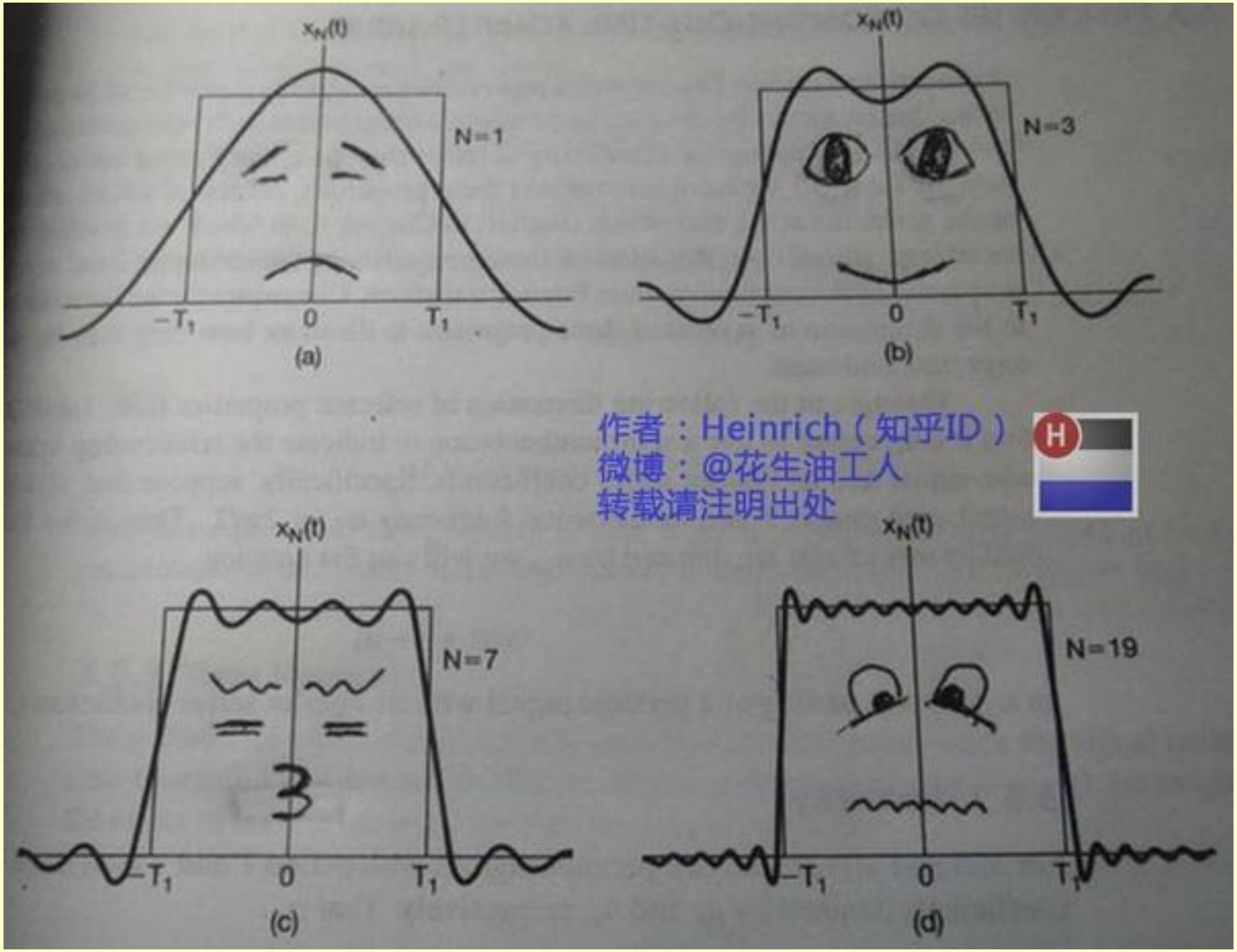
Chapter 3

Discrete-time signals in the frequency domain

Discrete-Time Signals in the Frequency Domain

- The frequency-domain representation of a discrete-time sequence is the discrete-time Fourier transform (DTFT)
- This transform maps a time-domain sequence into a continuous function of the frequency variable ω
- We first review briefly the continuous-time Fourier transform (CTFT)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N=3
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(b)
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$x_N(t)$
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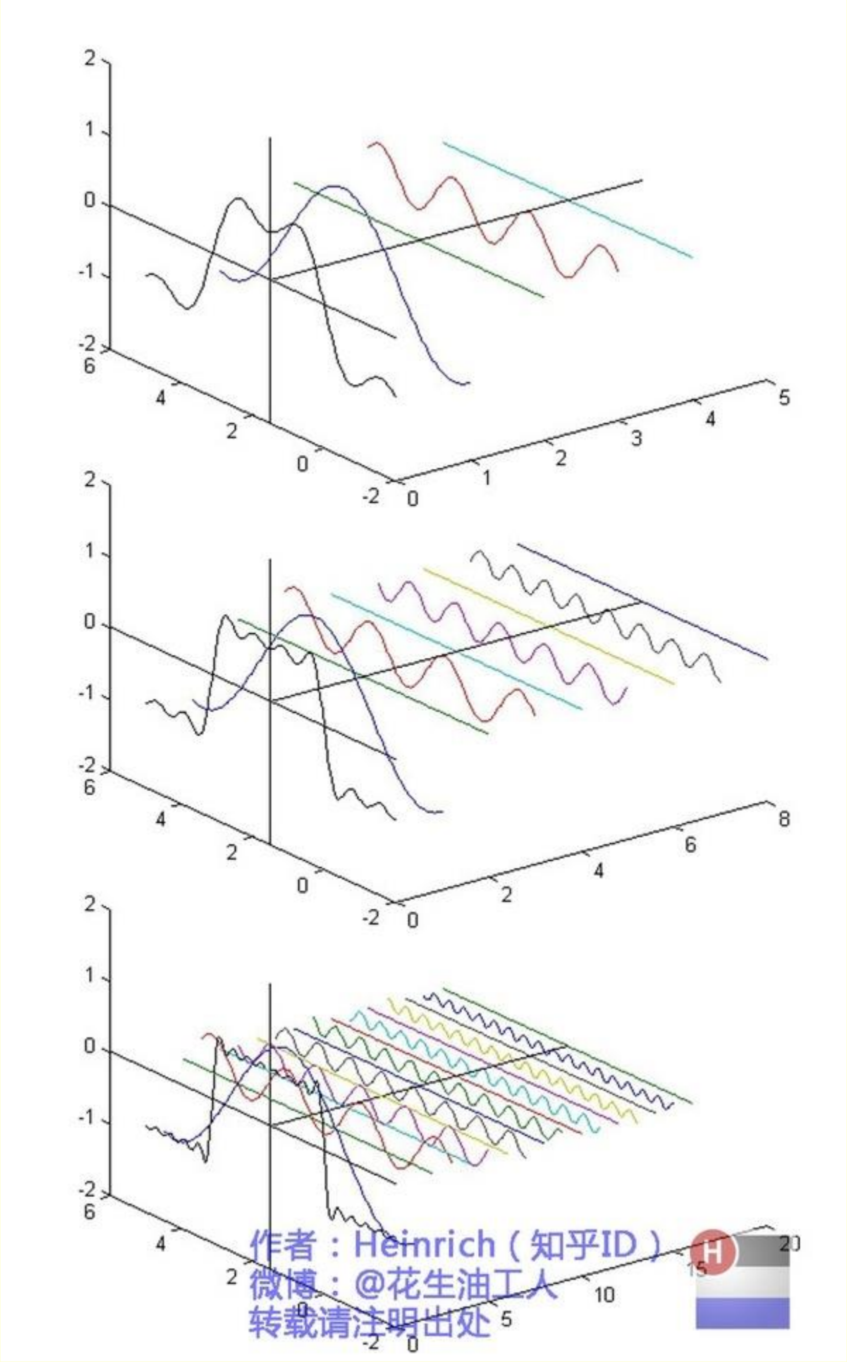
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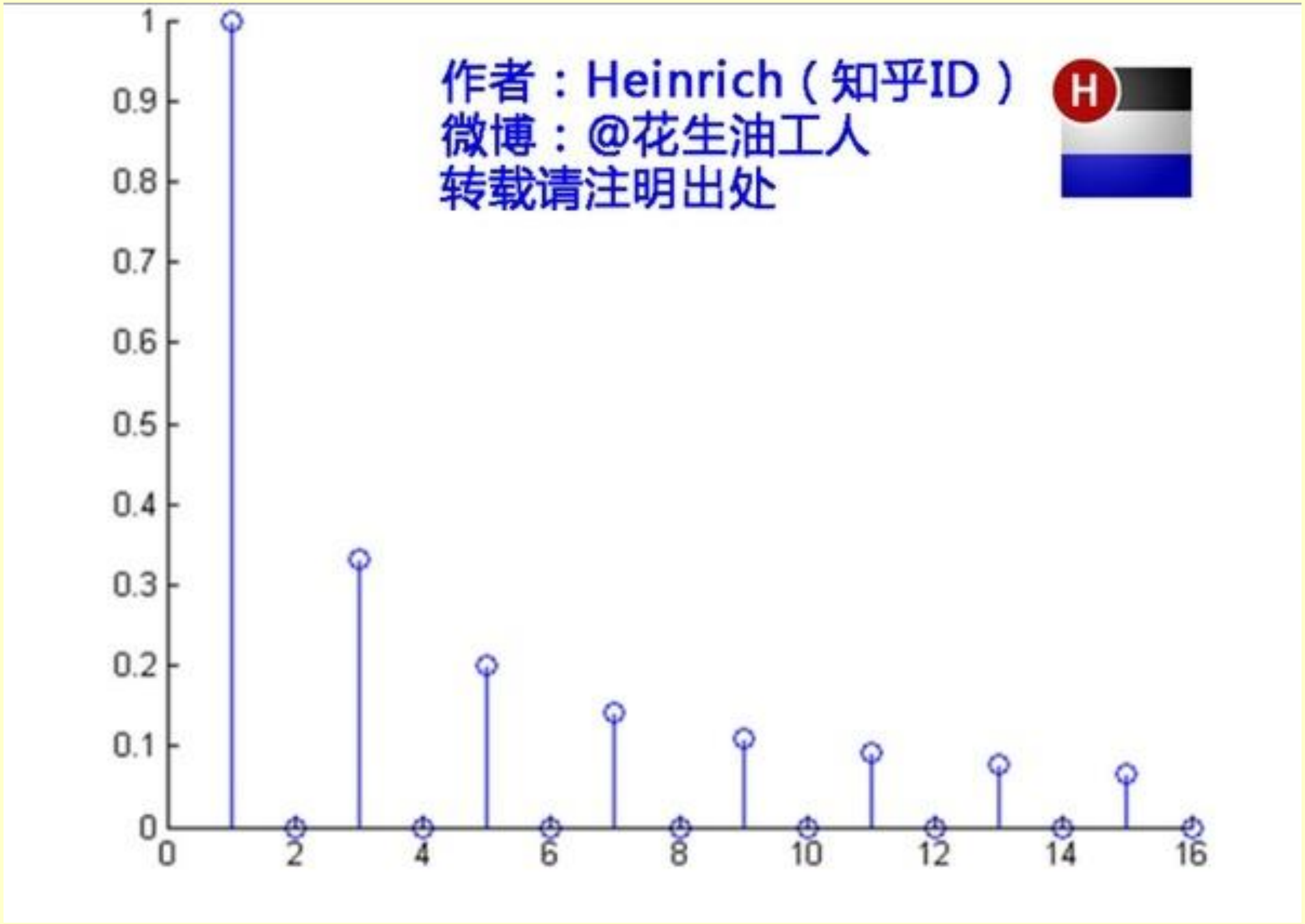
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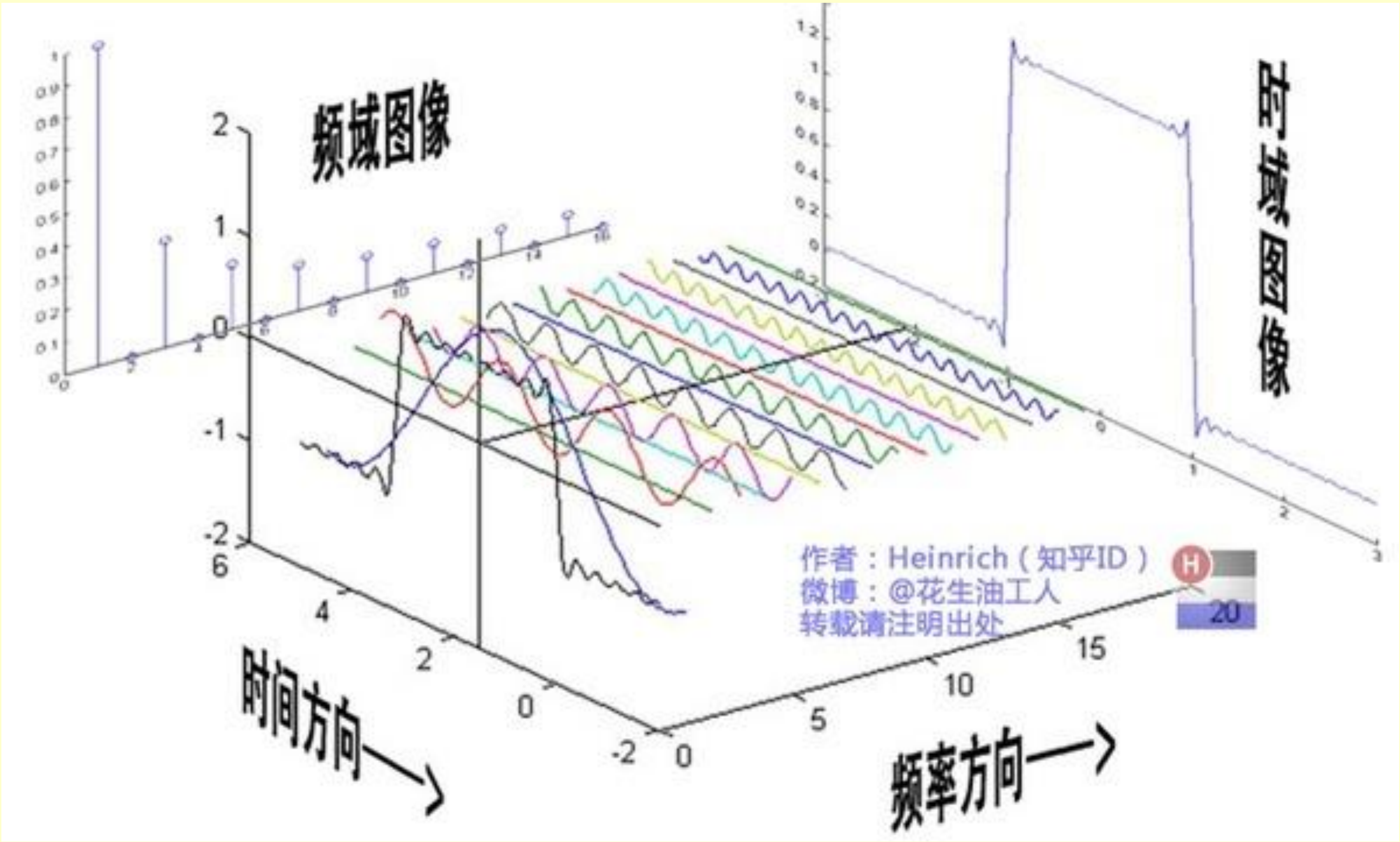

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时域图像
时间方向
频率方向
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Continuous-Time Fourier Transform

- **Definition** – The inverse CTFT of a Fourier transform $X_a(j\Omega)$ is given by

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(j\Omega) e^{j\Omega t} \, d\Omega$$

- Often referred to as the Fourier integral
- A CTFT pair will be denoted as

$$x_a(t) \overset{\text{CTFT}}{\leftrightarrow} X_a(j\Omega)$$

Continuous-Time Fourier Transform

- **Definition** – The CTFT of a continuous-time signal $x_a(t)$ is given by

$$X_a(j\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

- Often referred to as the Fourier spectrum or simply the spectrum of the continuous-time signal

Continuous-Time Fourier Transform

- Ω is real and denotes the continuous-time angular frequency variable in radians/sec if the unit of the independent variable t is in seconds
- In general, the CTFT is a complex function of Ω in the range $-\infty < \Omega < \infty$
- It can be expressed in the polar form as

$$X_a(j\Omega) = |X_a(j\Omega)| e^{j\theta_a(\Omega)}$$

where

$$\theta_a(\Omega) = \arg\{X_a(j\Omega)\}$$

Continuous-Time Fourier Transform

- The quantity $|X_a(j\Omega)|$ is called the magnitude spectrum and the quantity $\theta_a(\Omega)$ is called the phase spectrum
- Both spectrums are real functions of Ω
- In general, the CTFT $X_a(j\Omega)$ exists if $x_a(t)$ satisfies the Dirichlet conditions given on the next slide

Continuous-Time Fourier Transform

Dirichlet Conditions

- (a) The signal $x_a(t)$ has a finite number of discontinuities and a finite number of maxima and minima in any finite interval
- (b) The signal is absolutely integrable, i.e.,

$$\int_{-\infty}^{\infty} |x_a(t)|\, dt < \infty$$

Continuous-Time Fourier Transform

- If the Dirichlet conditions are satisfied, then

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(j\Omega)e^{j\Omega t}\,d\Omega$$

converges to $x_a(t)$ at all values of t except at values of t where $x_a(t)$ has discontinuities

- It can be shown that if $x_a(t)$ is absolutely integrable, then $|X_a(j\Omega)| < \infty$ proving the existence of the CTFT

Fourier Transform of typical signals

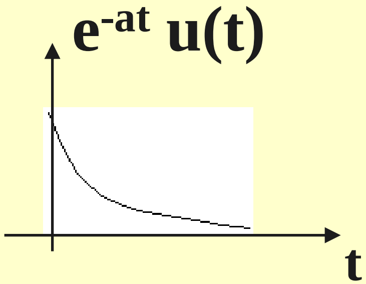
- Exponential $f(t) = e^{-at}\, u(t) \quad a>0$

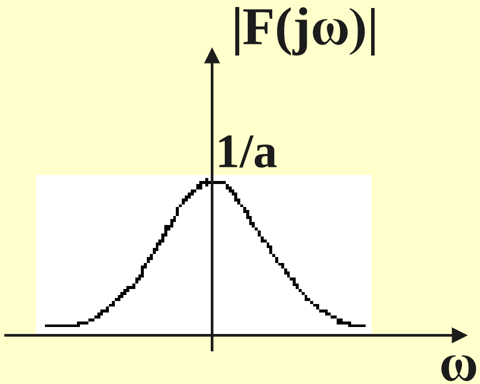
$$F(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt$$

$$= \frac{1}{a + j\omega}$$

$$|F(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$

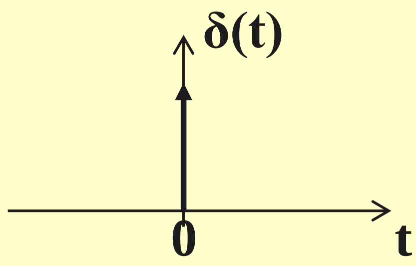
$$\varphi(\omega) = -tg^{-1}\frac{\omega}{a}$$

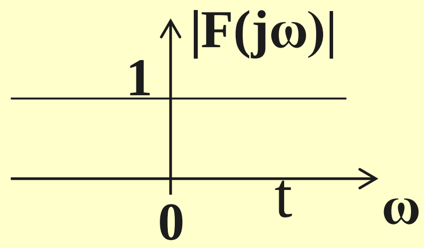




Fourier Transform of typical signals

- Unit impulse δ(t)





$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Unit impulse has uniform frequency density in whole frequency range, that means it has infinite wide band.

Fourier Transform of typical signals

- Constant 1

$$1 \longleftrightarrow 2\pi\delta(\omega)$$

This result could be got directly based on the symmetry of Fourier Transform.

Constant 1 represents direct current signal, and its spectrum is non-zero only at $\omega = 0$, which is a $\delta(\omega)$

Fourier Transform of typical signals

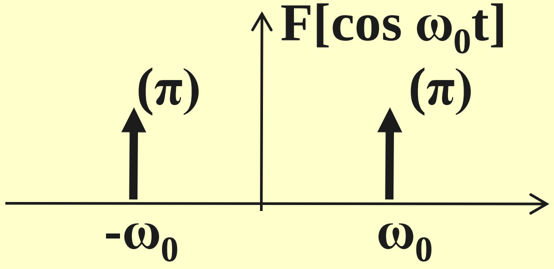
- Sin and cos function

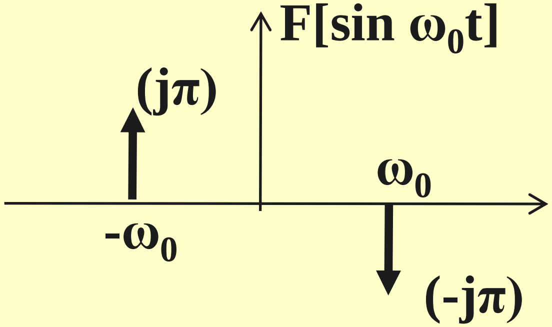
Based on the transform pair 1 $\longleftrightarrow$ $2\pi\delta(\omega)$ and $\delta(t)$ $\longleftrightarrow$ 1, we have some important conclusions:

$$\mathbf{F}[e^{j\omega_0 t}] = \int_0^{\infty} e^{-j\omega_0 t} e^{-j\omega t} dt = \int_0^{\infty} e^{-j(\omega-\omega_0)t} dt = 2\pi\delta(\omega-\omega_0)$$

$$\mathbf{F}[\cos\omega_0 t] = \mathbf{F}[(e^{j\omega_0 t} + e^{-j\omega_0 t})/2] = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

$$\mathbf{F}[\sin \omega_0 t] = \mathbf{F}[(e^{j\omega_0 t} - e^{-j\omega_0 t})/2j] = j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$



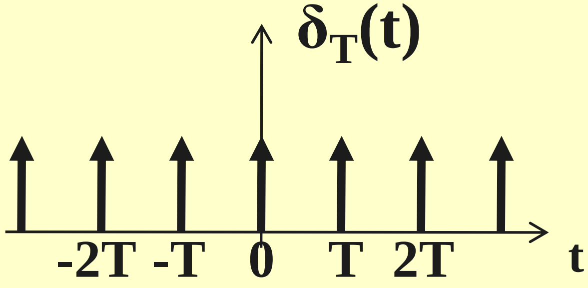


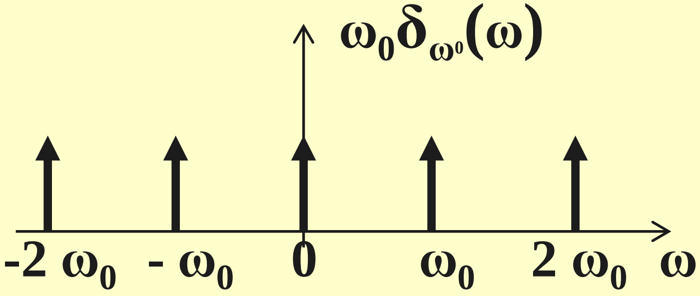
Fourier Transform of typical signals

- Unit impulse sequence

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$F[\delta_T(t)] = \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$
$$= \omega_0 \delta_{\omega_0}(\omega)$$





$$\omega_0 = 2\pi/T$$

Energy Density Spectrum

- The total energy $\mathcal{E}_x$ of a finite energy continuous-time complex signal $x_a(t)$ is given by

$$\mathcal{E}_x = \int_{-\infty}^{\infty} |x_a(t)|^2 dt = \int_{-\infty}^{\infty} x_a(t) x_a^*(t) dt$$

- The above expression can be rewritten as

$$\mathcal{E}_x = \int_{-\infty}^{\infty} x_a(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X_a^*(j\Omega) e^{-j\Omega t} d\Omega \right] dt$$

Energy Density Spectrum

- Interchanging the order of the integration we get

$$\mathcal{E}_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega)\left[\int_{-\infty}^{\infty} x_a(t)e^{-j\Omega t}\,dt\right]d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_a^*(j\Omega)X_a(j\Omega)\,d\Omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|X_a(j\Omega)\right|^2 d\Omega$$

Energy Density Spectrum

- Hence

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_a(j\Omega)|^2 \, d\Omega$$

- The above relation is more commonly known as the Parseval's theorem for finite-energy continuous-time signals

Energy Density Spectrum

- The quantity $\left|X_a(j\Omega)\right|^2$ is called the energy density spectrum of $x_a(t)$ and usually denoted as

$$S_{xx}(\Omega) = \left|X_a(j\Omega)\right|^2$$

- The energy over a specified range of frequencies $\Omega_a \leq \Omega \leq \Omega_b$ can be computed using

$$\mathcal{E}_{x,r} = \frac{1}{2\pi}\int_{\Omega_a}^{\Omega_b} S_{xx}(\Omega)\, d\Omega$$

1

Discrete-Time Fourier Transform

- Definition - The **discrete-time Fourier transform** (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

where ω is a continuous variable in the range $-\infty < \omega < \infty$

Discrete-Time Fourier Transform

- The infinite series $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ may or may not converge
- If it converges for all values of ω, then the DTFT $X(e^{j\omega})$ exists
- In general, $X(e^{j\omega})$ is a complex function of the real variable ω and can be written as

$$X(e^{j\omega}) = X_{re}(e^{j\omega}) + j\,X_{im}(e^{j\omega})$$

Discrete-Time Fourier Transform

- $X_{re}(e^{j\omega})$ and $X_{im}(e^{j\omega})$ are, respectively, the real and imaginary parts of $X(e^{j\omega})$, and are real functions of ω
- $X(e^{j\omega})$ can alternately be expressed as

$$X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{j\theta(\omega)}$$

where

$$\theta(\omega) = \arg\{X(e^{j\omega})\}$$

Discrete-Time Fourier Transform

- $\left|X(e^{j\omega})\right|$ is called the **magnitude function**
- $\theta(\omega)$ is called the **phase function**
- Both quantities are again real functions of ω
- In many applications, the DTFT is called the **Fourier spectrum**
- Likewise, $\left|X(e^{j\omega})\right|$ and $\theta(\omega)$ are called the **magnitude** and **phase spectra**

Discrete-Time Fourier Transform

- For a real sequence $x[n]$, $|X(e^{j\omega})|$ and $X_{re}(e^{j\omega})$ are even functions of ω, whereas, $\theta(\omega)$ and $X_{im}(e^{j\omega})$ are odd functions of ω
- Note: $X(e^{j\omega}) = |X(e^{j\omega})| e^{j\theta(\omega+2\pi k)}$

$$= |X(e^{j\omega})| e^{j\theta(\omega)}$$

for any integer k

- The phase function $\theta(\omega)$ cannot be uniquely specified for any DTFT

Discrete-Time Fourier Transform

- Unless otherwise stated, we shall assume that the phase function $\theta(\omega)$ is restricted to the following range of values:

$$-\pi \leq \theta(\omega) < \pi$$

called the **principal value**

Discrete-Time Fourier Transform

- Example - The DTFT of the unit sample sequence $\delta[n]$ is given by

$$\Delta(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = \delta[0] = 1$$

- Example - Consider the causal sequence

$$x[n] = \alpha^n \mu[n], \quad |\alpha| < 1$$

Discrete-Time Fourier Transform

- Its DTFT is given by

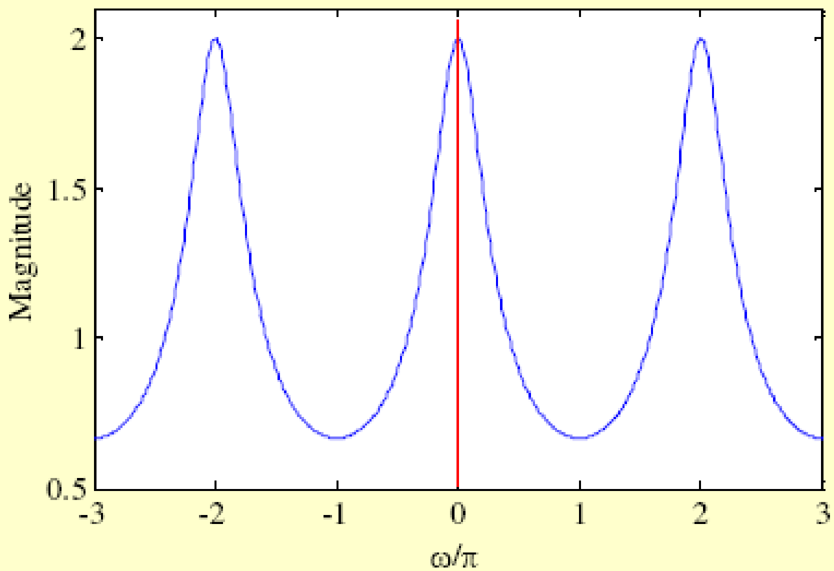
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

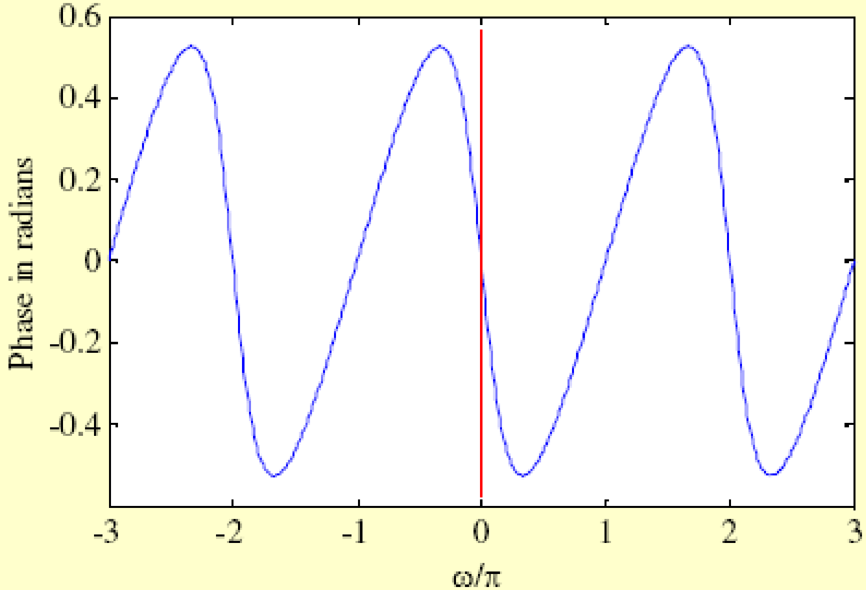
as $\left|\alpha e^{-j\omega}\right| = |\alpha| < 1$

Discrete-Time Fourier Transform

- The magnitude and phase of the DTFT $X(e^{j\omega}) = 1/(1-0.5e^{-j\omega})$ are shown below



$$\left|X(e^{j\omega})\right| = \left|X(e^{-j\omega})\right|$$



$$\theta(\omega) = -\theta(-\omega)$$

Discrete-Time Fourier Transform

- The DTFT $X(e^{j\omega})$ of a sequence $x[n]$ is a continuous function of ω
- It is also a periodic function of ω with a period 2π:

$$X(e^{j(\omega_o+2\pi k)}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega_o+2\pi k)n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_o n}e^{-j2\pi kn} = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega_o n} = X(e^{j\omega_o})$$

Discrete-Time Fourier Transform

- Therefore

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

represents the Fourier series representation of the periodic function

- As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

Discrete-Time Fourier Transform

- **Inverse discrete-time Fourier transform**:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Proof:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell} \right) e^{j\omega n} d\omega$$

Discrete-Time Fourier Transform

- The order of integration and summation can be interchanged if the summation inside the brackets converges uniformly, i.e. $X(e^{j\omega})$ exists
- Then $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{\ell=-\infty}^{\infty} x[\ell]e^{-j\omega\ell}\right)e^{j\omega n}d\omega$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j\omega(n-\ell)}d\omega\right) = \sum_{\ell=-\infty}^{\infty} x[\ell]\frac{\sin\pi(n-\ell)}{\pi(n-\ell)}$$

Discrete-Time Fourier Transform

- Now $$\frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \begin{cases} 1, & n=\ell \\ 0, & n \neq \ell \end{cases}$$
$$= \delta[n-\ell]$$
- Hence

$$\sum_{\ell=-\infty}^{\infty} x[\ell] \frac{\sin \pi(n-\ell)}{\pi(n-\ell)} = \sum_{\ell=-\infty}^{\infty} x[\ell]\delta[n-\ell] = x[n]$$

Discrete-Time Fourier Transform

- **Convergence Condition** - An infinite series of the form

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

may or may not converge

- Let

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n]e^{-j\omega n}$$

Discrete-Time Fourier Transform

- Then for uniform convergence of $X(e^{j\omega})$,

$$\lim_{K\to\infty}\left|X(e^{j\omega}) - X_K(e^{j\omega})\right| = 0$$

- Now, if $x[n]$ is an absolutely summable sequence, i.e., if

$$\sum_{n=-\infty}^{\infty}\left|x[n]\right| < \infty$$

Discrete-Time Fourier Transform

- Then

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \le \sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$$

for all values of ω

- Thus, the absolute summability of $x[n]$ is a sufficient condition for the existence of the DTFT $X(e^{j\omega})$

Discrete-Time Fourier Transform

- Example - The sequence $x[n] = \alpha^n \mu[n]$ for $|\alpha| < 1$ is absolutely summable as

$$\sum_{n=-\infty}^{\infty} |\alpha^n| \mu[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|} < \infty$$

and its DTFT $X(e^{j\omega})$ therefore converges to $1/(1-\alpha\, e^{-j\omega})$ uniformly

Discrete-Time Fourier Transform

- Since

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 \le \left(\sum_{n=-\infty}^{\infty} |x[n]| \right)^2,$$

  an absolutely summable sequence has always a finite energy

- However, a finite-energy sequence is not necessarily absolutely summable

Discrete-Time Fourier Transform

- Example - The sequence

$$x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & n \leq 0 \end{cases}$$

has a finite energy equal to

$$\mathcal{E}_x = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

- But, $x[n]$ is not absolutely summable

Discrete-Time Fourier Transform

- To represent a finite energy sequence $x[n]$ that is not absolutely summable by a DTFT $X(e^{j\omega})$, it is necessary to consider a **mean-square convergence** of $X(e^{j\omega})$:

$$\lim_{K\to\infty} \int_{-\pi}^{\pi} \left| X(e^{j\omega}) - X_K(e^{j\omega}) \right|^2 d\omega = 0$$

where

$$X_K(e^{j\omega}) = \sum_{n=-K}^{K} x[n] e^{-j\omega n}$$

Discrete-Time Fourier Transform

- Here, the total energy of the error

$$X(e^{j\omega}) - X_K(e^{j\omega})$$

must approach zero at each value of ω as K goes to ∞

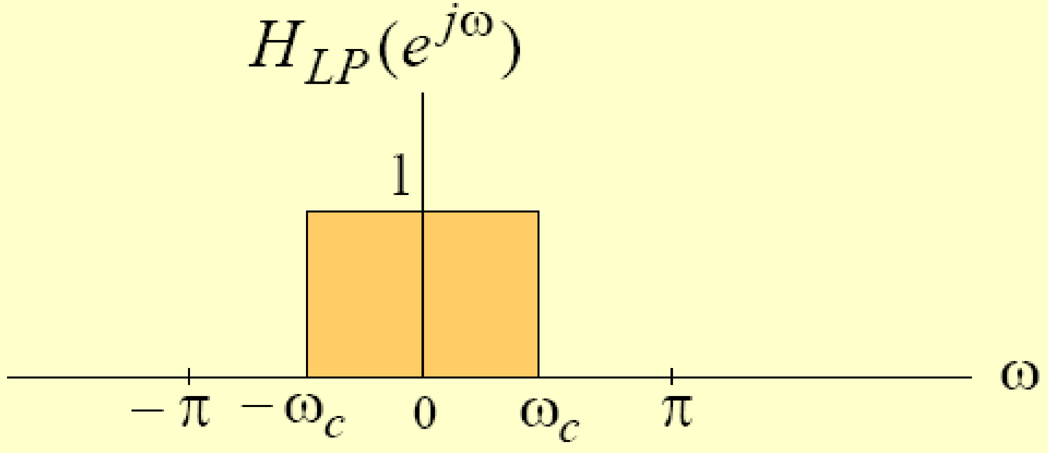
- In such a case, the absolute value of the error $\left|X(e^{j\omega}) - X_K(e^{j\omega})\right|$ may not go to zero as K goes to ∞ and the DTFT is no longer bounded

Discrete-Time Fourier Transform

- Example - Consider the DTFT

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

shown below



Discrete-Time Fourier Transform

- The inverse DTFT of $H_{LP}(e^{j\omega})$ is given by

$$h_{LP}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\left(\frac{e^{j\omega_c n}}{jn} - \frac{e^{-j\omega_c n}}{jn}\right) = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- The energy of $h_{LP}[n]$ is given by ω_c / π
- $\Rightarrow$ $h_{LP}[n]$ is a finite-energy sequence, but it is not absolutely summable

Discrete-Time Fourier Transform

- As a result

$$\sum_{n=-K}^{K} h_{LP}[n]e^{-j\omega n} = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

does not uniformly converge to $H_{LP}(e^{j\omega})$ for all values of ω, but converges to $H_{LP}(e^{j\omega})$ in the mean-square sense

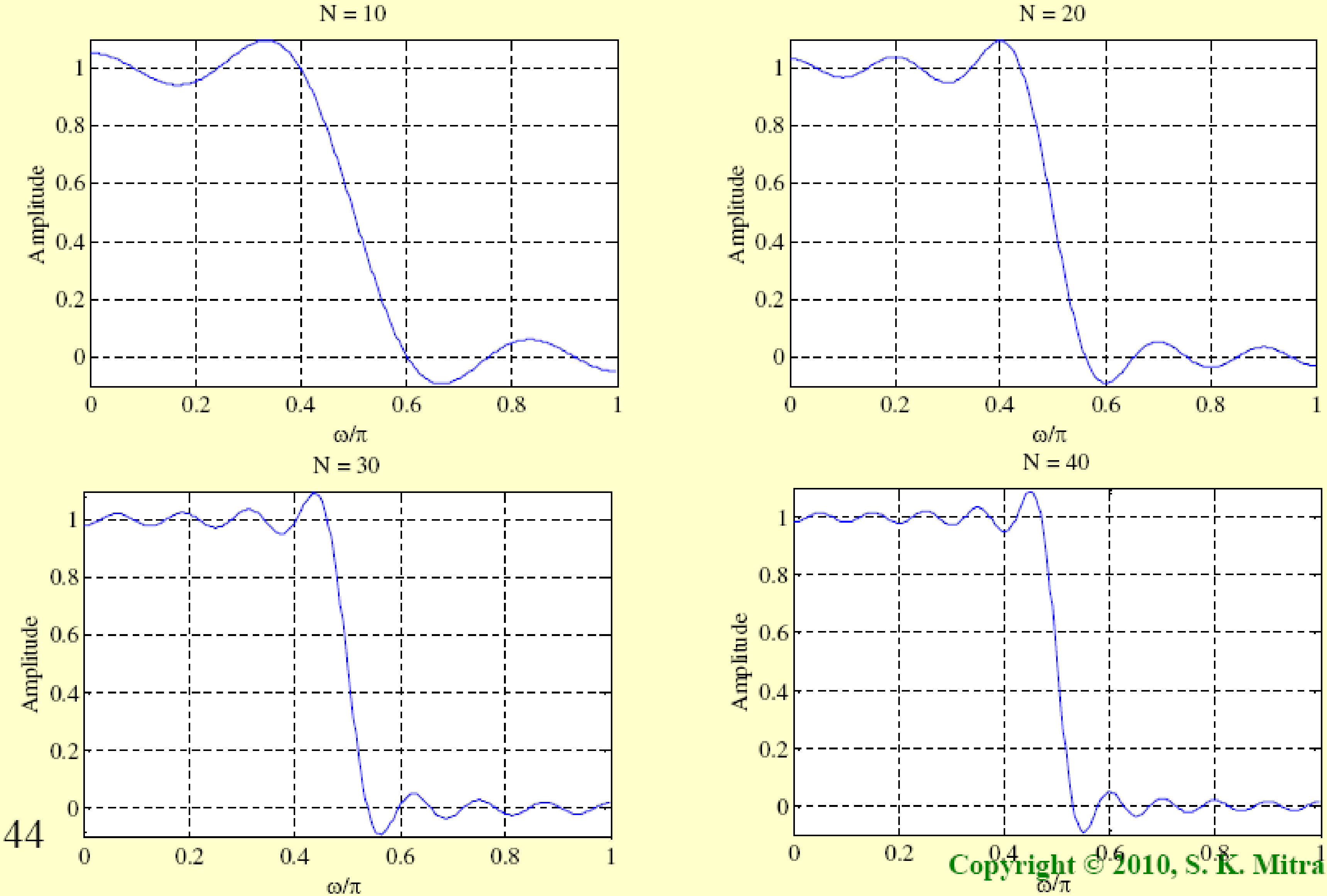
Discrete-Time Fourier Transform

- The mean-square convergence property of the sequence $h_{LP}[n]$ can be further illustrated by examining the plot of the function

$$H_{LP,K}(e^{j\omega}) = \sum_{n=-K}^{K} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

for various values of K as shown next

Discrete-Time Fourier Transform



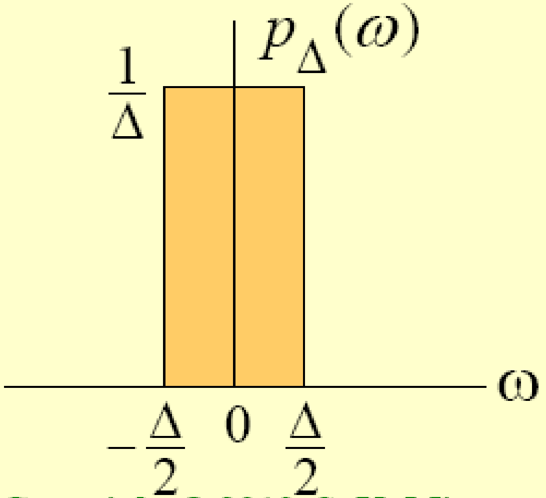
Discrete-Time Fourier Transform

- The DTFT can also be defined for a certain class of sequences which are neither absolutely summable nor square summable
- Examples of such sequences are the unit step sequence $\mu[n]$, the sinusoidal sequence $\cos(\omega_o n + \phi)$ and the exponential sequence $A\alpha^n$
- For this type of sequences, a DTFT representation is possible using the **Dirac delta function** $\delta(\omega)$

Discrete-Time Fourier Transform

- A Dirac delta function $\delta(\omega)$ is a function of ω with infinite height, zero width, and unit area
- It is the limiting form of a unit area pulse function $p_\Delta(\omega)$ as Δ goes to zero satisfying

$$\lim_{\Delta\to 0}\int_{-\infty}^{\infty} p_\Delta(\omega)d\omega = \int_{-\infty}^{\infty}\delta(\omega)d\omega$$



Discrete-Time Fourier Transform

- Example - Consider the complex exponential sequence

$$x[n] = e^{j\omega_o n}$$

- Its DTFT is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

where $\delta(\omega)$ is an impulse function of ω and

$$-\pi \leq \omega_o \leq \pi$$

Discrete-Time Fourier Transform

- The function

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$$

is a periodic function of ω with a period 2π and is called a **periodic impulse train**

- To verify that $X(e^{j\omega})$ given above is indeed the DTFT of $x[n] = e^{j\omega_o n}$ we compute the inverse DTFT of $X(e^{j\omega})$

Discrete-Time Fourier Transform

- Thus

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k) e^{j\omega n} d\omega$$

$$= \int_{-\pi}^{\pi} \delta(\omega - \omega_o) e^{j\omega n} d\omega = e^{j\omega_o n}$$

where we have used the sampling property of the impulse function $\delta(\omega)$

Commonly Used DTFT Pairs

Sequence		**DTFT**
$\delta[n]$	$\leftrightarrow$	1
1	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega + 2\pi k)$
$e^{j\omega_o n}$	$\leftrightarrow$	$\sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_o + 2\pi k)$
$\mu[n]$	$\leftrightarrow$	$\dfrac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$
$\mu[n],\ (\lvert\alpha\rvert < 1)$	$\leftrightarrow$	$\dfrac{1}{1 - \alpha e^{-j\omega}}$

DTFT Properties and Theorems

- There are a number of important properties and theorems of the DTFT that are useful in signal processing applications
- These are listed here without proof
- Their proofs are quite straightforward
- We illustrate the applications of some of the DTFT properties

Table 3.1: DTFT Properties: Symmetry Relations

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega})$
$x[-n]$	$X(e^{-j\omega})$
$x^*[-n]$	$X^*(e^{j\omega})$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{cs}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) + X^*(e^{-j\omega})\}$
$j\mathrm{Im}\{x[n]\}$	$X_{\mathrm{ca}}(e^{j\omega}) = \frac{1}{2}\{X(e^{j\omega}) - X^*(e^{-j\omega})\}$
$x_{\mathrm{cs}}[n]$	$X_{\mathrm{re}}(e^{j\omega})$
$x_{\mathrm{ca}}[n]$	$jX_{\mathrm{im}}(e^{j\omega})$

Note: $X_{\mathrm{cs}}(e^{j\omega})$ and $X_{\mathrm{ca}}(e^{j\omega})$ are the conjugate-symmetric and conjugate-antisymmetric parts of $X(e^{j\omega})$, respectively. Likewise, $x_{\mathrm{cs}}[n]$ and $x_{\mathrm{ca}}[n]$ are the conjugate-symmetric and conjugate-antisymmetric parts of $x[n]$, respectively.

$x[n]$: A complex sequence

Table 3.2: DTFT Properties: Symmetry Relations

Sequence	Discrete-Time Fourier Transform
$x[n]$	$X(e^{j\omega}) = X_{re}(e^{j\omega}) + jX_{im}(e^{j\omega})$
$x_{ev}[n]$	$X_{re}(e^{j\omega})$
$x_{od}[n]$	$jX_{im}(e^{j\omega})$
	$X(e^{j\omega}) = X^*(e^{-j\omega})$
	$X_{re}(e^{j\omega}) = X_{re}(e^{-j\omega})$
Symmetry relations	$X_{im}(e^{j\omega}) = -X_{im}(e^{-j\omega})$
	$\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert$
	$\arg\{X(e^{j\omega})\} = -\arg\{X(e^{-j\omega})\}$

Note: $x_{ev}[n]$ and $x_{od}[n]$ denote the even and odd parts of $x[n]$, respectively.

$x[n]$: A real sequence

Table 3.4 DTFT Theorems

Theorems	Sequence	DTFT
	$g[n]$ $h[n]$	$G(e^{j\omega})$ $H(e^{j\omega})$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(e^{j\omega}) + \beta H(e^{j\omega})$
Time-shifting	$g[n - n_o]$	$e^{-j\omega n_o} G(e^{j\omega})$
Frequency-shifting	$e^{j\omega_o n} g[n]$	$G\left(e^{j(\omega-\omega_o)}\right)$
Differentiation in frequency	$ng[n]$	$j\dfrac{dG(e^{j\omega})}{d\omega}$
Convolution	$g[n] \circledast h[n]$	$G(e^{j\omega})H(e^{j\omega})$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\theta})H(e^{j(\omega-\theta)})\,d\theta$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(e^{j\omega})H^*(e^{j\omega})\,d\omega$	

DTFT Theorems

- Example - Determine the DTFT $Y(e^{j\omega})$ of

$$y[n] = (n+1)\alpha^n \mu[n], \ |\alpha| < 1$$

- Let $x[n] = \alpha^n \mu[n], |\alpha| < 1$
- We can therefore write

$$y[n] = n\,x[n] + x[n]$$

- From Table 3.3, the DTFT of $x[n]$ is given by

$$X(e^{j\omega}) = \frac{1}{1 - \alpha\, e^{-j\omega}}$$

DTFT Theorems

- Using the differentiation theorem of the DTFT given in Table 3.4, we observe that the DTFT of $nx[n]$ is given by

$$j\frac{dX(e^{j\omega})}{d\omega} = j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2}$$

- Next using the linearity theorem of the DTFT given in Table 3.4 we arrive at

$$Y(e^{j\omega}) = \frac{\alpha e^{-j\omega}}{(1-\alpha e^{-j\omega})^2} + \frac{1}{1-\alpha e^{-j\omega}} = \frac{1}{(1-\alpha e^{-j\omega})^2}$$

DTFT Theorems

- Example - Determine the DTFT $V(e^{j\omega})$ of the sequence $v[n]$ defined by

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

- From Table 3.3, the DTFT of $\delta[n]$ is 1
- Using the time-shifting theorem of the DTFT given in Table 3.4 we observe that the DTFT of $\delta[n-1]$ is $e^{-j\omega}$ and the DTFT of $v[n-1]$ is $e^{-j\omega}V(e^{j\omega})$

DTFT Theorems

- Using the linearity theorem of Table 3.4 we then obtain the frequency-domain representation of

$$d_0 v[n] + d_1 v[n-1] = p_0 \delta[n] + p_1 \delta[n-1]$$

as

$$d_0 V(e^{j\omega}) + d_1 e^{-j\omega} V(e^{j\omega}) = p_0 + p_1 e^{-j\omega}$$

- Solving the above equation we get

$$V(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega}}{d_0 + d_1 e^{-j\omega}}$$

Linear Convolution Using DTFT

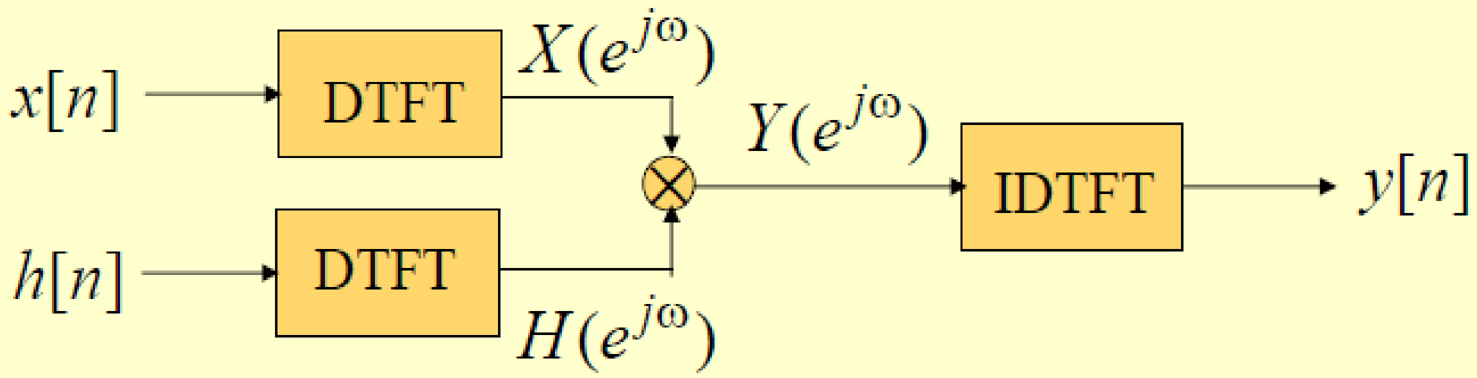
- An important property of the DTFT is given by the convolution theorem in Table 3.4
- It states that if $y[n] = x[n] \circledast h[n]$, then the DTFT $Y(e^{j\omega})$ of $y[n]$ is given by

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- An implication of this result is that the linear convolution $y[n]$ of the sequences $x[n]$ and $h[n]$ can be performed as follows:

Linear Convolution Using DTFT

- 1) Compute the DTFTs $X(e^{j\omega})$ and $H(e^{j\omega})$ of the sequences $x[n]$ and $h[n]$, respectively
- 2) Form the DTFT $Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$
- 3) Compute the IDFT $y[n]$ of $Y(e^{j\omega})$



§ 3.4 Energy Density Spectrum of a Discrete-Time Sequence

- **The total energy of a finite-energy sequence *g*[*n*] is given by**

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2$$

- **From Parseval's relation we observe that**

$$\mathcal{E}_g = \sum_{n=-\infty}^{\infty} \left| g[n] \right|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| G(e^{j\omega}) \right|^2 d\omega$$

§ 3.4 Energy Density Spectrum of a Discrete-Time Sequence

- **The quantity**

$$S_{gg}(\omega) = \left|G(e^{j\omega})\right|^2$$

is called the energy density spectrum

- **The area under this curve in the range $-\pi \le \omega \le \pi$ divided by 2π is the energy of the sequence**

§ 3.6 DTFT Computation Using MATLAB

- **The *Signal Processing Toolbox* in MATLAB includes a number of M-files to aid in the DTFT-based analysis of discrete-time signals.**
- **The function that can be used as:**

  1) **Freqz()**
  2) **Abs()**
  3) **Angle()**
  4) **Real(), imag()**
  5) **Unwrap()**

§ 3.6 DTFT Computation Using MATLAB

- The function **freqz** can be used to compute the values of the DTFT of a sequence, described as a rational function in the form of

$$X(e^{j\omega}) = \frac{p_0 + p_1 e^{-j\omega} + \dots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \dots + d_N e^{-j\omega N}}$$

at a prescribed set of discrete frequency points $\omega = \omega_l$

§ 3.6 DTFT Computation Using MATLAB

- For example, the statement

  H = freqz(num, den, w)

  returns the frequency response values as a vector H of a DTFT defined in terms of the vectors num and den containing the coefficients $\{p_i\}$ and $\{d_i\}$, respectively at a prescribed set of frequencies between 0 and 2π given by the vector w

§ 3.6 DTFT Computation Using MATLAB

- **There are several other forms of the function freqz**
- **The Program 3_1 in the text can be used to compute the values of the DTFT of a real sequence**
- **It computes the real and imaginary parts, and the magnitude and phase of the DTFT**

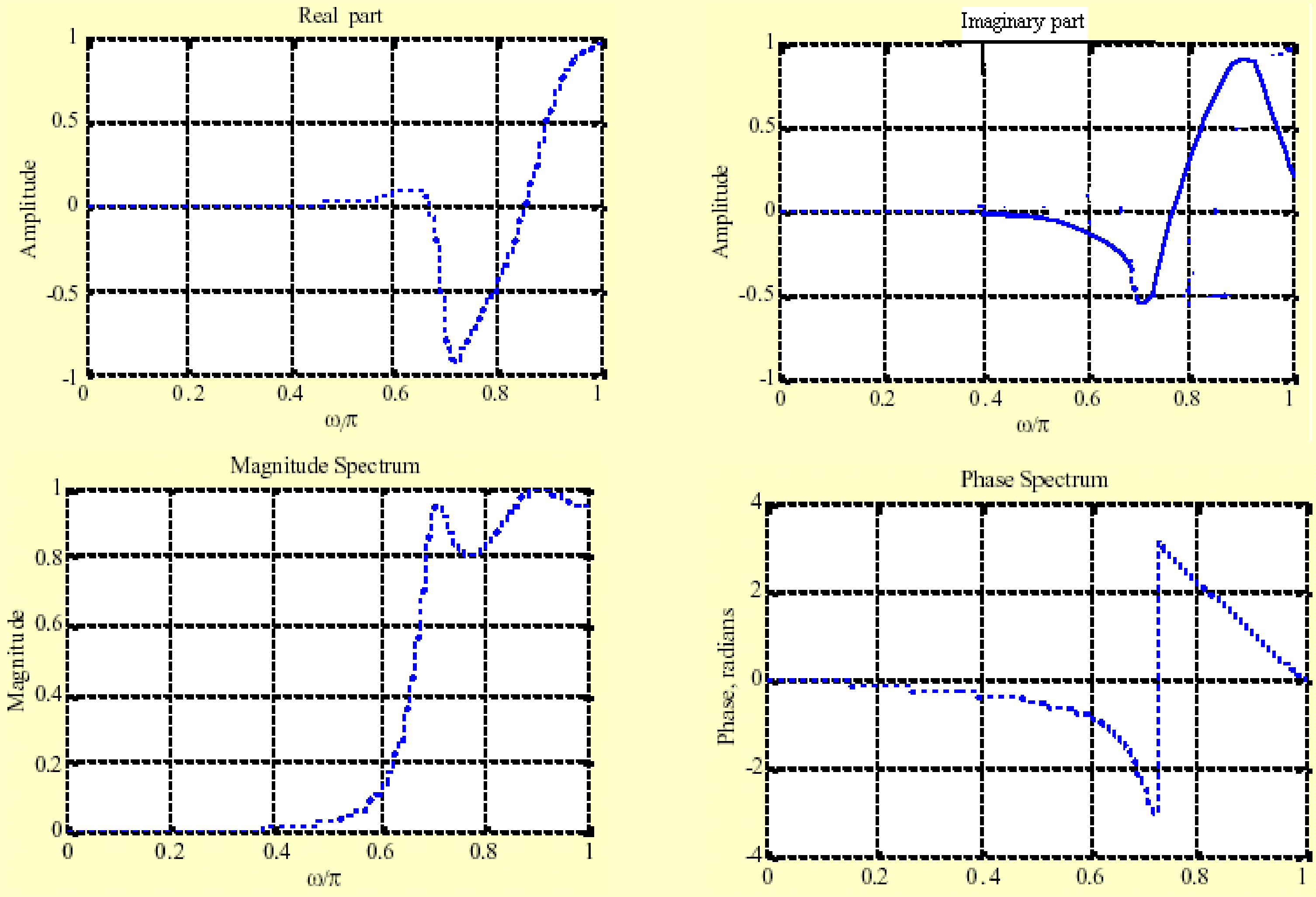
§ 3.6 DTFT Computation Using MATLAB

- **<u>Example</u> - Plots of the real and imaginary parts, and the magnitude and phase of the DTFT**

$$X(e^{j\omega}) = \frac{\begin{array}{c}0.008 - 0.033e^{-j\omega} + 0.05e^{-j2\omega} \\ -0.033e^{-j3\omega} + 0.008e^{-j4\omega}\end{array}}{\begin{array}{c}1 + 2.37e^{-j\omega} + 2.7e^{-j2\omega} \\ +1.6e^{-j3\omega} + 0.41e^{-j4\omega}\end{array}}$$

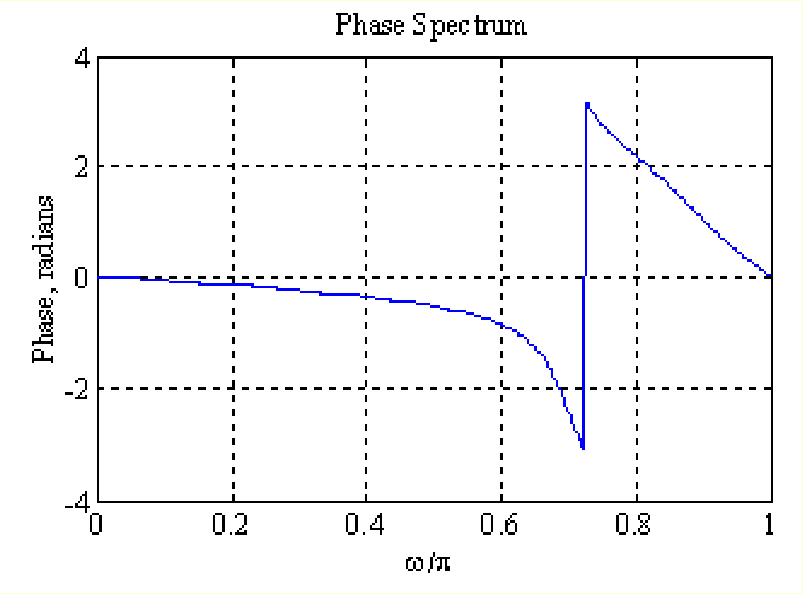
are shown on the next slide

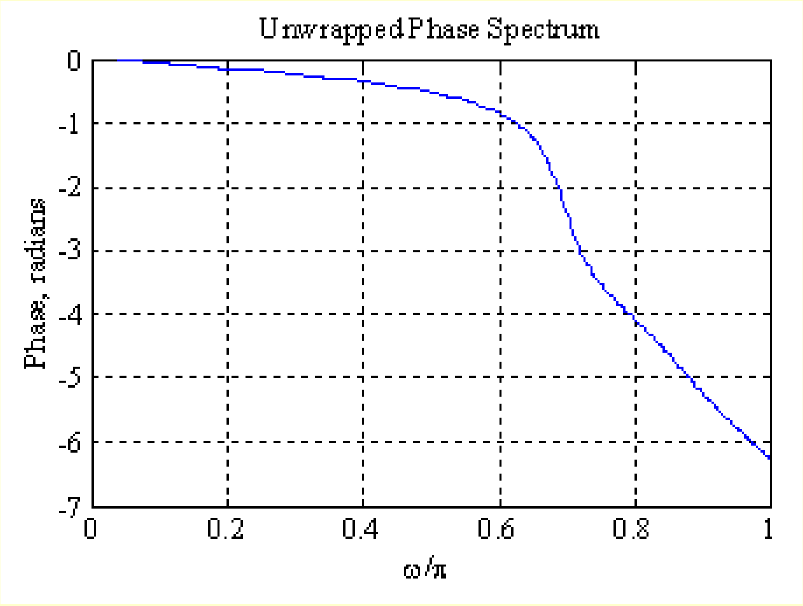
§ 3.6 DTFT Computation Using MATLAB



§ 3.6 DTFT Computation Using MATLAB

- **Note: The phase spectrum displays a discontinuity of 2π at $\omega = 0.72$**
- **This discontinuity can be removed using the function unwrap as indicated below**





Digital Processing of Continuous-Time Signals

- Digital processing of a continuous-time signal involves the following basic steps:

  (1) Conversion of the continuous-time signal into a discrete-time signal,

  (2) Processing of the discrete-time signal,

  (3) Conversion of the processed discrete-time signal back into a continuous-time signal

Digital Processing of Continuous-Time Signals

- **Conversion of a continuous-time signal into digital form is carried out by an analog-to-digital (A/D) converter**
- **The reverse operation of converting a digital signal into a continuous-time signal is performed by a digital-to-analog (D/A) converter**

Digital Processing of Continuous-Time Signals

- **Since the A/D conversion takes a finite amount of time, a sample-and-hold (S/H) circuit is used to ensure that the analog signal at the input of the A/D converter remains constant in amplitude until the conversion is complete to minimize the error in its representation**

Digital Processing of Continuous-Time Signals

- To prevent aliasing, an analog anti-aliasing filter is employed before the S/H circuit
- To smooth the output signal of the D/A converter, which has a staircase-like waveform, an analog reconstruction filter is used

Digital Processing of Continuous-Time Signals

Complete block-diagram

- Since both the anti-aliasing filter and the reconstruction filter are analog lowpass filters, we review first the theory behind the design of such filters
- Also, the most widely used IIR digital filter design method is based on the conversion of an analog lowpass prototype

Sampling of Continuous-time Signals

- **As indicated earlier, discrete-time signals in many applications are generated by sampling continuous-time signals**
- **We have seen earlier that identical discrete-time signals may result from the sampling of more than one distinct continuous-time function**

Sampling of Continuous-time Signals

- **In fact, there exists an infinite number of continuous-time signals, which when sampled lead to the same discrete-time signal**
- **However, under certain conditions, it is possible to relate a unique continuous-time signal to a given discrete-time signal**

Sampling of Continuous-time Signals

- **If these conditions hold, then it is possible to recover the original continuous-time signal from its sampled values**
- **We next develop this correspondence and the associated conditions**

Sampling of Continuous-time Signals

- **Let $g_a(t)$ be a continuous-time signal that is sampled uniformly at $t = nT$, generating the sequence $g[n]$ where**

$$g[n] = g_a(nT), \qquad -\infty < n < \infty$$

with T being the sampling period

- **The reciprocal of T is called the sampling frequency F_T, i.e., $F_T = 1/T$**

Sampling of Continuous-time Signals

- **Now, the frequency-domain representation of $g_a(t)$ is given by its continuos-time Fourier transform (CTFT):**

$$G_a(j\Omega) = \int_{-\infty}^{\infty} g_a(t) e^{-j\Omega t} dt$$

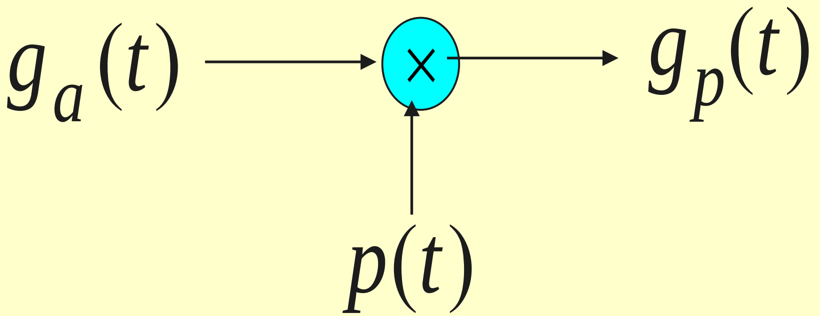
- **The frequency-domain representation of *g*[*n*] is given by its discrete-time Fourier transform (DTFT):**

$$G(e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] e^{-j\omega n}$$

Effect of Sampling in the Frequency Domain

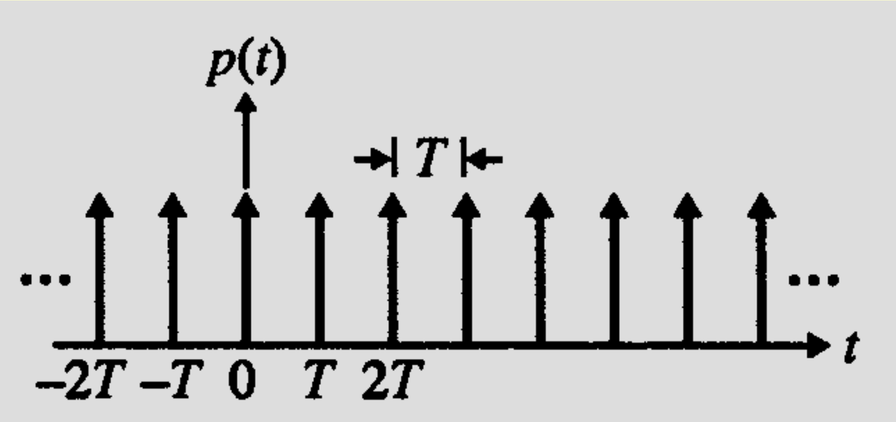
- **To establish the relation between $G_a(j\Omega)$ and $G(e^{j\omega})$, we treat the sampling operation mathematically as a multiplication of $g_a(t)$ by a periodic impulse train *p*(*t*):**

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



Effect of Sampling in the Frequency Domain

- ***p*(*t*) consists of a train of ideal impulses with a period *T* as shown below**

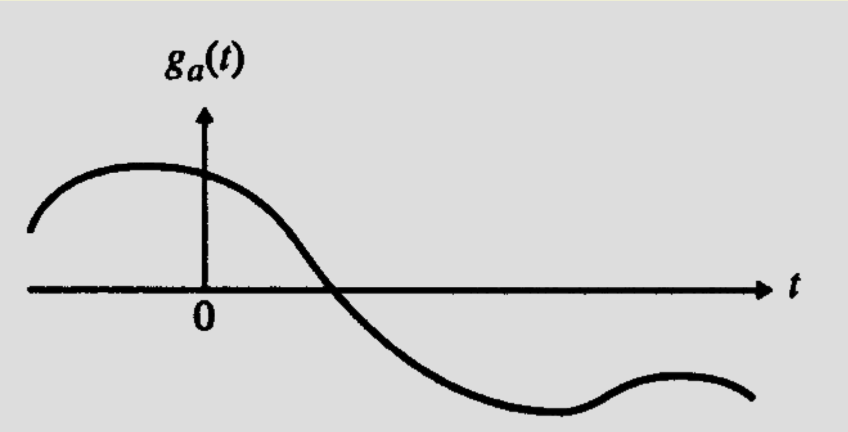


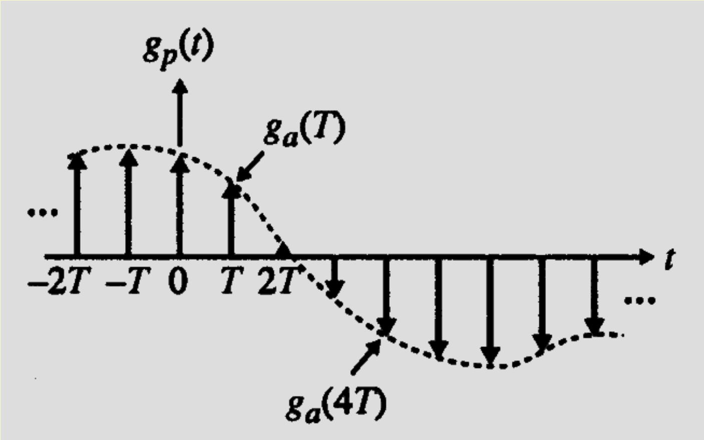
- **The multiplication operation yields an impulse train:**

$$g_p(t) = g_a(t)\,p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\,\delta(t - nT)$$

Effect of Sampling in the Frequency Domain

- $g_p(t)$ is a continuous-time signal consisting of a train of uniformly spaced impulses with the impulse at $t = nT$ weighted by the sampled value $g_a(nT)$ of $g_a(t)$ at that instant $t=nT$





Effect of Sampling in the Frequency Domain

- **There are two different forms of $G_p(j\Omega)$:**
- **One form is given by the weighted sum of the CTFTs of $\delta(t-nT)$:**

$$G_p(j\Omega) = \sum_{n=-\infty}^{\infty} g_a(nT) e^{-j\Omega nT}$$

• To derive the second form, we note that *p*(*t*) can be expressed as a Fourier series:

$$p(t) = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j(2\pi/T)kt} = \frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j\Omega_T kt}$$

where $\Omega_T = 2\pi / T$

Effect of Sampling in the Frequency Domain

- **The impulse train $g_p(t)$ therefore can be expressed as**

$$g_p(t) = \left(\frac{1}{T} \sum_{k=-\infty}^{\infty} e^{j\Omega_T kt} \right) \cdot g_a(t)$$

- **From the frequency-shifting property of the CTFT, the CTFT of $e^{j\Omega_T kt} g_a(t)$ is given by $G_a(j(\Omega - k\Omega_T))$**

Effect of Sampling in the Frequency Domain

- **Hence, an alternative form of the CTFT of $g_p(t)$ is given by**

$$G_p(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} G_a\big(j(\Omega - k\Omega_T)\big)$$

- **Therefore, $G_p(j\Omega)$ is a periodic function of Ω consisting of a sum of shifted and scaled replicas of $G_a(j\Omega)$, shifted by integer multiples of Ω_T and scaled by 1/T**

Effect of Sampling in the Frequency Domain

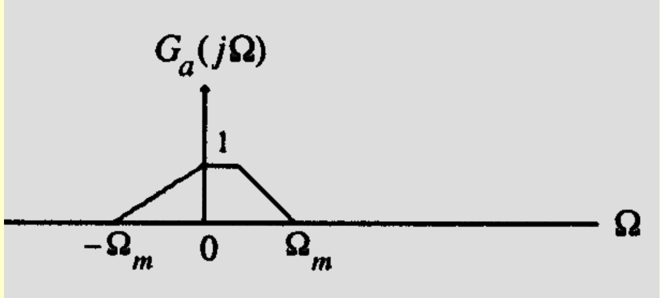
- **The term on the RHS of the previous equation for *k* = 0 is the baseband portion of $G_p(j\Omega)$, and each of the remaining terms are the frequency translated portions of $G_p(j\Omega)$**
- **The frequency range**

$$-\frac{\Omega_T}{2} \leq \Omega \leq \frac{\Omega_T}{2}$$

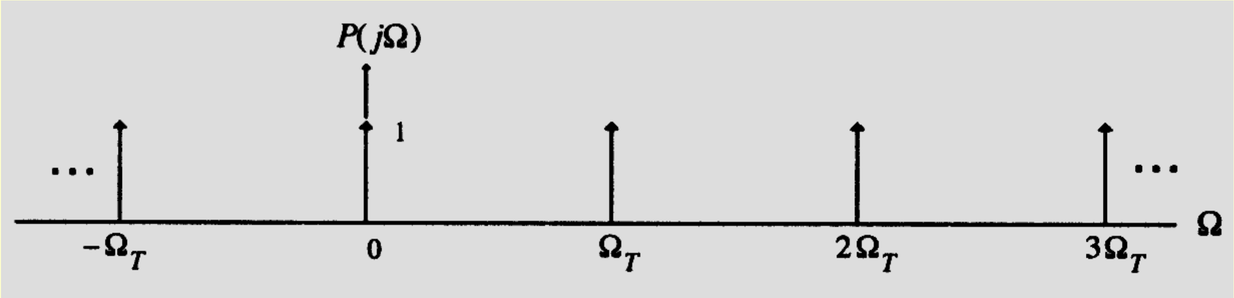
is called the baseband or Nyquist band

Effect of Sampling in the Frequency Domain

- **Assume $g_a(t)$ is a band-limited signal with a CTFT $G_a(j\Omega)$ as shown below**

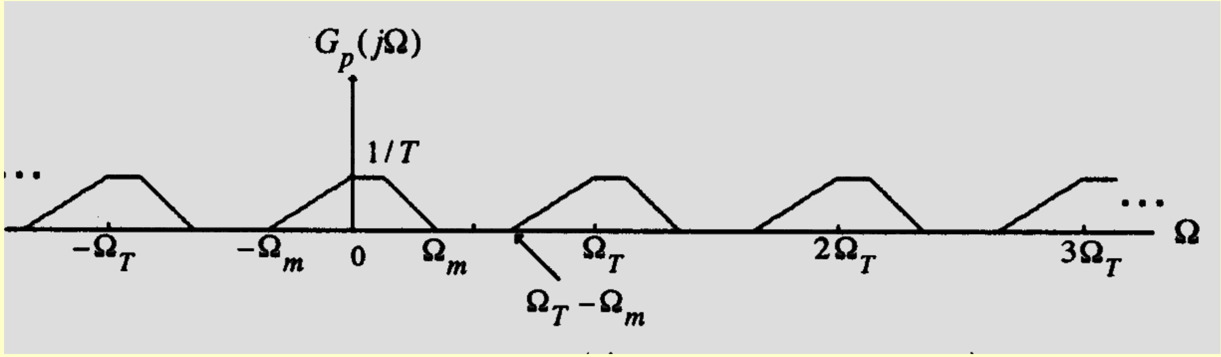


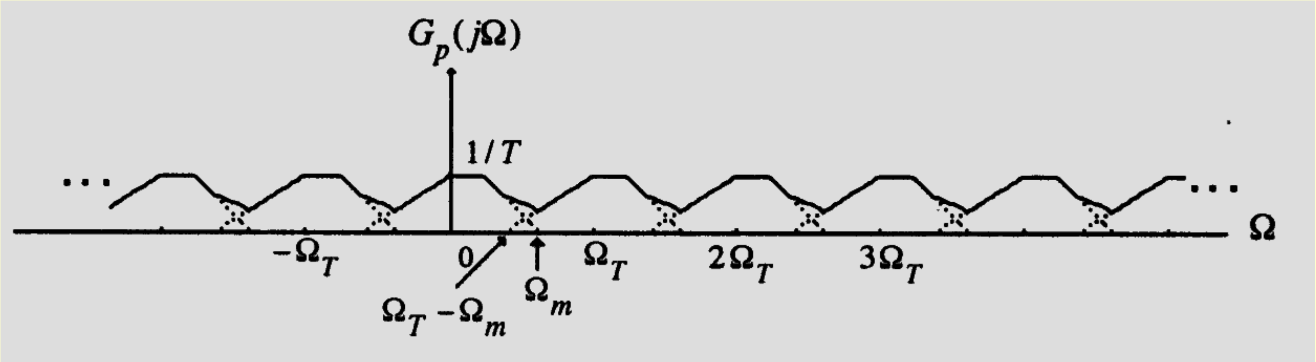
- **The spectrum $P(j\Omega)$ of p(t) having a sampling period $T=2\pi/\Omega_T$ is indicated below**



Effect of Sampling in the Frequency Domain

- Two possible spectra of $G_p(j\Omega)$ are shown below



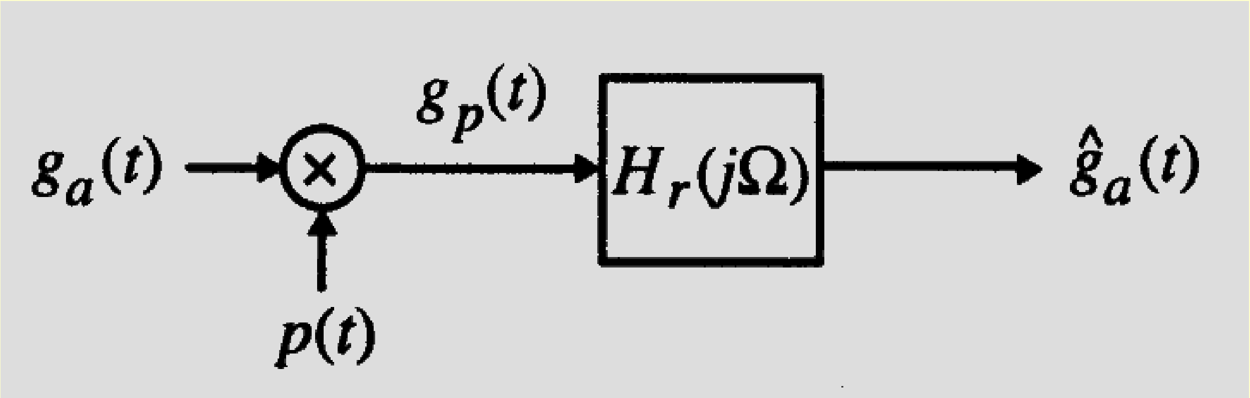


Effect of Sampling in the Frequency Domain

- It is evident from the top figure on the previous slide that if $\Omega_T > 2\,\Omega_m$, there is no overlap between the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$
- On the other hand, as indicated by the figure on the bottom, if $\Omega_T < 2\,\Omega_m$, there is an overlap of the spectra of the shifted replicas of $G_a(j\Omega)$ generating $G_p(j\Omega)$

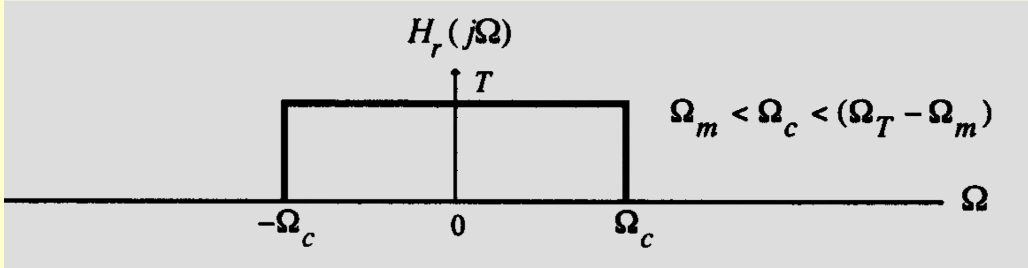
Effect of Sampling in the Frequency Domain

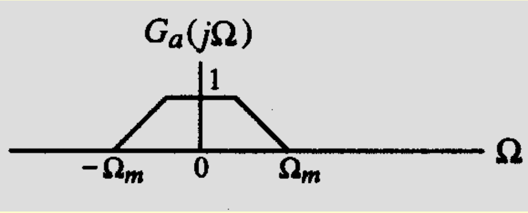
- If $\Omega_T > 2\,\Omega_m$, $g_a(t)$ can be recovered exactly from $g_p(t)$ by passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain T and a cutoff frequency Ω_c greater than Ω_m and less than $\Omega_T - \Omega_m$ as shown below

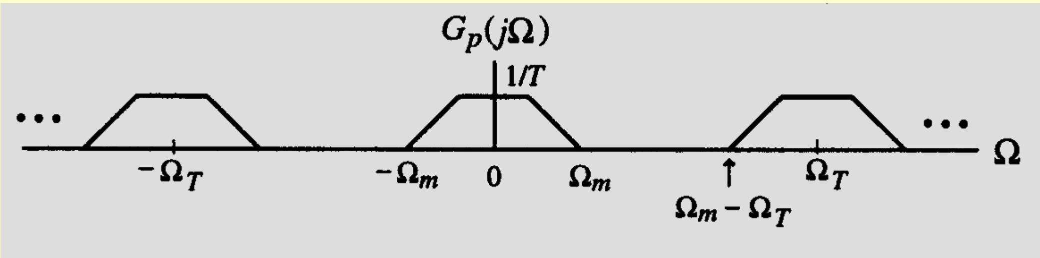


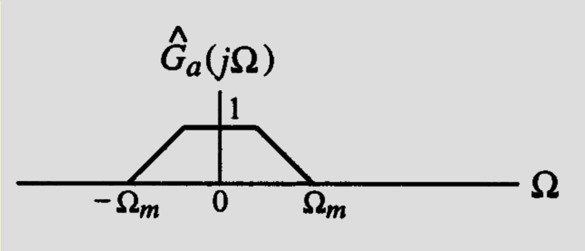
Effect of Sampling in the Frequency Domain

- The spectra of the filter and pertinent signals are shown below









Effect of Sampling in the Frequency Domain

- **On the other hand, if $\Omega_T < 2\,\Omega_m$, due to the overlap of the shifted replicas of $G_a(j\Omega)$, the spectrum $G_p(j\Omega)$ cannot be separated by filtering to recover $G_a(j\Omega)$ because of the distortion caused by a part of the replicas immediately outside the baseband folded back or aliased into the baseband**

Effect of Sampling in the Frequency Domain

Sampling theorem - **Let $g_a(t)$ be a band-limited signal with CTFT $G_a(j\Omega)=0$ for**

$$|\Omega| > \Omega_m$$

Then $g_a(t)$ is uniquely determined by its samples $g_a(nT)$, $-\infty \le n \le \infty$ if

$$\Omega_T \ge 2\,\Omega_m$$

where $\Omega_T = 2\pi/T$

Effect of Sampling in the Frequency Domain

- The condition $\Omega_T \geq 2\,\Omega_m$ is often referred to as the **Nyquist condition**

- The frequency $\Omega_T/2$ is usually referred to as the **folding frequency**

Effect of Sampling in the Frequency Domain

- **Given $\{g_a(nT)\}$, we can recover exactly $g_a(t)$ by generating an impulse train**

$$g_p(t) = \sum_{n=-\infty}^{\infty} g_a(nT)\delta(t - nT)$$

and then passing it through an ideal lowpass filter $H_r(j\Omega)$ with a gain *T* and a cutoff frequency Ω_c satisfying

$$\Omega_m < \Omega_c < (\Omega_T - \Omega_m)$$

Effect of Sampling in the Frequency Domain

- The highest frequency Ω_m contained in $g_a(t)$ is usually called the **Nyquist frequency** since it determines the minimum sampling frequency $\Omega_T = 2\Omega_m$ that must be used to fully recover $g_a(t)$ from its sampled version
- The frequency $2\Omega_m$ is called the **Nyquist rate**

Effect of Sampling in the Frequency Domain

- **Oversampling - The sampling frequency is higher than the Nyquist rate**
- **Undersampling - The sampling frequency is lower than the Nyquist rate**
- **Critical sampling - The sampling frequency is equal to the Nyquist rate**
- **Note: A pure sinusoid may not be recoverable from its critically sampled version**

Effect of Sampling in the Frequency Domain

- **In digital telephony, a 3.4 kHz signal bandwidth is acceptable for telephone conversation**
- **Here, a sampling rate of 8 kHz, which is greater than twice the signal bandwidth, is used**

Effect of Sampling in the Frequency Domain

- **In high-quality analog music signal processing, a bandwidth of 20 kHz has been determined to preserve the fidelity**
- **Hence, in compact disc (CD) music systems, a sampling rate of 44.1 kHz, which is slightly higher than twice the signal bandwidth, is used**

Effect of Sampling in the Frequency Domain

- **<u>Example</u> - Consider the three continuous-time sinusoidal signals:**

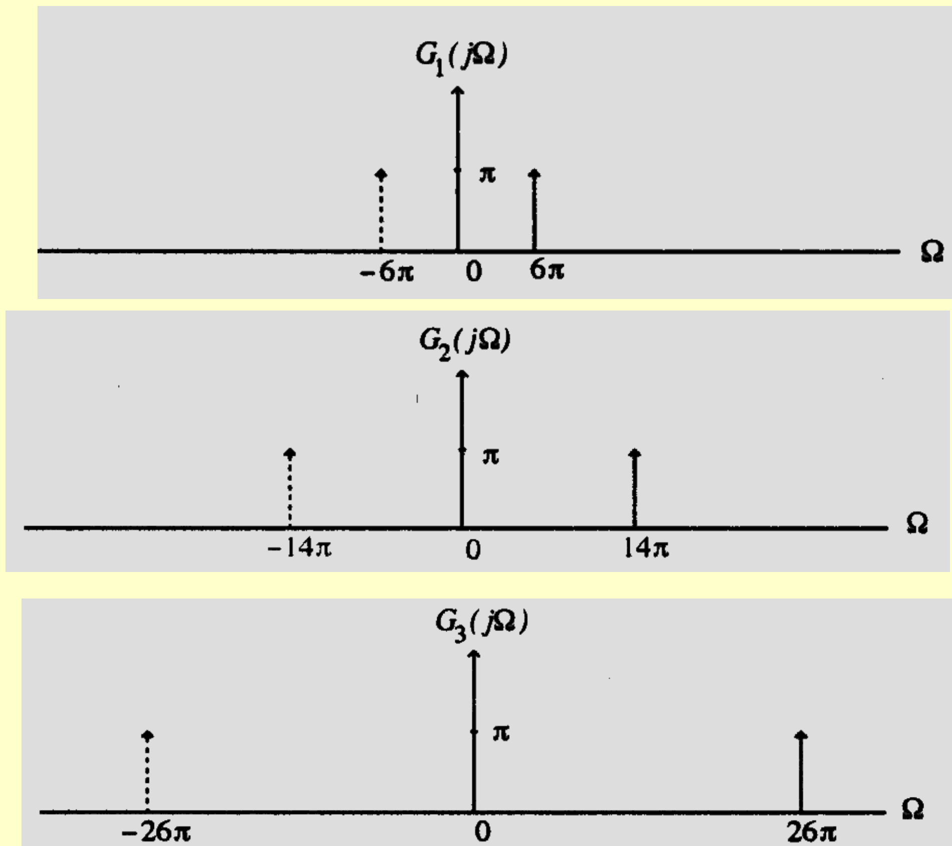
$$g_1(t) = \cos(6\pi t)$$
$$g_2(t) = \cos(14\pi t)$$
$$g_3(t) = \cos(26\pi t)$$

- **Their corresponding CTFTs are:**

$$G_1(j\Omega) = \pi[\delta(\Omega - 6\pi) + \delta(\Omega + 6\pi)]$$
$$G_2(j\Omega) = \pi[\delta(\Omega - 14\pi) + \delta(\Omega + 14\pi)]$$
$$G_3(j\Omega) = \pi[\delta(\Omega - 26\pi) + \delta(\Omega + 26\pi)]$$

Effect of Sampling in the Frequency Domain

- These three transforms are plotted below



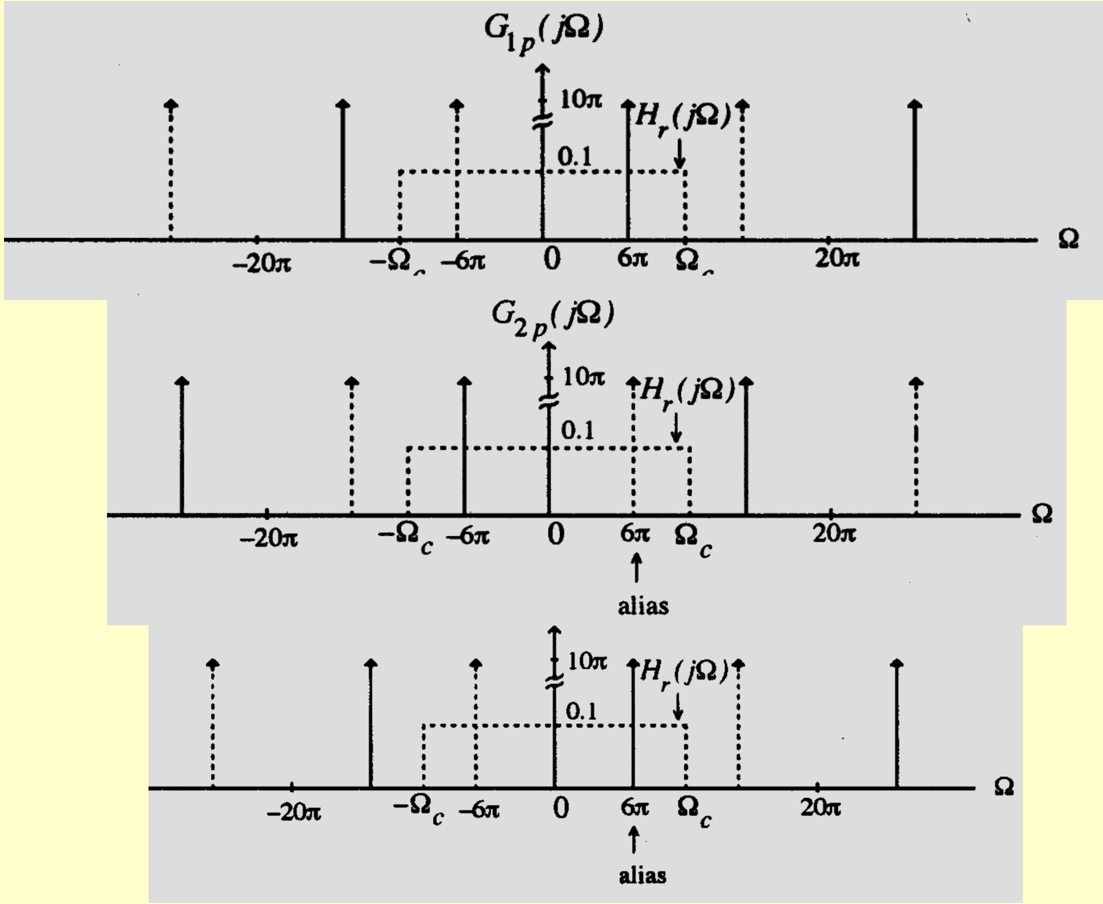
Effect of Sampling in the Frequency Domain

- These continuous-time signals sampled at a rate of T = 0.1 sec, i.e., with a sampling frequency $\Omega_T = 20\pi$ rad/sec
- The sampling process generates the continuous-time impulse trains, $g_{1p}(t)$, $g_{2p}(t)$, and $g_{3p}(t)$
- Their corresponding CTFTs are given by

$$G_{\ell p}(j\Omega) = 10\sum_{k=-\infty}^{\infty} G_{\ell}\big(j(\Omega - k\Omega_T)\big), \quad 1 \le \ell \le 3$$

Effect of Sampling in the Frequency Domain

- Plots of the 3 CTFTs are shown below



Effect of Sampling in the Frequency Domain

- **These figures also indicate by dotted lines the frequency response of an ideal lowpass filter with a cutoff at $\Omega_c = \Omega_T/2 = 10\pi$ and a gain T=0.1**
- **The CTFTs of the lowpass filter output are also shown in these three figures**
- **In the case of $g_1(t)$, the sampling rate satisfies the Nyquist condition, hence no aliasing**

Effect of Sampling in the Frequency Domain

- Moreover, the reconstructed output is precisely the original continuous-time signal
- In the other two cases, the sampling rate does not satisfy the Nyquist condition, resulting in aliasing and the filter outputs are all equal to $\cos(6\pi t)$

Effect of Sampling in the Frequency Domain

- Now, the CTFT $G_p(j\,\Omega)$ is a periodic function of Ω with a period $\Omega_T = 2\pi/T$

- Because of the mapping, the DTFT $G(e^{j\omega})$ is a periodic function of ω with a period 2π

Recovery of the Analog Signal

- **We now derive the expression for the output $\hat{g}_a(t)$ of the ideal lowpass reconstruction filter $H_r(j\Omega)$ as a function of the samples $g[n]$**

➢ **The impulse response $h_r(t)$ of the lowpass reconstruction filter is obtained by taking the inverse DTFT of $H_r(j\Omega)$**

$$H_r(j\Omega) = \begin{cases} T, & |\Omega| \le \Omega_c \\ 0, & |\Omega| > \Omega_c \end{cases}$$

Recovery of the Analog Signal

➢ **Thus, the impulse response is given by**

$$h_r(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_r(j\Omega)e^{j\Omega t}d\Omega = \frac{T}{2\pi}\int_{-\Omega_c}^{\Omega_c} e^{j\Omega t}d\Omega$$

$$= \frac{\sin(\Omega_c t)}{\Omega_T t/2}, \qquad -\infty \le t \le \infty$$

- **The input to the lowpass filter is the impulse train $g_p(t)$:**

$$g_p(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t-nT)$$

Recovery of the Analog Signal

- **Therefore, the output $\hat{g}_a(t)$ of the ideal lowpass filter is given by:**

$$\hat{g}_a(t) = h_r(t) \circledast g_p(t) = \sum_{n=-\infty}^{\infty} g[n] h_r(t - nT)$$

Substituting $h_r(t)=\sin(\Omega_c t)/(\Omega_T t/2)$ in the above and assuming for simplicity

$\Omega_c = \Omega_T/2 = \pi/T$, we get

$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin[\pi(t - nT)/T]}{\pi(t - nT)/T}$$

which is called Poisson sum formula

Recovery of the Analog Signal

- The ideal bandlimited interpolation process is illustrated below

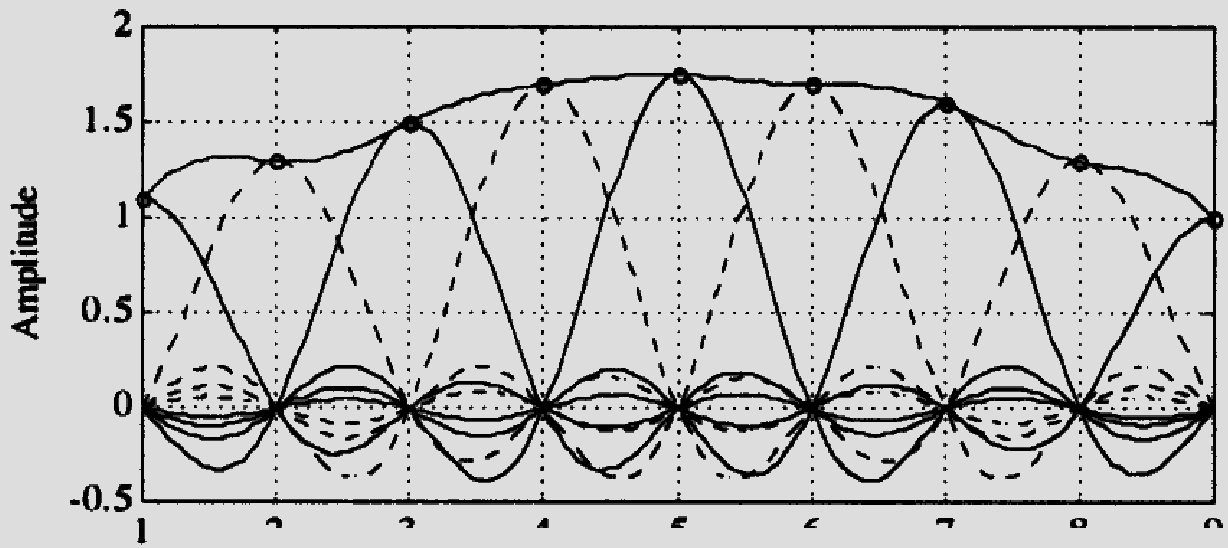
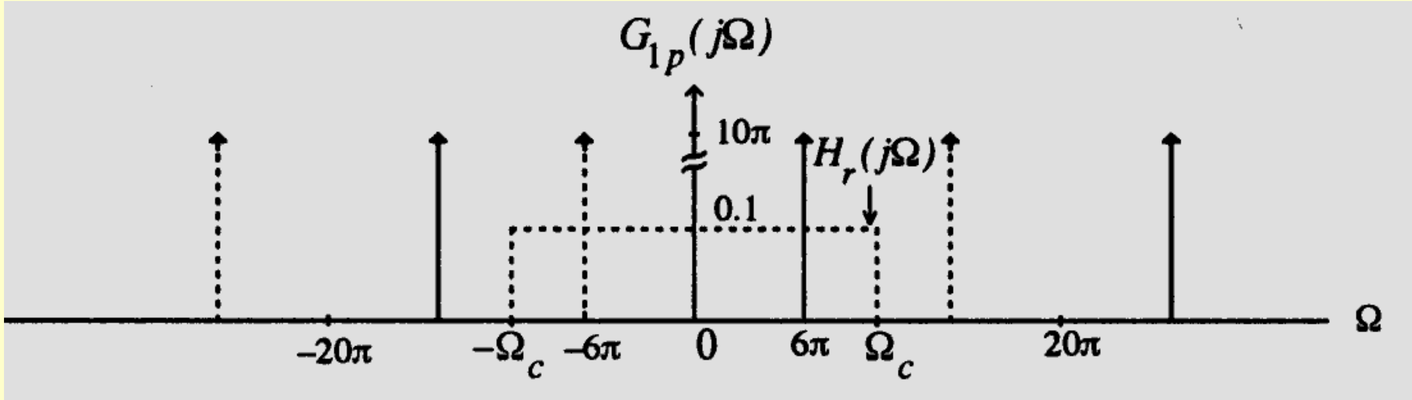


Illustration of Poisson sum formula

Implication of the Sampling Process

- Consider again the three continuous-time signals: $g_1(t)=\cos(6\pi t)$, $g_2(t)=\cos(14\pi t)$, and $g_3(t)=\cos(26\pi t)$
- The plot of the CTFT $G_{1p}(j\Omega)$ of the sampled version $g_{1p}(t)$ of $g_1(t)$ is shown below



Implication of the Sampling Process

- **From the plot, it is apparent that we can recover any of its frequency-translated versions $\cos[(20k \pm 6)\pi t]$ outside the baseband by passing $g_{1p}(t)$ through an ideal analog bandpass filter with a passband centered at $\Omega = (20k \pm 6)\pi$**

Implication of the Sampling Process

- **For example, to recover the signal cos(34πt), it will be necessary to employ a bandpass filter with a frequency response**

$$H_r(j\Omega) = \begin{cases} 0.1, & (34-\Delta)\pi \le |\Omega| \le (34+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

where Δ is a small number

Implication of the Sampling Process

- **Likewise, we can recover the aliased baseband component cos(6π*t*) from the sampled version of either $g_{2p}(t)$ or $g_{3p}(t)$ by passing it through an ideal lowpass filter with a frequency response**

$$H_r(j\Omega) = \begin{cases} 0.1, & (6-\Delta)\pi \le |\Omega| \le (6+\Delta)\pi \\ 0, & \text{otherwise} \end{cases}$$

Implication of the Sampling Process

- **There is no aliasing distortion unless the original continuous-time signal also contains the component cos(6π*t*)**
- **Similarly, from either $g_{2p}(t)$ or $g_{3p}(t)$ we can recover any one of the frequency-translated versions, including the parent continuous-time signal $g_2(t)$ or $g_3(t)$ as the case may be, by employing suitable filters**

Sampling of Bandpass Signals

- **The conditions developed earlier for the unique representation of a continuous-time signal by the discrete-time signal obtained by uniform sampling assumed that the continuous-time signal is bandlimited in the frequency range from DC to some frequency Ω_T**
- **Such a continuous-time signal is commonly referred to as a lowpass signal**

Sampling of Bandpass Signals

- There are applications where the continuous-time signal is bandlimited to a higher frequency range $\Omega_L \le |\Omega| \le \Omega_H$ with $\Omega_L > 0$
- Such a signal is usually referred to as the **bandpass signal**
- To prevent aliasing a bandpass signal can of course be sampled at a rate greater than twice the highest frequency, i.e. by ensuring

$$\Omega_T \ge 2\,\Omega_H$$

Sampling of Bandpass Signals

- However, due to the bandpass spectrum of the continuous-time signal, the spectrum of the discrete-time signal obtained by sampling will have spectral gaps with no signal components present in these gaps
- Moreover, if Ω_H is very large, the sampling rate also has to be very large which may not be practical in some situations

Sampling of Bandpass Signals

- A more practical approach is to use under-sampling
- Let $\Delta\Omega = \Omega_H - \Omega_L$ define the bandwidth of the bandpass signal
- Assume first that the highest frequency Ω_H contained in the signal is an integer multiple of the bandwidth, i.e.,

$$\Omega_H = M(\Delta\Omega)$$

Sampling of Bandpass Signals

- We choose the sampling frequency Ω_T to satisfy the condition

$$\Omega_T = 2(\Delta\Omega) = 2\Omega_H/M$$

which is smaller than $2\Omega_H$, the **Nyquist rate**

- Substitute the above expression in

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\big(j(\Omega - k\Omega_T)\big)$$

Sampling of Bandpass Signals

- **This leads to**

$$G_p(j\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} G_a\left(j\Omega - j2k(\Delta\Omega)\right)$$

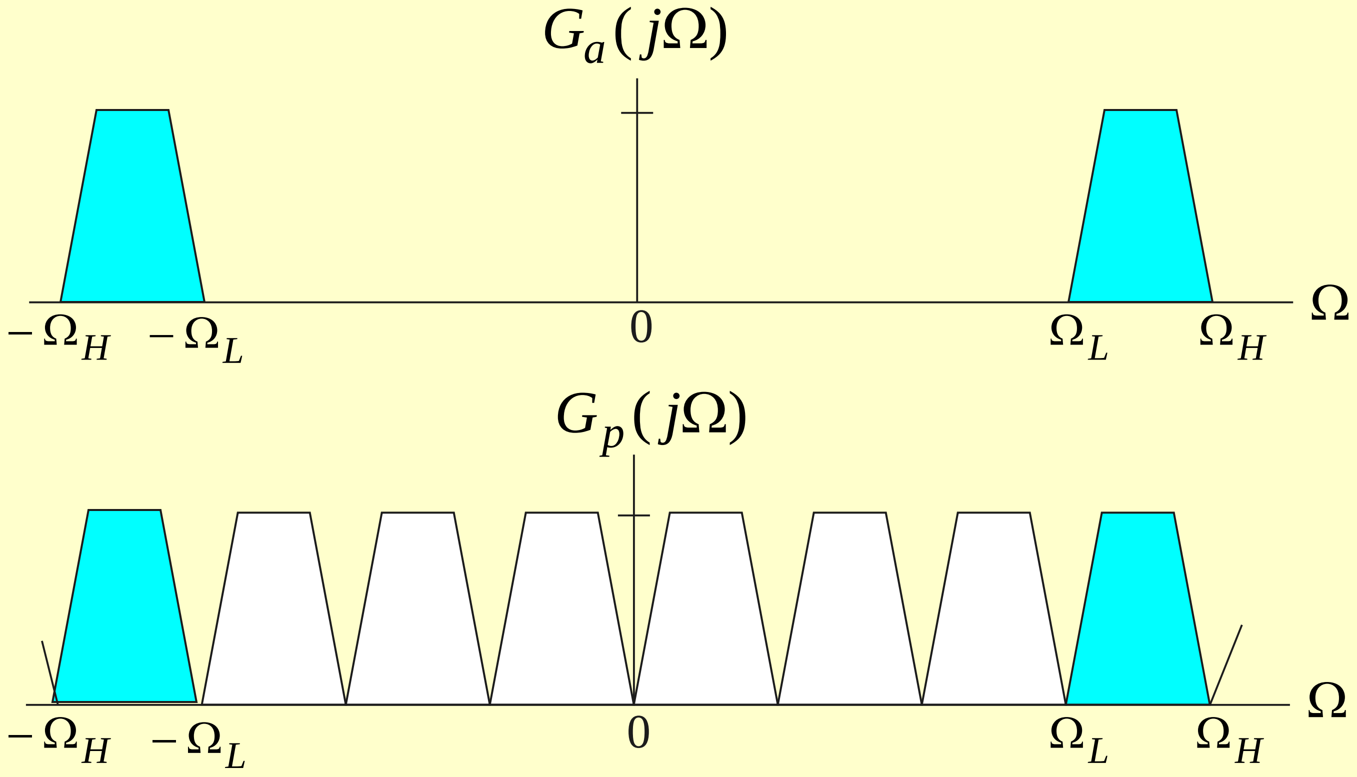
As before, $G_p(j\Omega)$ consists of a sum of $G_a(j\Omega)$ and replicas of $G_p(j\Omega)$ shifted by integer multiples of twice the bandwidth $\Delta\Omega$ and scaled by 1/T

- **The amount of shift for each value of k ensures that there will be no overlap between all shifted replicas**

→ **no aliasing**

Sampling of Bandpass Signals

- **Figure below illustrate the idea behind**



Sampling of Bandpass Signals

- As can be seen, $g_a(t)$ can be recovered from $g_p(t)$ by passing it through an ideal bandpass filter with a passband given by $\Omega_L \leq |\Omega| \leq \Omega_H$ and a gain of T
- Note: Any of the replicas in the lower frequency bands can be retained by passing through bandpass filters with passbands

  $\Omega_L - k(\Delta\Omega) \leq |\Omega| \leq \Omega_H - k(\Delta\Omega)$, $1 \leq k \leq M-1$
  providing a translation to lower frequency ranges

Exercise 3.16

- Determine the DTFT of each of the following sequences:

  (a) $x_1[n] = \alpha^n \mu[n-1], \qquad |\alpha| < 1$

  (c) $x_3[n] = \alpha^n \mu[n+1], \qquad |\alpha| < 1$

Exercise 3.60

- A 4.0s long segment of a continuous-time signal is uniformly sampled without aliasing and generating a finite-length sequence containing 8500 samples. What is the highest frequency component that could be present in the continuous-time signal?

Exercise 3.61

- A continuous-time signal x(t) is composed of a linear combination of sinusoidal signals of frequencies 300Hz, 500Hz, 1.2kHz, 2.15kHz, and 3.5kHz. The signal x(t) is sampled at a 3.0-kHz rate, and the sampled sequence is passed an ideal lowpass filter with a cutoff frequency of 900Hz, generating a continuous-time signal y(t). What are the frequency components present in the reconstructed signal y(t)?