补充题：1. 令 $\varphi(R)=\frac{1}{2\pi R}\int_{L_R}u(x,y)\,ds \xlongequal[y=y_0+R\sin\theta]{x=x_0+R\cos\theta} \frac{1}{2\pi}\int_0^{2\pi}u(x_0+R\cos\theta,$

$y_0+R\sin\theta)\,d\theta \Rightarrow \varphi'(R)=\frac{1}{2\pi}\int_0^{2\pi}u_x(x_0+R\cos\theta,y_0+R\sin\theta)(\cancel{-\sin\theta}\ \cos\theta)+u_y(x_0+R\cos\theta,$

$y_0+R\sin\theta)(\cancel{\cos\theta}\ \sin\theta)\,d\theta=\frac{1}{2\pi R}\int_{L_R}u_x(x,y)\,dy \pm u_y(x,y)\,dx$

$\xlongequal{\text{格林}} \frac{1}{2\pi R}\int_{B((x_0,y_0),R)}u_{xx}+u_{yy}\,dxdy=0 \Rightarrow \varphi(R)$ 为常值

从而 $\varphi(R)=\lim_{R\to 0^+}\varphi(R)=u(x_0,y_0)$

2. 若 $u(x,y)$ 在 (x_0,y_0) 内取到最大值，则 $\forall\ 0<R<\operatorname{dist}((x_0,y_0),\partial D)$

$u(x_0,y_0)=\frac{1}{2\pi R}\int_{L_R}u(x,y)\,ds \le \frac{1}{2\pi R}\int_{L_R}u(x_0,y_0)\,ds=u(x_0,y_0)$ ⊛

故 $u(x,y)\equiv u(x_0,y_0)$，$(x,y)\in L_R$，由 D 的连通性知 $\forall(x,y)\ne(x_0,y_0)$

存在一条落于 D 内的曲线 l 连结 (x_0,y_0) 与 (x,y)，记 $d=\operatorname{dist}(l,\partial D)$

则 $\forall(x,y)\in l$，$B((x,y),\cancel{d}\ s)\subset D \Longrightarrow B((x_0,y_0),d)\cap l$ 上 $u\equiv u(x_0,y_0)$
$0<s<d$

以又 $\{B((x,y),\cancel{s}\frac{d}{2})\}_{(x,y)\in l}$ 为紧集 l 的一个开覆盖，故存在 l 上有限个

点 $(x_1,y_1),\cdots,(x_n,y_n)$ 使得 $l\subset\bigcup_{i=1}^{n}B((x_i,y_i),\frac{d}{2})$ 且不妨设 ~~因~~

$(x_{i+1},y_{i+1})\in B((x_i,y_i),d)$ 不断对 (x_i,y_i) 重复 ⊛ 中讨论可得

$\forall(x,y)\in l$，$u(x,y)\equiv u(x_0,y_0)$，从而 $u(x,y)\equiv u(x_0,y_0)$，$(x,y)\in D$

这与 $u(x,y)$ 非常数矛盾！而 $-u$ 也是调和函数不可在 D 内取最大值从而 u 不可在 D 内取最小值.

11.3.6

(1) 原式 $\xlongequal[y=a\sin\theta]{x=a\cos\theta}\int_{\pi}^{0}\frac{-a\sin\theta(-a\sin\theta)+a\cos\theta(a\cos\theta)}{a^2}\,d\theta=-\pi$

(2) 记 L_1 为从 B 到 $(\frac{1}{2},0)$ 的直线段，L_2 为从 $(\frac{1}{2},0)$ 沿圆周 $y=\sqrt{\frac{1}{4}-x^2}$ 到 $(-\frac{1}{2},0)$ 的圆弧，L_3 为从 $(-\frac{1}{2},0)$ 到 A 的直线段

故 $\int_{L+L_1+L_2+L_3}\frac{-y\,dx+x\,dy}{x^2+y^2}\overset{\text{Green}}{\underset{\text{公式}}{=\!=}}\iint_Z \left(\frac{x}{x^2+y^2}\right)_x-\left(\frac{-y}{x^2+y^2}\right)_y dx\,dy=0$

而 $\int_{L_1}\frac{-y\,dx+x\,dy}{x^2+y^2}=\int_{L_3}\frac{-y\,dx+x\,dy}{x^2+y^2}=0$，$\int_{L_2}\frac{-y\,dx+x\,dy}{x^2+y^2}\overset{(1)}{=\!=}\pi$

故 $\int_L\frac{-y\,dx+x\,dy}{x^2+y^2}=-\int_{L_1+L_2+L_3}\frac{-y\,dx+x\,dy}{x^2+y^2}=-\pi$

11.3.7 (1) $\oint_L\frac{\partial f}{\partial n}ds=\oint_L f_1\vec{n}_1+f_2n_2\,ds=\oint_L f_1\,dy-f_2\,dx$

$\overset{\text{Green}}{\underset{\text{公式}}{=\!=}}\iint_D f_{11}-(-f_2)_2\,dx\,dy=\iint_D\Delta f\,dx\,dy.$

(2) $\oint_L\cos(a,n)\,ds=\oint_L\vec{a}\cdot\vec{n}.\,ds=\oint_L\frac{\partial(a_1x+a_2y)}{\partial\vec{n}}ds=0$

(3) 同(1)证明有 $\oint_L\vec{F}\cdot\vec{n}\,ds=\oint_L-F_2\,dx+F_1\,dy=\iint_D \operatorname{div}\vec{F}\,dx\,dy$

故 $\oint_L(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n})ds=\oint_L(v\nabla u-u\nabla v)\vec{n}\,ds=\iint_D \operatorname{div}(v\nabla u-u\nabla v)dxdy$

$=\iint_D(v\Delta u-u\Delta v)\,dx\,dy$

11.4.1

(2) $r(\theta,y)=(R\cos\theta,y,R\sin\theta)$，$r_\theta=(-R\sin\theta,0,R\cos\theta)$，$r_y=(0,1,0)$

$(-\frac{\pi}{2}\le\theta\le\frac{\pi}{2},0\le y\le h)$，则原式 $=\iint_D-\begin{vmatrix}0&0&yR^2\sin\theta\cos\theta\\-R\sin\theta&0&R\cos\theta\\0&1&0\end{vmatrix}d\theta\,dy$

$=+R^3\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^2\theta\cos\theta\,d\theta\cdot\int_0^h y\,dy=\frac{1}{3}R^3h^2$

(4) $r(\theta,\varphi)=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$ $(0\le\theta\le\frac{\pi}{2},0\le\varphi\le2\pi)$

$\frac{\partial(z,x)}{\partial(\theta,\varphi)}=\begin{vmatrix}-\sin\theta&0\\\cos\theta\cos\varphi&-\sin\theta\sin\varphi\end{vmatrix}=\sin^2\theta\sin\varphi$

故原式 $=\int_0^{\frac{\pi}{2}}d\theta\int_0^{2\pi}\sin^3\theta\cos\theta\sin^2\varphi\,d\varphi=\frac{\pi}{4}$

(6) $r(\theta,z)=(z\cos\theta,\ z\sin\theta,\ z)\quad (0\le\theta\le 2\pi,\ 0\le z\le 1)$

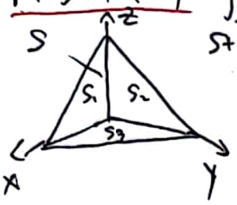
原式 $= \int_0^{2\pi} d\theta \int_0^1 -\begin{vmatrix} z\sin\theta - z & z - z\cos\theta & z\cos\theta - z\sin\theta \\ -z\sin\theta & z\cos\theta & 0 \\ \cos\theta & \sin\theta & 1\end{vmatrix} dz$

$= \int_0^{2\pi}(\cos\theta-\sin\theta)\,d\theta\cdot\int_0^1 2z^2\,dz = 0$

(7) 记 $D=\{(x,y,0)\mid x^2+y^2\le a^2\}$，则 $\iint_S = \int_{S+D} x^2z^2\,dydz + x^2y\,dzdx + y^2z\,dxdy \overset{Gauss}{\underset{公式}{=}} \iiint_\Omega (z^2+x^2+y^2)\,dxdydz = \frac12\iiint_{B(0,a)} x^2+y^2+z^2\,dxdydz$

$= \frac12\int_0^a ds \int_{\partial B(0,s)} s^2\, d\sigma_s = \frac12\int_0^a 4\pi s^4\, ds = \frac{2\pi}{5}a^5$

11.5.1 (2) $\iint_{S+S_1+S_2+S_3} \_\_ \overset{Gauss}{=} \iiint_V (y+z+x)\,dxdydz = \cancel{\int_0^1 ds\int_{x+y+z=s} s\, d\sigma_s} = \cancel{\frac{\sqrt3}{16}}\ \frac18$



而 $\cancel{S}\iint_{S_1+S_2+S_3} \_\_ = 0 \Rightarrow$ 原式 $=\cancel{\frac{\sqrt3}{16}}\ \frac18$ （以下"—"均代表题目中的被积式）

(5) 记 $D=\{(x,y,1)\mid x^2+y^2\le1\}$，则 $\iint_{S+D}\_\_ \overset{Gauss}{=} \iiint_V 3\,dxdydz$ （D取上侧）

$=\int_0^1 dz\iint_{x^2+y^2\le z} 3\,dxdy = \int_0^1 3\pi z\, dz = \frac32\pi$

而 $\iint_D \_\_ = \iint_{x^2+y^2\le1} (\cancel{x^2+y^2}\,1 - y)\,dxdy \overset{x=r\cos\theta}{\underset{y=r\sin\theta}{=}} \int_0^1 dr\int_0^{2\pi}(\cancel{r^2}\,1 - r\sin\theta)\, r\, d\theta = \pi$

故原式 $=\iint_{S+D}\_\_ - \iint_D\_\_ = \frac{3\pi}{2}-\pi=\frac\pi2$.

11.5.3 设 ~~V_1为$BS(0$~~ S_1为$B(0,\frac12)$表面，取指向原心一侧 $(r=\sqrt{x^2+y^2+z^2})$

则 $\iint_{S_1}\_\_ = \iint_{S_1}\frac1{r^3}(x,y,z)\cdot -\frac1r(x,y,z)\,dS = \iint_{S_1} -4\, dS = -4\pi$

而 $\iint_{S+S_1} \overset{Gauss}{=} \iiint_\Omega \left(\frac{x}{r^3}\right)_x+\left(\frac{y}{r^3}\right)_y+\left(\frac{z}{r^3}\right)_z\, dxdydz = 0$

故 原式 $=\iint_{S+S_1}\_\_ - \int_{S_1} = 4\pi$

11.5.4 $0=\oint\!\!\oint_S \text{—} \xlongequal{\text{Gauss}} \iiint_V (xf)_x-(xyf)_y-(e^{2x}z)_z \, dxdydz$

$=\iiint_V [xf'(x)+(1-x)f(x)-e^{2x}]\,dxdydz$ 在 $x>0$ 中任一区域 V 成立

故 $xf'(x)+(1-x)f(x)-e^{2x}=0,\ \forall x>0 \Rightarrow f(x)=\dfrac{e^x(e^x-1)}{x}+\dfrac{(C+1)e^x}{x}$

由 $\lim\limits_{x\to0^+}\dfrac{e^x}{x}$ 不存在 知 $C=-1$, $f(x)=\dfrac{e^x(e^x-1)}{x}$

11.5.7 $\iint_S \cos\langle c,n\rangle\, dS=\dfrac{1}{|c|}\iint_S \vec{c}\cdot\vec{n}\,dS=\dfrac{1}{|c|}\iiint_V \operatorname{div}\vec{c}\,dxdydz=0$

11.5.9 (2) 设 $x=a\cos t$, $y=a\sin t$, $z=h-h\cos t$ (从 $(a,0,0)$ 回到原处时 t 从 0 变化到 2π, 故原式 $=\int_0^{2\pi}(a\sin t-h+h\cos t)(-a\sin t)+(h-h\cos t-a\cos t)(a\cos t)+(a\cos t-a\sin t)(h\sin t)\,dt=\int_0^{2\pi}[-a^2-ha+ha(\sin t+\cos t)]\,dt$

$=-2\pi a(h+a)$

(4) 设 Σ 为 $y=z$ 被 $x^2+y^2\le 2y$ 所截部分, 则由 Stokes 公式, $\vec{n}=(0,\frac{1}{\sqrt2},-\frac{1}{\sqrt2})$

原式 $=\iint_\Sigma \begin{vmatrix} 0 & \frac{1}{\sqrt2} & -\frac{1}{\sqrt2} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} dS=\dfrac{1}{\sqrt2}\iint_\Sigma (y-z)\,dS=0$

11.5.10 (1) 圆面法向量 $\vec{n}=(0,0,1)$, 则 $\oint_L \text{—}=\iint_S \begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} d\sigma$

$=\iint_S -3x^2y^2\,d\sigma \xlongequal[y=r\sin\theta]{x=r\cos\theta} \int_0^R dr\int_0^{2\pi} -3r^4\cos^2\theta\sin^2\theta\cdot r\,d\theta=-\frac{\pi}{8}R^6$

(2) 半球面法向量 $\vec{n}=\frac{1}{R}(x,y,z)$, 同理有 $\oint_L \text{—}=\iint_S \operatorname{rot}\vec{v}\cdot\vec{n}\,d\sigma$

$=\dfrac{1}{R}\iint_S (x-3x^2y^2z)\,d\sigma \xlongequal[\substack{y=R\sin\theta\sin\varphi \\ z=R\cos\theta}]{x=R\sin\theta\cos\varphi} \int_0^{\frac{\pi}{2}}d\theta\int_0^{2\pi}\frac{1}{R}(R\sin\theta\cos\varphi-3R^5\sin^4\theta\cos\theta\cos^2\varphi\sin^2\varphi)R^2\sin\theta\,d\theta d\varphi=-\frac{\pi}{8}R^6$ 结果相同

11.7.3 (3) 由 $\nabla\times\vec{v}=(R_y-Q_z)i+(P_z-R_x)j+(Q_x-P_y)k=0$

知 v 为有势场, $\varphi(r,\theta,\phi)=\int_{(0,0,0)}^{(r,\theta,\phi)} r^2\vec{r}\,d\vec{r}+C=\frac{1}{4}r^4+C$

11.7.4. $\nabla\times\vec{F} = \left((2-2a)x,\ (1-a)y+(3a-3)z+5-5a\right) = 0 \Rightarrow a=1$

故 $\varphi(x,y,z) = \int_{(0,0,0)}^{(x,y,z)} \vec{F}\cdot d\vec{r} + C = \int_{(0,0,0)}^{(x,y,z)} (x^2+5y+3yz)\,dx + (5x+3xz-2)\,dy$
$+(3xy-4z)\,dz = \int_0^x x^2\,dx + \int_0^y (5x-2)\,dy + \int_0^z (3xy-4z)\,dz + C$
$= \frac{1}{3}x^3 + (5x-2)y + (3xyz - 2z^2) + C$

11.7.5 (1) $\begin{cases} u_x = 3x^2+6xy^2 \\ u_y = 6x^2y-4y^3 \end{cases} \Rightarrow u(x,y) = x^3+3x^2y^2-y^4+C$

11.7.6 (1) $P=(x-y)$, $Q=y-x \Rightarrow Q_x = P_y \Rightarrow$ 与路径无关

故 $\int_{(0,0)}^{(1,1)} = \underbrace{\int_{(0,0)}^{(0,1)} + \int_{(0,1)}^{(1,1)}}_{\text{直线段}} = \int_0^1 y\,dy + \int_0^1 (x-1)\,dx = 0$

与路径无关

(3) 当 $(x,y)\neq(0,0)$ 时, $\int_{(1,0)}^{(6,3)}$ —— $= \int_{(1,0)}^{(6,3)} d\left(\sqrt{x^2+y^2}\right) = \sqrt{x^2+y^2}\Big|_{(1,0)}^{(6,3)} = 3\sqrt{5}-1$

11.7.7 (2) 原式 $\xlongequal[dt=\frac{1}{t}(xdx+ydy+zdz)]{\sqrt{x^2+y^2+z^2}=t} \int_{t_0}^{t_0} t f(t)\,dt = 0 \qquad t_0 = \sqrt{x_0^2+y_0^2+z_0^2}$

11.7.11 $Q_x = (2xy)_y = 2x \Rightarrow Q(x,y) = x^2+f(y)$

且 $\int_{(0,0)}^{(x,y)}$ — $= \int_{(0,0)}^{(x,0)}$ — $+ \int_{(x,0)}^{(x,y)}$ — $= \int_0^y \cancel{Q}\,(x^2+f(\bar{y}))\,d\bar{y} = x^2y + \int_0^y f(\bar{y})\,d\bar{y}$

$\Rightarrow t^2 + \int_0^1 f(y)\,dy = t + \int_0^t f(y)\,dy \xRightarrow{\text{求导}} 2t = 1+f(t) \Rightarrow Q(x,y) = x^2+2y-1$

11.7.13 $(e^x\sin y + x^2y + f(x)y)_y = (f'(x)+e^x\cos y+2x)_x$

$\Rightarrow x^2+f(x) + e^x\cos y = f''(x)+e^x\cos y+2 \Rightarrow f''(x)-f(x) = x^2-2$

$\Rightarrow f(x) = -x^2+e^x-e^{-x}$ 由 $\begin{cases} u_x = e^x\sin y + x^2y + (-x^2+e^x-e^{-x})y \\ u_y = -2x+e^x+e^{-x}+e^x\cos y+2x \end{cases}$

知 $u(x,y) = e^x\sin y + (e^x+e^{-x})y + C$ 由 $du=0 \Rightarrow u=C$

从而方程解为 $e^x\sin y + (e^x+e^{-x})y = C$.