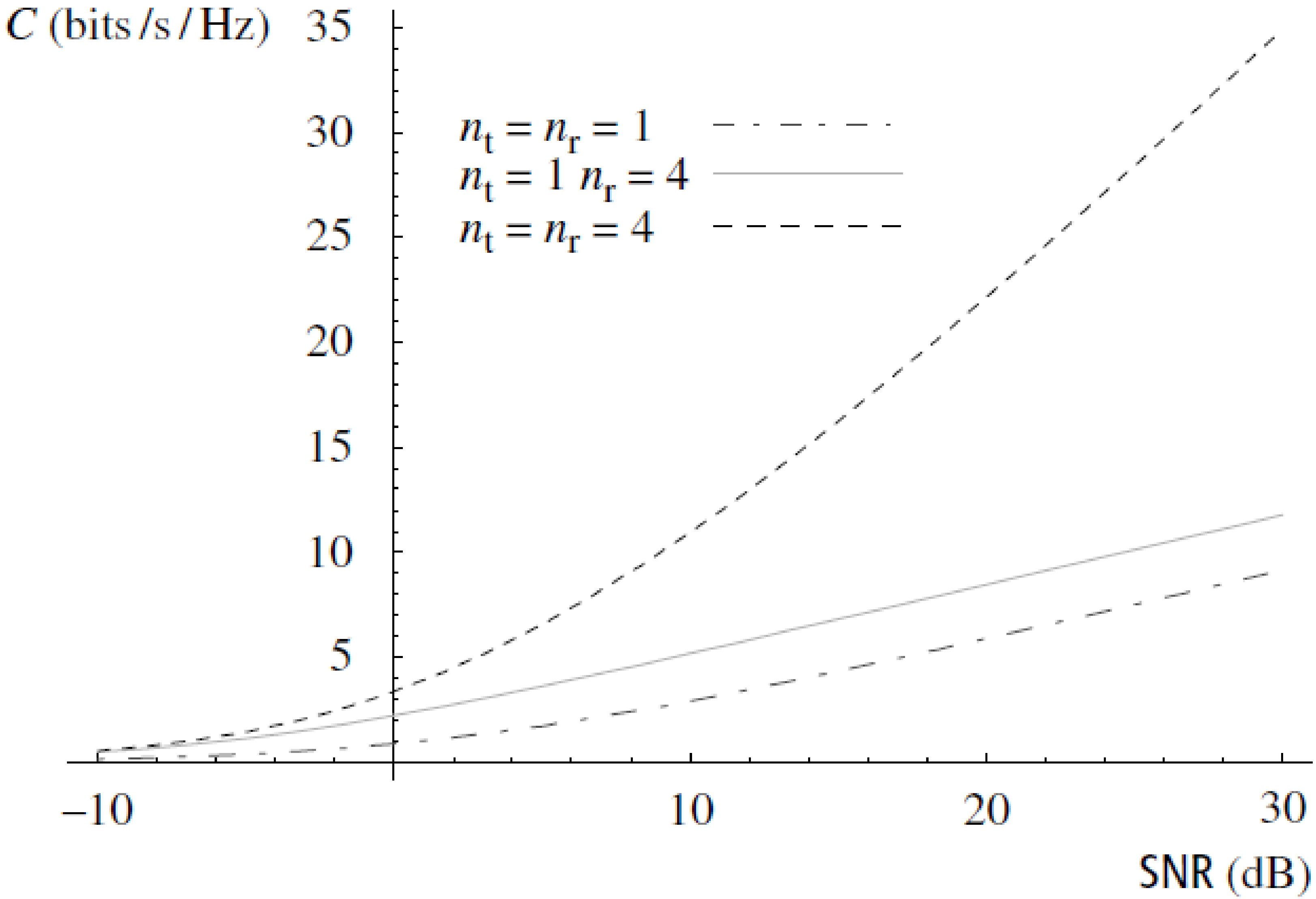
High SNR regime

For i.i.d. Rayleigh fading model, at high SNR, the capacity is

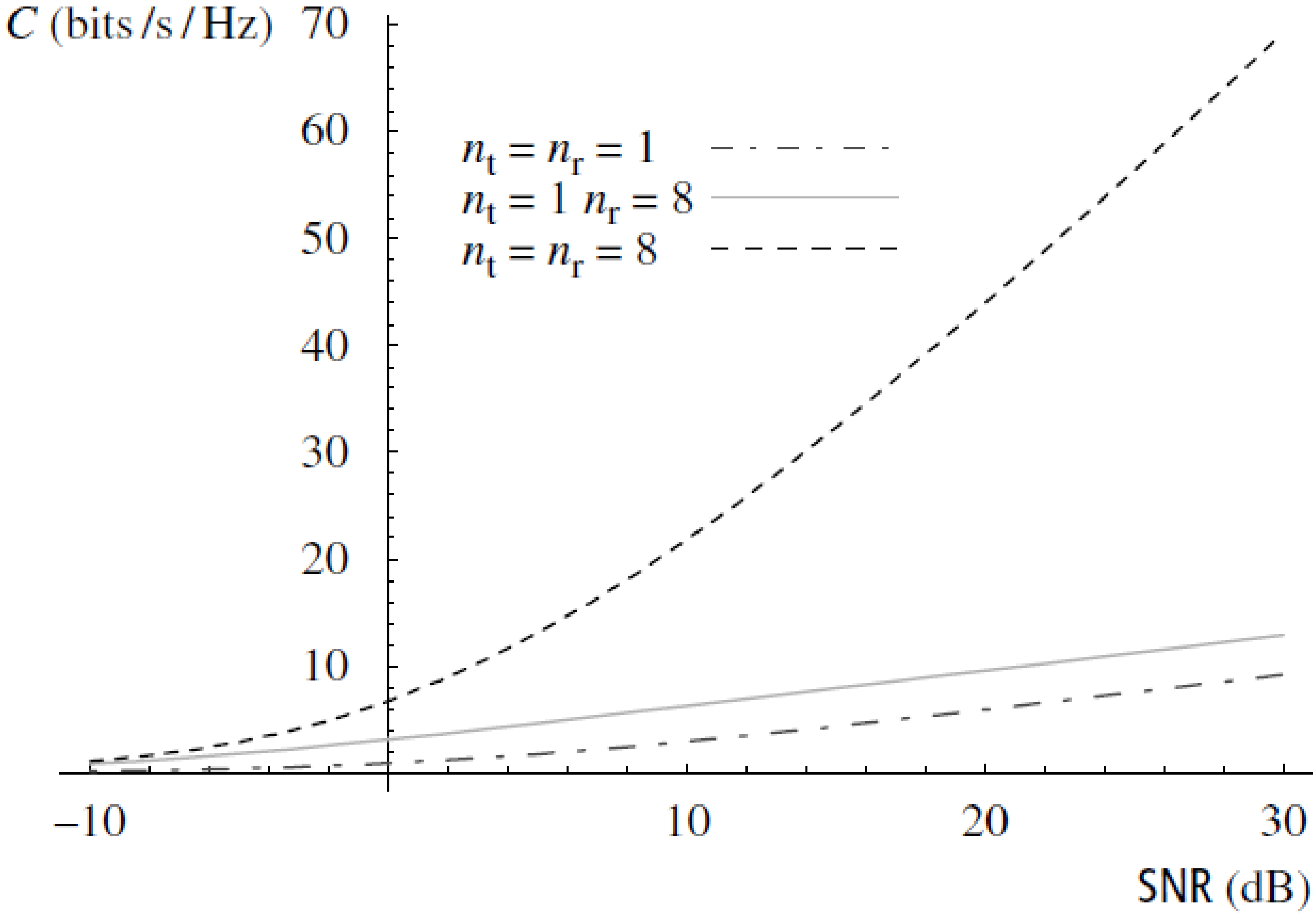
$$C \approx n_{\min} \log \frac{\mathsf{SNR}}{n_{\mathrm{t}}} + \sum_{i=1}^{n_{\min}} \mathbb{E}[\log \lambda_i^2],$$

The asymptotic slope of capacity versus SNR in dB scale is proportional to n, which means that the capacity scales with SNR like n log SNR.

Fast Fading Capacity for I.I.D. Rayleigh Fading



Fast Fading Capacity for I.I.D. Rayleigh Fading



Low SNR regime

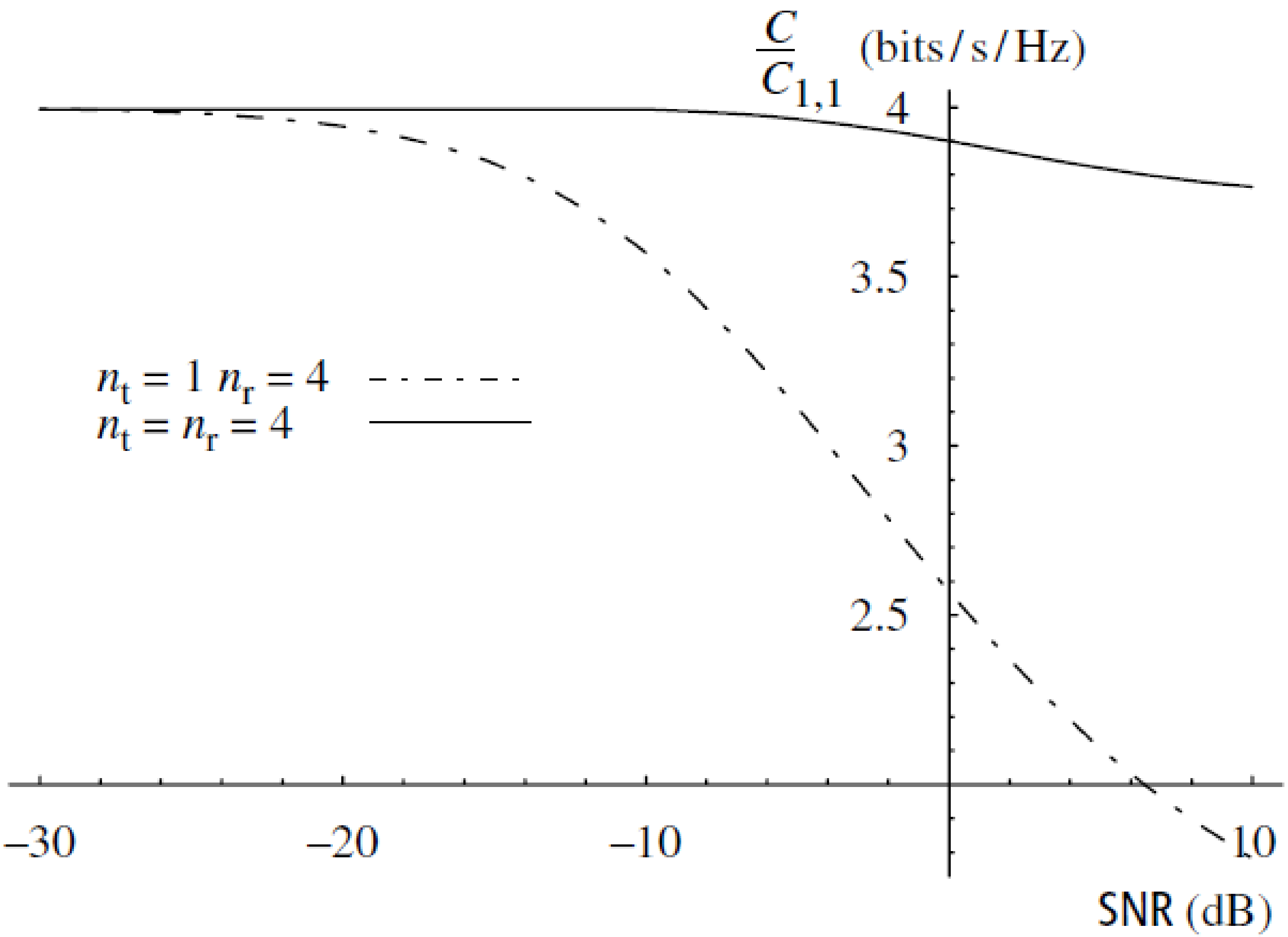
For i.i.d. Rayleigh fading model, at Low SNR, the capacity is

$$
\begin{aligned}
C &= \sum_{i=1}^{n_{\min}} \mathbb{E}\left[\log\left(1+\frac{\mathsf{SNR}}{n_{\mathrm{t}}}\lambda_i^2\right)\right] \\
&\approx \sum_{i=1}^{n_{\min}} \frac{\mathsf{SNR}}{n_{\mathrm{t}}}\mathbb{E}\left[\lambda_i^2\right]\log_2 \mathrm{e} \\
&= \frac{\mathsf{SNR}}{n_{\mathrm{t}}}\mathbb{E}[\mathrm{Tr}[\mathbf{H}\mathbf{H}^*]]\log_2 \mathrm{e} \\
&= \frac{\mathsf{SNR}}{n_{\mathrm{t}}}\mathbb{E}\left[\sum_{i,j}|h_{ij}|^2\right]\log_2 \mathrm{e} \\
&= n_{\mathrm{r}}\mathsf{SNR}\log_2 \mathrm{e}\ \text{bits/s/Hz}.
\end{aligned}
$$

Thus, at low SNR, an n_t by n_r system yields a power gain of n_r over a single antenna system.

Fast Fading Capacity: Low SNR

n_r – fold power gain at low SNR



Fast fading MIMO channel with full CSI

Decompose the channel matrix as

$$\mathbf{H}[m] = \mathbf{U}[m]\mathbf{\Lambda}[m]\mathbf{V}[m]^*.$$

A parallel channel

$$\tilde{y}_i[m] = \lambda_i[m]\tilde{x}_i[m] + \tilde{w}_i[m], \qquad i = 1, \ldots, n_{\min}.$$

waterfilling policy

$$P^*(\lambda) = \left(\mu - \frac{N_0}{\lambda^2}\right)^+.$$

capacity

$$C = \sum_{i=1}^{n_{\min}} \mathbb{E}\left[\log\left(1 + \frac{P^*(\lambda_i)\lambda_i^2}{N_0}\right)\right]$$

Linear Receiver

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$$

Linear MMSE (minimum mean squared error) receiver D

$$\tilde{x} = Dy = D\left(Hx + w\right)$$

Mean squared error (MSE)

$$MSE = E\left\{trace\left[\left(x - \tilde{x}\right)\left(x - \tilde{x}\right)^{*}\right]\right\}$$

Optimal D to minimize MSE is $D = H^{*}\left(HH^{*} + N_0 I\right)^{-1}$

Part 2: OFDM

- Refer to <<Fundamentals of Wireless Communication>> Chapter 3.4.4

Frequency-Selective Channel

Frequency-Selective Channel:

$$y[m] = \sum_{\ell} h_\ell[m] x[m-\ell] + w[m]$$

Assuming a finite number of non-zero taps L:

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell x[m-\ell] + w[m].$$

We transmit over only a finite duration, say Nc symbols:

$$\mathbf{d} = [d[0], d[1], \ldots, d[N_c - 1]]^t$$

Cyclic Prefix (CP)

A *prefix* of length L−1 consisting of data symbols:

$$d[N_c-L+1], d[N_c-L+2], \ldots, d[N_c-1]$$

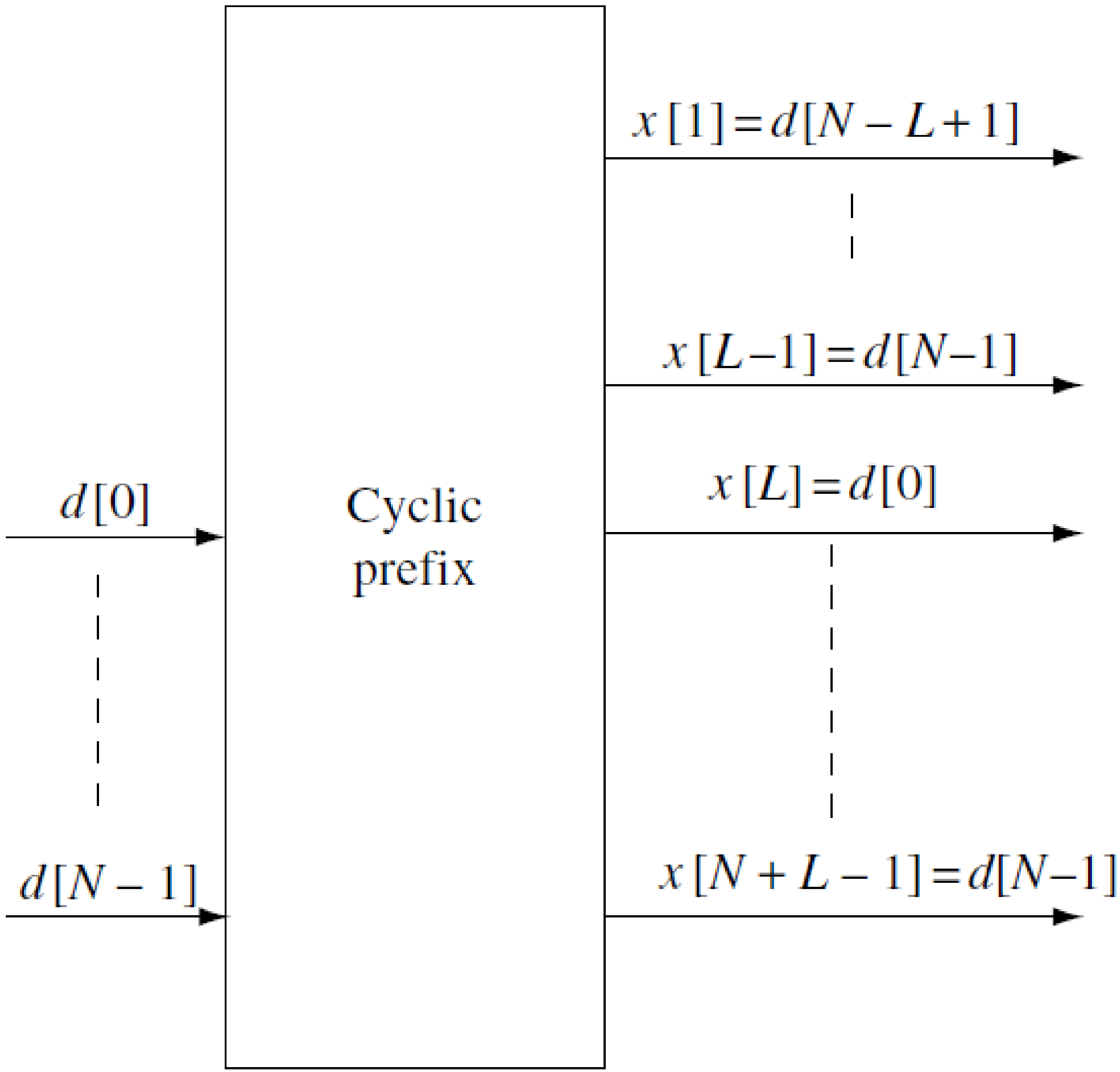
An Nc+L−1 input block:

$$\mathbf{x}=[d[N_c-L+1], d[N_c-L+2], \ldots, d[N_c-1], d[0], d[1], \ldots, d[N_c-1]]^t$$

The output to the channel:

$$y[m]=\sum_{\ell=0}^{L-1} h_\ell x[m-\ell]+w[m], \qquad m=1, \ldots, N_c+L-1$$

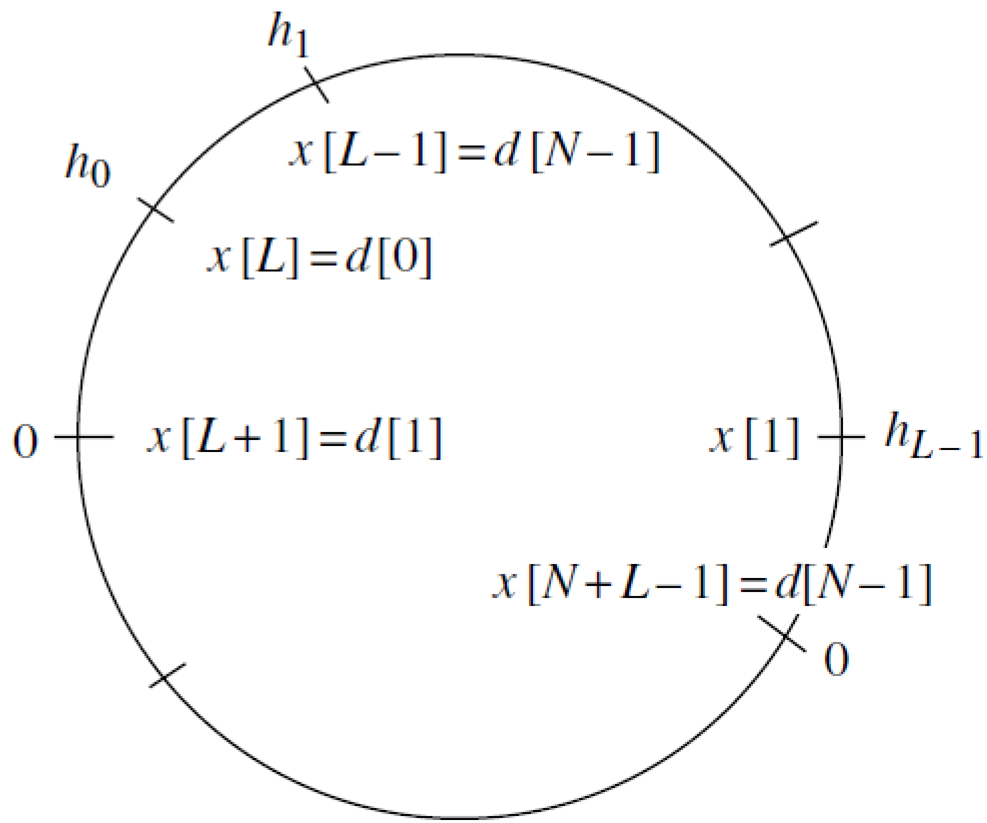
Cyclic Prefix (CP)



Removing Cyclic Prefix

The ISI extends over the first L−1 symbols and the receiver ignores it by considering only the output over the time interval m ∈ [L, Nc+L−1]:

$$y[m] = \sum_{\ell=0}^{L-1} h_\ell d[(m - L - \ell) \text{ modulo } N_c] + w[m].$$



Cyclic Convolution

Denoting the output of length Nc by

$$\mathbf{y} = [y[L], \ldots, y[N_c + L - 1]]^t$$

The channel by a vector of length Nc

$$\mathbf{h} = [h_0, h_1, \ldots, h_{L-1}, 0, \ldots, 0]^t$$

The output by *cyclic convolution*:

$$\mathbf{y} = \mathbf{h} \otimes \mathbf{d} + \mathbf{w}$$

Discrete Fourier transform (DFT)

DFT of d:

$$\tilde{d}_n := \frac{1}{\sqrt{N_c}} \sum_{m=0}^{N_c-1} d[m] \exp\left(\frac{-j2\pi nm}{N_c}\right), \qquad n = 0, \ldots, N-1$$

DFT of both sides of $\mathbf{y} = \mathbf{h} \otimes \mathbf{d} + \mathbf{w}$

$$\mathrm{DFT}(\mathbf{h} \otimes \mathbf{d})_n = \sqrt{N_c}\,\mathrm{DFT}(\mathbf{h})_n \cdot \mathrm{DFT}(\mathbf{d})_n, \qquad n = 0, \ldots, N_c - 1$$

Nc parallel subchannel:

$$\tilde{y}_n = \tilde{h}_n \tilde{d}_n + \tilde{w}_n, \qquad n = 0, \ldots, N_c - 1$$

Connection between the frequency-selective channel and the MIMO channel (1)

The circular convolution peration u = h⊗d can be viewed as a linear transformation:

$$\mathbf{u} = \mathbf{C}\mathbf{d}$$

where

$$\mathbf{C} := \begin{bmatrix} h_0 & 0 & \cdot & 0 & h_{L-1} & h_{L-2} & \cdot & h_1 \\ h_1 & h_0 & 0 & \cdot & 0 & h_{L-1} & \cdot & h_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & 0 & h_{L-1} & h_{L-2} & \cdot & h_1 & h_0 \end{bmatrix}$$

C is a *circulant matrix.*

Connection between the frequency-selective channel and the MIMO channel (2)

The DFT of d can be represented as an Nc-length vector Ud, where U is the unitary matrix with its (k,n)th entry equal to

$$\frac{1}{\sqrt{N_c}} \exp\left(\frac{-j2\pi kn}{N_c}\right), \qquad k, n = 0, \ldots, N_c - 1$$

Then $\mathrm{DFT}(\mathbf{h} \otimes \mathbf{d})_n = \sqrt{N_c}\,\mathrm{DFT}(\mathbf{h})_n \cdot \mathrm{DFT}(\mathbf{d})_n$ is equivalent to

$$\mathbf{Uu} = \Lambda \mathbf{Ud}$$

where

$$\Lambda_{nn} = \tilde{h}_n := \left(\sqrt{N_c}\mathbf{Uh}\right)_n, \qquad n = 0, \ldots, N_c - 1$$

Connection between the frequency-selective channel and the MIMO channel (3)

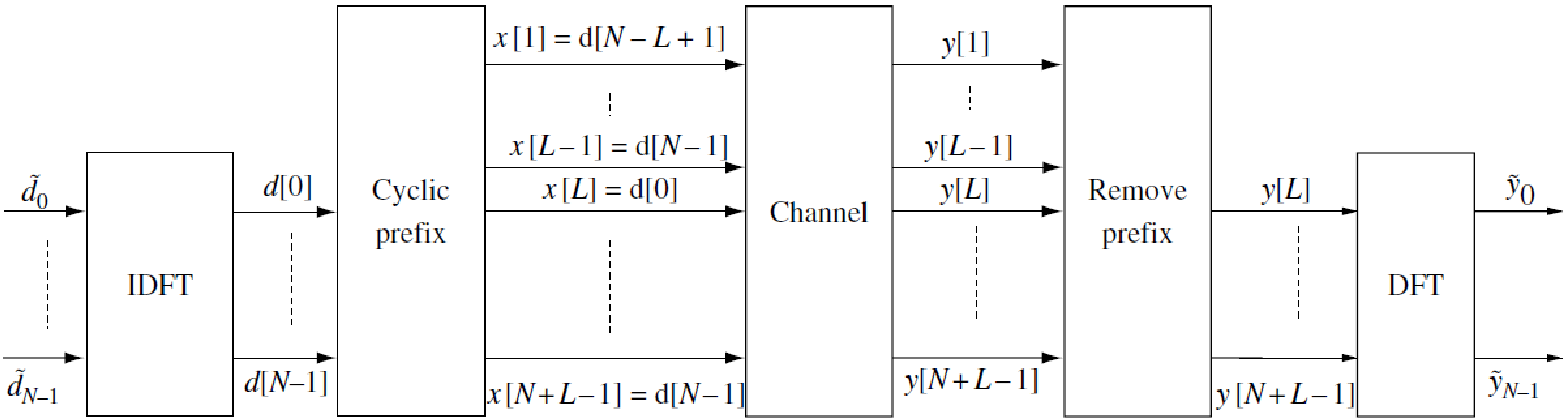
We come to the conclusion that

$$\mathbf{C} = \mathbf{U}^{-1}\Lambda\mathbf{U}.$$

Then $\mathbf{y} = \mathbf{h} \otimes \mathbf{d} + \mathbf{w}$ is equivalent to

$$\mathbf{y} = \mathbf{C}\mathbf{d} + \mathbf{w} = \mathbf{U}^{-1}\Lambda\mathbf{U}\mathbf{d} + \mathbf{w}.$$

The OFDM transmission and reception schemes



Homework

- **1. If** $H = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $K_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, **the noise power N_0=0dB, compute the capacity of the AWGN MIMO channel.**
- **2. For the MIMO channel** $\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}$ **, how to parallelize it?**

- **Tips: For Q1, see P16; For Q2, see P 6-8**
- **Requirements: Submitted next week.**