10.3

3.(1) $\iiint_V (x^2+y^2)dxdydz = \int_0^2 dz \iint_{x^2+y^2\le 2z} (x^2+y^2)dxdy$

$= \int_0^2 dz \int_0^{\sqrt{2z}} dr \int_0^{2\pi} r^2\cdot r\, d\theta = \int_0^2 2\pi z^2 dz = \frac{16\pi}{3}$

3.(2) $\iiint_V \sqrt{x^2+y^2}\, dxdydz = \int_0^1 dz \iint_{x^2+y^2\le z^2} \sqrt{x^2+y^2}\, dxdy$

$= \int_0^1 dz \int_0^z dr \int_0^{2\pi} r\cdot r\, d\theta = \frac{2\pi}{3}\int_0^1 z^3 dz = \frac{\pi}{6}$

3.(5) $\iiint_V x^2 dxdydz = \int_0^1 dz \int_{\frac{z}{2}}^{\sqrt{z}} dy \int_{\frac{z}{2}}^{z} x^2 dx$

$= \int_0^1 dz \int_{\frac{z}{2}}^{\sqrt{z}} \frac{7}{24} z^3 dy = \int_0^1 \frac{7}{48} z^{\frac{7}{2}} dz = \frac{7}{216}$

3.(8) 由对称性可知 $\iiint_V z e^{-(x^2+y^2+z^2)} dxdydz = 0$

故原积分 $= \iiint_V |x| e^{-(x^2+y^2+z^2)} dxdydz$

$= 8\times \int_1^2 dr \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} r|\sin\theta\cos\varphi| e^{-r^2}\cdot r^2\sin\theta\, d\theta$

$= 8\int_1^2 r^3 e^{-r^2} dr \int_0^{\frac{\pi}{2}} \cos\varphi\, d\varphi \int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$

$= 8\times\left(\frac{1}{e} - \frac{5}{2e^4}\right)\times 1\times\frac{\pi}{4}$

$= \left(\frac{2}{e} - \frac{5}{e^4}\right)\pi$

4.(1)

$\iiint_V (x+y)dxdydz = \iiint_V (-x-y)dxdydz$.

$\Rightarrow \iiint_V (x+y)dxdydz = 0$

4.(4) $\iiint_{V'} x^2 dxdydz = \iiint_{V'} y^2 dxdydz = \iiint_{V'} z^2 dxdydz$.

$\Rightarrow \iiint_{V'} (x^2+y^2)dxdydz = \frac{2}{3}\iiint_{V'} (x^2+y^2+z^2)dxdydz$

$= \frac{2}{3}\int_r^R dr \int_0^{2\pi} d\varphi \int_0^{\pi} r^2\cdot r^2 \sin\theta\, d\theta = \frac{8}{15}\pi(R^5 - r^5)$

其中 $V' = \{(x,y,z) \mid r^2 \le x^2+y^2+z^2 \le R^2\}$

又因为 $\iiint_V (x^2+y^2)\,dxdydz = \frac{1}{2}\iiint_{V'}(x^2+y^2)\,dxdydz$

故原积分 $= \frac{4\pi}{15}(R^5-r^5)$

5.(7) $V_1 = \int_0^{\frac{\sqrt{2}}{2}c} dz \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le\frac{z^2}{c^2}} dxdy = \int_0^{\frac{\sqrt{2}}{2}c} \pi ab\cdot\frac{z^2}{c^2} = \frac{\sqrt{2}}{12}\pi abc$

$V_2 = \int_{\frac{\sqrt{2}}{2}c}^{c} dz \iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1-\frac{z^2}{c^2}} = \int_{\frac{\sqrt{2}}{2}c}^{c} \pi ab\cdot(1-\frac{z^2}{c^2}) = (\frac{2}{3}-\frac{5}{12}\sqrt{2})\pi abc$

$V_1+V_2 = (\frac{2}{3}-\frac{\sqrt{2}}{3})\pi abc$，此为 $z\ge0$ 部分的体积.

由对称性可知 $V = 2(V_1+V_2) = \frac{4-2\sqrt{2}}{3}\pi abc$

5.(8) 由对称性可知 $V=4V_1$，其中 V_1 是该立体在第一卦限的体积.

$V_1 = \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} d\theta \int_0^{a\sqrt{\cos\varphi\sin\theta}} r^2\sin\theta\, dr = \frac{a^3}{3}\int_0^{\frac{\pi}{2}}\cos\varphi\, d\varphi\int_0^{\frac{\pi}{2}}\sin^2\theta\, d\theta$

$= \frac{a^3\pi}{12}$

故 $V = 4V_1 = \frac{a^3\pi}{3}$

8. 利用柱坐标：$x=r\cos\theta$，$y=r\sin\theta$，$z=z$.

则 $\iiint_{x^2+y^2+z^2\le1} f(z)\,dxdydz = \int_{-1}^{1}dz\int_0^{\sqrt{1-z^2}}dr\int_0^{2\pi}f(z)\cdot r\,d\theta$

$= 2\pi\int_{-1}^{1}dz\int_0^{\sqrt{1-z^2}}f(z)\cdot r\,dr = \pi\int_{-1}^{1}f(z)(1-z^2)\,dz$

6. $x^2+y^2+z^2\le x+y+z \Longleftrightarrow (x-\frac{1}{2})^2+(y-\frac{1}{2})^2+(z-\frac{1}{2})^2<\frac{3}{4}$

该球体体积 $V=\frac{4}{3}\pi(\frac{\sqrt{3}}{2})^3=\frac{\sqrt{3}}{2}\pi$

于是 $f(x,y,z)$ 在球内的平均值为 $\frac{\iiint_\Omega x^2+y^2+z^2\,dxdydz}{\frac{\sqrt{3}}{2}\pi}$

令 $u=x-\frac{1}{2}$，$v=y-\frac{1}{2}$，$w=z-\frac{1}{2}$

则 $\iiint_\Omega x^2+y^2+z^2\,dxdydz = \iiint_{u^2+v^2+w^2\le\frac{3}{4}} u^2+v^2+w^2+u+v+w+\frac{3}{4}\,dudvdw$

$= \iiint_{u^2+v^2+w^2\le\frac{3}{4}} u^2+v^2+w^2+\frac{3}{4}\,dudvdw = \frac{9\sqrt{3}\pi}{40}+\frac{3}{4}\times\frac{\sqrt{3}}{2}\pi$

故平均值 $=\frac{9}{20}+\frac{3}{4}=\frac{6}{5}$

15: 设质心坐标为 (x_0, y_0, z_0)，由对称性易知 $x_0 = y_0 = 0$

$$z_0 = \frac{\iiint_V z\,dxdydz}{\iiint_V dxdydz} = \frac{\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} dxdy \int_{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}^{c} z\,dz}{\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} dxdy \int_{c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}^{c} dz}$$

$$= \frac{\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1} \frac{c^2}{2}-\frac{c^2}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)dxdy}{\iint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1}\left(c-c\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\right)dxdy} = \frac{\int_0^{2\pi}d\theta\int_0^1 \frac{1}{2}c^2(1-r^2)\cdot abr\,dr}{\int_0^{2\pi}d\theta\int_0^1 c(1-r)\cdot abr\,dr}$$

$$= \frac{\frac{1}{4}abc^2\pi}{\frac{1}{3}abc\pi} = \frac{3}{4}c$$

故质心坐标为 $(0, 0, \frac{3}{4}c)$

18.(2) 取原点为球心，z轴为轴线，则转动惯量为：

$$I = \iiint_V \rho(x^2+y^2)\,dxdydz = \frac{2\rho}{3}\iiint_V (x^2+y^2+z^2)\,dxdydz$$

$$= \frac{2\rho}{3}\int_0^R dr\int_0^{\pi} d\theta\int_0^{2\pi} r^2\cdot r^2\sin\theta\, d\varphi = \frac{8\rho}{15}\pi R^5 = \frac{2}{5}MR^2$$

其中M是该球体的质量

2: 令 $x_i = a_i y_i$, $(i=1,2,\cdots,n)$. $\left|\frac{\partial(x_1,\cdots,x_n)}{\partial(y_1,\cdots,y_n)}\right| = a_1a_2\cdots a_n$

$$\mu(V_n) = \int_{V_n} dx_1dx_2\cdots dx_n = a_1a_2\cdots a_n\int_{S_n} dy_1dy_2\cdots dy_n = a_1a_2\cdots a_n\,\mu(S_n)$$

$= \frac{a_1a_2\cdots a_n}{n!}$，其中 S_n 是边长为1的n维单形.

4. 利用归纳法. 当 $n=1$ 时显然成立. 假设当 $n=k$ 等式成立.

即 $\int_0^a dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_{k-1}} f(x_1)f(x_2)\cdots f(x_k)dx_k = \frac{1}{k!}\left[\int_0^a f(t)dt\right]^k$

则 $\int_0^a dx_1\int_0^{x_1}dx_2\cdots\int_0^{x_k} f(x_{k+1})f(x_2)\cdots f(x_{k+1})dx_{k+1}$

$= \int_0^a f(x_1)dx_1\left[\int_0^{x_1}dx_2\cdots\int_0^{x_k} f(x_2)f(x_3)\cdots f(x_{k+1})dx_{k+1}\right]$

$= \int_0^a f(x_1)\frac{1}{k!}\left[\int_0^{x_1} f(t)dt\right]^k dx_1 = \frac{1}{(k+1)!}\left[\int_0^{x} f(t)dt\right]^{k+1}\Big|_0^a = \frac{1}{(k+1)!}\left[\int_0^a f(t)dt\right]^{k+1}$

1.(2) $I=\int_0^1\sqrt{9+36t^2+36t^4}\,dt=3\int_0^1\sqrt{(1+2t^2)^2}\,dt=3\int_0^1(1+2t^2)\,dt=5$

(5) 令 $x=4at^2$，由 $4x^2=3(z^2-y^2)=3(z+y)(z-y)$ 知 $|y|<|z|$

于是当 $z\geq 0$ 时 有 $z+y\geq 0$，$z-y\geq 0$．进一步地，$z+y=4at$，$z-y=\frac{16}{3}at^3$

也即 $y=\frac{8}{3}at^3-2at$，$z=\frac{8}{3}at^3+2at$．

$S=\int_0^{t(x,y,z)}\sqrt{(8at)^2+(8at^2-2a)^2+(8at^2+2a)^2}\,dt=\sqrt{2}\int_0^{t(x,y,z)}8at^2+2a\,dt$

$=\sqrt{2}\left(\frac{8}{3}at^3+2at\right)\Big|_0^{t(x,y,z)}=\sqrt{2}z$．

2.(3) OA：$\begin{cases}x=t\\y=0\end{cases}$ $t\in[0,1]$．AB：$\begin{cases}x=1-t\\y=t\end{cases}$ $t\in[0,1]$ BO：$\begin{cases}x=0\\y=1-t\end{cases}$ $t\in[0,1]$．

$I=\int_0^1t\,dt+\int_0^1(1-t+t)\cdot\sqrt{2}\,dt+\int_0^1(1-t)\,dt=\frac{1}{2}+\sqrt{2}+\frac{1}{2}=\sqrt{2}+1$

(5) $I=\int_{AB}+\int_{BC}=\int_0^1(1+t+0)\cdot dt+\int_0^{2\pi}(\cos t+\sin t+t)\sqrt{\cos^2t+\sin^2t+1}\,dt$

$=\frac{3}{2}+2\sqrt{2}\pi^2$

(9). 双纽线在极坐标下的方程为 $r^2=a^2\cos 2\theta$

于是 $\begin{cases}x=a\sqrt{\cos 2\theta}\cos\theta\\y=a\sqrt{\cos 2\theta}\sin\theta\end{cases}$ $\theta\in[-\frac{\pi}{4},\frac{\pi}{4}]$，$ds=\sqrt{[r'(\theta)]^2+r^2(\theta)}\,d\theta=\frac{a}{\sqrt{\cos 2\theta}}\,d\theta$．

故 $\int_L x\sqrt{x^2-y^2}\,ds=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a^2\cos^{\frac{3}{2}}2\theta\cos\theta\cdot\frac{a}{\sqrt{\cos 2\theta}}\,d\theta=a^3\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)\cos\theta\,d\theta$

$=\frac{2\sqrt{2}}{3}a^3$

(11). 由对称性知 $\int_L x^2ds=\frac{1}{3}\int_L(x^2+y^2+z^2)\,ds=\frac{1}{3}\int_L a^2ds=\frac{1}{3}a^2\cdot 2\pi a=\frac{2\pi a^3}{3}$

(12) $I=\frac{1}{2}\int_L[(x+y+z)^2-(x^2+y^2+z^2)]\,ds=-\frac{1}{2}a^2\cdot 2\pi a=-\pi a^3$

3. $\rho(x,y,z)=\frac{2}{x^2+y^2+z^2}$

$M=\int_L\rho(x,y,z)\,ds=\int_0^{t_0}\frac{1}{e^{2t}}\cdot\sqrt{3}e^t\,dt=\int_0^{t_0}\frac{\sqrt{3}}{e^t}\,dt=\sqrt{3}(1-e^{-t_0})$