



第九章 多元函数微分学

- 多变量函数的连续性
- 多变量函数的微分
- 隐函数定理和逆映射定理
- **空间曲线与曲面**
- Taylor公式与极值
- 向量场的微商
- 微分形式

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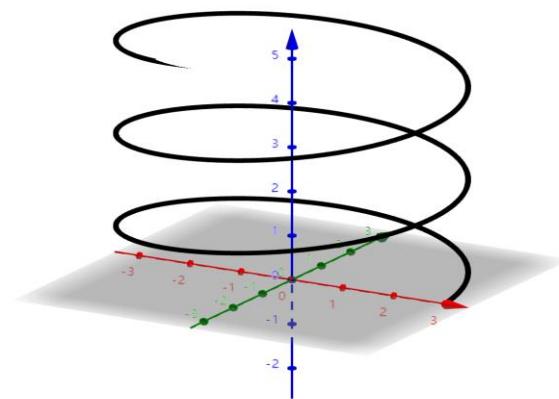
空间曲线的**参数方程**:

$$\begin{cases} x = x(t) \\ y = y(t), \quad t \in [\alpha, \beta] \\ z = z(t) \end{cases}$$

或表示为**向径式方程**:

$$\mathbf{r}(t) = (x(t), y(t), z(t)), t \in [\alpha, \beta].$$

例: 螺旋线 $\mathbf{r}(t) = (a \cos \lambda t, a \sin \lambda t, z(t) = bt), t \in [\alpha, \beta]$.



参数曲线在 $M_0 = (x(t_0), y(t_0), z(t_0))$ 处的**切向量**为：

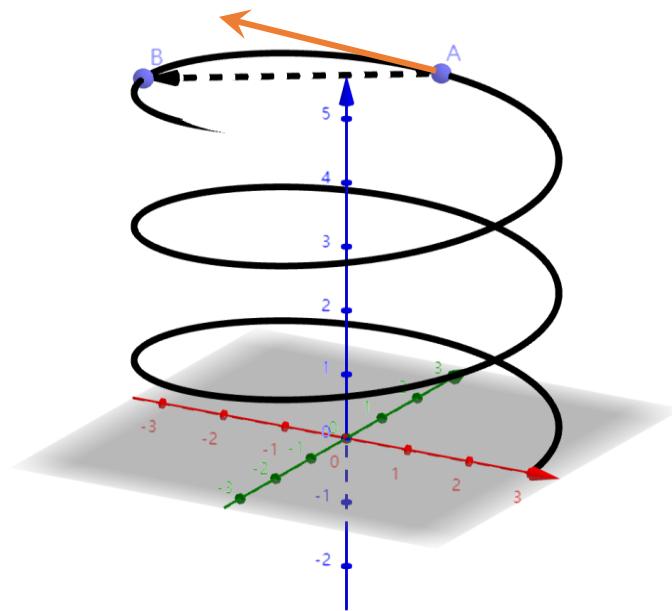
$$\boldsymbol{r}'(t_0) = (x'(t_0), y'(t_0), z'(t_0)),$$

过 M_0 **切线方程**为：

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}.$$

过 M_0 **法平面方程**为：

$$x'(t_0)(x - x(t_0)) + y'(t_0)(y - y(t_0)) + z'(t_0)(z - z(t_0)) = 0.$$



例：螺旋线 ($a, b > 0$)

$$x(t) = a \cos t, y(t) = a \sin t, z(t) = bt, (-\infty < t < +\infty)$$

在每点的切向量与 z 轴夹角是常数.

证明：曲线在参数 t 处的切向量为

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = (-a \sin t, a \cos t, b).$$

z 轴上的方向向量为 $\mathbf{k} = (0, 0, 1)$, 与切向量夹角的余弦为

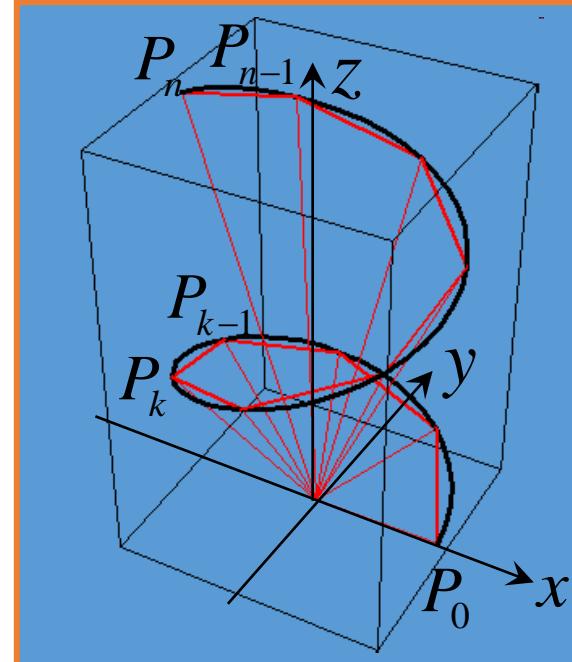
$$\cos \theta = \frac{\mathbf{r}'(t) \cdot \mathbf{k}}{\|\mathbf{r}'(t)\| \|\mathbf{k}\|} = \frac{b}{\sqrt{a^2 + b^2}}, \text{ 与 } t \text{ 无关.}$$

例: $C: \mathbf{r} = \mathbf{r}(t) (0 \leq t \leq 3\pi)$

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{1}{3}t \right)$$

$$l(T) = \sum_{i=1}^n | \mathbf{r}(t_i) - \mathbf{r}(t_{i-1}) |$$

n	σ_n
10	9.60794
50	9.92136
100	9.93128
200	9.93376



在光滑曲线 $L: \mathbf{r} = \mathbf{r}(t), t \in [\alpha, \beta]$ 上作内接折线，对应分割：

$$T: \alpha = t_0 < t_1 < t_2 < \cdots < t_{i-1} < t_i < \cdots < t_{n-1} < t_n = \beta$$

折线长度为：

$$\begin{aligned} l(T) &= \sum_{i=1}^n | \mathbf{r}(t_i) - \mathbf{r}(t_{i-1}) | \\ &= \sum_{i=1}^n \sqrt{[x(t_i) - x(t_{i-1})]^2 + [y(t_i) - y(t_{i-1})]^2 + [z(t_i) - z(t_{i-1})]^2} \\ &= \sum_{i=1}^n \sqrt{x'^2(\xi_i) + y'^2(\eta_i) + z'^2(\zeta_i)} \Delta t_i \end{aligned}$$

其中 $t_{i-1} < \xi_i, \eta_i, \zeta_i < t_i, \Delta t_i = t_i - t_{i-1}$.

由于 $x'(t), y'(t), z'(t)$ 连续, 故 $f(\xi, \eta, \zeta) = \sqrt{x'^2(\xi) + y'^2(\eta) + z'^2(\zeta)}$ 一致连续. 于是 $\forall \varepsilon > 0, \exists \delta > 0, s.t. \|T\| := \max_{1 \leq i \leq n} \Delta t_i < \delta$ 时,

对任意 $(\xi_i, \eta_i, \zeta_i) \in [t_{i-1}, t_i]^3 (i=1, 2, \dots, n)$ 有:

$$\left| \sqrt{x'^2(\xi_i) + y'^2(\eta_i) + z'^2(\zeta_i)} - \sqrt{x'^2(t_i) + y'^2(t_i) + z'^2(t_i)} \right| < \varepsilon$$

$$\Rightarrow \left| l(T) - \sum_{i=1}^n |\mathbf{r}'(t_i)| \Delta t_i \right| < \varepsilon \sum_{i=1}^n \Delta t_i = \varepsilon(\beta - \alpha).$$

再令 $\|T\| \rightarrow 0$ 知:

$$\lim_{\|T\| \rightarrow 0} l(T) = \lim_{\|T\| \rightarrow 0} \sum_{i=1}^n |\mathbf{r}'(t_i)| \Delta t_i = \int_{\alpha}^{\beta} |r'(t)| dt.$$

定义：光滑曲线 $L: r = r(t), t \in [\alpha, \beta]$ 的长度定义为：

$$\begin{aligned} l &= \lim_{\|T\| \rightarrow 0} l(T) = \lim_{\|T\| \rightarrow 0} \sum_{i=1}^n |r(t_i) - r(t_{i-1})| \\ &= \int_{\alpha}^{\beta} |r'(t)| dt = \int_{\alpha}^{\beta} \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} dt. \end{aligned}$$

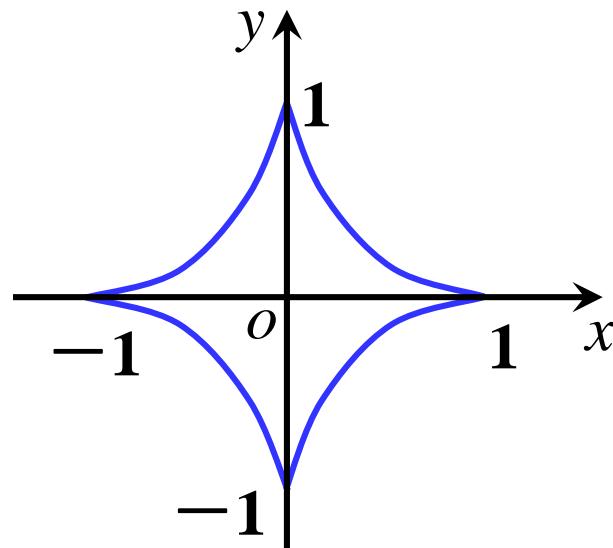
特别地，平面曲线 $r(t) = (x(t), y(t), 0), t \in [\alpha, \beta]$ 弧长为

$$l = \int_{\alpha}^{\beta} \sqrt{x'^2(t) + y'^2(t)} dt.$$

当曲线为 $y = f(x)$ 时，有参数表示 $r(x) = (x, f(x), 0), x \in [\alpha, \beta]$

$$\Rightarrow l = \int_{\alpha}^{\beta} \sqrt{1 + f'^2(x)} dx.$$

例：求星形线 $x = \cos^3 t, y = \sin^3 t$ ($0 \leq t \leq 2\pi$) 的弧长.



设 L 是正则曲线 ($r'(t) \neq 0$), 从起点到 $M(t)$ 的弧 \widehat{AM} 的弧长为

$$s(t) = \int_{\alpha}^t \sqrt{x'^2(\tau) + y'^2(\tau) + z'^2(\tau)} d\tau = \int_{\alpha}^t |r'(\tau)| d\tau.$$

由 $\frac{ds}{dt} = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} = |r'(t)| > 0$ 知 $s(t)$ 严格单调增,

于是有反函数 $t = t(s)$. 曲线可以表示为

$$r = (x(t), y(t), z(t)) = (\bar{x}(s), \bar{y}(s), \bar{z}(s)).$$

以曲线固有的几何量 s 为参数的参数表示, 称为曲线的**自然方程**.

例如：螺旋线 $r(t) = (a \cos t, a \sin t, bt)$ ($0 \leq t \leq 2\pi$).

$$\Rightarrow s(t) = \int_0^t |r'(\tau)| d\tau = \sqrt{a^2 + b^2} t.$$

$$\Rightarrow t = t(s) = \frac{s}{\sqrt{a^2 + b^2}}.$$

于是其有自然方程表示：

$$r(s) = \left(a \cos \frac{s}{\sqrt{a^2 + b^2}}, a \sin \frac{s}{\sqrt{a^2 + b^2}}, b \frac{s}{\sqrt{a^2 + b^2}} \right) (0 \leq s \leq 2\pi\sqrt{a^2 + b^2}).$$

考虑正则曲线 $L: \mathbf{r} = \mathbf{r}(t) \in C^2[\alpha, \beta]$ ，设 s 为弧长参数，则：

$$\dot{\mathbf{r}}(s) = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{dt} \frac{dt}{ds} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|}$$

为**单位切向量**。于是 $\dot{\mathbf{r}} \cdot \dot{\mathbf{r}} = 1 \Rightarrow \frac{d}{ds}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = 2\dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = 0$

$\ddot{\mathbf{r}}$: **主法向量**; $\kappa = \dot{\mathbf{r}} \times \ddot{\mathbf{r}}$: **副法向量**.

设曲线 L 上一段长 Δs 的弧 $\widehat{M_1 M_2}$, 弧上向量转角为 $\Delta\alpha$

$$\kappa = \lim_{\Delta s \rightarrow 0} \left| \frac{\Delta\alpha}{\Delta s} \right| = \lim_{\Delta s \rightarrow 0} \left| \frac{\dot{\Delta \mathbf{r}}}{\Delta s} \right| = |\ddot{\mathbf{r}}| = \kappa$$

称为曲线的**曲率**。

考虑一般参数正则曲线 $L: r = r(t) \in C^2[\alpha, \beta]$,

主法向量为:

$$\ddot{r} = \frac{d\dot{r}}{ds} = \frac{d}{dt} \left(\frac{\dot{r}(t)}{|\dot{r}(t)|} \right) \frac{dt}{ds} = \frac{1}{|\dot{r}(t)|} \left\{ \frac{\ddot{r}(t)}{|\dot{r}(t)|} + \left[\frac{d}{dt} \frac{1}{|\dot{r}(t)|} \right] \dot{r}(t) \right\}.$$

副法向量为: $\kappa = \dot{r} \times \ddot{r} = \frac{\dot{r}(t)}{|\dot{r}(t)|} \times \ddot{r} = \frac{\dot{r}(t) \times \ddot{r}(t)}{|\dot{r}(t)|^3}.$

曲率为: $\kappa = \frac{|\dot{r}(t) \times \ddot{r}(t)|}{|\dot{r}(t)|^3}.$

注: 曲线曲率恒0 $\Leftrightarrow r = r(t) = r_0 + tv, t \in \mathbb{R}$.

例：求螺旋线 $r(t) = (a \cos t, a \sin t, bt)$ ($-\infty < t < +\infty$) 的曲率.

解：求导得

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = (-a \sin t, a \cos t, b),$$

$$\mathbf{r}''(t) = (x''(t), y''(t), z''(t)) = (-a \cos t, -a \sin t, 0) \left| \mathbf{r}'(t) \right|$$

$$= \sqrt{a^2 + b^2} (-a \cos t, -a \sin t, 0),$$

$$\left| \mathbf{r}'(t) \times \mathbf{r}''(t) \right| = a \sqrt{a^2 + b^2}, \quad \kappa = \frac{\left| \mathbf{r}'(t) \times \mathbf{r}''(t) \right|}{\left| \mathbf{r}'(t) \right|^3} = \frac{a}{a^2 + b^2}.$$

定义在区域 $D \subset \mathbb{R}^2$ 上的二元向量值函数

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)), \quad (u, v) \in D$$

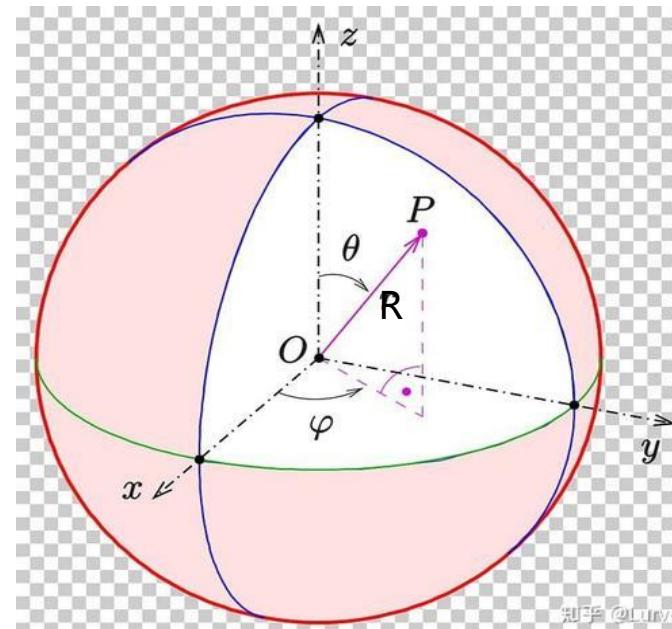
确定了空间中一个曲面，称之为曲面的**向径表示**. 等价于：

$$\begin{cases} x = x(u, v) \\ y = y(u, v), \quad (u, v) \in D, \\ z = z(u, v). \end{cases}$$

称之为曲面的**参数方程**.

例如，球面参数方程

$$\begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases}$$



曲面 $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, $(u, v) \in D$

曲面上的 u -曲线:

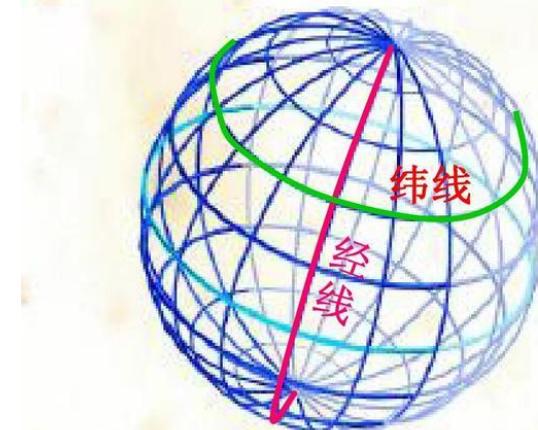
$$\mathbf{r}_1(u) = \mathbf{r}(u, v_0) = (x(u, v_0), y(u, v_0), z(u, v_0))$$

曲面上的 v -曲线:

$$\mathbf{r}_2(v) = \mathbf{r}(u_0, v) = (x(u_0, v), y(u_0, v), z(u_0, v))$$

整张曲面可看成由不同的 u -曲线和 v -曲线交织而成.

例如, 球面参数方程

$$\begin{cases} x = R \sin \theta \cos \varphi \\ y = R \sin \theta \sin \varphi \\ z = R \cos \theta \end{cases}$$


设 $S: r(u, v)$ 为光滑曲面, $M_0 = r(u_0, v_0) \in S$.

$$\vec{r}'_u = \frac{\partial \vec{r}}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right), \quad \vec{r}'_v = \frac{\partial \vec{r}}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).$$

则 S 在 M_0 处的切平面法向为:

$$\vec{n}(u_0, v_0) = \vec{r}'_u(u_0, v_0) \times \vec{r}'_v(u_0, v_0) = \left(\frac{\partial(y, z)}{\partial(u, v)}, \frac{\partial(z, x)}{\partial(u, v)}, \frac{\partial(x, y)}{\partial(u, v)} \right).$$

切平面方程为:

$$\frac{\partial(y, z)}{\partial(u, v)}(X - x(u, v)) + \frac{\partial(z, x)}{\partial(u, v)}(Y - y(u, v)) + \frac{\partial(x, y)}{\partial(u, v)}(Z - z(u, v)) = 0.$$

特别的，对显式曲面 $z = f(x, y) \quad ((x, y) \in D)$, 可转为参数表示

$$\mathbf{r} = \mathbf{r}(x, y) = (x, y, f(x, y)), \quad (x, y) \in D.$$

于是, $\mathbf{r}'_x = (1, 0, f'_x), \mathbf{r}'_y = (0, 1, f'_y) \Rightarrow \mathbf{n} = \mathbf{r}'_x \times \mathbf{r}'_y = (-f'_x, -f'_y, 1).$

曲面 S 在其上一点 $M_0 = (x_0, y_0, f(x_0, y_0))$ 处切平面方程为:

$$z - f(x_0, y_0) = f'_x(x_0, y_0)(x - x_0) + f'_y(x_0, y_0)(y - y_0).$$

例: 平面 π 过直线 $L: \begin{cases} x + y + b = 0 \\ x + ay - z - 3 = 0 \end{cases}$ 且与曲面 $z = x^2 + y^2$

在点 $(1, 2, 5)$ 处相切, 求 a, b .

设有平面隐式曲线 $F(x, y) = 0$, $\text{grad}(F) \neq \vec{0}$. (x, y) 为曲线上一点.

1. 切线方程为 $(X - x)F'_x + (Y - y)F'_y = 0$

$$2. \text{ 曲率 } \kappa = \frac{f''(x)}{\left[1 + f'^2(x)\right]^{3/2}} = \frac{-F_y'^2 F_{xx}'' + 2F_x' F_y' F_{xy}'' - F_x'^2 F_{yy}''}{\left(\sqrt{F_x'^2 + F_y'^2}\right)^3}$$

3. 拐点方程 $F_y'^2 F_{xx}'' - 2F_x' F_y' F_{xy}'' + F_x'^2 F_{yy}'' = 0$.

4. 法向量 $\text{grad}(F)$.

空间隐式曲面 $S: F(x, y, z) = 0, \quad \text{grad}(F) \neq \vec{0}.$ $M_0(x_0, y_0, z_0) \in S.$

1. M_0 处法向量为 $\text{grad}(F) = (F'_x, F'_y, F'_z).$

2. M_0 处切平面为 $F'_x(X - x_0) + F'_y(Y - y_0) + F'_z(Z - z_0) = 0.$

空间隐式曲线 $\begin{cases} F(x, y, z) = 0 \\ G(x, y, z) = 0 \end{cases}, \quad M_0(x_0, y_0, z_0)$ 为曲线上一点.

M_0 处切线方向为: $l = \text{grad}(F) \times \text{grad}(G).$

例: 求曲线 $\begin{cases} x^2 + y^2 + z^2 - 4a^2 = 0 \\ x^2 + y^2 - 2ax = 0 \end{cases}$ 在 $M_0(a, a, \sqrt{2}a)$ 处的切线与法平面.