数字图像处理
Digital Image Processing

Image as A Function (3)

Discrete Sampling of A Function

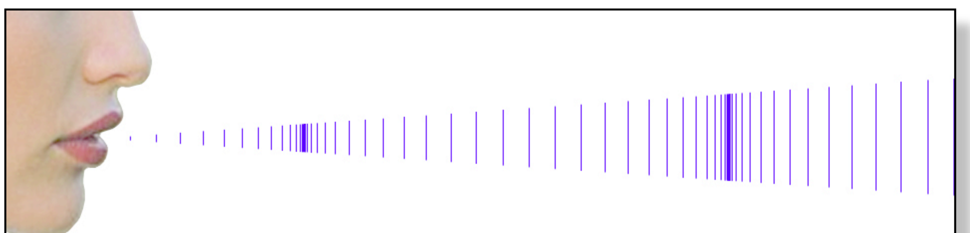
Fourier Transformation

Most slides are courtesy of Juyong Zhang

Signal:

A measurable phenomenon that changes over time or throughout space.

sound

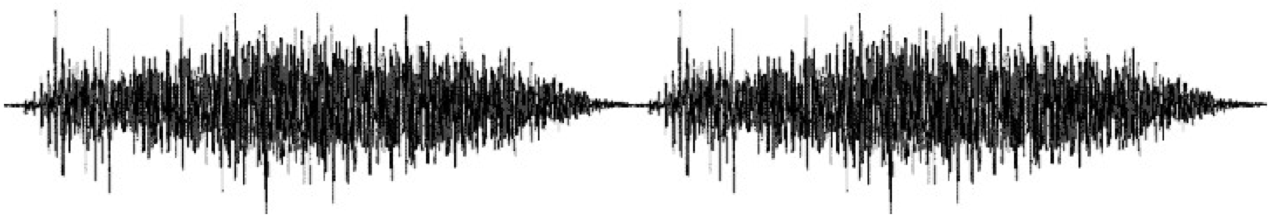


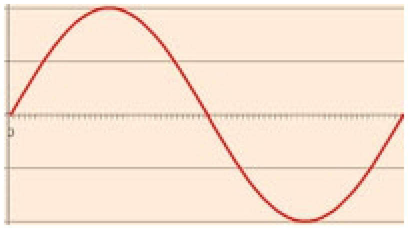
code

011010001011011101100101100001

image

Sound Signal





Space-Time Representation

Frequency-Domain Representation

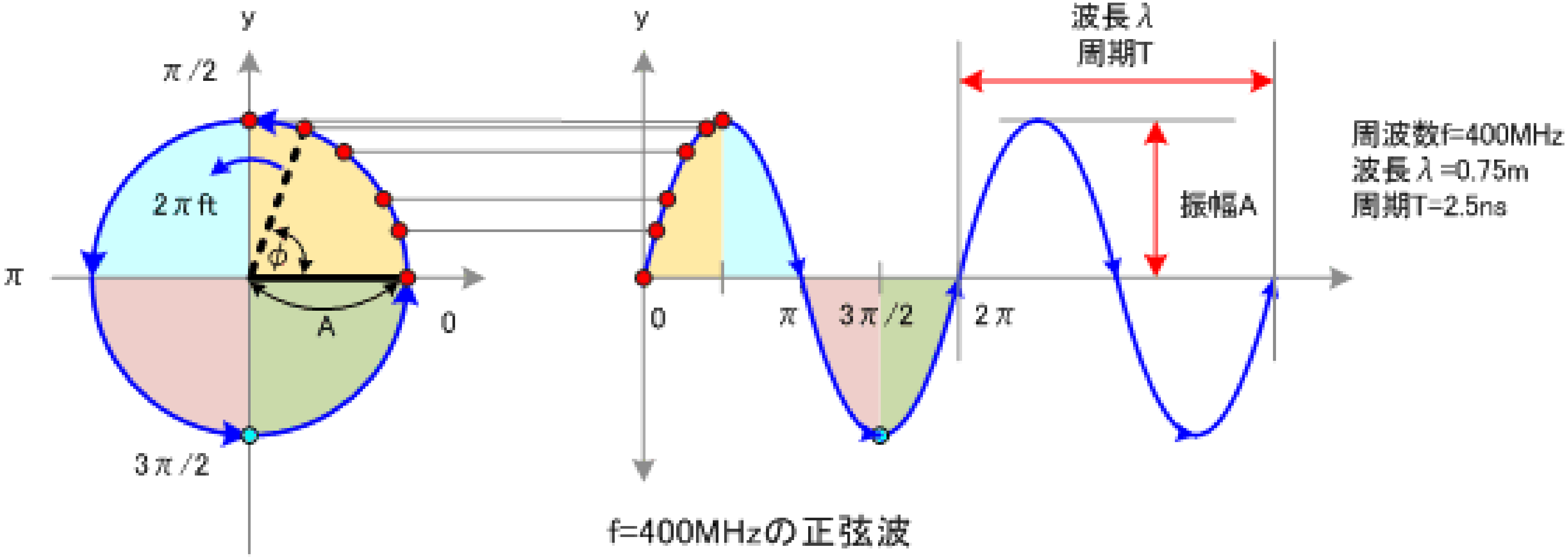
Signals: Space-Time vs. Frequency-Domain Representation

Space/time representation: a graph of the measurements with respect to a point in time and/or positions in space.

Fact: signals undulate (otherwise they'd contain no information).

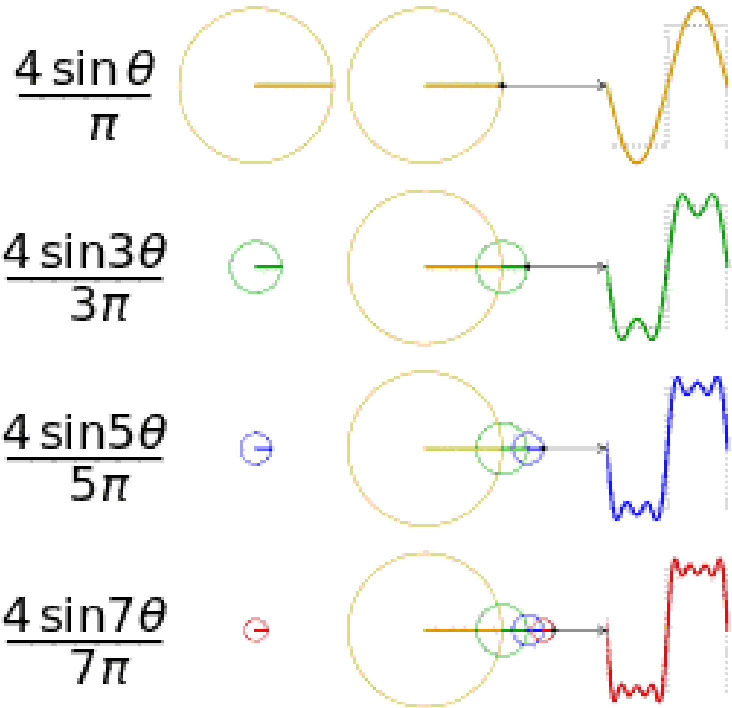
Frequency-domain representation: an exact description of a signal *in terms of* its undulations.

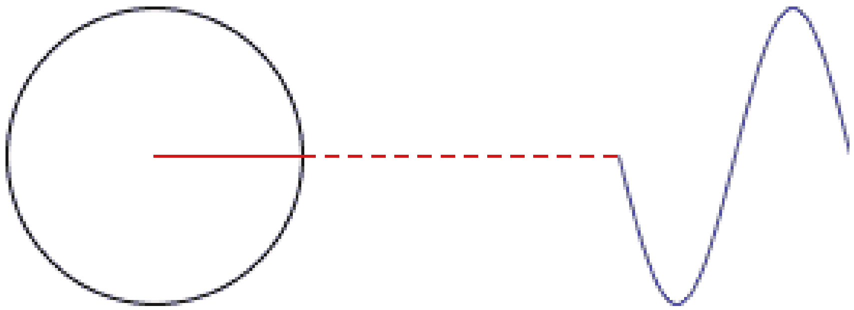
Sinusoids



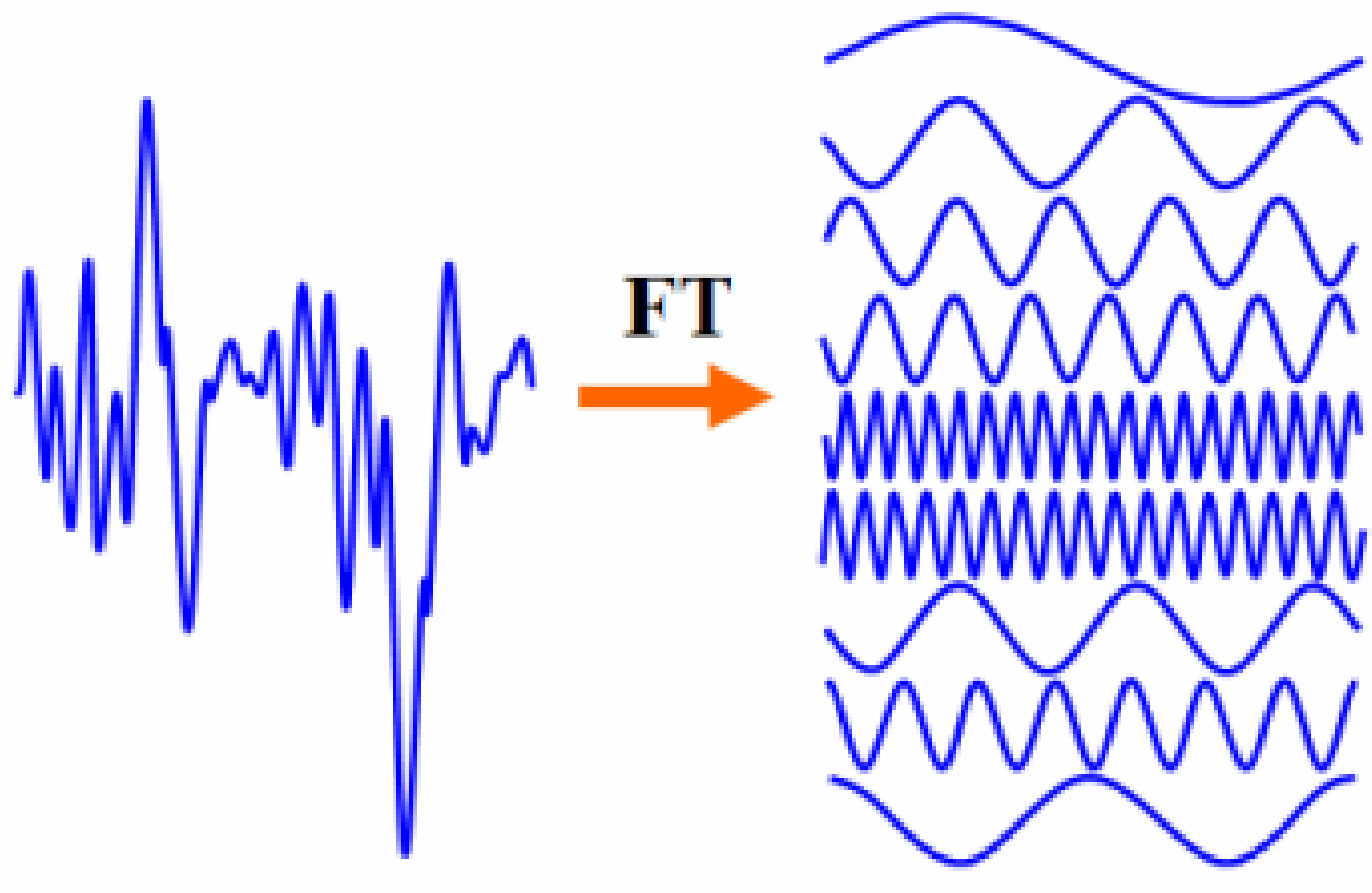
y
π /2
2 π ft
φ
π
A
0
3 π /2
y
0
π
3 π /2
2 π
波長 λ
周期T
振幅A
周波数f=400MHz
波長 λ =0.75m
周期T=2.5ns
f=400MHzの正弦波

Sum of Sinusoids

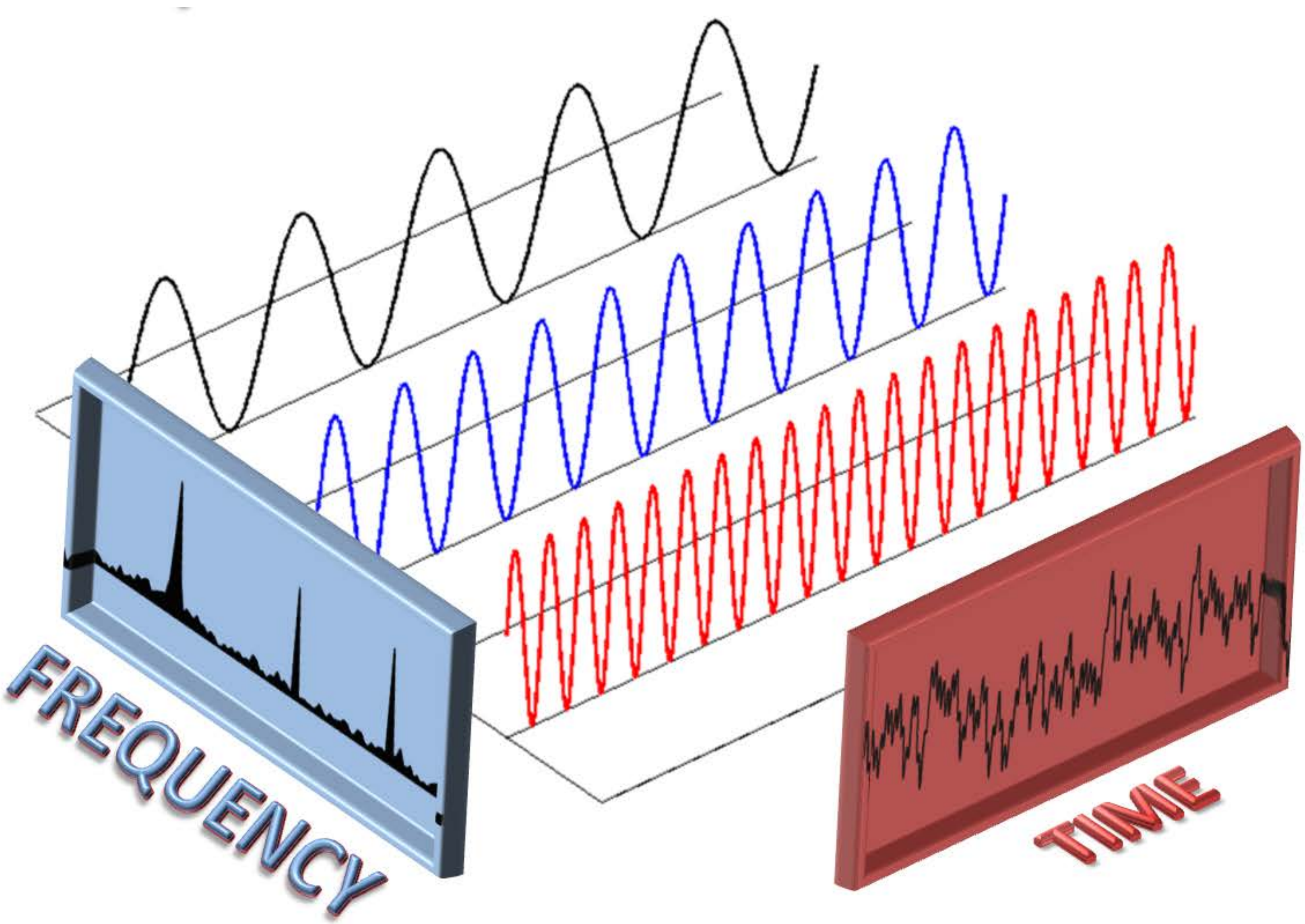




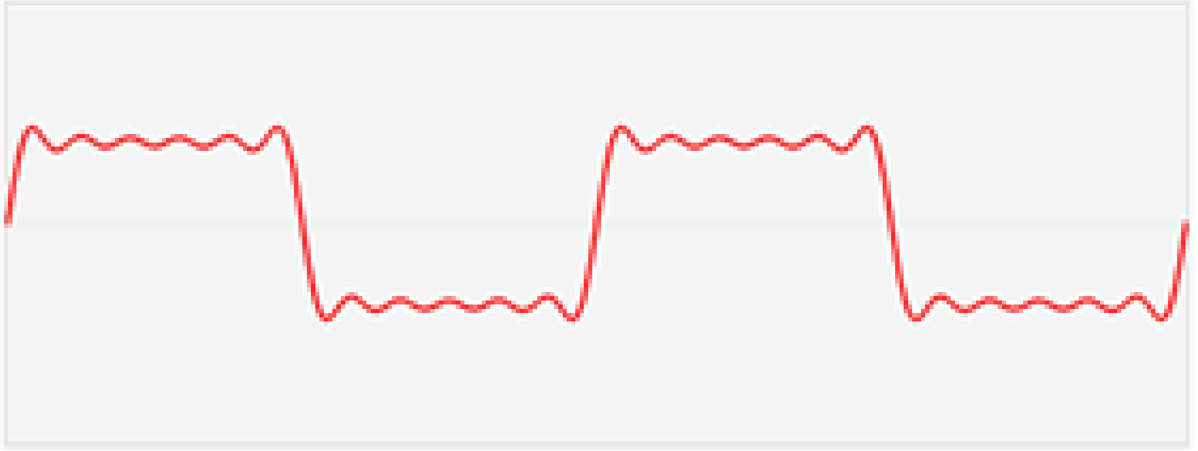
Fourier Transformation



Frequency-Domain Representation



Animation



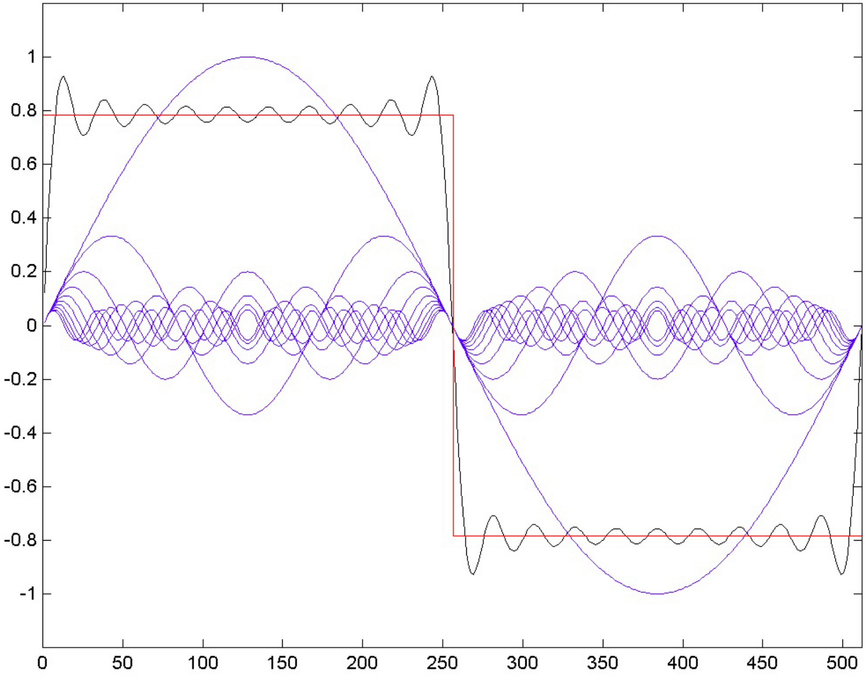
Fact: Any Real Signal has a Frequency-Domain Representation

Odd-order harmonics

$$\mathrm{sq}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \sin\left[\frac{2\pi}{\lambda}(2n+1)t\right]$$

The modes shown (blue) sum to the rippling square wave (black).

As the number of modes in the sum becomes large, it approaches a square wave (red).



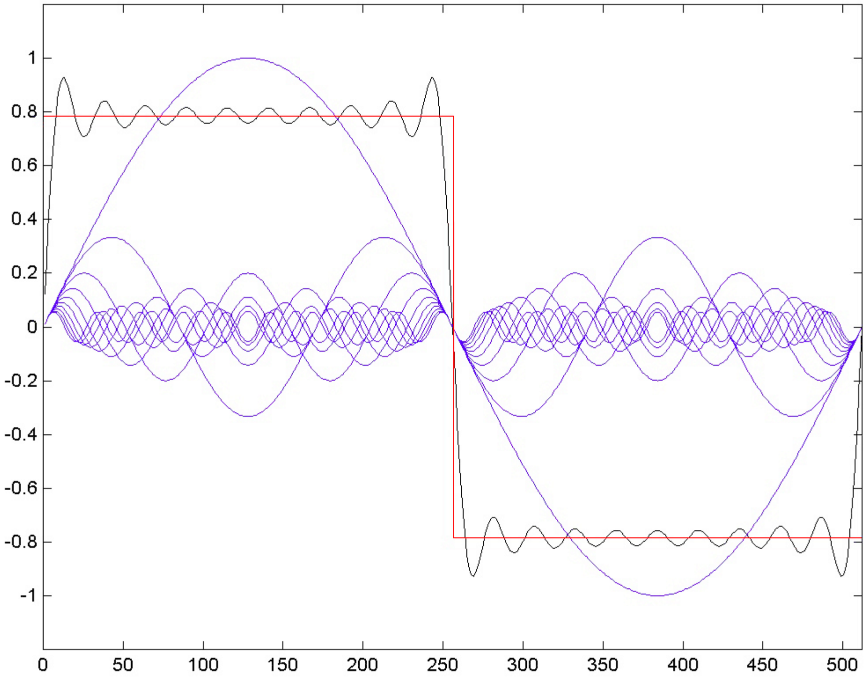
Frequency-Domain Representation

Any periodic signal can be described by a sum of sinusoids.

$$\mathrm{sq}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{2n+1} \sin\left[\frac{2\pi}{\lambda}(2n+1)\, t\right]$$

The sinusoids are called "basis functions".

The multipliers are called "Fourier coefficients".



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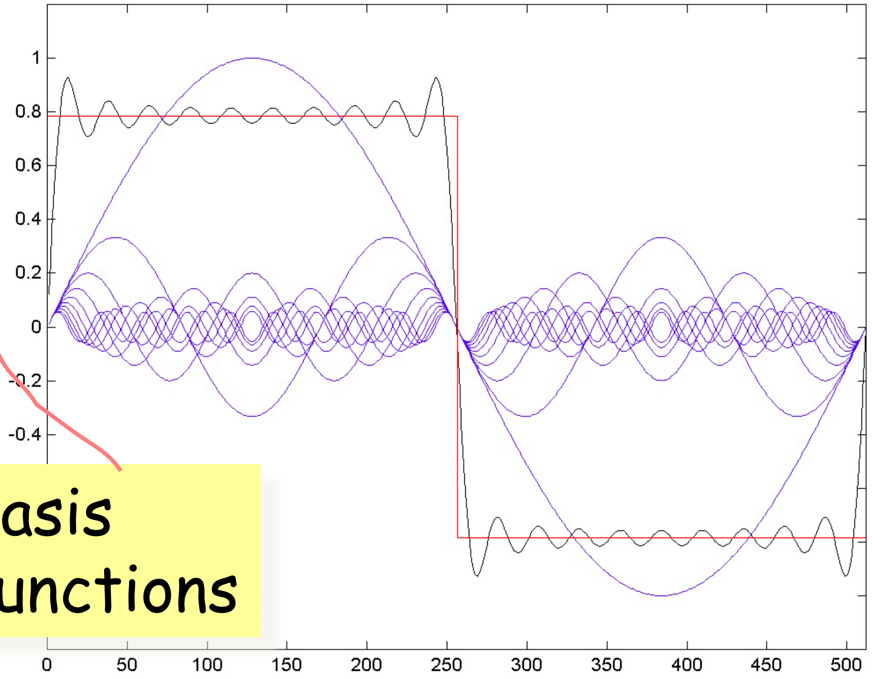
Basis functions



Frequency-Domain Representation

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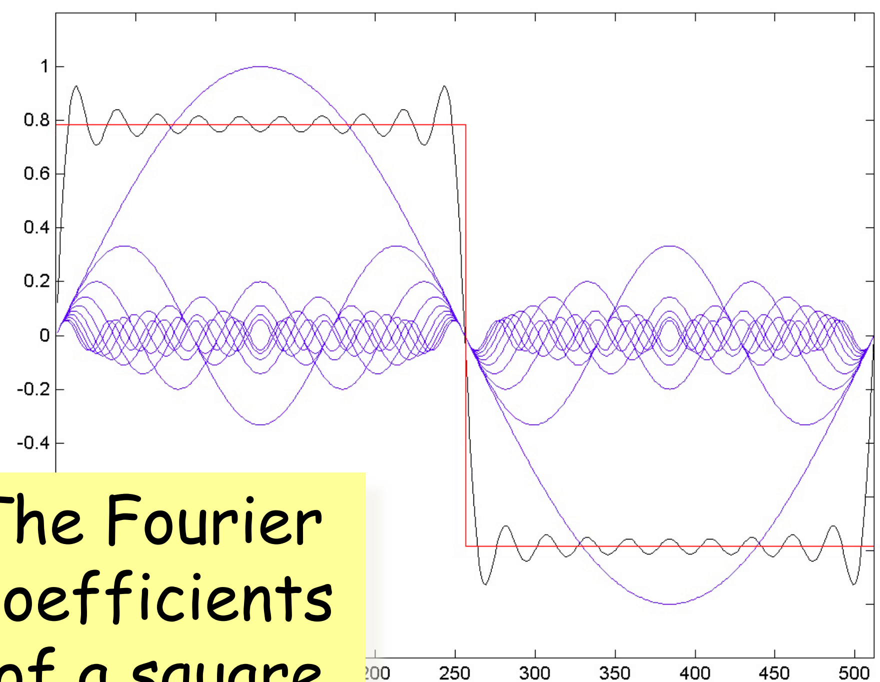
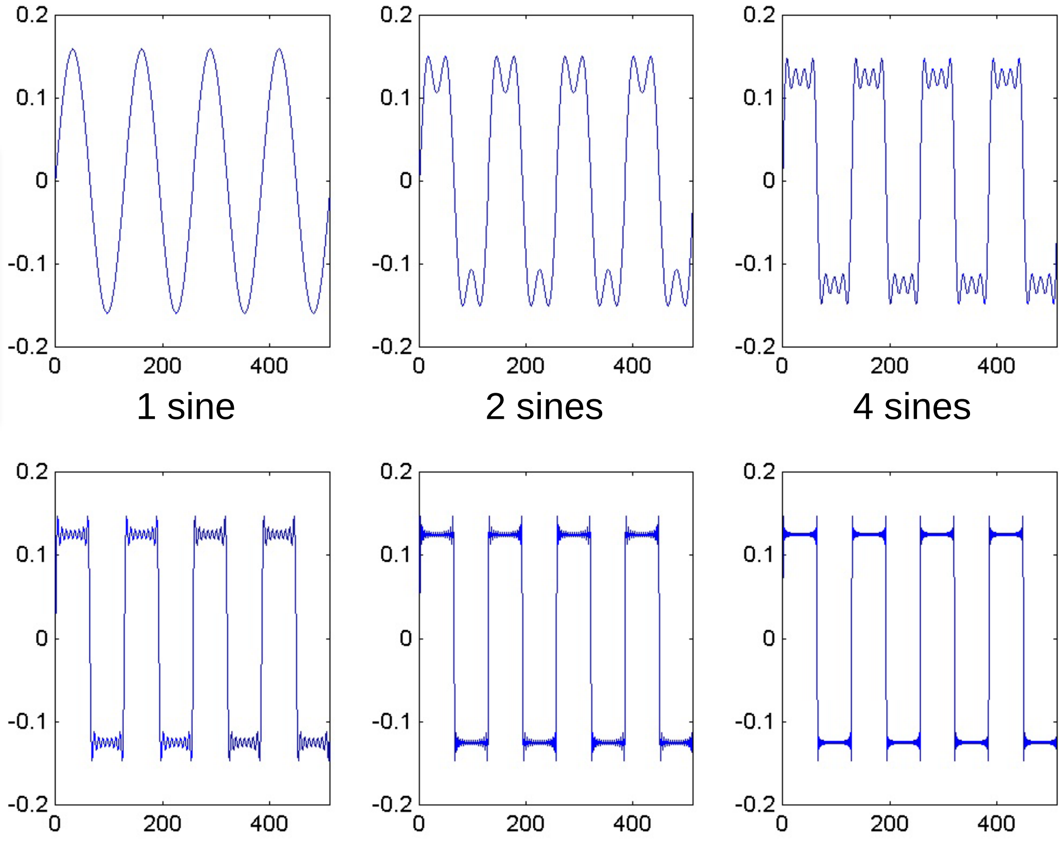


The Fourier coefficients (of a square wave).

Example: Partial Sums of a Square Wave

The limit of the given sequence of partial sums[1] is exactly a square wave

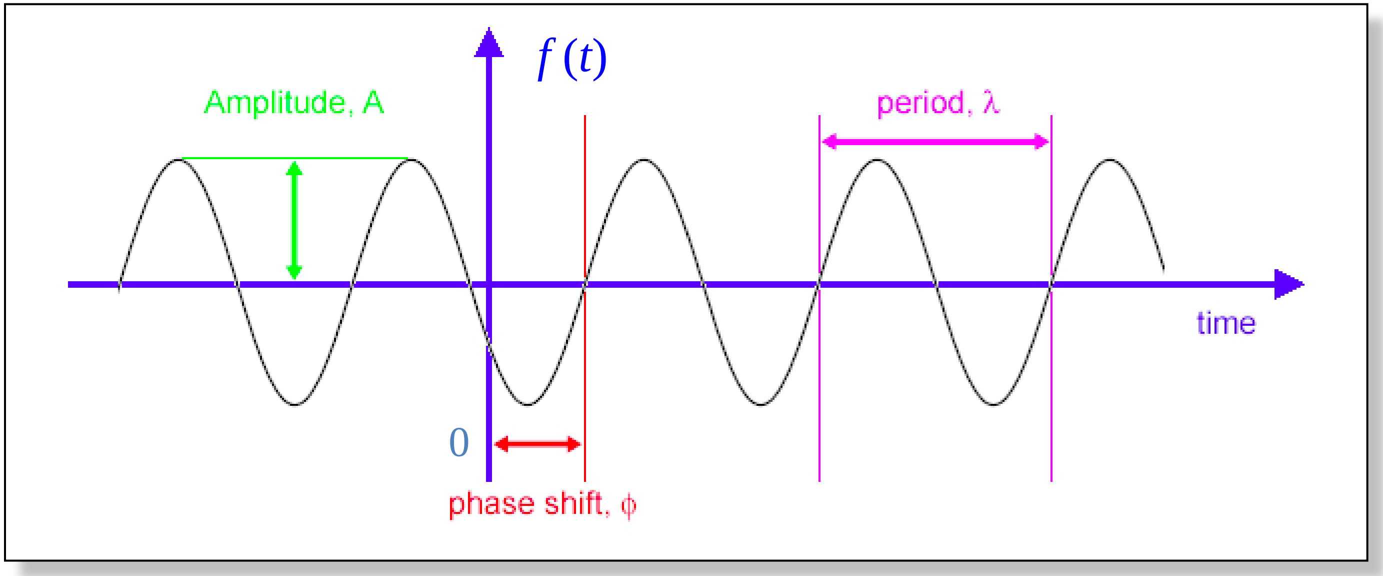


1 sine, 2 sines, 4 sines

8 sines, 16 sines, 32 sines

[1] the limit as n approaches infinity of the sum of n sines.

Anatomy of a Sinusoid



$$f(t) = A\sin\left(\frac{2\pi}{\lambda}t - \phi\right)$$

$1/\lambda$ is the frequency of the sinusoid (Hz).
$2\pi/\lambda$ is the angular frequency (radians/s).

The Inner Product: a Measure of Similarity

The similarity between functions f and g on the interval $\left(-\lambda/2, \lambda/2\right)$ can be defined by

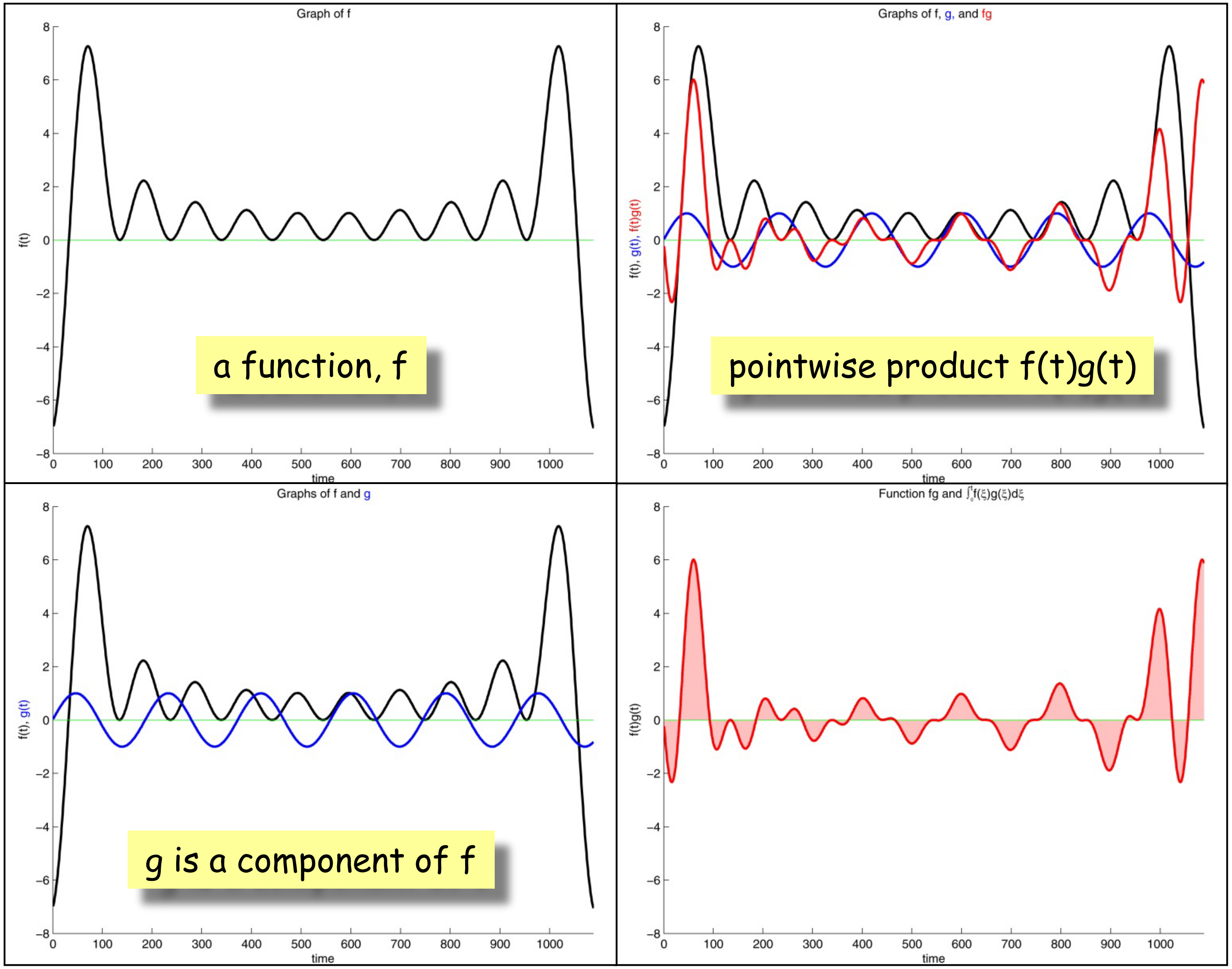
$$\langle f, g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\, g^*(t)\, dt$$

where $g^*(t)$ is the complex conjugate of $g(t)$.

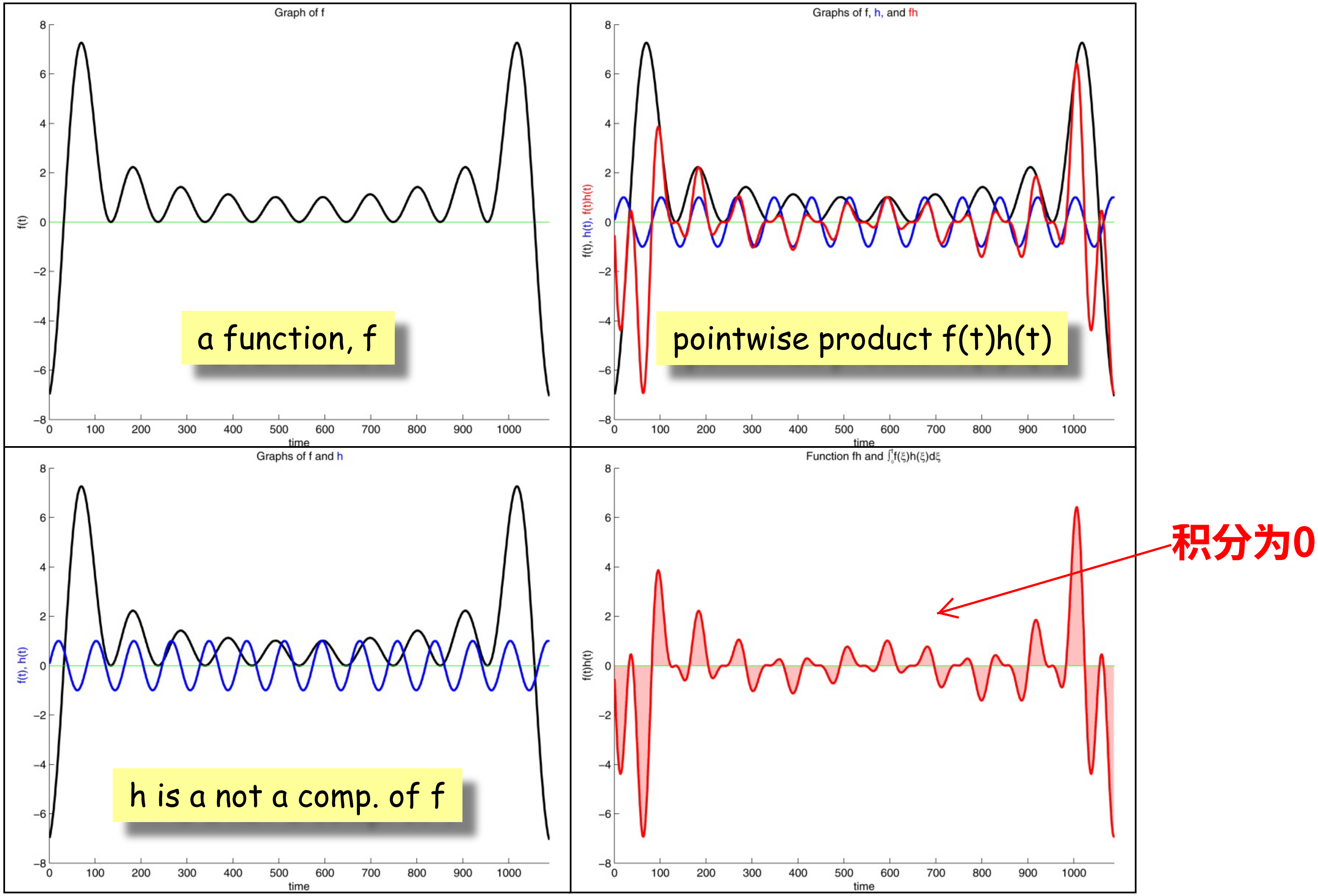
This number, called the *inner product of* f and g, can also be thought of as the amount of g in f or as the projection of f onto g.

If f and g have the same energy, then their inner product is maximal if $f = g$. On the other hand if $\langle f, g\rangle = 0$, then f and g have nothing in common.

Inner Products



Inner Products



Inner Product of a Periodic Function and a Sinusoid

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \sin\left(\tfrac{2\pi}{\lambda} t\right) dt$$

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \cos\left(\tfrac{2\pi}{\lambda} t\right) dt$$

$$\begin{aligned}\langle f, g \rangle &= \int_{-\lambda/2}^{\lambda/2} f(t) \left[\cos\left(\tfrac{2\pi}{\lambda} t\right) - i \sin\left(\tfrac{2\pi}{\lambda} t\right)\right] dt \\ &= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\frac{2\pi}{\lambda} t} dt \\ &= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\omega t} dt\end{aligned}$$

3 different representations

$$e^{-i\frac{2\pi}{\lambda} t} = \cos\left(\tfrac{2\pi}{\lambda} t\right) - i \sin\left(\tfrac{2\pi}{\lambda} t\right)$$

$$\omega = \frac{2\pi}{\lambda}$$

Inner Product of a Periodic Function and a Sinusoid

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \sin\left(\tfrac{2\pi}{\lambda} t\right) dt$$

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t) \cos\left(\tfrac{2\pi}{\lambda} t\right) dt$$

real number results yield the amplitude of that sinusoid in the function.

$$\langle f, g \rangle = \int_{-\lambda/2}^{\lambda/2} f(t \ldots t)\Big] dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\frac{2\pi}{\lambda}t} dt$$

$$= \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\omega t} dt$$

Inner Product of a Periodic Function and a Sinusoid

$$\langle f,g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\sin\left(\tfrac{2\pi}{\lambda}t\right)dt$$

$$\langle f,g\rangle = \int_{-\lambda/2}^{\lambda/2} f(t)\cos\left(\tfrac{2\pi}{\lambda}t\right)dt$$

$$\begin{aligned}\langle f,g\rangle &= \int_{-\lambda/2}^{\lambda/2} f(t)\left[\cos\left(\tfrac{2\pi}{\lambda}t\right)-i\sin\left(\tfrac{2\pi}{\lambda}t\right)\right]dt\\ &= \int_{-\lambda/2}^{\lambda/2} f(t)e^{-i\frac{2\pi}{\lambda}t}\,dt\\ &= \int_{-\lambda/2}^{\lambda/2} f(t)e^{-i\omega t}\,dt\end{aligned}$$

Complex number result yields the amplitude and phase of that sinusoid in the function.

The Fourier Series

is the decomposition of a λ-periodic signal into a sum of sinusoids.

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n}{\lambda} t\right) + B_n \sin\left(\frac{2\pi n}{\lambda} t\right)$$

$$\text{periodic: } \exists \lambda \in \mathbb{R} \text{ such that } f(t \pm n\lambda) = f(t)$$

$$A_n = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t) \left[\cos\left(\frac{2\pi n}{\lambda} t\right)\right] dt \text{ for } n \geq 0$$

$$B_n = \frac{2}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t) \left[\sin\left(\frac{2\pi n}{\lambda} t\right)\right] dt \text{ for } n \geq 0$$

The representation of a function by its Fourier Series is the sum of sinusoidal "basis functions" multiplied by coefficients.

Fourier coefficients are generated by taking the inner product of the function with the basis.

The basis functions correspond to modes of vibration.

The Fourier Series

can also be written in terms of complex exponentials

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{+i\frac{2\pi n}{\lambda}t} = \sum_{n=-\infty}^{\infty} |C_n| e^{+i\left(\frac{2\pi n}{\lambda}t + \phi_n\right)}$$

$$= \sum_{n=-\infty}^{\infty} |C_n| \cos\left(\frac{2\pi n}{\lambda}t + \phi_n\right) + |C_n| \sin\left(\frac{2\pi n}{\lambda}t + \phi_n\right)$$

i

$$C_n = |C_n| e^{+i\phi_n} = \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t) e^{-i\frac{2\pi n}{\lambda}t}\, dt$$

$$= \frac{1}{\lambda} \int_{-\lambda/2}^{\lambda/2} f(t) \left[\cos\left(\frac{2\pi n}{\lambda}t\right) - \sin\left(\frac{2\pi n}{\lambda}t\right)\right] dt$$

i

$$i = \sqrt{-1}$$

$$e^{\pm ix} = \cos x + i \sin x$$

$$f(t + n\lambda) = f(t)$$

for all integers n

The Fourier Series

Cont'd. on next page.

Relationship between the real and the complex Fourier Series

$$
\begin{aligned}
f(t) &= \sum_{n=0}^{\infty}\left[A_n \cos\omega_n t + B_n \sin\omega_n t\right], \quad \text{where } \omega_n = \frac{2\pi n}{\lambda} \\
&= \frac{2}{\lambda}\sum_{n=0}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\cos\omega_n\eta\, d\eta \cos\omega_n t + \int_{-\lambda/2}^{\lambda/2} f(\eta)\sin\omega_n\eta\, d\eta \sin\omega_n t\right] \\
&= \frac{2}{\lambda}\sum_{n=0}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)\left[\cos\omega_n\eta \cos\omega_n t + \cancel{f(\eta)}\sin\omega_n\eta \sin\omega_n t\right] d\eta \\
&= \frac{1}{\lambda}\sum_{n=0}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)\cos(\omega_n\eta - \omega_n t)\, d\eta
\end{aligned}
$$

1. cos是偶函数
2. 第三步—》第四步，2—》1
3. 故有 n=-infinity not n=0

The sine-plus-cosine form results from the projection of f onto a cosine that is in phase with the current time.

Relationship between the real and the complex Fourier Series (cont'd.)

Cont'd. on next page.

Claim: $$0=\sum_{n=-\infty}^{\infty}\sin\left(\omega_n\eta-\omega_n t\right).$$

注意：sin(x)是奇函数，故claim成立

Therefore: $$\int_{-\lambda/2}^{\lambda/2}\left[f(\eta)\sum_{n=-\infty}^{\infty}\sin\left(\omega_n\eta-\omega_n t\right)\right]d\eta=0.$$

Thus: $$-i\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2}f(\eta)\sin\left(\omega_n\eta-\omega_n t\right)d\eta\right]=0.$$

Then add zero to the equation at the end of the previous page:

$$f(t)=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2}f(\eta)\cos\left(\omega_n\eta-\omega_n t\right)d\eta\right]-i\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2}f(\eta)\sin\left(\omega_n\eta-\omega_n t\right)d\eta\right]$$

Relationship between the real and the complex Fourier Series (cont'd.)

周期函数傅氏级数实表达和复指数表达的关系
1. 实表达，p24
n=0 —》 +infinity
2. 复指数表达
n=-infinity —》 +infinity

$$f(t)=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\cos(\omega_n\eta-\omega_n t)\,d\eta\right]-i\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\sin(\omega_n\eta-\omega_n t)\,d\eta\right]$$

$$=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)\left[\cos\omega_n(\eta-t)-i\sin\omega_n(\eta-t)\right]d\eta$$

$$=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)e^{-i\omega_n(\eta-t)}\,d\eta$$

$$=\sum_{n=-\infty}^{\infty}\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(\eta)e^{-i\frac{2\pi n}{\lambda}\eta}\,d\eta\, e^{+i\frac{2\pi n}{\lambda}t}$$

$$=\sum_{n=-\infty}^{\infty}C_n e^{+i\frac{2\pi n}{\lambda}t}=\sum_{n=-\infty}^{\infty}|C_n|e^{i\phi_n}e^{+i\frac{2\pi n}{\lambda}t}=\sum_{n=-\infty}^{\infty}|C_n|e^{+i\left(\frac{2\pi n}{\lambda}t+\phi_n\right)}$$

Then some algebraic manipulations lead to the result.

Relationship between the real and the complex Fourier Series (cont'd.)

$$f(t)=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\boxed{\cos(\omega_n\eta-\omega_n t)}\,d\eta\right]-i\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\left[\int_{-\lambda/2}^{\lambda/2} f(\eta)\boxed{\sin(\omega_n\eta-\omega_n t)}\,d\eta\right]$$

$$=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)\boxed{\left[\cos\omega_n(\eta-t)-i\sin\omega_n(\eta-t)\right]}\,d\eta$$

$$=\frac{1}{\lambda}\sum_{n=-\infty}^{\infty}\int_{-\lambda/2}^{\lambda/2} f(\eta)\boxed{e^{-i\omega_n(\eta-t)}}\,d\eta$$

$$=\sum_{n=-\infty}^{\infty}\boxed{\frac{1}{\lambda}\int_{-\lambda/2}^{\lambda/2} f(\eta)e^{-i\frac{2\pi n}{\lambda}\eta}\,d\eta}\,e^{+i\frac{2\pi n}{\lambda}t}$$

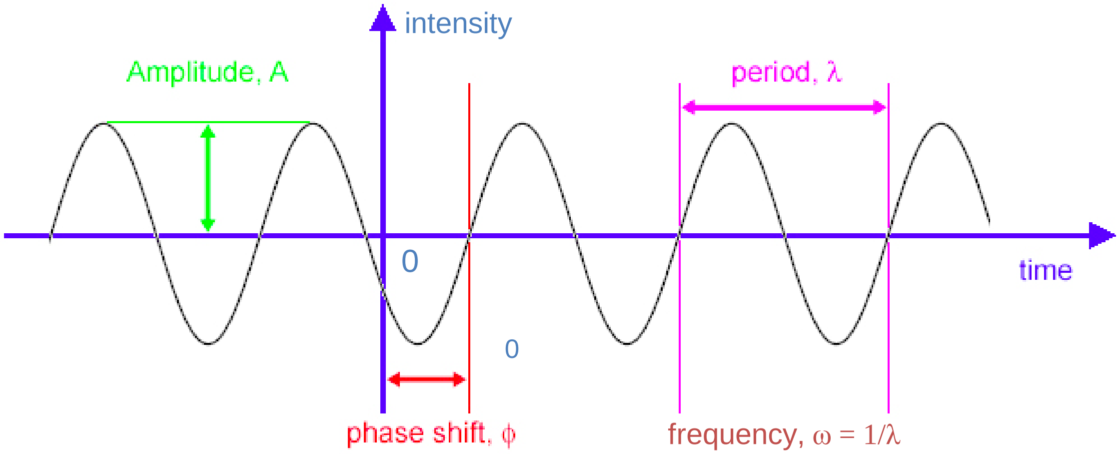
$$=\sum_{n=-\infty}^{\infty}\boxed{C_n}e^{+i\frac{2\pi n}{\lambda}t}=\sum_{n=-\infty}^{\infty}|C_n|e^{i\phi_n}e^{+i\frac{2\pi n}{\lambda}t}=\sum_{n=-\infty}^{\infty}|C_n|e^{+i\left(\frac{2\pi n}{\lambda}t+\phi_n\right)}$$

Then some algebraic manipulations lead to the result.

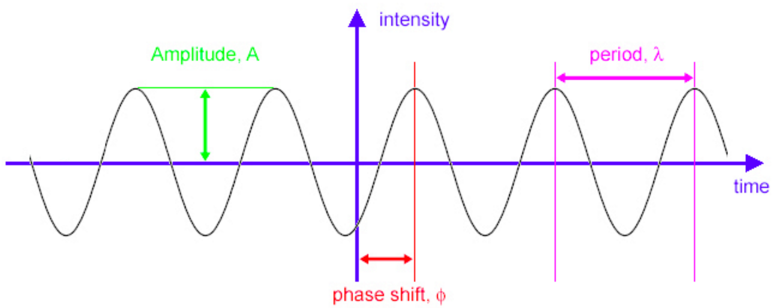
Why are Fourier Coefficients Complex Numbers?

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{+i\frac{2\pi n}{\lambda}t} \quad \text{where} \quad C_n = \left| C_n \right| e^{+i\phi_n}.$$

C_n represents the amplitude, $A = |C_n|$, and relative phase, ϕ, of that part of the original signal, $f(t)$, that is a sinusoid of frequency $\omega_n = n / \lambda$.

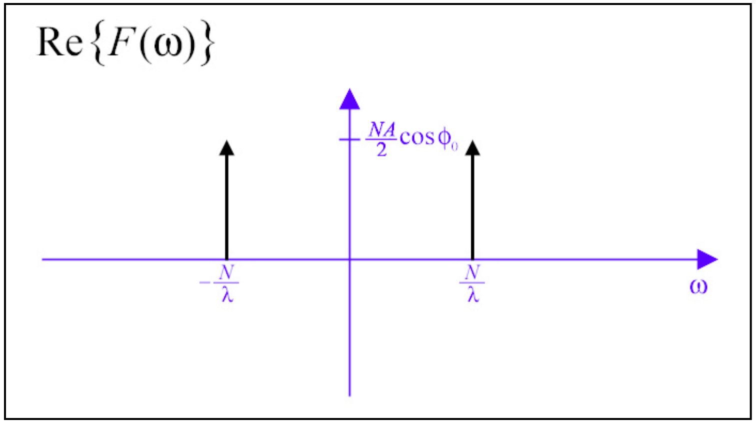


What about real + imaginary?

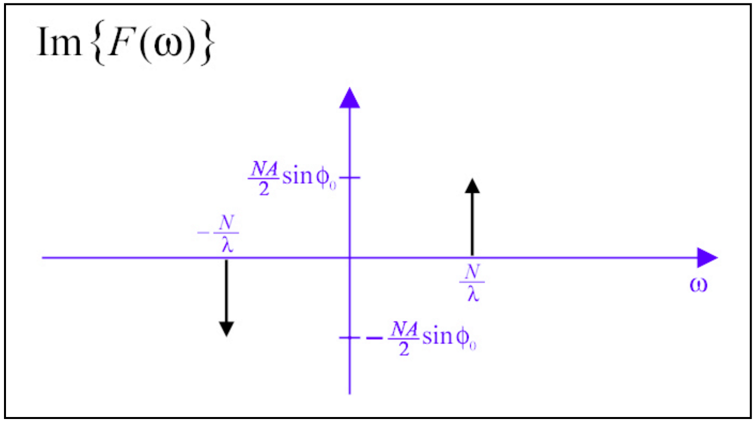


$$f(n) = A\cos[(2\pi n/\lambda) - \phi_0],\ \text{for } n = 0, \ldots, N\text{-}1.$$

$\Im$



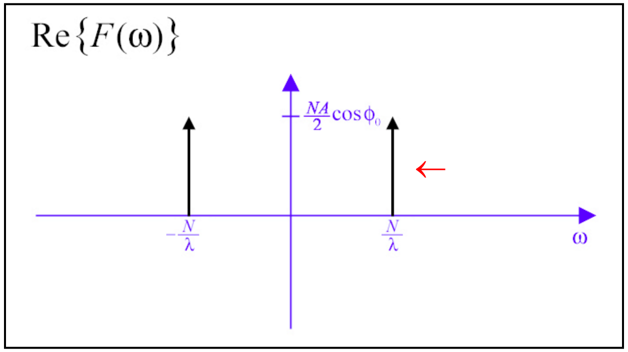
$$F(\omega) = \left(\tfrac{NA}{2}\cos\varphi\right)\left[\delta(\omega + N/\lambda) + \delta(\omega - N/\lambda)\right]$$
$$+ i\left(\tfrac{NA}{2}\sin\varphi\right)\left[-\delta(\omega + N/\lambda) + \delta(\omega - N/\lambda)\right]$$

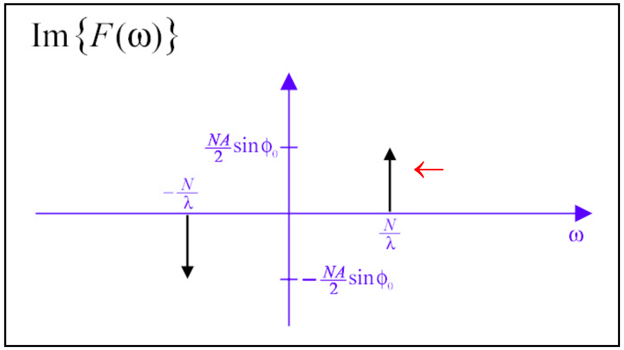


The FS of a cosine is a pair of impulses with complex amplitudes

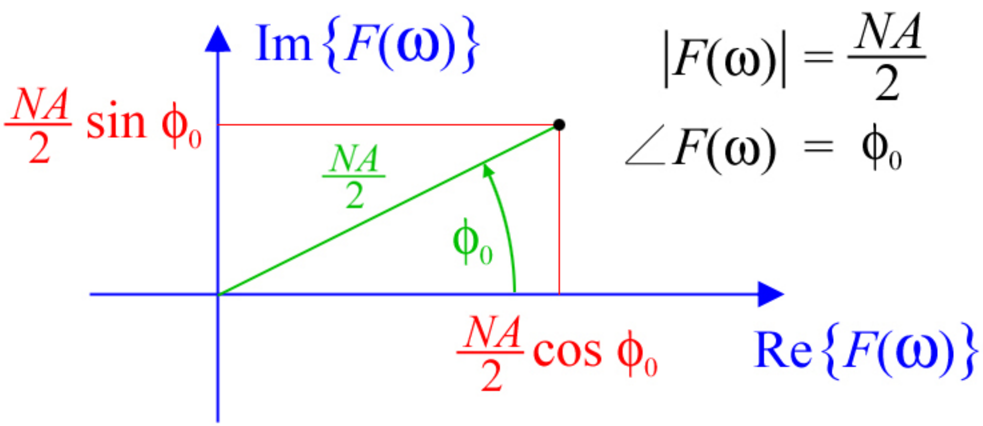
Real + Imaginary to Magnitude & Phase

The real and imaginary parts at the positive frequency, N/λ ...





Complex Value at $\omega = +N/\lambda$

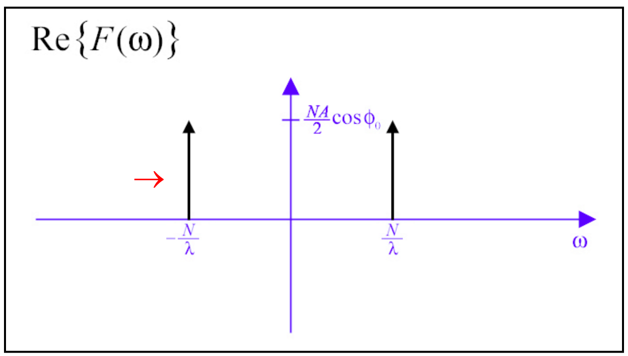


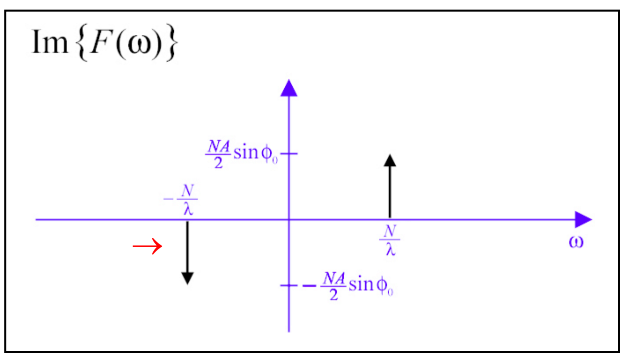
$$F(\omega = +N/\lambda) = \frac{NA}{2}\cos\phi_0 + i\,\frac{NA}{2}\sin\phi_0$$

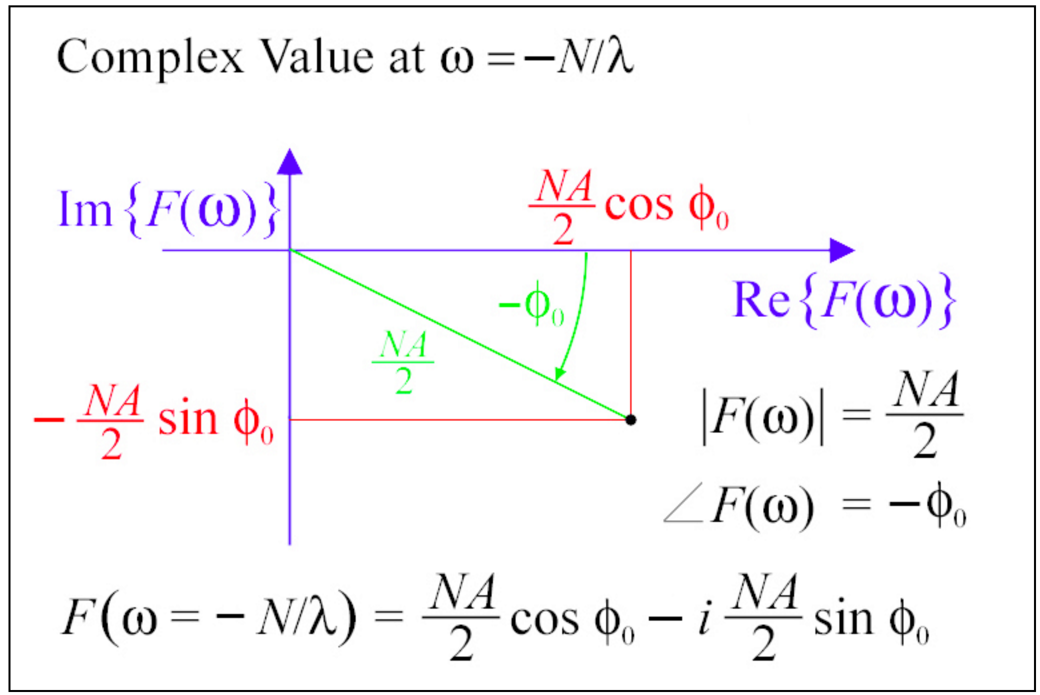
... form a magnitude, NA/2, and a phase, ϕ_0.

Real + Imaginary to Magnitude & Phase

The real and imaginary parts at the negative frequency, -N/λ ...







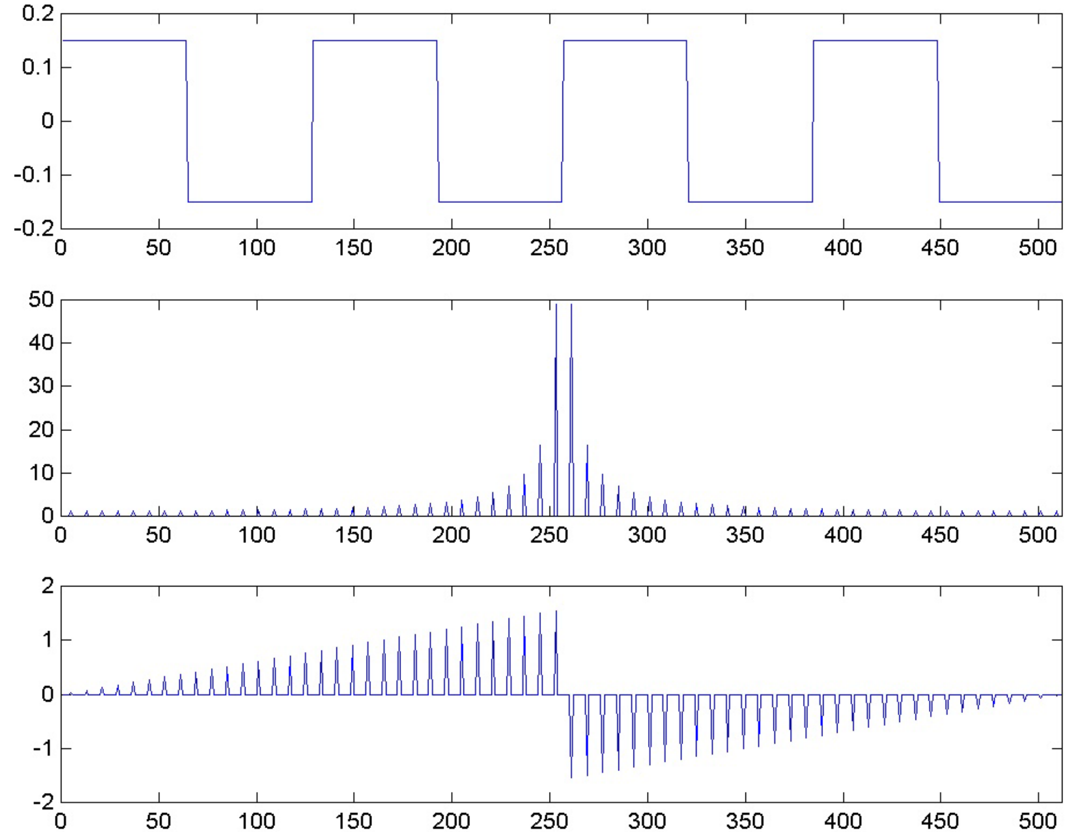
... form a magnitude, NA/2, and a phase, $-\phi_0$.

Fourier Series of a Square Wave

Time-domain signal

Fourier magnitude

Fourier phase



The Fourier Transform

is the decomposition of a *nonperiodic* signal into a continuous sum* of sinusoids.

$$F(\omega) = \left| F(\omega) \right| e^{i\Phi(\omega)} = \int_{-\infty}^{\infty} f(t)\, e^{i2\pi\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \left[\cos(2\pi\omega t) + i \sin(2\pi\omega t) \right] dt$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i2\pi\omega t}\, d\omega = \int_{-\infty}^{\infty} \left| F(\omega) \right| e^{-i(2\pi\omega t + \Phi(\omega))}\, d\omega$$

$$= \int_{-\infty}^{\infty} F(\omega) \left[\cos(2\pi\omega t) - i \sin(2\pi\omega t) \right] d\omega$$

$$= \int_{-\infty}^{\infty} \left| F(\omega) \right| \left[\cos(2\pi\omega t + \Phi(\omega)) - i \sin(2\pi\omega t + \Phi(\omega)) \right] d\omega$$

* *i.e.*, an integral.

The Discrete Fourier Transform

A discrete signal, $\left\{ h_k \,\middle|\, k = 0,1,2,\ldots,N-1 \right\}$, of finite length N can be repre - sented as a weighted sum of N sinusoids, $\left\{ e^{-i2\pi kn/N} \,\middle|\, n = 0,1,2,\ldots,N-1 \right\}$ through

$$h_k = \sum_{n=0}^{N-1} H_n e^{i2\pi kn/N}$$

where the set, $\left\{ H_n \,\middle|\, n = 0,1,2,\ldots,N-1 \right\}$, are the Fourier coefficients defined as the projection of the original signal onto sinusoid, n, given by :

$$H_n = \frac{1}{N} \sum_{k=0}^{N-1} h_k e^{-i2\pi kn/N}$$

数字频率和模拟频率

1. 连续傅立叶展开：

$$f(x)=\int_{-\infty}^{\infty}F(u)e^{j2\pi ux}du$$，模拟频率：$u\in[-\infty,\ \infty]$

2. 离散傅立叶展开：

$$f(n)=\sum_{u=0}^{N-1}F(u)e^{j\frac{2\pi un}{N}}$$，数字频率：$u\in[0,1,\cdots,N-1]$

3. 数字频率在$[0,1,\cdots,N-1]$间共轭对称，即$F(u)=F^{*}(N-1-u)$，最大数字频率在$u=\lfloor N/2\rfloor$

4. 数字频率对应的模拟频率取决于离散信号采样周期T，有最大模拟频率为$f_{\max}=1/2T$，u对应的模拟频率为$\cancel{u(f_{\max}/N)}$ （HZ）

u*f_max/(N/2)
N点频谱内有2个对称谱，
N/2处为最大频率

The 2D Fourier Transform

Primary Uses of the FT in Image Processing:

- ☐ Explains why down-sampling can add distortion to an image and shows how to avoid it.
- ☐ Useful for certain types of noise reduction, deblurring, and other types of image restoration.
- ☐ For feature detection and enhancement, especially edge detection.

注意：
1. 这里复指数上的w最好为f
2. <f(t),g(t)>—》f和g的共轭积分

The Fourier Transform: Discussion

The expressions

内积取复共轭

$$F(\omega)=\int_{-\infty}^{\infty} f(t)\,e^{-i2\pi\omega t}\,dt=\left\langle f(t),\ e^{+i2\pi\omega t}\right\rangle$$

continuous signals defined over all real numbers

and

$$H_n=\frac{1}{N}\sum_{\cancel{n}=0}^{N-1} h_k e^{-i2\pi k n/N}=\left\langle h_k,\ e^{+i2\pi k n/N}\right\rangle$$

k

discrete signals with N terms or samples.

for the Fourier coefficients are “inner products” which can be thought of as measures of the similarity between the functions $f(t)$ and $e^{+i2\pi\omega t}$ for $t\in(-\infty,\infty)$ or between the sequences $\{h_k\}_{k=0}^{N-1}$ and $\{e^{+i2\pi k n/N}\}_{k=0}^{N-1}$.

The Fourier Transform: Discussion (cont'd.)

In the context of inner products, the complex exponentials

$$\left\{ e^{-i2\pi\omega t} \;\middle|\; \omega \in \Re \text{ and } \omega \in (-\infty, \infty) \right\} \text{ and } \left\{ e^{-i2\pi kn/N} \;\middle|\; \ldots, -2, -1, 0, 1, 2, \ldots \right\}$$

are called "orthogonal sets" since they have the property:

$$\left\langle e^{-i2\pi\omega_1 t}, e^{-i2\pi\omega_2 t} \right\rangle = \int_{-\infty}^{\infty} e^{-i2\pi\omega_1 t} \cdot e^{+i2\pi\omega_2 t}\, dt = \begin{cases} \infty, & \text{if } \omega_1 = \omega_2 \\ 0, & \text{if } \omega_1 \neq \omega_2 \end{cases}$$

$$\left\langle e^{-i2\pi jn/N}, e^{-i2\pi kn/N} \right\rangle = \sum_{n=0}^{N-1} e^{-i2\pi jn/N} \cdot e^{+i2\pi kn/N} = \begin{cases} c, & \text{if } j = k \\ 0, & \text{if } j \neq k \end{cases},$$

注意：
1. 应该写成exp(-i2*pai*ft)比较合适
2. 当f1=f2时—》exp（0）如果在T=1/f1内积分为1，在无穷区间积分

The function sets are called "orthogonal basis sets"

They are called "basis sets" since for any function[1], $f(t)$, of a real variable there exists a complex-valued function F(w), and for any sequence[1], h_k, there exist complex numbers, H_n, such that

$$f(t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-i2\pi\omega t}\, d\omega \quad \text{and} \quad h_k = \sum_{n=0}^{N-1} H_n e^{-i2\pi kn/N}.$$

[1] with finite energy.

The Fourier Transform: Discussion (cont'd.)

Consider the 2-dimensional functions

$$\left\{ e^{-i2\pi(ux+vy)} \,\middle|\, u,v,x,y \in \Re \right\} \text{ and } \left\{ e^{-i2\pi\left(\frac{jm}{M}+\frac{kn}{N}\right)} \,\middle|\, j,m \in 0,\ldots,M-1,\ k,n \in 0,\ldots,N-1 \right\}$$

These are, likewise, orthogonal:

$$\left\langle e^{-i2\pi(u_1x+v_1y)},\ e^{-i2\pi(u_2x+v_2y)} \right\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i2\pi(u_1x+v_1y)} \cdot e^{+i2\pi(u_2x+v_2y)}\,dxdy$$

$$= \begin{cases} \infty, & \text{if } u_1=u_2 \text{ and } v_1=v_2 \\ 0, & \text{otherwise} \end{cases},$$

$$\left\langle e^{-i2\pi\left(\frac{j_1m}{M}+\frac{k_1n}{N}\right)},\ e^{-i2\pi\left(\frac{j_2m}{M}+\frac{k_2n}{N}\right)} \right\rangle = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} e^{-i2\pi\left(\frac{j_1m}{M}+\frac{k_1n}{N}\right)} \cdot e^{+i2\pi\left(\frac{j_2m}{M}+\frac{k_2n}{N}\right)}$$

$$= \begin{cases} c, & \text{if } j_1=j_2 \text{ and } k_1=k_2 \\ 0, & \text{otherwise} \end{cases}.$$

The Fourier Transform: Discussion (cont'd.)

Therefore

$$\left\{ e^{-i2\pi(ux+vy)} \,\middle|\, u,v,x,y \in \mathbb{R} \right\} \text{ and } \left\{ e^{-i2\pi\left(\frac{jm}{M}+\frac{kn}{N}\right)} \,\middle|\, j,k,m,n,M \in \mathbb{Z} \right\}$$

are orthogonal basis sets. This suggests that function $f(x,y)$ defined on the real plane, and sequence $\{\{ h_{mn} \}\}$ for integers m and n have analogous Fourier representations,

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{+i2\pi(ux+vy)}\, du\, dv \quad \text{and} \quad h_{mn} = \sum_{j=0}^{M-1}\sum_{k=0}^{N-1} H_{jk} e^{+i2\pi\left(\frac{jm}{M}+\frac{kn}{N}\right)}.$$

where the Fourier coefficients are given by

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{-i2\pi(ux+vy)}\, dx\, dy \quad \text{and} \quad H_{jk} = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} h_{mn} e^{-i2\pi\left(\frac{jm}{M}+\frac{kn}{N}\right)}.$$

(True for finite energy functions $f(x,y)$ and $\{\{ h_{mn} \}\}$.)

注意：只有有限能量的f(x,y)和h_mn有FT

Continuous Fourier Transform

$$\mathbf{I}(r,c) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varsigma(v,u)\, e^{+i2\pi(vr+uc)}\, du\, dv$$

$$\varsigma(v,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{I}(r,c)\, e^{-i2\pi(vr+uc)}\, dc\, dr$$

The continuous Fourier transform assumes a continuous image exists in a finite region of an infinite plane.

Discrete Fourier Transform

The discrete Fourier transform assumes a digital image exists on a closed surface, a torus.

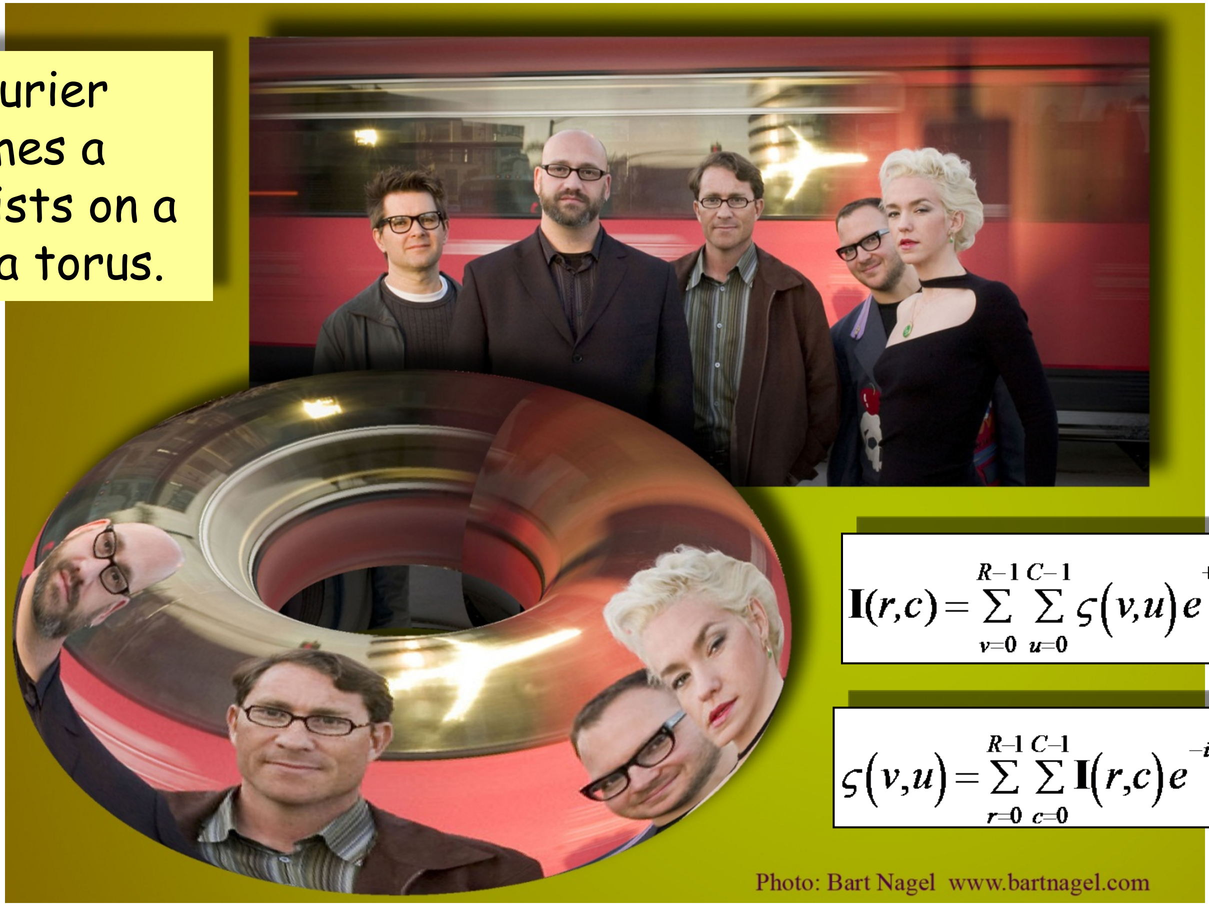


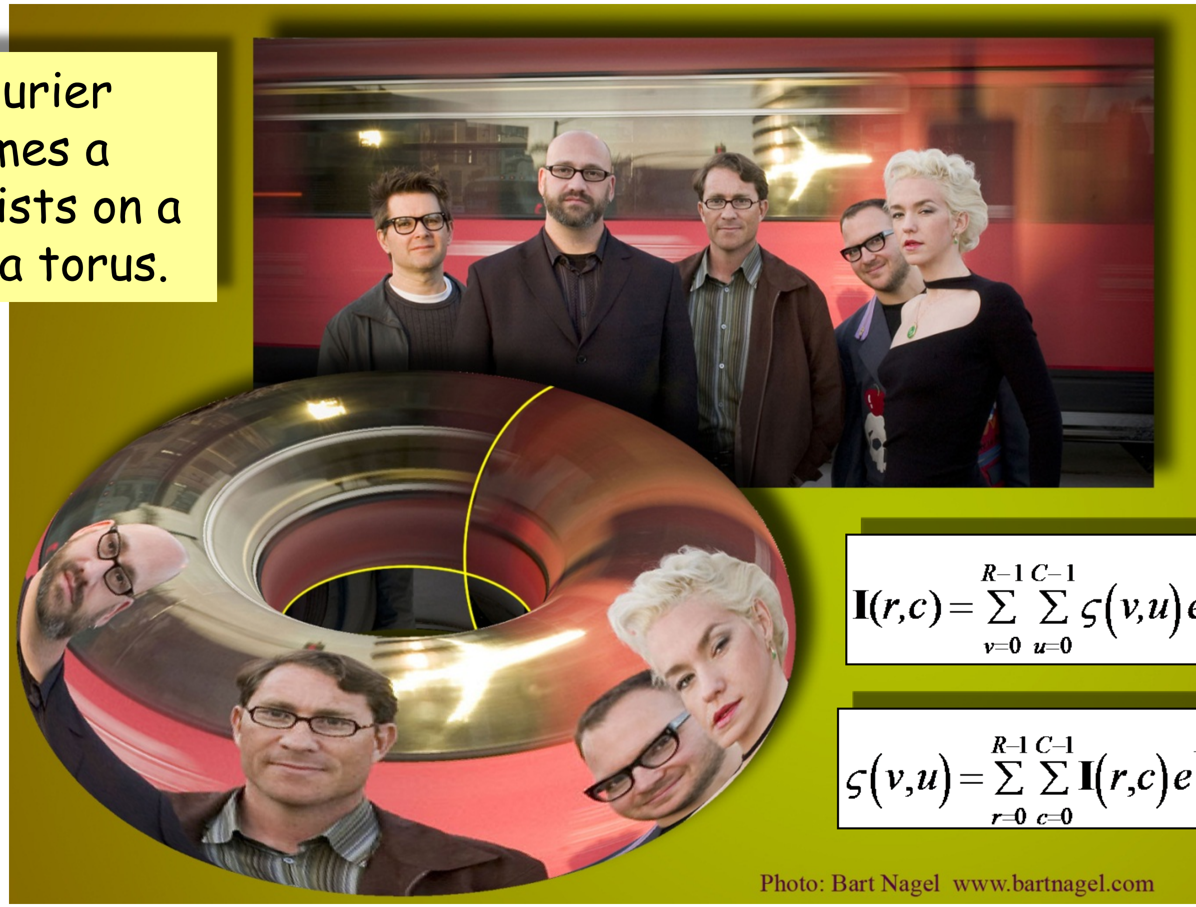
Photo: Bart Nagel www.bartnagel.com

$$\mathbf{I}(r,c) = \sum_{v=0}^{R-1} \sum_{u=0}^{C-1} \varsigma(v,u)\, e^{+i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

$$\varsigma(v,u) = \sum_{r=0}^{R-1} \sum_{c=0}^{C-1} \mathbf{I}(r,c)\, e^{-i2\pi\left(\frac{rv}{R}+\frac{cu}{C}\right)}$$

Discrete Fourier Transform

The discrete Fourier transform assumes a digital image exists on a closed surface, a torus.



$$\mathbf{I}(r,c)=\sum_{v=0}^{R-1}\sum_{u=0}^{C-1}\varsigma(v,u)e^{+i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

$$\varsigma(v,u)=\sum_{r=0}^{R-1}\sum_{c=0}^{C-1}\mathbf{I}(r,c)e^{-i2\pi\left(\frac{rv}{R}+\frac{cu}{C}\right)}$$

相当于水平和垂直方向循环重合（周期），形成一个圆环

The 2D Fourier Transform of a Digital Image

Let $\mathbf{I}(r,c)$ be a single-band (intensity) digital image with R rows and C columns. Then, $\mathbf{I}(r,c)$ has Fourier representation

$$\mathbf{I}(r,c) = \sum_{u=0}^{R-1}\sum_{v=0}^{C-1} \varsigma(v,u)\, e^{+i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)},$$

these complex exponentials are 2D sinusoids.

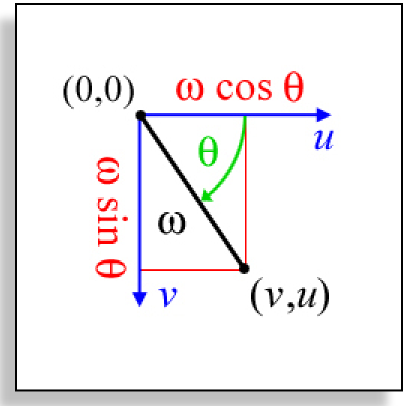
where

$$\varsigma(v,u) = \frac{1}{RC}\sum_{r=0}^{R-1}\sum_{c=0}^{C-1} \mathbf{I}(r,c)\, e^{-i2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)}$$

are the R x C Fourier coefficients.

在2D情况下，频率对(v,u)实际上是一个有方向的频率w在垂直和水平方向的分量

What are 2D sinusoids?



To simplify the situation assume $R = C = N$. Then

$$e^{\pm i 2\pi\left(\frac{vr}{R}+\frac{uc}{C}\right)} = e^{\pm i\frac{2\pi}{N}(vr+uc)} = e^{\pm i\frac{2\pi\omega}{N}(r\sin\theta + c\cos\theta)},$$

where

$$v=\omega\sin\theta,\quad u=\omega\cos\theta,\quad \omega=\sqrt{v^2+u^2},\quad \text{and}\quad \theta=\tan^{-1}\left(\tfrac{v}{u}\right).$$

Write

$$\lambda=\frac{N}{\omega},$$

Note: since images are indexed by row & col with r down and c to the right, θ is positive in the clockwise direction.

Then by Euler's relation,

$$e^{\pm i 2\pi\frac{1}{\lambda}(r\sin\theta+c\cos\theta)} = \cos\left[\tfrac{2\pi}{\lambda}\left(r\sin\theta+c\cos\theta\right)\right] \pm i\sin\left[\tfrac{2\pi}{\lambda}\left(r\sin\theta+c\cos\theta\right)\right].$$

Cont'd. on next page.

What are 2D sinusoids? (cont'd.)

Both the real part of this,

$$\mathrm{Re}\left\{ e^{\pm i2\pi\frac{1}{\lambda}(r\sin\theta + c\cos\theta)} \right\} = +\cos\left[\tfrac{2\pi}{\lambda}\left(r\sin\theta + c\cos\theta\right)\right]$$

and the imaginary part,

$$\mathrm{Im}\left\{ e^{\pm i2\pi\frac{1}{\lambda}(r\sin\theta + c\cos\theta)} \right\} = \pm\sin\left[\tfrac{2\pi}{\lambda}\boxed{\left(r\sin\theta + c\cos\theta\right)}\right]$$

are sinusoidal "gratings" of unit amplitude, period λ and direction θ.

方向为ci，周期为lamda的正弦隔栅

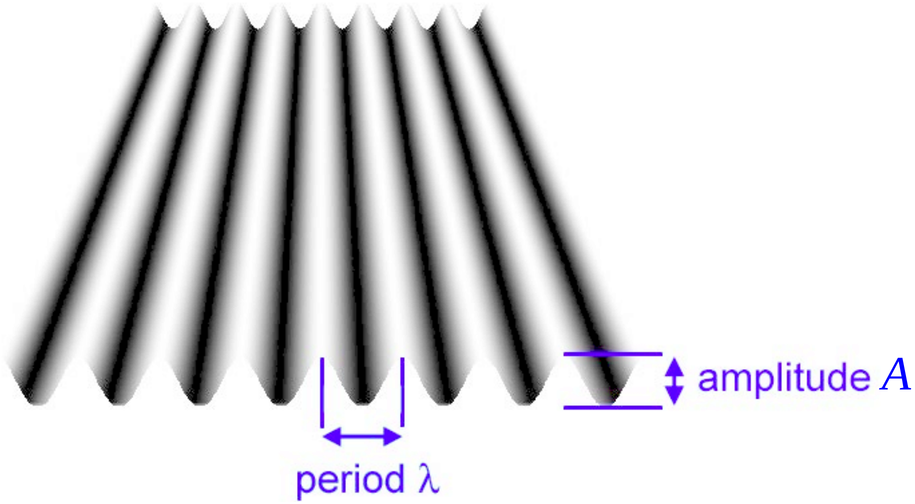
Then $\frac{2\pi\omega}{N}$ is the radian frequency, and $\frac{\omega}{N}$ the frequency, of the $\boxed{\text{wavefront}}$

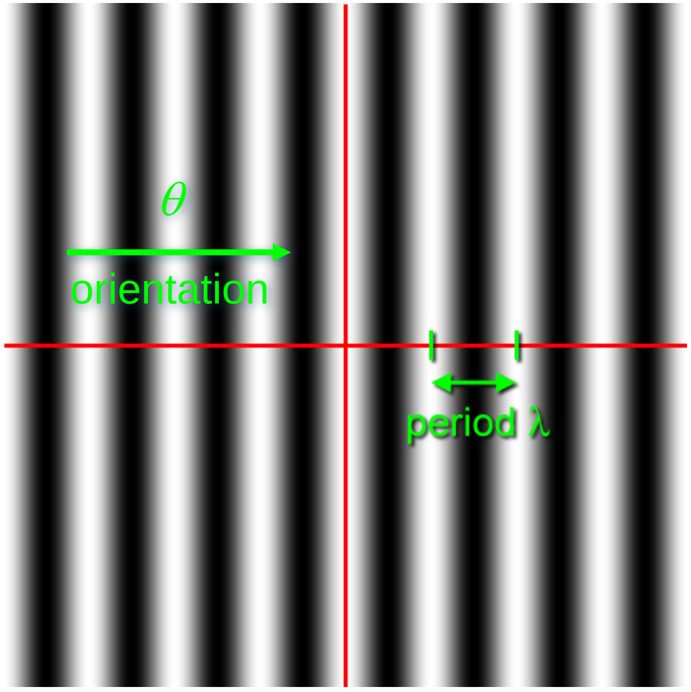
and $\lambda = \frac{N}{\omega}$ is the wavelength in pixels in the wavefront direction.

2D Sinusoids:

$$\mathbf{I}(r,c)=\frac{A}{2}\left\{\cos\left[\frac{2\pi}{\lambda}(r\cdot\sin\theta+c\cdot\cos\theta)+\varphi\right]+1\right\}$$

... are plane waves with grayscale amplitudes, periods in terms of lengths, ...

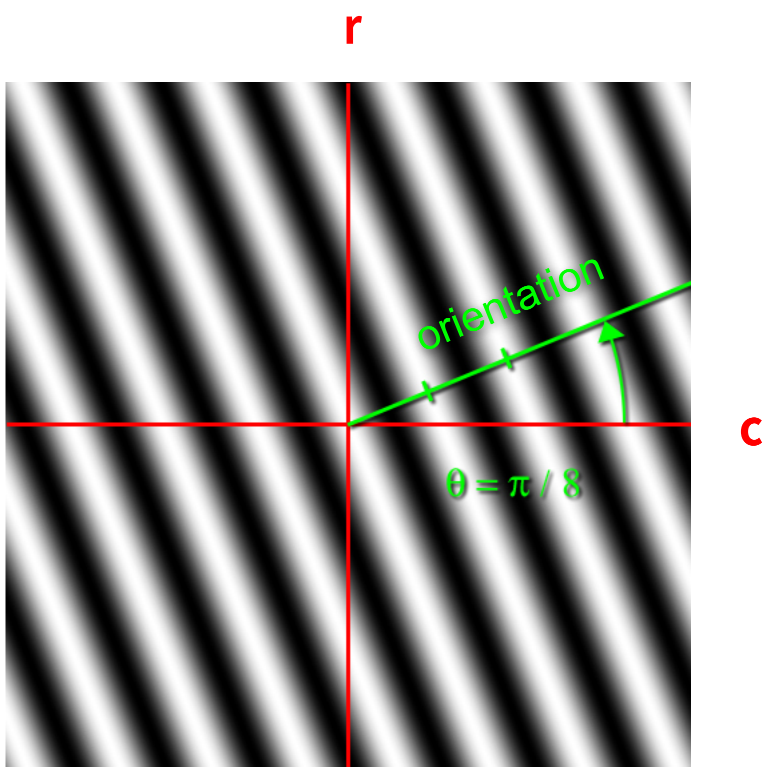


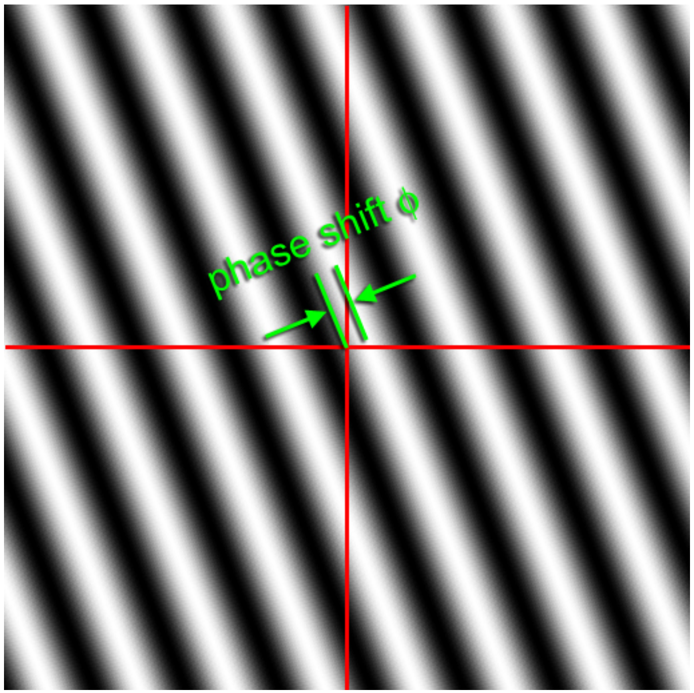


ϕ = phase shift

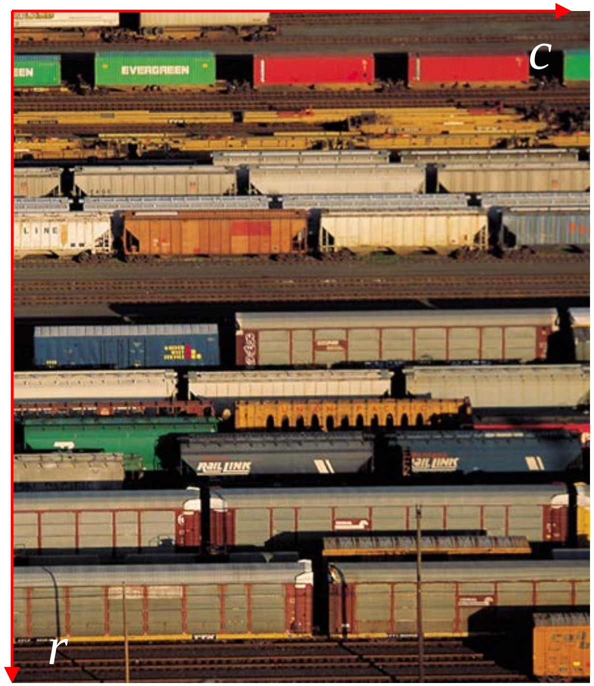
2D Sinusoids:

... specific orientations,
and phase shifts.

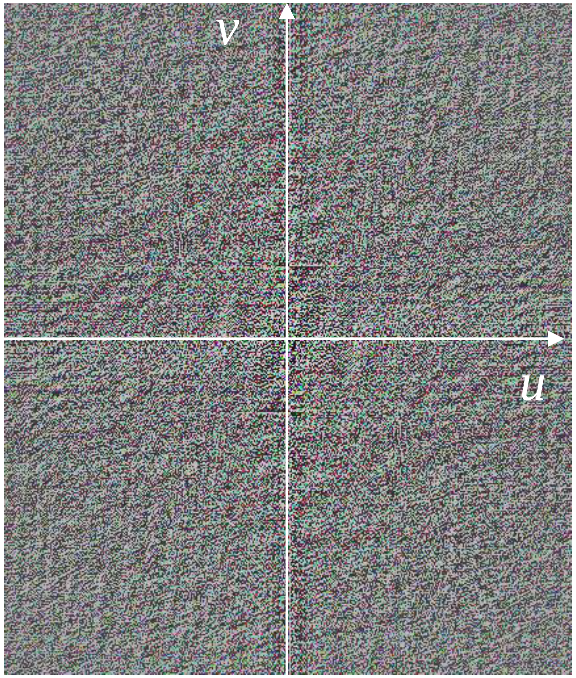




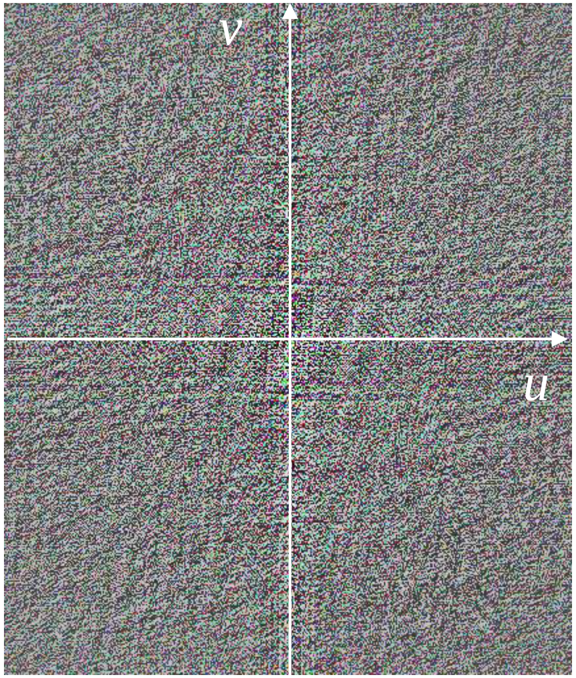
The Fourier Transform of an Image



$\mathbf{I}$



$\mathrm{Re}[\mathfrak{F}\{\mathbf{I}\}]$



$\mathrm{Im}[\mathfrak{F}\{\mathbf{I}\}]$

Points on the Fourier Plane (of a Digital Image)

In the Fourier transform of an $R \times C$ digital image, positions u and v indicate the number of repetitions of the sinusoid in those directions. Therefore the wavelengths along the column and row axes are

$$\lambda_u = \frac{C}{u} \quad \text{and} \quad \lambda_v = \frac{R}{v} \quad \text{pixels},$$

and the wavelength in the wavefront direction is

$$\lambda_{\mathrm{wf}} = \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2}.$$

The frequency is the fraction of the sinusoid traversed over one pixel,

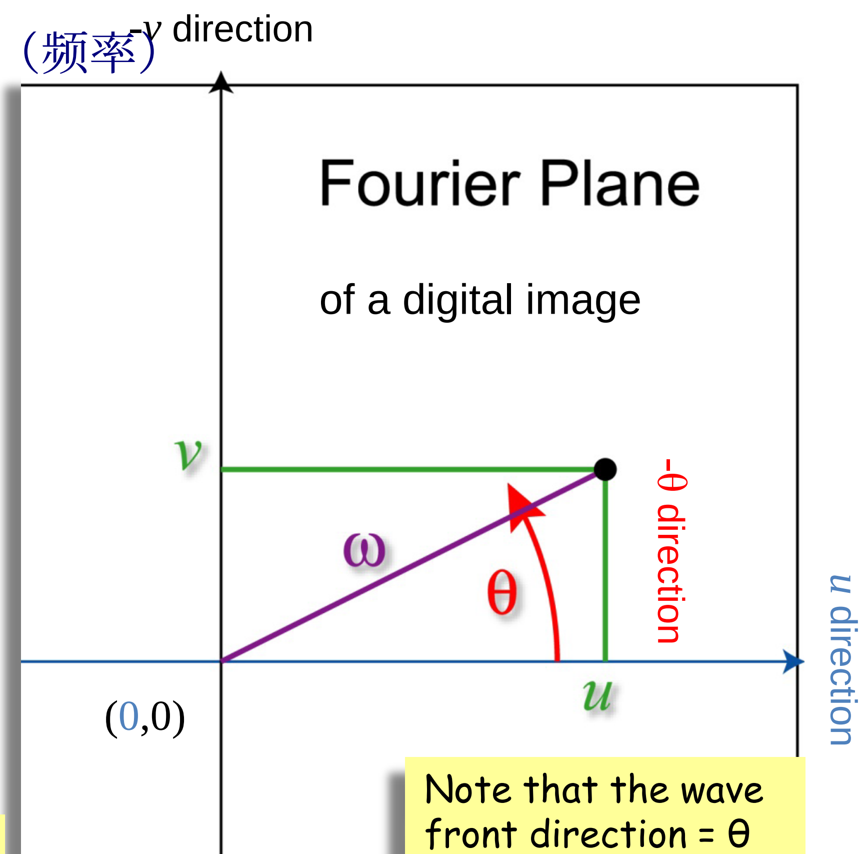
$$\omega_u = \frac{u}{C}, \ \omega_v = \frac{v}{R}, \text{ and}$$

$$\omega_{\mathrm{wf}} = 1 \Big/ \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2} \quad \text{cycles.}$$

The wavefront direction is given by

$$\theta_{\mathrm{wf}} = \tan^{-1}\left(\frac{\omega_v}{\omega_u}\right) = \boxed{\tan^{-1}\left(\frac{v\,C}{u\,R}\right)}.$$

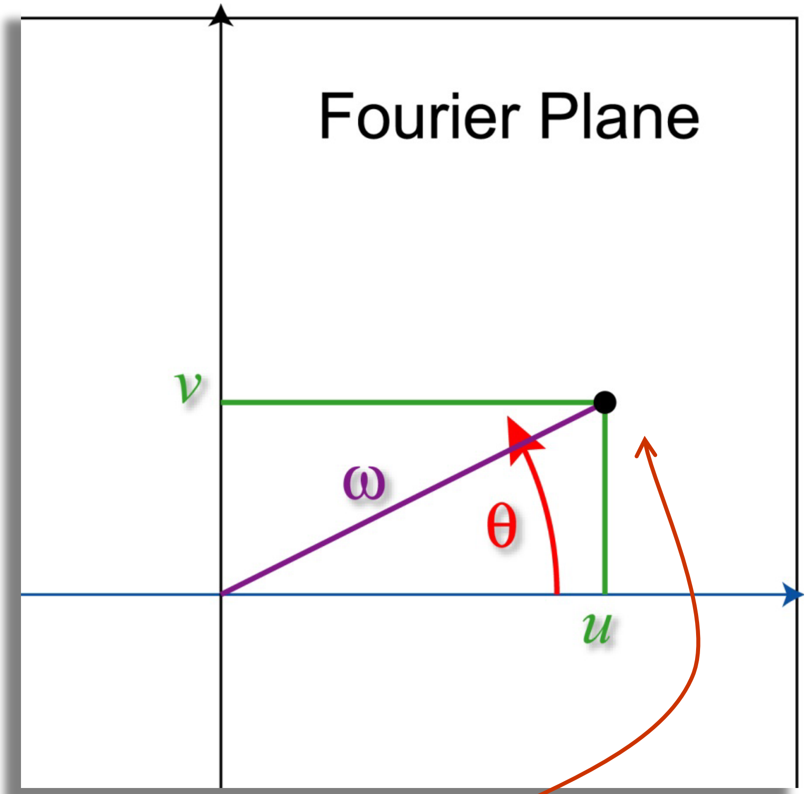
$\frac{\text{row freq.}}{\text{column freq.}}$

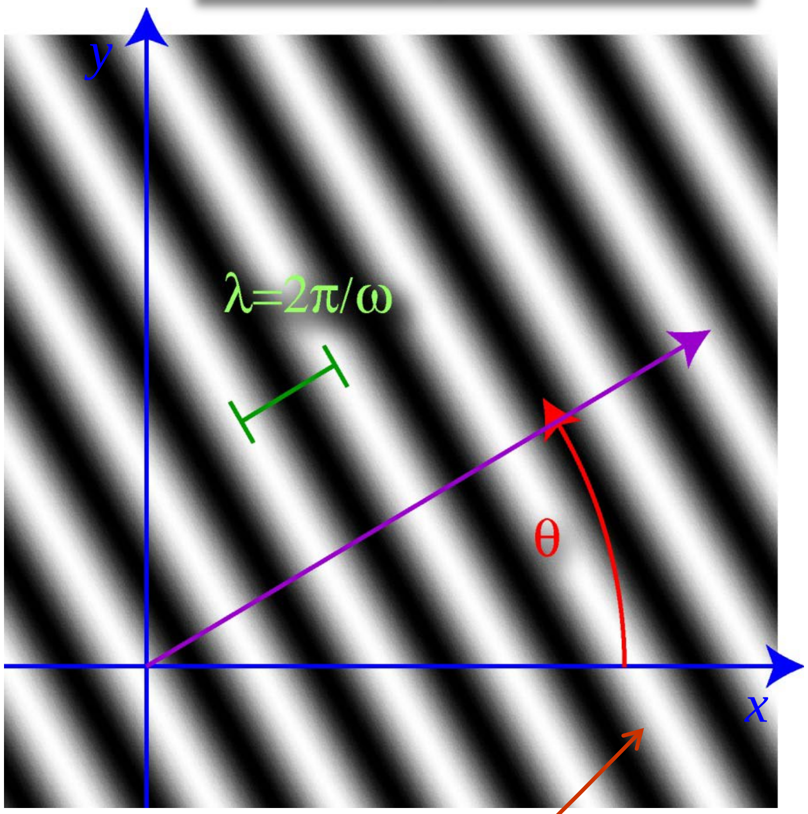


Note that the wave front direction = θ only if R=C.

Points on the Fourier Plane

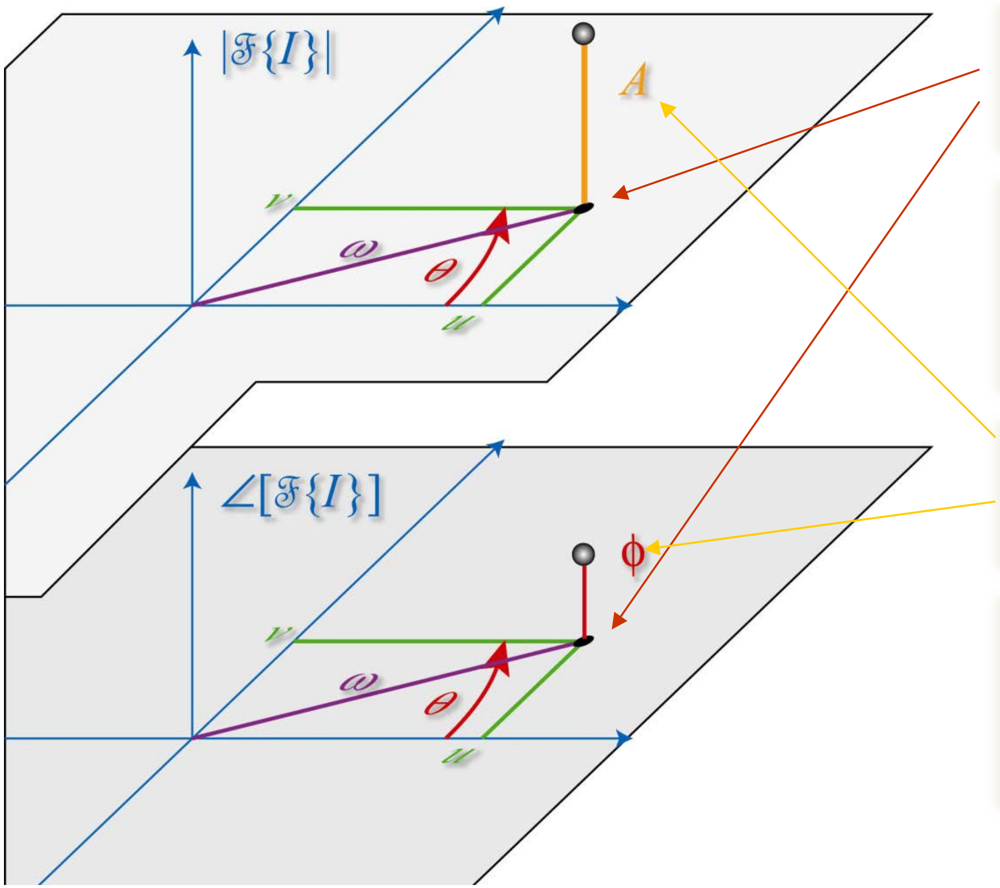
Note that θ is the wavefront direction only if R=C.





This point represents this particular sinusoidal grating

The Fourier Coefficient at (u,v)



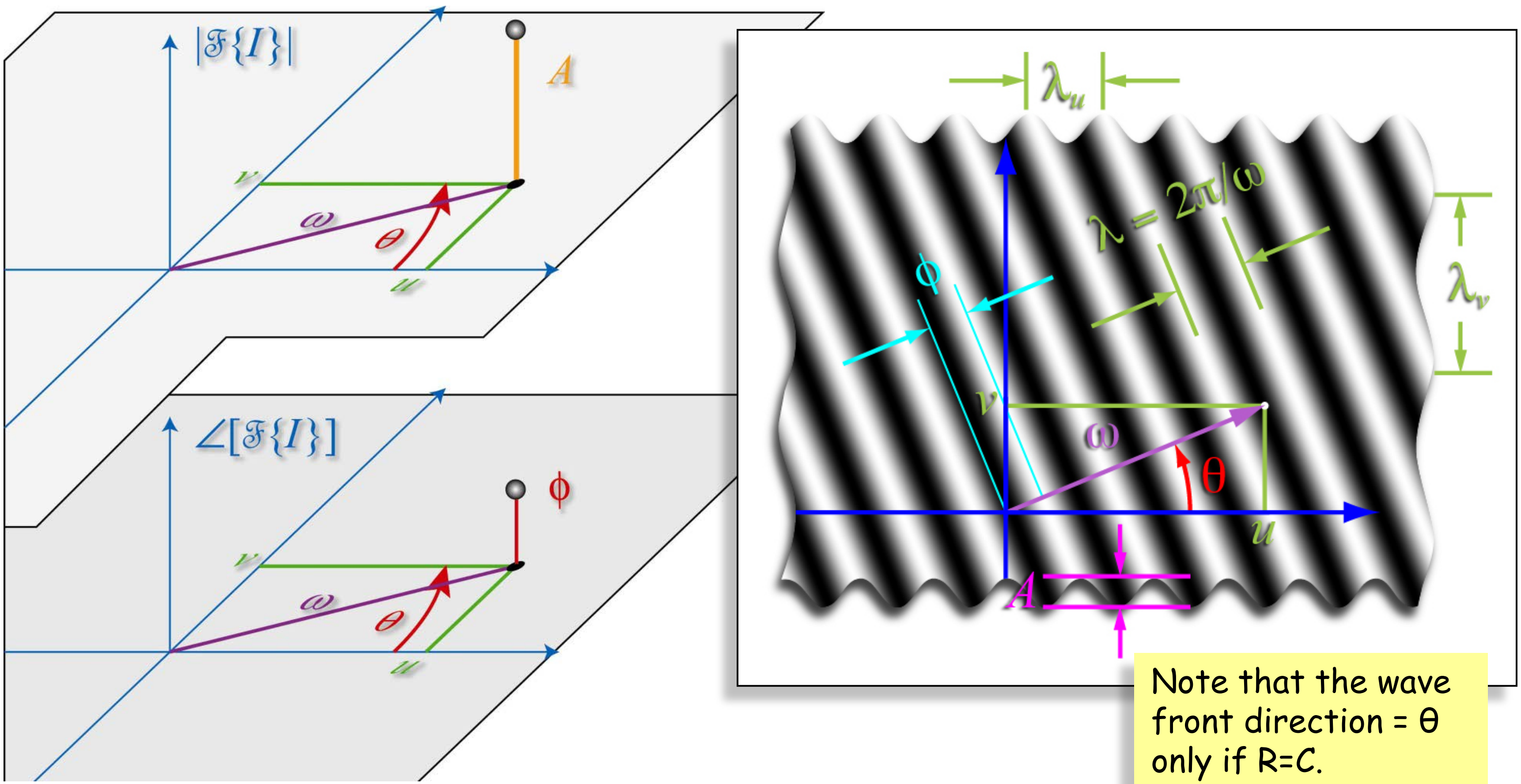
So, the point (u,v) on the Fourier plane…

…represents a sinusoidal grating of frequency ω and orientation θ.*

The complex value, $F(u,v)$, of the FT at point (u,v)…

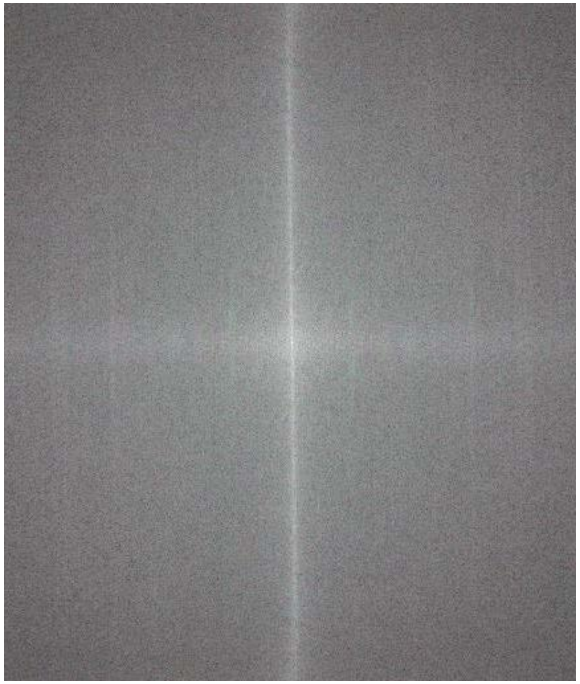
…represents the amplitude, A, and the phase offset, ϕ, of the sinusoid.

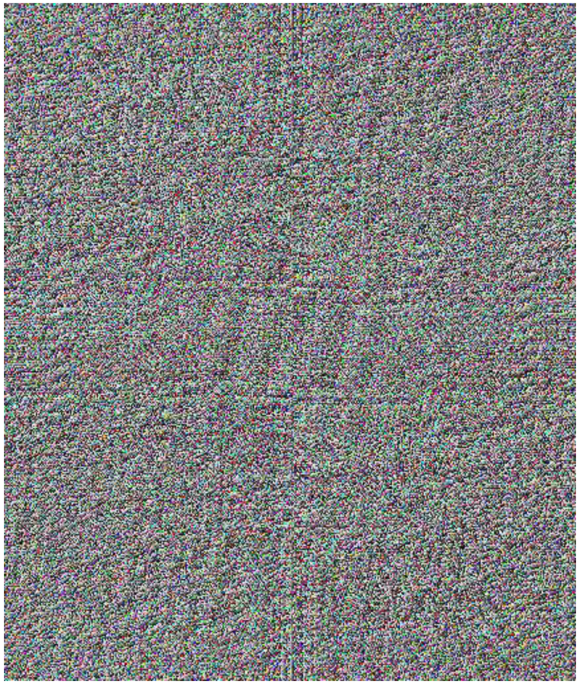
The Sinusoid from the Fourier Coeff. at (*u*,*v*)



Note that the wave front direction = θ only if R=C.

FT of an Image (Magnitude + Phase)

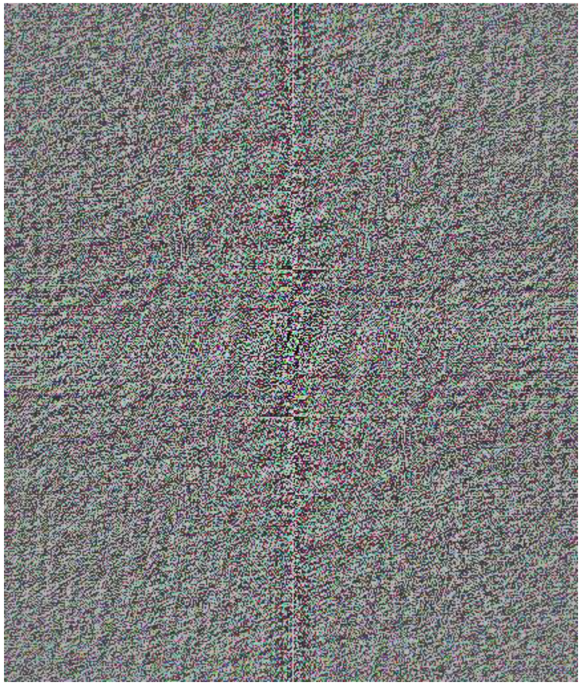




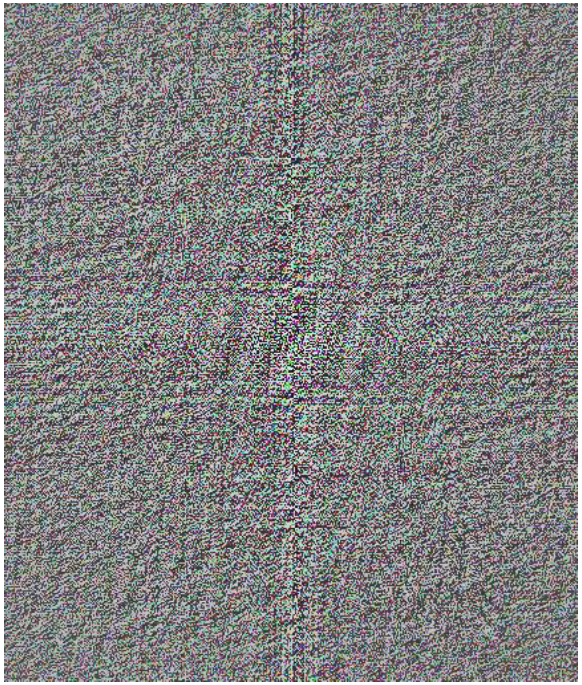
$\mathbf{I}$ $\quad$ $\log\{|\mathfrak{F}\{\mathbf{I}\}|^2+1\}$ $\quad$ $\angle[\mathfrak{F}\{\mathbf{I}\}]$

FT of an Image (Real + Imaginary)

$\mathbf{I}$



$\mathrm{Re}[\mathcal{F}\{\mathbf{I}\}]$



$\mathrm{Im}[\mathcal{F}\{\mathbf{I}\}]$

The Power Spectrum

For display, the log of the power spectrum is often used.

The power spectrum of a signal is the square of the magnitude of its Fourier Transform.

$$
\begin{aligned}
\left|\varsigma(u,v)\right|^2 &= \varsigma(u,v)\varsigma^*(u,v) \\
&= \left[\mathrm{Re}\,\varsigma(u,v) + i\,\mathrm{Im}\,\varsigma(u,v)\right]\left[\mathrm{Re}\,\varsigma(u,v) - i\,\mathrm{Im}\,\varsigma(u,v)\right] \\
&= \left[\mathrm{Re}\,\varsigma(u,v)\right]^2 + \left[\mathrm{Im}\,\varsigma(u,v)\right]^2 .
\end{aligned}
$$

At each location (u,v) it indicates the squared intensity of the frequency component with period $\lambda = 1\big/\sqrt{u^2+v^2}$ and orientation $\theta = \tan^{-1}(v/u)$.

```
For display in Matlab:
PS = fftshift(2*log(abs(fft2(I))+1));
```

On the Computation of the Power Spectrum

The power spectrum (PS) is defined by $\mathrm{PS}(\mathrm{I}) = \left| \varsigma\left\{ \mathrm{I}(u,v) \right\} \right|^2.$

We take the base-e logarithm of the PS in order to view it. Otherwise its dynamic range could be too large to see everything at once. We add 1 to it first so that the minimum value of the result is 0 rather than–infinity, which it would be if there were any zeros in the PS. Recall that

$\log(f^2) = 2\log(f)$.

Multiplying by 2 is not necessary if you are generating a PS for viewing, since you'll probably have to scale it into the range 0-255 anyway. It is much easier to see the structures in a Fourier plane if the origin is in the center. Therefore we usually perform an fftshift on the PS before it is displayed.

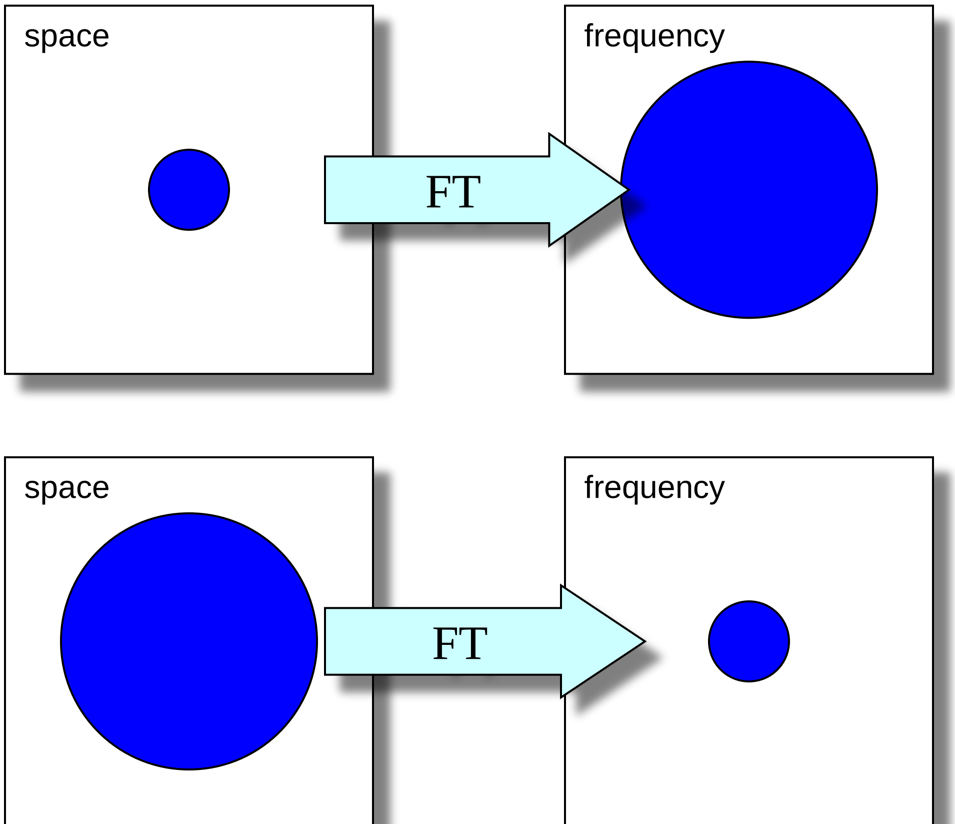
```
>> PS = fftshift(log(abs(fft2(I))+1));
>> M  = max(PS(:));
>> image(uint8(255*(PS/M)));
```

If the PS is being calculated for later computational use -- for example the autocorrelation of a function is the inverse FT of the PS of the function -- it should be calculated by

```
>> PS = abs(fft2(I)).^2;
```

测不准原理：
1. 一种信号表达如果有较好的频率分辨率，则其空间分辨率较差。反之亦然。
2. 信号的空域表达和频域表达是2种极端的表达
3. 小波变换是时-频表达，折衷

The Uncertainty Relation

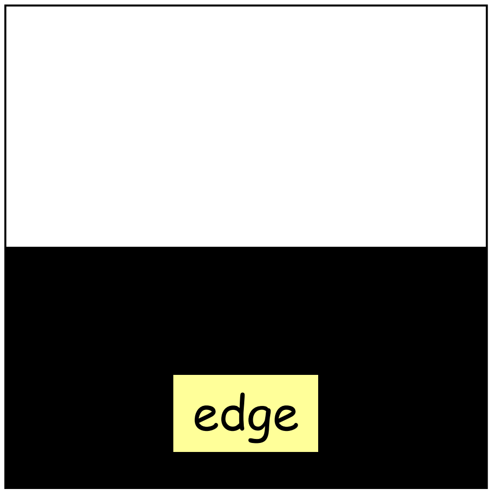


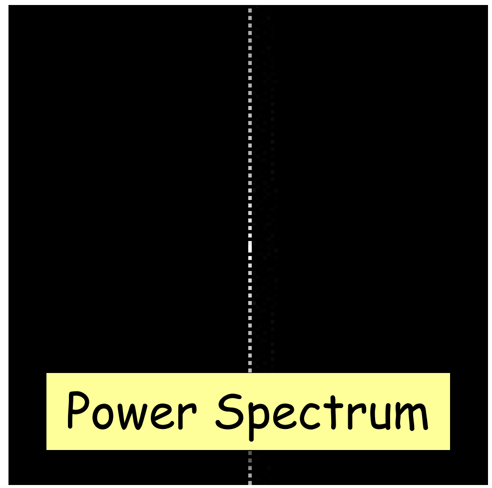
If $\Delta x \, \Delta y$ is the extent of the object in space and if $\Delta u \, \Delta v$ is its extent in frequency then,

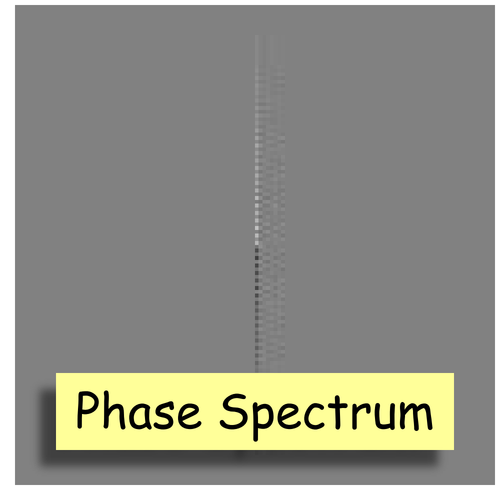
$$\Delta x \, \Delta y \cdot \Delta u \, \Delta v \geq \frac{1}{16\pi^2}$$

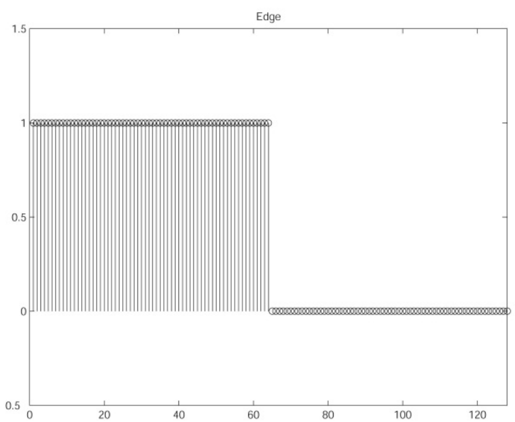
A small object in space has a large frequency extent and vice-versa.

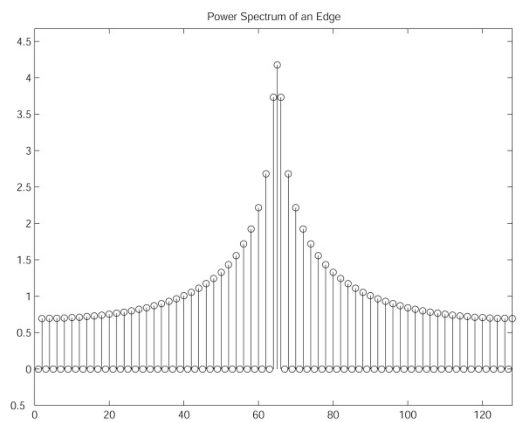
The Fourier Transform of an Edge

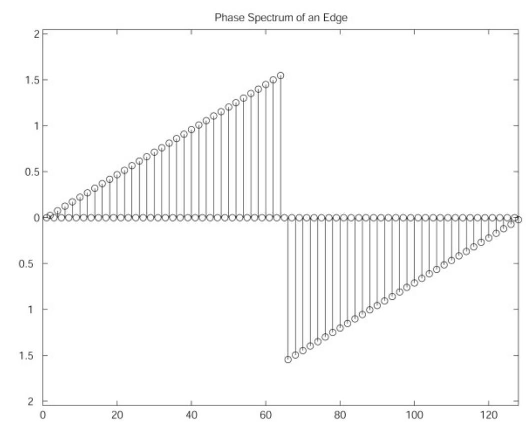




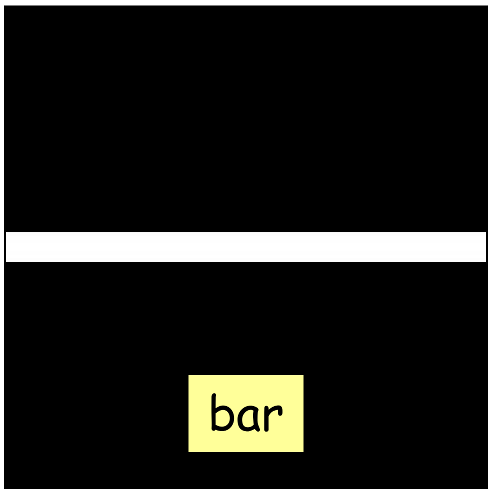


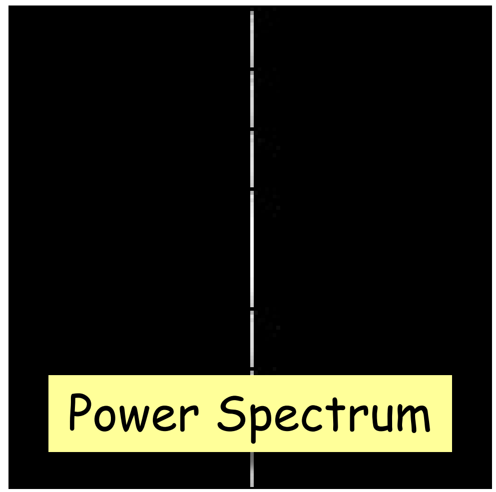


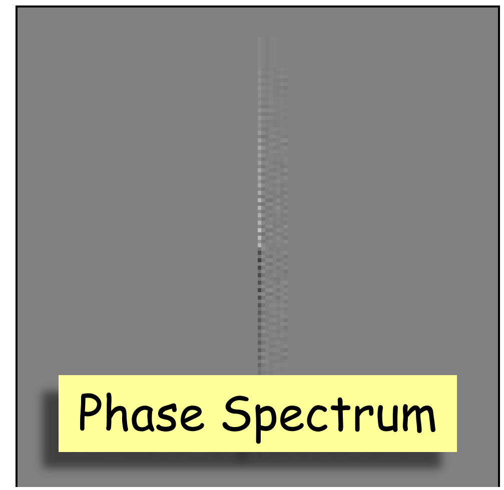


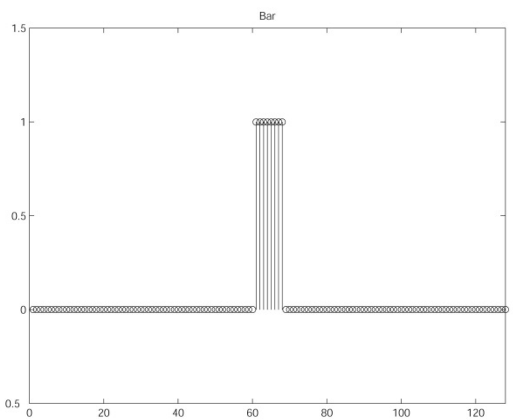


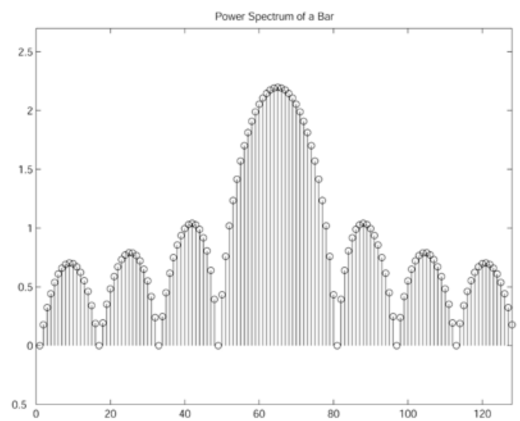
The Fourier Transform of a Bar

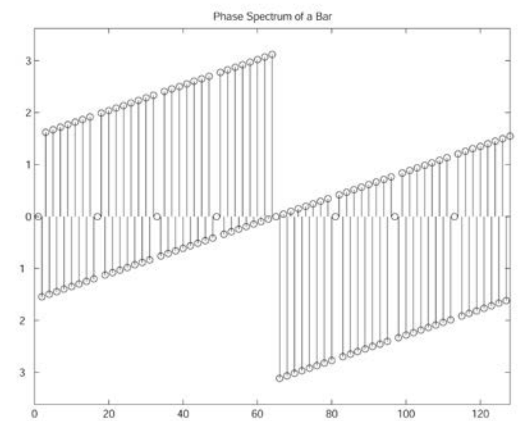












Coordinate Origin of the FFT

Center = (floor(R/2)+1, floor(C/2)+1)

Even | Odd | Even | Odd

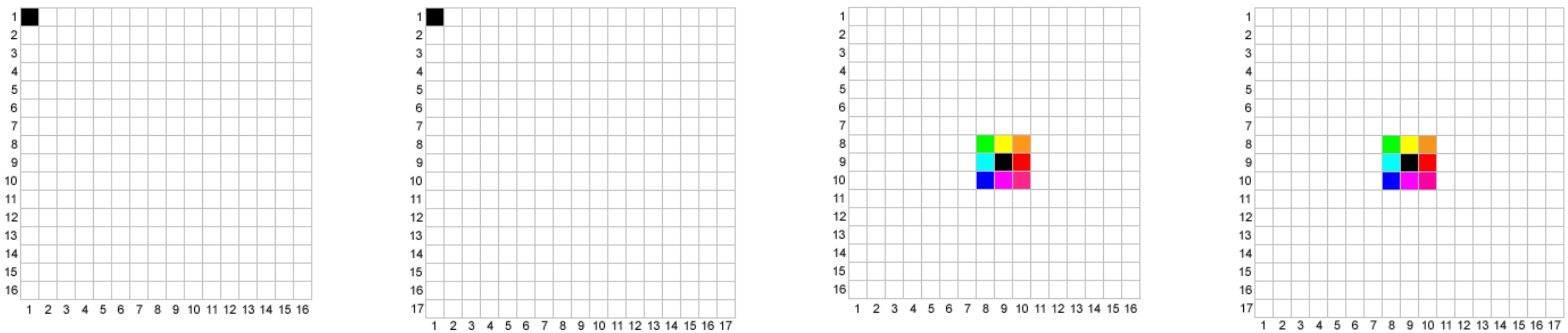
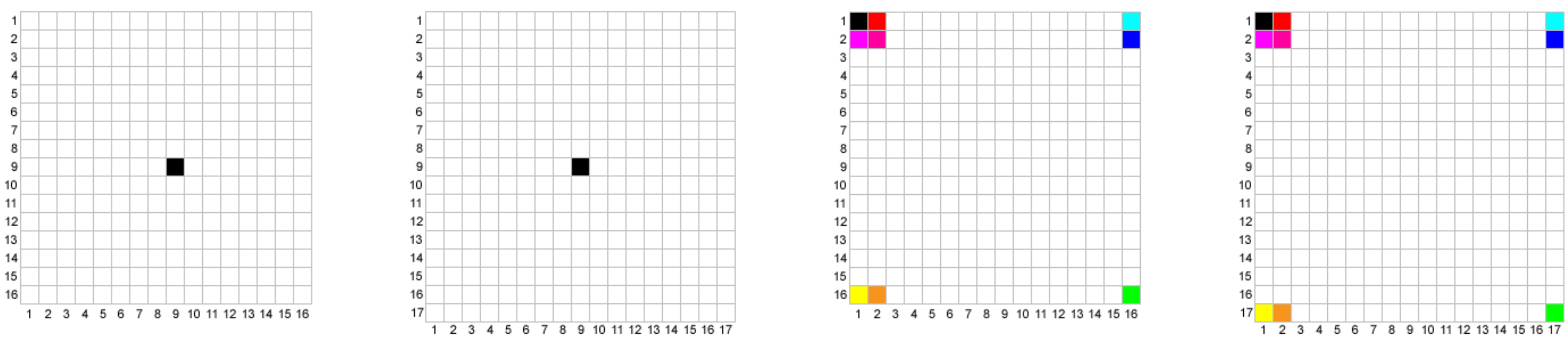


Image Origin | Image Origin | Weight Matrix Origin | Weight Matrix Origin



After FFT shift | After FFT shift | After IFFT shift | After IFFT shift

Matlab’s `fftshift` and `ifftshift`

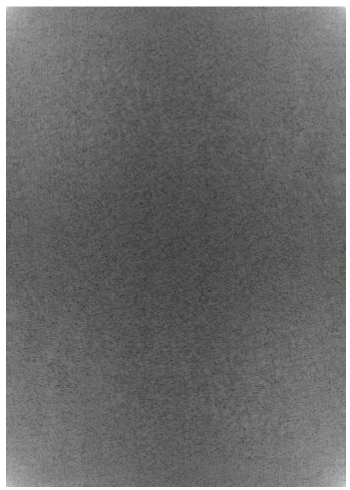
i — 逆变换

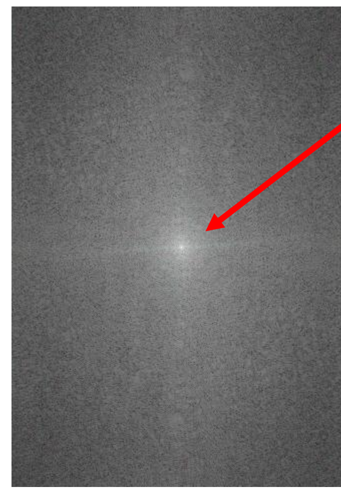
`I = ifftshift(J):`

`J = fftshift(I):`

origin

from FFT2
or ifftshift





origin

after fftshift

$\mathbf{J}\,(\lfloor R/2 \rfloor + 1, \lfloor C/2 \rfloor + 1) \rightarrow \mathbf{I}\,(1,1)$

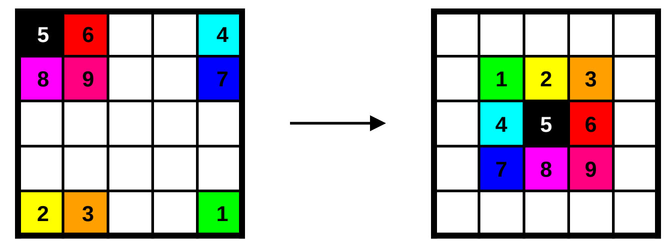
$\mathbf{I}\,(1,1) \rightarrow \mathbf{J}\,(\lfloor R/2 \rfloor + 1, \lfloor C/2 \rfloor + 1)$

where $\lfloor x \rfloor$ = `floor(x)` = the largest integer smaller than x.

Matlab's `fftshift` and `ifftshift`

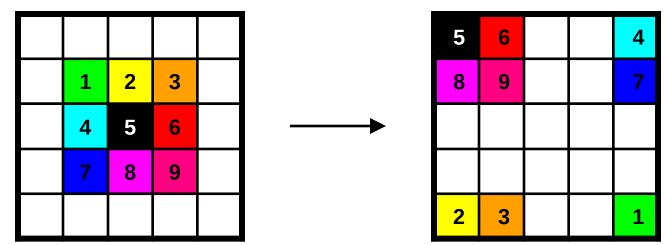
`J = fftshift(I):`

$$\mathbf{I}\,(1,1) \rightarrow \mathbf{J}\,(\lfloor R/2 \rfloor +1, \lfloor C/2 \rfloor +1)$$



`I = ifftshift(J):`

$$\mathbf{J}\,(\lfloor R/2 \rfloor +1, \lfloor C/2 \rfloor +1) \rightarrow \mathbf{I}\,(1,1)$$

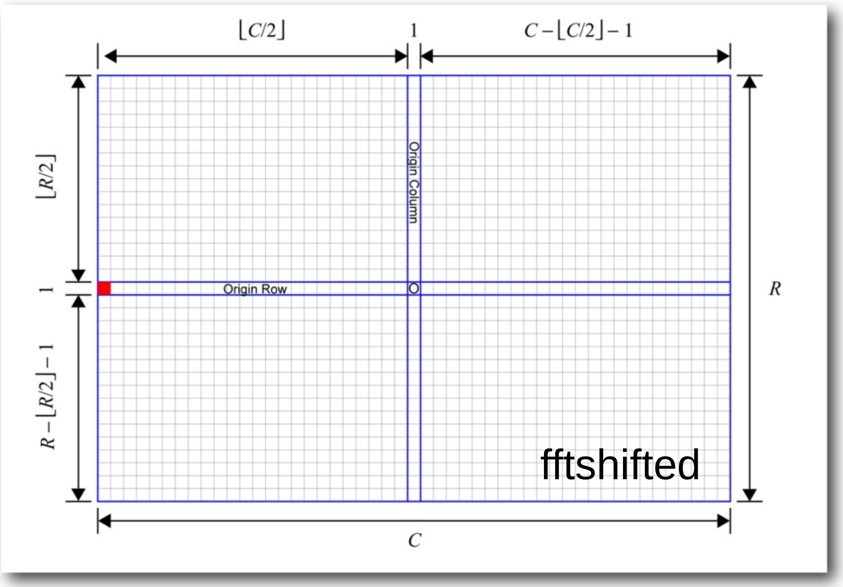


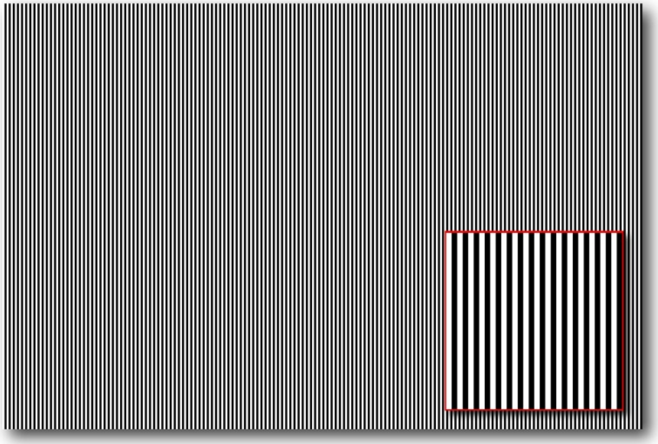
where $\lfloor x \rfloor$ = `floor(x)` = the largest integer smaller than x.

Inverse FFTs of Impulses

1. 频域脉冲，垂直频率u=0，水平频率v/=0
2. 水平方向有变化

“horizontal” is the wavefront direction.



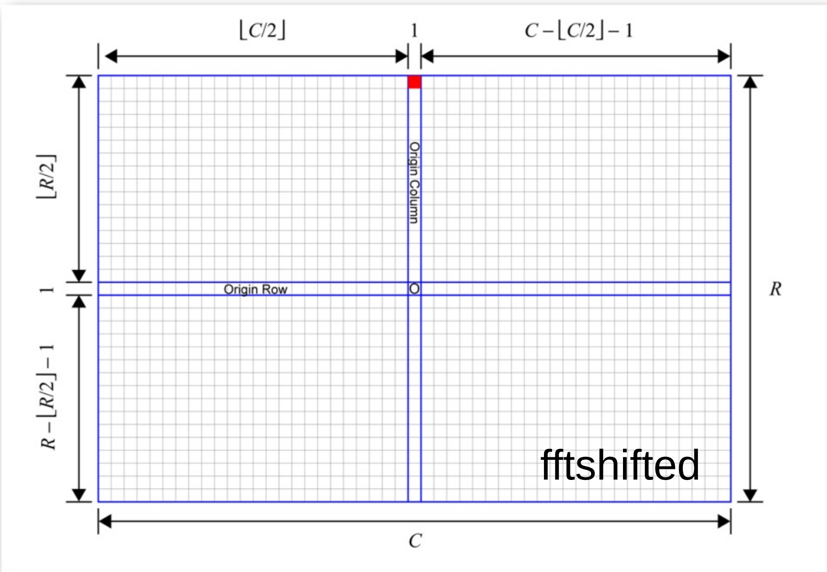


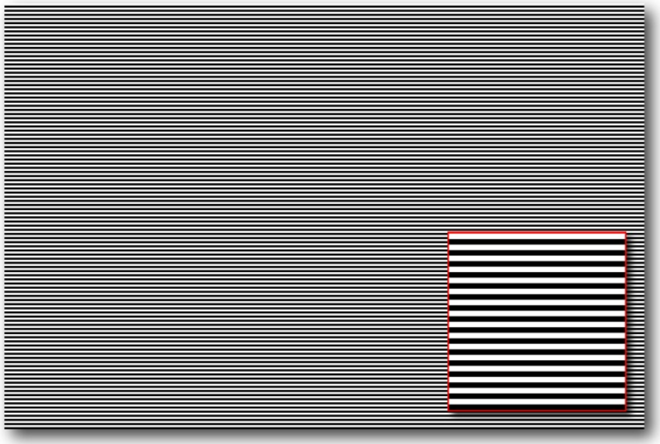
highest-possible-frequency horizontal sinusoid (*C* is even)

Inverse FFTs of Impulses

1. 水平频率=0，垂直频率/=0
2. 垂直方向有变化

"vertical" is the wavefront direction.

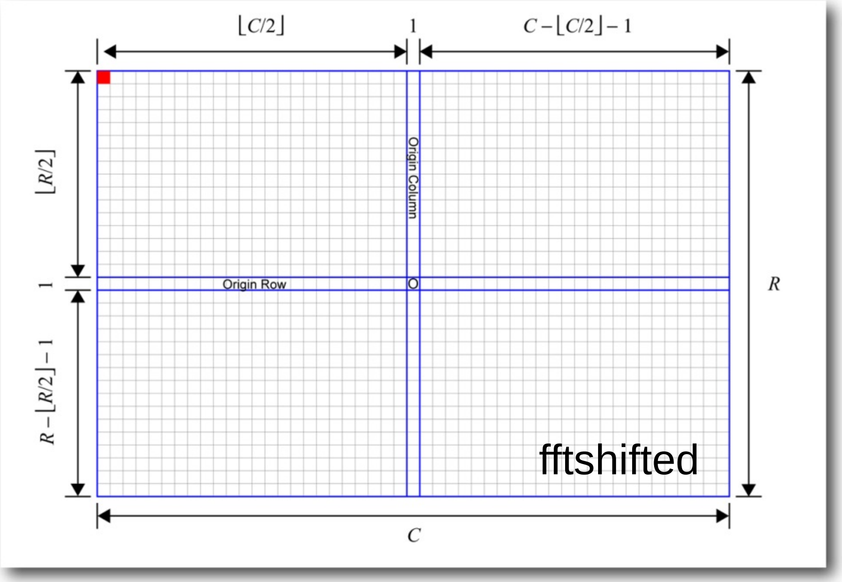


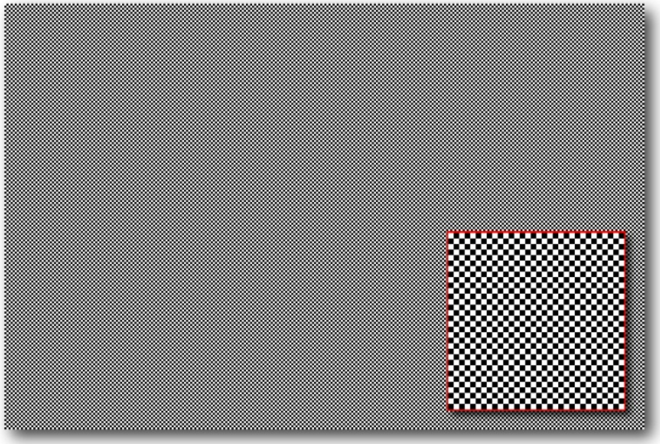


highest-possible-frequency vertical sinusoid (*R* is even)

Inverse FFTs of Impulses

a checker-board pattern.

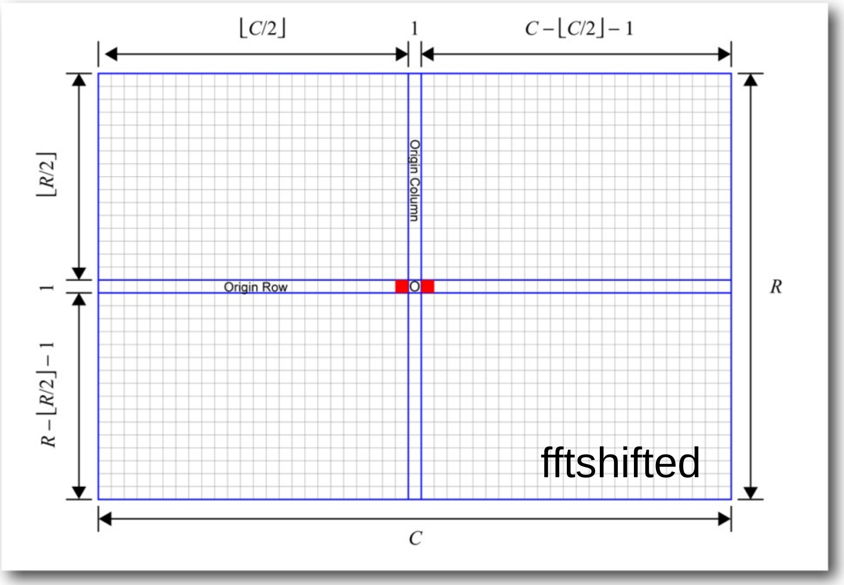


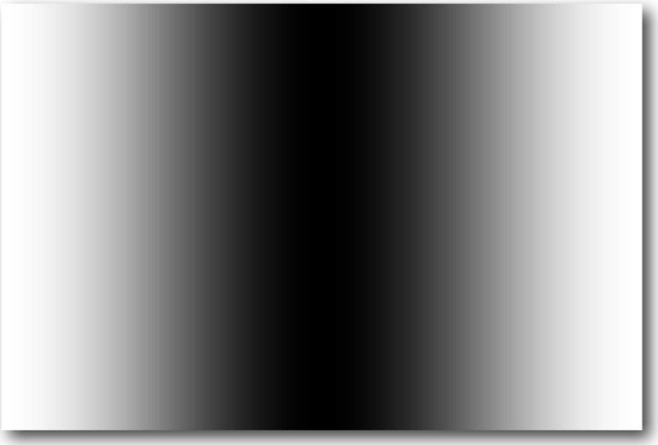


highest-possible-freq horizontal+vertical sinusoid (R & C even)

Inverse FFTs of Impulses

"horizontal" is the wavefront direction.

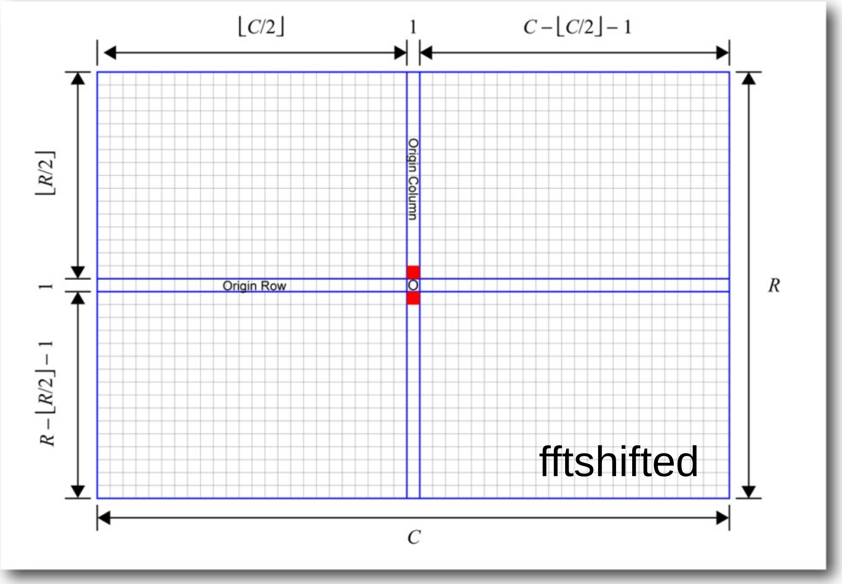


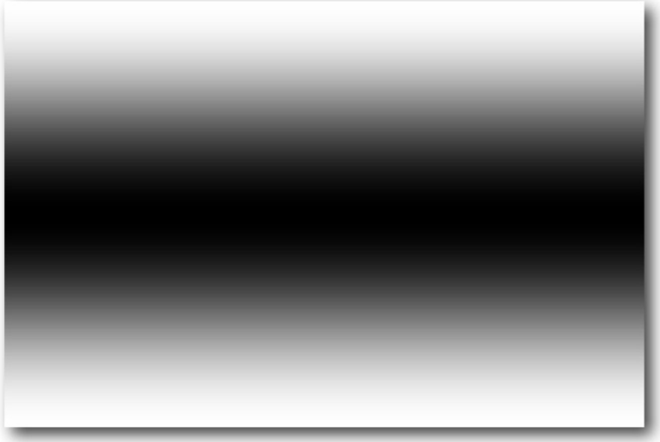


lowest-possible-frequency horizontal sinusoid

Inverse FFTs of Impulses

“vertical” is the wavefront direction.

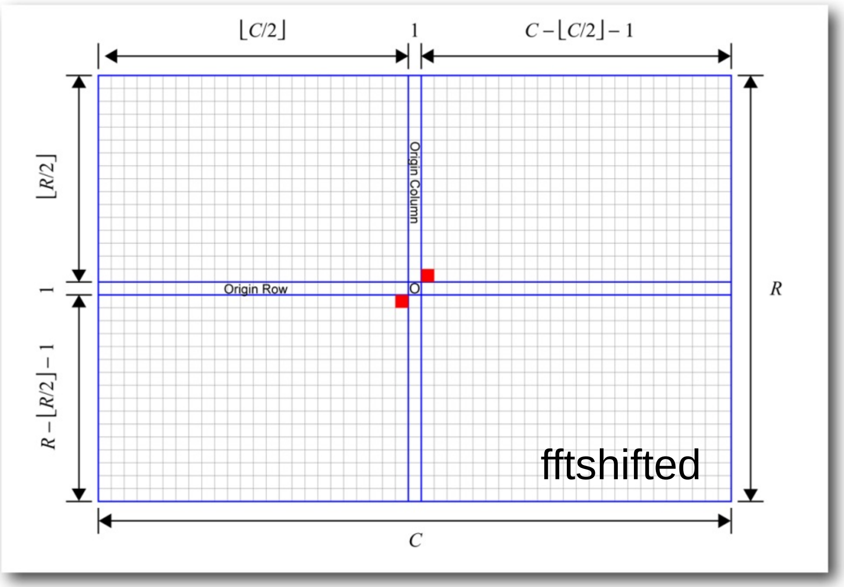


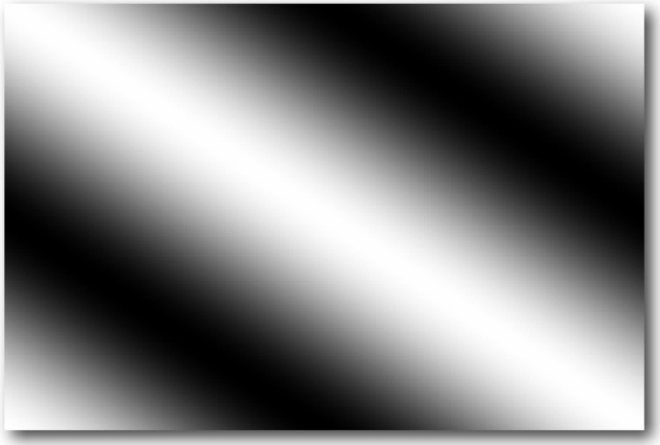


lowest-possible-frequency vertical sinusoid

Inverse FFTs of Impulses

"negative diagonal" is the wavefront direction.

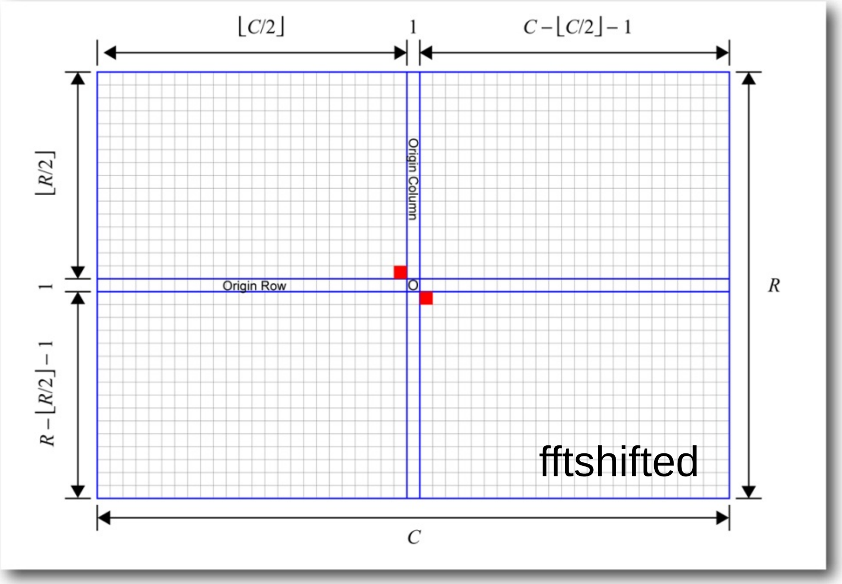


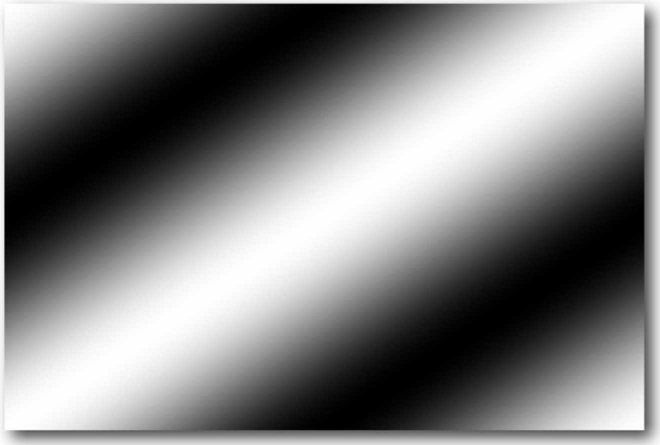


lowest-possible-frequency negative diagonal sinusoid

Inverse FFTs of Impulses

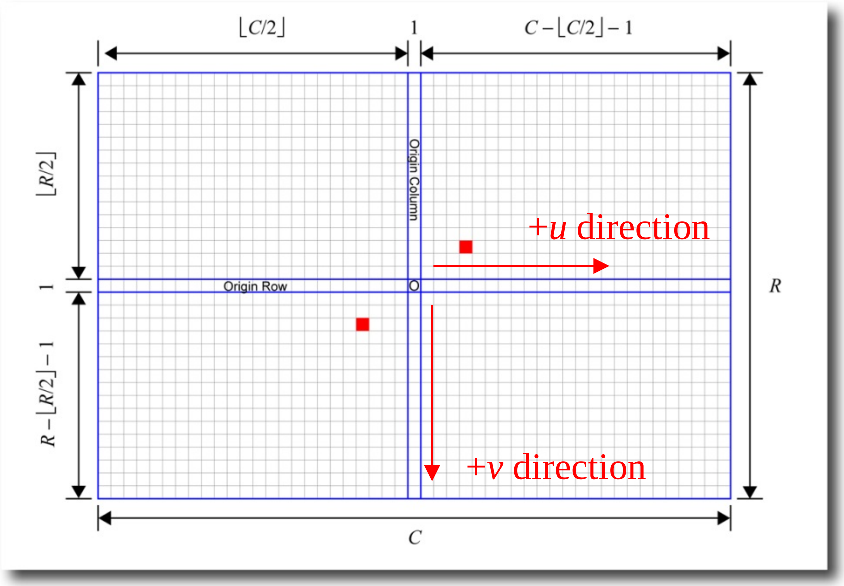
"positive diagonal" is the wavefront direction.

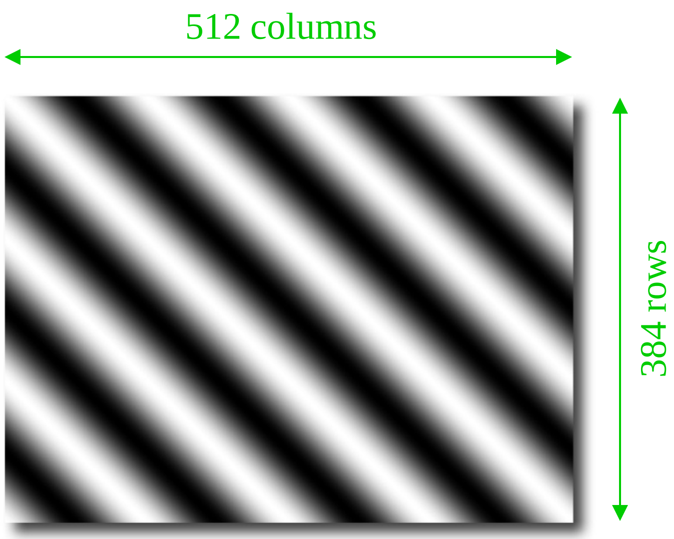




lowest-possible-frequency positive diagonal sinusoid

Frequencies and Wavelengths in the Fourier Plane





Note this …

… and this.

frequencies: $(u,v) = (4,3)$; wavelengths: $(\lambda_u, \lambda_v) = (128,128)$

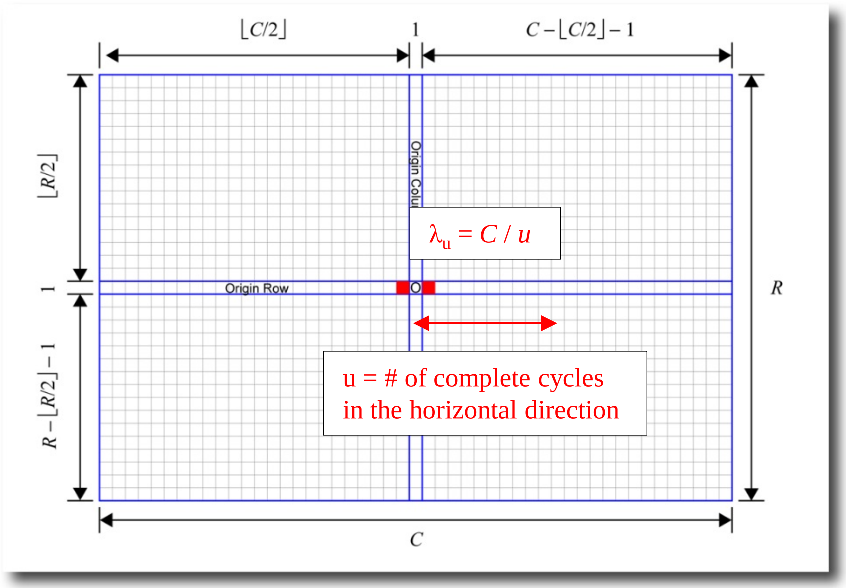
How can that be?

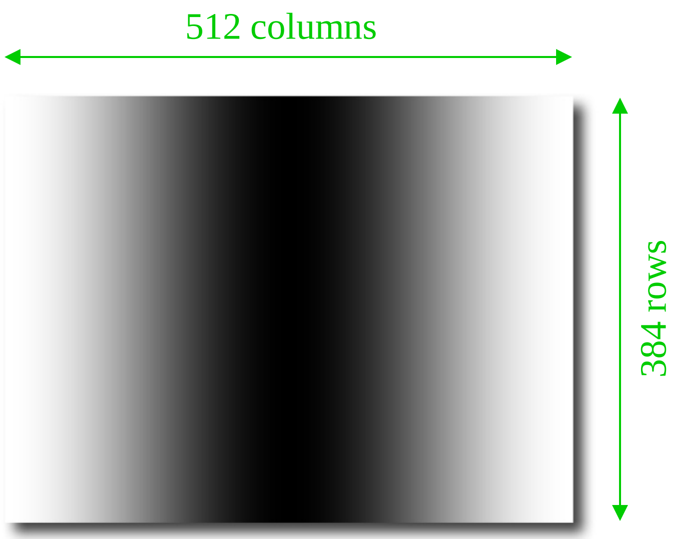
还记得前面的公式？

C/4

R/3

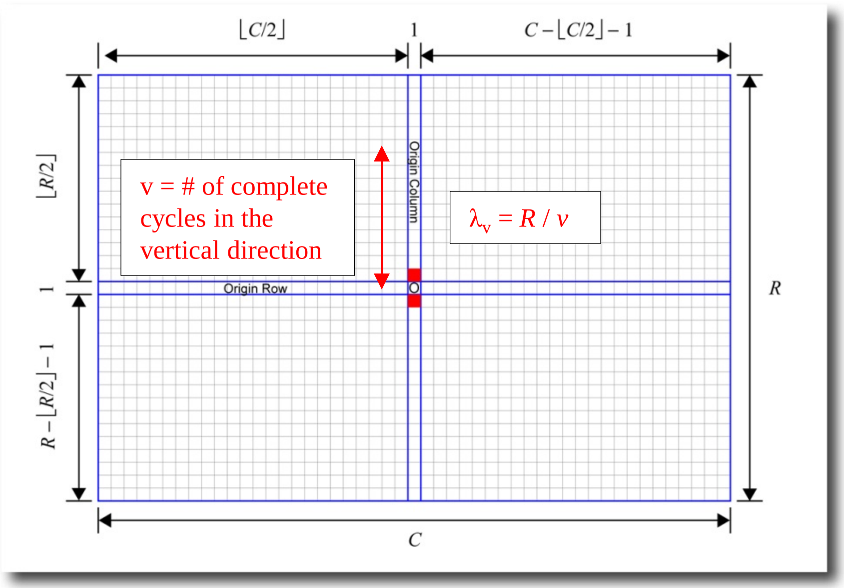
Frequencies and Wavelengths in the Fourier Plane

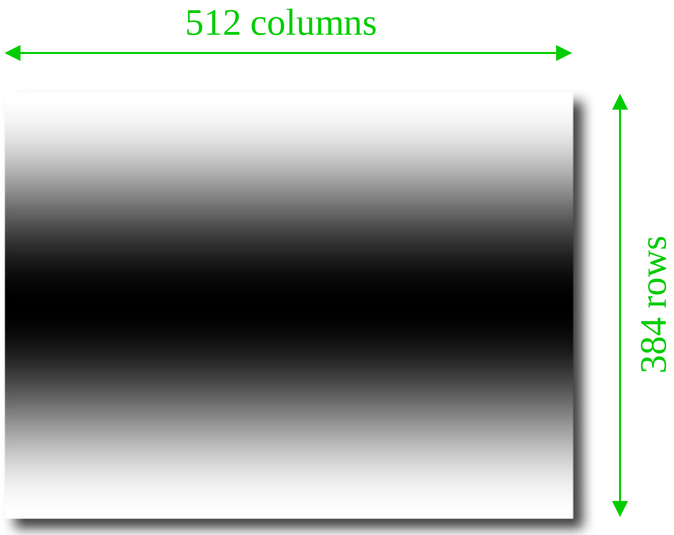




frequencies: $(u,v) = (1,0)$; wavelength: $\lambda_u = 512$

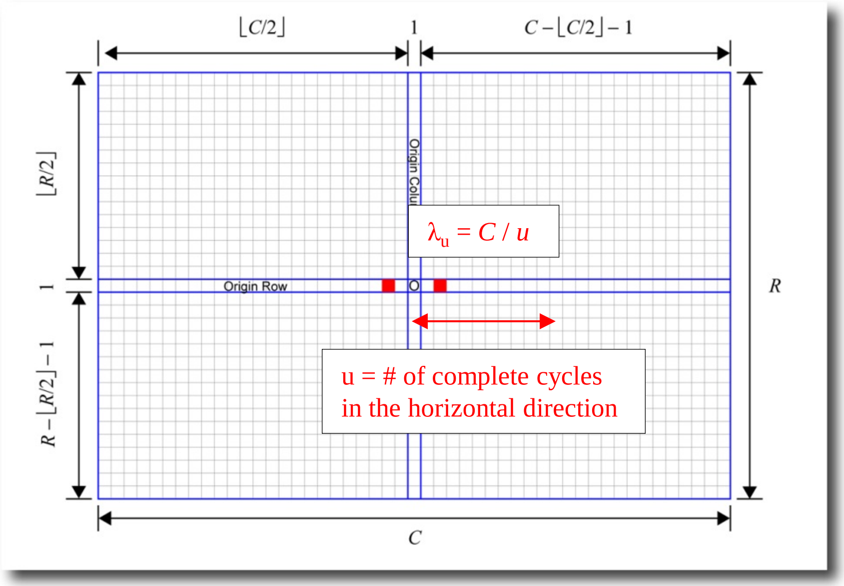
Frequencies and Wavelengths in the Fourier Plane

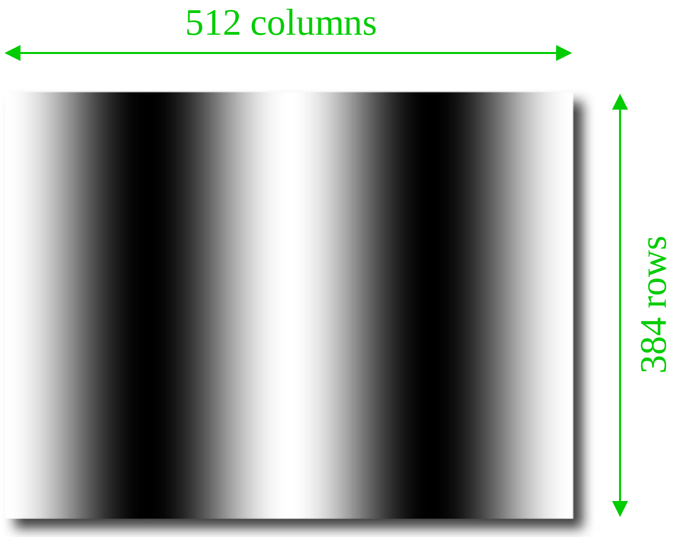




frequencies: $(u,v) = (0,1)$; wavelength: $\lambda_v = 384$

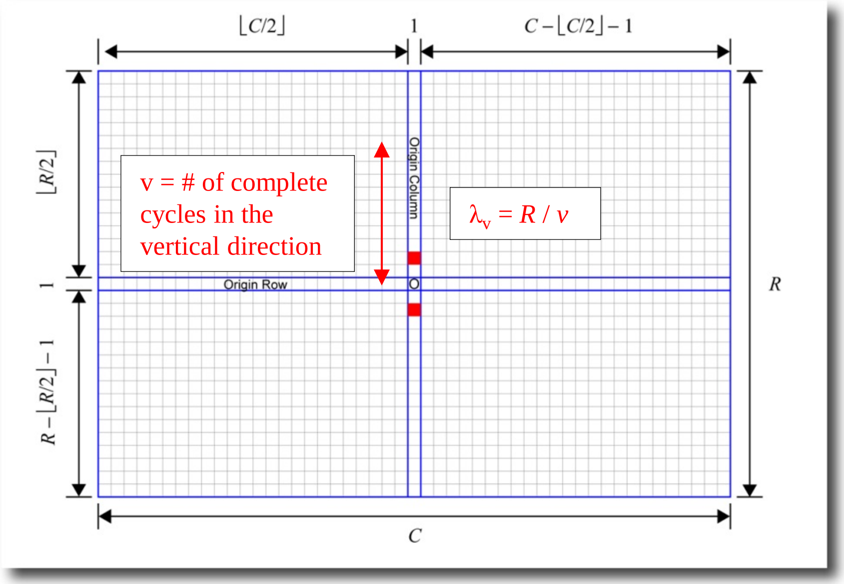
Frequencies and Wavelengths in the Fourier Plane

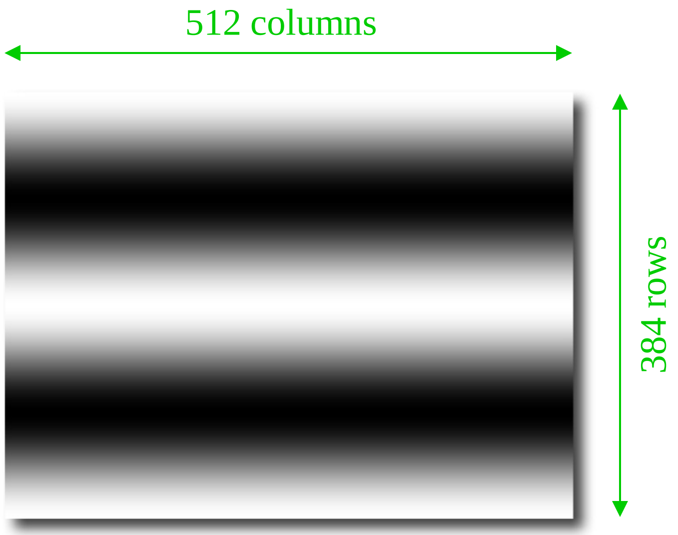




frequencies: $(u,v) = (2,0)$; wavelength: $\lambda_u = 256$

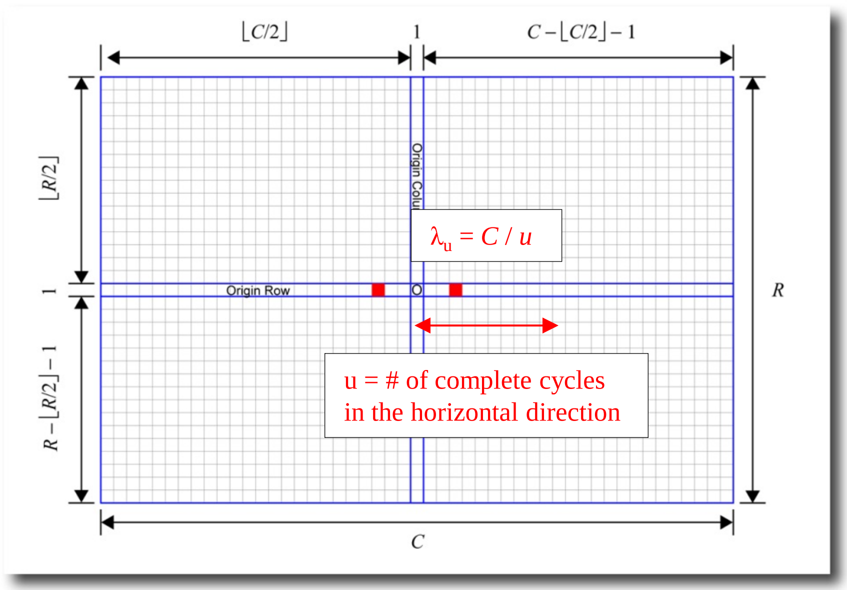
Frequencies and Wavelengths in the Fourier Plane

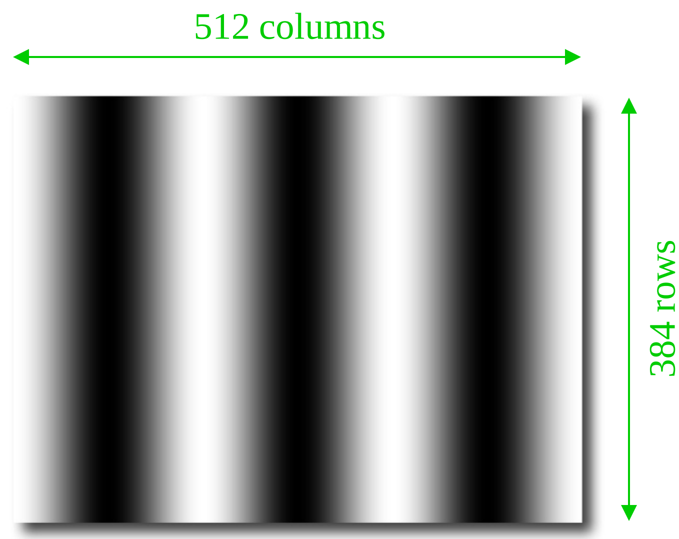




frequencies: $(u,v) = (0,2)$; wavelength: $\lambda_v = 192$

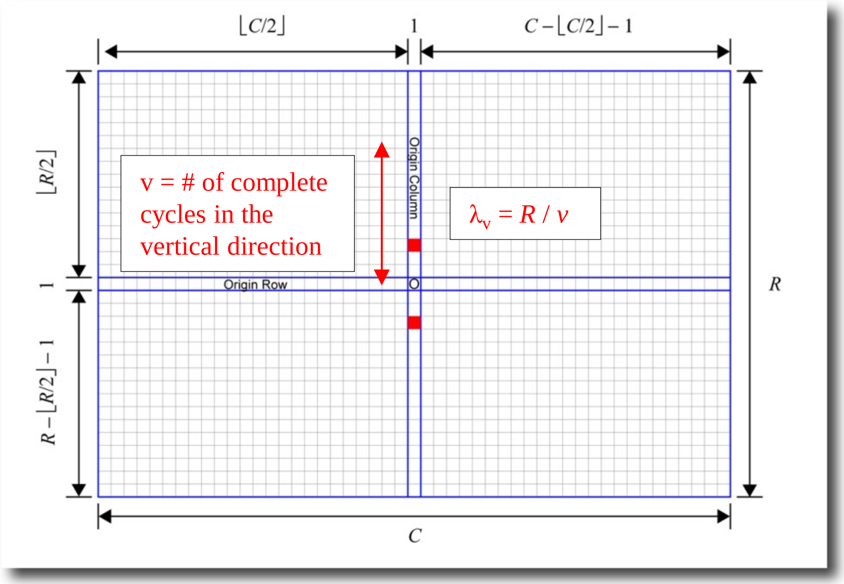
Frequencies and Wavelengths in the Fourier Plane

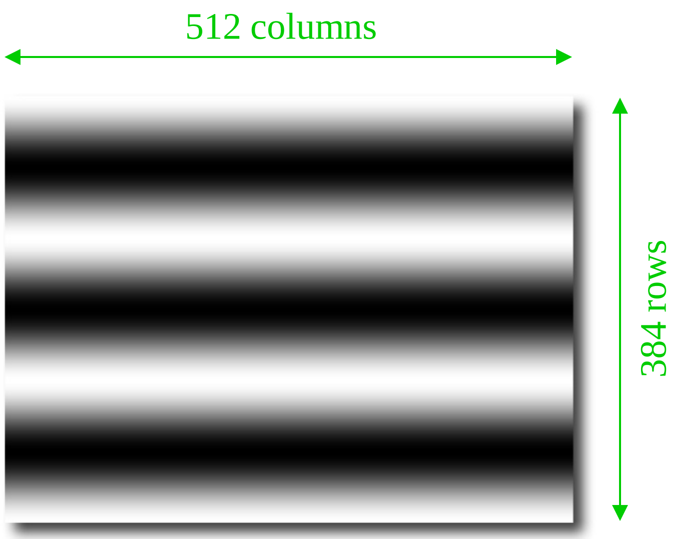




frequencies: $(u,v) = (3,0)$; wavelength: $\lambda_u = 170\frac{2}{3}$

Frequencies and Wavelengths in the Fourier Plane

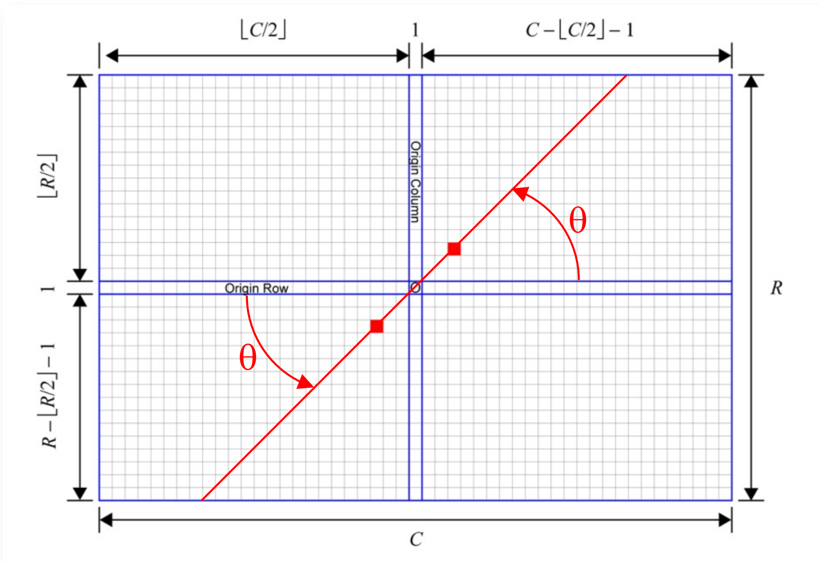


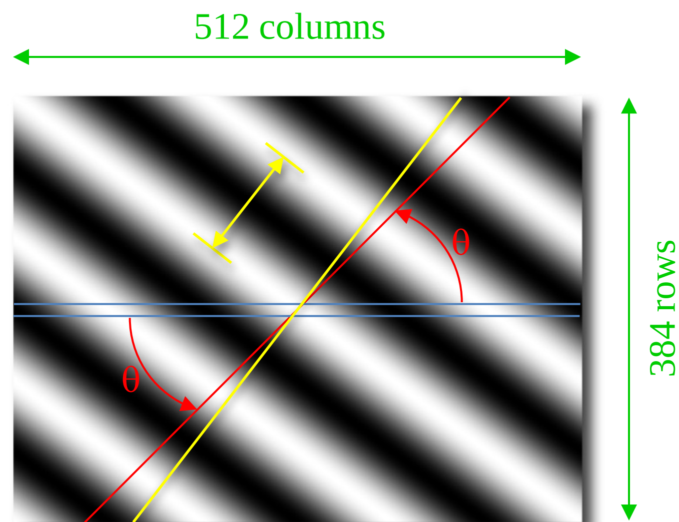


frequencies: $(u,v) = (0,3)$; wavelength: $\lambda_v = 128$

In the Fourier plane of a **square image**, the orientation of the line through the point pair = the orientation of the wave front in the image. **Not so for a non-square image.**

In the F plane the angle is -45° in this image it's about -53° (yellow line). That's because the fraction of R covered by one pixel is 4/3 the fraction of C covered by one pixel.





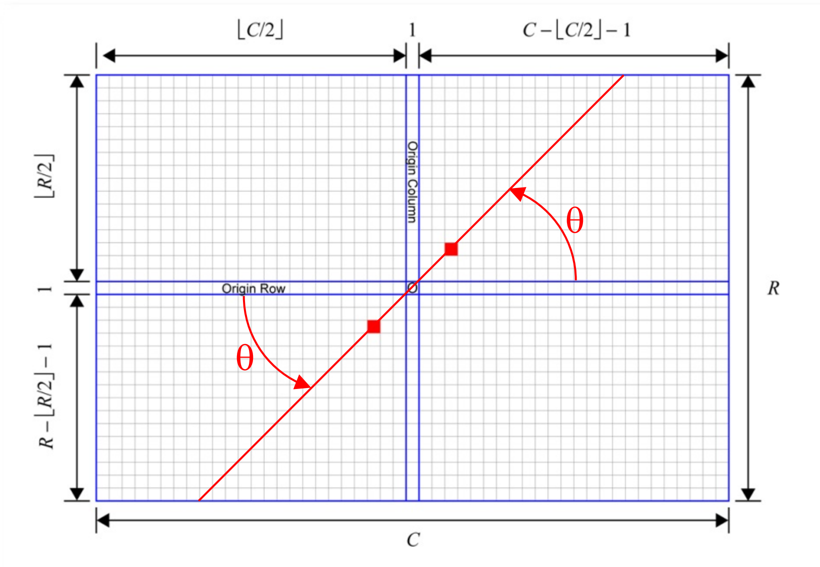
Also as a result, the wavelength is 213⅓.

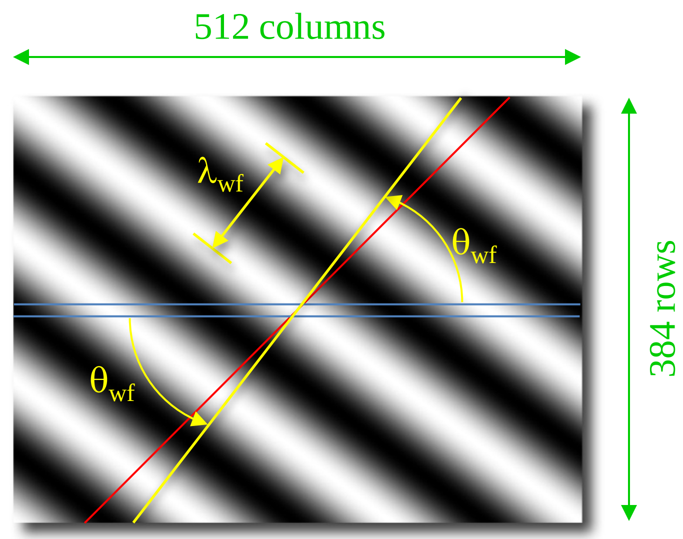
frequencies: $(u,v) = (3,3)$; wavelengths: $(\lambda_u, \lambda_v) = (170\tfrac{2}{3}, 128)$

In general the slope of the wavefront direction in the image is given by (v/R) / (u/C). Therefore its angle is

$$\theta_{\mathrm{wf}} = \tan^{-1}\left(\frac{vC}{uR}\right),$$

d Wavelengths in the Fourier Plane



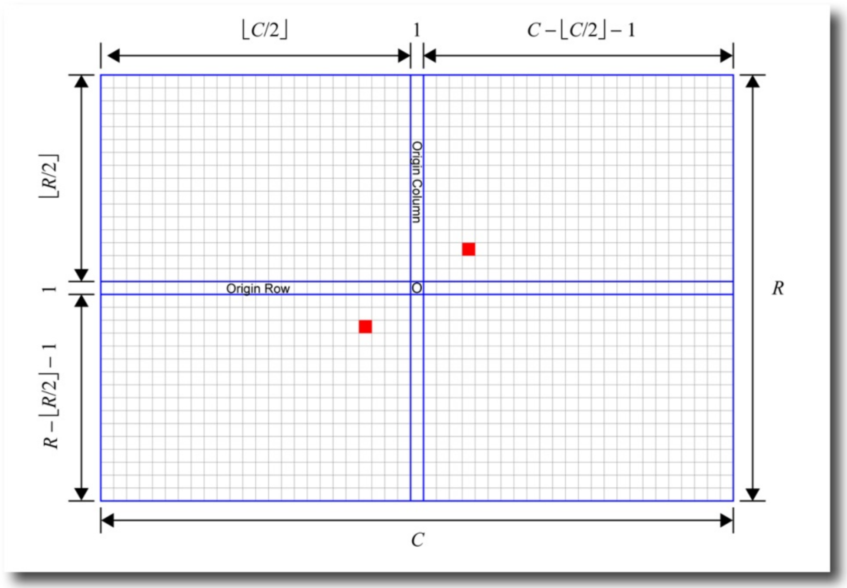


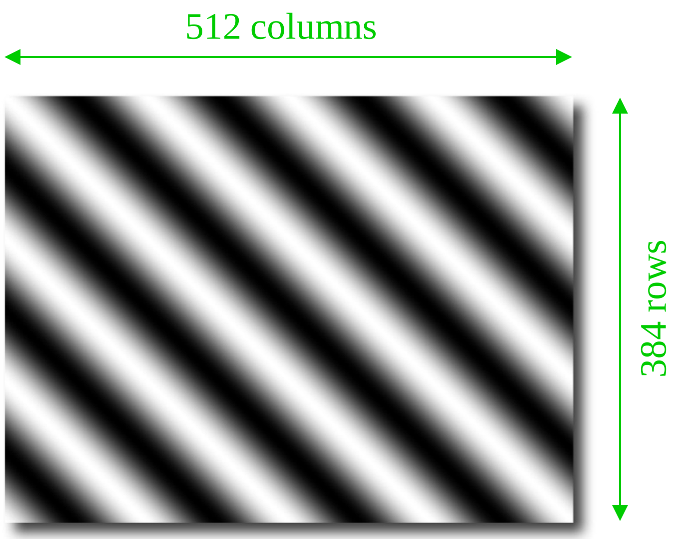
frequencies: $(u,v) = (3,3)$; wavelengths: $(\lambda_u, \lambda_v) = (170⅔, 128)$

and the wavelength is:

$$\lambda_{\mathrm{wf}} = \sqrt{\left(\frac{C}{u}\right)^2 + \left(\frac{R}{v}\right)^2},$$

Frequencies and Wavelengths in the Fourier Plane



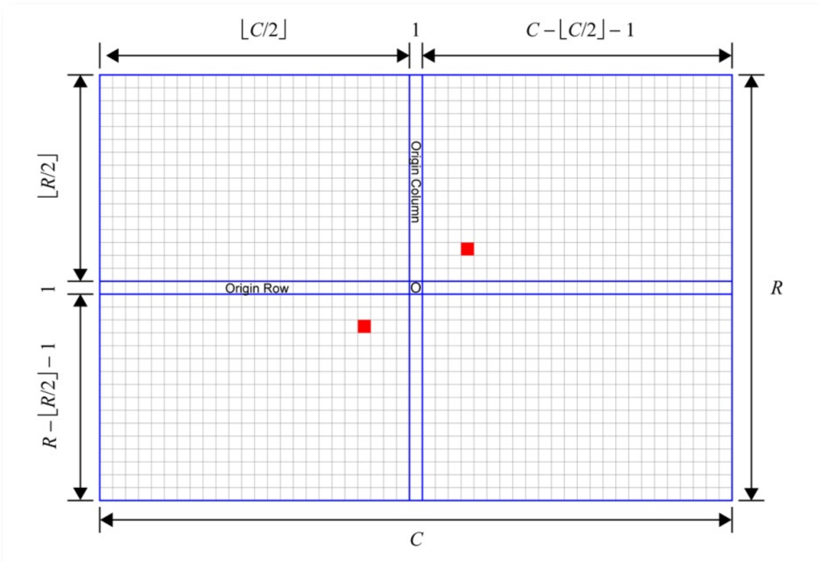


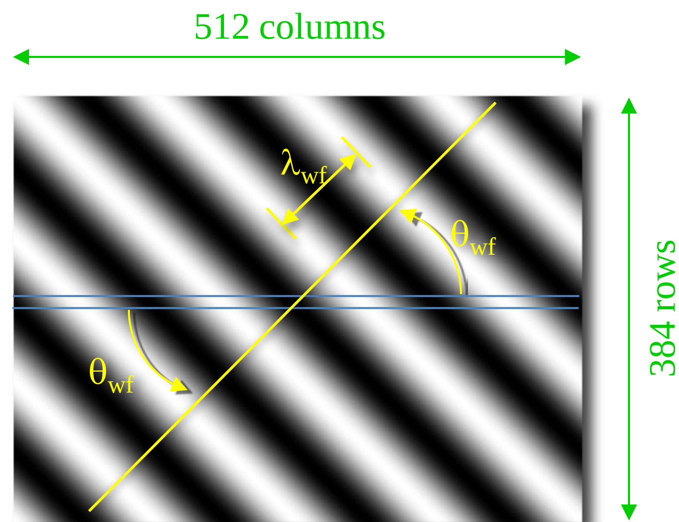
frequencies: $(u,v) = (4,3)$; wavelengths: $(\lambda_u, \lambda_v) = (128,128)$

Fr Fourier Plane

The ratio R/C = ¾ in this image. Therefore at frequency (4,3) the wave front angle is

$$\theta_{\mathrm{wf}} = \tan^{-1}\left(\frac{3\cdot 512}{4\cdot 384}\right) = \tan^{-1}\left(\frac{3\cdot 4}{4\cdot 3}\right) = \tan^{-1}(1) = 45^{\circ},$$





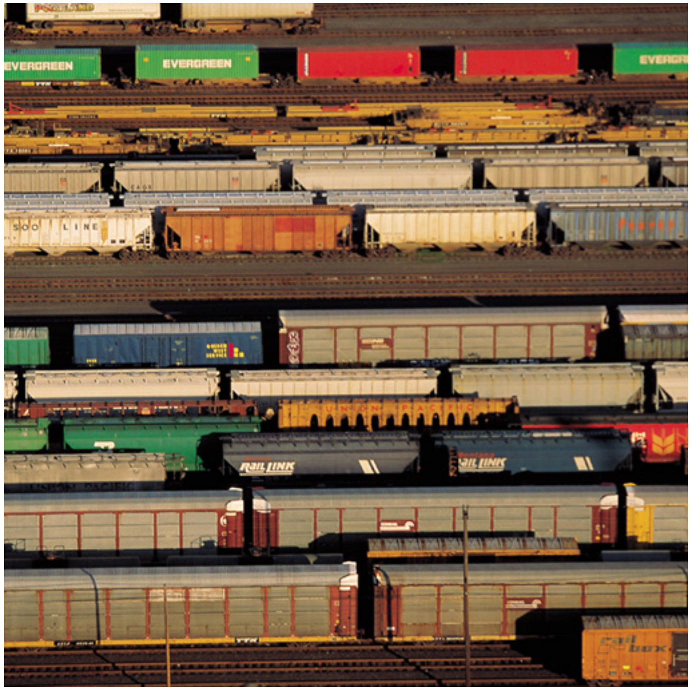
frequencies: $(u,v) = (4,3$

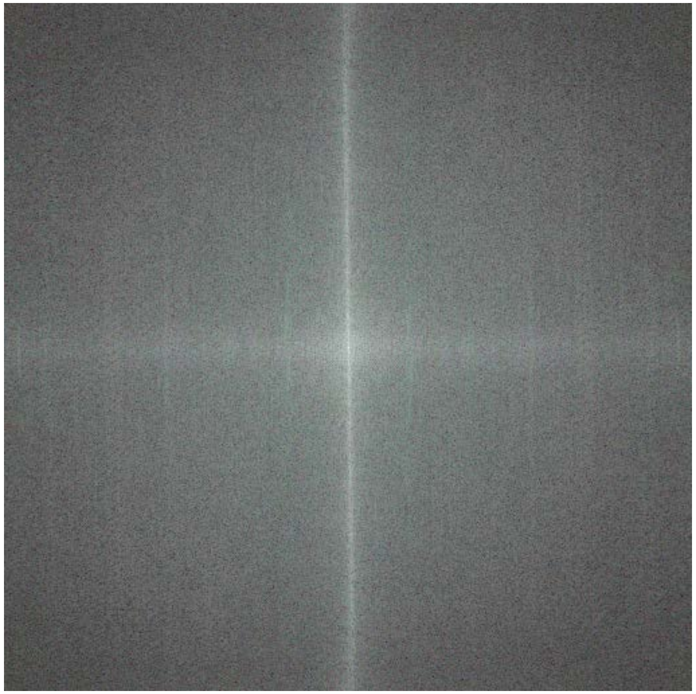
and the wavelength is

$$\lambda_{\mathrm{wf}} = \sqrt{\left(\frac{512}{4}\right)^2 + \left(\frac{384}{3}\right)^2} = \sqrt{2\cdot 128^2} = 128\sqrt{2},$$

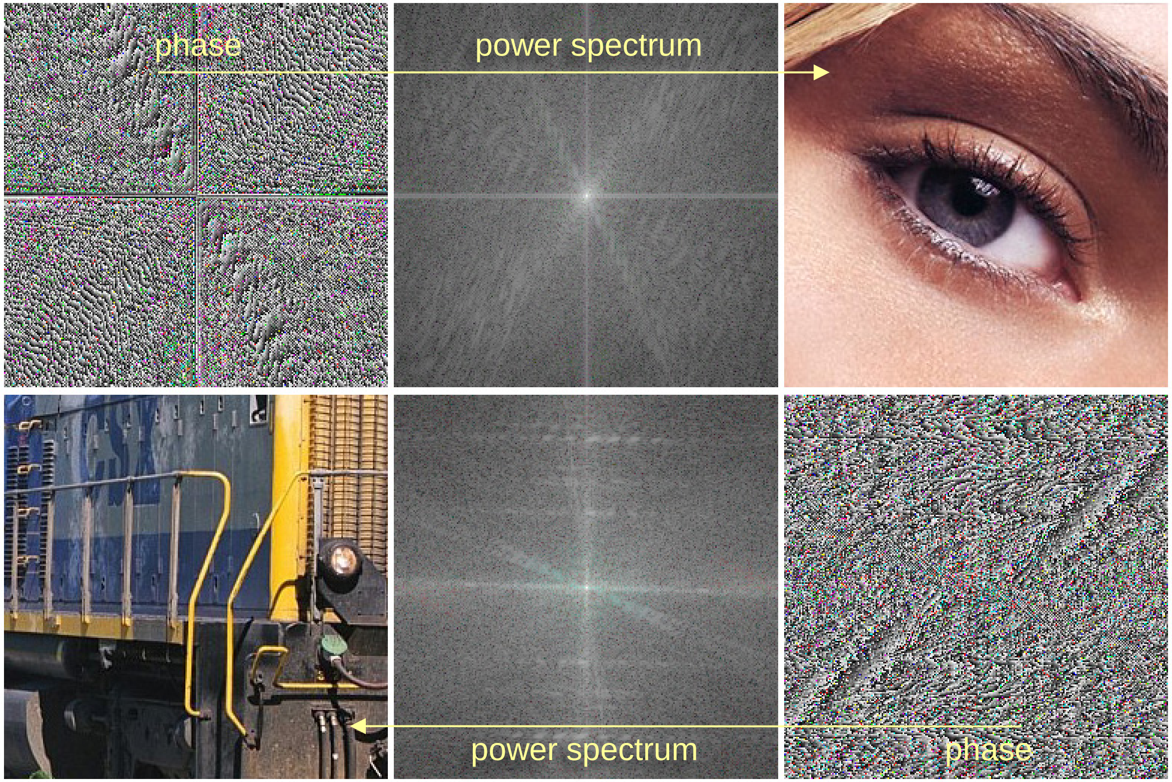
从图像功率谱图可以读出：
垂直方向的频率成份较大

Power Spectrum of an Image





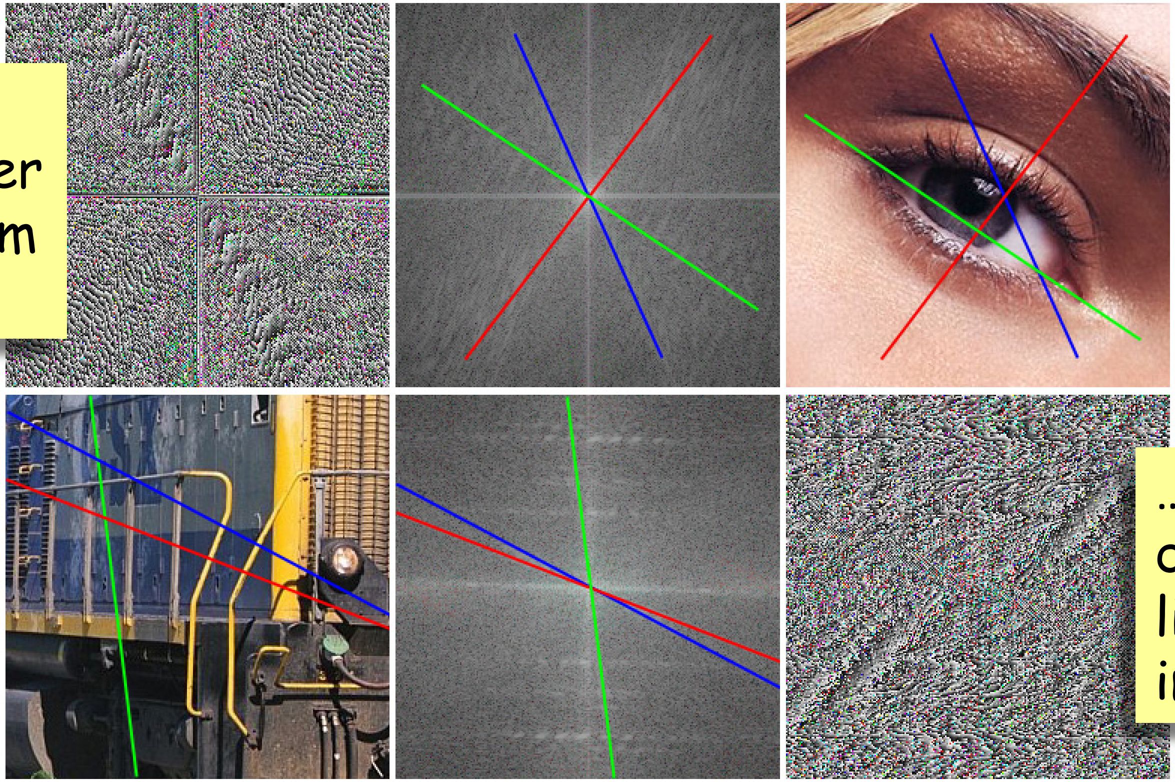
Relationship between Image and FT



1. 幅度谱中的明亮线和图像中的轮毂是垂直的
2. 幅度谱中的明亮线反映了灰度的变化-边缘

Features in the FT and in the Image

Lines in the Power Spectrum are …



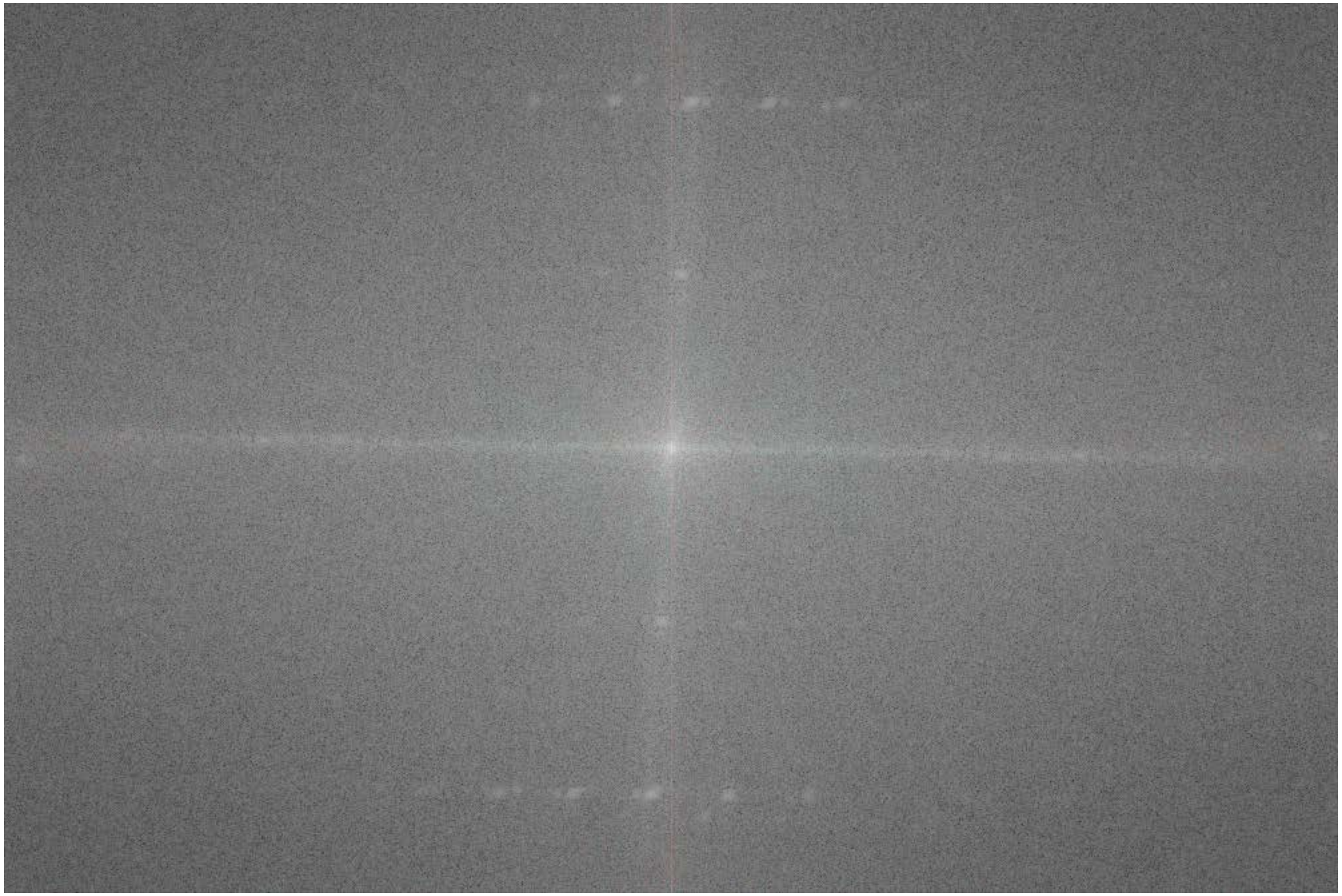
… perpen-dicular to lines in the image.

Fourier Magnitude and Phase

I

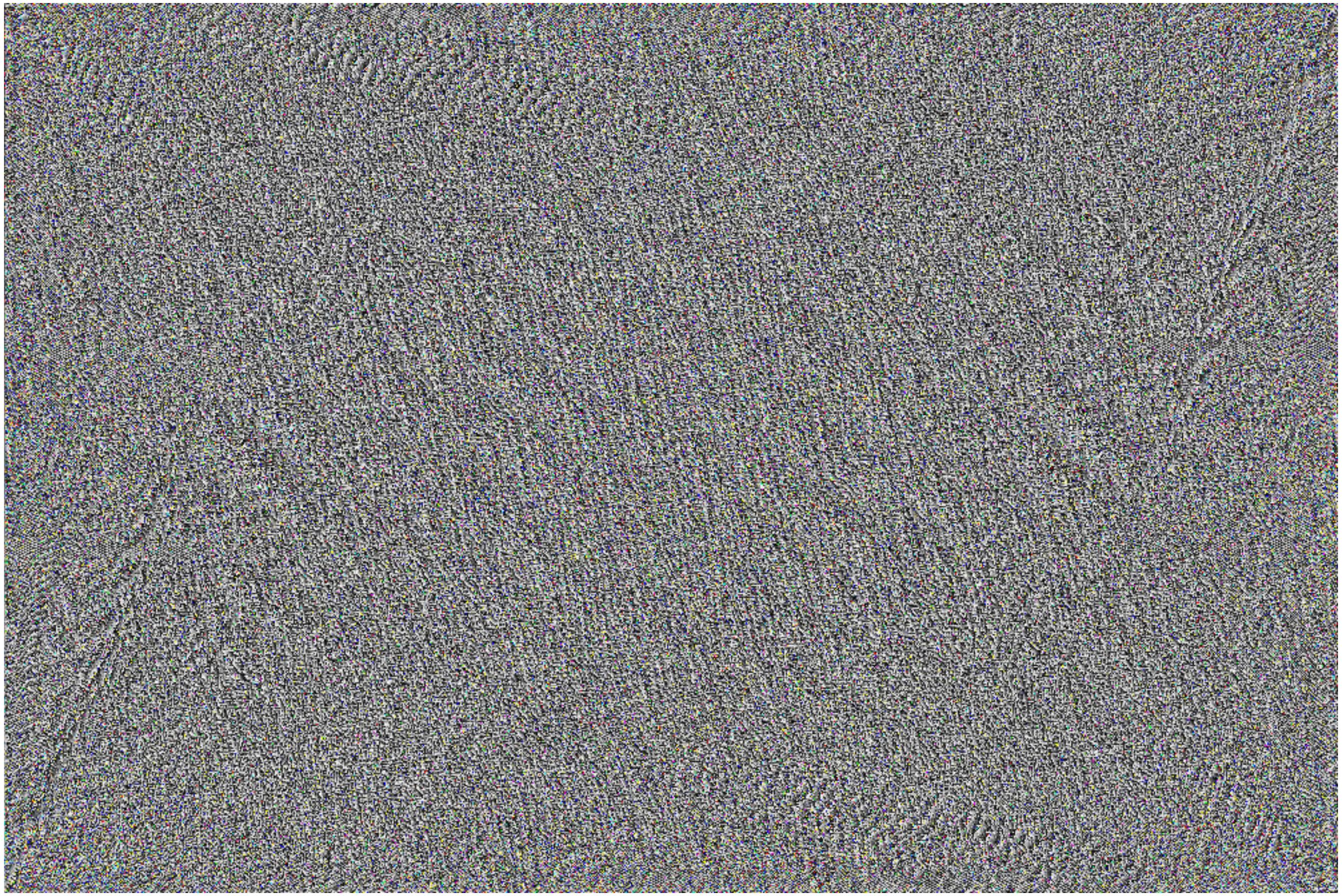
Fourier Magnitude

$\log\left|\varsigma\{\mathbf{I}\}\right|$



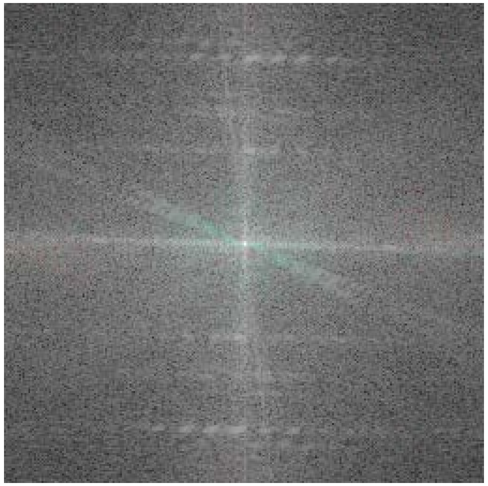
Fourier Phase

$\angle \mathcal{G}\{\mathbf{I}\}$

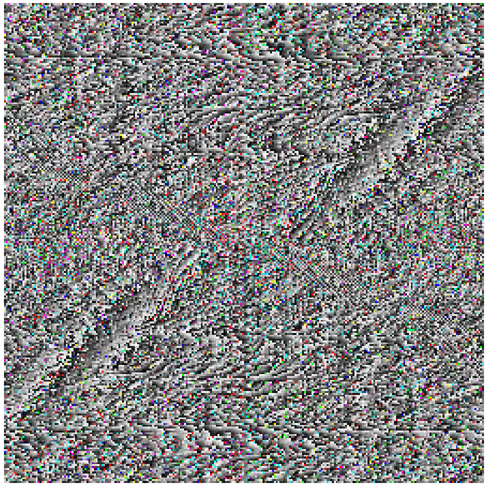


Q: Which contains more visually relevant information; magnitude or phase?

original image

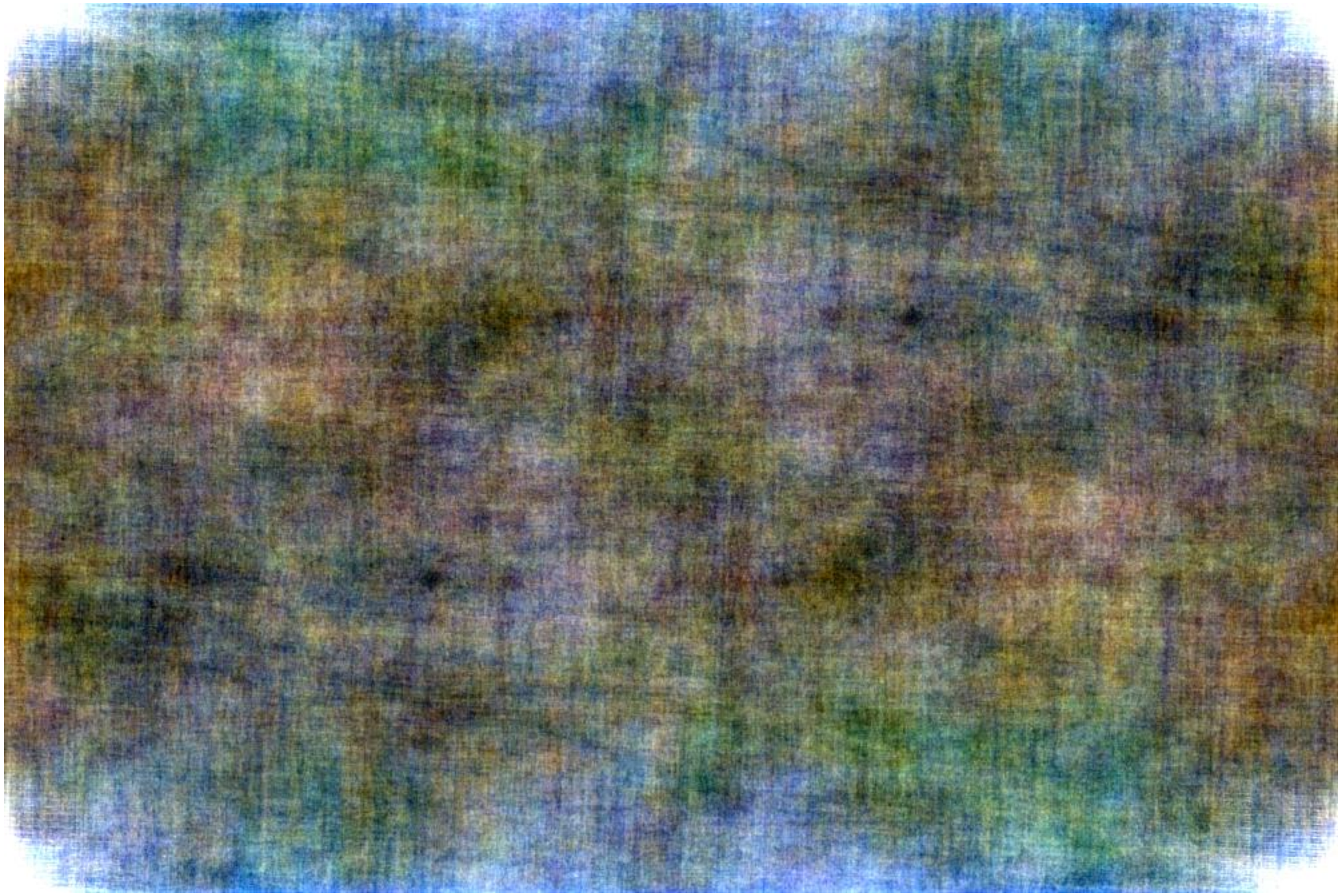


Fourier log magnitude



Fourier phase

Magnitude Only Reconstruction

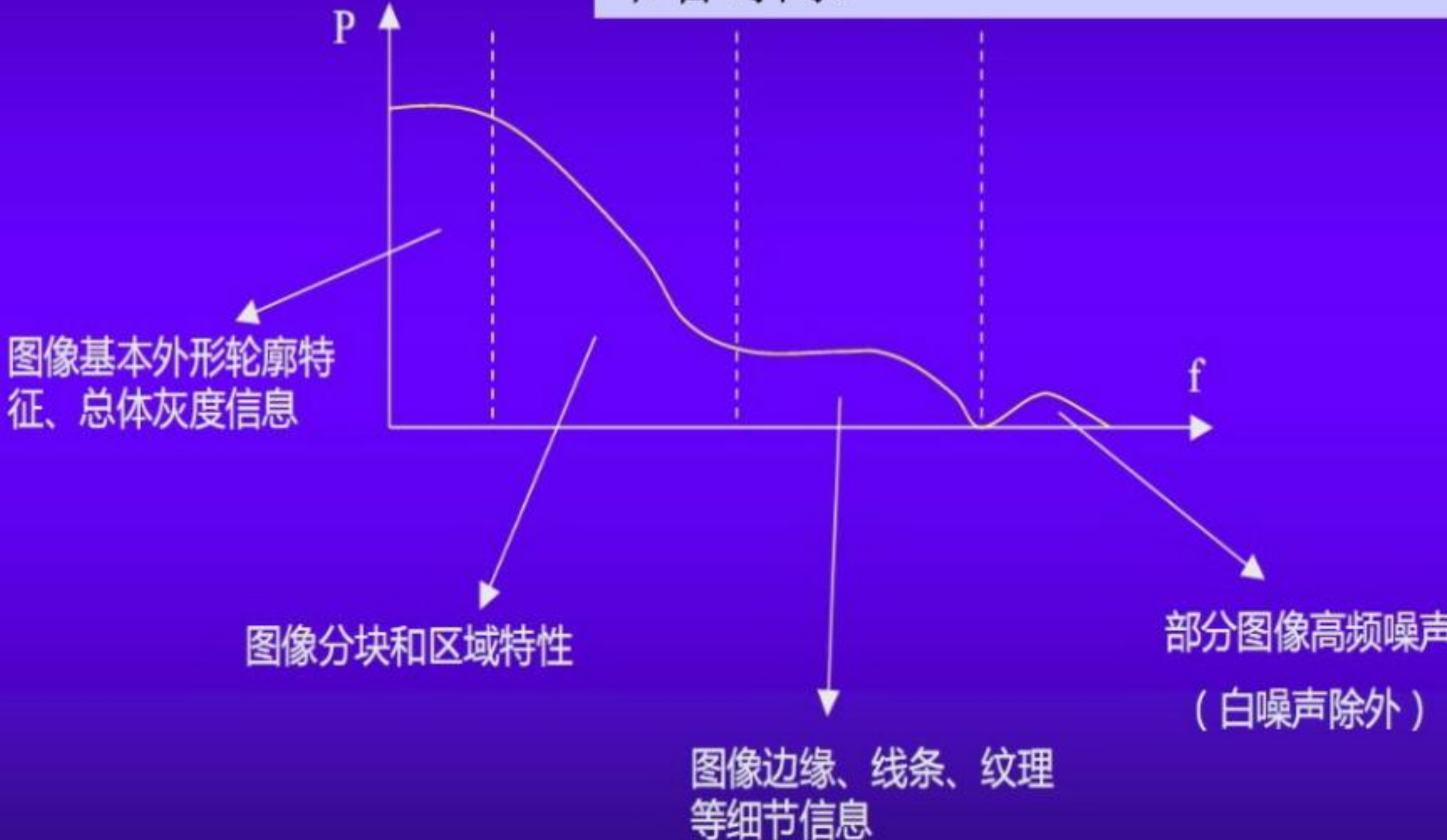


Phase Only Reconstruction

为什么要做图像变换？

❖变换后的图象，大部分能量都分布于低频谱段，这对以后图象的压缩、传输都比较有利。使得运算次数减少，节省时间。

图像信号的频域模型



怎么把信号分解为频域？

- 傅里叶，法国数学家、物理学家（1768-1830）
- 《热分析理论》

傅里叶变换

任何周期函数都可以表示为不同频率的正弦和/或余弦函数之和

傅里叶变换的意义

- 回答了频域信息如何表示这个基本问题
- 表示法里面最重要的特征之一
- 带来了信号处理领域的一场革命

一维连续傅里叶及其反变换

- 连续函数***f*(*x*)**的傅立叶变换***F*(*u*)**:

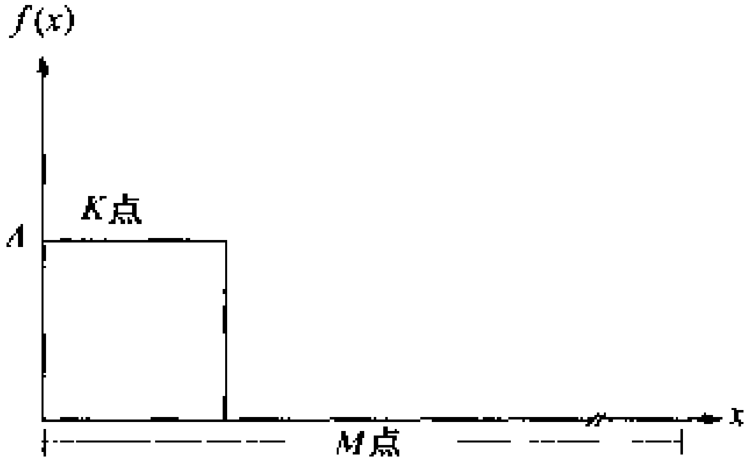
$$F(u) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi ux} dx$$

- 傅立叶变换***F*(*u*)**的反变换:

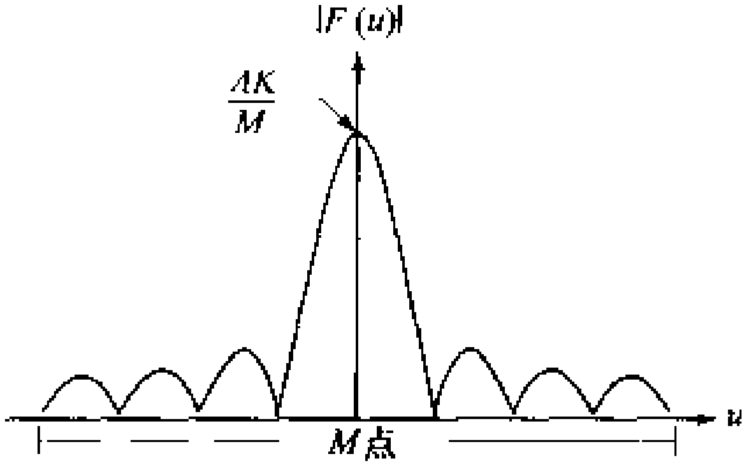
$$f(x) = \int_{-\infty}^{\infty} F(u) e^{j2\pi ux} du$$

一维连续傅立叶变换

- 矩形函数f(x)
- 傅里叶谱函数F(x)



(a)



(b)

一维连续傅立叶变换

- 计算公式

解： $F(u)=\int_{-\infty}^{\infty} f(x)e^{-j2\pi ux}dx$

$$=\int_0^X Ae^{-j2\pi ux}dx=\frac{-A}{j2\pi u}\left[e^{-j2\pi ux}\right]_0^X$$

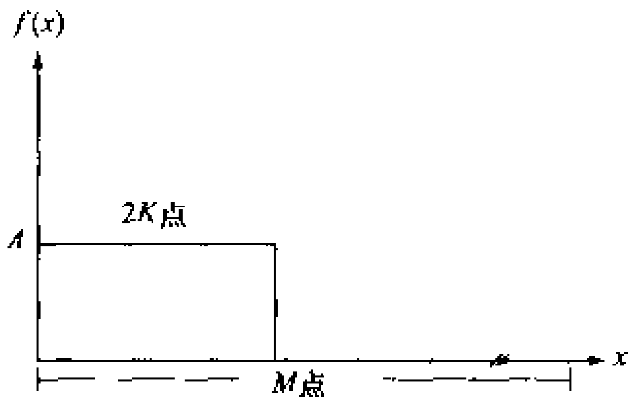
$$=\frac{-A}{j2\pi u}\left[e^{-j2\pi uX}-1\right]=\frac{-A}{j2\pi u}\left[e^{-j\pi uX}-e^{j\pi uX}\right]e^{-j\pi uX}$$

$$=\frac{A}{\pi u}\sin(\pi uX)e^{-j\pi uX}$$

$$|F(u)|=AX\left|\frac{\sin(\pi uX)}{\pi uX}\right|$$

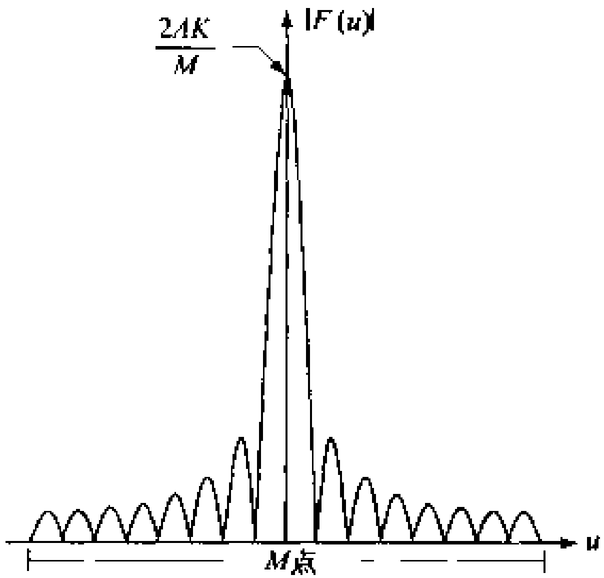
一维连续傅立叶变换

- 矩形函数f(x)



(c)

- 傅里叶谱函数F(x)



(d)

点数增加，傅里叶谱变窄

一维离散傅里叶变换DFT及其反变换

- 离散函数$f(x)$(其中**x**，**u**=0,1,2,…,N-1)的傅立叶变换:

$$F(u)=\sum_{x=0}^{N-1} f(x)e^{-j2\pi ux/N}$$

计算F(u):
1)在指数项中代入
u=0，然后将所有x 值相加
，得到F(0);
2) u=1，对所有x 的相加，
得到F(1);
3) 对所有M 个u 重复此
过程，得到全部完整的FT。

•$F(u)$的反变换的反变换:

$$f(x)=\frac{1}{N}\sum_{x=0}^{N-1} F(u)e^{j2\pi ux/N}$$

- 离散傅里叶变换及其反变换总存在。
- 用欧拉公式得 $e^{j\theta} = \cos\theta + j\sin\theta$

$$F(u) = \sum_{x=0}^{N-1} f(x)[\cos 2\pi ux / N - j\sin 2\pi ux / N]$$

每个F(u) 由$f(x)$与对应频率的正弦和余弦乘积和组成;

u 值决定了变换的频率成份，因此，**F(u)** 覆盖的域 **(u**值**)** 称为频率域，其中每一项都被称为**FT** 的频率分量。与$f(x)$ 的"时间域"和"时间成份"相对应。

一些常用的傅立叶变换函数

函数	$f(t)$	$F(u)$
高斯	$e^{-\pi t^2}$	$e^{-\pi u^2}$
矩形脉冲	$\Pi(\mathrm{t})$	$\sin(\pi u)/\pi u$
三角脉冲	$\Lambda(t)$	$\sin^2(\pi u)/(\pi u)^2$
冲激	$\delta(t)$	1
单位阶跃	$u(t)$	$[\delta(u) - j/\pi u]/2$
余弦	$\cos(2\pi ft)$	$[\delta(u+f) + \delta(u-f)]/2$
正弦	$\sin(2\pi ft)$	$j[\delta(u+f) + \delta(u-f)]/2$
复指数	$e^{2\pi ft}$	$\delta(u-f)$

傅里叶变换的作用

- 傅里叶变换将信号分成不同频率成份。类似光学中的分色棱镜把白光按波长(频率)分成不同颜色，称数学棱镜。

- 傅里叶变换的成份：直流分量和交流分量

- 信号变化的快慢与频率域的频率有关。噪声、边缘、跳跃部分代表图像的高频分量；背景区域和慢变部分代表图像的低频分量

二维连续傅里叶变换

1） 定义

$$F(u)=\int_{-\infty}^{\infty}f(x)e^{-j2\pi ux}dx \qquad F(u,v)=\int_{-\infty}^{\infty}f(x,y)e^{-j2\pi(ux+vy)}dxdy$$

2） 逆傅里叶变换

$$f(x)=\int_{-\infty}^{\infty}F(u)e^{j2\pi ux}du \qquad f(x,y)=\int_{-\infty}^{\infty}F(u,v)e^{j2\pi(ux+vy)}dudv$$

3） 傅里叶变换特征参数

$$F(u,v)=R(u,v)+jI(u,v)$$

频谱/幅度谱/模 $|F(u,v)|=\sqrt{R^2(u,v)+I^2(u,v)}$

能量谱/功率谱 $P(u,v)=|F(u,v)|^2=R^2(u,v)+I^2(u,v)$

相位谱 $\phi(u,v)=\arctan\dfrac{I(u,v)}{R(u,v)}$

二维DFT傅里叶变换

- 一个图像尺寸为$M\times N$的函数$f(x,y)$的离散傅立叶变换$F(u,v)$:

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

1/MN

- $F(u,v)$的反变换:

$$f(x,y)=\cancel{\frac{1}{MN}}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{j2\pi(ux/M+vy/N)}$$

二维DFT傅里叶变换

- $(u,v)=(0,0)$位置的傅里叶变换值为

$$F(0,0)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)=\bar{f}(x,y)$$

即$f(x,y)$ 的均值，原点(0,0) 的傅里叶变换是图像的平均灰度。$F(0,0)$ 称为频率谱的直流分量(系数)，其它$F(u,v)$ 值称为交流分量(交流系数)。

二维离散傅里叶变换

1） 定义

$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)e^{-j2\pi(ux/M+vy/N)}$$

$u=0,1,\cdots M-1$

$v=0,1,\cdots N-1$

2） 逆傅里叶变换

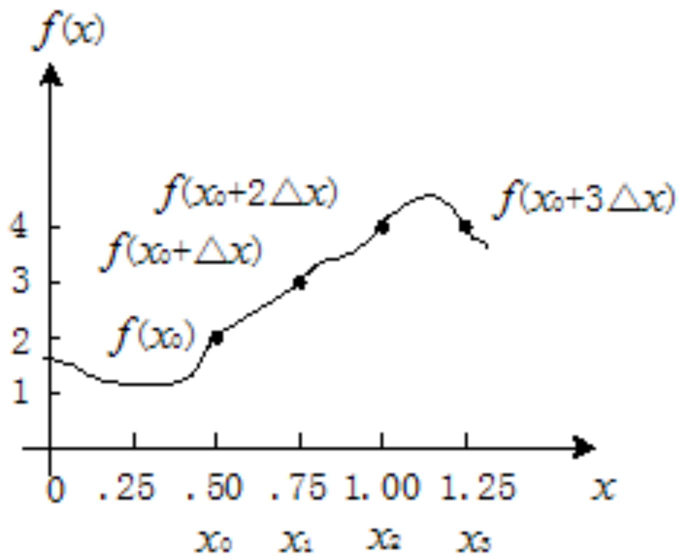
$$f(x,y)=\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)e^{+j2\pi(ux/M+vy/N)}$$

$x=0,1,\cdots M-1$

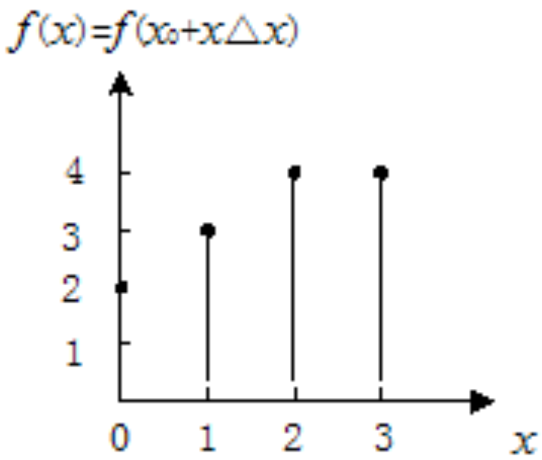
$y=0,1,\cdots N-1$

离散的情况下，傅里叶变换和逆傅里叶变换始终存在。

例 设一函数如图（a）所示，如果将此函数在自变量 $x_0 = 0.5, x_1 = 0.75, x_2 = 1.00, x_3 = 1.25$ 取样并重新定义为图（b）离散函数，求其傅里叶变换。



（a）



（b）

$$F(0) = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp\{0\}$$

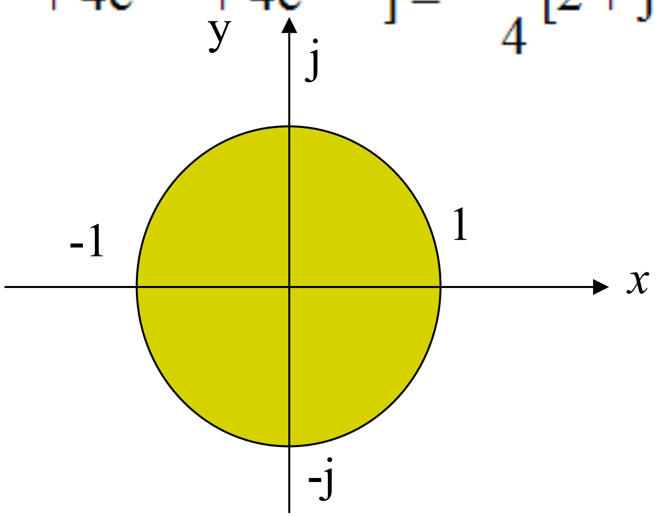
$$= (1/4)[f(0) + f(1) + f(2) + f(3)] = (1/4)[2 + 3 + 4 + 4] = 3.25$$

$$F(1) = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp\{-j2\pi x/4\} = \frac{1}{4}[2e^{0} + 3e^{-j\pi/2} + 4e^{-j\pi} + 4e^{-j3\pi/2}] = \frac{1}{4}[-2 + j]$$

$$F(2) = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp\{-j4\pi x/N\} = \frac{1}{4}[2e^{0} + 3e^{-j\pi} + 4e^{-j2\pi} + 4e^{-j3\pi}] = -\frac{1}{4}[1 + j0]$$

$$F(3) = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp\{-j6\pi x/4\} = \frac{1}{4}[2e^{0} + 3e^{-j3\pi 3\pi} + 4e^{-j3\pi} + 4e^{-j9\pi 9\pi}] = -\frac{1}{4}[2 + j]$$

$$F(u) = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

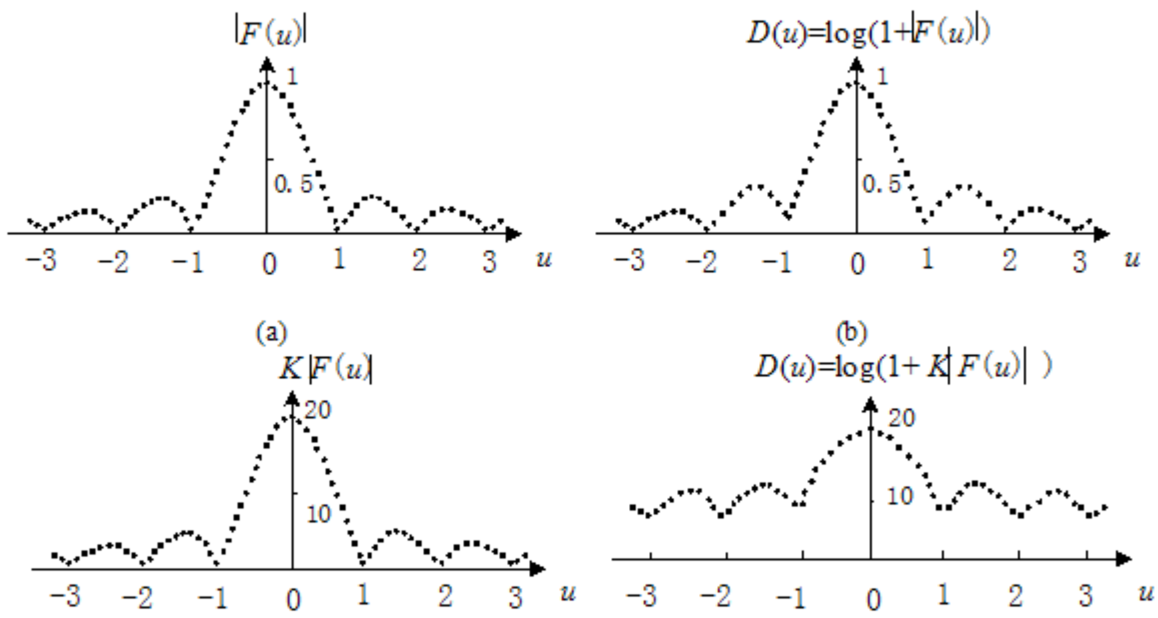


图像的频谱幅度随频率增大而迅速衰减

许多图像的傅里叶频谱的幅度随着频率的增大而迅速减小，这使得在显示与观察一副图像的频谱时遇到困难。但以图像的形式显示它们时，其高频项变得越来越不清楚。

解决办法：

对数化

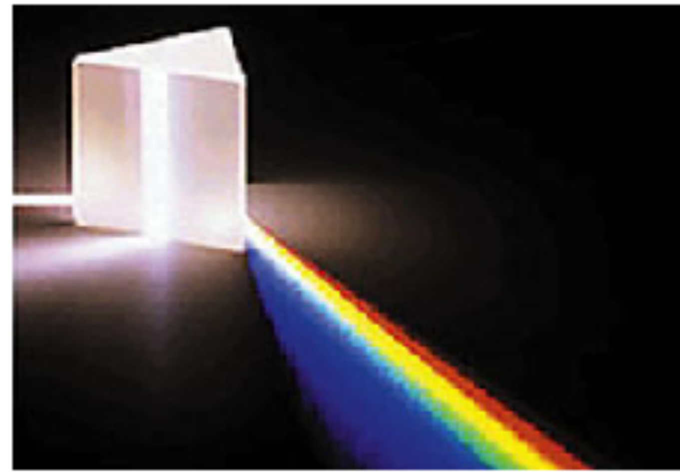


傅里叶变换的意义

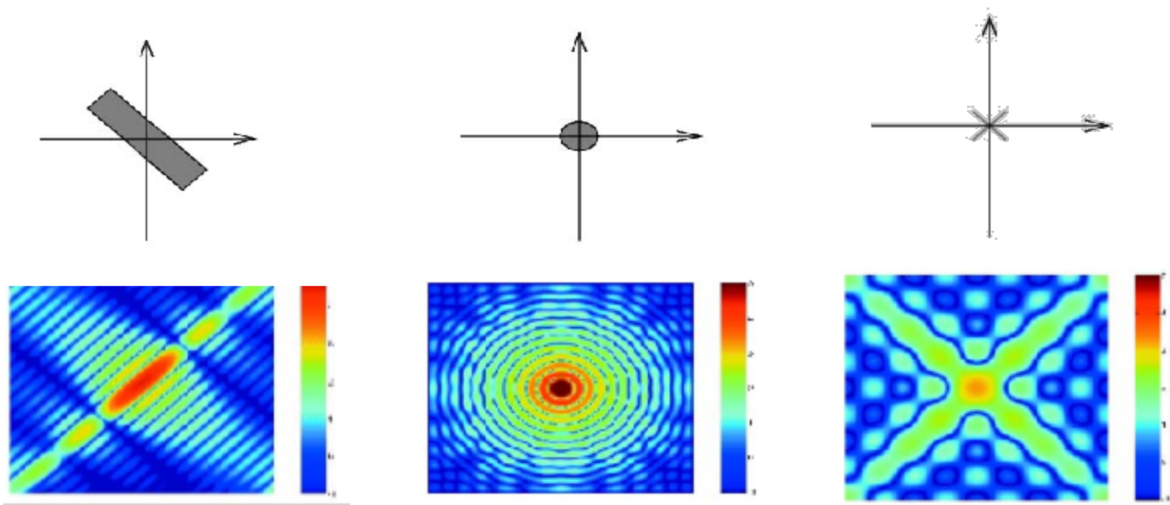
傅里叶变换好比一个玻璃棱镜

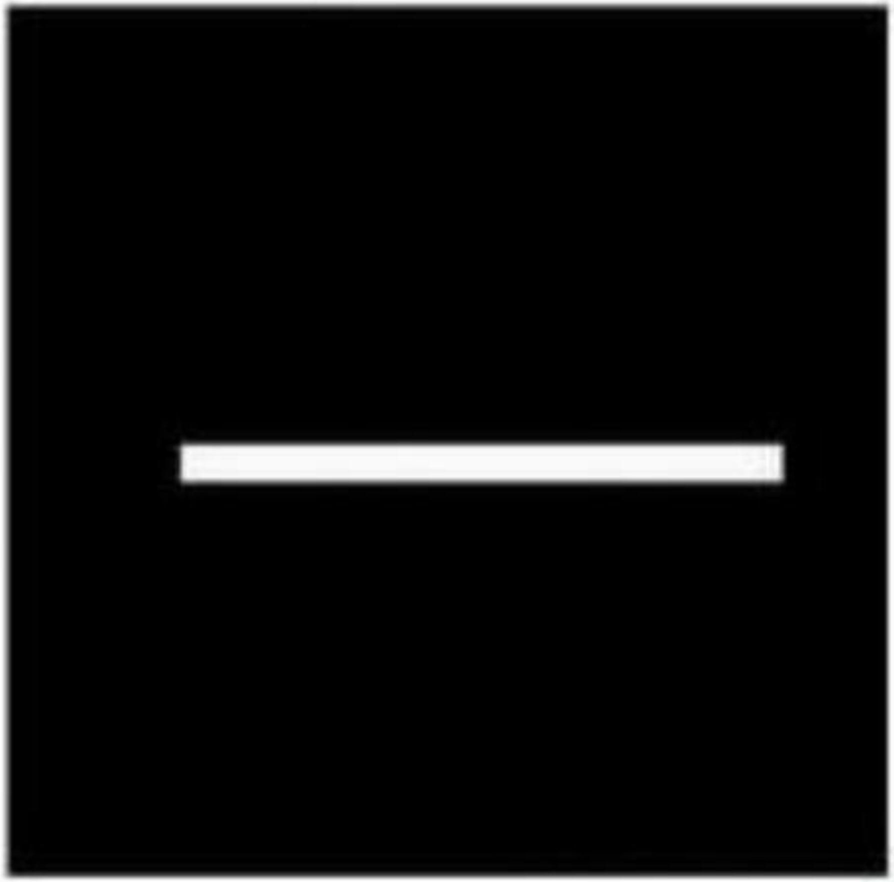
棱镜是可以将光分成不同颜色的物理仪器，每个成分的颜色由波长决定。

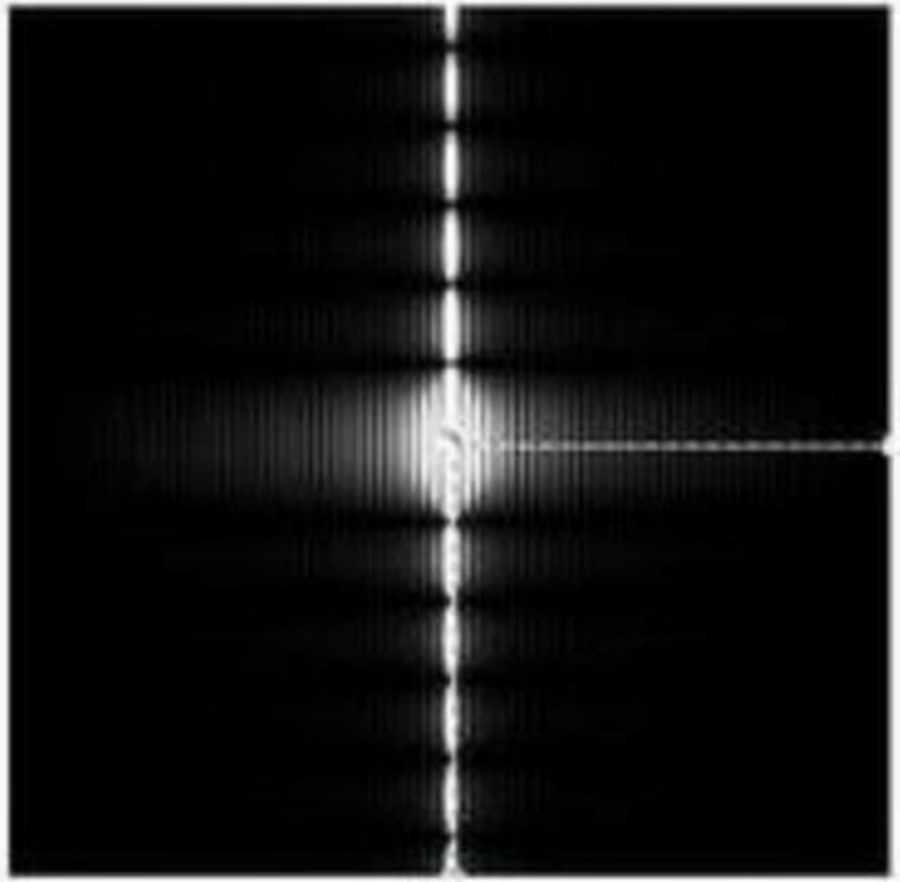
傅里叶变换可看做是“数学中的棱镜”，将函数基于频率分成不同的成分。



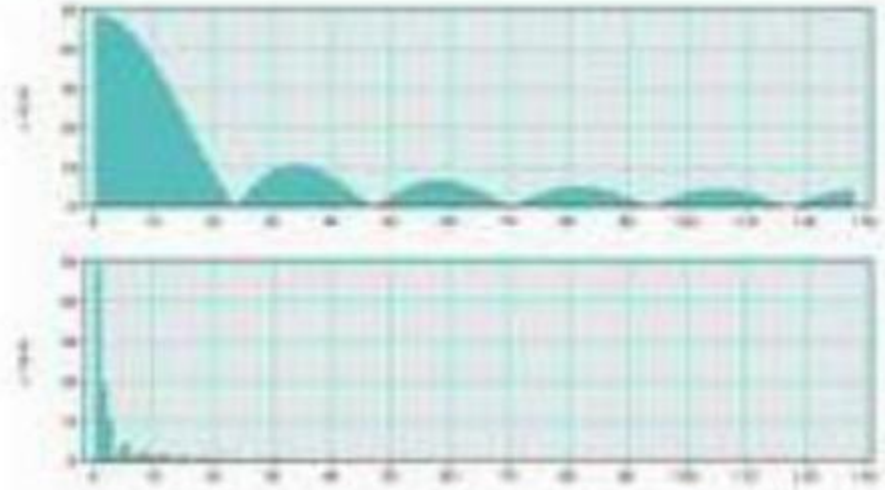
一些图像的傅里叶变换

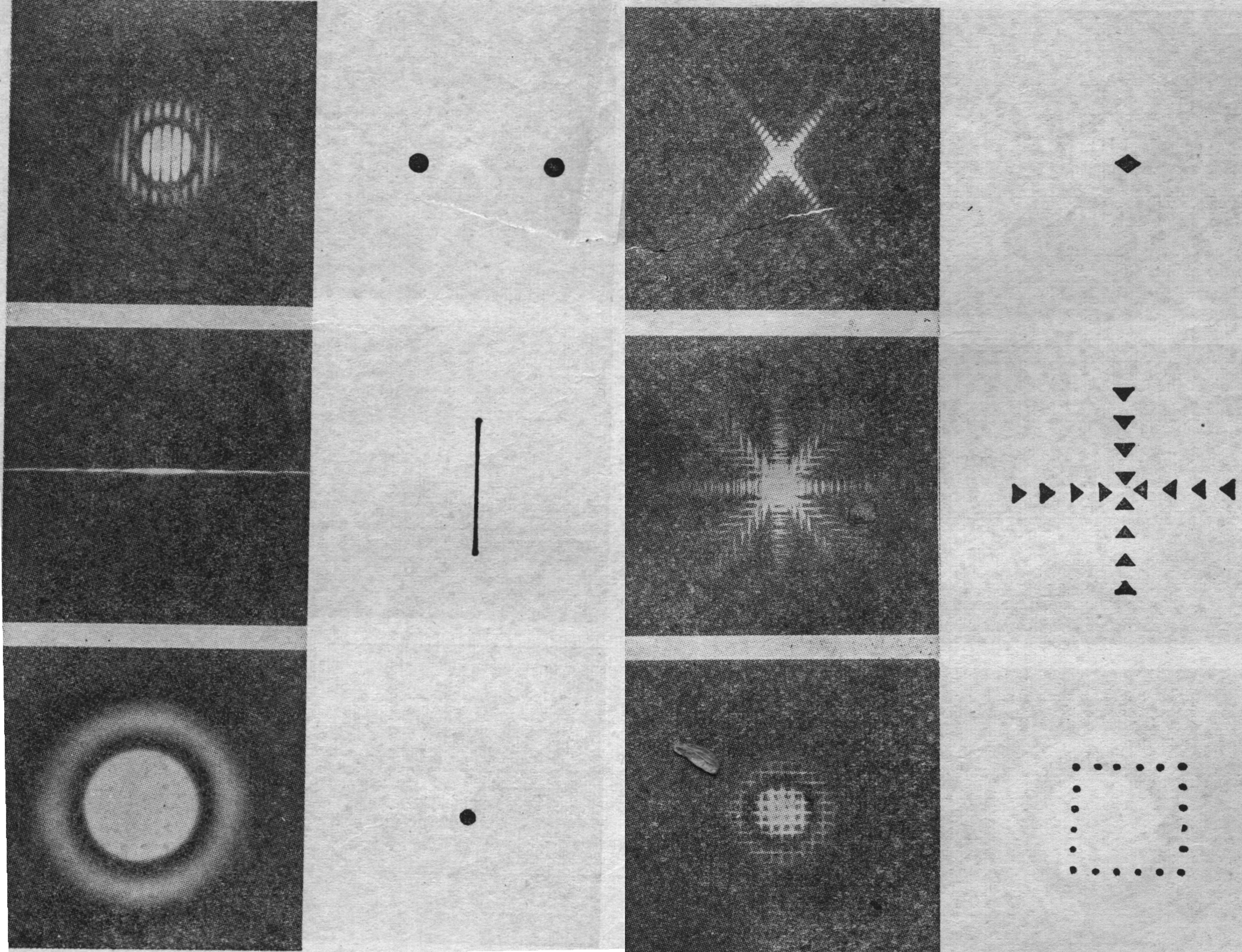






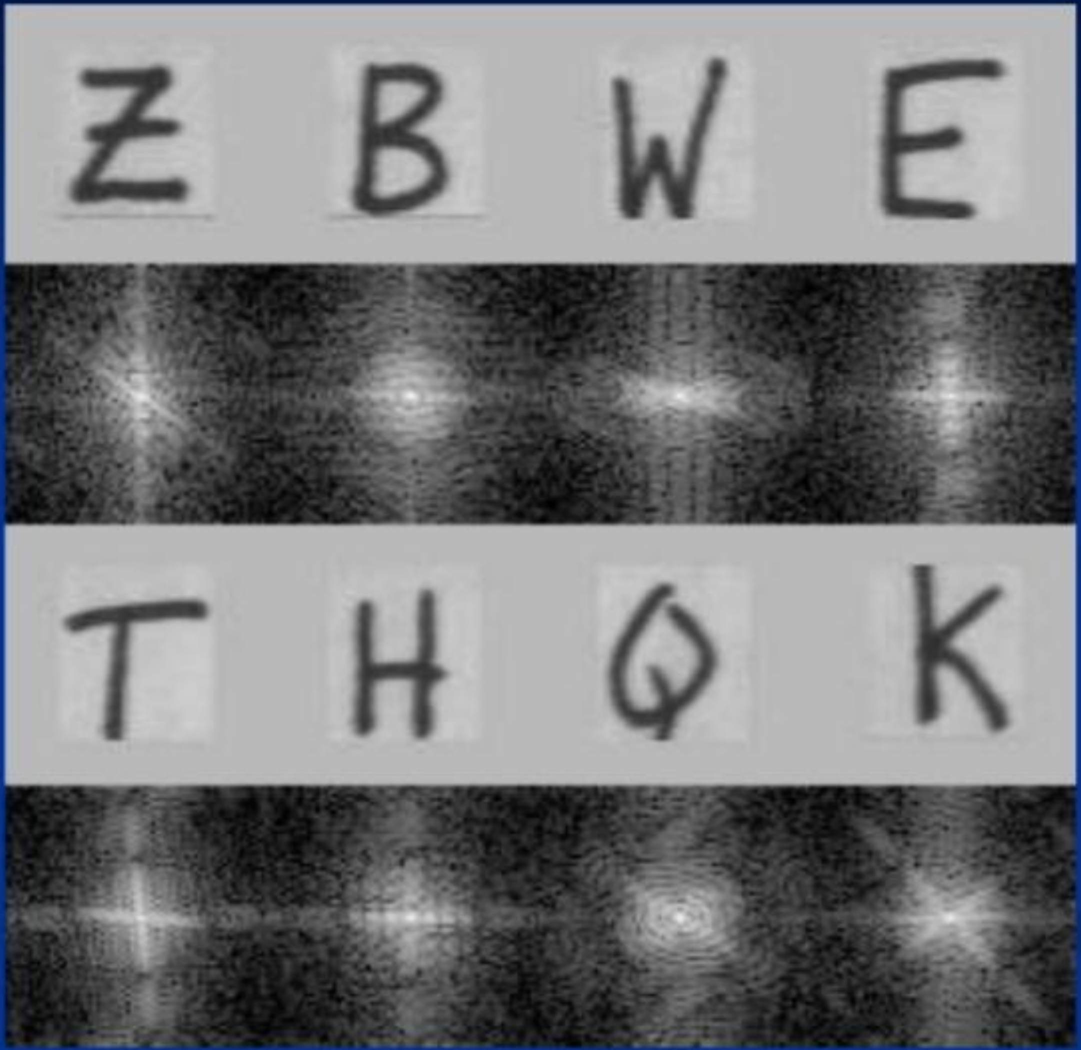
在垂直方向上需要更多的频率分量，所以它的波峰比较宽，而水平方向上的波峰比较窄





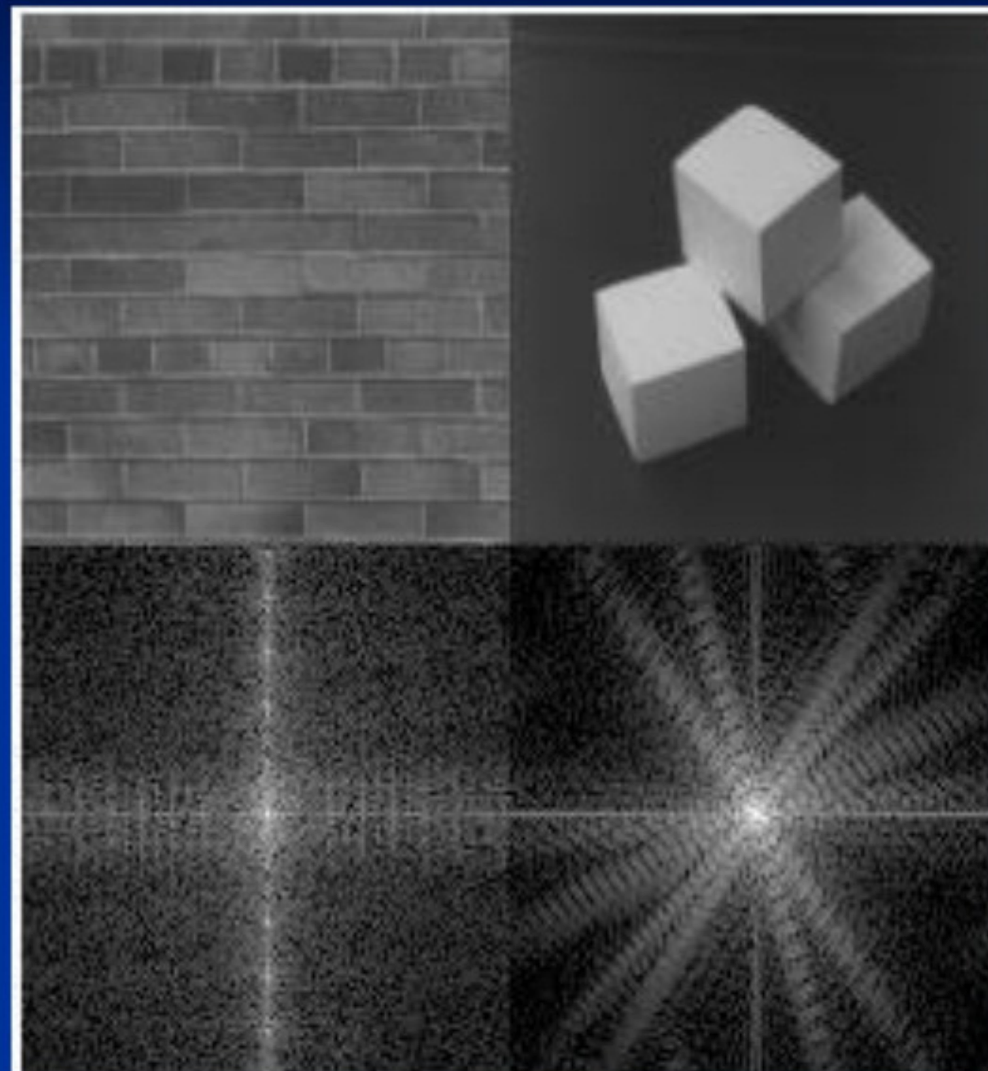
有趣的幅度谱

- 从幅度谱中我们可以看出明亮线和原始图像中对应的轮廓线是垂直的。如果原始图像中有圆形区域那么幅度谱中也呈圆形分布



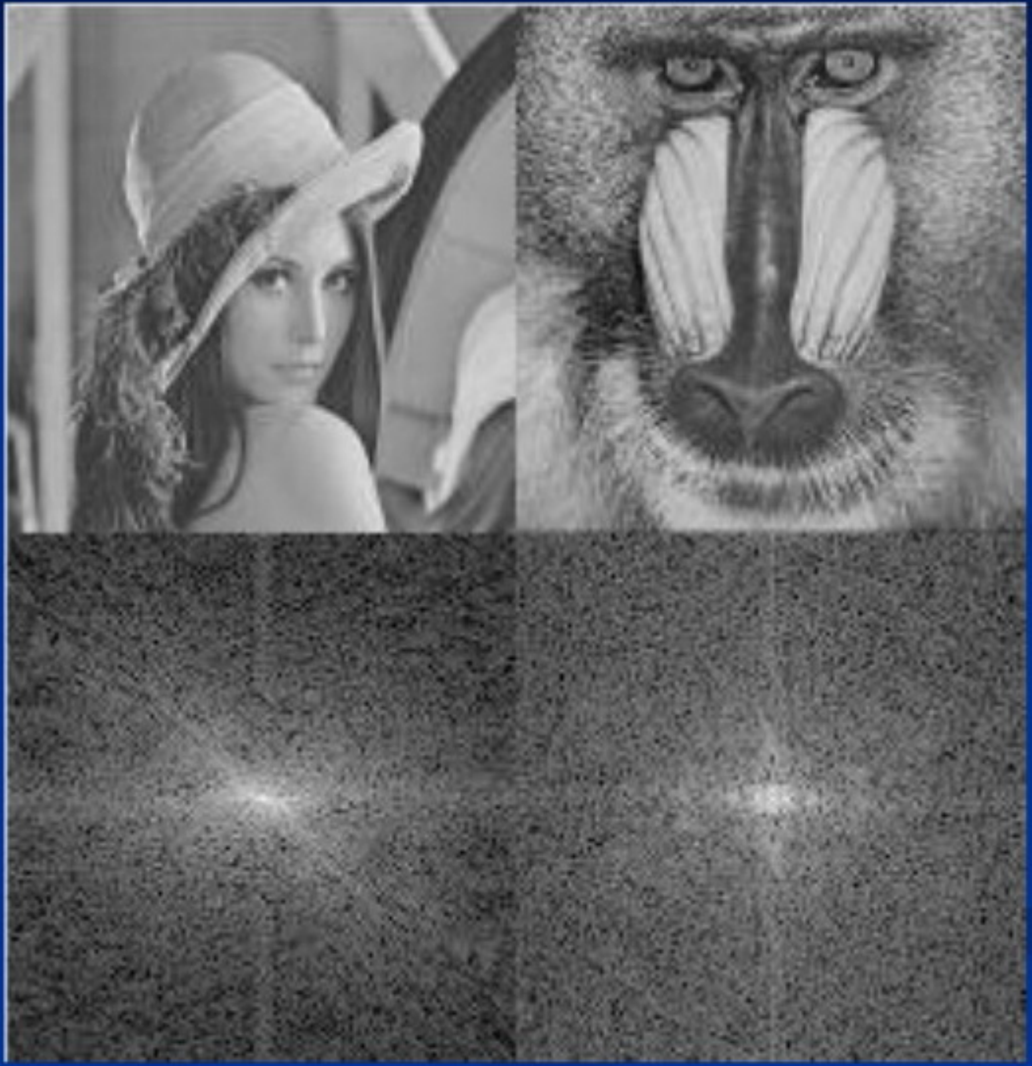
有趣的幅度谱

- 从幅度谱中我们可以看出明亮线反映出原始图像的灰度级变化，这正是图像的轮廓边

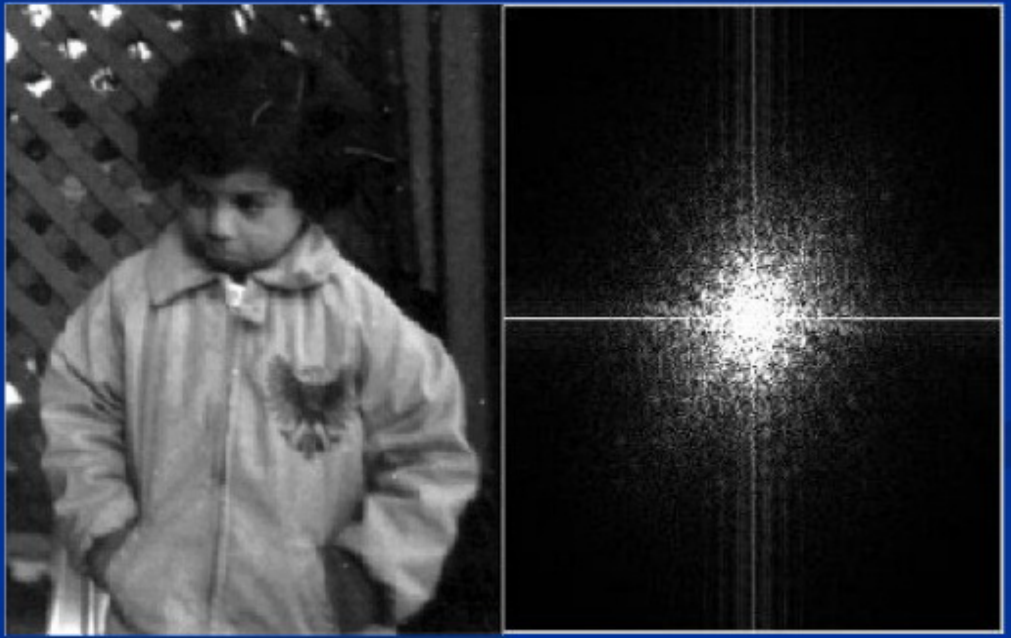


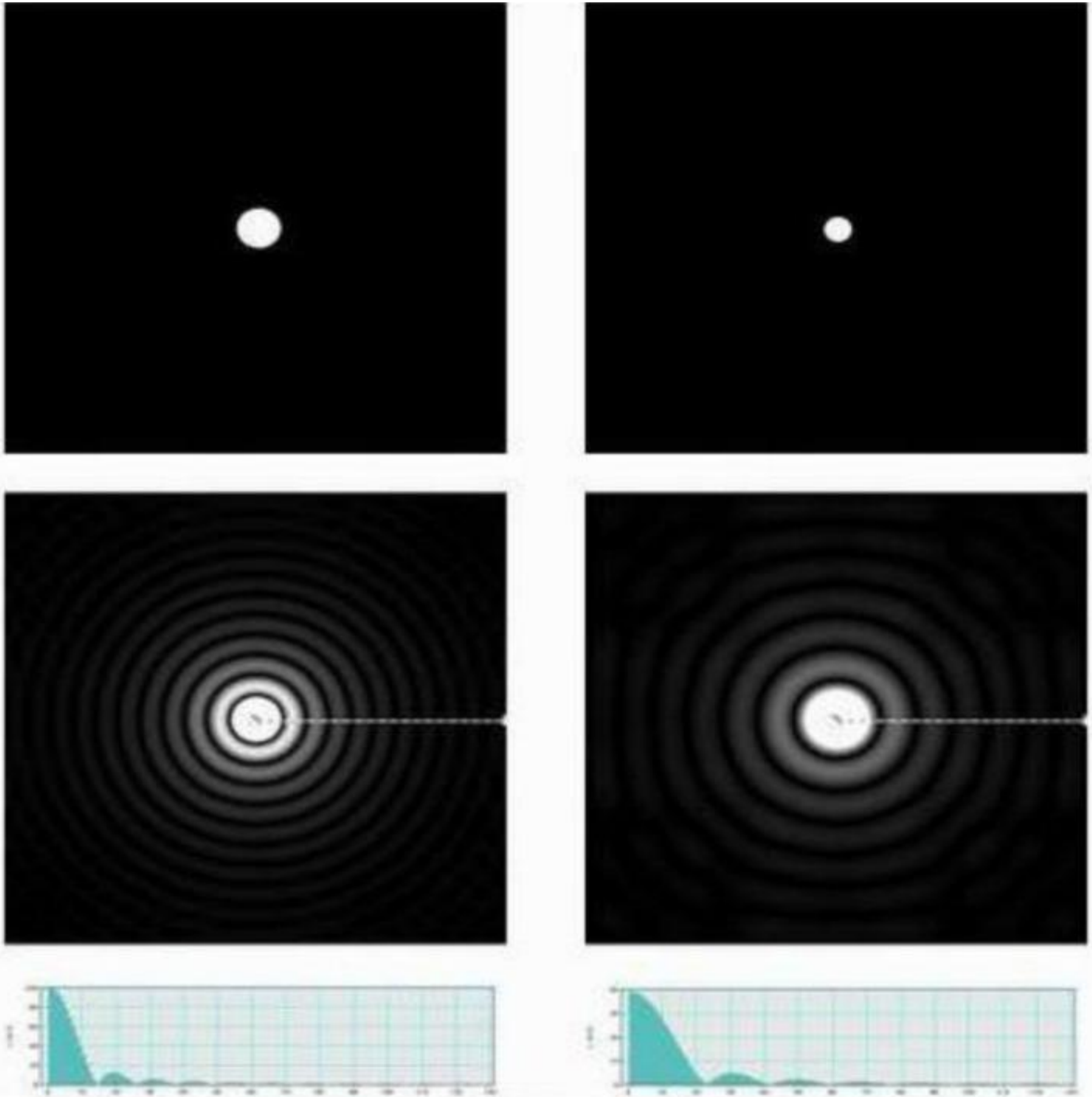
有趣的幅度谱

- 这些图像没有特定的结构，左上角到右下角有一条斜线，它可能是由帽子和头发之间的边线产生的
- 两个图像都存在一些小边界



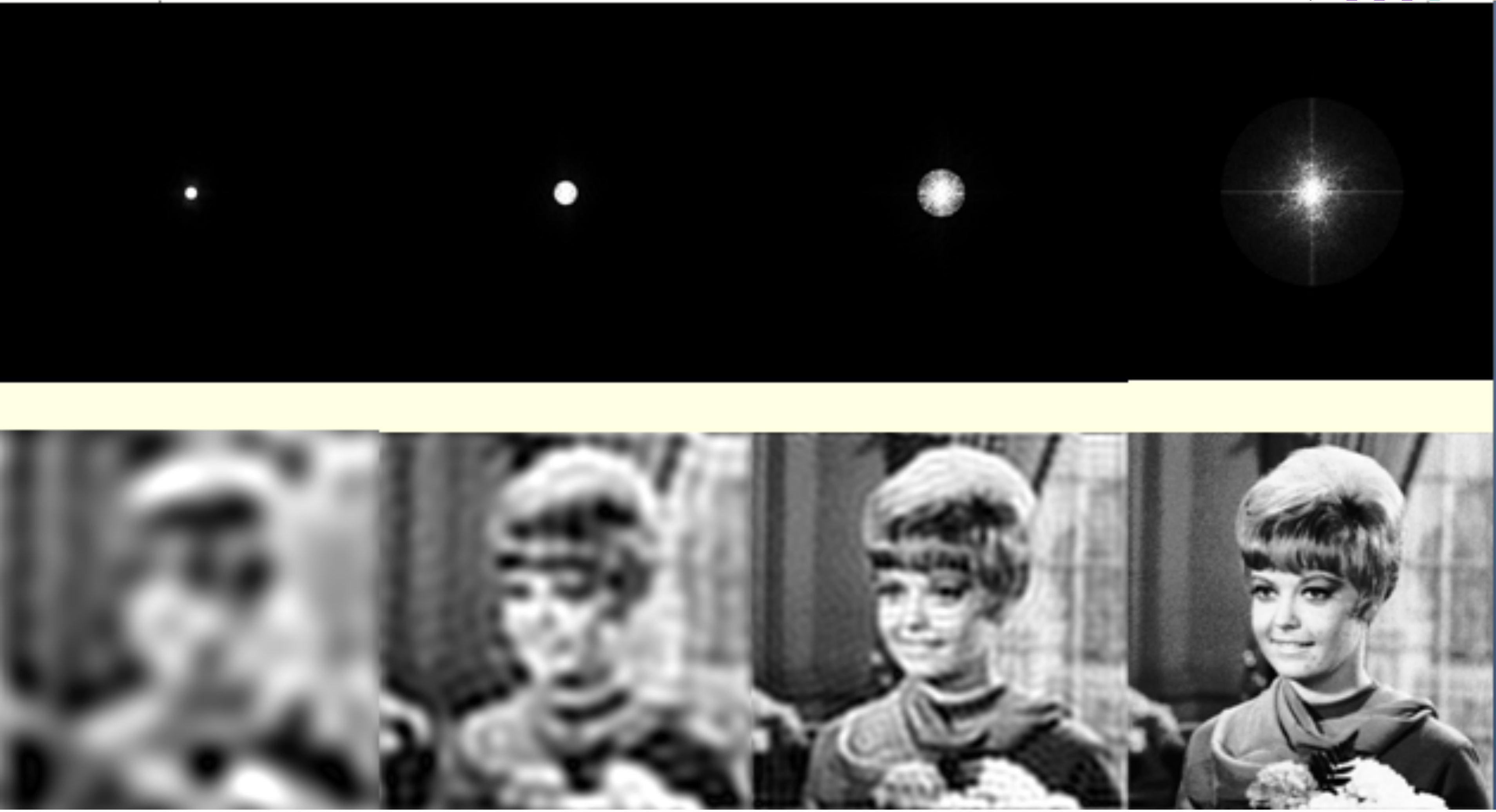
频谱图中暗的点数更多，那么实际图像是比较柔和的，反之，如果频谱图中亮的点数多，那么实际图像一定是尖锐的，边界分明且边界两边像素差异较大的



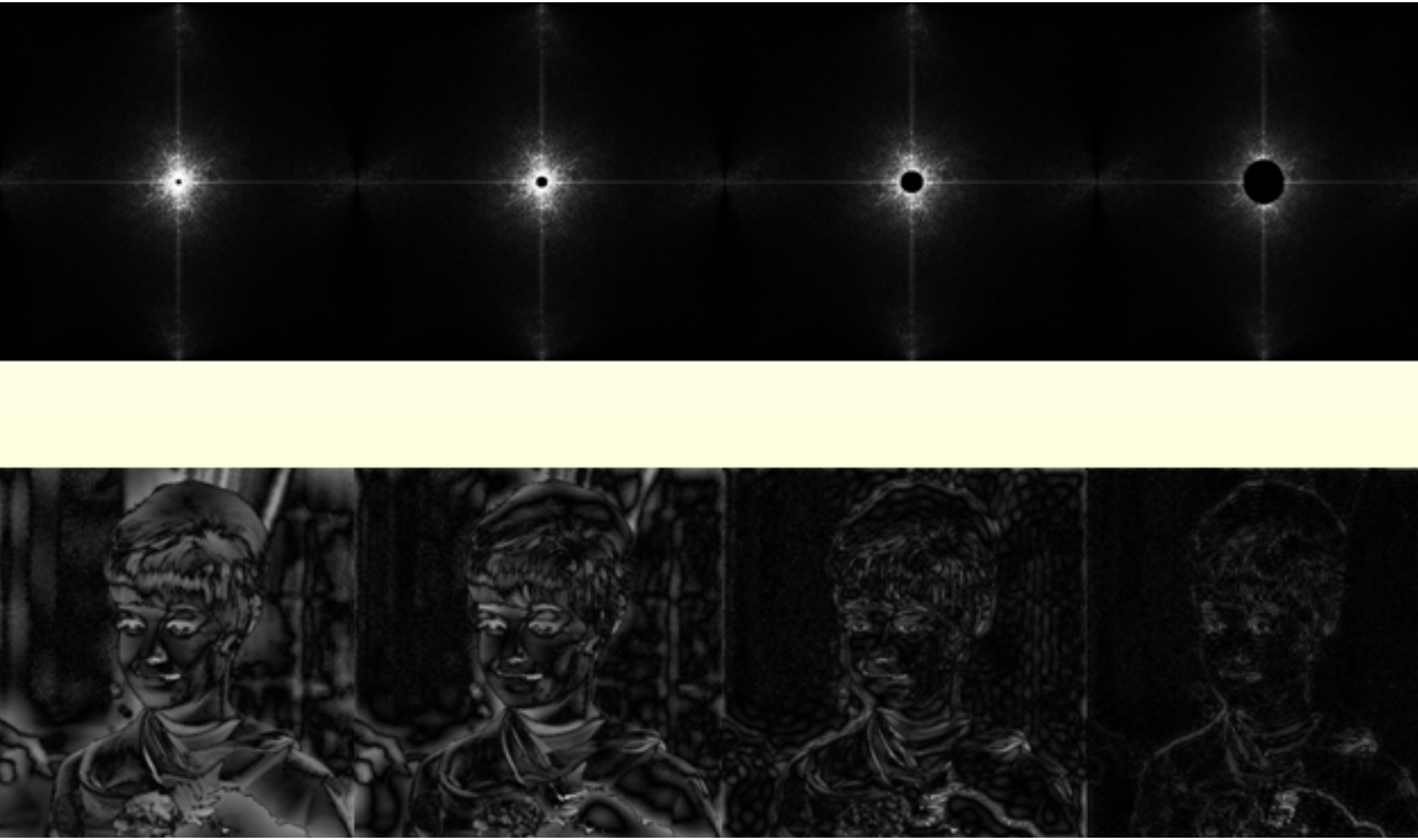


可以看到大圆经傅立叶变换之后，其圆环小；而小圆经傅立叶变换之后，其圆环反而大。

因为越尖锐变换越剧烈的信号总包含着更多的频率成分。

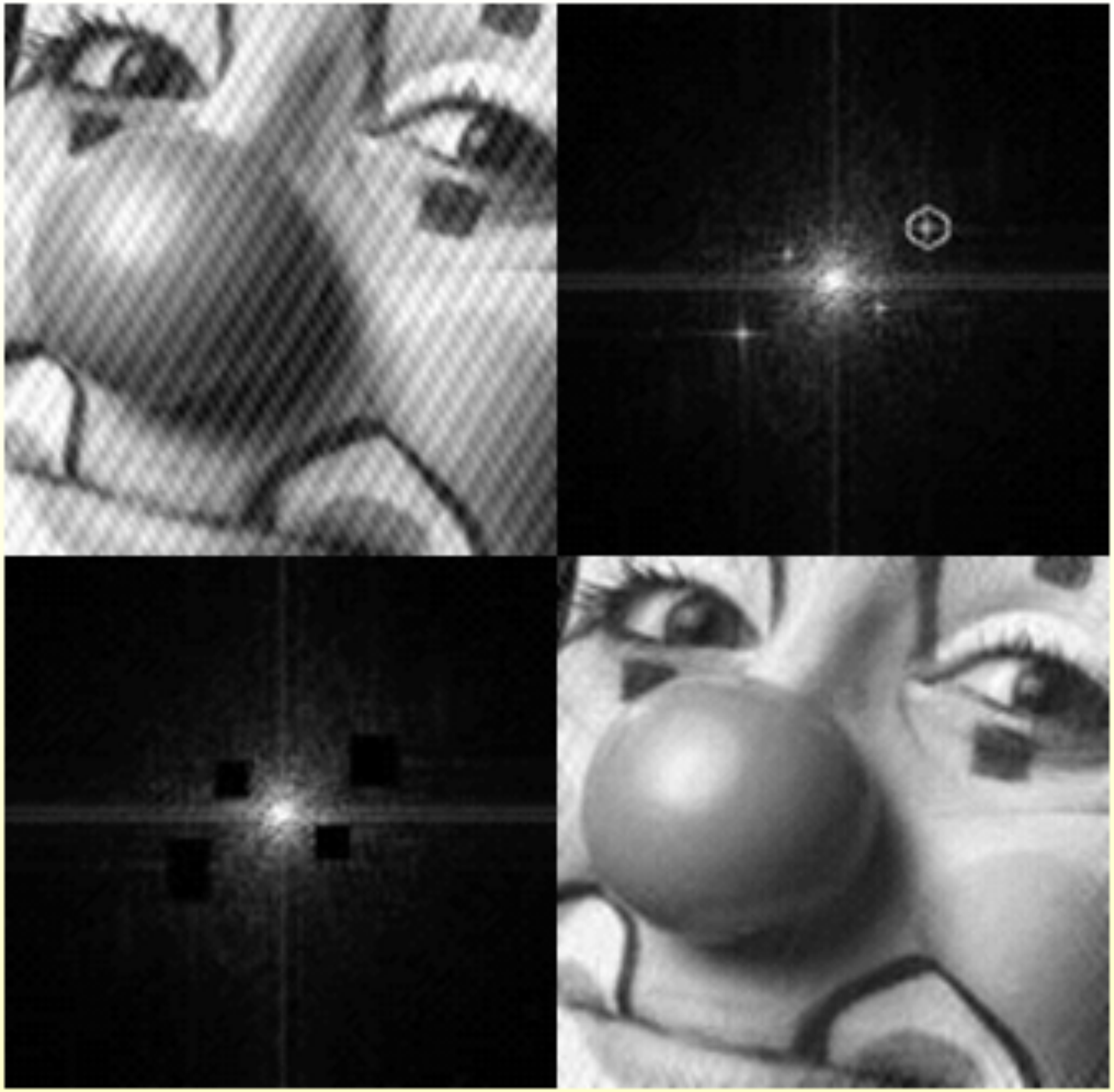


高频部分（距离中心较远区域）越多，图像细节越丰富。反之图像越平滑，显得模糊



高频部分（距离中心较远区域）对应于图像的细节（边缘，线条等）。低频部分（中心部分）对应于图像的整体信息。

通过去除图像频域谱的噪声点，增强图像



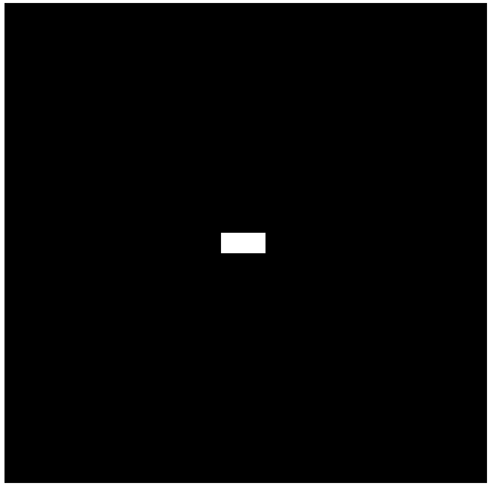
例题：对一幅图像实施二维DFT，显示并观察其频谱。
解：源程序及运行结果如下：

```
%对单缝进行快速傅里叶变换，以三种方式显示频谱，
%即：直接显示（坐标原点在左上角）；把坐标原点平
%移至中心后显示；以对数方式显示。
f=zeros(512,512);
f(246:266,230:276)=1;
subplot(221),imshow(f,[]),title('单狭缝图像')
F=fft2(f); %对图像进行快速傅里叶变换
S=abs(F);
subplot(222)
imshow(S,[])  %显示幅度谱
title('幅度谱（频谱坐标原点在坐上角）')
```

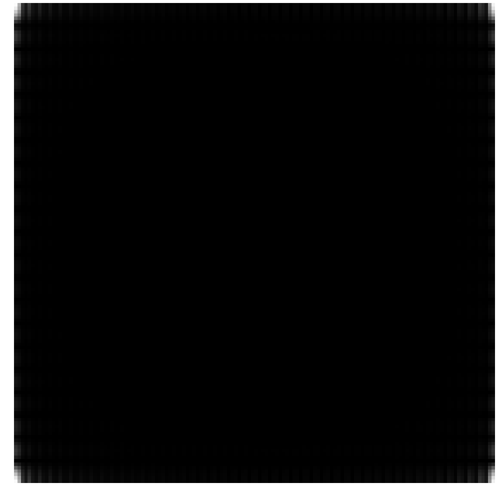
```
Fc=fftshift(F); %把频谱坐标原点由左上角移至屏幕中央
subplot(223)
Fd=abs(Fc);
imshow(Fd,[])
ratio=max(Fd(:))/min(Fd(:))
 %ratio = 2.3306e+007,动态范围太大，显示器无法正常显示
title('幅度谱（频谱坐标原点在屏幕中央）')
S2=log(1+abs(Fc));
subplot(224)
imshow(S2,[])
title('以对数方式显示频谱')
```

运行上面程序后，结果如下：

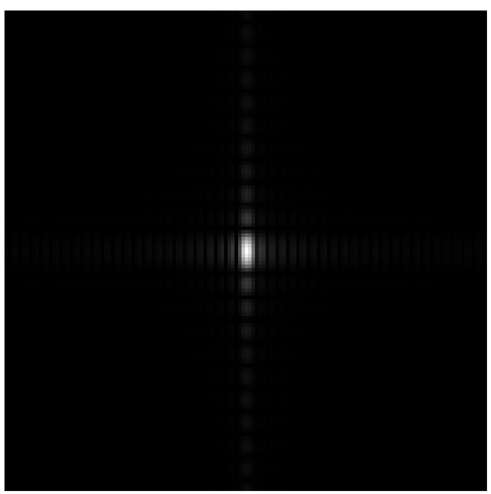
单狭缝图像



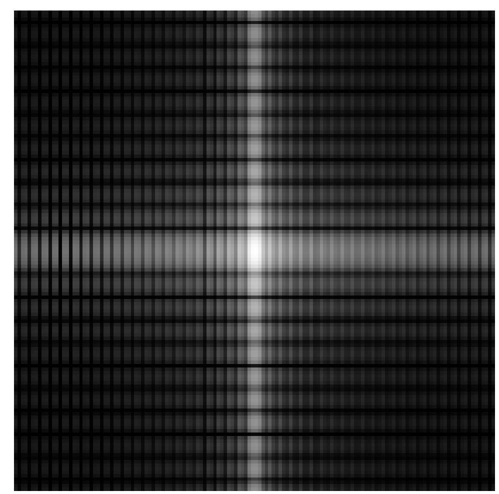
幅度谱（频谱坐标原点在坐上角）



幅度谱（频谱坐标原点在屏幕中央）



以对数方式显示频谱



二维离散傅里叶变换的性质

➢ 线性组合律

$$\begin{cases} f_1(x,y) \Leftrightarrow F_1(u,v) \\ f_2(x,y) \Leftrightarrow F_2(u,v) \end{cases} \Rightarrow c_1 f_1(x,y) + c_2 f_2(x,y) \Leftrightarrow c_1 F_1(u,v) + c_2 F_2(u,v)$$

证明：

$$DFT\left[c_1 f_1(x,y) + c_2 f_2(x,y)\right]$$

$$= \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left[c_1 f_1(x,y) + c_2 f_2(x,y)\right] \cdot e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= c_1 \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_1(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} + c_2 \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f_2(x,y) e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= c_1 F_1(u,v) + c_2 F_2(u,v)$$

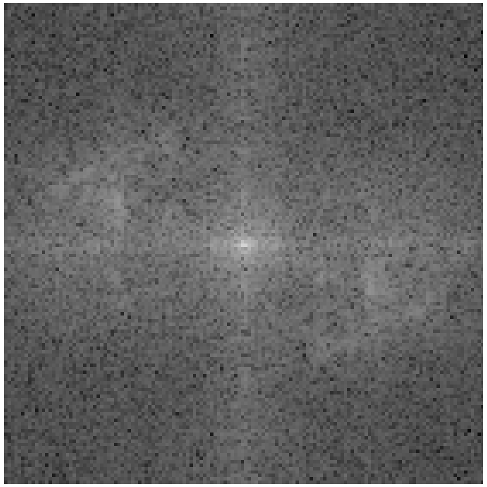
```
%imagelinear.m
%该程序验证了二维DFT的线性性质

f=imread('D:\chenpc\data\thry\chpt4\Fig4.04(a).jpg');
g=imread('D:\chenpc\data\thry\chpt4\Fig4.30(a).jpg');
[m,n]=size(g);
f(m,n)=0;
f=im2double(f);
g=im2double(g);
subplot(221)
imshow(f,[])
title('f')
subplot(222)
imshow(g,[])
title('g')
```

```
F=fftshift(fft2(f));
G=fftshift(fft2(g));
subplot(223)
imshow(log(abs(F+G)),[])
FG=fftshift(fft2(f+g));
title('DFT(f)+DFT(g)')
subplot(224)
imshow(log(abs(FG)),[])
title('DFT(f+g)')
```

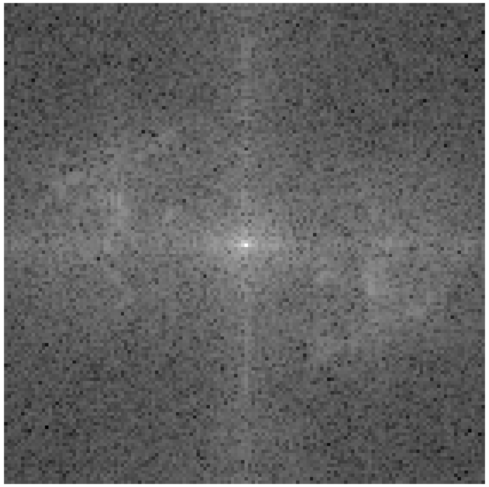
f

g

DFT(f)+DFT(g)



DFT(f+g)



➢ 可分离性

二维DFT可视为由沿x,y方向的两个一维DFT所构成。

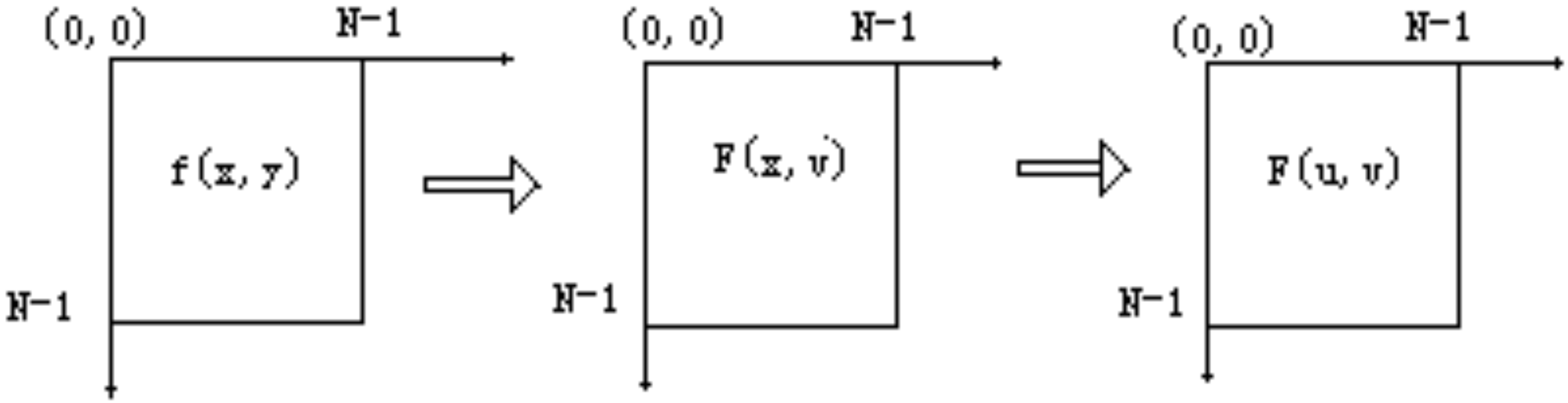
$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\cdot e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$
$$=\sum_{x=0}^{M-1}\left[\sum_{y=0}^{N-1}f(x,y)\cdot e^{-j2\pi\frac{vy}{N}}\right]\cdot e^{-j2\pi\frac{ux}{M}}$$
$$=\sum_{x=0}^{M-1}F(x,v)\cdot e^{-j2\pi\frac{ux}{M}}$$

$$f(x,y)=\frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F(u,v)\cdot e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$
$$=\frac{1}{M}\sum_{u=0}^{M-1}\left[\frac{1}{N}\sum_{v=0}^{N-1}F(u,v)\cdot e^{j2\pi\frac{vy}{N}}\right]\cdot e^{j2\pi\frac{ux}{M}}$$
$$=\frac{1}{M}\sum_{u=0}^{M-1}F(u,y)\cdot e^{j2\pi\frac{ux}{M}}$$

其中：

$$\begin{cases}F(x,v)=\sum_{y=0}^{N-1}f(x,y)\cdot e^{-j2\pi\frac{vy}{N}}\sim y\text{方向的}DFT\\ F(u,v)=\sum_{x=0}^{M-1}F(x,v)\cdot e^{-j2\pi\frac{ux}{M}}\sim x\text{方向的}DFT\end{cases}$$

$$\begin{cases}F(u,y)=\frac{1}{N}\sum_{v=0}^{N-1}F(u,v)\cdot e^{j2\pi\frac{vy}{N}}\sim y\text{方向的}IDFT\\ f(x,y)=\frac{1}{M}\sum_{u=0}^{M-1}F(u,y)\cdot e^{j2\pi\frac{ux}{M}}\sim x\text{方向的}IDFT\end{cases}$$



例题：编程验证二维离散傅里叶变换可分离为两个一维离散傅里叶变换。

解：

%myseparable.m

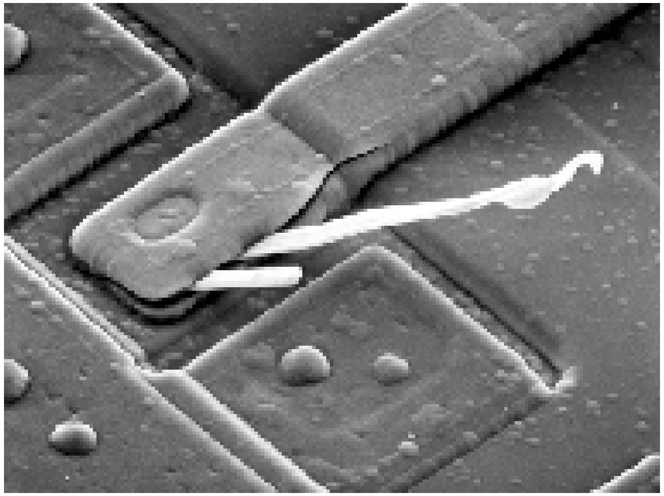
%该程序验证了二维DFT的可分离性质

%该程序产生了冈萨雷斯《数字图像处理》（第二版）

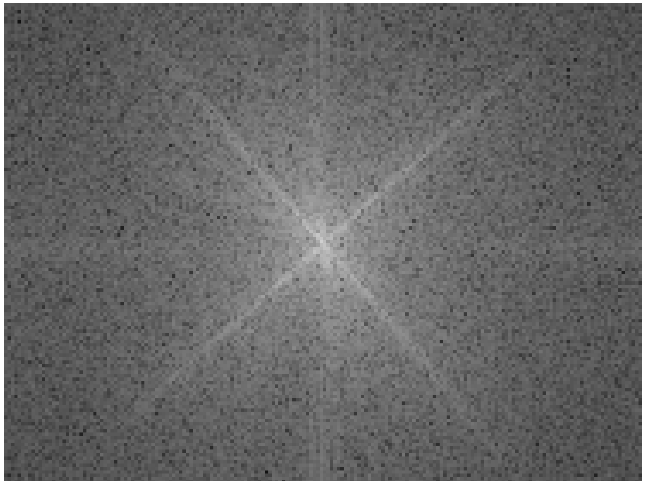
%P125 图4.4

```
f=imread('D:\chenpc\data\thry\chpt4\Fig4.04(a).jpg');
subplot(211)
imshow(f,[])
title('原图')
F=fftshift(fft2(f));
subplot(223)
imshow(log(1+abs(F)),[])
title('用fft2实现二维离散傅里叶变换')
[m,n]=size(f);
F=fft(f);        %沿x方向求离散傅里叶变换
G=fft(F')';      %沿y方向求离散傅里叶变换
F=fftshift(G);
subplot(224)
imshow(log(1+abs(F)),[])
title('用fft实现二维离散傅里叶变换')
```

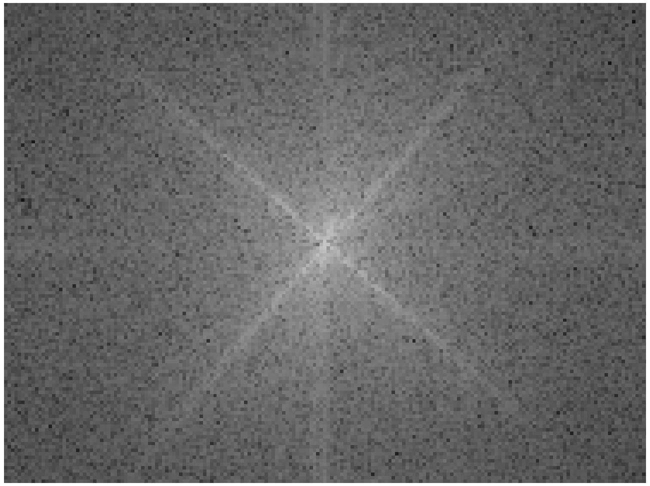
原 图



用 fft2实 现 二 维 离 散 傅 里 叶 变 换



用 fft实 现 二 维 离 散 傅 里 叶 变 换



➢ 平移性

$$f(x,y)\Leftrightarrow F(u,v)\Rightarrow\begin{cases}f(x,y)\cdot e^{j2\pi\left[\frac{u_0x}{M}+\frac{v_0y}{N}\right]}\Leftrightarrow F(u-u_0,v-v_0)\\ f(x-x_0,y-y_0)\Leftrightarrow F(u,v)\cdot e^{-j2\pi\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)}\end{cases}$$

证明：

（1）频域移位

$$DFT\left[f(x,y)\cdot e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}\right]$$
$$=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\cdot e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)}\cdot e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$
$$=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\cdot e^{-j2\pi\left(\frac{(u-u_0)x}{M}+\frac{(v-v_0)y}{N}\right)}$$
$$=F(u-u_0,v-v_0)$$

结论：

$$f(x,y)\cdot e^{j2\pi\left(\frac{u_0x}{M}+\frac{v_0y}{N}\right)} \Leftrightarrow F(u-u_0, v-v_0)$$

当 $u_0=\dfrac{M}{2}, v_0=\dfrac{N}{2}$

$$e^{j2\pi(u_0x/M+v_0y/N)} = e^{j\pi(x+y)} = (-1)^{x+y}$$

$$\Rightarrow f(x,y)\cdot(-1)^{x+y} \Leftrightarrow F\left(u-\frac{M}{2}, v-\frac{N}{2}\right)$$

即如果需要将频域的坐标原点从显示屏起始点（0，0）移至显示屏的中心点只要将f(x,y)乘以$(-1)^{x+y}$因子再进行傅里叶变换即可实现。

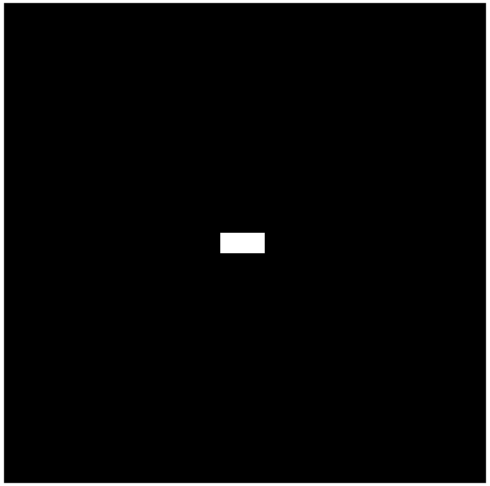
例题：利用$(-1)^{x+y}$对单缝图像$f(x,y)$进行调制，实现把频谱坐标原点移至屏幕正中央的目标。

解：完成本题的源程序为：

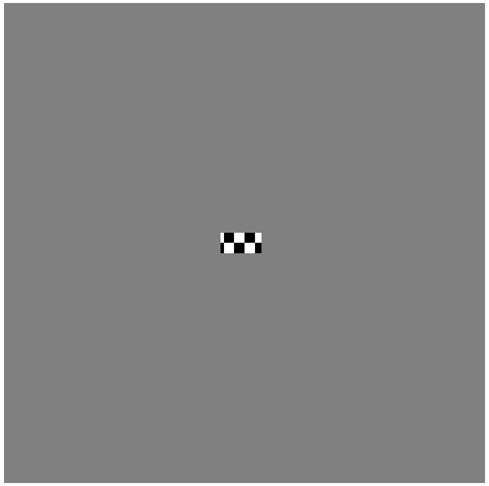
%在傅里叶变换之前，把函数乘以(-1) $^{x+y}$，相当于把频谱
%坐标原点移至屏幕窗口正中央。

```
f(512,512)=0;
f=mat2gray(f);
[Y,X]=meshgrid(1:512,1:512);
f(246:266,230:276)=1;
g=f.*(-1).^(X+Y);
subplot(221),imshow(f,[]),title('原图像f(x,y)')
subplot(222),imshow(g,[]),title('空域调制图像
g(x,y)=f(x,y)*(-1)^{x+y}')
F=fft2(f);
subplot(223),imshow(log(1+abs(F)),[]),title('f(x,y)的傅里叶频
谱')
G=fft2(g);
subplot(224),imshow(log(1+abs(G)),[]),title('g(x,y)的傅里叶
频谱')
```

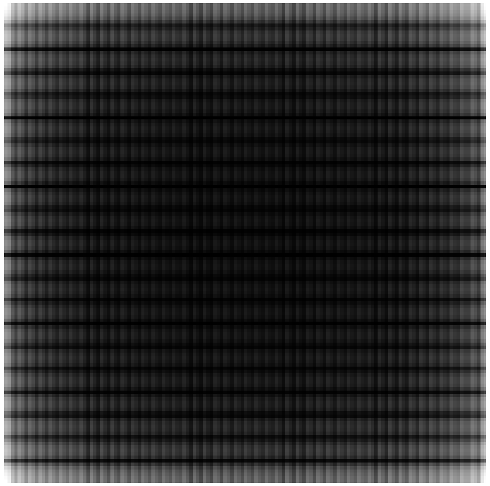
原 图 像 f(x,y)



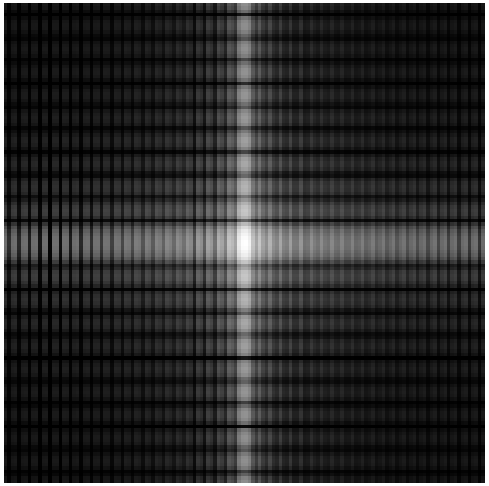
空 域 调 制 图 像 $g(x,y)=f(x,y)*(-1)^{x+y}$



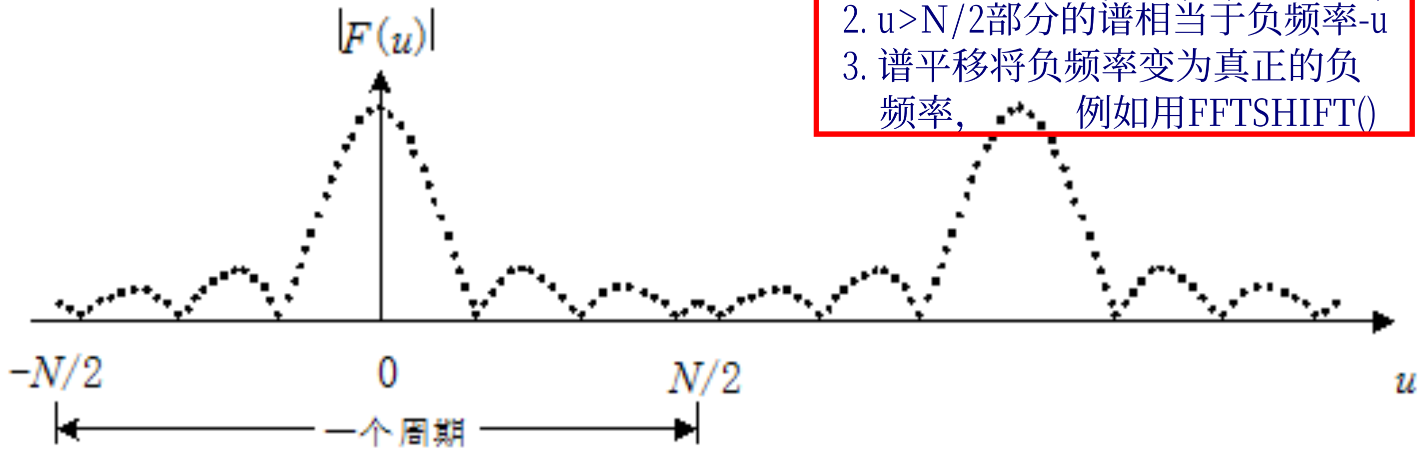
f(x,y)的 傅 里 叶 频 谱



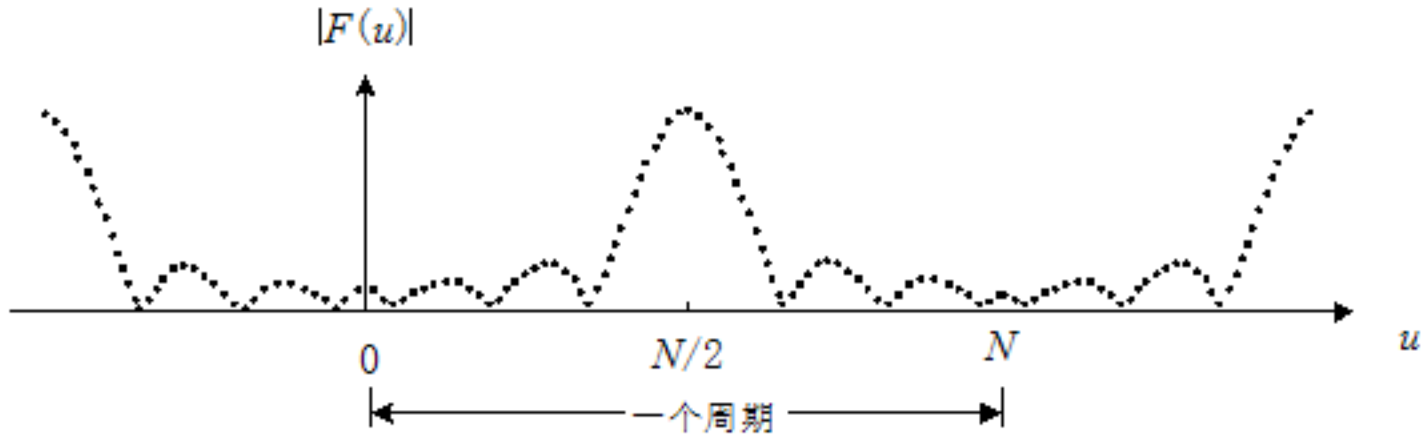
g(x,y)的 傅 里 叶 频 谱



1. $F(u)=F^*(-u)$，或者$F(u)=F^*(N-1-u)$，幅度谱相同，$|F(u)|=|F(N-1-u)|$
2. $u>N/2$部分的谱相当于负频率$-u$
3. 谱平移将负频率变为真正的负频率， 例如用FFTSHIFT()



（a） 在[0 N-1]周期中有两个背靠背半周期



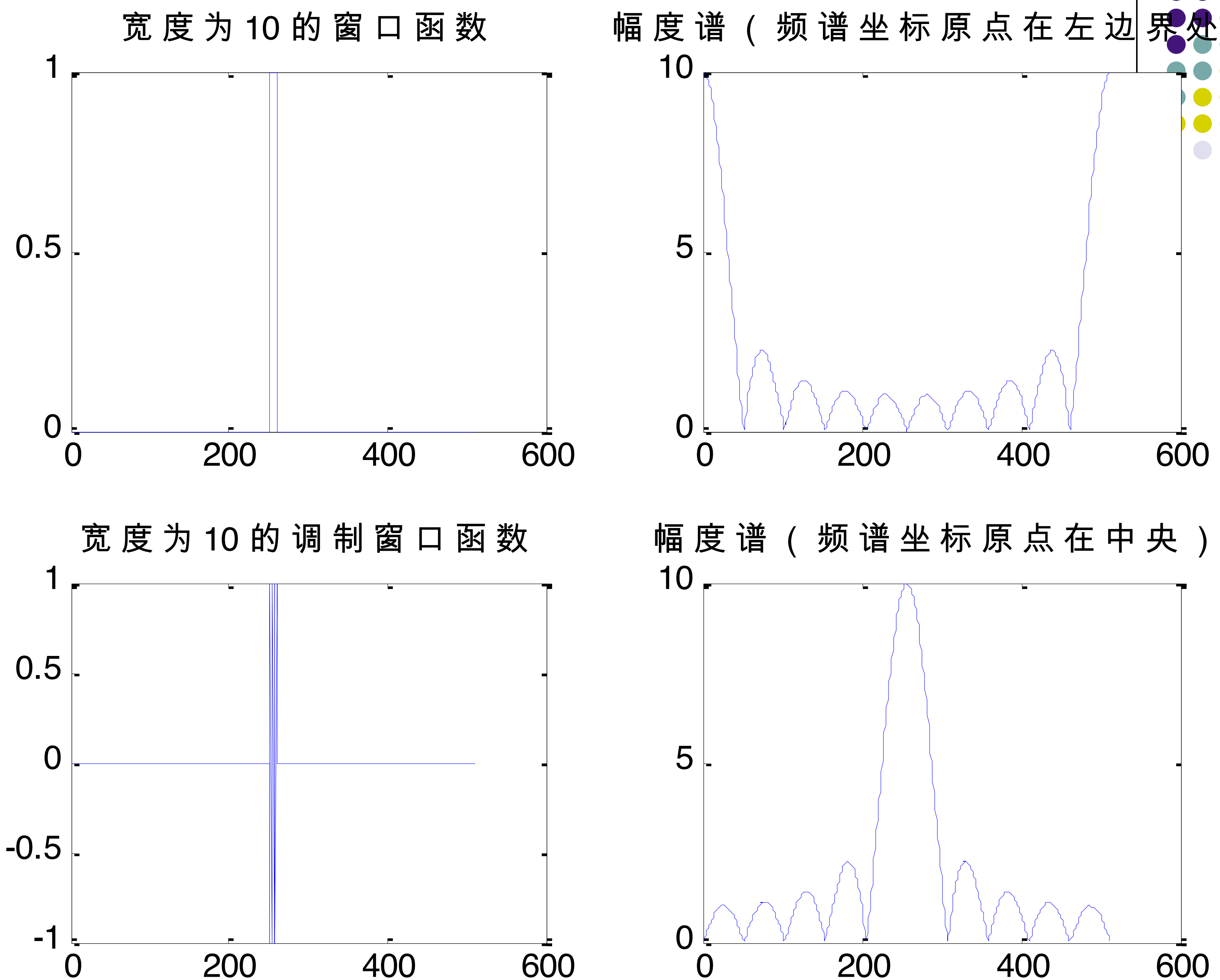
（b） 同一区间内有一个完整的周期

这就意味着，坐标原点移到了频谱图像的中间位置，这一点十分重要，尤其是对以后的图像显示和滤波处理。

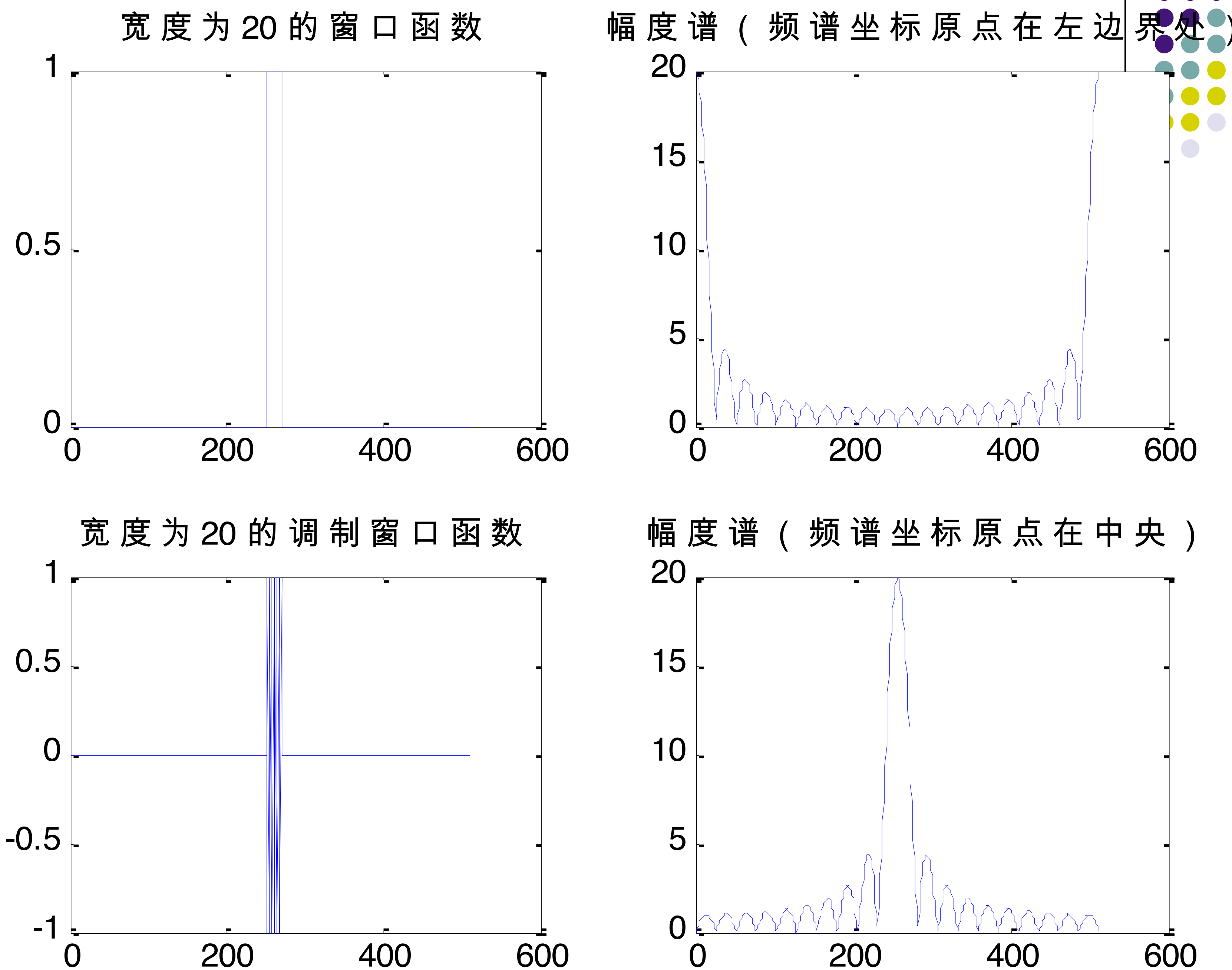
例题：利用$(-1)^x$对$f(x)$曲线进行调制，达到平移频域坐标原点至屏幕正中央的目的。

%以一维情况为例，说明空域调制对应着频域坐标原点移位。
f(1:512)=0;
f(251:260)=1;　 %产生宽度为10的窗口函数
subplot(221),plot(f),title('宽度为10 的窗口函数')
F=fft(f,512);　　%进行快速傅里叶变换，延拓周期周期为512
subplot(222)
plot(abs(F))　%绘幅度频谱（频谱坐标原点在左边界处）
title('幅度谱（频谱坐标原点在左边界处）')
x=251:260;
f(251:260)=(-1).^x;　 %把曲线f(x)乘以(-1)^x，可以把频谱
%坐标原点移至屏幕正中央
subplot(223),plot(f),title('宽度为10 的调制窗口函数')

```
F=fft(f,512);       %进行快速傅里叶变换
subplot(224);
plot(abs(F))  %直接显示幅度频谱（频谱坐标原点在正中央）
title('幅度谱（频谱坐标原点在中央）')
figure
f(1:512)=0;
f(251:270)=1;    %产生宽度为20的窗口函数
subplot(221),plot(f),title('宽度为20 的窗口函数')
F=fft(f,512);       %进行快速傅里叶变换，延拓周期周期为512
subplot(222)
plot(abs(F))   %绘幅度频谱（频谱坐标原点在左边界处）
title('幅度谱（频谱坐标原点在左边界处）')
x=251:270;
f(251:270)=(-1).^x;    %把曲线f(x)乘以(-1)^x，可以把频谱坐标原点移至屏幕正中央
subplot(223),plot(f),title('宽度为20 的调制窗口函数')
F=fft(f,512);       %进行快速傅里叶变换
subplot(224);
plot(abs(F))  %直接显示幅度频谱（频谱坐标原点在正中央）
title('幅度谱（频谱坐标原点在中央）')
```



宽度为10的窗口函数
幅度谱（频谱坐标原点在左边界处）
宽度为10的调制窗口函数
幅度谱（频谱坐标原点在中央）
0
200
400
600
1
0.5
0
-0.5
-1
10
5



宽度为20的窗口函数
幅度谱（频谱坐标原点在左边界处）
宽度为20的调制窗口函数
幅度谱（频谱坐标原点在中央）
0
0.5
1
-0.5
-1
5
10
15
20
200
400
600

（2）空域移位：

$$DFT[f(x-x^0, y-y^0)]$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f\left(x-x_0, y-y_0\right)\cdot e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f\left(x-x_0, y-y_0\right)\cdot e^{-j2\pi\left(\frac{u\left(x-x_0+x_0\right)}{M}+\frac{v\left(y-y_0+y_0\right)}{N}\right)}$$

$$= e^{-j\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)\cdot 2\pi}\cdot\sum_{x=-x_0}^{M-1-x_0}\sum_{y=-y_0}^{N-1-y_0} f\left(x,y\right)\cdot e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= e^{-j\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)\cdot 2\pi}\cdot\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f\left(x,y\right)\cdot e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

$$= e^{-j\left(\frac{ux_0}{M}+\frac{vy_0}{N}\right)\cdot 2\pi}\cdot F\left(u,v\right)$$

➢ 周期性和共轭对称性

周期性：

$$\begin{cases} F(u,v) = F(u+mM, v+nN) \\ f(x,y) = f(x+mM, y+nN) \end{cases}$$

$$(m,n = 0, \pm 1, \pm 2, \ldots)$$

共轭对称性：

$$F(u,v) = F^*(-u,-v)$$

$$|F(u,v)| = |F(-u,-v)|$$

注意：

1. 幅度谱对称

2. 对于N*M之FFT，有

   |F(u,v)|=|F(N-1-u, M-1-v)|

证明：　（1）周期性：

$$\begin{cases} F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\cdot e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \\ f(x,y)=\frac{1}{MN}\cdot\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\cdot e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \end{cases}$$

$$e^{-j2\pi m}=1 \longrightarrow \begin{cases} F(u+mM,v+nN)=F(u,v) \\ f(x+mM,y+nN)=f(x,y) \end{cases}$$

(2) 共轭对称性：

$$F(u,v)=\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\cdot e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}=\left\{\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\cdot e^{-j2\pi\left[\frac{(-u)x}{M}+\frac{(-v)y}{N}\right]}\right\}^*$$

$$=F^*(-u,-v)$$

$|F(u,v)|=|F(-u,-v)|$，即$F(u,v)$关于原点对称

➢ 旋转不变性

$f(x,y) \leftrightarrow F(u,v)$ $\xrightarrow{\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases},\ \begin{cases} u = \omega\cos\varphi \\ v = \omega\sin\varphi \end{cases}}$ $f(r,\theta) \leftrightarrow F(\omega,\varphi)$

$\longrightarrow$ $f(r,\theta+\theta_0) \leftrightarrow F(\omega,\varphi+\theta_0)$

证明：

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\cdot e^{-j2\pi(ux+vy)}\cdot dxdy$$

若 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta, \end{cases}$ $\begin{cases} u = \omega\cos\varphi \\ v = \omega\sin\varphi, \end{cases}$ 则：

$$F(\omega,\varphi) = \int_0^{\infty}\int_0^{2\pi} f(r,\theta)\cdot e^{-j2\pi\cdot\omega\cdot r\cdot\cos(\varphi-\theta)}\cdot r\cdot dr\cdot d\theta$$

θ'=θ-θ0 —》θ=θ'+θ0

$$F(\omega,\varphi+\theta_0)=\int_0^{\infty}\int_0^{2\pi} f(r,\theta)\cdot e^{-j2\pi r\omega\cos[\varphi-(\theta-\theta_0)]}\cdot r dr d\theta$$

$$=\int_0^{\infty}\int_{-\theta_0}^{2\pi-\theta_0} f(r,\theta+\theta_0)\cdot e^{-j2\pi r\omega\cos(\varphi-\theta)}\cdot r dr d\theta$$

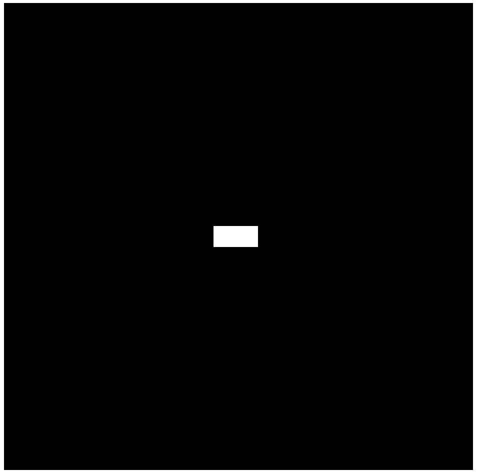
$f(r,\theta)=f(r,\theta+2\pi)$

$$=\int_0^{\infty}\int_0^{2\pi} f(r,\theta+\theta_0)\cdot e^{-j2\pi r\omega\cos(\varphi-\theta)}\cdot r dr d\theta$$

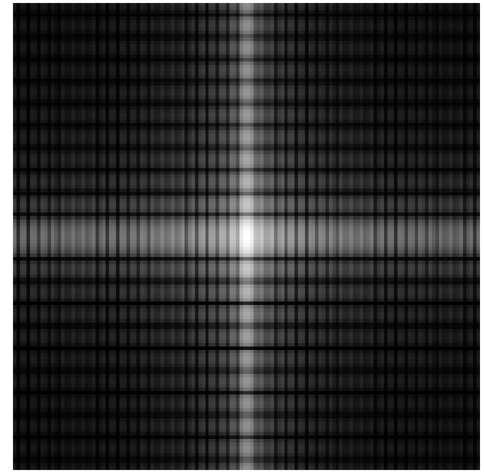
注：为看清问题的实质、简化旋转不变性的证明，以上用二维连续傅里叶变换进行证明。实际上，由连续积分公式进行离散化处理，即可得到离散公式，证明可参照连续情况进行。

```
f=zeros(512,512);
f(246:266,230:276)=1;
subplot(221);
imshow(f,[])
title('原图')
F=fftshift(fft2(f));
subplot(222);
imshow(log(1+abs(F)),[])
title('原图的频谱')
f=imrotate(f,45,'bilinear','crop');
subplot(223)
imshow(f,[])
title('旋转45^0图')
Fc=fftshift(fft2(f));
subplot(224);
imshow(log(1+abs(Fc)),[])
title('旋转图的频谱')
```

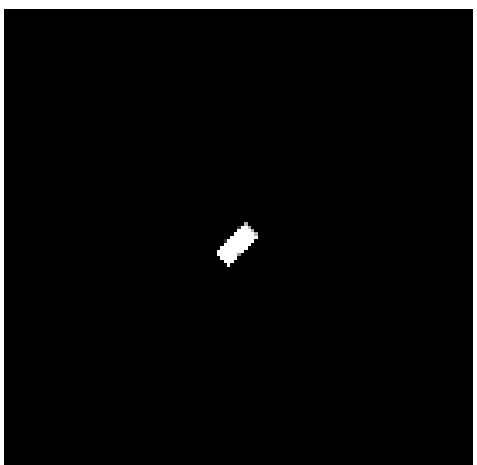
原图



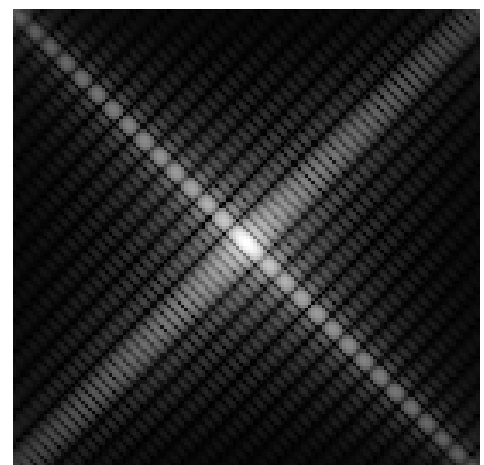
原图的频谱



旋转45^0图



旋转图的频谱



一维离散傅立叶变换

- 快速傅立叶变换FFT

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1}f(x)e^{-j\frac{2\pi ux}{N}}$$

$$DFT\text{计算复杂度}=N^2\text{次乘法}+N(N-1)\text{次加法}$$

$$\text{对于}N=2^n\text{幂时有快速算法}$$

$$FFT\text{计算复杂度}=N\lg_2 N$$

- 时域分组：将旋转因子W（由$e^{-j2\pi x/N}$构成的矩阵）中把x不断分解为奇偶表达式；
- 频域分组：将u不断分解为奇偶表达式。

旋转因子 W_N^{km} 的性质

1)周期性

$$W_N^{(k+N)m} = W_N^{k(m+N)} = W_N^{km}$$

2) 对称性

$$W_N^{mk+\frac{N}{2}} = -W_N^{mk} \qquad \left(W_N^{km}\right)^* = W_N^{-mk}$$

3)可约性

$$W_N^{mk} = W_{nN}^{nmk}$$

$$W_N^{mk} = W_{N/n}^{mk/n}, \quad N/n\text{为整数}$$

旋转因子定义:

$$W_N^u = \exp(-j2\pi u / N)$$

周期性证明:

$$\begin{aligned} W_N^{(k+N)m} &= \exp[-j2\pi(km + Nm) / N] \\ &= \exp(-j2\pi km / N - j2\pi m) \\ &= W_N^{km} \exp(-j2\pi m) \\ &= W_N^{km} \end{aligned}$$

快速离散傅立叶变换

$N=2^m$幂，$f(x)$分解为$f(2x)$和$f(2x+1)$:

$$F(u)=\frac{1}{N}\sum_{x=0}^{N-1}f(x)W_N^{ux}$$

$$=\frac{1}{2}\left[\frac{2}{N}\sum_{x=0}^{N/2-1}f(2x)W_N^{2ux}+\frac{2}{N}\sum_{x=0}^{N/2-1}f(2x+1)W_N^{u(2x+1)}\right]$$

注意x的取值范围

$$=\frac{1}{2}\left[\frac{2}{N}\sum_{x=0}^{N/2-1}f(2x)W_{N/2}^{ux}+\frac{2}{N}\sum_{x=0}^{N/2-1}f(2x+1)W_{N/2}^{ux}W_N^{u}\right]$$

N/2 点的DFT

$$=\frac{1}{2}\left[F_e(u)+W_N^uF_o(u)\right]$$

$$F\left(u+\frac{N}{2}\right)=\frac{1}{2}\left[F_e\left(u+\frac{N}{2}\right)+W_N^{u+N/2}F_o\left(u+\frac{N}{2}\right)\right]$$

快速离散傅立叶变换

注意：此时是N/2点，周期为N/2

$$F_e\left(u+\frac{N}{2}\right)=F_e(u),F_o\left(u+\frac{N}{2}\right)=F_o(u)$$

$$W_N^{u+N/2}=W_N^u W_N^{N/2}=W_N^u e^{-j\frac{2\pi}{N}\frac{N}{2}}=W_N^u e^{-j\pi}=-W_N^u$$

$$\therefore F\left(u+\frac{N}{2}\right)=\frac{1}{2}\left[F_e(u)-W_N^u F_o(u)\right]$$

因此F_e和F_o中的x继续分解，直到2点。

$$F_0\sim F_7\Rightarrow F_0\sim F_3\Rightarrow F_0\sim F_1\Rightarrow F_0=f_0$$

快速傅里叶变换（FFT）原理

- **FFT算法——基本思想**

1) 将变换公式分解为奇数项和偶数项之和

$$F(u)\big|_{0\ldots N-1} = F(u)\big|_{0\ldots M-1} + F(u+M)\big|_{0\ldots M-1}$$

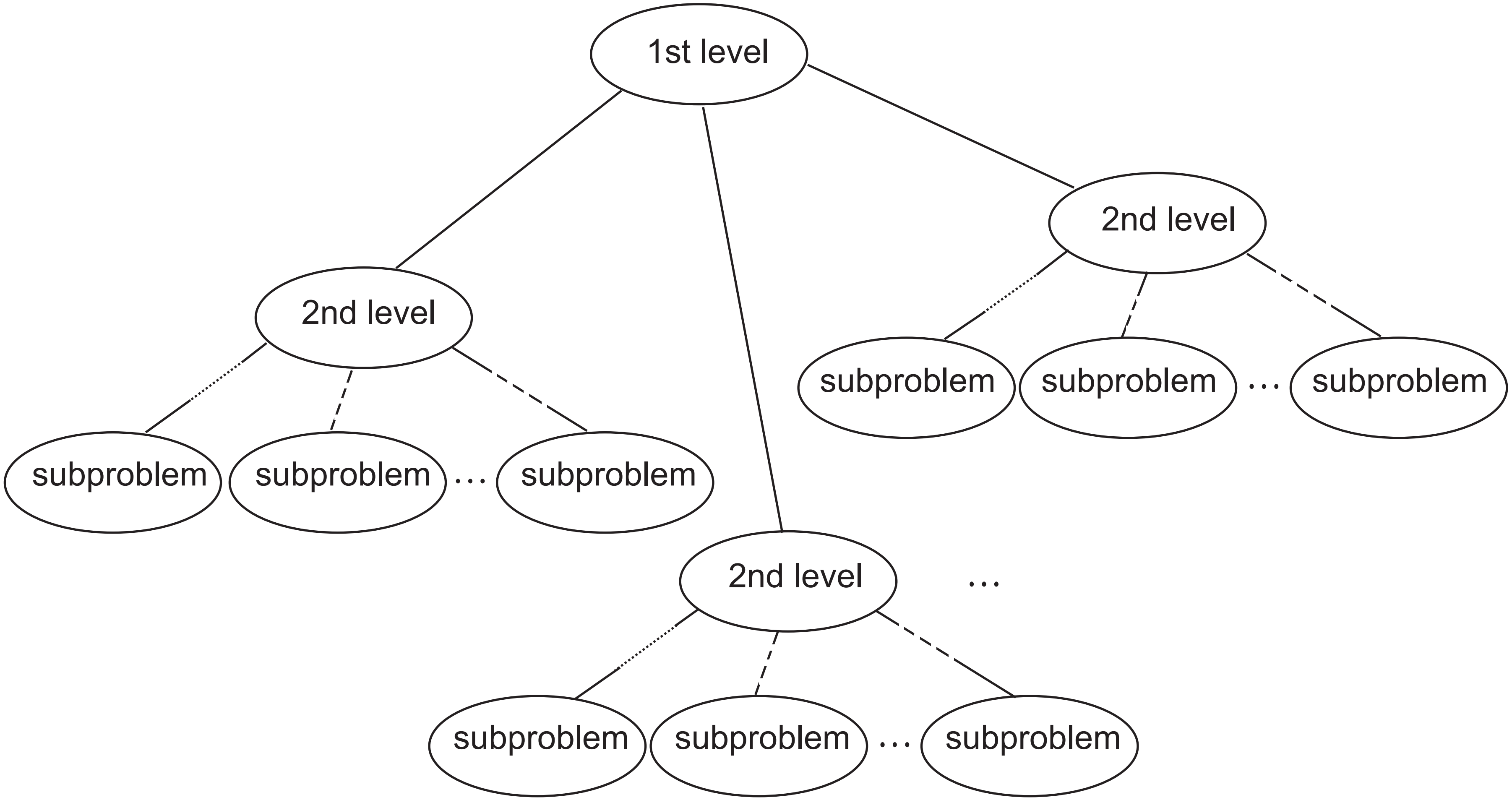
$$= \frac{1}{2}[F_e(u) + w_N^u F_o(u)] + \frac{1}{2}[F_e(u) - w_N^u F_o(u)]$$

基本分解公式：a+b*w, a-b*w, — 蝶形运算

2) 不断地将原函数分为奇数项和偶数项之和，最终得到需要的结果

FFT是将复杂的运算变成重复两个数相加（减）的简单运算

Divide-and-Conquer Technique



■ 快速傅里叶变换（FFT）原理

例：设对一个函数进行快速**Fourier**变换，函数序列为：

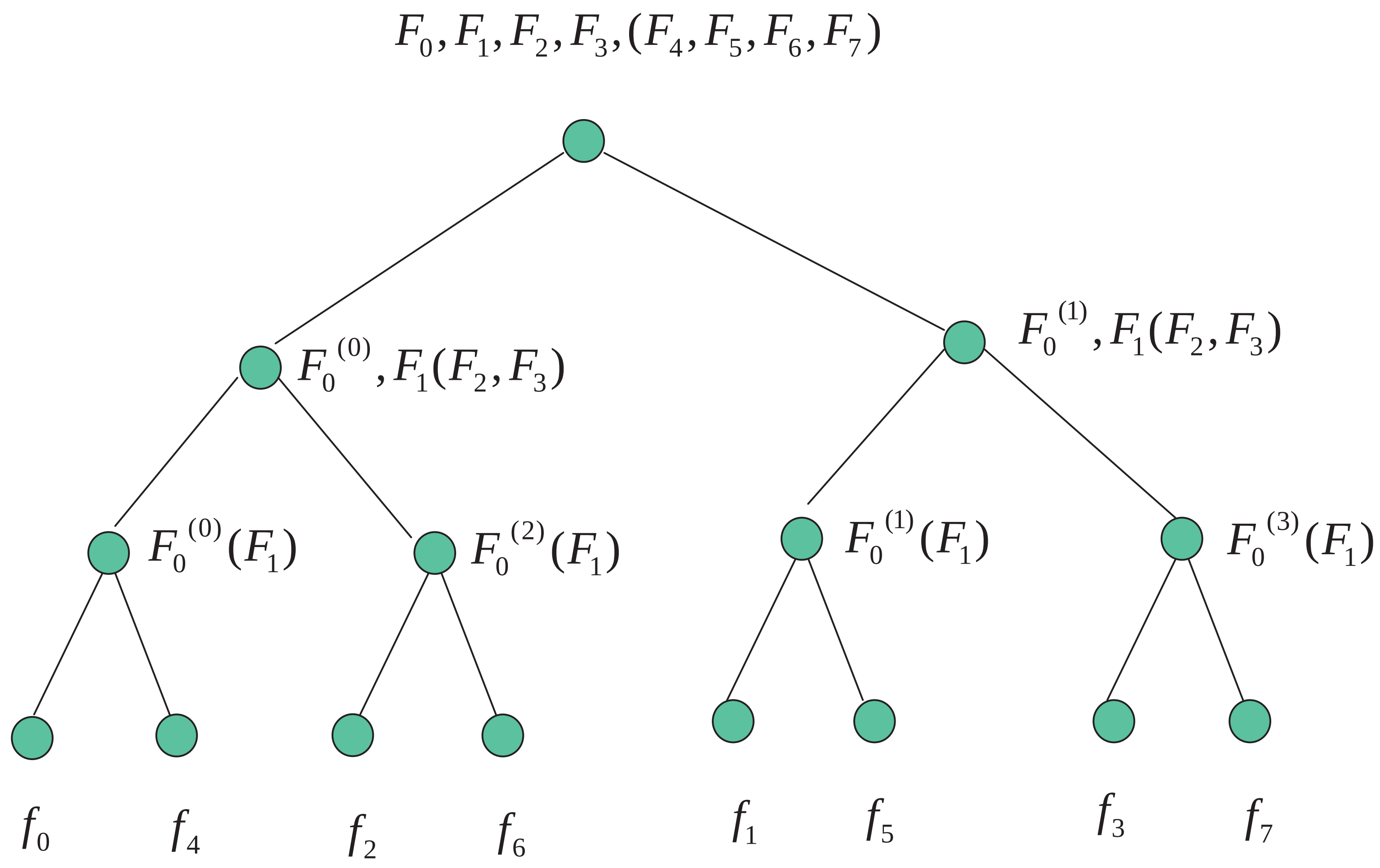
$$f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7$$

分成偶数、奇数部分：

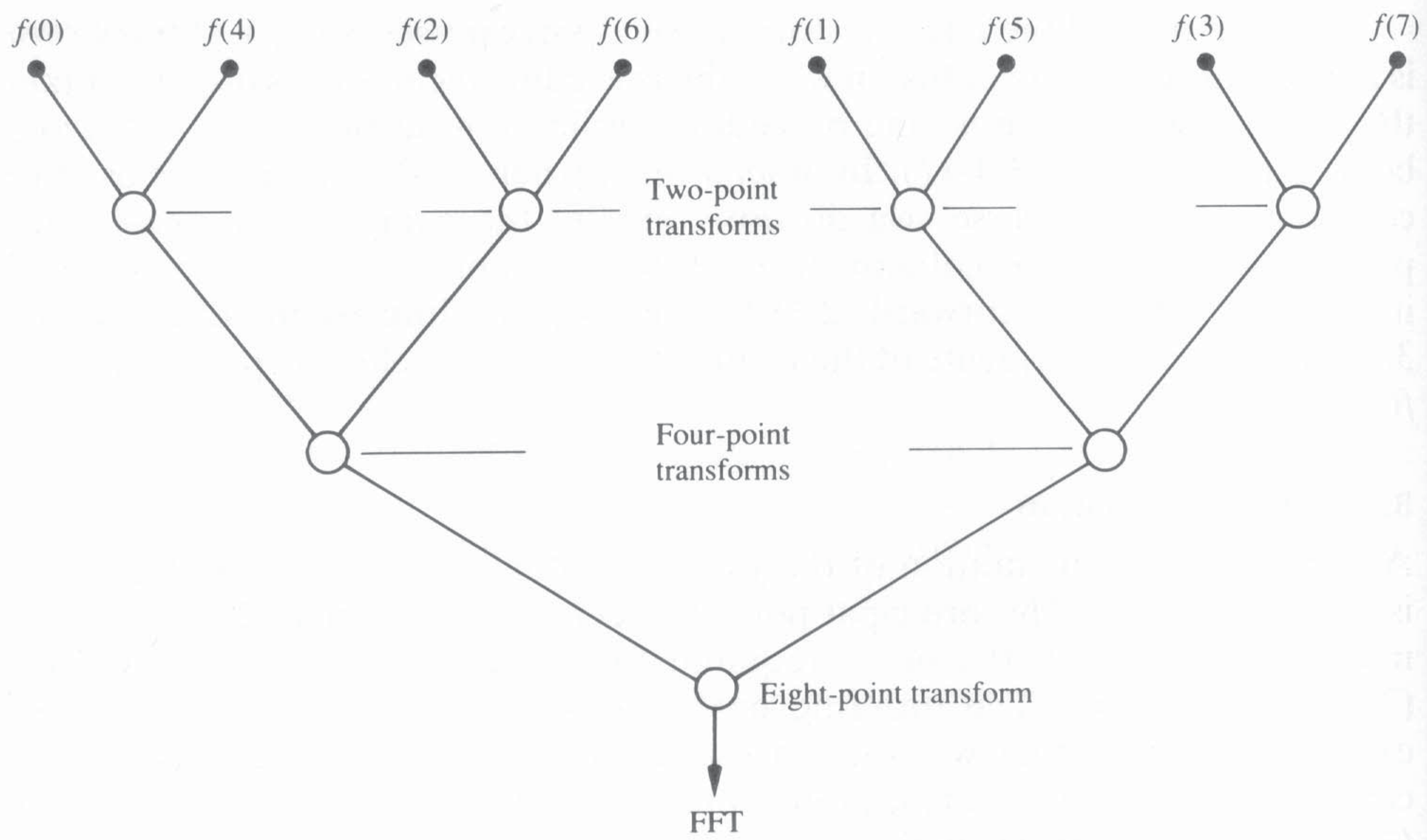
f_0, f_2, f_4, f_6 | f_1, f_3, f_5, f_7

↓ ↓

f_0, f_4 | f_2, f_6 f_1, f_5 | f_3, f_7



$F_0, F_1, F_2, F_3, (F_4, F_5, F_6, F_7)$
$F_0^{(0)}, F_1(F_2, F_3)$
$F_0^{(1)}, F_1(F_2, F_3)$
$F_0^{(0)}(F_1)$
$F_0^{(2)}(F_1)$
$F_0^{(1)}(F_1)$
$F_0^{(3)}(F_1)$
f_0
f_4
f_2
f_6
f_1
f_5
f_3
f_7



f(0)
f(4)
f(2)
f(6)
f(1)
f(5)
f(3)
f(7)
Two-point transforms
Four-point transforms
Eight-point transform
FFT

直接面向f(x)蝶形计算得到中间结果（2点）

$f_0, f_1, f_2, f_3, f_4, f_5, f_6, f_7$

偶数区: f_0, f_4, f_2, f_6

$$F^{(0)}(0)=\tfrac{1}{2}\cdot\left[f_0+w_2^0 f_4\right]$$
$$F^{(0)}(1)=\tfrac{1}{2}\cdot\left[f_0-w_2^0 f_4\right]$$
$$F^{(2)}(0)=\tfrac{1}{2}\cdot\left[f_2+w_2^0 f_6\right]$$
$$F^{(2)}(1)=\tfrac{1}{2}\cdot\left[f_2-w_2^0 f_6\right]$$

奇数区: f_1, f_5, f_3, f_7

$$F^{(1)}(0)=\tfrac{1}{2}\cdot\left[f_1+w_2^0 f_5\right]$$
$$F^{(1)}(1)=\tfrac{1}{2}\cdot\left[f_1-w_2^0 f_5\right]$$
$$F^{(3)}(0)=\tfrac{1}{2}\cdot\left[f_3+w_2^0 f_7\right]$$
$$F^{(3)}(1)=\tfrac{1}{2}\cdot\left[f_3-w_2^0 f_7\right]$$

第二步中间结果，间隔2

$$F^{(0)}(0)=\tfrac{1}{2}\cdot\left[F^{(0)}(0)+w_4^0 F^{(2)}(0)\right]$$
$$F^{(0)}(1)=\tfrac{1}{2}\cdot\left[F^{(0)}(1)+w_4^1 F^{(2)}(1)\right]$$
$$F^{(0)}(2)=\tfrac{1}{2}\cdot\left[F^{(0)}(0)-w_4^0 F^{(2)}(0)\right]$$
$$F^{(0)}(3)=\tfrac{1}{2}\cdot\left[F^{(0)}(1)-w_4^1 F^{(2)}(1)\right]$$

$$F^{(1)}(0)=\tfrac{1}{2}\cdot\left[F^{(1)}(0)+w_4^0 F^{(3)}(0)\right]$$
$$F^{(1)}(1)=\tfrac{1}{2}\cdot\left[F^{(1)}(1)+w_4^1 F^{(3)}(1)\right]$$
$$F^{(1)}(2)=\tfrac{1}{2}\cdot\left[F^{(1)}(0)-w_4^0 F^{(3)}(0)\right]$$
$$F^{(1)}(3)=\tfrac{1}{2}\cdot\left[F^{(1)}(1)-w_4^1 F^{(3)}(1)\right]$$

$$\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \end{bmatrix}$$

第三步最终结果，间隔4

$$F(0)=\tfrac{1}{2}\cdot\left[F^{(0)}(0)+w_8^0F^{(1)}(0)\right]$$

$$F(1)=\tfrac{1}{2}\cdot\left[F^{(0)}(1)+w_8^1F^{(1)}(1)\right]$$

$$F(2)=\tfrac{1}{2}\cdot\left[F^{(0)}(2)+w_8^2F^{(1)}(2)\right]$$

$$F(3)=\tfrac{1}{2}\cdot\left[F^{(0)}(3)+w_8^3F^{(1)}(3)\right]$$

$$F(4)=\tfrac{1}{2}\cdot\left[F^{(0)}(0)-w_8^0F^{(1)}(0)\right]$$

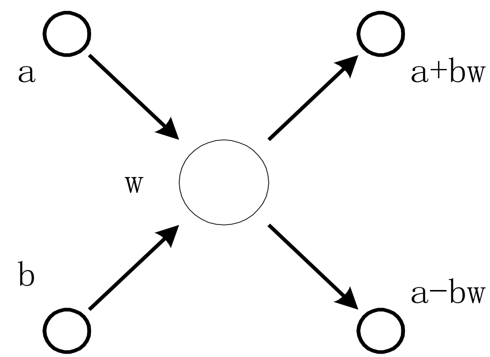
$$F(5)=\tfrac{1}{2}\cdot\left[F^{(0)}(1)-w_8^1F^{(1)}(1)\right]$$

$$F(6)=\tfrac{1}{2}\cdot\left[F^{(0)}(2)-w_8^2F^{(1)}(2)\right]$$

$$F(7)=\tfrac{1}{2}\cdot\left[F^{(0)}(3)-w_8^3F^{(1)}(3)\right]$$

快速离散傅立叶变换

- 蝶形图



- 显然计算一次蝶形需1次乘法和2次加（减）法。

对于$N=2^m$点的DFT，每轮有$N/2$个蝶形，

总共有$\dfrac{N}{2}\times m=\dfrac{N}{2}\times\log_2 N$个蝶形。

总共有$\dfrac{N}{2}\times\log_2 N$次乘法和$N\log_2 N$加法。

逆向FFT算法

- 算法思想：用正向变换计算逆向变换

设$F(u)=\mathrm{FFT}[f(x)]$

于是有：

$$f(x)=\mathrm{FFT}^{-1}[F(u)]=N\cdot\{\mathrm{FFT}[F^{*}(u)]\}^{*}$$

即：对F(u)取共轭，做正向FFT变换，其结果取共轭后再乘以N，得到f(x)。

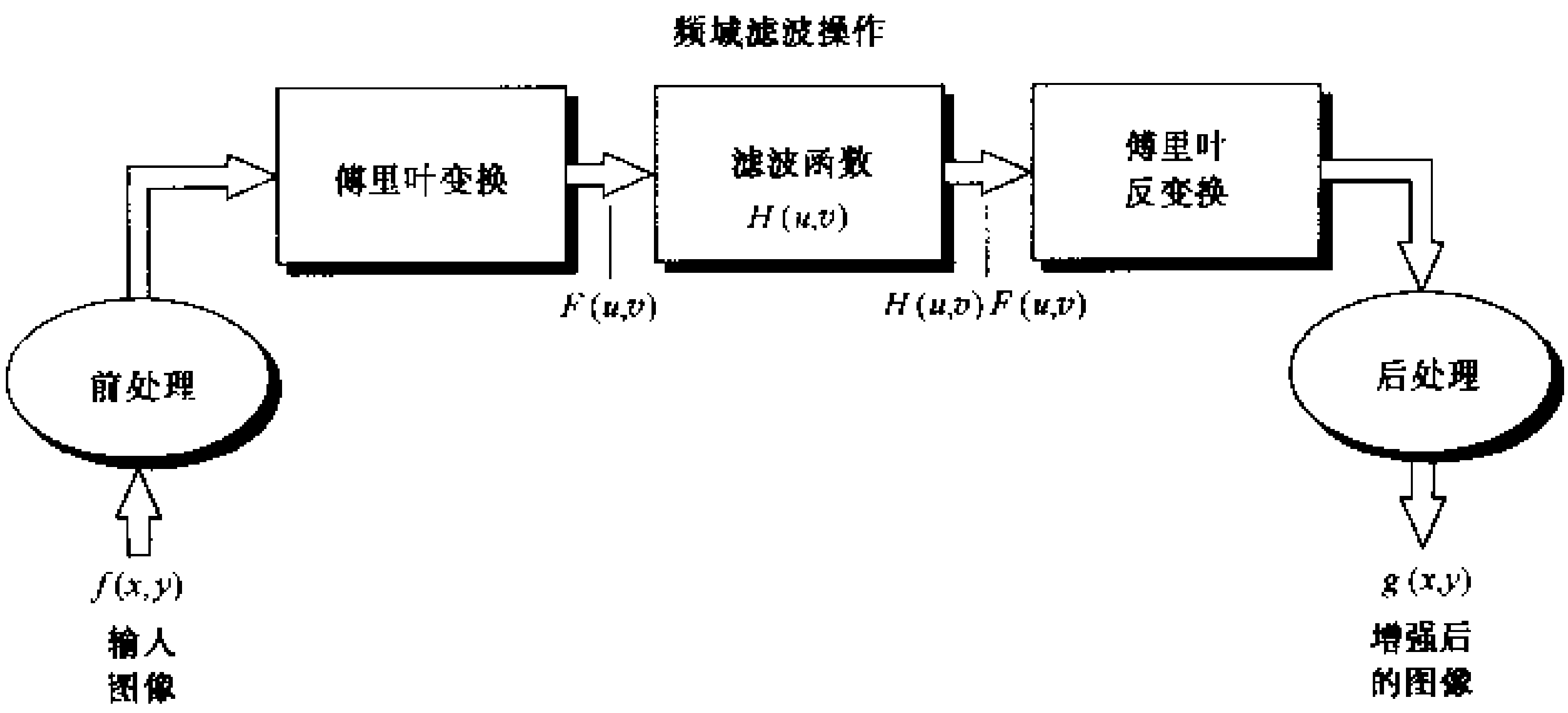
■ 二维快速Fourier变换:

利用傅里叶变换的分离性质，对二维FFT进行2次的一维FFT变换

$$F(u,v) = FFT_{行}\{FFT_{列}[f(x,y)]\}$$

频域滤波

- 基本步骤



频域滤波操作
傅里叶变换
滤波函数
H(u,v)
傅里叶
反变换
F(u,v)
H(u,v)F(u,v)
前处理
后处理
f(x,y)
输入
图像
g(x,y)
增强后
的图像

频域滤波

- 基本步骤

频域进行滤波操作，相当于在空间域做了何种操作？

1. 用$(-1)^{x+y}$乘以输入图像来进行中心变换，如式(4.2.21)所示。
2. 由(1)计算图像的 DFT，即 $F(u,v)$。
3. 用滤波器函数 $H(u,v)$乘以 $F(u,v)$。
4. 计算(3)中结果的反 DFT。
5. 得到(4)中结果的实部。
6. 用$(-1)^{x+y}$乘以(5)中的结果。

频域滤波（与灰度级函数非常类似）：

$$G(u,v) = H(u,v)F(u,v)$$

卷积　也称为空间域滤波

- 离散一维卷积

对于两个长度为m和n的序列$f(i)$和$g(j)$，

$$h(i) = f(i) * g(i) = \sum_j f(j) g(i-j)$$

给出长度为$N = m + n - 1$的输出序列。

$$\mathbf{h} = \mathbf{g} \bullet \mathbf{f} = \begin{bmatrix} \mathbf{g_p}(1) & \mathbf{g_p}(\mathbf{N}) & \cdots & \mathbf{g_p}(2) \\ \mathbf{g_p}(2) & \mathbf{g_p}(1) & \cdots & \mathbf{g_p}(3) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{g_p}(\mathbf{N}) & \mathbf{g_p}(\mathbf{N}-1) & \cdots & \mathbf{g_p}(1) \end{bmatrix} \begin{bmatrix} \mathbf{f_p}(1) \\ \mathbf{f_p}(2) \\ \vdots \\ \mathbf{f_p}(\mathbf{N}) \end{bmatrix}$$

卷积

- 二维卷积和离散二维卷积
  - 二维卷积定义

$$h(x,y) = f * g = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)\,g(x-u,y-v)\,dudv$$

  - 离散二维卷积定义

$$H = F * G$$

$$H(i,j) = \sum_{m}\sum_{n} F(m,n)\,G(i-m,j-n)$$

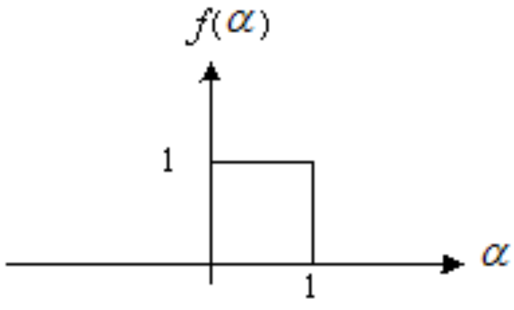
➢ 离散卷积定理

1）连续卷积 $f(x)*g(x)=\int_{-\infty}^{\infty}f(a)g(x-a)da$

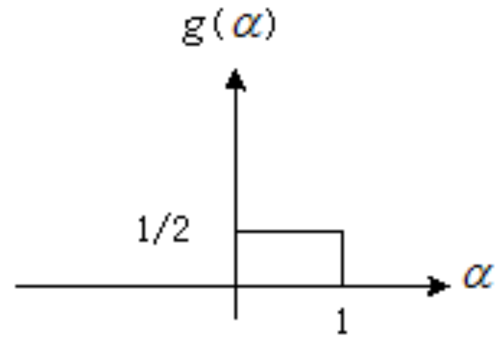
$$f(x,y)*g(x,y)=\int_{-\infty}^{\infty}f(u,v)g(x-u,y-v)dudv$$

例1　求以下两个函数的卷积

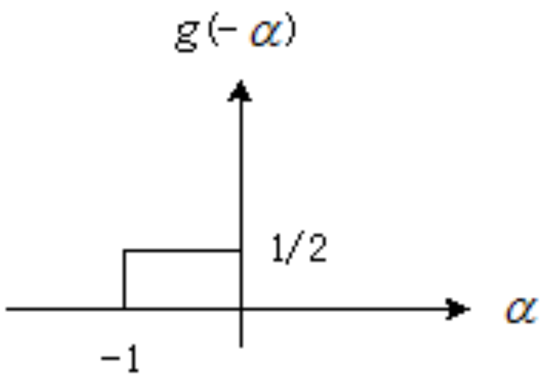
$$f(x)=\begin{cases}1 & 0\le x\le 1\\ 0 & \text{其它}\end{cases}\qquad g(x)=\begin{cases}1/2 & 0\le x\le 1\\ 0 & \text{其它}\end{cases}$$



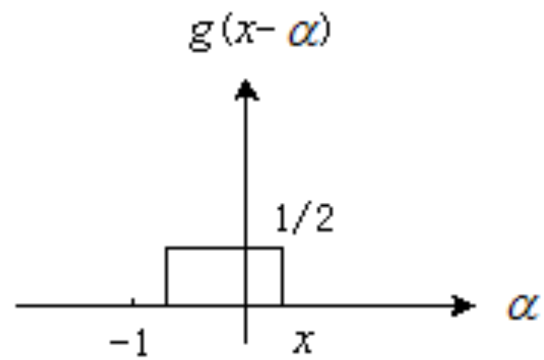
(a)



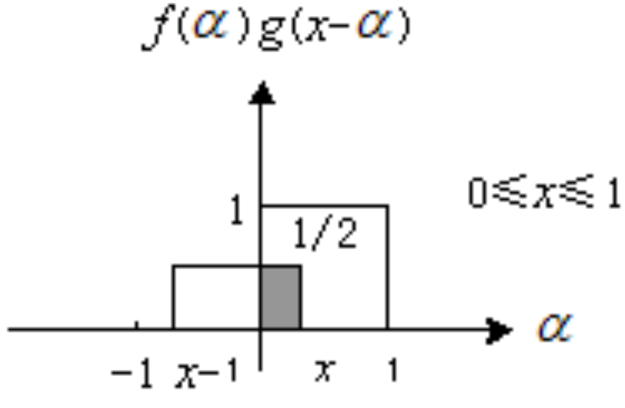
(b)



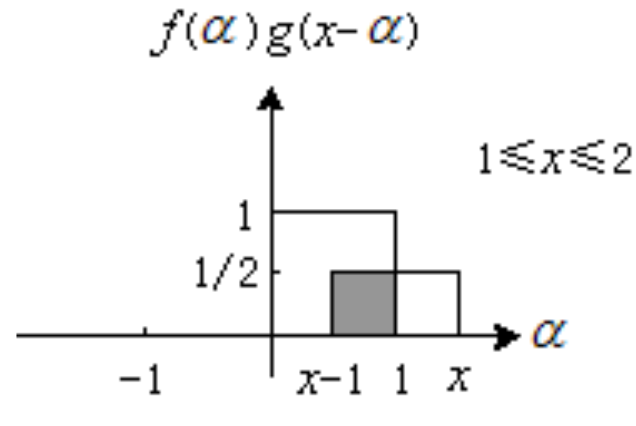
(c)



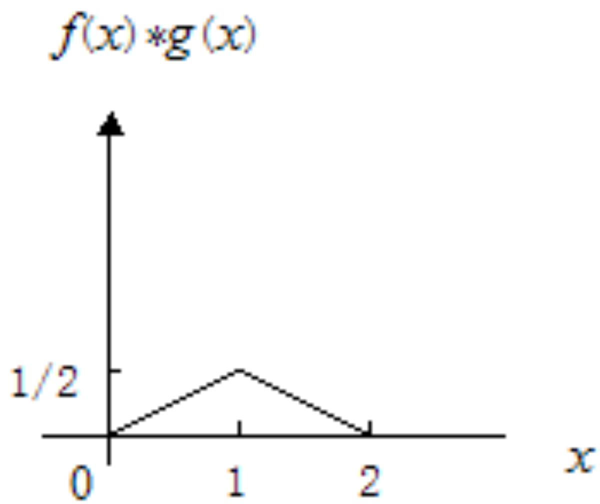
(d)



(e)



(f)



(g)

$$f(x)*g(x)=\begin{cases} x/2 & 0\le x\le 1 \\ 1-x/2 & 1\le x\le 2 \\ 0 & \text{其它} \end{cases}$$

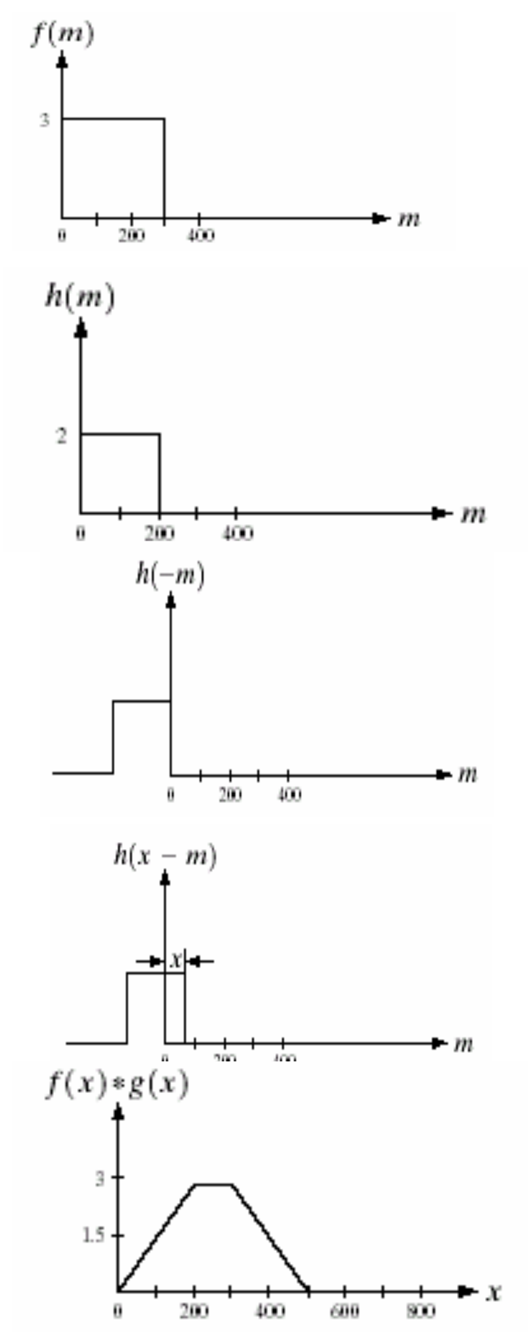
2）离散卷积定理

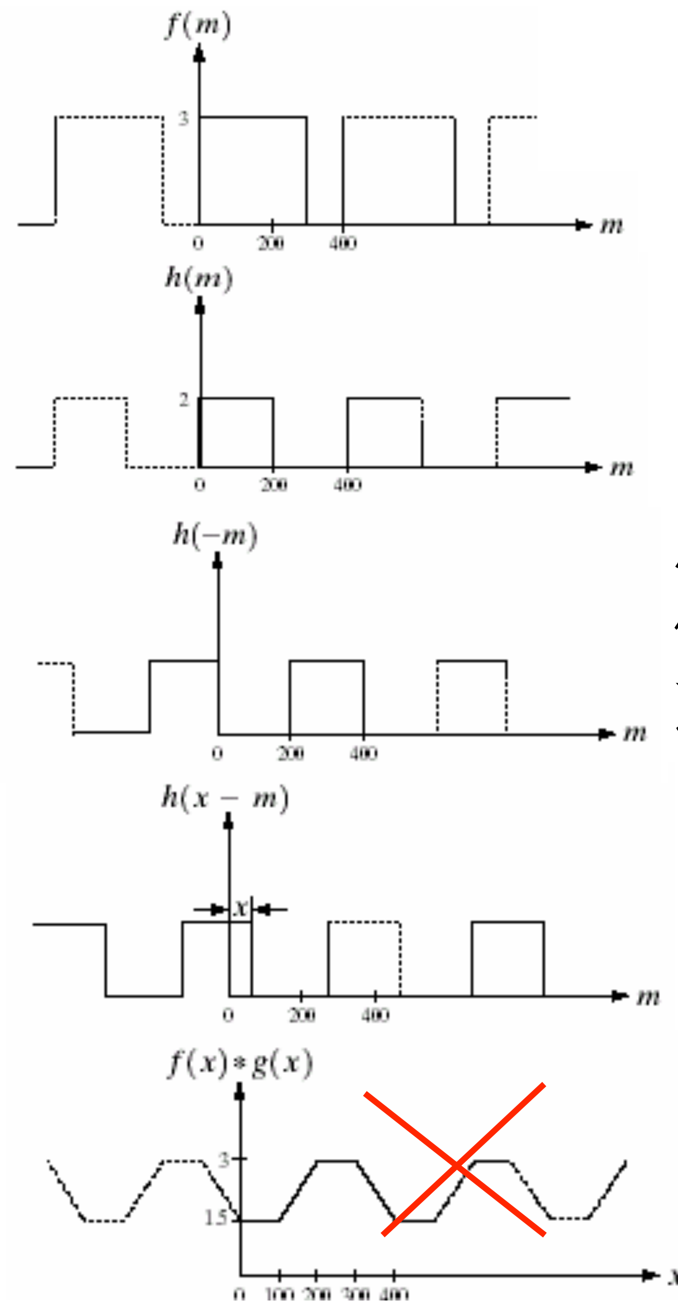
离散卷积定义：
$$f_e(x) * g_e(x) = \sum_{m=9}^{M-1} f_e(m) g_e(x-m)$$

$$f(x,y) * g(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n) g(x-m,y-n)$$

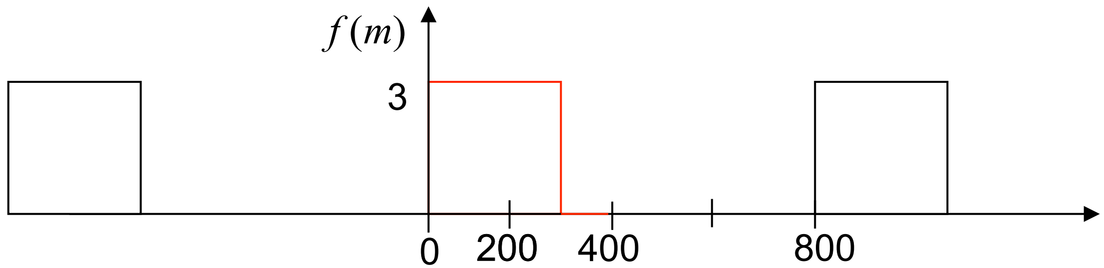
如果待卷积函数的大小一样怎么办?

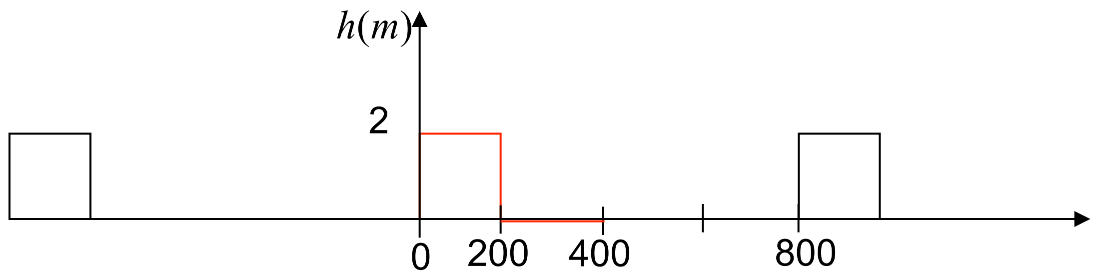
为防止频谱混叠误差，需对离散的二维函数补零，即周期延拓，对两个函数同时添加零，使它们具有相同的周期。

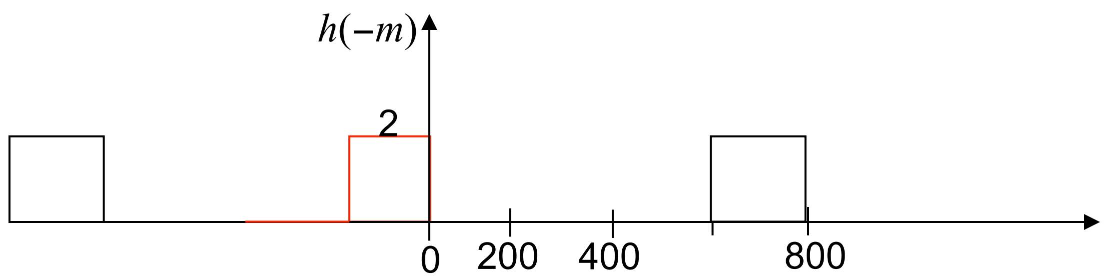




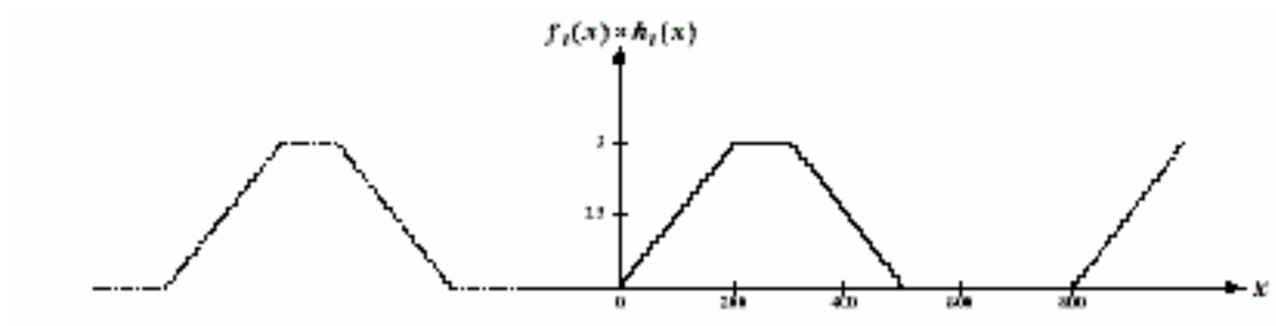
虽然进行了周期延拓，但两者周期不等







延拓至相同的周期再进行计算



周期延拓

$f(x,y)$的大小为 $A \times B$ $g(x,y)$的大小为$C \times D$

$$f_e(x,y)=\begin{cases} f(x,y) & 0 \le x \le A-1, 0 \le y \le B-1 \\ 0 & A \le x \le M-1, B \le y \le N-1 \end{cases}$$

$$g_e(x,y)=\begin{cases} g(x,y) & 0 \le x \le C-1, 0 \le y \le D-1 \\ 0 & C \le x \le M-1, D \le y \le N-1 \end{cases}$$

$$M \ge A+C-1, N \ge C+D-1$$

$$z_e(x,y)=f_e(x,y)*g_e(x,y)$$

空间域滤波和频域滤波的关系

$$\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} s(x,y)\delta(x,y) = s(0,0)$$

$$\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} \delta(x,y)e^{-j2\pi(ux/M+vy/N)} = \frac{1}{MN}$$

$$\delta(x,y) * h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} \delta(m,n)h(x-m,y-n) = \frac{h(x,y)}{MN}$$

$$f(x,y) * h(x,y) \Rightarrow F(u,v)H(u,v)$$

$$\delta(x,y) * h(x,y) \Rightarrow H(u,v)$$

$$h(x,y) \Rightarrow H(u,v)$$

频域进行滤波操作相当于空间域做卷积操作。

空间域和频域的滤波器构成傅里叶变换对

$$\begin{cases} f(x,y) \Leftrightarrow F(u,v) \\ g(x,y) \Leftrightarrow G(u,v) \end{cases} \Rightarrow \begin{cases} f(x,y) * g(x,y) \Leftrightarrow F(u,v) \cdot G(u,v) \\ f(x,y) \cdot g(x,y) \Leftrightarrow \dfrac{1}{MN} F(u,v) * G(u,v) \end{cases}$$

证明：（1）空域卷积和

$$\begin{aligned} & DFT\left[f(x,y) * g(x,y)\right] \\ & = DFT\left[\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n) \cdot g(x-m, y-n)\right] \\ & = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n) \cdot DFT\left[g(x-m, y-n)\right] \\ & = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n) \cdot e^{-j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)} \cdot G(u,v) \\ & = F(u,v) \cdot G(u,v) \end{aligned}$$

注意：
1.对（x,y）做卷积，f(m,n)不变
2. 空域平移傅氏变换的性质

（2）频域卷积和：

$$
\begin{aligned}
& DFT\left[f(x,y)\cdot g(x,y)\right] \\
&= DFT\left[\frac{1}{MN}\cdot\sum_{u'=0}^{M-1}\sum_{v'=0}^{N-1}F(u',v')\cdot e^{j2\pi\left(\frac{u'x}{M}+\frac{v'y}{N}\right)}\cdot g(x,y)\right] \\
&= \frac{1}{MN}\cdot\sum_{u'=0}^{M-1}\sum_{v'=0}^{N-1}F(u',v')\cdot DFT\left[e^{j2\pi\left(\frac{u'x}{M}+\frac{v'y}{N}\right)}\cdot g(x,y)\right] \\
&= \frac{1}{MN}\cdot\sum_{u'=0}^{M-1}\sum_{v'=0}^{N-1}F(u',v')\cdot G(u-u',v-v') \\
&= \frac{1}{MN}\cdot F(u,v)*G(u,v)
\end{aligned}
$$

相关定理

$$f(x,y)\circ g(x,y)=\int_{-\infty}^{\infty} f(\alpha,\beta)g(x+\beta,y+\beta)d\alpha d\beta$$

$$f(x,y)\circ g(x,y)\Rightarrow F(u,v)G^*(u,v)$$

$$f(x,y)\circ g^*(x,y)\Rightarrow F(u,v)G(u,v)$$

证明：

$$
\begin{aligned}
&DFT\left[f(x,y)\circ g(x,y)\right] \\
&= DFT\left[\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\cdot g(x+m,y+n)\right] \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\cdot DFT\left[g(x+m,y+n)\right] \\
&= \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\cdot e^{j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}\cdot G(u,v) \\
&= \left[\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)\cdot e^{-j2\pi\left(\frac{mu}{M}+\frac{nv}{N}\right)}\right]^{*}\cdot G(u,v) \\
&= F^{*}(u,v)\cdot G(u,v)
\end{aligned}
$$

Thanks!