Chapter 5

Finite-Length Discrete Transforms

Discrete-Time Fourier Transform

- Definition - The **discrete-time Fourier transform** (DTFT) $X(e^{j\omega})$ of a sequence $x[n]$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

where ω is a continuous variable in the range $-\infty < \omega < \infty$

The Discrete Fourier Transform (DFT)

- **DTFT is the Fourier Transform of discrete-time sequence. It is discrete in time domain and its spectrum is periodical, but continue which cannot be processed by computer which could only process digital signals in both sides, that means the signals in both sides must be both discrete and periodical.**

Discrete Fourier Transform (DFT)

Time domain **Frequency domain**

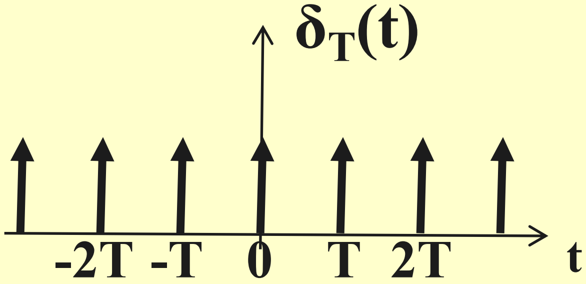
Continue Aperiodical ← CTFT → Continue Aperiodical

Discrete Aperiodical ← DTFT → Periodical Continue

Discrete Periodical ← DFT → Periodical Discrete

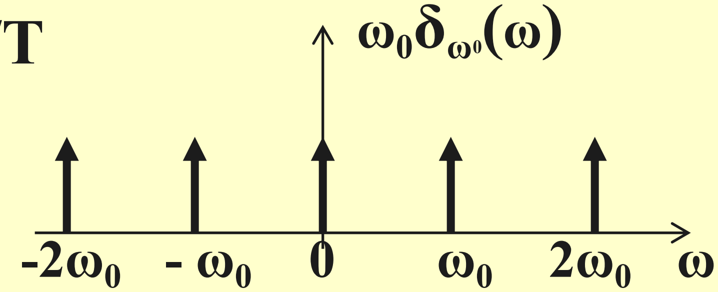
Typical DFT Pair

$\delta_T(t) \longleftrightarrow \omega_0\delta_{\omega_0}(\omega)$



$\omega_0 = 2\pi/T$

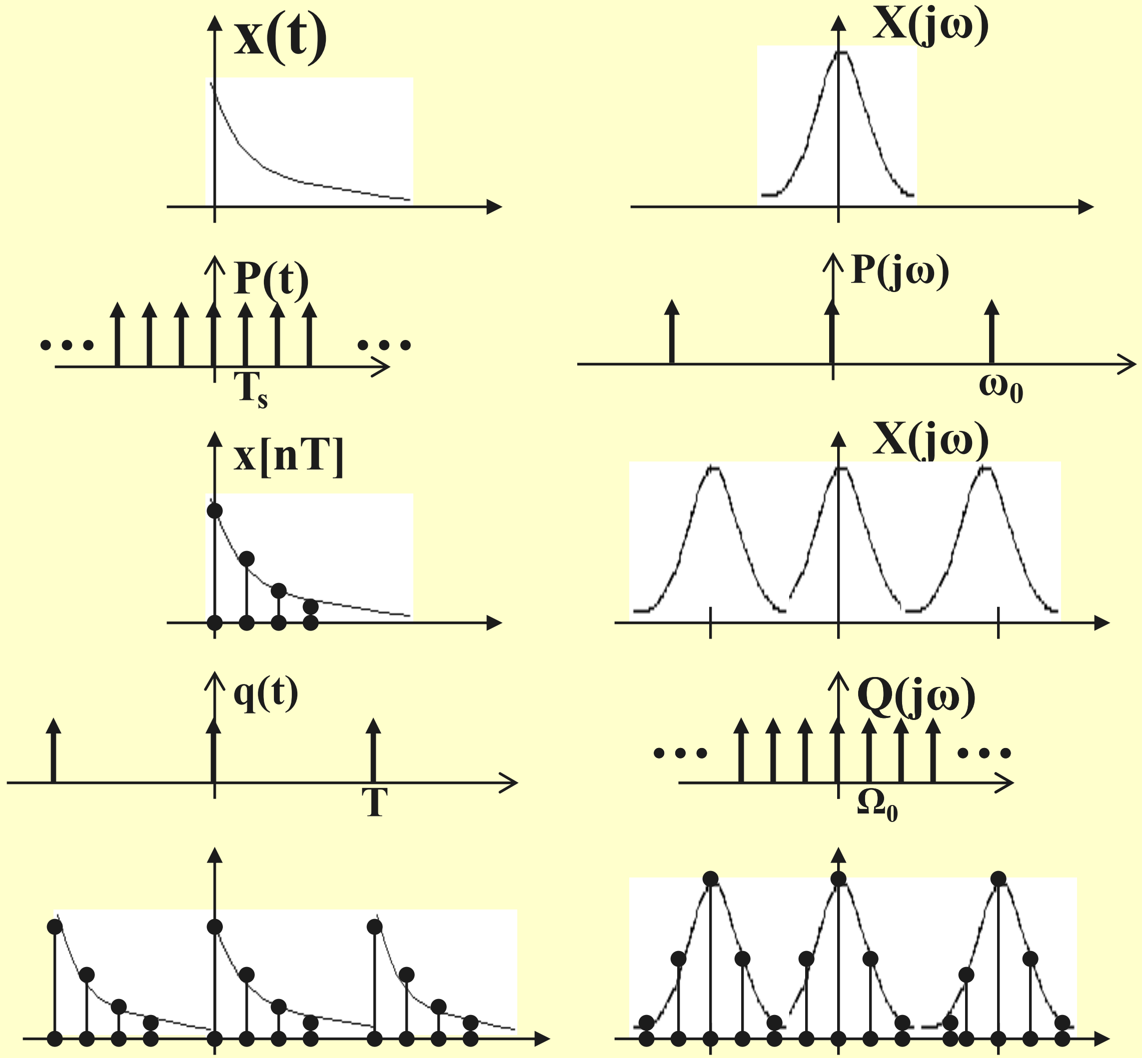
$\longleftrightarrow$



- **In DFT, the signals in both sides are discrete, so it is the only transform pair which can be processed by computer.**

- **The signals in both sides are periodical, so the processing could be in one period, which is important because (1) the number of calculation is limited, which is necessary for computer; (2) all of the signal information could be kept in one period, which is necessary for accurate processing.**

Make a signal discrete and periodical



CTFT

$\omega_0 = 2\pi / T_s$

DTFT

$\Omega_0 = 2\pi / T$

DFT

Make a signal discrete and periodical

- The engineering signals are often continue and aperiodical. If we want to process the signals with DFT, we have to make the signals discrete and periodical.

- Sampling to make the signal discrete
- Make the signal periodical:
  - If x[n] is a limited length N-point sequence, see it as one period of a periodical signal that means extend it to a periodical
  - If x[n] is an infinite length sequence, cut-off its tail to make a N-point sequence, then do the periodic extending. The tail cutting-off will introduce distortion. We must develop truncation algorithm to reduce the error, which is windowing.

Definition (DFT)

- **Definition - The simplest relation which can be obtained by uniformly sampling DTFT $X(e^{j\omega})$ of a length-N sequence x[n], defined for $0 \le n \le N-1$ on the ω-axis between $0 \le \omega \le 2\pi$ at $\omega_k = 2\pi k/N$, $0 \le k \le N-1$**
- **From the definition of the DTFT have**

$$X[k] = X(e^{j\omega})\Big|_{\omega = 2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad 0 \le k \le N-1$$

Definition (DFT)

- **<u>Note</u>: X[k] is also a length-N sequence in the frequency domain**
- **The sequence X[k] is called the discrete Fourier transform (DFT) of the sequence x[n]**
- **Using the notation $W_N = e^{-j2\pi/N}$, the DFT is usually expressed as:**

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \; 0 \le k \le N-1$$

Definition (DFT)

- The **inverse discrete Fourier transform (IDFT)** is given by

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn},\ \ 0\le n\le N-1$$

Discrete-Time Fourier Transform

- Therefore

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

represents the Fourier series representation of the periodic function

- As a result, the Fourier coefficients $x[n]$ can be computed from $X(e^{j\omega})$ using the Fourier integral

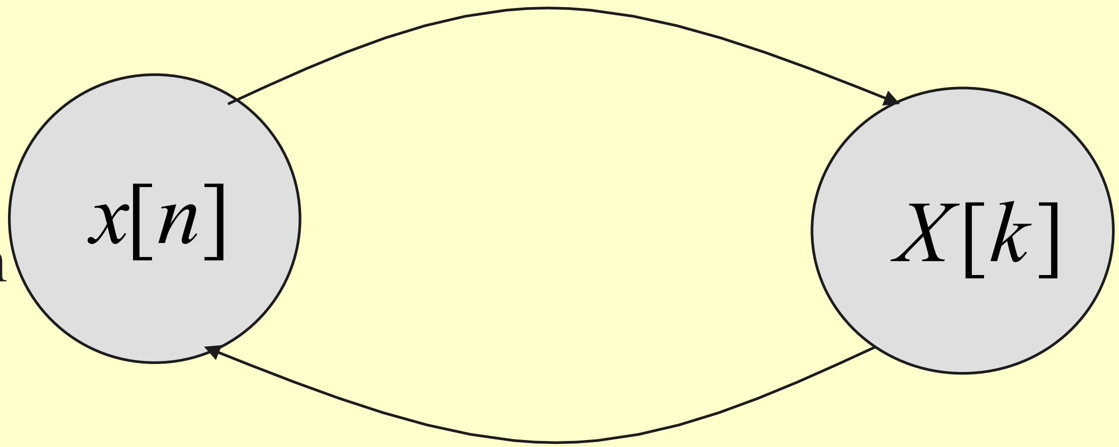
$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Definition (DFT)

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}, \quad 0 \le k \le N-1$$

DFT: analysis equation

time domain



frequency domain

IDFT: synthesis equation

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N}, \quad 0 \le n \le N-1$$

Definition (DFT)

- **Example 5.1 - Consider the length-*N* sequence**

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le N-1 \end{cases}$$

Its *N*-point DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1$$

$$0 \le k \le N-1$$

Definition (DFT)

- Now consider the length-N sequence

$$y[n] = \begin{cases} 1, & n = m \\ 0, & 0 \le n \le m-1,\ m+1 \le n \le N-1 \end{cases}$$

Its N-point DFT is given by

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{kn} = y[m] W_N^{km} = W_N^{km}$$

$W_N = e^{-j2\pi/N}$

$$0 \le k \le N-1$$

Definition (DFT)

- **Example 5.2-** **Consider the length-*N* sequence defined for 0 ≤n ≤N-1**

$$g[n] = \cos(2\pi rn/N),\ \ 0 \le r \le N-1$$

Using a trigonometric identity we can write 三角恒等式

$$g[n] = \frac{1}{2}\left(e^{j2\pi rn/N} + e^{-j2\pi rn/N}\right)$$

$$= \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

Definition (DFT)

- **The N-point DFT of $g[n]$ is thus given by**

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn}$$

$$= \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n} \right),$$

$$0 \le k \le N-1$$

Definition (DFT)

- **Making use of the identity**

$$\sum_{n=0}^{N-1} W_N^{-(k-\ \ )n} = \begin{cases} N, \text{ for } k - \quad = rN, & r \text{ an integer} \\ 0, \quad \text{otherwise} \end{cases}$$

we get

$$G[k] = \begin{cases} N/2, & \text{for } k = r \\ N/2, & \text{for } k = N - r \\ 0, & \text{otherwise} \end{cases}$$

$$0 \le k \le N-1$$

Definition (DFT)

- symmetry

$$(W_N^{nk})^* = W_N^{-nk}$$

- periodicity

$$W_N^{nk} = W_N^{(n+N)k} = W_N^{n(k+N)}$$

- divisibility

$$W_N^{nk} = W_{mN}^{mnk} = W_{N/m}^{nk/m}$$

Definition (DFT)

- **Example- Consider the length-8 sequence defined for** $x[n]=\begin{cases}1, 0\le n\le 5\\ 0, 6\le n\le 7\end{cases}$ **, its 8-point DFT is X[k], Determine the values of X[0], X[4] and** $\sum_{k=0}^{7} X[k]$

$$\because X[k]=\sum_{n=0}^{N-1} x[n]W_N^{kn},\ 0\le k\le N-1$$

$$\Rightarrow X[0]=\sum_{n=0}^{7} x[n]W_N^{0}=\sum_{n=0}^{7} x[n]=6$$

Definition (DFT)

$$\Rightarrow X[4] = \sum_{n=0}^{7} x[n] W_8^{4n} = \sum_{n=0}^{7} x[n] W_2^{n} =$$

$$\sum_{n=0}^{7} x[n] e^{-j\frac{2\pi}{2}n} = \sum_{n=0}^{7} x[n] e^{-j\pi n} = \sum_{n=0}^{7} x[n] (-1)^n = 0$$

$$\mathrm{Q}\ x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn},\ \ 0 \le n \le N-1$$

$$\Rightarrow x[0] = \frac{1}{8} \sum_{k=0}^{7} X[k] W_8^{0} = \frac{1}{8} \sum_{k=0}^{7} X[k]$$

$$\Rightarrow \sum_{k=0}^{7} X[k] = 8x[0] = 8$$

Matrix Relations

- The DFT samples defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

Can be expressed in matrix form as

$$X = D_N x$$

where

$$X = [X[0]\;\; X[1] \ldots .\; X[N-1]]^T$$

$$x = [x[0]\;\; x[1] \ldots .\; x[N-1]]^T$$

Matrix Relations

- D_N **is the N×N DFT matrix given by**

$$D_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

Matrix Relations

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \ldots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & W_N^1 & W_N^2 & \ldots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \ldots & W_N^{2(N-1)} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \ldots & W_N^{(N-1)^2} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \ldots \\ x[N-1] \end{bmatrix}$$

Matrix Relations

- **Likewise, the IDFT relation given by**

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}, \quad 0 \le n \le N-1$$

Can be expressed in matrix form as

$$x = D_N^{-1} X$$

Where D_N^{-1} **is N×N IDFT matrix**

Matrix Relations

- **where**

$$D_N^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \dots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \dots & W_N^{-2(N-1)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \dots & W_N^{-(N-1)^2} \end{bmatrix}$$

Note

$$D_N^{-1} = \frac{1}{N} D_N^{*}$$

Matrix Relations

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \ldots \\ x[N-1] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 & \\ 1 & W_N^{-1} & W_N^{-2} & \ldots & W_N^{-(N-1)} & \\ 1 & W_N^{-2} & W_N^{-4} & \ldots & W_N^{-2(N-1)} & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \ldots & W_N^{-(N-1)^2} & \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ \ldots \\ X[N-1] \end{bmatrix}$$

DFT Computation Using MATLAB

- The functions to compute the DFT and the IDFT are FFT and IFFT
- These functions make use of FFT algorithms which are computationally highly efficient compared to the direct computation
- Programs 5_1.m and 5_2.m illustrate the use of these functions

Relation Between the DTFT and the DFT, and Their Inverses

- **We now examine the relation between the DTFT and the N-point DFT of a length-N sequence, and the relation between the DTFT of length-M sequence and the N-point DFT obtained by sampling the DTFT.**

Relation with the DTFT

- **The DTFT $X(e^{j\omega})$ of the length-N sequence x[n], defined for 0≤n≤N-1, is given by**

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=0}^{N-1} x[n]e^{-j\omega n}$$

- **By uniformly sampling $X(e^{j\omega})$ at N equally spaced frequency $\omega_k = 2\pi k / N, \quad 0 \le k \le N-1,$ on the ω-axis between 0 ≤ ω<2π, we get**

$$X(e^{j\omega})\big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n]e^{-j2\pi k/N} = X[k],$$

$$0 \le k \le N-1$$

Relation with the DTFT

- **We observe that the N-point DFT sequence X[k] is precisely the set of frequency samples of the DTFT $X(e^{j\omega})$ of the length-N sequence x[n] at N equally spaced frequencies:**

$$\omega_k = 2\pi k / N, \quad 0 \le k \le N-1.$$

Numerical Computation of the DTFT Using the DFT

- **A practical approach to the numerical computation of the DTFT of a finite-length sequence**
- **Let** $X(e^{j\omega})$ **be the DTFT of a length-*N* sequence** $x[n]$
- **We wish to evaluate** $X(e^{j\omega})$ **on a dense grid of frequencies** $\omega_k = 2\pi k / M$

  $0 \le k \le M-1$
- **where** $M >> N$**:**

Numerical Computation of the DTFT Using the DFT

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/M}$$

- Define a new sequence

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

- Then

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x_e[n] \;\; e^{-j2\pi kn/M}$$

Numerical Computation of the DTFT Using the DFT

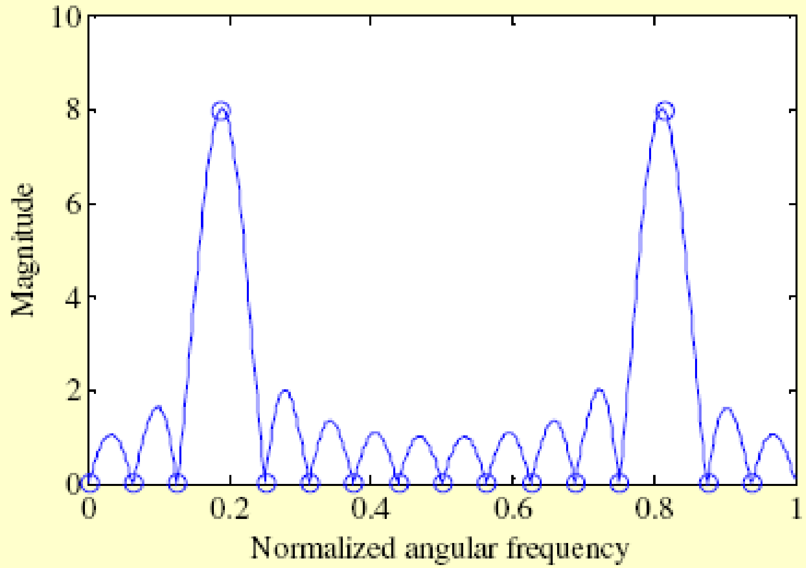
- **Thus $X(e^{j\omega_k})$ is essentially an *M*-point DFT $X_e[k]$ of the length-*M* sequence $x_e[n]$**
- **The DFT $X_e[k]$ can be computed very efficiently using the FFT algorithm if *M* is an integer power of 2**
- **The function freqz employs this approach to evaluate the frequency response at a prescribed set of frequencies of a DTFT expressed as a rational function of $e^{-j\omega}$**

DFT Computation Using MATLAB

- Example - Program 5_3.m can be used to compute the DFT and the DTFT of the sequence

$$x[n] = \cos(6\pi n/16), \quad 0 \le n \le 15$$

as shown below



○ indicates DFT samples

DTFT from DFT by Interpolation

- The *N*-point DFT $X[k]$ of a length-*N* sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at *N* uniformly spaced frequency points

$$\omega = \omega_k = 2\pi k/N, \quad 0 \le k \le N-1$$

- Given the *N*-point DFT $X[k]$ of a length-*N* sequence $x[n]$, its DTFT $X(e^{j\omega})$ can be uniquely determined from $X[k]$

DTFT from DFT by Interpolation

- **Thus**

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k / N)n}$$

DTFT from DFT by Interpolation

- **It can readily be shown that**

$$X(e^{j\omega}) =$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} \cdot e^{-j[(\omega - 2\pi k/N)][(N-1)/2]}$$

Sampling the DTFT

- Consider a sequence $x[n]$ with a DTFT $X(e^{j\omega})$
- We sample $X(e^{j\omega})$ at N equally spaced points $\omega_k = 2\pi k / N$, $0 \le k \le N-1$ developing the N frequency samples $\{X(e^{j\omega_k})\}$
- These N frequency samples can be considered as an N-point DFT $Y[k]$ whose N-point IDFT is a length-N sequence $y[n]$, $0 \le n \le N-1$

Sampling the DTFT

- Now $X(e^{j\omega}) = \sum_{=-\infty}^{\infty} x[\ ]e^{-j\omega}$

- Thus $Y[k] = X(e^{j\omega_k}) = X(e^{j2\pi k/N})$

$$= \sum_{=-\infty}^{\infty} x[\ ]e^{-j2\pi k\ /N} = \sum_{=-\infty}^{\infty} x[\ ]W_N^{k}$$

- An IDFT of $Y[k]$ yields

$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} Y[k]W_N^{-kn}$$

Sampling the DTFT

- i.e. $$y[n] = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{=-\infty}^{\infty} x[\;\;]W_N^{k\;\;} W_N^{-kn}$$

$$= \sum_{=-\infty}^{\infty} x[\;\;]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-\;\;)}\right]$$

- Making use of the identity

$$\frac{1}{N}\sum_{n=0}^{N-1} W_N^{-k(n-r)} = \begin{cases} 1, & \text{for } r = n + mN \\ 0, & \text{otherwise} \end{cases}$$

Sampling the DTFT

we arrive at the desired relation

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \;\; 0 \le n \le N-1$$

- Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0 \le n \le N-1$

Sampling the DTFT

- To apply

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1$$

  to finite-length sequences, we assume that the samples outside the specified range are zeros
- Thus if $x[n]$ is a length-M sequence with $M \le N$, then $y[n] = x[n]$ for $0 \le n \le N-1$

Sampling the DTFT

- If $M > N$, there is a ***time-domain aliasing*** of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$
- Example - Let $\{x[n]\} = \{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$
- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = 2\pi k/4$, $0 \le k \le 3$ and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

Sampling the DTFT

$$y[n] = x[n] + x[n+4] + x[n-4], \quad 0 \le n \le 3$$

i.e.

$$\{y[n]\} = \{\underset{\uparrow}{4} \quad 6 \quad 2 \quad 3\}$$

- $\{x[n]\}$ cannot be recovered from $\{y[n]\}$

Table 5.1: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[\langle -k\rangle_N]$
$x^*[\langle -n\rangle_N]$	$X^*[k]$
$\mathrm{Re}\{x[n]\}$	$X_{\mathrm{pcs}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] + X^*[\langle -k\rangle_N]\}$
$j\,\mathrm{Im}\{x[n]\}$	$X_{\mathrm{pca}}[k] = \frac{1}{2}\{X[\langle k\rangle_N] - X^*[\langle -k\rangle_N]\}$
$x_{\mathrm{pcs}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{pca}}[n]$	$j\,\mathrm{Im}\{X[k]\}$

Note: $x_{\mathrm{pcs}}[n]$ and $x_{\mathrm{pca}}[n]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $x[n]$, respectively. Likewise, $X_{\mathrm{pcs}}[k]$ and $X_{\mathrm{pca}}[k]$ are the periodic conjugate-symmetric and periodic conjugate-antisymmetric parts of $X[k]$, respectively.

$x[n]$ is a complex sequence

Table 5.2: DFT Properties: Symmetry Relations

Length-N Sequence	N-point DFT
$x[n]$	$X[k] = \mathrm{Re}\{X[k]\} + j\,\mathrm{Im}\{X[k]\}$
$x_{\mathrm{pe}}[n]$	$\mathrm{Re}\{X[k]\}$
$x_{\mathrm{po}}[n]$	$j\,\mathrm{Im}\{X[k]\}$
Symmetry relations	$X[k] = X^*[\langle -k\rangle_N]$ $\mathrm{Re}\,X[k] = \mathrm{Re}\,X[\langle -k\rangle_N]$ $\mathrm{Im}\,X[k] = -\,\mathrm{Im}\,X[\langle -k\rangle_N]$ $\lvert X[k]\rvert = \lvert X[\langle -k\rangle_N]\rvert$ $\arg X[k] = -\arg X[\langle -k\rangle_N]$

Note: $x_{\mathrm{pe}}[n]$ and $x_{\mathrm{po}}[n]$ are the periodic even and periodic odd parts of $x[n]$, respectively.

$x[n]$ is a real sequence

DFT Theorems

- Like the DTFT, the DFT also satisfies a number of properties that are useful in signal processing applications
- Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different
- A summary of the DFT properties are given in tables in the following slides

DFT Theorems

Type of Property	length-N sequence	N-point DFT
	$g[n]$	$G[k]$
	$h[n]$	$H[k]$
Linearity	$\alpha g[n]+\beta h[n]$	$\alpha G[k]+\beta H[k]$
Circular Time-shifting	$g[\langle n-n_0\rangle_N]$	$W_N^{kn_0} G[k]$
Frequency-shifting	$W_N^{-k_0 n} g[n]$	$G[\langle k-k_0\rangle_N]$
Duality	$G[n]$	$N[g\langle -k\rangle_N]$
Circular Convolution	$\sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N]$	$G[k]H[k]$
Modulation	$g[n]h[n]$	$\frac{1}{N}\sum_{m=0}^{N-1} G[m]H[\langle k-m\rangle_N]$
Parseval's relation	$\sum_{n=0}^{N-1} \|x[n]\|^2 = \frac{1}{N}\sum_{k=0}^{N-1} \|X[k]\|^2$	

Operations on Finite-Length Sequences

- Like the DTFT, the DFT also satisfies a number of properties that are useful processing applications. Some of these properties are essentially identical to those of the DTFT, while some others are somewhat different.
- We first point out the differences between two important operations on sequences and then review the properties of the DFT.

Circular Shift of a Sequence

- **This property is analogous to the time-shifting property of the DTFT , but with a subtle difference**
- **Consider length-*N* sequences defined for**

$$0 \leq n \leq N-1$$

- **Sample values of such sequences are equal to zero for values of $n < 0$ and**

$n \geq N$

Circular Shift of a Sequence

- **If x[n] is such a sequence, then for any arbitrary integer n_0 , the shifted sequence**

$$x_1[n] = x[n - n_0]$$

  is no longer defined for the range $0 \le n \le N-1$

- **We thus need to define another type of a shift that will always keep the shifted sequence in the range $0 \le n \le N-1$**

Circular Shift of a Sequence

- **The desired shift, called the circular shift, is defined using a modulo operation:**

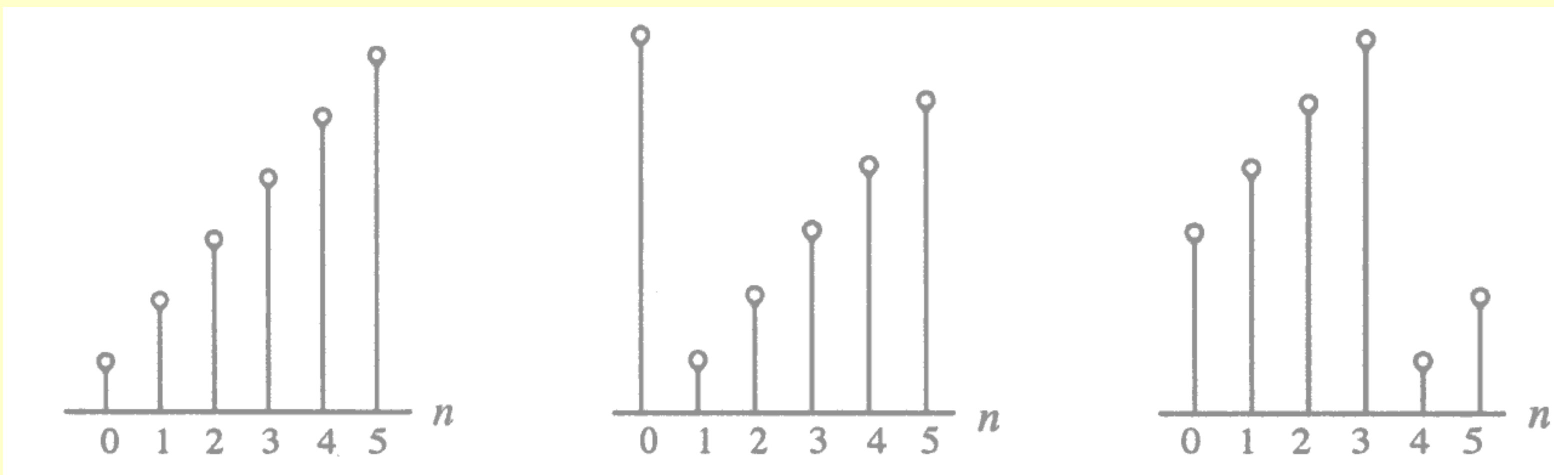
$$x_c[n] = x[\langle n - n_o \rangle_N]$$

For $n_0>0$ (right circular shift), the above equation implies

$$x_c[n] = \begin{cases} x[n-n_o], & \text{for } n_o \le n \le N-1 \\ x[N-n_o+n], & \text{for } 0 \le n < n_o \end{cases}$$

Circular Shift of a Sequence

➢ **Illustration of the concept of a circular shift**



$x[n]$

$x[\langle n-1\rangle_6]$

$= x[\langle n+5\rangle_6]$

$x[\langle n-4\rangle_6]$

$= x[\langle n+2\rangle_6]$

Circular Shift of a Sequence

- **As can be seen from the previous figure, a right circular shift by n_0 is equivalent to a left circular shift by $N-n_0$ sample periods**
- **A circular shift by an integer number n_0 greater than N is equivalent to a circular shift by $\langle n_0 \rangle_N$**

Circular Convolution

- **This operation is analogous to linear convolution, but with a subtle difference**
- **Consider two length-N sequences, g[n] and h[n], respectively**
- **Their linear convolution results in a length-(2N-1) sequence $y_L[n]$ given by**

$$y_L[n] = \sum_{m=0}^{N-1} g[m]h[n-m], \quad 0 \le n \le 2N-2$$

Circular Convolution

- **In computing $y_L[n]$ we have assumed that both length-N sequences have been zero-padded to extend their lengths to 2N-1**
- **The longer form of $y_L[n]$ results from the time-reversal of the sequence $h[n]$ and its linear shift to the right**
- **The first nonzero value of $y_L[n]$ is $y_L[0]=g[0]h[0]$, and the last nonzero value is $y_L[2N-2]=g[N-1]h[N-1]$**

Circular Convolution

- To develop a convolution-like operation resulting in a **length-*N*** sequence $y_C[n]$, we need to define a circular time-reversal, and then apply a circular time-shift
- Resulting operation, called a **circular convolution**, is defined by

$$y_C[n] = \sum_{m=0}^{N-1} g[m]h[\langle n-m\rangle_N], \quad 0 \le n \le N-1$$

Circular Convolution

- **Since the operation defined involves two length-N sequences, it is often referred to as an N-point circular convolution, denoted as**

$$y[n] = g[n] \, \textcircled{N} \, h[n]$$

- **The circular convolution is commutative, i.e.**

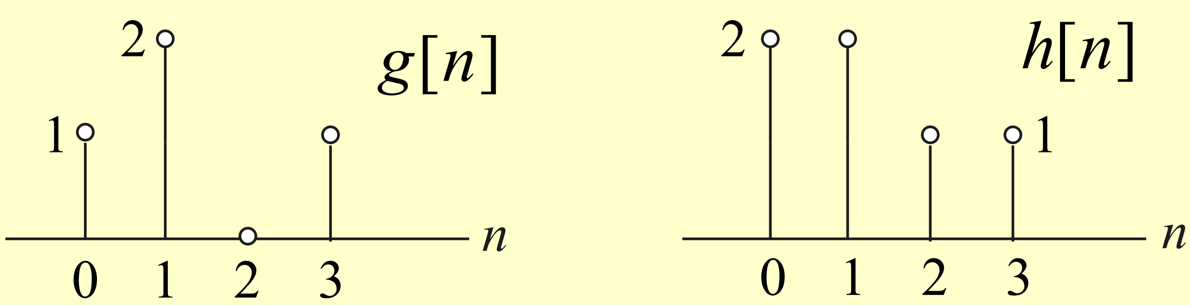
$$g[n] \, \textcircled{N} \, h[n] = h[n] \, \textcircled{N} \, g[n]$$

Circular Convolution

- **<u>Example</u> - Determine the 4-point circular convolution of the two length-4 sequences:**

$$\{g[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}, \quad \{h[n]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

as sketched below



Circular Convolution

- **The result is a length-4 sequence $y_C[n]$ given by**

$$y_C[n] = g[n] \, ④ \, h[n] = \sum_{m=0}^{3} g[m] h[\langle n-m \rangle_4],$$

$$0 \le n \le 3$$

From the above we observe

$$y_C[0] = \sum_{m=0}^{3} g[m] h[\langle -m \rangle_4]$$

$$= g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1]$$

$$= (1 \times 2) + (2 \times 1) + (0 \times 1) + (1 \times 2) = 6$$

Circular Convolution

- **Likewise** $y_C[1] = \sum_{m=0}^{3} g[m]h[\langle 1-m \rangle_4]$

$$= g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2]$$

$$= (1\times 2) + (2\times 2) + (0\times 1) + (1\times 1) = 7$$

$$y_C[2] = \sum_{m=0}^{3} g[m]h[\langle 2-m \rangle_4]$$

$$= g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3]$$

$$= (1\times 1) + (2\times 2) + (0\times 2) + (1\times 1) = 6$$

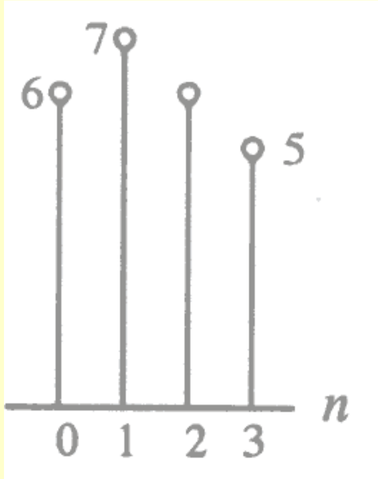
Circular Convolution

$$y_C[3] = \sum_{m=0}^{3} g[m]h[\langle 3-m \rangle_4]$$

$$= g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5$$

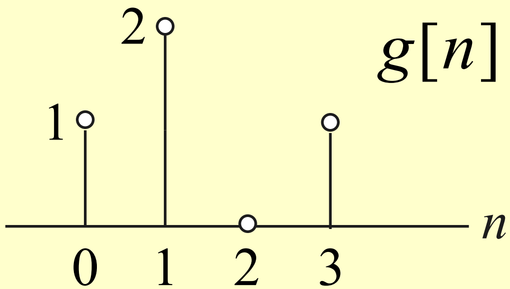
- **The circular convolution can also be computed using a DFT-based approach**

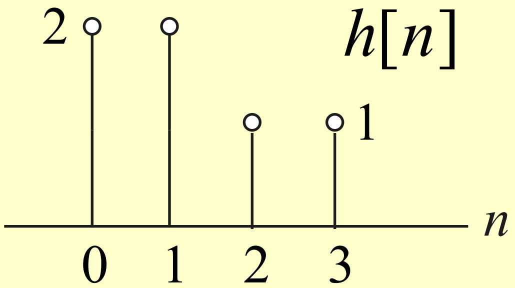


$y_C[n]$

Circular Convolution

- **<u>Example</u> - Consider the two length-4 sequences repeated below for convenience:**





The 4-point DFT G[k] of g[n] is given by

$$
\begin{aligned}
G[k] &= g[0] + g[1]e^{-j2\pi k/4} \\
&\quad + g[2]e^{-j4\pi k/4} + g[3]e^{-j6\pi k/4} \\
&= 1 + 2e^{-j\pi k/2} + e^{-j3\pi k/2}, \quad 0 \le k \le 3
\end{aligned}
$$

Circular Convolution

- **Therefore**

$$G[0] = 1 + 2 + 1 = 4,$$
$$G[1] = 1 - j2 + j = 1 - j,$$
$$G[2] = 1 - 2 - 1 = -2,$$
$$G[3] = 1 + j2 - j = 1 + j$$

Likewise,

$$H[k] = h[0] + h[1]e^{-j2\pi k/4} + h[2]e^{-j4\pi k/4} + h[3]e^{-j6\pi k/4}$$
$$= 2 + 2e^{-j\pi k/2} + e^{-j\pi k} + e^{-j3\pi k/2}, \;\; 0 \le k \le 3$$

Circular Convolution

- Hence,
$$H[0] = 2 + 2 + 1 + 1 = 6,$$
$$H[1] = 2 - j2 - 1 + j = 1 - j,$$
$$H[2] = 2 - 2 + 1 - 1 = 0,$$
$$H[3] = 2 + j2 - 1 - j = 1 + j$$

- The two 4-point DFTs can also be computed using the matrix relation given earlier

Circular Convolution

$$\begin{bmatrix} G[0] \\ G[1] \\ G[2] \\ G[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ g[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$\begin{bmatrix} H[0] \\ H[1] \\ H[2] \\ H[3] \end{bmatrix} = \mathbf{D}_4 \begin{bmatrix} h[0] \\ h[1] \\ h[2] \\ h[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$\mathbf{D}_4$ is the 4-point DFT matrix

Circular Convolution

- If $Y_C[k]$ denotes the 4-point DFT of $y_C[n]$ then from Table 5.3 we observe

$$Y_C[k] = G[k]H[k], \ \ 0 \le k \le 3$$

- Thus

$$\begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix} = \begin{bmatrix} G[0]H[0] \\ G[1]H[1] \\ G[2]H[2] \\ G[3]H[3] \end{bmatrix} = \begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix}$$

Circular Convolution

- The 4-point IDFT of $Y_C[k]$ yields

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ y_C[3] \end{bmatrix} = \frac{1}{4}\mathbf{D}_4^* \begin{bmatrix} Y_C[0] \\ Y_C[1] \\ Y_C[2] \\ Y_C[3] \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 24 \\ -j2 \\ 0 \\ j2 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix}$$

Circular Convolution

- Example - Now let us extended the two length-4 sequences to length 7 by appending each with three zero-valued samples, i.e.

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le 3 \\ 0, & 4 \le n \le 6 \end{cases}$$

Circular Convolution

- We next determine the 7-point circular convolution of $g_e[n]$ and $h_e[n]$:

$$y[n] = \sum_{m=0}^{6} g_e[m] h_e[\langle n-m \rangle_7], \quad 0 \le n \le 6$$

- From the above $y[0] = g_e[0]h_e[0] + g_e[1]h_e[6]$

$$+ g_e[3]h_e[4] + g_e[4]h_e[3] + g_e[5]h_e[2] + g_e[6]h_e[1]$$

$$= g[0]h[0] = 1 \times 2 = 2$$

Circular Convolution

- Continuing the process we arrive at

$$y[1] = g[0]h[1] + g[1]h[0] = (1\times 2) + (2\times 2) = 6,$$

$$y[2] = g[0]h[2] + g[1]h[1] + g[2]h[0]$$
$$= (1\times 1) + (2\times 2) + (0\times 2) = 5,$$

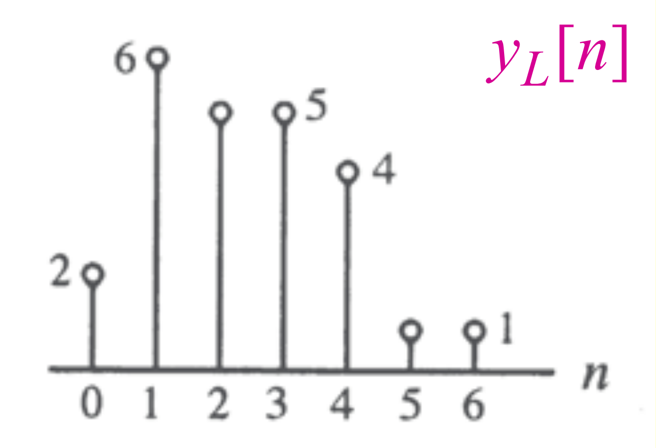
$$y[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$
$$= (1\times 1) + (2\times 1) + (0\times 2) + (1\times 2) = 5,$$

$$y[4] = g[1]h[3] + g[2]h[2] + g[3]h[1]$$
$$= (2\times 1) + (0\times 1) + (1\times 2) = 4,$$

Circular Convolution

$$y[5] = g[2]h[3] + g[3]h[2] = (0 \times 1) + (1 \times 1) = 1,$$
$$y[6] = g[3]h[3] = (1 \times 1) = 1$$

- As can be seen from the above that $y[n]$ is precisely the sequence $y_L[n]$ obtained by a linear convolution of $g[n]$ and $h[n]$



Circular Convolution

- The N-point circular convolution can be written in matrix form as

$$\begin{bmatrix} y_C[0] \\ y_C[1] \\ y_C[2] \\ \vdots \\ y_C[N-1] \end{bmatrix} = \begin{bmatrix} h[0] & h[N-1] & h[N-2] & \cdots & h[1] \\ h[1] & h[0] & h[N-1] & \cdots & h[2] \\ h[2] & h[1] & h[0] & \cdots & h[3] \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h[N-1] & h[N-2] & h[N-3] & \cdots & h[0] \end{bmatrix} \begin{bmatrix} g[0] \\ g[1] \\ g[2] \\ \vdots \\ g[N-1] \end{bmatrix}$$

- Note: The elements of each diagonal of the $N \times N$ matrix are equal

循环行列式

- Such a matrix is called a ***circulant matrix***

Circular Convolution

- Tabular Method
- We illustrate the method by an example
- Consider the evaluation of $y[n] = h[n] \circledast g[n]$ where $\{g[n]\}$ and $\{h[n]\}$ are length-4 sequences
- First, the samples of the two sequences are multiplied using the conventional multiplication method as shown on the next slide

Circular Convolution

n:	0	1	2	3	$\langle 4\rangle_4$	$\langle 5\rangle_4$	$\langle 6\rangle_4$
$g[n]$:	$g[0]$	$g[1]$	$g[2]$	$g[3]$			
$h[n]$	$h[0]$	$h[1]$	$h[2]$	$h[3]$			
	$g[0]h[0]$	$g[1]h[0]$	$g[2]h[0]$	$g[3]h[0]$			
		$g[0]h[1]$	$g[1]h[1]$	$g[2]h[1]$	$\boxed{g[3]h[1]}$		
			$g[0]h[2]$	$g[1]h[2]$	$\boxed{g[2]h[2]}$	$\boxed{g[3]h[2]}$	
				$g[0]h[3]$	$\boxed{g[1]h[3]}$	$\boxed{g[2]h[3]}$	$\boxed{g[3]h[3]}$

The partial products generated in the 2nd, 3rd, and 4th rows are circularly shifted to the left as indicated above

Circular Convolution

- The modified table after circular shifting is shown below

n:	0	1	2	3
$g[n]$:	$g[0]$	$g[1]$	$g[2]$	$g[3]$
$h[n]$:	$h[0]$	$h[1]$	$h[2]$	$h[3]$
	$g[0]h[0]$	$g[1]h[0]$	$g[2]h[0]$	$g[3]h[0]$
	$g[3]h[1]$	$g[0]h[1]$	$g[1]h[1]$	$g[2]h[1]$
	$g[2]h[2]$	$g[3]h[2]$	$g[0]h[2]$	$g[1]h[2]$
	$g[1]h[3]$	$g[2]h[3]$	$g[3]h[3]$	$g[0]h[3]$
$y_c[n]$:	$y_c[0]$	$y_c[1]$	$y_c[2]$	$y_c[3]$

- The samples of the sequence $\{y_c[n]\}$ are obtained by adding the 4 partial products in the column above of each sample

Circular Convolution

- Thus

$$y_c[0] = g[0]h[0] + g[3]h[1] + g[2]h[2] + g[1]h[3]$$

$$y_c[1] = g[1]h[0] + g[0]h[1] + g[3]h[2] + g[2]h[3]$$

$$y_c[2] = g[2]h[0] + g[1]h[1] + g[0]h[2] + g[3]h[3]$$

$$y_c[3] = g[3]h[0] + g[2]h[1] + g[1]h[2] + g[0]h[3]$$

Computation of the DFT of Real Sequences

- **In most practical applications, sequences of interest are real. In such cases, the symmetry properties of the DFT given in Tables 5.1 and 5.2 can be exploited to make the DFT computations more efficient.**
- **The N-point DFTs of two length-N real sequences can be computed from a single N-point DFT of a length-N complex sequence formed from the two length-N real sequences;**
- **The 2N-point DFT of a length-2N real sequence can be determined from a single N-point DFT of a length-N complex sequence formed from the length-2N real sequence.**

N-point DFTs of Two Real Sequences Using a Signal N-Point DFT

- **Let g[n] and h[n] be two real sequences of length N each, with G[k] and H[k] denoting their respective N-point DFTs. These two N-point DFTs can be computed efficiently using a signle N-point DFT X[k] of a complex length-N sequence defined by**

$$x[n] = g[n] + j\,h[n]$$

where, g[n]=Re{x[n]} and h[n]=Im{x[n]}

N-point DFTs of Two Real Sequences Using a Signal N-Point DFT

From Table 5.1, we arrive at

$$G[k] = \frac{1}{2}\{X[k] + X^*[<-k>_N]\}$$

$$H[k] = \frac{1}{2j}\{X[k] - X^*[<-k>_N]\}$$

Note that

$$X^*[<-k>_N] = X^*[<N-k>_N]$$

N-point DFTs of Two Real Sequences Using a Signal N-Point DFT

- **Example** – **Computation of the 4-point DFTs of the two length-4 real sequences:**

$$\{g[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}, \quad \{h[n]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

We can form a N-point complex sequence x[n]:

$$\begin{aligned} x[n] &= g[n] + j\,h[n] \\ &= \{\underset{\uparrow}{1+j2}, \quad 2+j2 \quad j \quad 1+j\} \end{aligned}$$

N-point DFTs of Two Real Sequences Using a Signal N-Point DFT

Its DFT is then:

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1+j2 \\ 2+j2 \\ j \\ 1+j \end{bmatrix} = \begin{bmatrix} 4+j6 \\ 2 \\ -2 \\ j2 \end{bmatrix}$$

From the above, we have

$$X^*[k] = [4-j6 \quad 2 \quad -2 \quad -j2]$$

Therefore

$$X^*[<4-k>_4] = [4-j6 \quad -j2 \quad -2 \quad 2]$$

Then we get

$$G[k] = \frac{1}{2}\{X[k] + X^*[< -k >_N]\}$$
$$= [4 \quad 1-j \quad -2 \quad 1+j]$$

$$H[k] = \frac{1}{2j}\{X[k] - X^*[< -k >_N]\}$$
$$= [6 \quad 1-j \quad 0 \quad -1+j]$$

2N-point DFT of a Real Sequences Using a Signal N-Point DFT

- **Let v[n] be a real sequences of length 2N with V[k] denoting its 2N-point DFT. Define two real sequences g[n] and h[n] of length-N each as**

$$g[n] = v[2n], \qquad h[n] = v[2n+1], \quad 0 \le n < N$$

With G[k] and H[k] denoting their N-point DFTs.

Therefore, the 2N-point DFT V[k] has

$$V[k] = G[<k>_N] + W_{2N}^{k} H[<k>_N], \quad 0 \le k \le 2N-1$$

2N-point DFT of a Real Sequences Using a Signal N-Point DFT

- Define a length-N complex sequence

$$\{x[n]\} = \{g[n]\} + j\{h[n]\}$$

with an N-point DFT $X[k]$

- Then as shown earlier

$$G[k] = \frac{1}{2}\{X[k] + X^*[\langle -k \rangle_N]\}$$

$$H[k] = \frac{1}{2j}\{X[k] - X^*[\langle -k \rangle_N]\}$$

2N-point DFT of a Real Sequences Using a Signal N-Point DFT

- Now $V[k] = \sum_{n=0}^{2N-1} v[n] W_{2N}^{nk}$

$$= \sum_{n=0}^{N-1} v[2n] W_{2N}^{2nk} + \sum_{n=0}^{N-1} v[2n+1] W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} g[n] W_{N}^{nk} + \sum_{n=0}^{N-1} h[n] W_{N}^{nk} W_{2N}^{k}$$

$$= \sum_{n=0}^{N-1} g[n] W_{N}^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} h[n] W_{N}^{nk}, \; 0 \le k \le 2N-1$$

2N-Point DFT of a Real Sequence Using an N-point DFT

- i.e.,

$$V[k] = G[\langle k \rangle_N] + W_{2N}^{k} H[\langle k \rangle_N], \quad 0 \le k \le 2N-1$$

- Example - Let us determine the 8-point DFT V[k] of the length-8 real sequence

$$\{v[n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 2 \quad 2 \quad 0 \quad 1 \quad 1 \quad 1\}$$

- We form two length-4 real sequences as follows

2*N*-Point DFT of a Real Sequence Using an *N*-point DFT

$$\{g[n]\} = \{v[2n]\} = \{\underset{\uparrow}{1} \quad 2 \quad 0 \quad 1\}$$

$$\{h[n]\} = \{v[2n+1]\} = \{\underset{\uparrow}{2} \quad 2 \quad 1 \quad 1\}$$

- Now

$$V[k] = G[\langle k \rangle_4] + W_8^k H[\langle k \rangle_4], \quad 0 \le k \le 7$$

- Substituting the values of the 4-point DFTs $G[k]$ and $H[k]$ computed earlier we get

2N-Point DFT of a Real Sequence Using an N-point DFT

$$V[0] = G[0] + H[0] = 4 + 6 = 10$$

$$V[1] = G[1] + W_8^1 H[1]$$

$$= (1 - j) + e^{-j\pi/4}(1 - j) = 1 - j2.4142$$

$$V[2] = G[2] + W_8^2 H[2] = -2 + e^{-j\pi/2} \cdot 0 = -2$$

$$V[3] = G[3] + W_8^3 H[3]$$

$$= (1 + j) + e^{-j3\pi/4}(1 + j) = 1 - j0.4142$$

$$V[4] = G[0] + W_8^4 H[0] = 4 + e^{-j\pi} \cdot 6 = -2$$

2N-Point DFT of a Real Sequence Using an N-point DFT

$$V[5] = G[1] + W_8^5 H[1]$$
$$= (1-j) + e^{-j5\pi/4}(1-j) = 1 + j0.4142$$

$$V[6] = G[2] + W_8^6 H[2] = -2 + e^{-j3\pi/2} \cdot 0 = -2$$

$$V[7] = G[3] + W_8^7 H[3]$$

$$= (1+j) + e^{-j7\pi/4}(1+j) = 1 + j2.4142$$

Linear Convolution Using the DFT

- **Linear convolution is a key operation in many signal processing applications**
- **Since a DFT can be efficiently implemented using FFT algorithms, it is of interest to develop methods for the implementation of linear convolution using the DFT**

Linear Convolution Using the DFT

- Let $g[n]$ and $h[n]$ be two finite-length sequences of length N and M, respectively
- Denote $\quad L = N + M - 1$
- Define two length-L sequences

$$g_e[n] = \begin{cases} g[n], & 0 \le n \le N-1 \\ 0, & N \le n \le L-1 \end{cases}$$

$$h_e[n] = \begin{cases} h[n], & 0 \le n \le M-1 \\ 0, & M \le n \le L-1 \end{cases}$$

Linear Convolution Using the DFT

- Then
  $$y_L[n] = g[n] \circledast h[n] = y_C[n] = g[n] \textcircled{\scriptsize L} h[n]$$
- The corresponding implementation scheme is illustrated below