Chapter 6

z-Transform

Convergence condition of DTFT

- $x[n]$ is an absolutely summable sequence

- mean-square convergence

$$\lim_{K\to\infty}\int_{-\pi}^{\pi}\left|X(e^{j\omega})-X_K(e^{j\omega})\right|^2 d\omega=0$$

- Dirac delta function $\delta(\omega)$

z-Transform

- The DTFT provides a frequency-domain representation of discrete-time signals and LTI discrete-time systems
- Because of the convergence condition, in many cases, the DTFT of a sequence may not exist
- As a result, it is not possible to make use of such frequency-domain characterization in these cases

z-Transform

- A generalization of the DTFT defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

  leads to the *z*-transform

- *z*-transform may exist for many sequences for which the DTFT does not exist
- Moreover, use of *z*-transform techniques permits simple algebraic manipulations

  代数运算

Definition and Properties

- Consequently, z-transform has become an important tool in the analysis and design of digital filters
- For a given sequence $g[n]$, its z-transform $G(z)$ is defined as

$$\boxed{G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n}}$$

where $z = \mathrm{Re}(z) + j\,\mathrm{Im}(z)$ is a complex variable

Definition and Properties

- If we let $z = r e^{j\omega}$, then the *z*-transform reduces to

$$G(r e^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

- The above can be interpreted as the DTFT of the modified sequence $\{g[n] r^{-n}\}$
- For $r = 1$ (i.e., $|z| = 1$), *z*-transform reduces to its DTFT, provided the latter exists

Definition and Properties

- The 周线, 等高线 contour $|z| = 1$ is a circle in the z-plane of unity radius and is called the ***unit circle***
- Like the DTFT, there are conditions on the convergence of the infinite series

$$\sum_{n=-\infty}^{\infty} g[n] z^{-n}$$

- For a given sequence, the set R of values of z for which its z-transform converges is called the ***region of convergence (ROC)***

Definition and Properties

- From our earlier discussion on the uniform convergence of the DTFT, it follows that the series

$$G(re^{j\omega}) = \sum_{n=-\infty}^{\infty} g[n]r^{-n}e^{-j\omega n}$$

converges if $\{g[n]r^{-n}\}$ is absolutely summable, i.e., if

$$\sum_{n=-\infty}^{\infty} \left| g[n]r^{-n} \right| < \infty$$

z-Transform

- If $\sum_{n=-\infty}^{\infty} |g[n]r^{-n}| < \infty$ for $r = \mathcal{R}_{g-}$ and $r = \mathcal{R}_{g+}$ with $0 \le \mathcal{R}_{g-} < \mathcal{R}_{g+} < \infty$ then the sequence $g[n]r^{-n}$ is absolutely summable for all values of r; that is,

$$\sum_{n=-\infty}^{\infty} |g[n]r^{-n}| < \infty$$

for all values of r in the range

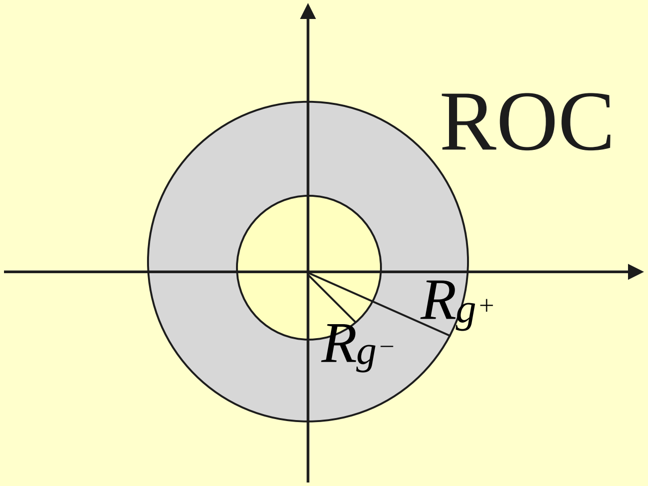
$$0 \le \mathcal{R}_{g-} \le r \le \mathcal{R}_{g+} < \infty$$

Definition and Properties

- In general, the ROC of a z-transform of a sequence $g[n]$ is an annular region of the z-plane:

$$R_{g^-} < |z| < R_{g^+}$$

where $0 \le R_{g^-} < R_{g^+} \le \infty$



Definition and Properties

- Example - Determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC
- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$
- The above power series converges to

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \text{for } |\alpha z^{-1}| < 1$$

- ROC is the annular region $|z| > |\alpha|$

Definition and Properties

- Example - The z-transform $\boldsymbol{\mu}(z)$ of the unit step sequence $\mu[n]$ can be obtained from

$$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha z^{-1}\right| < 1$$

by setting $\alpha = 1$:

$$\boldsymbol{\mu}(z) = \frac{1}{1-z^{-1}}, \quad \text{for } \left|z^{-1}\right| < 1$$

- ROC is the annular region $1 < |z| \leq \infty$

Definition and Properties

- Note: The unit step sequence $\mu[n]$ is not absolutely summable, and hence its DTFT does not converge uniformly

- Example - Consider the anti-causal sequence

$$y[n] = -\alpha^n \mu[-n-1]$$

Definition and Properties

- Its z-transform is given by

$$Y(z) = \sum_{n=-\infty}^{-1} -\alpha^{n} z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^{m}$$

$$= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^{m} = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha^{-1} z\right| < 1$$

- ROC is the annular region $|z| < |\alpha|$

Definition and Properties

- Note: The z-transforms of the two sequences $\alpha^n \mu[n]$ and $-\alpha^n \mu[-n-1]$ are identical even though the two parent sequences are different
- Only way a unique sequence can be associated with a z-transform is by specifying its ROC

Definition and Properties

- The DTFT $G(e^{j\omega})$ of a sequence $g[n]$ converges uniformly if and only if the ROC of the z-transform $G(z)$ of $g[n]$ includes the unit circle
- The existence of the DTFT does not always imply the existence of the z-transform

Definition and Properties

- Example - The finite energy sequence

$$h_{LP}[n] = \frac{\sin \omega_c n}{\pi n} = \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c n}{\pi}\right), \qquad -\infty < n < \infty$$

has a DTFT given by

$$H_{LP}(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

which converges in the mean-square sense

Definition and Properties

- However, $h_{LP}[n]$ does not have a z-transform as it is not absolutely summable for any value of r, i.e.
$$\sum_{n=-\infty}^{\infty} \left| h_{LP}[n]\, r^{-n} \right| = \infty \quad \forall\, r$$
- Some commonly used z-transform pairs are listed on the next slide

Commonly Used z-Transform Pairs

Sequence	z-Transform	ROC
$\delta[n]$	1	All values of z
$\mu[n]$	$\dfrac{1}{1 - z^{-1}}$	$\|z\| > 1$
$\alpha^n \mu[n]$	$\dfrac{1}{1 - \alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$(r^n \cos \omega_o n)\mu[n]$	$\dfrac{1 - (r \cos \omega_o) z^{-1}}{1 - (2r \cos \omega_o) z^{-1} + r^2 z^{-2}}$	$\|z\| > r$
$(r^n \sin \omega_o n)\mu[n]$	$\dfrac{(r \sin \omega_o) z^{-1}}{1 - (2r \cos \omega_o) z^{-1} + r^2 z^{-2}}$	$\|z\| > r$

Frequency response

$$\left(\sum_{k=0}^{N} d_k e^{-j\omega k}\right) Y(e^{j\omega}) = \left(\sum_{k=0}^{M} p_k e^{-j\omega k}\right) X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Rational *z*-Transforms

- In the case of LTI discrete-time systems we are concerned with in this course, all pertinent z-transforms are rational functions of z^{-1}
- That is, they are ratios of two polynomials in z^{-1}:

$$G(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + \cdots + p_{M-1} z^{-(M-1)} + p_M z^{-M}}{d_0 + d_1 z^{-1} + \cdots + d_{N-1} z^{-(N-1)} + d_N z^{-N}}$$

Rational *z*-Transforms

- The degree of the numerator polynomial $P(z)$ is M and the degree of the denominator polynomial $D(z)$ is N
- An alternate representation of a rational z-transform is as a ratio of two polynomials in z:

$$G(z) = z^{(N-M)} \frac{p_0 z^M + p_1 z^{M-1} + \cdots + p_{M-1} z + p_M}{d_0 z^N + d_1 z^{N-1} + \cdots + d_{N-1} z + d_N}$$

Rational *z*-Transforms

- A rational *z*-transform can be alternately written in factored form as

$$G(z) = \frac{p_0 \prod_{\ell=1}^{M} (1 - \xi_\ell z^{-1})}{d_0 \prod_{\ell=1}^{N} (1 - \lambda_\ell z^{-1})}$$

$$= z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

Rational z-Transforms

- At a root $z = \xi_\ell$ of the numerator polynomial $G(\xi_\ell) = 0$, and as a result, these values of z are known as the ***zeros*** of $G(z)$
- At a root $z = \lambda_\ell$ of the denominator polynomial $G(\lambda_\ell) \to \infty$, and as a result, these values of z are known as the ***poles*** of $G(z)$

Rational z-Transforms

- Consider

$$G(z) = z^{(N-M)} \frac{p_0 \prod_{\ell=1}^{M} (z - \xi_\ell)}{d_0 \prod_{\ell=1}^{N} (z - \lambda_\ell)}$$

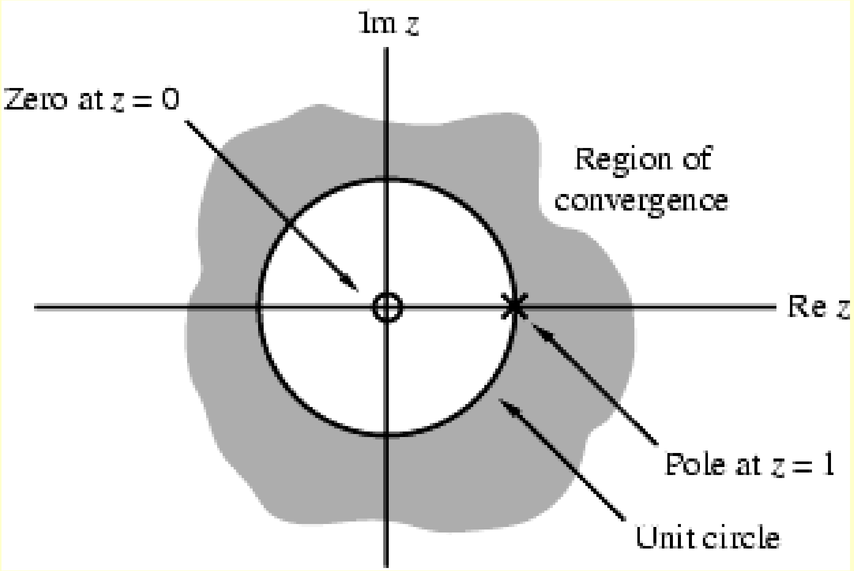
- Note $G(z)$ has M finite zeros and N finite poles
- If $N > M$ there are additional $N - M$ zeros at $z = 0$ (the origin in the z-plane)
- If $N < M$ there are additional $M - N$ poles at $z = 0$

Rational *z*-Transforms

- Example - The *z*-transform

$$\mu(z) = \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1$$

has a zero at $z = 0$ and a pole at $z = 1$

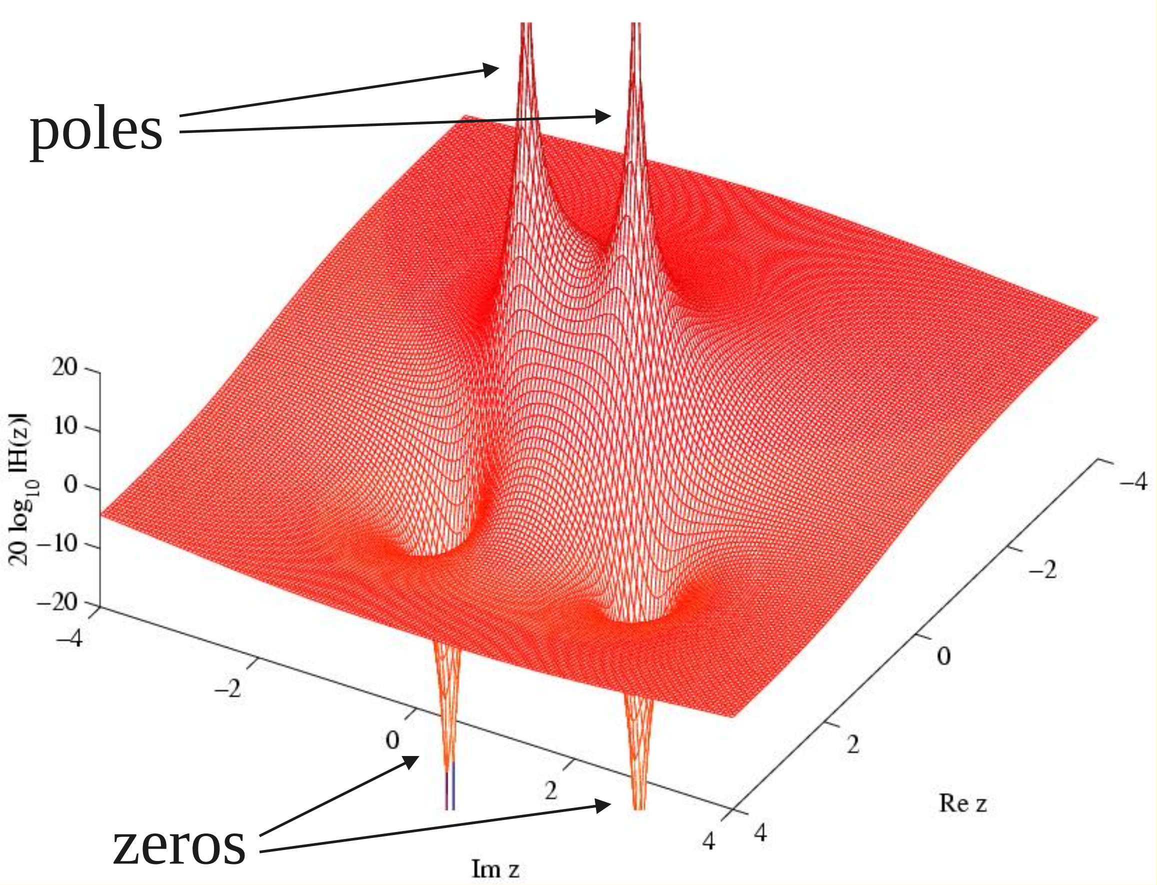


Rational z-Transforms

- A physical interpretation of the concepts of poles and zeros can be given by plotting the log-magnitude $20\log_{10}|G(z)|$ as shown on next slide for

$$G(z) = \frac{1-2.4z^{-1}+2.88z^{-2}}{1-0.8z^{-1}+0.64z^{-2}}$$

Rational *z*-Transforms



Rational z-Transforms

- Observe that the magnitude plot exhibits very large peaks around the points $z = 0.4 \pm j0.6928$ which are the poles of $G(z)$
- It also exhibits very narrow and deep wells around the location of the zeros at

$$z = 1.2 \pm j1.2$$

ROC of a Rational *z*-Transform

- ROC of a *z*-transform is an important concept
- Without the knowledge of the ROC, there is no unique relationship between a sequence and its *z*-transform
- Hence, the *z*-transform must always be specified with its ROC

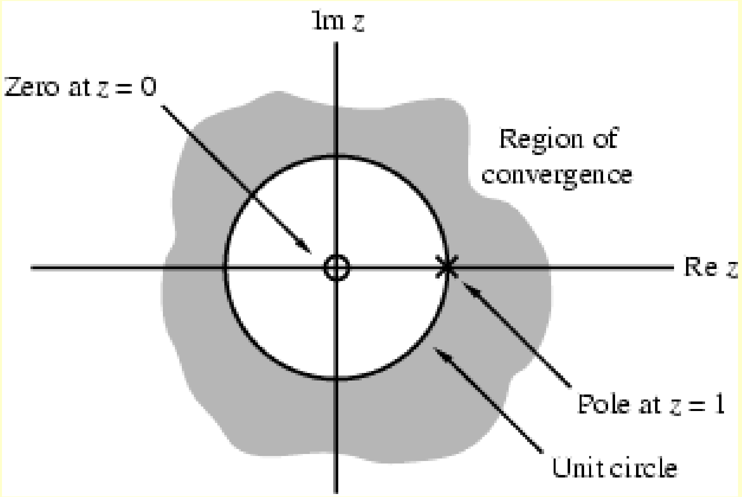
ROC of a Rational *z*-Transform

- Moreover, if the ROC of a *z*-transform includes the unit circle, the DTFT of the sequence is obtained by simply evaluating the *z*-transform on the unit circle
- There is a relationship between the ROC of the *z*-transform of the impulse response of a causal LTI discrete-time system and its BIBO stability

ROC of a Rational z-Transform

- The ROC of a rational z-transform is bounded by the locations of its poles
- To understand the relationship between the poles and the ROC, it is instructive to examine the pole-zero plot of a z-transform
- Consider again the pole-zero plot of the z-transform $\mu(z)$

ROC of a Rational *z*-Transform



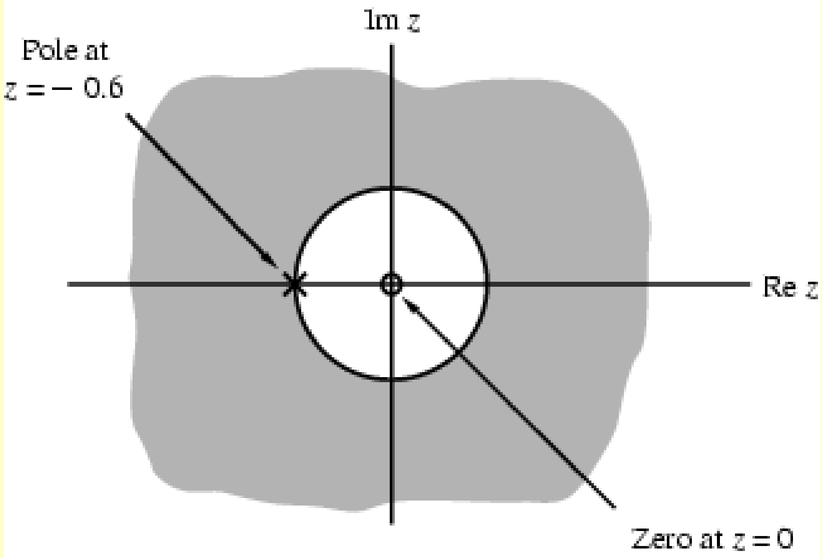
- In this plot, the ROC, shown as the shaded area, is the region of the *z*-plane just outside the circle centered at the origin and going through the pole at $z = 1$

ROC of a Rational z-Transform

- Example - The z-transform $H(z)$ of the sequence is given by $h[n] = (-0.6)^n \mu[n]$

$$H(z) = \frac{1}{1 + 0.6 z^{-1}},$$

$$|z| > 0.6$$



- Here the ROC is just outside the circle going through the point $z = -0.6$

ROC of a Rational z-Transform

- A sequence can be one of the following types: ***finite-length, right-sided, left-sided and two-sided***
- In general, the ROC depends on the type of the sequence of interest

ROC of a Rational z-Transform

- Example - Consider a ***finite-length sequence*** $g[n]$ defined for $-M \le n \le N$, where M and N are non-negative integers and $|g[n]| < \infty$
- Its z-transform is given by

$$G(z) = \sum_{n=-M}^{N} g[n]\, z^{-n} = \frac{\sum_{n=0}^{N+M} g[n-M]\, z^{N+M-n}}{z^{N}}$$

ROC of a Rational *z*-Transform

- Note: $G(z)$ has M poles at $z = \infty$ and N poles at $z = 0$
- As can be seen from the expression for $G(z)$, the z-transform of a finite-length bounded sequence converges everywhere in the z-plane except possibly at $z = 0$ and/or at $z = \infty$

ROC of a Rational z-Transform

- Example - A ***right-sided sequence*** with nonzero sample values for $n \geq 0$ is sometimes called a ***causal sequence***
- Consider a causal sequence $u_1[n]$
- Its z-transform is given by

$$U_1(z) = \sum_{n=0}^{\infty} u_1[n] z^{-n}$$

- Example - Determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC
- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$
- The above power series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } \left| \alpha z^{-1} \right| < 1$$

- ROC is the annular region $|z| > |\alpha|$

ROC of a Rational *z*-Transform

- It can be shown that $U_1(z)$ converges exterior to a circle $|z| = R_1$, including the point $z = \infty$
- On the other hand, a right-sided sequence $u_2[n]$ with nonzero sample values only for $n \geq -M$ with M nonnegative has a *z*-transform $U_2(z)$ with M poles at $z = \infty$
- The ROC of $U_2(z)$ is exterior to a circle $|z| = R_2$, excluding the point $z = \infty$

ROC of a Rational z-Transform

- Example - A ***left-sided sequence*** with nonzero sample values for $n \leq 0$ is sometimes called a ***anti-causal sequence***
- Consider an anti-causal sequence $v_1[n]$
- Its z-transform is given by

$$V_1(z) = \sum_{n=-\infty}^{0} v_1[n] z^{-n}$$

$$y[n] = -\alpha^n \mu[-n-1]$$

- Its z-transform is given by

$$Y(z) = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{m=1}^{\infty} \alpha^{-m} z^m$$

$$= -\alpha^{-1} z \sum_{m=0}^{\infty} \alpha^{-m} z^m = -\frac{\alpha^{-1} z}{1-\alpha^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}}, \quad \text{for } \left|\alpha^{-1} z\right| < 1$$

- ROC is the annular region

$$|z| < |\alpha|$$

ROC of a Rational *z*-Transform

- It can be shown that $V_1(z)$ converges interior to a circle $|z| = R_3$, including the point $z = 0$
- On the other hand, a left-sided sequence with nonzero sample values only for $n \le N$ with N nonnegative has a *z*-transform $V_2(z)$ with N poles at $z = 0$
- The ROC of $V_2(z)$ is interior to a circle $|z| = R_4$, excluding the point $z = 0$

ROC of a Rational z-Transform

- Example - The z-transform of a ***two-sided sequence*** $w[n]$ can be expressed as

$$W(z) = \sum_{n=-\infty}^{\infty} w[n] z^{-n} = \sum_{n=0}^{\infty} w[n] z^{-n} + \sum_{n=-\infty}^{-1} w[n] z^{-n}$$

- The first term on the RHS, $\sum_{n=0}^{\infty} w[n] z^{-n}$, can be interpreted as the z-transform of a right-sided sequence and it thus converges exterior to the circle $|z| = R_5$

ROC of a Rational *z*-Transform

- The second term on the RHS, $\sum_{n=-\infty}^{-1} w[n]z^{-n}$, can be interpreted as the *z*-transform of a left-sided sequence and it thus converges interior to the circle $|z| = R_6$
- If $R_5 < R_6$, there is an overlapping ROC given by $R_5 < |z| < R_6$
- If $R_5 > R_6$, there is no overlap and the *z*-transform does not exist

ROC of a Rational z-Transform

- Example - Consider the two-sided sequence

$$u[n] = \alpha^n$$

  where α can be either real or complex

- Its z-transform is given by

$$U(z) = \sum_{n=-\infty}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} + \sum_{n=-\infty}^{-1} \alpha^n z^{-n}$$

- The first term on the RHS converges for $|z| > |\alpha|$, whereas the second term converges for $|z| < |\alpha|$

ROC of a Rational *z*-Transform

- There is no overlap between these two regions
- Hence, the *z*-transform of $u[n] = \alpha^n$ does not exist!

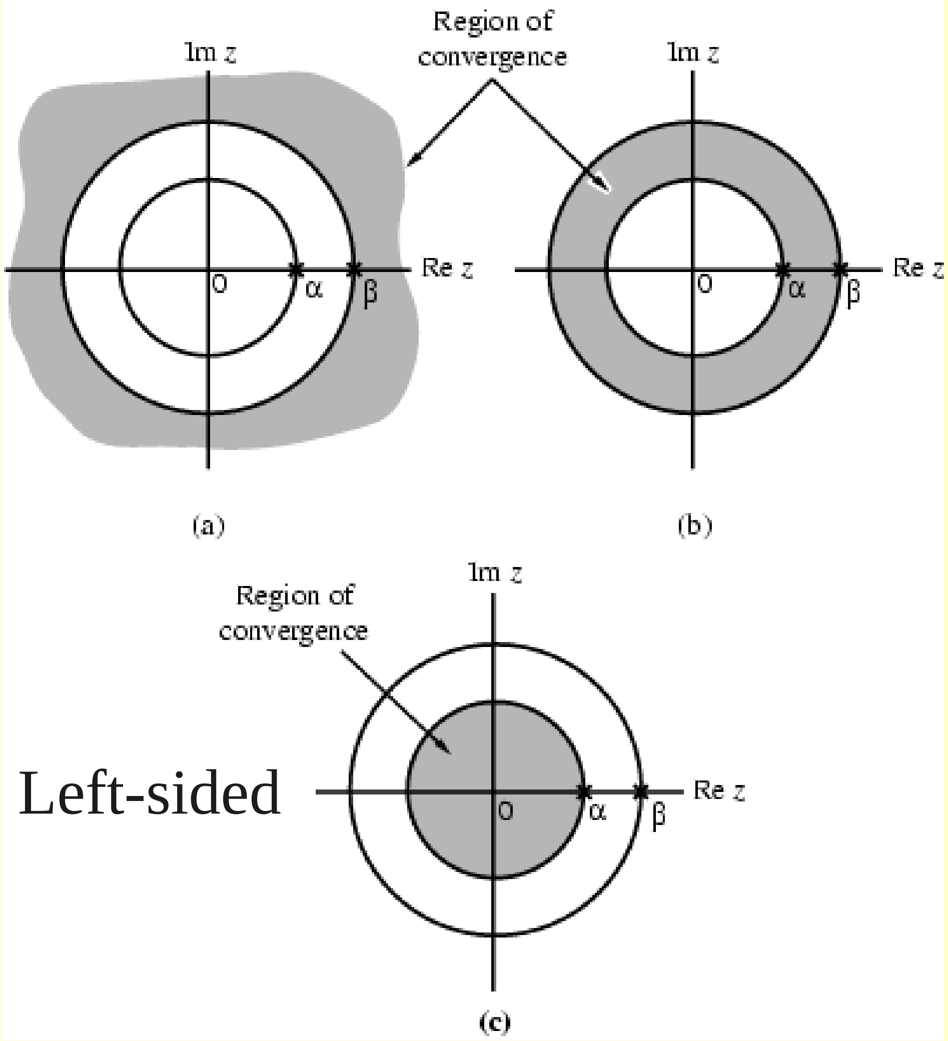
ROC of a Rational z-Transform

- The ROC of a rational z-transform cannot contain any poles and is bounded by the poles
- As an example, assume that a rational z-transform $X(z)$ has two simple poles at $z = \alpha$ and $z = \beta$ with $|\alpha| < |\beta|$
- There are three possible ROCs associated with $X(z)$

ROC of a Rational *z*-Transform

Right-sided

Two-sided



Left-sided

ROC of a Rational z-Transform

- In general, if the rational z-transform has N poles with R distinct magnitudes, then it has $R+1$ ROCs
- Thus, there are $R+1$ distinct sequences with the same z-transform
- Hence, a rational z-transform with a specified ROC has a unique sequence as its inverse z-transform

ROC of a Rational *z*-Transform

- The ROC of a rational *z*-transform can be easily determined using MATLAB

  `[z,p,k] = tf2zp(num,den)`

  determines the zeros, poles, and the gain constant of a rational *z*-transform with the numerator coefficients specified by the vector `num` and the denominator coefficients specified by the vector `den`

ROC of a Rational *z*-Transform

[num,den] = zp2tf(z,p,k) implements the reverse process

- The factored form of the *z*-transform can be obtained using sos = zp2sos(z,p,k)
- The above statement computes the coefficients of each second-order factor given as an $L \times 6$ matrix sos

ROC of a Rational *z*-Transform

$$sos = \begin{bmatrix} b_{01} & b_{11} & b_{21} & a_{01} & a_{11} & a_{12} \\ b_{02} & b_{12} & b_{22} & a_{02} & a_{12} & a_{22} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{0L} & b_{1L} & b_{2L} & a_{0L} & a_{1L} & a_{2L} \end{bmatrix}$$

where

$$G(z) = \prod_{k=1}^{L} \frac{b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2}}{a_{0k} + a_{1k} z^{-1} + a_{2k} z^{-2}}$$

ROC of a Rational z-Transform

- The ***pole-zero plot*** is determined using the function zplane
- The z-transform can be either described in terms of its zeros and poles:
  zplane(zeros,poles)

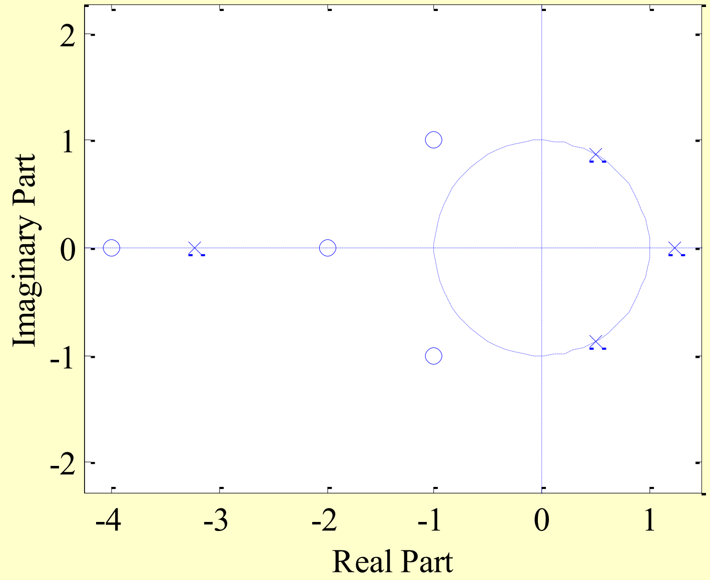
  or, it can be described in terms of its numerator and denominator coefficients:
  zplane(num,den)

ROC of a Rational z-Transform

- Example - The pole-zero plot of

$$G(z) = \frac{2z^4 + 16z^3 + 44z^2 + 56z + 32}{3z^4 + 3z^3 - 15z^2 + 18z - 12}$$

obtained using MATLAB is shown below



$\times$ – pole

o – zero

Inverse z-Transform

- ***General Expression***: Recall that, for $z = r\,e^{j\omega}$, the z-transform $G(z)$ given by

  $$G(z) = \sum_{n=-\infty}^{\infty} g[n] z^{-n} = \sum_{n=-\infty}^{\infty} g[n] r^{-n} e^{-j\omega n}$$

  is merely the DTFT of the modified sequence $g[n]r^{-n}$

- Accordingly, the inverse DTFT is thus given by

$$g[n]r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} G(re^{j\omega})\, e^{j\omega n} d\omega$$

General Expression

- By making a change of variable $z = r\,e^{j\omega}$, the previous equation can be converted into a contour integral given by

$$\boxed{g[n] = \frac{1}{2\pi j}\oint_{C'} G(z)\,z^{n-1}dz}$$

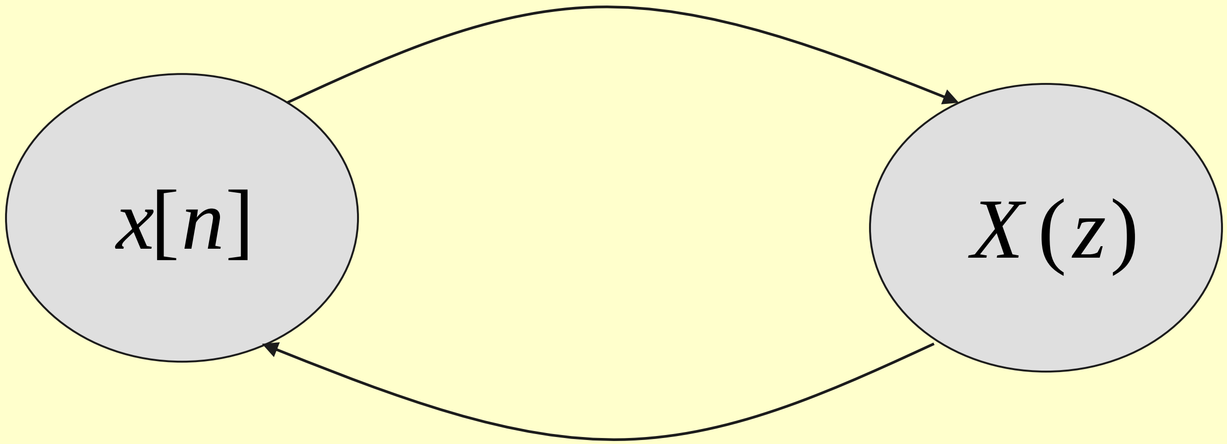
where C' is a counterclockwise contour of integration defined by $|z| = r$

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

z-Transform: analysis equation

time domain



z-domain

Inverse z-Transform: synthesis equation

$$x[n] = \frac{1}{2\pi j} \oint_{C'} X(z)\, z^{n-1} dz$$

General Expression

- But the integral remains unchanged when it is replaced with any contour C encircling the point $z = 0$ in the ROC of $G(z)$
- The contour integral can be evaluated using the ***Cauchy's residue theorem*** resulting in

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$$g[n] = \sum_{z_i \text{ inside } C} \mathrm{Res}(G(z)z^{n-1})_{|z=z_i}$$

- The above equation needs to be evaluated at all values of n and is not pursued here

Inverse Transform by Partial-Fraction Expansion

- A rational z-transform $G(z)$ with a causal inverse transform $g[n]$ has a ROC that is exterior to a circle
- Here it is more convenient to express $G(z)$ in a partial-fraction expansion form and then determine $g[n]$ by summing the inverse transform of the individual simpler terms in the expansion

Inverse Transform by Partial-Fraction Expansion

- A rational $G(z)$ can be expressed as

$$G(z) = \frac{P(z)}{D(z)} = \frac{\sum_{i=0}^{M} p_i z^{-i}}{\sum_{i=0}^{N} d_i z^{-i}}$$

- If $M \geq N$ then $G(z)$ can be re-expressed as

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \frac{P_1(z)}{D(z)}$$

where the degree of $P_1(z)$ is less than N

Inverse Transform by Partial-Fraction Expansion

- The rational function $P_1(z)/D(z)$ is called a ***proper fraction*** 真分式
- Example - Consider

$$G(z) = \frac{2 + 0.8z^{-1} + 0.5z^{-2} + 0.3z^{-3}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- By long division we arrive at

$$G(z) = -3.5 + 1.5z^{-1} + \frac{5.5 + 2.1z^{-1}}{1 + 0.8z^{-1} + 0.2z^{-2}}$$

- Example - Determine the z-transform $X(z)$ of the causal sequence $x[n] = \alpha^n \mu[n]$ and its ROC
- Now $X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu[n] z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n}$
- The above power series converges to

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad \text{for } \left| \alpha z^{-1} \right| < 1$$

- ROC is the annular region $|z| > |\alpha|$

Inverse Transform by Partial-Fraction Expansion

- Example - Let the z-transform $H(z)$ of a causal sequence $h[n]$ be given by

$$H(z) = \frac{z(z+2)}{(z-0.2)(z+0.6)} = \frac{1+2z^{-1}}{(1-0.2z^{-1})(1+0.6z^{-1})}$$

- A partial-fraction expansion of $H(z)$ is then of the form

$$H(z) = \frac{\rho_1}{1-0.2z^{-1}} + \frac{\rho_2}{1+0.6z^{-1}}$$

Inverse Transform by Partial-Fraction Expansion

- Now

$$\rho_1 = (1 - 0.2\,z^{-1})H(z)\Big|_{z=0.2} = \left.\frac{1+2\,z^{-1}}{1+0.6\,z^{-1}}\right|_{z=0.2} = 2.75$$

and

$$\rho_2 = (1 + 0.6\,z^{-1})H(z)\Big|_{z=-0.6} = \left.\frac{1+2\,z^{-1}}{1-0.2\,z^{-1}}\right|_{z=-0.6} = -1.75$$

Inverse Transform by Partial-Fraction Expansion

- Hence

$$H(z) = \frac{2.75}{1-0.2z^{-1}} - \frac{1.75}{1+0.6z^{-1}}$$

- The inverse transform of the above is therefore given by

$$h[n] = 2.75(0.2)^n \mu[n] - 1.75(-0.6)^n \mu[n]$$

Inverse Transform by Partial-Fraction Expansion

- ***Simple Poles:*** In most practical cases, the rational z-transform of interest $G(z)$ is a proper fraction with simple poles
- Let the poles of G(z) be at $z = \lambda_k, 1 \le k \le N$
- A partial-fraction expansion of $G(z)$ is then of the form

$$G(z) = \sum_{\ell=1}^{N} \left(\frac{\rho_\ell}{1 - \lambda_\ell z^{-1}} \right)$$

Inverse Transform by Partial-Fraction Expansion

- The constants ρ_ℓ in the partial-fraction expansion are called the ***residues*** and are given by

$$\boxed{\rho_\ell = (1-\lambda_\ell z^{-1})G(z)\Big|_{z=\lambda_\ell}}$$

- Each term of the sum in partial-fraction expansion has a ROC given by $|z| > |\lambda_\ell|$ and, thus, has an inverse transform of the form $\boxed{\rho_\ell(\lambda_\ell)^n \mu[n]}$

Inverse Transform by Partial-Fraction Expansion

- Therefore, the inverse transform $g[n]$ of $G(z)$ is given by

$$\boxed{g[n] = \sum_{\ell=1}^{N} \rho_\ell (\lambda_\ell)^n \mu[n]}$$

- Note: The above approach with a slight modification can also be used to determine the inverse of a rational z-transform of a noncausal sequence

Inverse Transform by Partial-Fraction Expansion

- ***Multiple Poles***: If $G(z)$ has multiple poles, the partial-fraction expansion is of slightly different form
- Let the pole at $z = \nu$ be of multiplicity L and the remaining $N - L$ poles be simple and at $z = \lambda_\ell, 1 \le \ell \le N - L$

Inverse Transform by Partial-Fraction Expansion

- Then the partial-fraction expansion of $G(z)$ is of the form

$$G(z) = \sum_{\ell=0}^{M-N} \eta_\ell z^{-\ell} + \sum_{\ell=1}^{N-L} \frac{\rho_\ell}{1-\lambda_\ell z^{-1}} + \sum_{i=1}^{L} \frac{\gamma_i}{(1-\nu z^{-1})^i}$$

where the constants γ_i are computed using

$$\gamma_i = \frac{1}{(L-i)!(-\nu)^{L-i}} \frac{d^{L-i}}{d(z^{-1})^{L-i}} \left[(1-\nu z^{-1})^L G(z)\right]_{z=\nu},$$
$$1 \le i \le L$$

- The residues ρ_ℓ are calculated as before

Partial-Fraction Expansion Using MATLAB

[r,p,k]= residuez(num,den) develops the partial-fraction expansion of a rational z-transform with numerator and denominator coefficients given by vectors num and den

- Vector r contains the residues
- Vector p contains the poles
- Vector k contains the constants η_ℓ

Partial-Fraction Expansion Using MATLAB

`[num,den]=residuez(r,p,k)` converts a z-transform expressed in a partial-fraction expansion form to its rational form

Inverse z-Transform via Long Division

- The z-transform $G(z)$ of a causal sequence $\{g[n]\}$ can be expanded in a power series in z^{-1}
- In the series expansion, the coefficient multiplying the term z^{-n} is then the n-th sample $g[n]$
- For a rational z-transform expressed as a ratio of polynomials in z^{-1}, the power series expansion can be obtained by long division

Inverse z-Transform via Long Division

- Example - Consider

$$H(z)=\frac{1+2z^{-1}}{1+0.4z^{-1}-0.12z^{-2}}$$

- Long division of the numerator by the denominator yields

$$H(z)=1+1.6z^{-1}-0.52z^{-2}+0.4z^{-3}-0.2224z^{-4}+\cdots$$

- As a result

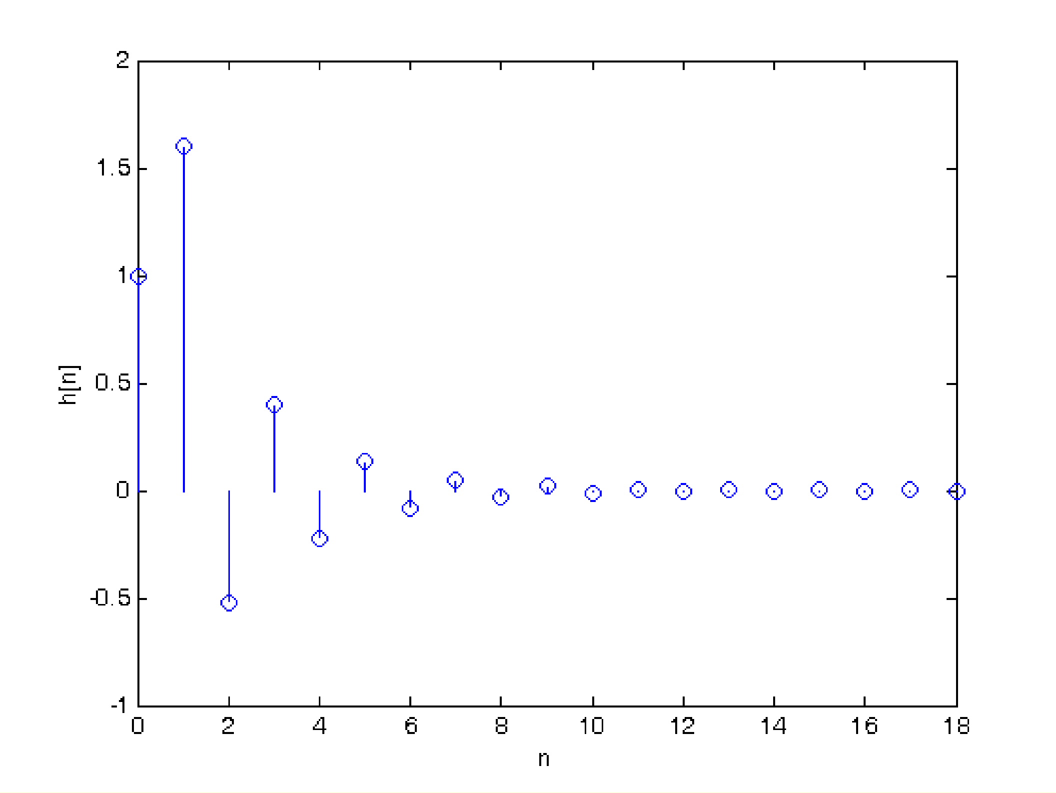
$$\{h[n]\}=\{\underset{\uparrow}{1}\quad 1.6\quad -0.52\quad 0.4\quad -0.2224\quad \cdots\},\quad n\geq 0$$

Inverse z-Transform Using MATLAB

- The function impz can be used to find the inverse of a rational z-transform $G(z)$
- The function computes the coefficients of the power series expansion of $G(z)$
- The number of coefficients can either be user specified or determined automatically

Inverse z-Transform Using MATLAB

```
>> num=[1 2];
>> den=[1 0.4 -0.12];
>> [h,t]=impz(num,den);
>> figure(1)
>> stem(t,h)
>> xlabel('n')
>> ylabel('h[n]')
```



$$h[n]=[1.0000 \quad 1.6000 \quad -0.5200 \quad 0.4000 \quad -0.2224 \ \ldots]$$

z-Transform Properties

Property	Sequence	z-Transform	ROC
	$g[n]$ $h[n]$	$G(z)$ $H(z)$	$\mathcal{R}_g$ $\mathcal{R}_h$
Conjugation	$g^*[n]$	$G^*(z^*)$	$\mathcal{R}_g$
Time-reversal	$g[-n]$	$G(1/z)$	$1/\mathcal{R}_g$
Linearity	$\alpha g[n] + \beta h[n]$	$\alpha G(z) + \beta H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Time-shifting	$g[n - n_o]$	$z^{-n_o} G(z)$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Multiplication by an exponential sequence	$\alpha^n g[n]$	$G(z/\alpha)$	$\lvert\alpha\rvert\mathcal{R}_g$
Differentiation of $G(z)$	$ng[n]$	$-z\dfrac{dG(z)}{dz}$	$\mathcal{R}_g$, except possibly the point $z = 0$ or ∞
Convolution	$g[n] \circledast h[n]$	$G(z)H(z)$	Includes $\mathcal{R}_g \cap \mathcal{R}_h$
Modulation	$g[n]h[n]$	$\frac{1}{2\pi j}\oint_C G(v)H(z/v)v^{-1}\,dv$	Includes $\mathcal{R}_g\mathcal{R}_h$
Parseval's relation	$\sum_{n=-\infty}^{\infty} g[n]h^*[n] = \frac{1}{2\pi j}\oint_C G(v)H^*(1/v^*)v^{-1}\,dv$		

Note: If $\mathcal{R}_g$ denotes the region $R_{g^-} < |z| < R_{g^+}$ and $\mathcal{R}_h$ denotes the region $R_{h^-} < |z| < R_{h^+}$, then $1/\mathcal{R}_g$ denotes the region $1/R_{g^+} < |z| < 1/R_{g^-}$ and $\mathcal{R}_g\mathcal{R}_h$ denotes the region $R_{g^-}R_{h^-} < |z| < R_{g^+}R_{h^+}$.

z-Transform Properties

- Example - Consider the two-sided sequence

$$v[n] = \alpha^n \mu[n] - \beta^n \mu[-n-1]$$

- Let $x[n] = \alpha^n \mu[n]$ and $y[n] = -\beta^n \mu[-n-1]$ with $X(z)$ and $Y(z)$ denoting, respectively, their z-transforms
- Now $$X(z) = \frac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|$$

and $$Y(z) = \frac{1}{1-\beta z^{-1}}, \quad |z| < |\beta|$$

z-Transform Properties

- Using the linearity property we arrive at

$$V(z) = X(z) + Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}$$

- The ROC of $V(z)$ is given by the overlap regions of $|z| > |\alpha|$ and $|z| < |\beta|$
- If $|\alpha| < |\beta|$, then there is an overlap and the ROC is an annular region $|\alpha| < |z| < |\beta|$
- If $|\alpha| > |\beta|$, then there is no overlap and $V(z)$ does not exist

z-Transform Properties

- Example - Determine the z-transform and its ROC of the causal sequence

$$x[n] = r^n (\cos \omega_o n) \mu[n]$$

- We can express $x[n] = v[n] + v^*[n]$ where

$$v[n] = \tfrac{1}{2} r^n e^{j\omega_o n} \mu[n] = \tfrac{1}{2} \alpha^n \mu[n]$$

- The z-transform of $v[n]$ is given by

$$V(z) = \tfrac{1}{2} \cdot \frac{1}{1 - \alpha z^{-1}} = \tfrac{1}{2} \cdot \frac{1}{1 - r e^{j\omega_o} z^{-1}}, \qquad |z| > |\alpha| = r$$

z-Transform Properties

- Using the conjugation property we obtain the z-transform of $v^*[n]$ as

$$V^*(z^*) = \frac{1}{2} \cdot \frac{1}{1 - \alpha^* z^{-1}} = \frac{1}{2} \cdot \frac{1}{1 - r e^{-j\omega_o} z^{-1}},$$

$$|z| > |\alpha|$$

- Finally, using the linearity property we get

$$X(z) = V(z) + V^*(z^*)$$

$$= \frac{1}{2}\left(\frac{1}{1 - r e^{j\omega_o} z^{-1}} + \frac{1}{1 - r e^{-j\omega_o} z^{-1}}\right)$$

z-Transform Properties

- or,

$$X(z) = \frac{1-(r\cos\omega_o)z^{-1}}{1-(2r\cos\omega_o)z^{-1}+r^2z^{-2}}, \quad |z|>r$$

- Example - Determine the z-transform $Y(z)$ and the ROC of the sequence

$$y[n] = (n+1)\alpha^n\mu[n]$$

- We can write $y[n] = n\,x[n] + x[n]$ where

$$x[n] = \alpha^n\mu[n]$$

z-Transform Properties

- Now, the z-transform $X(z)$ of $x[n] = \alpha^n \mu[n]$ is given by

$$X(z) = \frac{1}{1 - \alpha z^{-1}}, \quad |z| > |\alpha|$$

- Using the differentiation property, we arrive at the z-transform of $n\,x[n]$ as

$$-z\frac{d\,X(z)}{dz} = \frac{\alpha z^{-1}}{(1 - \alpha z^{-1})^2}, \qquad |z| > |\alpha|$$

z-Transform Properties

- Using the linearity property we finally obtain

$$Y(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$$

$$= \frac{1}{(1-\alpha z^{-1})^2}, \quad |z| > |\alpha|$$

Linear Convolution Using z-Transform

- Let $\{x[n]\}, 0 \le n \le L,$ denote a finite-length sequence of length $L+1$
- Let $\{h[n]\}, 0 \le n \le M$, denote a finite-length sequence of length $M+1$
- We shall evaluate $y[n] = x[n] \circledast h[\mathrm{n}]$ using *z*-transform
- Note: $\{y[n]\}$ is a sequence of length $L + M + 1$

Linear Convolution Using z-Transform

- Let $X(z)$ denote the z-transform of $\{x[n]\}$ which is a polynomial of degree L in z^{-1}, i.e.,

  $$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots + x[L]z^{-L}$$

- Let $H(z)$ denote the z-transform of $\{h[n]\}$ which is a polynomial of degree M in z^{-1}, i.e.,

  $$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \cdots + h[M]z^{-M}$$

Linear Convolution Using z-Transform

- From the convolution property of the *z*-transform it follows that the *z*-transform of $\{y[n]\}$ is simply given by $Y(z) = X(z)H(z)$ which is a polynomial of degree $L+M$ in z^{-1} i.e.,

$$Y(z) = y[0] + y[1]z^{-1} + y[2]z^{-2} + \cdots + y[L+M]z^{-(L+M)}$$

Linear Convolution Using z-Transform

where

$$y[n] = \sum_{k=0}^{L+M} x[k]h[n-k], \;\; 0 \le n \le L+M$$

- In the above we have assumed

$$x[n] = 0 \;\; \text{for } n > L$$

$$h[n] = 0 \;\; \text{for } n > M$$

Linear Convolution Using z-Transform

- **Example** – $X(z) = -2 + z^{-2} - z^{-3} + 3z^{-4}$

  $H(z) = 1 + 2z^{-1} - z^{-3}$

- Therefore

$$Y(z) = (-2 + z^{-2} - z^{-3} + 3z^{-4})(1 + 2z^{-2} - z^{-3})$$

$$= -2 + z^{-2} - z^{-3} + 3z^{-4} - 4z^{-1} + 2z^{-3}$$

$$- 2z^{-4} + 6z^{-5} + 2z^{-3} - z^{-5} + z^{-6} - 3z^{-7}$$

Linear Convolution Using z-Transform

$$= -2 + z^{-2} - z^{-3} + 3z^{-4} - 4z^{-1} + 2z^{-3}$$
$$- 2z^{-4} + 6z^{-5} + 2z^{-3} - z^{-5} + z^{-6} - 3z^{-7}$$

$$= -2 - 4z^{-1} + z^{-2} + (2z^{-3} + 2z^{-3} - z^{-3})$$
$$+ (3z^{-4} - 2z^{-4}) + (6z^{-5} - z^{-5}) + z^{-6} - 3z^{-7}$$

$$= -2 - 4z^{-1} + z^{-2} + 3z^{-3} + z^{-4}$$
$$+ 5z^{-5} + z^{-6} - 3z^{-7}$$

Linear Convolution Using z-Transform

- Hence

$$\{y[n]\} = \{-2, \quad -4, \quad 1, \quad 3, \quad 1, \quad 5, \quad 1, \quad -3\}$$

Circular Convolution Using z-Transform

- Let $\{x[n]\}$ and $\{h[n]\}$ be two length-N sequences defined for $0 \le n \le N-1$ with $X(z)$ and $H(z)$ denoting their z-transforms
- Let $y_C[n] = x[n] \circledN h[n]$ denote the N-point circular convolution of $x[n]$ and $h[n]$
- Let $y_L[n] = x[n] \circledast h[n]$ denote the linear convolution of $x[n]$ and $h[n]$

Circular Convolution Using z-Transform

- Let $Y_C(z)$ and $Y_L(z)$ denote the z-transforms of $y_C[n]$ and $y_L[n]$
- It can be shown that

$$Y_C(z) = \langle Y_L(z) \rangle_{(z^{-N}-1)}$$

- The modulo operation with respect to $z^{-N}-1$ is taken by setting $z^{-N}=1$

Circular Convolution Using z-Transform

- **Example** –

$$G(z) = g[0] + g[1]z^{-1} + g[2]z^{-2} + g[3]z^{-3}$$

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + h[3]z^{-3}$$

- Then

$$Y_L(z) = G(z)H(z)$$

$$= y_L[0] + y_L[1]z^{-1} + y_L[2]z^{-2} + y_L[3]z^{-3}$$

$$+ y_L[4]z^{-4} + y_L[5]z^{-5} + y_L[6]z^{-6}$$

Circular Convolution Using z-Transform

where

$$y_L[0] = g[0]h[0]$$

$$y_L[1] = g[0]h[1] + g[1]h[0]$$

$$y_L[2] = g[0]h[2] + g[1]h[1] + g[2]h[0]$$

$$y_L[3] = g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]$$

$$y_L[4] = g[1]h[3] + g[2]h[2] + g[3]h[1]$$

$$y_L[5] = g[2]h[3] + g[3]h[2]$$

$$y_L[6] = g[3]h[3]$$

Circular Convolution Using z-Transform

- Now $Y_C(z) = \langle Y_L(z) \rangle_{(z^{-4}-1)}$

$$= y_L[0] + y_L[1]z^{-1} + y_L[2]z^{-2} + y_L[3]z^{-3} + y_L[4] + y_L[5]z^{-1} + y_L[6]z^{-2}$$

$$= g[0]h[0] + \left(g[0]h[1] + g[1]h[0]\right)z^{-1} + \left(g[0]h[2] + g[1]h[1] + g[2]h[0]\right)z^{-2} + \left(g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]\right)z^{-3} + \left(g[1]h[3] + g[2]h[2] + g[3]h[1]\right) + \left(g[2]h[3] + g[3]h[2]\right)z^{-1} + g[3]h[3]z^{-2}$$

Circular Convolution Using z-Transform

$$= \underbrace{g[0]h[0] + g[1]h[3] + g[2]h[2] + g[3]h[1]}_{y_C[0]}$$

$$+ \underbrace{\left(g[0]h[1] + g[1]h[0] + g[2]h[3] + g[3]h[2]\right)}_{y_C[1]} z^{-1}$$

$$+ \underbrace{\left(g[0]h[2] + g[1]h[1] + g[2]h[0] + g[3]h[3]\right)}_{y_C[2]} z^{-2}$$

$$+ \underbrace{\left(g[0]h[3] + g[1]h[2] + g[2]h[1] + g[3]h[0]\right)}_{y_C[3]} z^{-3}$$

LTI Discrete-Time Systems in the Transform Domain

- An LTI discrete-time system is completely characterized in the time-domain by its implse response sequence {h[n]}
- Thus, the transform-domain representation of a discrete-time signal can also be equally applied to the transform-domain representation of an LTI discrete-time system

LTI Discrete-Time Systems in the Transform Domain

- **Such transform-domain representations provide additional insight into the behavior of such systems**
- **It is easier to design and implement these systems in the transform-domain for certain applications**
- **We consider now the use of the DTFT and the *z*-transform in developing the transform-domain representations of an LTI system**

LTI Discrete-Time Systems in the Transform Domain

- **In this course we shall be concerned with LTI discrete-time systems characterized by linear constant coefficient difference equations of the form:**

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

LTI Discrete-Time Systems in the Transform Domain

- **Applying the z-transform to both sides of the difference equation and making use of the linearity and the time-invariance properties we arrive at**

$$\sum_{k=0}^{N} d_k z^{-k} Y(z) = \sum_{k=0}^{M} p_k z^{-k} X(z)$$

where Y(z) and X(z) denote the z-transforms of y[n] and x[n] with associated ROCs, respectively

LTI Discrete-Time Systems in the Transform Domain

- **A more convenient form of the *z*-domain representation of the difference equation is given by**

$$\left(\sum_{k=0}^{N} d_k z^{-k}\right) Y(z) = \left(\sum_{k=0}^{M} p_k z^{-k}\right) X(z)$$

The Transfer Function

传输函数

- A generalization of the frequency response function
- The convolution sum description of an LTI discrete-time system with an impulse response $h[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

Definition

- Taking the *z*-transforms of both sides we get

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} h[k]x[n-k] \right) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{n=-\infty}^{\infty} x[n-k]z^{-n} \right)$$

$$= \sum_{k=-\infty}^{\infty} h[k] \left(\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-(\ell+k)} \right)$$

Definition

- Or, $Y(z) = \sum_{k=-\infty}^{\infty} h[k] \underbrace{\left(\sum_{\ell=-\infty}^{\infty} x[\ell] z^{-\ell} \right)}_{X(z)} z^{-k}$
- Therefore, $Y(z) = \underbrace{\left(\sum_{k=-\infty}^{\infty} h[k] z^{-k} \right)}_{H(z)} X(z)$
- Thus, $Y(z) = H(z)X(z)$

Definition

- Hence,

$$H(z) = Y(z) / X(z)$$

- The function $H(z)$, which is the z-transform of the impulse response $h[n]$ of the LTI system, is called the ***transfer function*** or the ***system function***
- The inverse z-transform of the transfer function $H(z)$ yields the impulse response $h[n]$

Transfer Function Expression

- Consider an LTI discrete-time system characterized by a difference equation

$$\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]$$

- Its transfer function is obtained by taking the *z*-transform of both sides of the above equation
- Thus

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

Transfer Function Expression

- Or, equivalently as

$$H(z) = z^{(N-M)} \frac{\sum_{k=0}^{M} p_k z^{M-k}}{\sum_{k=0}^{N} d_k z^{N-k}}$$

- An alternate form of the transfer function is given by

$$H(z) = \frac{p_0}{d_0} \cdot \frac{\prod_{k=1}^{M} (1 - \xi_k z^{-1})}{\prod_{k=1}^{N} (1 - \lambda_k z^{-1})}$$

Transfer Function Expression

- Or, equivalently as

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

- $\xi_1, \xi_2, \ldots, \xi_M$ are the finite ***zeros***, and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are the finite ***poles*** of $H(z)$
- If $N > M$, there are additional $(N-M)$ zeros at $z = 0$
- If $N < M$, there are additional $(M-N)$ poles at $z = 0$

Transfer Function Expression

- For a causal IIR digital filter, the impulse response is a causal sequence
- The ROC of the causal transfer function

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

  is thus exterior to a circle going through the pole farthest from the origin
- Thus the ROC is given by $|z| > \max_k |\lambda_k|$

Transfer Function Expression

- Example - Consider the M-point moving-average FIR filter with an impulse response

$$h[n] = \begin{cases} 1/M, & 0 \leq n \leq M-1 \\ 0, & \text{otherwise} \end{cases}$$

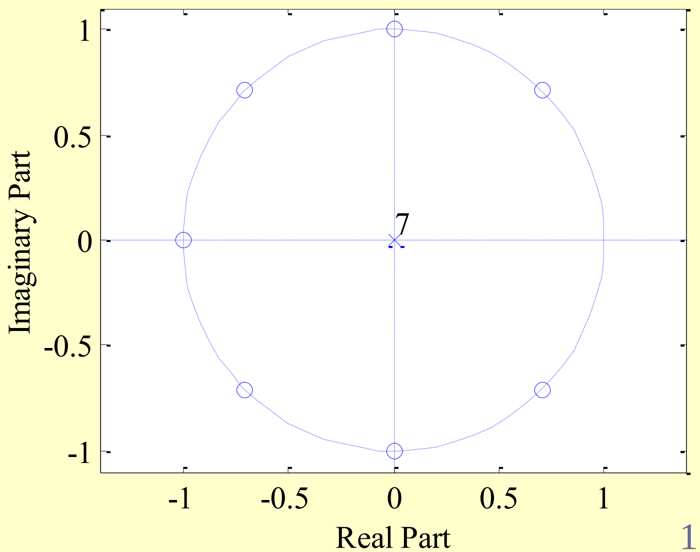
- Its transfer function is then given by

$$H(z) = \frac{1}{M}\sum_{n=0}^{M-1} z^{-n} = \frac{1-z^{-M}}{M(1-z^{-1})} = \frac{z^{M}-1}{M[z^{M-1}(z-1)]}$$

Transfer Function Expression

- The transfer function has M zeros on the unit circle at $z = e^{j2\pi k/M}$, $0 \le k \le M-1$
- There are $M-1$ poles at $z = 0$ and a single pole at $z = 1$
- The pole at $z = 1$ exactly cancels the zero at z = 1
- The ROC is the entire z-plane except $z = 0$

$M = 8$



Transfer Function Expression

- Example - A causal LTI IIR digital filter is described by a constant coefficient difference equation given by

$$y[n] = x[n-1] - 1.2\,x[n-2] + x[n-3] + 1.3\,y[n-1]$$
$$-1.04\,y[n-2] + 0.222\,y[n-3]$$

- Its transfer function is therefore given by

$$H(z) = \frac{z^{-1} - 1.2z^{-2} + z^{-3}}{1 - 1.3z^{-1} + 1.04z^{-2} - 0.222z^{-3}}$$

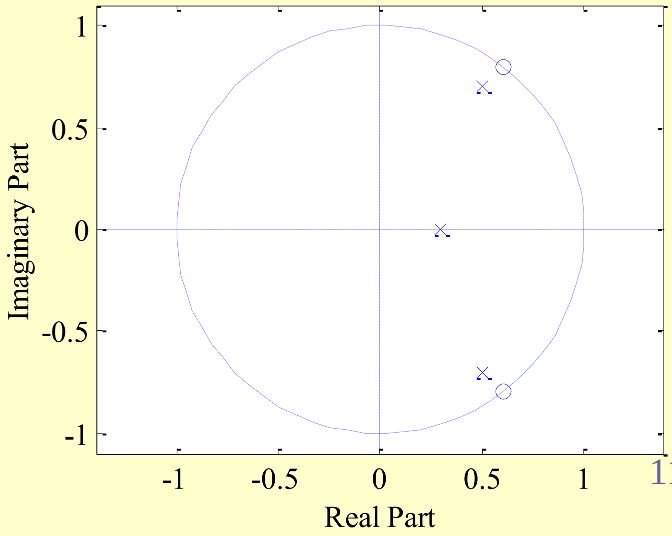
Transfer Function Expression

- Alternate forms:

$$H(z) = \frac{z^2 - 1.2z + 1}{z^3 - 1.3z^2 + 1.04z - 0.222}$$

$$= \frac{(z - 0.6 + j0.8)(z - 0.6 - j0.8)}{(z - 0.3)(z - 0.5 + j0.7)(z - 0.5 - j0.7)}$$

- Note: Poles farthest from $z = 0$ have a magnitude $\sqrt{0.74}$
- ROC: $|z| > \sqrt{0.74}$



Frequency Response from Transfer Function

- If the ROC of the transfer function $H(z)$ includes the unit circle, then the frequency response $H(e^{j\omega})$ of the LTI digital filter can be obtained simply as follows:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

- For a real coefficient transfer function $H(z)$ it can be shown that

$$\left|H(e^{j\omega})\right|^2 = H(e^{j\omega})H^*(e^{j\omega})$$
$$= H(e^{j\omega})H(e^{-j\omega}) = H(z)H(z^{-1})\big|_{z=e^{j\omega}}$$

Frequency Response from Transfer Function

- For a stable rational transfer function in the form

$$H(z) = \frac{p_0}{d_0} z^{(N-M)} \frac{\prod_{k=1}^{M}(z-\xi_k)}{\prod_{k=1}^{N}(z-\lambda_k)}$$

the factored form of the frequency response is given by

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)}$$

Frequency Response from Transfer Function

- It is convenient to visualize the contributions of the ***zero factor*** $(z-\xi_k)$ and the ***pole factor*** $(z-\lambda_k)$ from the factored form of the frequency response
- The magnitude function is given by

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \left|e^{j\omega(N-M)}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega}-\xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega}-\lambda_k\right|}$$

Frequency Response from Transfer Function

which reduces to

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - \xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - \lambda_k\right|}$$

- The phase response for a rational transfer function is of the form

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M) + \sum_{k=1}^{M}\arg(e^{j\omega} - \xi_k) - \sum_{k=1}^{N}\arg(e^{j\omega} - \lambda_k)$$

Frequency Response from Transfer Function

- The magnitude-squared function of a real-coefficient transfer function can be computed using

$$\left|H(e^{j\omega})\right|^2 = \left|\frac{p_0}{d_0}\right|^2 \frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)(e^{-j\omega}-\xi_k^*)}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)(e^{-j\omega}-\lambda_k^*)}$$

Geometric Interpretation of Frequency Response Computation

- The factored form of the frequency response

$$H(e^{j\omega}) = \frac{p_0}{d_0} e^{j\omega(N-M)} \frac{\prod_{k=1}^{M}(e^{j\omega} - \xi_k)}{\prod_{k=1}^{N}(e^{j\omega} - \lambda_k)}$$

is convenient to develop a geometric interpretation of the frequency response computation from the pole-zero plot as ω varies from 0 to 2π on the unit circle

Geometric Interpretation of Frequency Response Computation

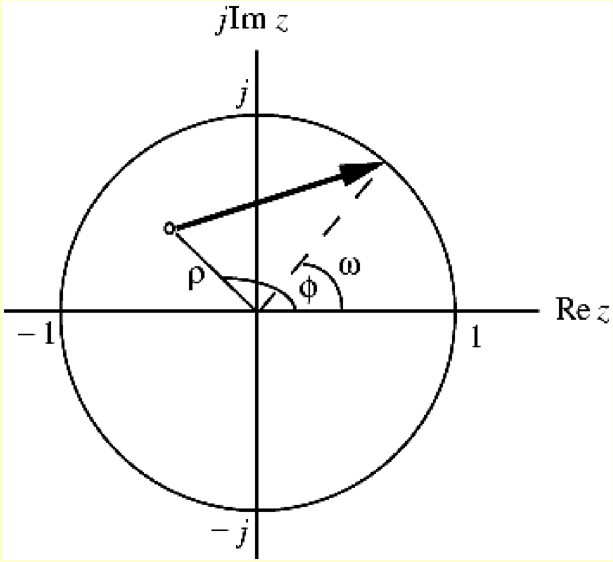
- The geometric interpretation can be used to obtain a sketch of the response as a function of the frequency
- A typical factor in the factored form of the frequency response is given by

$$(e^{j\omega} - \rho e^{j\phi})$$

where $\rho e^{j\phi}$ is a zero if it is zero factor or is a pole if it is a pole factor

Geometric Interpretation of Frequency Response Computation

- As shown below in the z-plane the factor $(e^{j\omega} - \rho e^{j\phi})$ represents a vector starting at the point $z = \rho e^{j\phi}$ and ending on the unit circle at $z = e^{j\omega}$



Geometric Interpretation of Frequency Response Computation

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- As ω is varied from 0 to 2π, the tip of the vector moves counterclockise from the point $z = 1$ tracing the unit circle and back to the point $z = 1$

Geometric Interpretation of Frequency Response Computation

- As indicated by

$$\left|H(e^{j\omega})\right| = \left|\frac{p_0}{d_0}\right| \frac{\prod_{k=1}^{M}\left|e^{j\omega} - \xi_k\right|}{\prod_{k=1}^{N}\left|e^{j\omega} - \lambda_k\right|}$$

the magnitude response $\left|H(e^{j\omega})\right|$ at a specific value of ω is given by the product of the magnitudes of all zero vectors divided by the product of the magnitudes of all pole vectors

Geometric Interpretation of Frequency Response Computation

- Likewise, from

$$\arg H(e^{j\omega}) = \arg(p_0/d_0) + \omega(N-M) + \sum_{k=1}^{M} \arg(e^{j\omega} - \xi_k) - \sum_{k=1}^{N} \arg(e^{j\omega} - \lambda_k)$$

we observe that the phase response at a specific value of ω is obtained by adding the phase of the term p_0/d_0 and the linear-phase term $\omega(N-M)$ to the sum of the angles of the zero vectors minus the angles of the pole vectors

Geometric Interpretation of Frequency Response Computation

- Thus, an approximate plot of the magnitude and phase responses of the transfer function of an LTI digital filter can be developed by examining the pole and zero locations
- Now, a zero (pole) vector has the smallest magnitude when $\omega = \phi$

Geometric Interpretation of Frequency Response Computation

- To highly attenuate signal components in a specified frequency range, we need to place zeros very close to or on the unit circle in this range
- Likewise, to highly emphasize signal components in a specified frequency range, we need to place poles very close to or on the unit circle in this range

Stability Condition in Terms of the Pole Locations

- A causal LTI digital filter is ***BIBO stable*** if and only if its impulse response $h[n]$ is absolutely summable, i.e.,

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- We now develop a stability condition in terms of the pole locations of the transfer function $H(z)$

Stability Condition in Terms of the Pole Locations

- The ROC of the z-transform $H(z)$ of the impulse response sequence $h[n]$ is defined by values of $|z| = r$ for which $h[n]r^{-n}$ is absolutely summable
- Thus, if the ROC includes the unit circle $|z| = 1$, then the digital filter is stable, and vice versa

Stability Condition in Terms of the Pole Locations

- In addition, for a stable and causal digital filter for which $h[n]$ is a right-sided sequence, the ROC will include the unit circle and entire z-plane including the point $z = \infty$
- An FIR digital filter with bounded impulse response is always stable

Stability Condition in Terms of the Pole Locations

- On the other hand, an IIR filter may be unstable if not designed properly
- In addition, an originally stable IIR filter characterized by infinite precision coefficients may become unstable when coefficients get quantized due to implementation

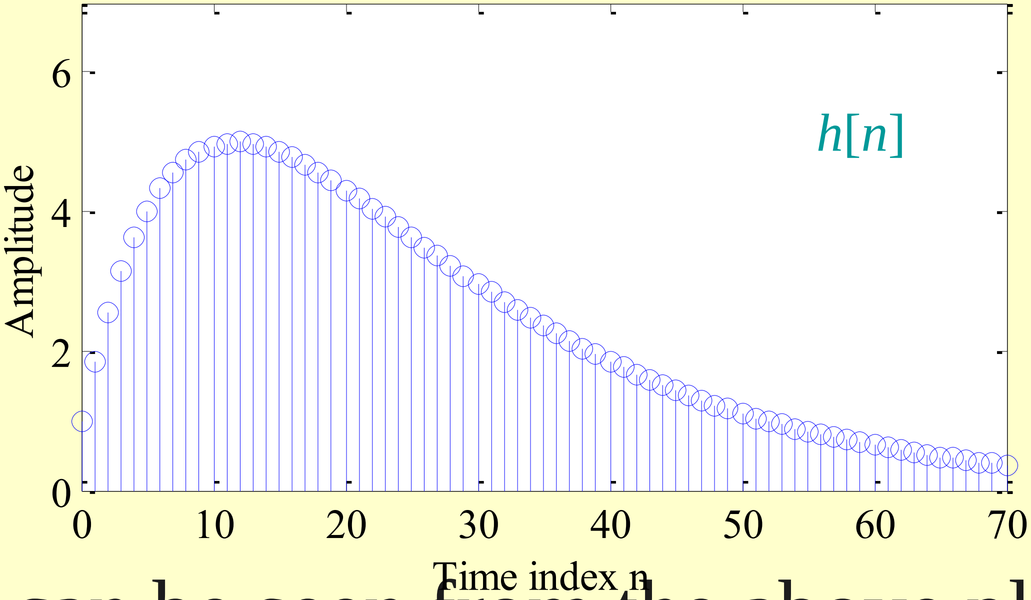
Stability Condition in Terms of the Pole Locations

- Example - Consider the causal IIR transfer function

$$H(z)=\frac{1}{1-1.845z^{-1}+0.850586z^{-2}}$$

- The plot of the impulse response coefficients is shown on the next slide

Stability Condition in Terms of the Pole Locations



- As can be seen from the above plot, the impulse response coefficient $h[n]$ decays rapidly to zero value as n increases

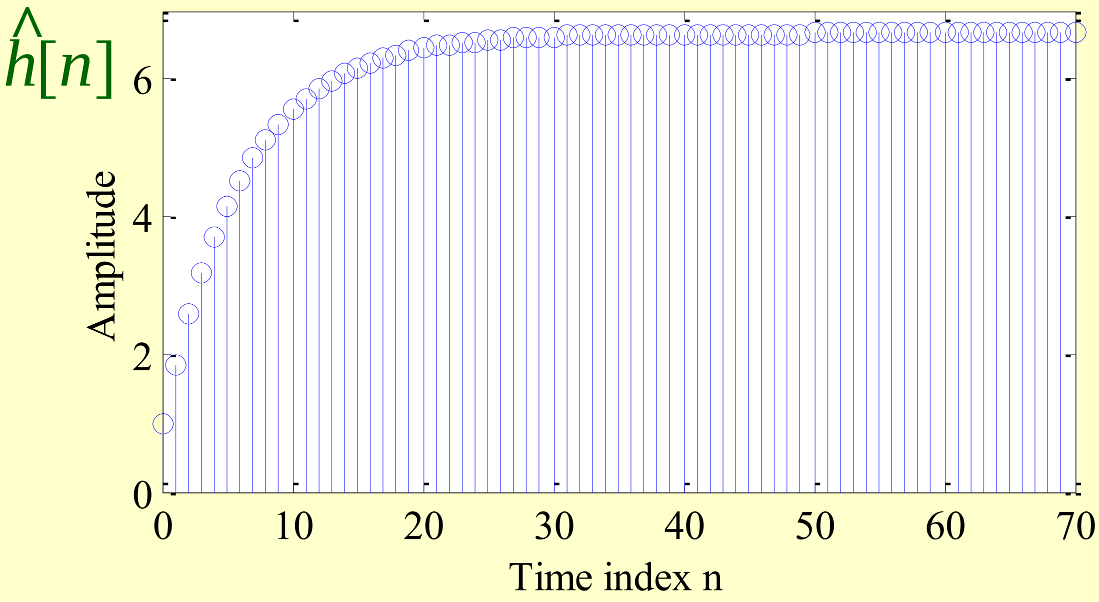
Stability Condition in Terms of the Pole Locations

- The absolute summability condition of $h[n]$ is satisfied
- Hence, $H(z)$ is a stable transfer function
- Now, consider the case when the transfer function coefficients are rounded to values with 2 digits after the decimal point:

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

Stability Condition in Terms of the Pole Locations

- A plot of the impulse response of $\hat{h}[n]$ is shown below



Stability Condition in Terms of the Pole Locations

- In this case, the impulse response coefficient $\hat{h}[n]$ increases rapidly to a constant value as n increases
- Hence, the absolute summability condition of is violated
- Thus, $\hat{H}(z)$ is an unstable transfer function

Stability Condition in Terms of the Pole Locations

- The stability testing of a IIR transfer function is therefore an important problem
- In most cases it is difficult to compute the infinite sum

$$S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

- For a causal IIR transfer function, the sum S can be computed approximately as

$$S_K = \sum_{n=0}^{K-1} |h[n]|$$

Stability Condition in Terms of the Pole Locations

- The partial sum is computed for increasing values of K until the difference between a series of consecutive values of $\mathcal{S}_K$ is smaller than some arbitrarily chosen small number, which is typically 10^{-6}
- For a transfer function of very high order this approach may not be satisfactory
- An alternate, easy-to-test, stability condition is developed next

Stability Condition in Terms of the Pole Locations

- Consider the causal IIR digital filter with a rational transfer function $H(z)$ given by

$$H(z) = \frac{\sum_{k=0}^{M} p_k z^{-k}}{\sum_{k=0}^{N} d_k z^{-k}}$$

- Its impulse response $\{h[n]\}$ is a right-sided sequence
- The ROC of $H(z)$ is exterior to a circle going through the pole farthest from $z = 0$

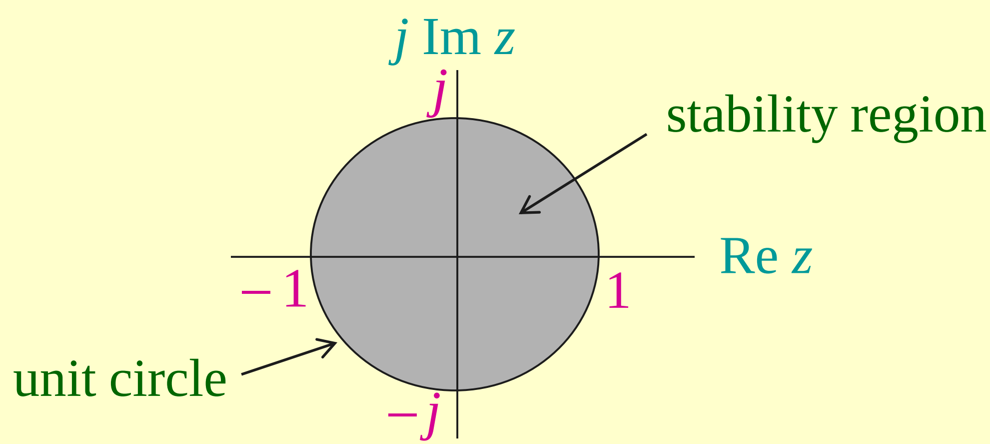
Stability Condition in Terms of the Pole Locations

- But stability requires that $\{h[n]\}$ be absolutely summable
- This in turn implies that the DTFT $H(e^{j\omega})$ of $\{h[n]\}$ exists
- Now, if the ROC of the z-transform $H(z)$ includes the unit circle, then

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}$$

Stability Condition in Terms of the Pole Locations

- Conclusion: All poles of a causal stable transfer function $H(z)$ must be strictly inside the unit circle
- The stability region (shown shaded) in the z-plane is shown below



Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$H(z) = \frac{1}{1-0.845z^{-1}+0.850586z^{-2}}$$

  is

$$H(z) = \frac{1}{(1-0.902z^{-1})(1-0.943z^{-1})}$$

  which has a real pole at $z = 0.902$ and a real pole at $z = 0.943$

- Since both poles are inside the unit circle, $H(z)$ is BIBO stable

Stability Condition in Terms of the Pole Locations

- Example - The factored form of

$$\hat{H}(z) = \frac{1}{1 - 1.85z^{-1} + 0.85z^{-2}}$$

is

$$\hat{H}(z) = \frac{1}{(1 - z^{-1})(1 - 0.85z^{-1})}$$

which has a real pole on the unit circle at $z = 1$ and the other pole inside the unit circle

- Since one pole is not inside but on the unit circle, $H(z)$ is unstable

Exercise 6.2

Determine the z-transform and the corresponding ROC of the following causal sequences:

(a) $x_a[n] = -\alpha\mu[-n-1]$

(c) $x_c[n] = \alpha^n \cos(\omega_o n)\mu[n]$

Exercise 6.44

The transfer function of a causal LTI discrete-time system is given by

$$H(z) = \frac{-1.5z^{-1} + 0.3z^{-2}}{1 + 0.25z^{-1} - 0.06z^{-2}}$$

(a) Determine the impulse response of the above system.

(b) Determine the output of the above system for all values of for an input

$$x[n] = 2.1(0.4)^n \mu[n] + 0.3(-0.3)^n \mu[n]$$