Chapter 9

IIR digital filter design

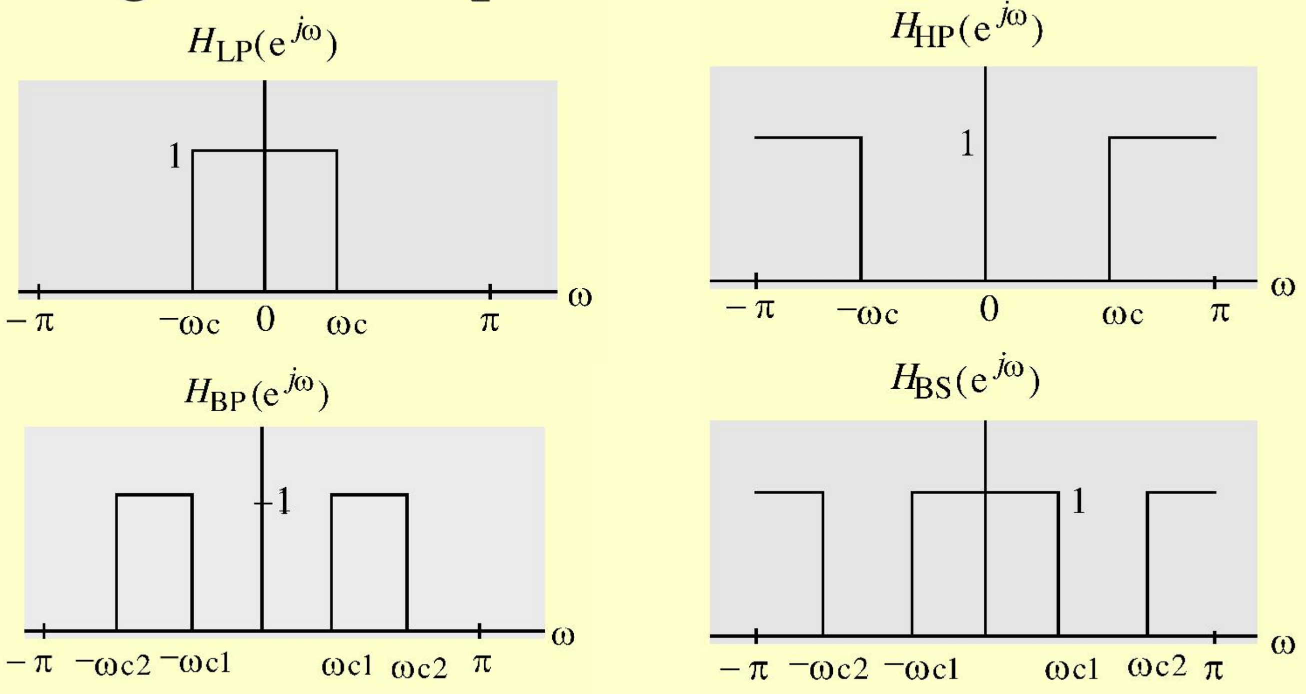
- Objective - Determination of a realizable transfer function G(z) approximating a given frequency response specification is an important step in the design of a digital filter
- If an IIR filter is desired, G(z) should be a stable real rational function
- Digital filter design is the process of deriving the transfer function G(z)

Digital Filter Specifications

- Usually, either the magnitude and/or the phase (delay) response is specified for the design of digital filter for most applications
- In some situations, the unit sample response or the step response may be specified
- In most practical applications, the problem of interest is the development of a realizable approximation to a given magnitude response specification

Digital Filter Specifications

- We discuss in this course only the magnitude approximation problem
- There are four basic types of ideal filters with magnitude responses as shown below

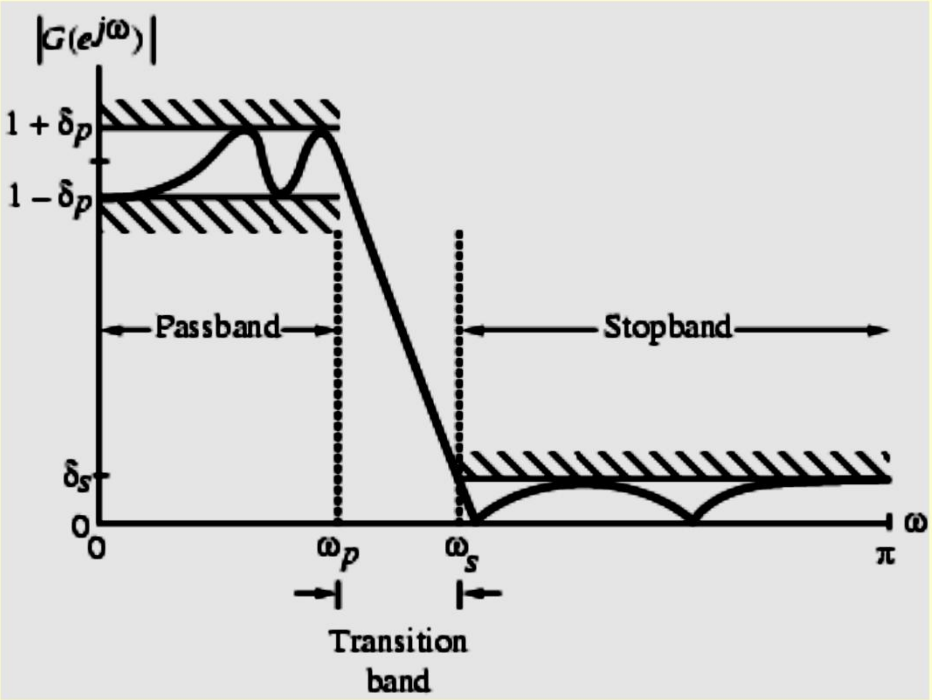


Digital Filter Specifications

- As the impulse response corresponding to each of these ideal filters is noncausal and of infinite length, these filters are not realizable
- In practice, the magnitude response specifications of a digital filter in the passband and in the stopband are given with some acceptable tolerances
- In addition, a transition band is specified between the passband and stopband

Digital Filter Specifications

- For example, the magnitude response $|G(e^{j\omega})|$ of a digital lowpass filter may be given as indicated below



Digital Filter Specifications

- As indicated in the figure, in the passband, defined by $0 \leq \omega \leq \omega_p$, we require that $|G(e^{j\omega})| \cong 1$ with an error $\pm\delta_p$, i.e.,

$$1 - \delta_p \leq |G(e^{j\omega})| \leq 1 + \delta_p, \quad |\omega| \leq \omega_p$$

- In the stopband, defined by $\omega_s \leq \omega \leq \pi$, we require that $|G(e^{j\omega})| \cong 0$ with an error δ_s

i.e., $|G(e^{j\omega})| \leq \delta_p, \quad \omega_s \leq |\omega| \leq \pi$

Digital Filter Specifications

- ω_p - passband edge frequency
- ω_s - stopband edge frequency
- δ_p - peak ripple value in the passband
- δ_s - peak ripple value in the stopband
- Since $G(e^{j\omega})$ is a periodic function of ω, and $|G(e^{j\omega})|$ of a real-coefficient digital filter is an even function of ω
- As a result, filter specifications are given only for the frequency range $0 \leq |\omega| \leq \pi$

Digital Filter Specifications

- Specifications are often given in terms of loss function $\mathcal{G}(\omega) = -20\log_{10}|G(e^{j\omega})|$ in dB
- Peak passband ripple

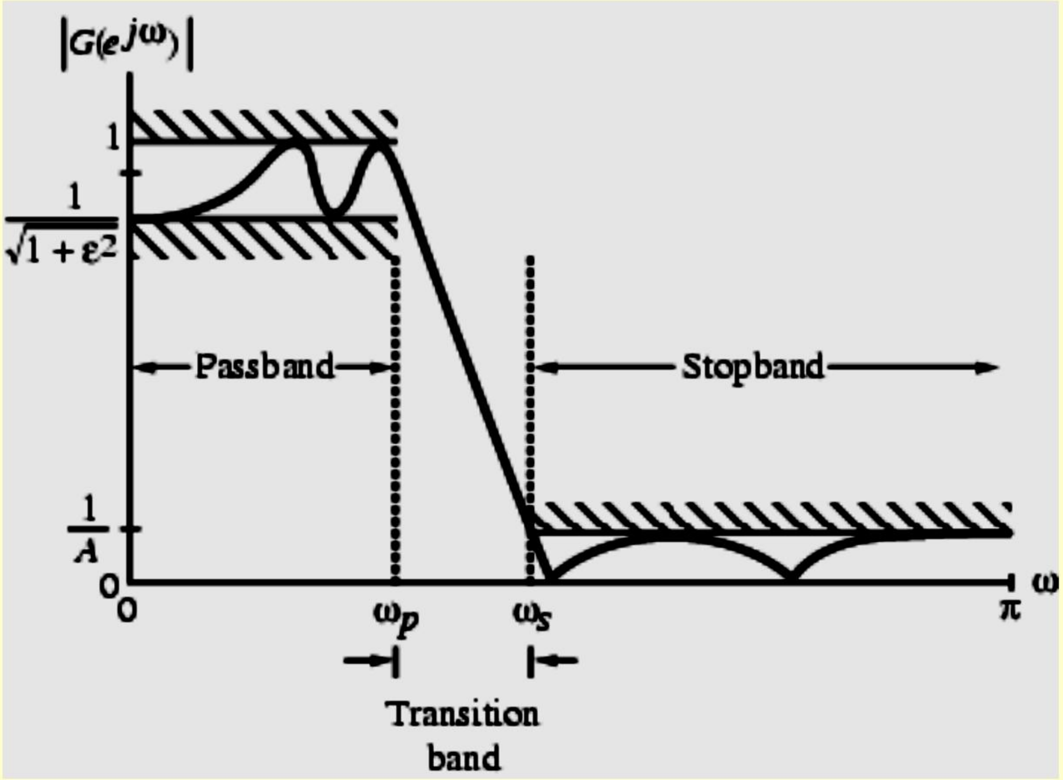
$$\alpha_p = -20\log_{10}(1-\delta_p) \text{ dB}$$

- Minimum stopband attenuation

$$\alpha_s = -20\log_{10}(\delta_s) \text{ dB}$$

Digital Filter Specifications

- Magnitude specifications may alternately be given in a normalized form as indicated below



Digital Filter Specifications

- Here, the maximum value of the magnitude in the passband is assumed to be unity

- $1/\sqrt{(1+\varepsilon^2)}$ - Maximum passband deviation, given by the minimum value of the magnitude in the passband

- 1/A - Maximum stopband magnitude

Digital Filter Specifications

- For the normalized specification, maximum value of the gain function or the minimum value of the loss function is 0 dB
- Maximum passband attenuation

$$\alpha_{max} = 20\log_{10}\left(\sqrt{1+\varepsilon^2}\right) \quad \text{dB}$$

- For $\delta_p \ll 1$, it can be shown that

$$\alpha_{max} \cong -20\log_{10}(1-2\delta_p) \quad \text{dB}$$

Filter Specifications

- Two additional parameters are defined -

(1) Transition ratio $k = \Omega_p / \Omega_s$

For a lowpass filter $k<1$

分辨参数

(2) Discrimination parameter $k_1 = \varepsilon / \sqrt{(A^2 - 1)}$

Usually $k_1 << 1$

Digital Filter Specifications

- In practice, passband edge frequency F_p and stopband edge frequency F_s are specified in Hz
- For digital filter design, normalized bandedge frequencies need to be computed from specifications in Hz using

$$\omega_p = \frac{\Omega_p}{F_T} = \frac{2\pi F_p}{F_T} = 2\pi F_p T$$

$$\omega_s = \frac{\Omega_s}{F_T} = \frac{2\pi F_s}{F_T} = 2\pi F_s T$$

Digital Filter Specifications

- Example - Let $F_p = 7$ kHz, $F_s = 3$ kHz, and $F_T = 25$ kHz
- Then

$$\omega_p = \frac{2\pi(7\times 10^3)}{25\times 10^3} = 0.56\pi$$

$$\omega_s = \frac{2\pi(3\times 10^3)}{25\times 10^3} = 0.24\pi$$

Selection of Filter Type

- The transfer function H(z) meeting the frequency response specifications should be a causal transfer function
- For IIR digital filter design, the IIR transfer function is a real rational function of z^{-1}:

$$H(z) = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} + \cdots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} + \cdots + d_N z^{-N}}$$

- H(z) must be a stable transfer function and must be of lowest order N for reduced computational complexity

Selection of Filter Type

- For FIR digital filter design, the FIR transfer function is a polynomial in z^{-1} with real coefficients:

$$H(z) = \sum_{n=0}^{N} h[n] z^{-n}$$

For reduced computational complexity, degree N of H(z) must be as small as possible

- If a linear phase is desired, the filter coefficients must satisfy the constraint:

$$h[n] = \pm h[N-n]$$

Selection of Filter Type

- **Advantages** in using an FIR filter -

  (1) Can be designed with exact linear phase,

  (2) Filter structure always stable with quantized coefficients

- **Disadvantages** in using an FIR filter - Order of an FIR filter, in most cases, is considerably higher than the order of an equivalent IIR filter meeting the same specifications, and FIR filter has thus higher computational complexity

Comparison of IIR and FIR filters:

IIR filter	FIR filter
$h(n)$ is infinite length	h(n) is finite length
H(z) is a rational function of z^{-1}	H(z) is a polynomial in z^{-1}
Nonlinear phase	Exact linear phase
Can not be computed with FFT	Can be computed with FFT
With a recursive structure	In general, is not a recursive structure
With a lower order of N_{IIR}	With a considerably higher order of N_{FIR}
Can be designed from analog prototype filter	Designed with the aid of computer

Digital Filter Design: Basic Approaches

- Most common approach to IIR filter design –
- (1) Convert the digital filter specifications into an analog prototype lowpass filter specifications
- (2) Determine the analog lowpass filter transfer function $H_a(s)$
- (3) Transform $H_a(s)$ into the desired digital transfer function $G(z)$

Digital Filter Design: Basic Approaches

- This approach has been widely used for the following reasons:

  (1) Analog approximation techniques are highly advanced

  (2) They usually yield closed-form solutions

  (3) Extensive tables are available for analog filter design

Butterworth Approximation

- The magnitude-square response of an N-th order analog lowpass Butterworth filter is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

First 2N - 1 derivatives of $|H_a(j\Omega)|^2$ at $\Omega = 0$ are equal to zero

- The Butterworth lowpass filter thus is said to have a maximally-flat magnitude at $\Omega = 0$

Butterworth Approximation

- Gain in dB is

  $G(\Omega)=10\log_{10}|H_a(j\Omega)|^2$

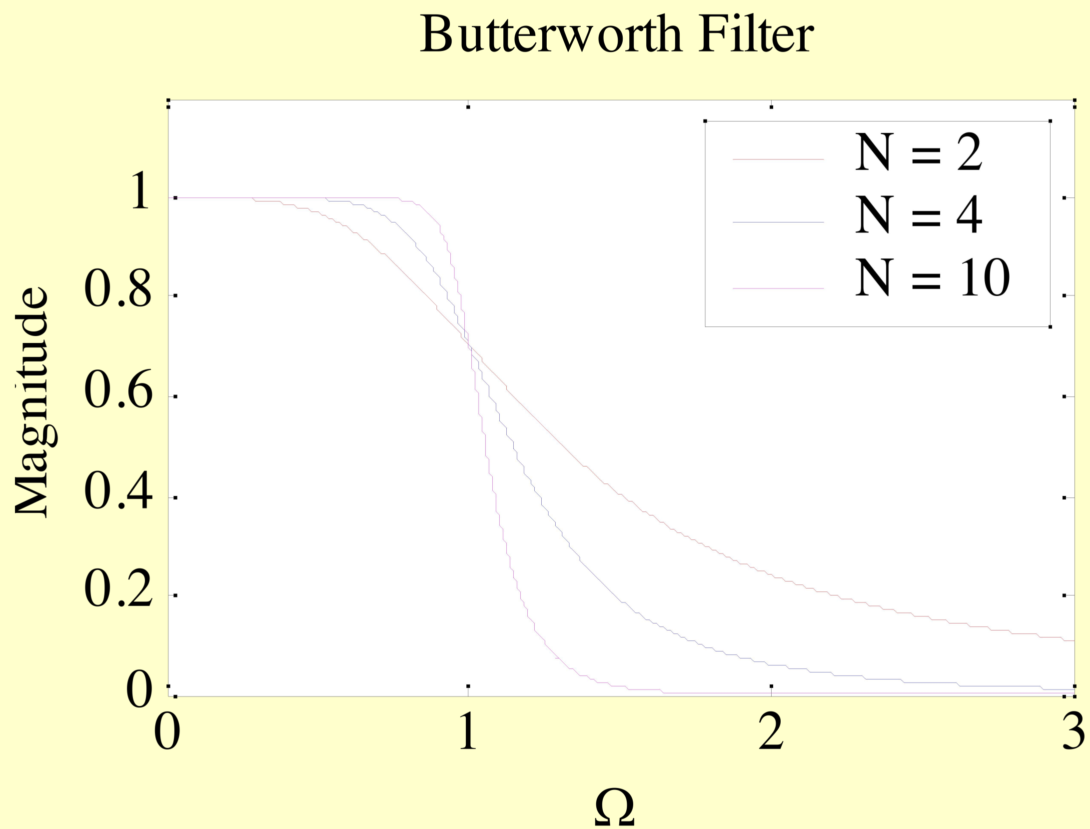
As $G(0)=0$ and

  $G(\Omega_c)=10\log_{10}(0.5)=-3.0103\cong-3$ dB

  Ω_c is called 3-dB cutoff frequency

Butterworth Approximation

- Typical magnitude responses with $\Omega_c = 1$



Butterworth Approximation

- Two parameters completely characterizing a Butterworth lowpass filter are Ω_c and N
- These are determined from the specified bandedges Ω_p and Ω_s , and minimum passband magnitude $1/\sqrt{(1+\varepsilon^2)}$, and maximum stopband ripple $1/A$

Butterworth Approximation

- Ω_c and N are thus determined from

$$|H_a(j\Omega_p)|^2 = \frac{1}{1+(\Omega_p/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2}$$

$$|H_a(j\Omega_s)|^2 = \frac{1}{1+(\Omega_s/\Omega_c)^{2N}} = \frac{1}{A^2}$$

- Solving the above we get

$$N = \frac{1}{2}\cdot\frac{\log_{10}[(A^2-1)/\varepsilon^2]}{\log_{10}(\Omega_s/\Omega_p)} = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)}$$

Butterworth Approximation

- Since order N must be an integer, value obtained is rounded up to the next highest integer
- This value of N is used next to determine Ω_c by satisfying either the stopband edge or the passband edge specification exactly
- If the stopband edge specification is satisfied, then the passband edge specification is exceeded providing a safety margin

Butterworth Approximation

- Example - Determine the lowest order of a Butterworth lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz

- Now

$$10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right)=-1$$

which yields $\varepsilon^2 = 0.25895$

and

$$10\log_{10}\left(\frac{1}{A^2}\right)=-40$$

which yields $A^2 = 10{,}000$

Butterworth Approximation

- Therefore $$\frac{1}{k_1} = \frac{\sqrt{A^2 - 1}}{\varepsilon} = 196.51334$$

  and $$\frac{1}{k} = \frac{\Omega_s}{\Omega_p} = 5$$

- Hence

$$N = \frac{\log_{10}(1/k_1)}{\log_{10}(1/k)} = 3.2811$$

- We choose $N = 4$

Chebyshev Approximation

- The magnitude-square response of an N-th order analog lowpass **Type 1 Chebyshev filter** is given by

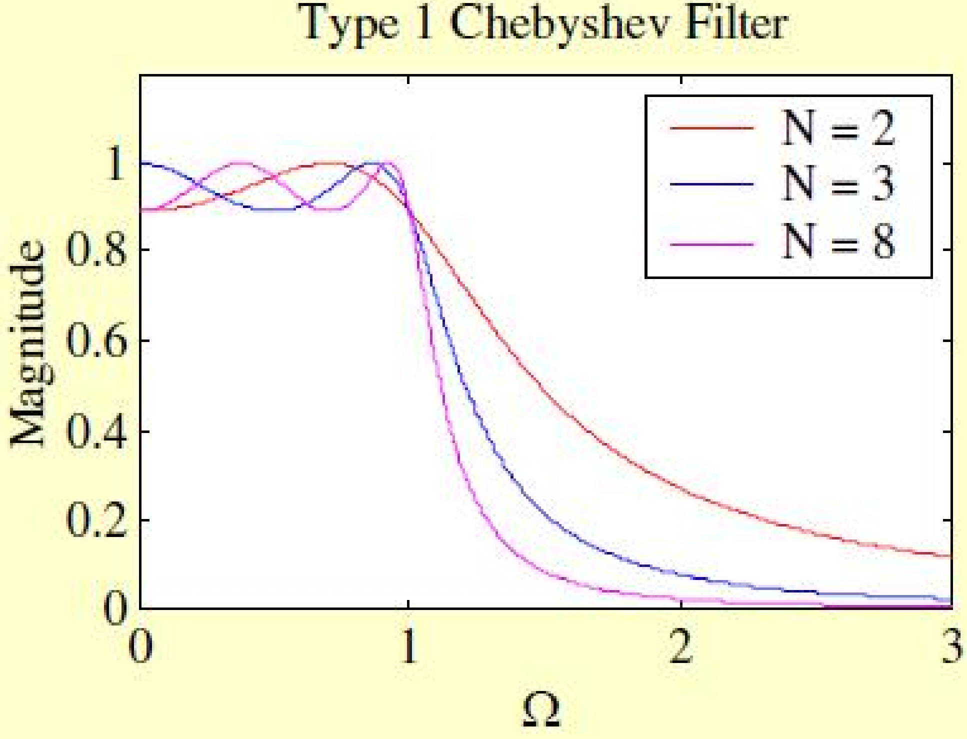
$$\left|H_a(s)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}$$

where $T_N(\Omega)$ is the **Chebyshev polynomial** of order N:

$$T_N(\Omega) = \begin{cases} \cos(N\cos^{-1}\Omega), & |\Omega| \le 1 \\ \cosh(N\cosh^{-1}\Omega), & |\Omega| > 1 \end{cases}$$

Chebyshev Approximation

- Typical magnitude response plots of the analog lowpass Type 1 Chebyshev filter are shown below



Chebyshev Approximation

- If at $\Omega = \Omega_s$ the magnitude is equal to $1/A$, then

$$\left|H_a(j\Omega_s)\right|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega_s/\Omega_p)} = \frac{1}{A^2}$$

- Solving the above we get

$$N = \frac{\cosh^{-1}(\sqrt{A^2-1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)}$$

- Order N is chosen as the nearest integer greater than or equal to the above value

Chebyshev Approximation

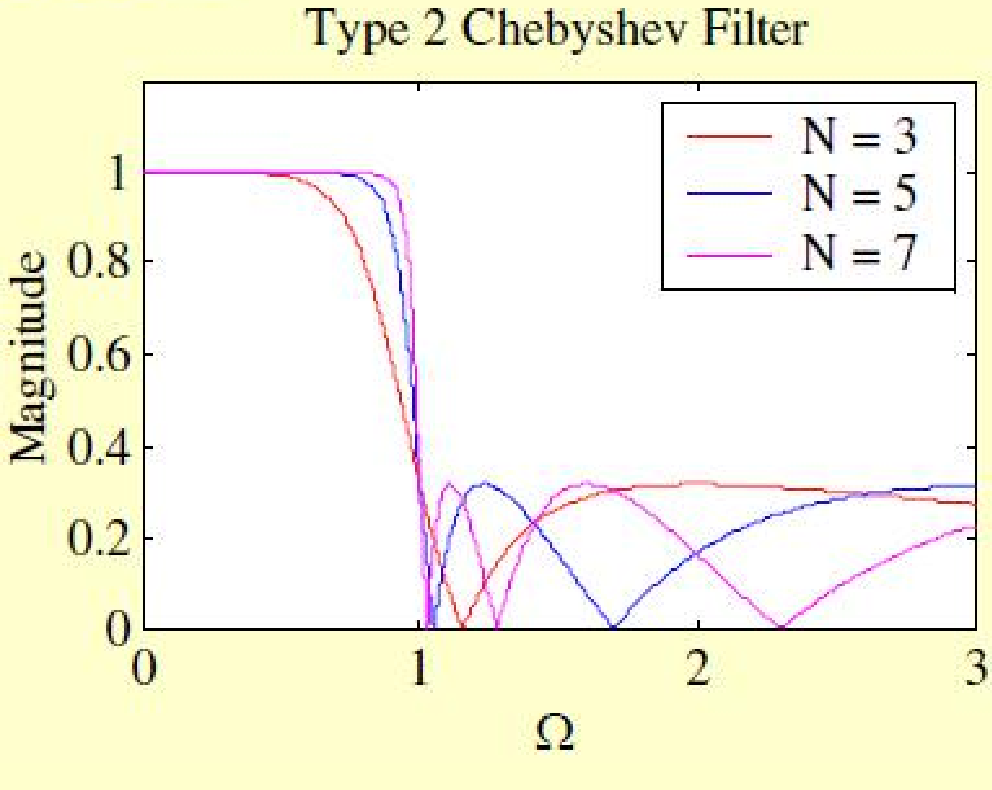
- The magnitude-square response of an N-th order analog lowpass **Type 2 Chebyshev** (also called **inverse Chebyshev**) **filter** is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[\dfrac{T_N(\Omega_s/\Omega_p)}{T_N(\Omega_s/\Omega)}\right]^2}$$

where $T_N(\Omega)$ is the **Chebyshev polynomial** of order N

Chebyshev Approximation

- Typical magnitude response plots of the analog lowpass Type 2 Chebyshev filter are shown below



Chebyshev Approximation

- The order N of the Type 2 Chebyshev filter is determined from given ε, Ω_s, and A using

$$N = \frac{\cosh^{-1}(\sqrt{A^2 - 1}/\varepsilon)}{\cosh^{-1}(\Omega_s/\Omega_p)} = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)}$$

- Example - Determine the lowest order of a Chebyshev lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz -

$$N = \frac{\cosh^{-1}(1/k_1)}{\cosh^{-1}(1/k)} = 2.6059$$

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Elliptic Approximation

- The square-magnitude response of an elliptic lowpass filter is given by

$$|H_a(j\Omega)|^2 = \frac{1}{1 + \varepsilon^2 R_N^2(\Omega/\Omega_p)}$$

where $R_N(\Omega)$ is a rational function of order N satisfying $R_N(1/\Omega) = 1/R_N(\Omega)$, with the roots of its numerator lying in the interval $0 < \Omega < 1$ and the roots of its denominator lying in the interval $1 < \Omega < \infty$

Elliptic Approximation

- For given Ω_p, Ω_s, ε, and A, the filter order can be estimated using

$$N \cong \frac{2\log_{10}(4/k_1)}{\log_{10}(1/\rho)}$$

where $k' = \sqrt{1-k^2}$

$$\rho_0 = \frac{1-\sqrt{k'}}{2(1+\sqrt{k'})}$$

$$\rho = \rho_0 + 2(\rho_0)^5 + 15(\rho_0)^9 + 150(\rho_0)^{13}$$

Elliptic Approximation

- Example - Determine the lowest order of a elliptic lowpass filter with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz
  Note: $k = 0.2$ and $1/k_1 = 196.5134$
- Substituting these values we get

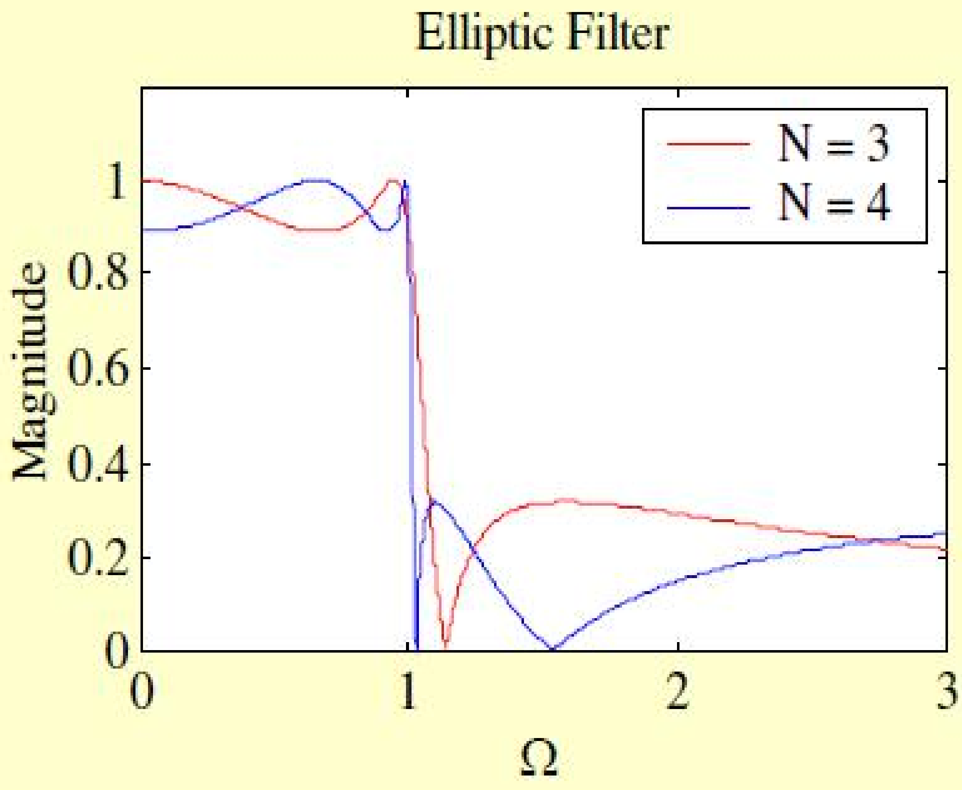
$$k' = 0.979796, \qquad \rho_0 = 0.00255135,$$

$$\rho = 0.0025513525$$

- and hence $N = 2.23308$
- Choose $N = 3$

Elliptic Approximation

- Typical magnitude response plots with $\Omega_p = 1$ are shown below



Analog Lowpass Filter Design

- Example - Design an elliptic lowpass filter of lowest order with a 1-dB cutoff frequency at 1 kHz and a minimum attenuation of 40 dB at 5 kHz
- Code fragments used

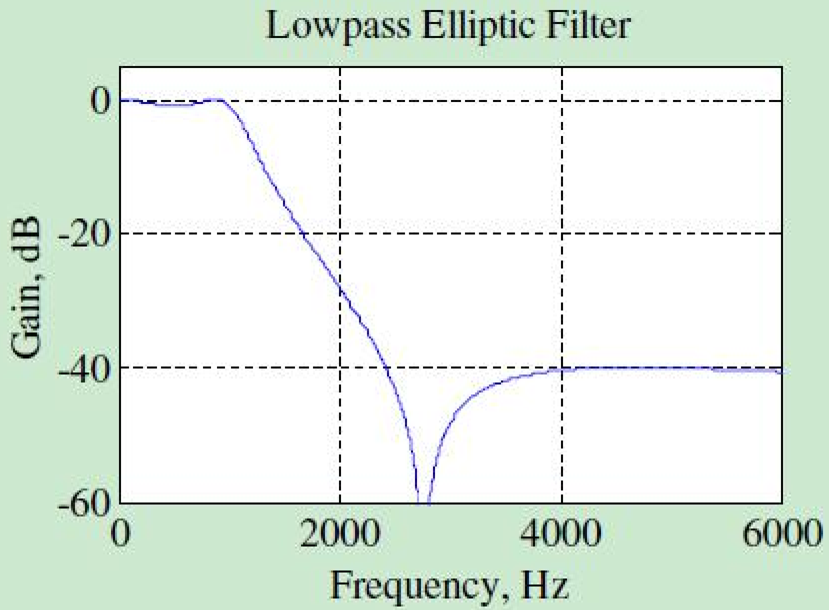
```
[N, Wn] = ellipord(Wp, Ws, Rp, Rs, 's');
[b, a] = ellip(N, Rp, Rs, Wn, 's');
```

with

```
Wp = 2*pi*1000;
Ws = 2*pi*5000;
Rp = 1;
Rs = 40;
```

Analog Lowpass Filter Design

- Gain plot



Design of Analog Highpass, Bandpass and Bandstop Filters

- Steps involved in the design process:

  Step 1 - Develop of specifications of a prototype analog lowpass filter $H_{LP}(s)$ from specifications of desired analog filter $H_D(s)$ using a frequency transformation

  Step 2 - Design the prototype analog lowpass filter

  Step 3 - Determine the transfer function $H_D(s)$ of desired analog filter by applying the inverse frequency transformation to $H_{LP}(s)$

Design of Analog Highpass, Bandpass and Bandstop Filters

- Let s denote the Laplace transform variable of prototype analog lowpass filter $H_{LP}(s)$ and $\hat{s}$ denote the Laplace transform variable of desired analog filter $H_D(\hat{s})$
- The mapping from s-domain to $\hat{s}$-domain is given by the invertible transformation

$$s = F(\hat{s})$$

- Then $H_D(\hat{s}) = H_{LP}(s)\big|_{s=F(\hat{s})}$

$$H_{LP}(s) = H_D(\hat{s})\big|_{\hat{s}=F^{-1}(s)}$$

Analog Highpass Filter Design

- Spectral Transformation:

$$s = \frac{\Omega_p \hat{\Omega}_p}{\hat{s}}$$

where Ω_p is the passband edge frequency of $H_{LP}(s)$ and $\hat{\Omega}_p$ is the passband edge frequency of $H_{HP}(\hat{s})$

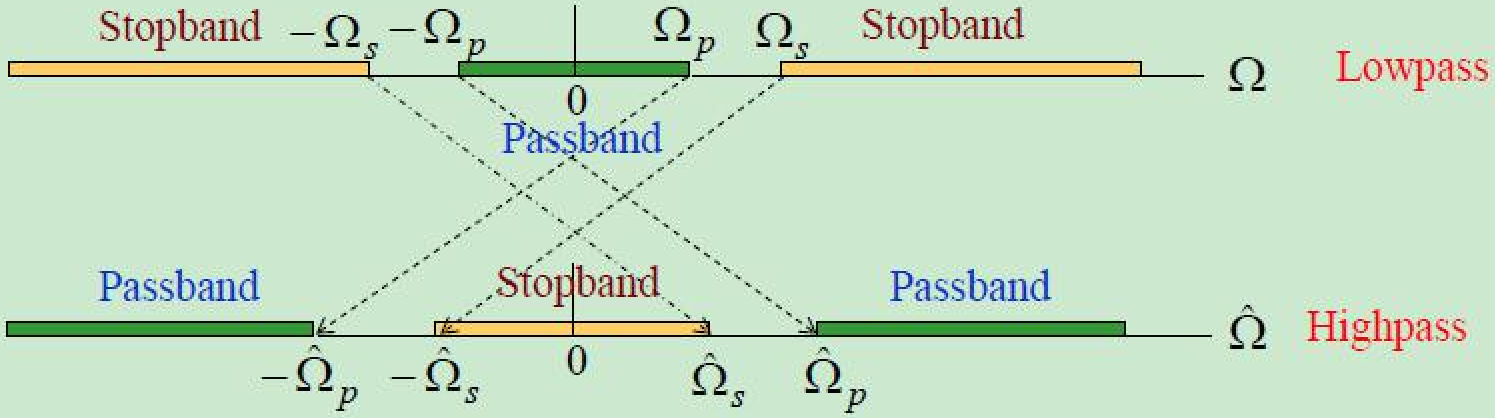
- On the imaginary axis the transformation is

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

Analog Highpass Filter Design

$$\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$$

Ω



Analog Highpass Filter Design

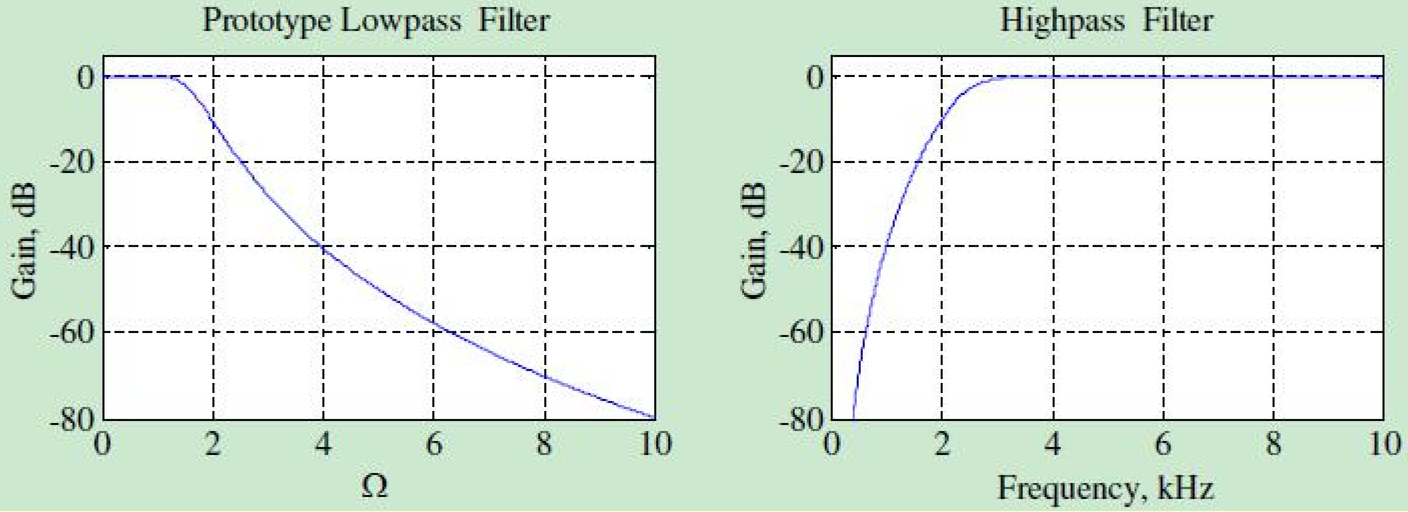
- Example - Design an analog Butterworth highpass filter with the specifications:
  $\hat{F}_p = 4$ kHz, $\hat{F}_s = 1$ kHz, $\alpha_p = 0.1$ dB,
  $\alpha_s = 40$ dB
- Choose $\Omega_p = 1$
- Then
$$\Omega_s = \frac{2\pi\hat{F}_p}{2\pi\hat{F}_s} = \frac{\hat{F}_p}{\hat{F}_s} = \frac{4000}{1000} = 4$$
- Analog lowpass filter specifications: $\Omega_p = 1$,
  $\Omega_s = 4$, $\alpha_p = 0.1$ dB, $\alpha_s = 40$ dB

Analog Highpass Filter Design

- Code fragments used

```
[N, Wn] = buttord(1, 4, 0.1, 40, 's');
[B, A] = butter(N, Wn, 's');
[num, den] = lp2hp(B, A, 2*pi*4000);
```

- Gain plots



Analog Bandpass Filter Design

- Spectral Transformation

$$s = \Omega_p \frac{\hat{s}^2 + \hat{\Omega}_o^2}{\hat{s}(\hat{\Omega}_{p2} - \hat{\Omega}_{p1})}$$

where Ω_p is the passband edge frequency of $H_{LP}(s)$, and $\hat{\Omega}_{p1}$ and $\hat{\Omega}_{p2}$ are the lower and upper passband edge frequencies of desired bandpass filter $H_{BP}(\hat{s})$

Analog Bandpass Filter Design

- On the imaginary axis the transformation is

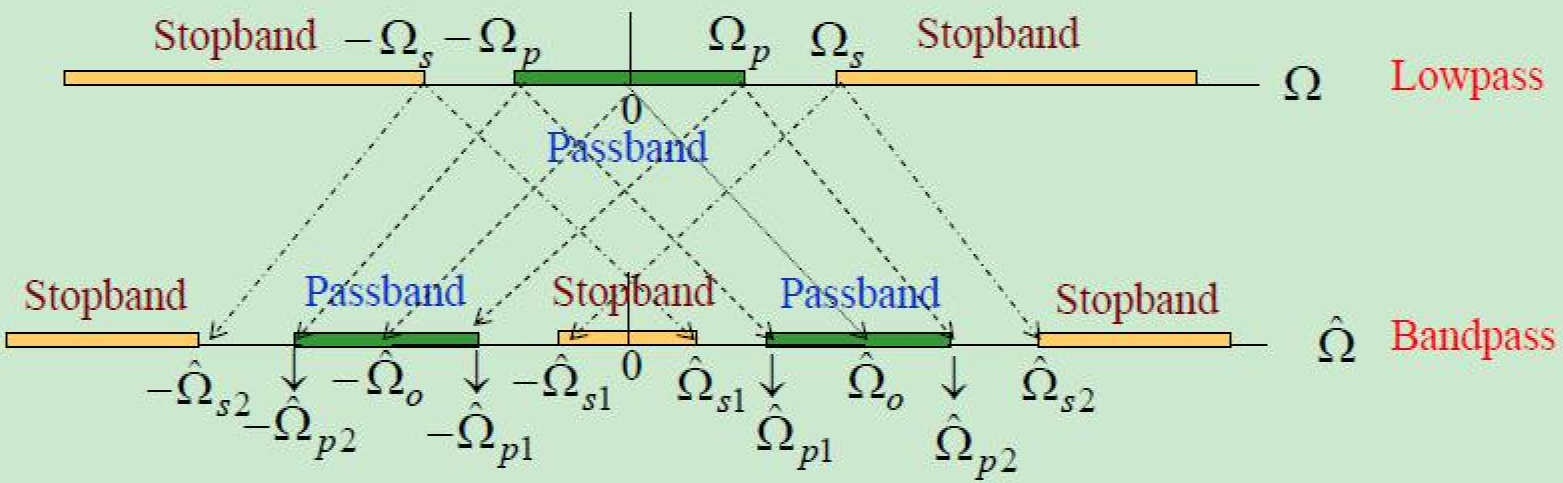
$$\Omega = -\Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega} B_w}$$

where $B_w = \hat{\Omega}_{p2} - \hat{\Omega}_{p1}$ is the width of passband and $\hat{\Omega}_o$ is the **passband center frequency** of the bandpass filter

- Passband edge frequency $\pm\Omega_p$ is mapped into $\mp\hat{\Omega}_{p1}$ and $\pm\hat{\Omega}_{p2}$, lower and upper passband edge frequencies

Analog Bandpass Filter Design

$$\Omega = -\Omega_p \frac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega} B_w}$$



Analog Bandpass Filter Design

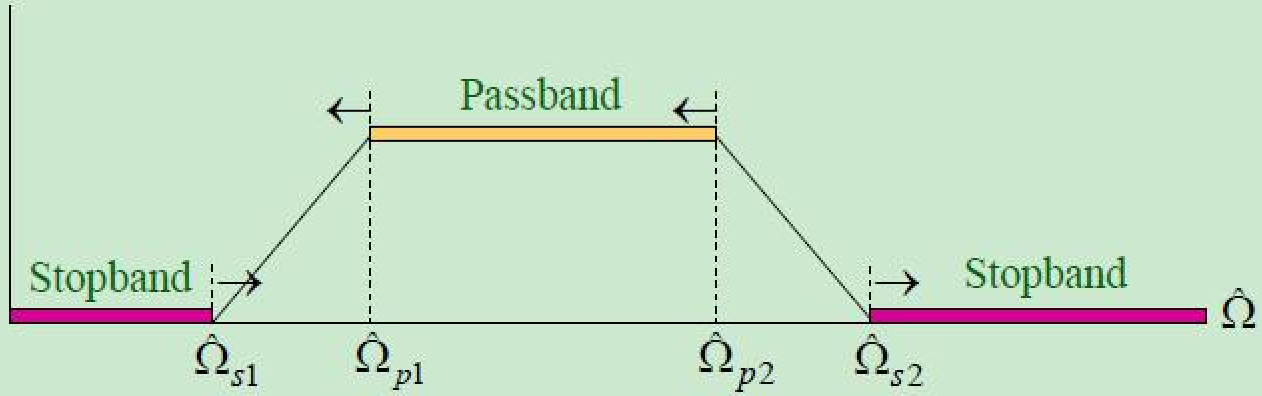
- Stopband edge frequency $\pm\Omega_s$ is mapped into $\mp\hat{\Omega}_{s1}$ and $\pm\hat{\Omega}_{s2}$, lower and upper stopband edge frequencies
- Also,

$$\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$$

- If bandedge frequencies do not satisfy the above condition, then one of the frequencies needs to be changed to a new value so that the condition is satisfied

Analog Bandpass Filter Design

- **Case 1**: $\hat{\Omega}_{p1}\hat{\Omega}_{p2} > \hat{\Omega}_{s1}\hat{\Omega}_{s2}$
  To make $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can either increase any one of the stopband edges or decrease any one of the passband edges as shown below



Analog Bandpass Filter Design

(1) Decrease $\hat{\Omega}_{p1}$ to $\hat{\Omega}_{s1}\hat{\Omega}_{s2}/\hat{\Omega}_{p2}$

⇨ larger passband and shorter leftmost transition band

(2) Increase $\hat{\Omega}_{s1}$ to $\hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s2}$

⇨ No change in passband and shorter leftmost transition band

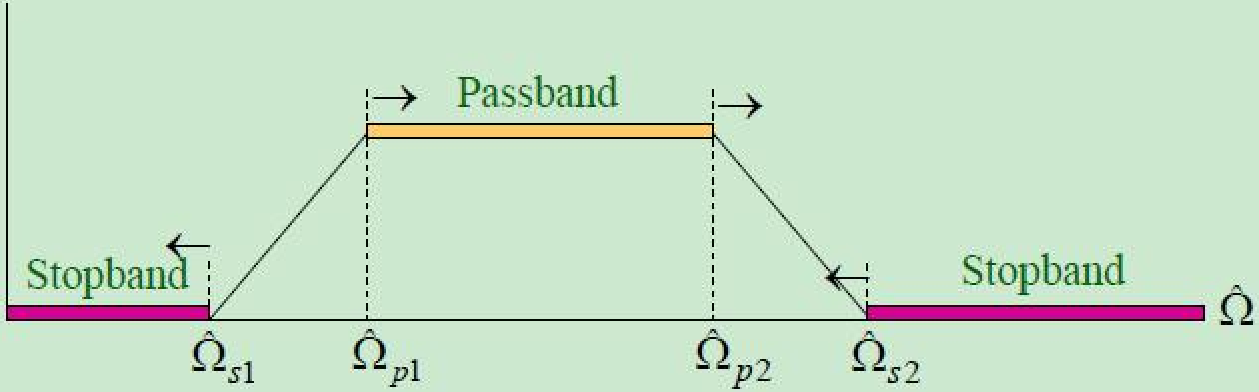
Analog Bandpass Filter Design

- Note: The condition $\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ can also be satisfied by decreasing $\hat{\Omega}_{p2}$ which is not acceptable as the passband is reduced from the desired value
- Alternately, the condition can be satisfied by increasing $\hat{\Omega}_{s2}$ which is not acceptable as the upper stop band is reduced from the desired value

Analog Bandpass Filter Design

- **Case 2**: $\hat{\Omega}_{p1}\hat{\Omega}_{p2} < \hat{\Omega}_{s1}\hat{\Omega}_{s2}$

  To make $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can either decrease any one of the stopband edges or increase any one of the passband edges as shown below



Analog Bandpass Filter Design

(1) Increase $\hat{\Omega}_{p2}$ to $\hat{\Omega}_{s1}\hat{\Omega}_{s2}/\hat{\Omega}_{p1}$

$\Rightarrow$ larger passband and shorter rightmost transition band

(2) Decrease $\hat{\Omega}_{s2}$ to $\hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s1}$

$\Rightarrow$ No change in passband and shorter rightmost transition band

Analog Bandpass Filter Design

- Note: The condition $\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ can also be satisfied by increasing $\hat{\Omega}_{p1}$ which is not acceptable as the passband is reduced from the desired value
- Alternately, the condition can be satisfied by decreasing $\hat{\Omega}_{s1}$ which is not acceptable as the lower stopband is reduced from the desired value

Analog Bandpass Filter Design

- Example - Design an analog elliptic bandpass filter with the specifications:

  $\hat{F}_{p1} = 4$ kHz, $\hat{F}_{p2} = 7$ kHz, $\hat{F}_{s1} = 3$ kHz

  $\hat{F}_{s2} = 8$ kHz, $\alpha_p = 1$ dB, $\alpha_s = 22$ dB

- Now $\hat{F}_{p1}\hat{F}_{p2} = 28 \times 10^6$ and $\hat{F}_{s1}\hat{F}_{s2} = 24 \times 10^6$
- Since $\hat{F}_{p1}\hat{F}_{p2} > \hat{F}_{s1}\hat{F}_{s2}$ we choose

$$\hat{F}_{p1} = \frac{\hat{F}_{s1}\hat{F}_{s2}}{\hat{F}_{p2}} = \frac{24}{7} = 3.42857 \text{ kHz}$$

Analog Bandpass Filter Design

- Bandwidth $B_w = 7 - \frac{24}{7} = \frac{25}{7}$ kHz
- We choose $\Omega_p = 1$

$$\Omega_s = \frac{\hat{F}_{s1}\hat{F}_{s2} - (\hat{F}_{s1})^2}{\hat{F}_{s1} \cdot B_w} = \frac{24-9}{(25/7)\times 3} = 1.4$$

- Analog lowpass filter specifications:

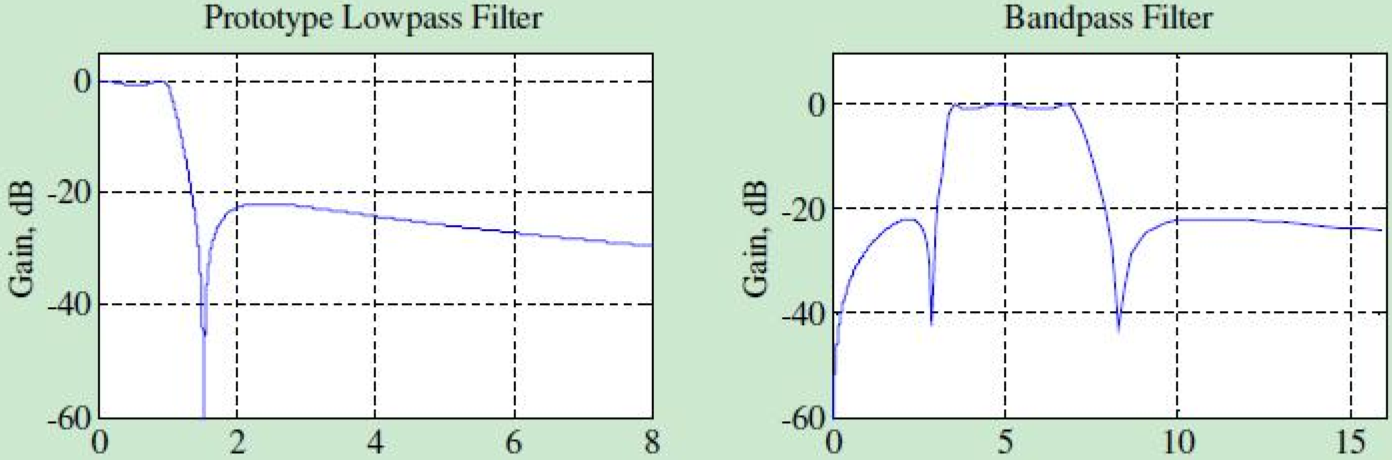
$\Omega_p = 1$, $\Omega_s = 1.4$, $\alpha_s = 22$ dB, $\alpha_p = 1$ dB

Analog Bandpass Filter Design

- Code fragments used

```
[N, Wn] = ellipord(1, 1.4, 1, 22, 's');
[B, A] = ellip(N, 1, 22, Wn, 's');
[num, den]
      = lp2bp(B, A, 2*pi*4.8989795, 2*pi*25/7);
```

- Gain plot



Analog Bandstop Filter Design

- Spectral Transformation

$$s = \Omega_s \frac{\hat{s}(\hat{\Omega}_{s2} - \hat{\Omega}_{s1})}{\hat{s}^2 + \hat{\Omega}_o^2}$$

where Ω_s is the stopband edge frequency of $H_{LP}(s)$, and $\hat{\Omega}_{s1}$ and $\hat{\Omega}_{s2}$ are the lower and upper stopband edge frequencies of the desired bandstop filter $H_{BS}(\hat{s})$

Analog Bandstop Filter Design

- On the imaginary axis the transformation is

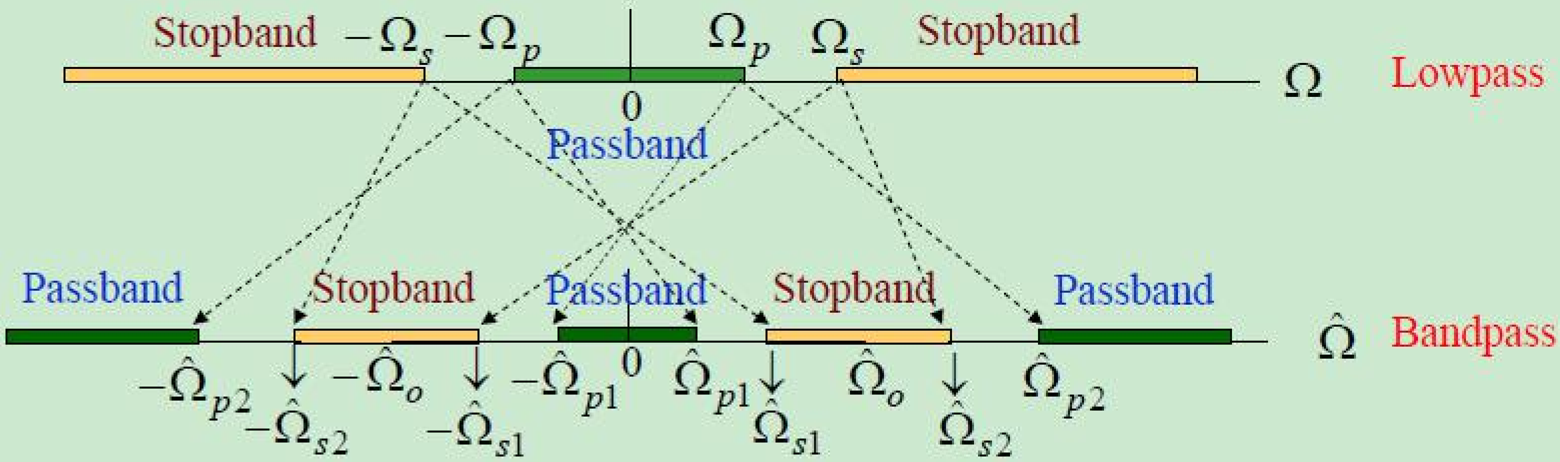
$$\Omega = \Omega_s \frac{\hat{\Omega} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}^2}$$

where $B_w = \hat{\Omega}_{s2} - \hat{\Omega}_{s1}$ is the width of stopband and $\hat{\Omega}_o$ is the **stopband center frequency** of the bandstop filter

- Stopband edge frequency $\pm\Omega_s$ is mapped into $\mp\hat{\Omega}_{s1}$ and $\pm\hat{\Omega}_{s2}$, lower and upper stopband edge frequencies

Analog Bandstop Filter Design

- Passband edge frequency $\pm\Omega_p$ is mapped into $\mp\hat{\Omega}_{p1}$ and $\pm\hat{\Omega}_{p2}$, lower and upper passband edge frequencies



Analog Bandstop Filter Design

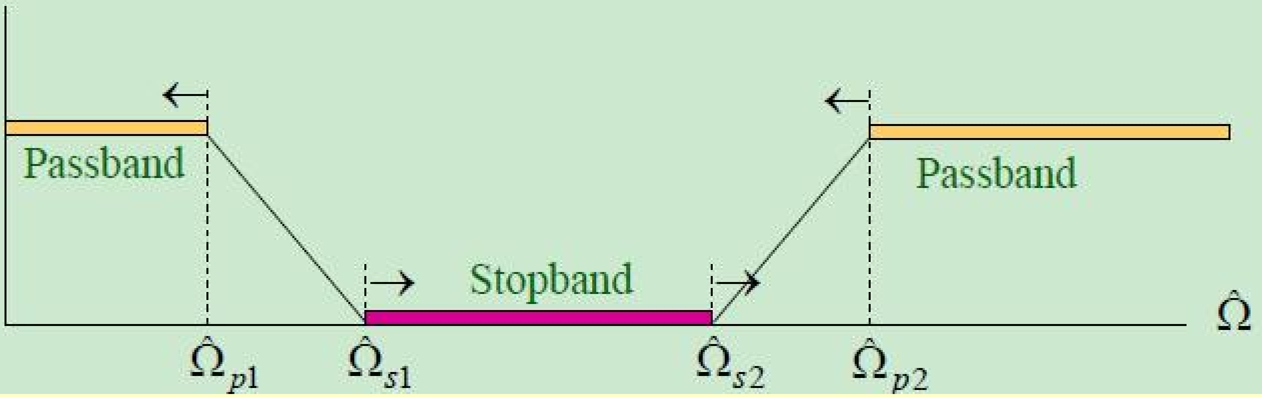
- Also,

$$\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$$

- If bandedge frequencies do not satisfy the above condition, then one of the frequencies needs to be changed to a new value so that the condition is satisfied

Analog Bandstop Filter Design

- **Case 1**: $\hat{\Omega}_{p1}\hat{\Omega}_{p2} > \hat{\Omega}_{s1}\hat{\Omega}_{s2}$
- To make $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can either increase any one of the stopband edges or decrease any one of the passband edges as shown below



Analog Bandstop Filter Design

(1) Decrease $\hat{\Omega}_{p2}$ to $\hat{\Omega}_{s1}\hat{\Omega}_{s2}/\hat{\Omega}_{p2}$

⇨ larger high-frequency passband and shorter rightmost transition band

(2) Increase $\hat{\Omega}_{s2}$ to $\hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s2}$

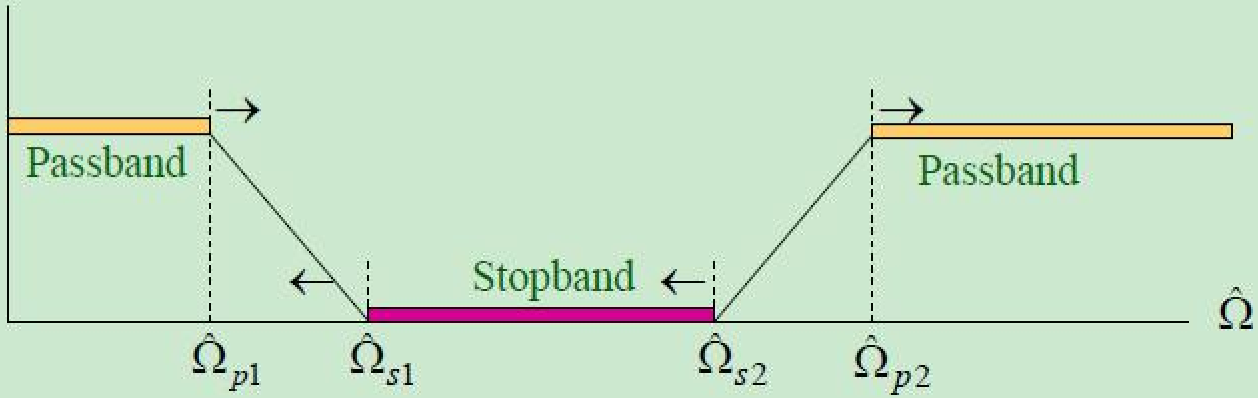
⇨ No change in passbands and shorter rightmost transition band

Analog Bandstop Filter Design

- Note: The condition $\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ can also be satisfied by decreasing which is not acceptable as the low-$\hat{\Omega}_{p1}$ frequency passband is reduced from the desired value
- Alternately, the condition can be satisfied by increasing $\hat{\Omega}_{s1}$ which is not acceptable as the stopband is reduced from the desired value

Analog Bandstop Filter Design

- **Case 2**: $\hat{\Omega}_{p1}\hat{\Omega}_{p2} < \hat{\Omega}_{s1}\hat{\Omega}_{s2}$
- To make $\hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ we can either decrease any one of the stopband edges or increase any one of the passband edges as shown below



Analog Bandstop Filter Design

(1) Increase $\hat{\Omega}_{p1}$ to $\hat{\Omega}_{s1}\hat{\Omega}_{s2}/\hat{\Omega}_{p1}$

⇨ larger passband and shorter leftmost transition band

(2) Decrease $\hat{\Omega}_{s1}$ to $\hat{\Omega}_{p1}\hat{\Omega}_{p2}/\hat{\Omega}_{s1}$

⇨ No change in passbands and shorter leftmost transition band

Analog Bandstop Filter Design

- Note: The condition $\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = \hat{\Omega}_{s1}\hat{\Omega}_{s2}$ can also be satisfied by increasing $\hat{\Omega}_{p2}$ which is not acceptable as the high-frequency passband is decreased from the desired value
- Alternately, the condition can be satisfied by decreasing $\hat{\Omega}_{s2}$ which is not acceptable as the stopband is decreased

IIR Digital Filter Design: Bilinear Transformation Method

- **Bilinear** transformation

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \qquad z = \frac{2/T+s}{2/T-s} = \frac{1+s}{1-s}\bigg|_{T=2}$$

Above transformation maps a single point in the s-plane to a unique point in the z-plane and vice-versa

- Relation between G(z) and $H_a(s)$ is then given by

$$G(z) = H_a(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

IIR Digital Filter Design: Bilinear Transformation Method

- Digital filter design consists of 3 steps:

  (1) Develop the specifications of $H_a(s)$ by applying the inverse bilinear transformation to specifications of $G(z)$

  (2) Design $H_a(s)$

  (3) Determine $G(z)$ by applying bilinear transformation to $H_a(s)$

- As a result, the parameter T has no effect on $G(z)$ and $T = 2$ is chosen for convenience

Bilinear Transformation

- Inverse bilinear transformation for $T = 2$ is

$$z = \frac{1+s}{1-s}$$

- For $s = \sigma_o + j\Omega_o$

$$z = \frac{(1+\sigma_o) + j\Omega_o}{(1-\sigma_o) - j\Omega_o} \quad \Rightarrow |z|^2 = \frac{(1+\sigma_o)^2 + \Omega_o^2}{(1-\sigma_o)^2 + \Omega_o^2}$$

- Thus,

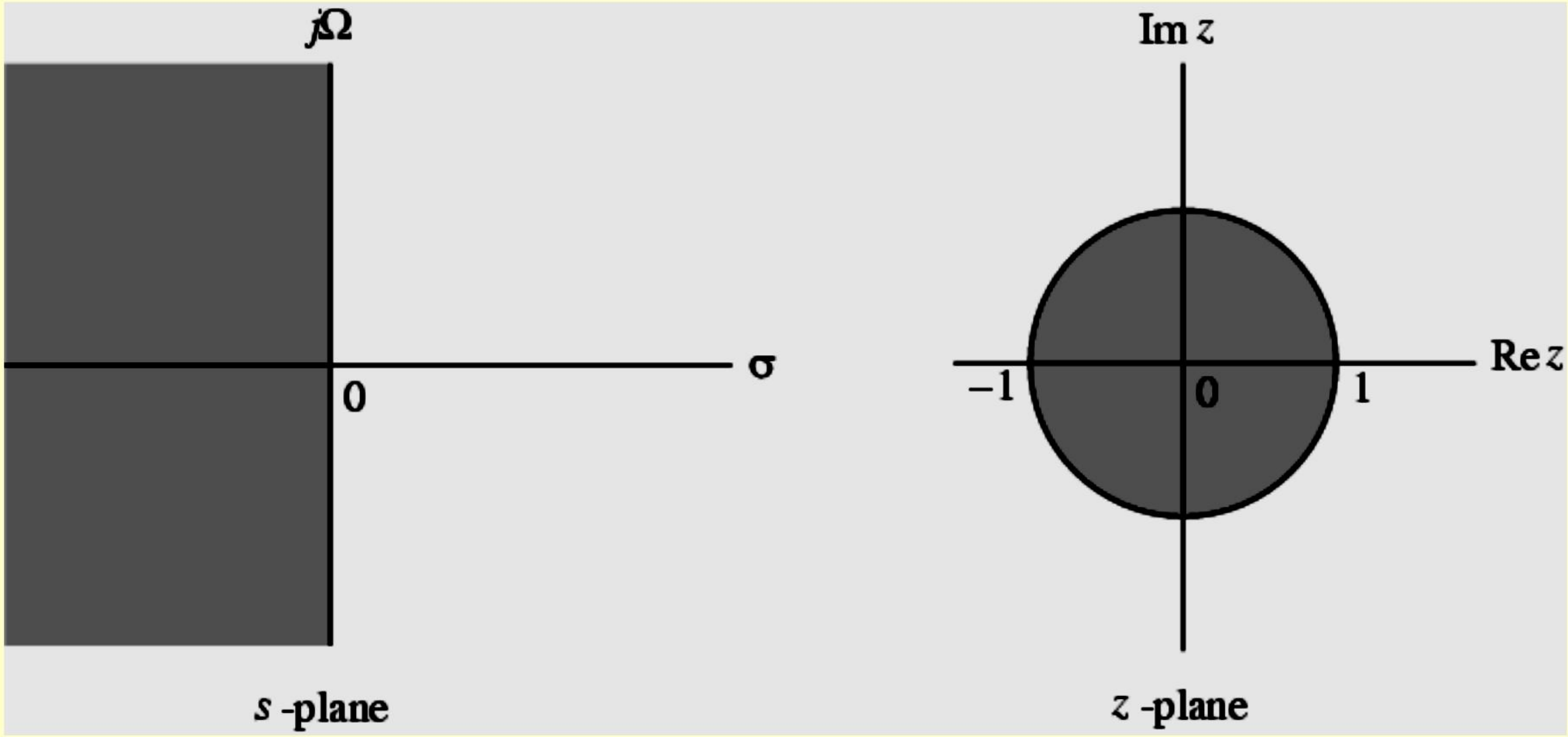
$$\sigma_o = 0 \rightarrow |z| = 1$$

$$\sigma_o < 0 \rightarrow |z| < 1$$

$$\sigma_o > 0 \rightarrow |z| > 1$$

IIR Digital Filter Design: Bilinear Transformation Method

- Mapping of s-plane into the z-plane



Digital Filter Design: Basic Approaches

- An analog transfer function to be denoted as

$$H_a(s) = P_a(s) / D_a(s)$$

where the subscript "a" specifically indicates the analog domain

- A digital transfer function derived from $H_a(s)$ shall be denoted as

$$G(z) = P(z)/D(z)$$

Digital Filter Design: Basic Approaches

- Basic idea behind the conversion of $H_a(s)$ into G(z) is to apply a mapping from the *s*-domain to the *z*-domain so that essential properties of the analog frequency response are preserved

- Thus mapping function should be such that:
  - Imaginary ($j\Omega$) axis in the *s*-plane be mapped onto the unit circle of the *z*-plane
  - A stable analog transfer function be mapped into a stable digital transfer function

Design of Low-Order Digital Filters

- Example - Consider

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c}$$

Applying bilinear transformation to the above we get the transfer function of a first-order digital lowpass Butterworth filter

$$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} = \frac{\Omega_c(1+z^{-1})}{(1-z^{-1}) + \Omega_c(1+z^{-1})}$$

Design of Low-Order Digital Filters

- Rearranging terms we get

$$G(z) = \frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha z^{-1}}$$

where

$$\alpha = \frac{1-\Omega_c}{1+\Omega_c} = \frac{1-\tan(\omega_c/2)}{1+\tan(\omega_c/2)}$$

Design of Low-Order Digital Filters

- <u>Example</u> - Consider the second-order analog notch transfer function

$$H_a(s) = \frac{s^2 + \Omega_o^2}{s^2 + Bs + \Omega_o^2}$$

for which $|H_a(j\Omega_0)| = 0$

$$|H_a(j0)| = |H_a(j\infty)| = 1$$

➢ Ω_0 is called the notch frequency

➢ If $|H_a(j\Omega_2)| = |H_a(j\Omega_1)| = 1/\sqrt{2}$ then $B = \Omega_2 - \Omega_1$ is the 3-dB notch bandwidth

Design of Low-Order Digital Filters

- **Then** $G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$

$$= \frac{(1+\Omega_o^2) - 2(1-\Omega_o^2)z^{-1} + (1+\Omega_o^2)z^{-2}}{(1+\Omega_o^2+B) - 2(1-\Omega_o^2)z^{-1} + (1+\Omega_o^2-B)z^{-2}}$$

$$= \frac{1+\alpha}{2} \cdot \frac{1-2\beta z^{-1}+z^{-2}}{1-2\beta(1+\alpha)z^{-1}+\alpha z^{-2}}$$

where $$\alpha = \frac{1+\Omega_o^2-B}{1+\Omega_o^2+B} = \frac{1-\tan(B_w/2)}{1+\tan(B_w/2)}$$

$$\beta = \frac{1-\Omega_o^2}{1+\Omega_o^2} = \cos\omega_o$$

Design of Low-Order Digital Filters

- Example - Design a 2nd-order digital notch filter operating at a sampling rate of 400 Hz with a notch frequency at 60 Hz, 3-dB notch bandwidth of 6 Hz
- Thus $\omega_0 = 2\pi(60/400) = 0.3\,\pi$

$$B_w = 2\pi(6/400) = 0.03\,\pi$$

- From the above values we get

$$\alpha = 0.90993$$

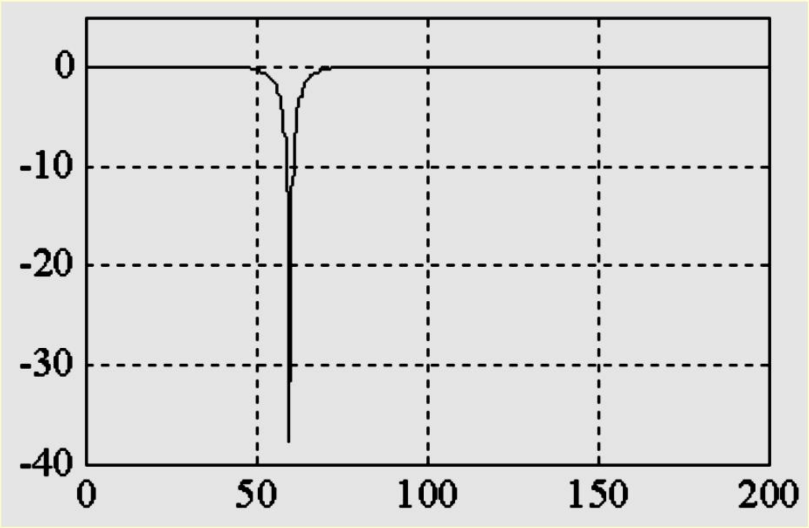
$$\beta = 0.587785$$

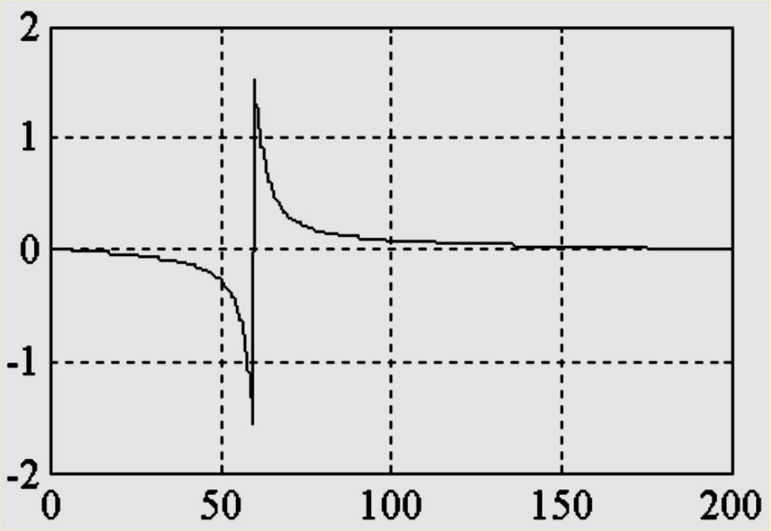
Design of Low-Order Digital Filters

Thus

$$G(z) = \frac{0.954965 - 1.1226287\, z^{-1} + 0.954965\, z^{-2}}{1 - 1.1226287\, z^{-1} + 0.90993\, z^{-2}}$$

The gain and phase responses are shown below





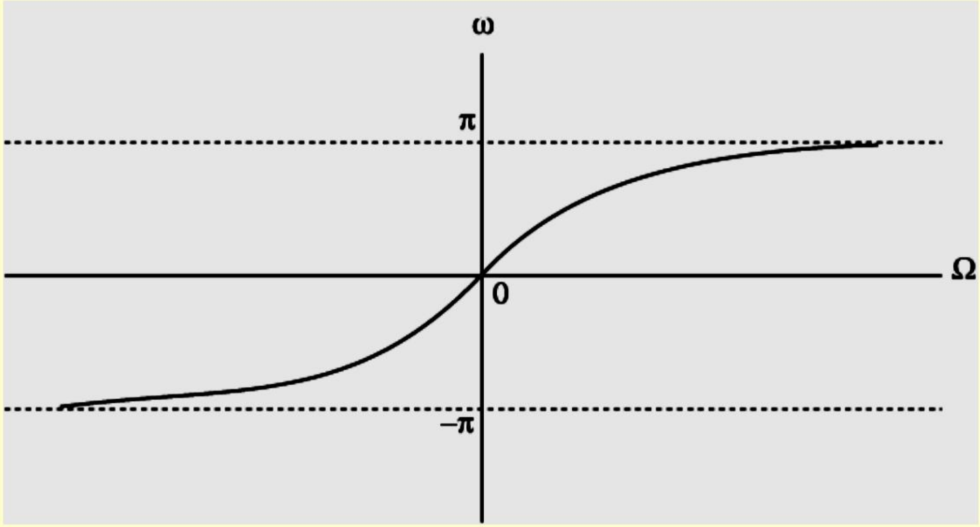
IIR Digital Filter Design: Bilinear Transformation Method

- For $z=e^{j\omega}$ with T = 2 we have

$$j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega/2}-e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2}+e^{-j\omega/2})}$$

$$= \frac{j2\sin(\omega/2)}{2\cos(\omega/2)} = j\tan(\omega/2)$$

or $\Omega=\tan(\omega/2)$

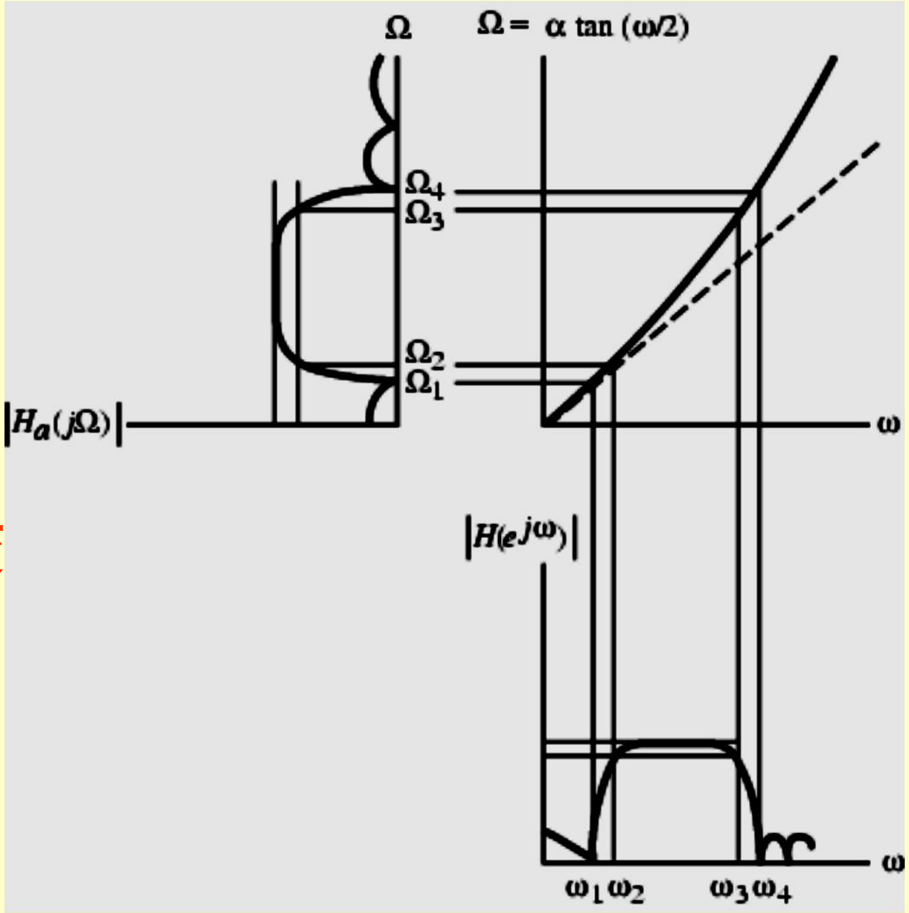


IIR Digital Filter Design: Bilinear Transformation Method

- Mapping is highly nonlinear
- Complete negative imaginary axis in the *s*-plane from $\Omega = -\infty$ to $\Omega = 0$ is mapped into the lower half of the unit circle in the *z*-plane from $z = -1$ to $z = 1$
- Complete positive imaginary axis in the *s*-plane from $\Omega = 0$ to $\Omega = \infty$ is mapped into the upper half of the unit circle in the *z*-plane from $z = 1$ to $z = -1$

IIR Digital Filter Design: Bilinear Transformation Method

- Nonlinear mapping introduces a distortion in the frequency axis called **frequency warping** 频率畸变
- Effect of warping shown right



Design of Low-Order Digital Filters

- Steps in the design of a digital filter -

  (1) Prewarp (ω_p, ω_s) to find their analog equivalents (Ω_p, Ω_s) 预畸变

  (2) Design the analog filter $H_a(s)$

  (3) Design the digital filter $G(z)$ by applying bilinear transformation to $H_a(s)$

- Transformation does not preserve phase response of analog filter

Design of Lowpass IIR Digital Filters

Example - Design a lowpass Butterworth digital filter with $\omega_p = 0.25\pi$, $\omega_s = 0.55\pi$, $\alpha_p \le 0.5$ dB, and $\alpha_s \ge 15$ dB

Solution:

1) If $|G(e^{j0})| = 1$, this implies

$$10\log_{10}|G(e^{j0.25\pi})|^2 = 10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right) \ge -0.5$$

$$10\log_{10}|G(e^{j0.55\pi})|^2 = 10\log_{10}\frac{1}{A^2} \le -15$$

Design of Lowpass IIR Digital Filters

Thus， we get

$\varepsilon^2=0.1220185$, $A^2=31.622777$

2) Prewarping, we get

$\Omega_p=\tan(\omega_p/2)=\tan(0.25\pi/2)=0.4142136$

$\Omega_s=\tan(\omega_s/2)=\tan(0.55\pi/2)=1.1708496$

The inverse transition ratio is

$1/k = \Omega_s/\Omega_p = 2.8266809$

The inverse discrimination ratio is

$1/k_1 = \sqrt{(A^2-1)}/\varepsilon = 15.841979$

Design of Lowpass IIR Digital Filters

3) Compute the order of analog lowpass filter:

$$N=\frac{1}{2}\frac{\log_{10}[(A^2-1)/\varepsilon^2]}{\log_{10}(\Omega_s/\Omega_p)}=\frac{\log_{10}(1/k1)}{\log_{10}(1/k)}=2.6586997$$

Choose N = 3

4) To determine Ω_c , we use

$|H_a(j\Omega_p)|^2 = 1/[1+(\Omega_p/\Omega_c)^{2N}] = 1/(1+\varepsilon^2)$, or

$|H_a(j\Omega_s)|^2 = 1/[1+(\Omega_s/\Omega_c)^{2N}] = 1/A^2$

Design of Lowpass IIR Digital Filters

We then get

$\Omega_c = 1.419915(\Omega_p) = 0.588148$

5) 3rd-order lowpass Butterworth transfer function for $\Omega_c = 1$ is

$H_{an}(s) = 1/[(s+1)(s^2+s+1)]$

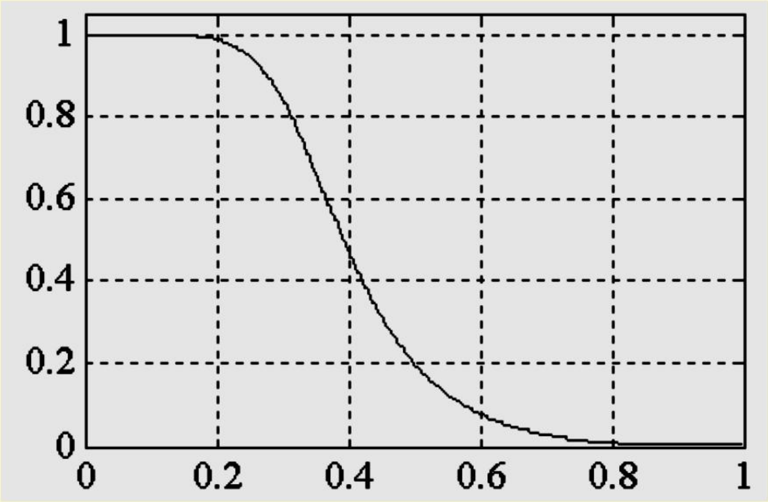
6) Denormalizing to get $\Omega_c = 0.588148$,we arrive at $H_a(s) = H_{an}(s/0.588148)$

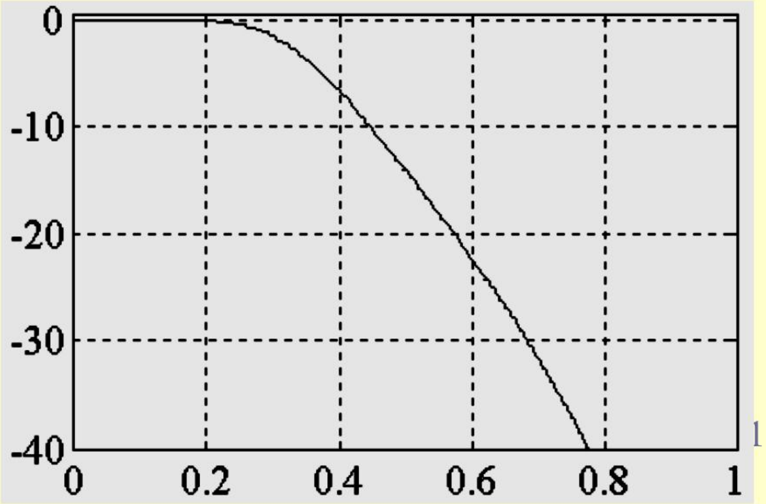
Design of Lowpass IIR Digital Filters

7) Applying bilinear transformation to $H_a(s)$ we get the desired digital transfer function

$$G(z) = H_a(s)\Big|_{s=\frac{1-z^{-1}}{1+z^{-1}}}$$

Magnitude and gain responses of $G(z)$ shown below:





IIR Highpass, Bandpass, and Bandstop Digital Filter Design

- First Approach -

  (1) Prewarp digital frequency specifications of desired digital filter $G_D(z)$ to arrive at frequency specifications of analog filter $H_D(s)$ of same type

  (2) Convert frequency specifications of $H_D(s)$ into that of prototype analog lowpass filter $H_{LP}(s)$

  (3) Design analog lowpass filter $H_{LP}(s)$

IIR Highpass, Bandpass, and Bandstop Digital Filter Design

(4) Convert $H_{LP}(s)$ into $H_D(s)$ using inverse frequency transformation used in Step 2

(5) Design desired digital filter $G_D(z)$ by applying bilinear transformation to $H_{LP}(s)$

IIR Highpass, Bandpass, and Bandstop Digital Filter Design

- Second Approach -

  (1) Prewarp digital frequency specifications of desired digital filter $G_D(z)$ to arrive at frequency specifications of analog filter $H_D(s)$ of same type

  (2) Convert frequency specifications of $H_D(s)$ into that of prototype analog lowpass filter $H_{LP}(s)$

IIR Highpass, Bandpass, and Bandstop Digital Filter Design

(3) Design analog lowpass filter $H_{LP}(s)$

(4) Convert $H_{LP}(s)$ into an IIR digital transfer function $G_{LP}(z)$ using bilinear transformation

(5) Transform $G_{LP}(z)$ into the desired digital transfer function $G_D(z)$

- We illustrate the first approach

IIR Highpass Digital Filter Design

- Design of a Type 1 Chebyshev IIR digital highpass filter
- Specifications: F_p=700Hz, F_s = 500Hz, α_p=1 dB, α_s=32dB, F_T=2 kHz
- Normalized angular bandedge frequencies

$$\omega_p = 2\pi F_p / F_T = 2\pi \times 700/2000 = 0.7\pi$$

$$\omega_s = 2\pi F_s / F_T = 2\pi \times 500/2000 = 0.5\pi$$

IIR Highpass Digital Filter Design

- Prewarping these frequencies we get

$$\hat{\Omega}_p = \tan(\omega_p / 2) = 1.9626105$$

$$\hat{\Omega}_s = \tan(\omega_s / 2) = 1.0$$

For the prototype analog lowpass filter choose

$\Omega_p = 1$

Using $\Omega = -\frac{\Omega_p \hat{\Omega}_p}{\hat{\Omega}}$ we get $\Omega_s = 1.962105$

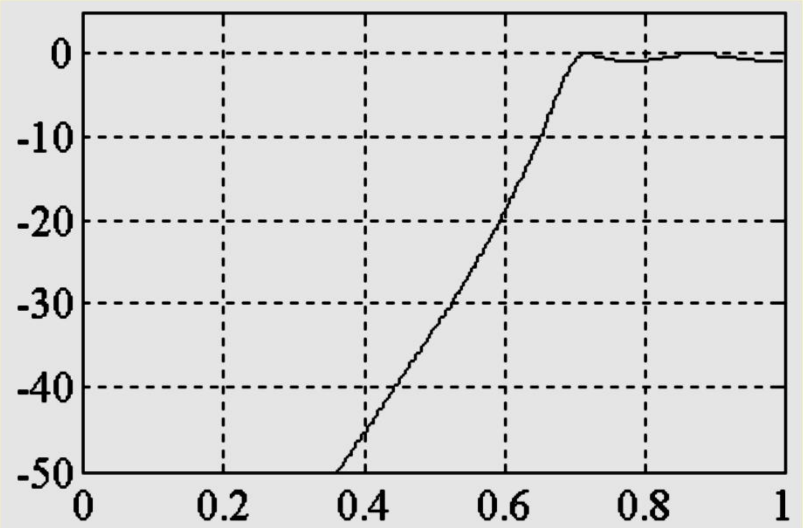
- Analog lowpass filter specifications: $\Omega_p = 1$,

$\Omega_s = 1.926105$, α_p=1 dB, α_s=32 dB

IIR Highpass Digital Filter Design

- MATLAB code fragments used for the design

```
[N, Wn] = cheb1ord(1, 1.9626105, 1, 32, 's')
[B, A] = cheby1(N, 1, Wn, 's');
[BT, AT] = lp2hp(B, A, 1.9626105);
[num, den] = bilinear(BT, AT, 0.5);
```



IIR Bandpass Digital Filter Design

- Design of a Butterworth IIR digital bandpass filter
- Specifications: ω_{p1}=0.45π , ω_{p1}=0.65π ,

ω_{s1}=0.3π, ω_{s2}=0.75π, α_p=1 dB, α_s=40 dB

- Prewarping we get

$$\hat{\Omega}_{p1} = \tan(\omega_{p1}/2) = 0.8540807$$

$$\hat{\Omega}_{p2} = \tan(\omega_{p2}/2) = 1.6318517$$

$$\hat{\Omega}_{s1} = \tan(\omega_{s1}/2) = 0.5095254$$

$$\hat{\Omega}_{s2} = \tan(\omega_{s2}/2) = 2.41421356$$

IIR Bandpass Digital Filter Design

- Width of passband $B_w = \hat{\Omega}_{p2} - \hat{\Omega}_{p1} = 0.777771$

$$\hat{\Omega}_o^2 = \hat{\Omega}_{p1}\hat{\Omega}_{p2} = 1.393733$$

$$\hat{\Omega}_{s1}\hat{\Omega}_{s2} = 1.23010325 \neq \hat{\Omega}_o^2$$

- We therefore modify $\hat{\Omega}_{s1}$ so that $\hat{\Omega}_{s1}$ and $\hat{\Omega}_{s2}$ exhibit geometric symmetry with respect to $\hat{\Omega}_o^2$

- We set $\hat{\Omega}_{s1} = 0.5773031$

For the prototype analog lowpass filter we choose $\Omega_p = 1$

IIR Bandpass Digital Filter Design

- Using $\Omega = -\Omega_p \dfrac{\hat{\Omega}_o^2 - \hat{\Omega}^2}{\hat{\Omega} B_w}$ we get

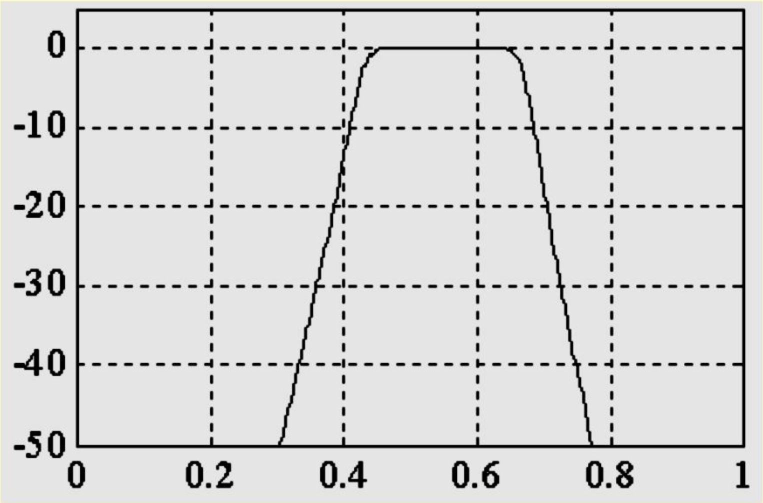
$$\Omega_s = \frac{1.393733 - 0.3332788}{0.5773031 \times 0.777771} = 2.3617627$$

➢ Specifications of prototype analog Butterworth lowpass filter: $\Omega_p = 1$, $\Omega_s = 2.3617627$, $\alpha_p = 1$ dB, $\alpha_s = 40$ dB

IIR Bandpass Digital Filter Design

- MATLAB code fragments used for the design

```
[N, Wn] = buttord(1, 2.3617627, 1, 40, 's')
[B, A] = butter(N, Wn, 's');
[BT, AT] = lp2bp(B, A, 1.1805647, 0.777771);
[num, den] = bilinear(BT, AT, 0.5);
```



IIR Bandstop Digital Filter Design

- Design of an elliptic IIR digital bandstop filter
- Specifications: ω_{s1}=0.45π, ω_{s2}=0.65π , ω_{p1}=0.3π, ω_{p2}=0.75π, α_p=1 dB, α_s=40 dB
- Prewarping we get

$$\hat{\Omega}_{s1} = 0.8540806, \quad \hat{\Omega}_{s2} = 1.6318517,$$
$$\hat{\Omega}_{p1} = 0.5095254, \quad \hat{\Omega}_{p2} = 2.4142136$$

- Width of stopband

$$\hat{\Omega}_o^2 = \hat{\Omega}_{s2}\hat{\Omega}_{s1} = 1.393733$$
$$\hat{\Omega}_{p2}\hat{\Omega}_{p1} = 1.230103 \neq \hat{\Omega}_o^2$$

IIR Bandstop Digital Filter Design

- We therefore modify $\hat{\Omega}_{s1}$ so that $\hat{\Omega}_{s1}$ and $\hat{\Omega}_{s2}$ exhibit geometric symmetry with respect to $\hat{\Omega}_o^2$

- We set $\hat{\Omega}_{p1} = 0.577303$

➢ For the prototype analog lowpass filter we choose $\Omega_s = 1$

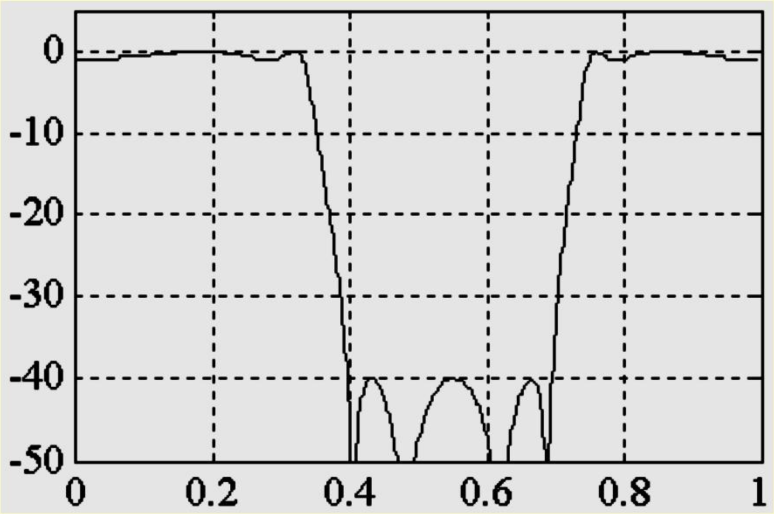
- Using $\Omega = \Omega_s \dfrac{\hat{\Omega} B_w}{\hat{\Omega}_o^2 - \hat{\Omega}^2}$ we get

$$\Omega_p = \frac{0.5095254 \times 0.777771}{1.393733 - 0.3332787} = 0.4234126$$

IIR Bandstop Digital Filter Design

- MATLAB code fragments used for the design

```
[N, Wn] = ellipord(0.4234126, 1, 1, 40, 's');
[B, A] = ellip(N, 1, 40, Wn, 's');
[BT, AT] = lp2bs(B, A, 1.1805647, 0.777771);
[num, den] = bilinear(BT, AT, 0.5);
```



IIR Filter Design Using Matlab

- Order Estimation -
- For IIR filter design using bilinear transformation, MATLAB statements to determine the order and bandedge are:

```
[N, Wn] = buttord(Wp, Ws, Rp, Rs);
[N, Wn] = cheb1ord(Wp, Ws, Rp, Rs);
[N, Wn] = cheb2ord(Wp, Ws, Rp, Rs);
[N, Wn] = ellipord(Wp, Ws, Rp, Rs);
```

IIR Digital Filter Design Using MATLAB

- Example - Determine the minimum order of a Type 2 Chebyshev digital highpass filter with the following specifications:

  $F_p = 1$ kHz, $F_s = 0.6$ kHz, $F_T = 4$ kHz,

  $\alpha_p = 1$ dB, $\alpha_s = 40$ dB

- Here, $\text{Wp} = 2 \times 1/4 = 0.5$, $\text{Ws} = 2 \times 0.6/4 = 0.3$
- Using the statement

  [N, Wn] = cheb2ord(0.5, 0.3, 1, 40);

  we get N = 5 and Wn = 0.3224

IIR Digital Filter Design Using MATLAB

- Filter Design -
- For IIR filter design using bilinear transformation, MATLAB statements to use are:

  [b, a] = butter(N, Wn)

  [b, a] = cheby1(N, Rp, Wn)

  [b, a] = cheby2(N, Rs, Wn)

  [b, a] = ellip(N, Rp, Rs, Wn)

IIR Digital Filter Design Using MATLAB

- The form of transfer function obtained is

$$G(z) = \frac{B(z)}{A(z)} = \frac{b(1) + b(2)z^{-1} + \cdots + b(N+1)z^{-N}}{1 + a(2)z^{-1} + \cdots + a(N+1)z^{-N}}$$

- The frequency response can be computed using the M-file freqz(b, a, w) where w is a set of specified angular frequencies
- It generates a set of complex frequency response samples from which magnitude and/or phase response samples cn be computed

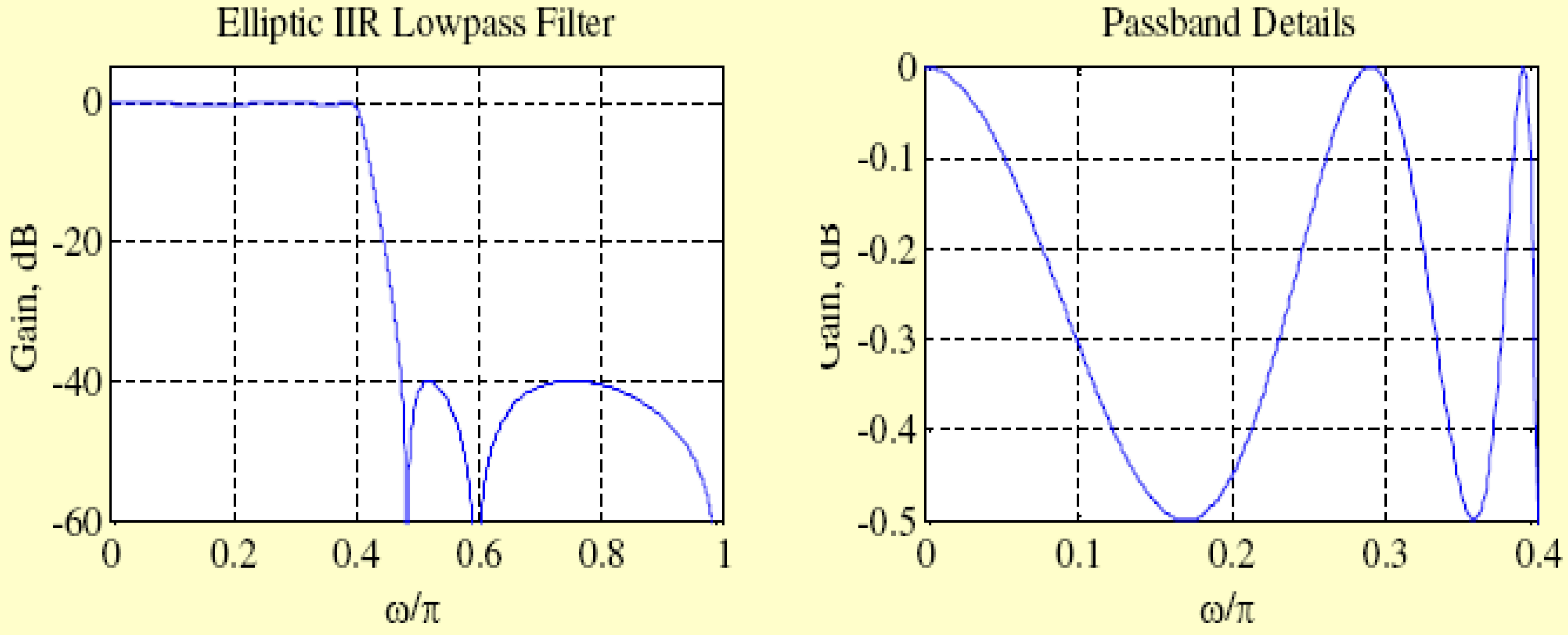
IIR Digital Filter Design Using MATLAB

- Example - Design an elliptic IIR lowpass filter with the specifications: $F_p = 0.8$ kHz, $F_s = 1$ kHz, $F_T = 4$ kHz, $\alpha_p = 0.5$dB, $\alpha_s = 40$ dB
- Here, $\omega_p = 2\pi F_p / F_T = 0.4\pi$, $\omega_s = 2\pi F_s / F_T = 0.5\pi$
- Code fragments used are:

```
[N,Wn] = ellipord(0.4, 0.5, 0.5, 40);
[b, a] = ellip(N, 0.5, 40, Wn);
```

IIR Digital Filter Design Using MATLAB

- Gain response plot is shown below:



Spectral Transformations of IIR Digital Filters

- <u>Objective</u> - Transform a given lowpass digital transfer function $G_L(z)$ to another digital transfer function $G_D(\hat{z})$ that could be a lowpass, highpass, bandpass or bandstop filter
- z^{-1} has been used to denote the unit delay in the prototype lowpass filter GL(z) and $\hat{z}^{-1}$ to denote the unit delay in the transformed filter $G_D(\hat{z})$ to avoid confusion

Lowpass-to-Lowpass Spectral Transformation

- To transform a lowpass filter $G_L(z)$ with a cutoff frequency ω_c to another lowpass filter $G_D(\hat{z})$ with a cutoff frequency $\hat{\omega}_c$, the transformation is

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{1 - \alpha\,\hat{z}}{\hat{z} - \alpha}$$

where α is a function of the two specified cutoff frequencies

Lowpass-to-Lowpass Spectral Transformation

- On the unit circle we have

$$e^{-j\omega} = \frac{e^{-j\hat{\omega}} - \alpha}{1 - \alpha e^{-j\hat{\omega}}}$$

From the above we get

$$e^{-j\omega} \mp 1 = \frac{e^{-j\hat{\omega}} - \alpha}{1 - \alpha e^{-j\hat{\omega}}} \mp 1 = (1 \pm \alpha) \cdot \frac{e^{-j\hat{\omega}} - 1}{1 - \alpha e^{-j\hat{\omega}}}$$

- Taking the ratios of the above two expressions

$$\tan(\omega/2) = \left(\frac{1+\alpha}{1-\alpha}\right) \tan(\hat{\omega}/2)$$

Lowpass-to-Lowpass Spectral Transformation

- Solving we get

$$\alpha = \frac{\sin\left((\omega_c - \hat{\omega}_c)/2\right)}{\sin\left((\omega_c + \hat{\omega}_c)/2\right)}$$

Example: Consider the lowpass digital filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593\,z^{-1})(1-0.6763\,z^{-1}+0.3917\,z^{-2})}$$

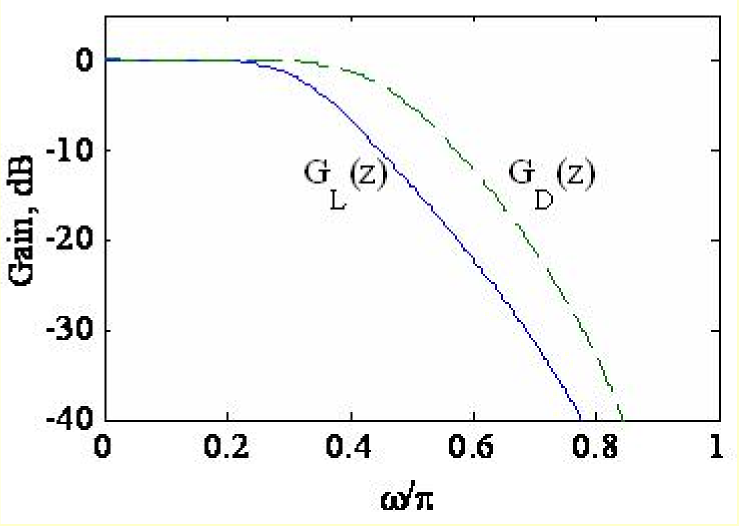
which has a passband from dc to 0.25π with a 0.5 dB ripple

Redesign the above filter to move the passband edge to 0.35π

Lowpass-to-Lowpass Spectral Transformation

- Here
$$\alpha = -\frac{\sin(0.05\pi)}{\sin(0.3\pi)} = -0.1934$$

Hence, the desired lowpass transfer function is
$$G_D(\hat{z}) = G_L(z)\Big|_{z^{-1} = \frac{\hat{z}^{-1} + 0.1934}{1 + 0.1934\,\hat{z}^{-1}}}$$



Lowpass-to-Lowpass Spectral Transformation

- The lowpass-to-lowpass transformation

$$z^{-1} = \frac{1}{F(\hat{z})} = \frac{1 - \alpha\,\hat{z}}{\hat{z} - \alpha}$$

can also be used as highpass-to-highpass, bandpass-to-bandpass and bandstop-to-bandstop transformations

Lowpass-to-Highpass Spectral Transformation

- Desired transformation

$$z^{-1} = -\frac{\hat{z}^{-1} + \alpha}{1 + \alpha\,\hat{z}^{-1}}$$

- The transformation parameter α is given by

$$\alpha = -\frac{\cos\left((\omega_c + \hat{\omega}_c)/2\right)}{\cos\left((\omega_c - \hat{\omega}_c)/2\right)}$$

where ω_c is the cutoff frequency of the lowpass filter and $\hat{\omega}_c$ is the cutoff frequency of the desired highpass filter

Lowpass-to-Highpass Spectral Transformation

- Example - Transform the lowpass filter

$$G_L(z) = \frac{0.0662(1+z^{-1})^3}{(1-0.2593\,z^{-1})(1-0.6763\,z^{-1}+0.3917z^{-2})}$$

with a passband edge at 0.25π to a highpass filter with a passband edge at 0.55π

Here $\alpha = -\cos(0.4\pi)/\cos(0.15\pi) = -0.3468$

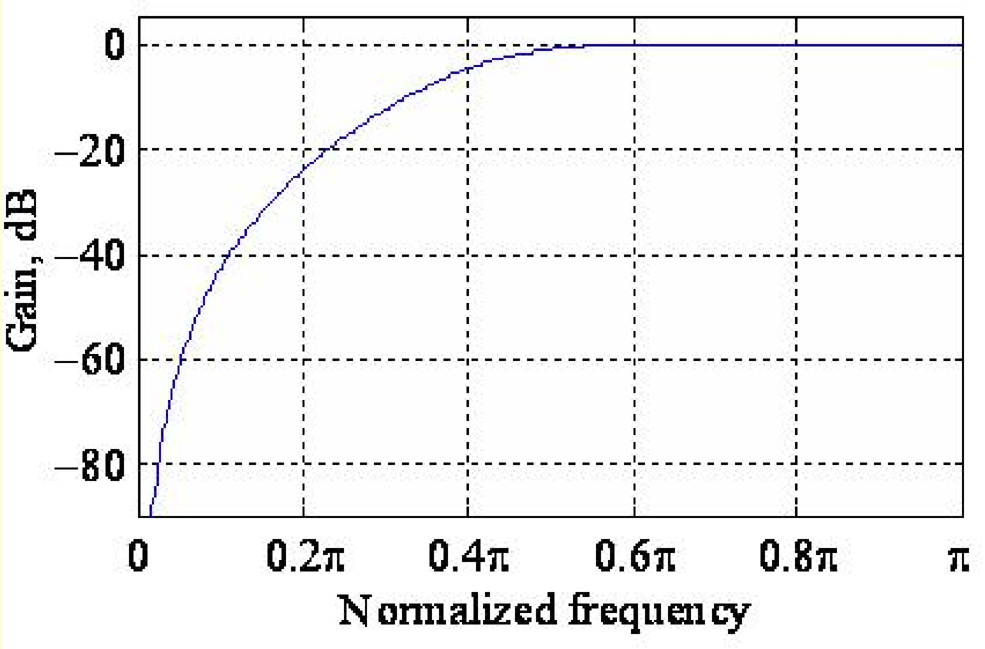
- The desired transformation is

$$z^{-1} = -\frac{\hat{z}^{-1} - 0.3468}{1 - 0.3468\hat{z}^{-1}}$$

Lowpass-to-Highpass Spectral Transformation

- The desired highpass filter is

$$G_D(\hat{z}) = G(z)\Big|_{z^{-1} = -\frac{\hat{z}^{-1} - 0.3468}{1 - 0.3468\,\hat{z}^{-1}}}$$



Lowpass-to-Highpass Spectral Transformation

- The lowpass-to-highpass transformation can also be used to transform a highpass filter with a cutoff at ω_c to a lowpass filter with a cutoff at $\hat{\omega}_c$

and transform a bandpass filter with a center frequency at ω_o to a bandstop filter with a center frequency at $\hat{\omega}_o$

Lowpass-to-Bandpass Spectral Transformation

- Desired transformation

$$z^{-1} = -\frac{\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + \frac{\beta-1}{\beta+1}}{\frac{\beta-1}{\beta+1}\hat{z}^{-2} - \frac{2\alpha\beta}{\beta+1}\hat{z}^{-1} + 1}$$

Lowpass-to-Bandpass Spectral Transformation

- The parameters α and β are given by

$$\alpha = \frac{\cos\big((\hat{\omega}_{c2} + \hat{\omega}_{c1})/2\big)}{\cos\big((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\big)}$$

$$\beta = \cot\big((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\big)\tan(\omega_c/2)$$

where ω_c is the cutoff frequency of the lowpass filter, and $\hat{\omega}_{c1}$ and $\hat{\omega}_{c2}$ are the desired upper and lower cutoff frequencies of the bandpass filter

Lowpass-to-Bandpass Spectral Transformation

Special Case - The transformation can be simplified if $\omega_c = \hat{\omega}_{c2} - \hat{\omega}_{c1}$

Then the transformation reduces to

$$z^{-1} = -\hat{z}^{-1} \frac{\hat{z}^{-1} - \alpha}{1 - \alpha \hat{z}^{-1}}$$

where $\alpha = \cos\hat{\omega}_o$ with $\hat{\omega}_o$ denoting the desired center frequency of the bandpass filter

Lowpass-to-Bandstop Spectral Transformation

- Desired transformation

$$z^{-1} = \frac{\hat{z}^{-2} - \frac{2\alpha\beta}{1+\beta}\hat{z}^{-1} + \frac{1-\beta}{1+\beta}}{\frac{1-\beta}{1+\beta}\hat{z}^{-2} - \frac{2\alpha\beta}{1+\beta}\hat{z}^{-1} + 1}$$

Lowpass-to-Bandstop Spectral Transformation

- The parameters α and β are given by

$$\alpha = \frac{\cos\left((\hat{\omega}_{c2} + \hat{\omega}_{c1})/2\right)}{\cos\left((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\right)}$$

$$\beta = \tan\left((\hat{\omega}_{c2} - \hat{\omega}_{c1})/2\right)\tan(\omega_c/2)$$

where ω_c is the cutoff frequency of the lowpass filter, and $\hat{\omega}_{c1}$ and $\hat{\omega}_{c2}$ are the desired upper and lower cutoff frequencies of the bandstop filter

Generation of Allpass Function Using MATLAB

- The allpass function needed for the spectral transformation from a specified lowpass transfer function to a desired highpass or bandpass or bandstop transfer function can be generated using MATLAB

Generation of Allpass Function Using MATLAB

- **Lowpass-to-Highpass Transformation**
- Basic form:

```
[AllpassNum,AllpassDen] =
allpasslp2hp(wold,wnew)
```

where `wold` is the specified angular bandedge frequency of the original lowpass filter, and `wnew` is the desired angular bandedge frequency of the highpass filter

Generation of Allpass Function Using MATLAB

- **Lowpass-to-Bandpass Transformation**
- Basic form:

```
[AllpassNum,AllpassDen] =
allpasslp2bp(wold,wnew)
```

where `wold` is the specified angular bandedge frequency of the original lowpass filter, and `wnew` is the desired angular bandedge frequency of the bandpass filter

Generation of Allpass Function Using MATLAB

- **Lowpass-to-Bandstop Transformation**
- Basic form:

```
[AllpassNum,AllpassDen] =
allpasslp2bs(wold,wnew)
```

where `wold` is the specified angular bandedge frequency of the original lowpass filter, and `wnew` is the desired angular bandedge frequency of the bandstop filter

Generation of Allpass Function Using MATLAB

- **Lowpass-to-Highpass Example –**
  wold = 0.25π, wnew = 0.55π
- The MATLAB statement

```
[APnum, APden]
= allpasslp2hp(0.25, 0.55)
```

yields the mapping

$$z^{-1} \rightarrow \frac{-z^{-1} + 0.3468}{-0.3468z^{-1} + 1}$$

Spectral Transformation Using MATLAB

- The pertinent M-files are `iirlp2lp`, `iirlp2hp`, `iirlp2bp`, and `iirlp2bs`
- **Lowpass-to-Highpass Example –**

$$G_{LP}(z) = \frac{0.066(1+z^{-1})^3}{1-0.9353z^{-1}+0.5669z^{-2}-0.1015z^{-3}}$$

Passband edge wold $= 0.25\pi$

Desired passband edge of highpass filter

wnew $= 0.55\pi$

Spectral Transformation Using MATLAB

- The MATLAB code fragments used are

```
b = 0.066*[1 3 3 1];
a = [1.00  -0.9353  0.5669 -0.1015];
[num,den,APnum,APden]
= iirlp2hp(b,a,0.25,0.55);
```

- The desired highpass filter obtained is

$$G_{HP}(z) = \frac{0.218(1-z^{-1})^3}{1-0.3521z^{-1}+0.3661z^{-2}-0.0329z^{-3}}$$