9.5.10 (1) u ~~表示~~ $= x^2+b^2(1-\frac{x}{a})^2=(1+\frac{b^2}{a^2})x^2-\frac{2b^2}{a}x+b^2$

故 u 有极小值 $=\frac{-\frac{4b^4}{a^2}}{4(1+\frac{b^2}{a^2})}+b^2=\frac{a^2b^2}{a^2+b^2}$，无极大值

9.5.11 (4) 在 $x=0,y=0$ 两边上 z 均为 0

在 $x+y=6$ 上 $(0\le x\le 6)$ $z=-2x^2(6-x)\in[\cancel{-16}\ -64,0]$

若在内部取到最值，则 $z_x=y[2x(4-x-y)+x^2(-1)]=0$

$z_y=x^2[(4-x-y)+y(-1)]=0$ $\Rightarrow$ 该点只能为 $(2,1)$ 对应 z 为 4

综上 $z_{max}=4$，$z_{min}=\cancel{-16}-64$.

9.5.13 由 $x^2+2y^2=6-2x^2-y^2$ 知 $x^2+y^2=2$

故 $z=2(x^2+y^2)-x^2\le 4$ 取最大值的点 $(0,\pm\sqrt{2},4)$

$z=(x^2+y^2)+y^2\ge 2$ 取最小值的点 $(\pm\sqrt{2},0,2)$

9.5.18 考虑 $d(x,y)=(x-x_1)^2+(y-y_1)^2+\cdots+(x-x_n)^2+(y-y_n)^2$

$=nx^2-2(x_1+\cdots+x_n)x+\sum_{i=1}^{n}x_i^2+ny^2-2(\sum_{i=1}^{n}y_i)\cdot y+\sum_{i=1}^{n}y_i^2$

故 $d(x,y)$ 取最小值的点为 $\left(\frac{\sum_{i=1}^{n}x_i}{n},\frac{\sum_{i=1}^{n}y_i}{n}\right)$ 即为所求.

9.5.21 记 $f(x,y,z)=\sqrt{x}+\sqrt{y}+\sqrt{z}-\sqrt{a}$

(1) $\nabla f=\frac{1}{2}(\frac{1}{\sqrt{x}},\frac{1}{\sqrt{y}},\frac{1}{\sqrt{z}})=\vec{n}$ 从而过点 (x_0,y_0,z_0) 的切平面

为 $(x-x_0)\frac{1}{2}\frac{1}{\sqrt{x_0}}+(y-y_0)\frac{1}{2\sqrt{y_0}}+(z-z_0)\frac{1}{2\sqrt{z_0}}=0$ 即 $\frac{x}{\sqrt{x_0}}+\frac{y}{\sqrt{y_0}}+\frac{z}{\sqrt{z_0}}=\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0}$

$=\sqrt{a}$ 从而截距之和为 $\sqrt{a}(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=a$.

(2) 设三截距分别为 d,b,c，则 $b+c+d=a$

$V_{四面体}=\frac{1}{6}bcd\le\frac{1}{6}\left(\frac{b+c+d}{3}\right)^3=\frac{a^3}{162}$ 当 $b=c=d=\frac{a}{3}$ 时取等

即对应切平面为 $\frac{3x}{a}+\frac{3y}{a}+\frac{3z}{a}=1$

9.6.2. $v = w \times r = (w_2 z - w_3 y) i + (w_3 x - w_1 z) j + (w_1 y - w_2 x) k$

$$\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = 2w_1 i + 2w_2 j + 2w_3 k = 2w$$

9.6.3 (1) $\nabla v = (6x, 3y^2 + z^2, xy - 6xz)\Big|_{(1,2,2)} = (6, 16, -14)$

(2) $\nabla v = (2x\sin y, 2y\sin(xz), xy\cos(\cos z))(-\sin z))$

9.6.5 (1) $\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2zi - 2xj - 2yz$

(2) $\nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} = (1 + 2e^y z - 1) i - (xe^y + 1) k$

9.6.9 $\nabla \times \nabla \phi = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \phi_x & \phi_y & \phi_z \end{vmatrix} = (\phi_{zy} - \phi_{yz}) i + (\phi_{xz} - \phi_{zx}) j + (\phi_{yx} - \phi_{xy}) k = 0$

$\nabla \times a = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix} = [(a_3)_y - (a_2)_z] i + [(a_1)_z - (a_3)_x] j + [(a_2)_x - (a_1)_y] k$

$\nabla \cdot (\nabla \times a) = (a_3)_{yx} - (a_2)_{zx} + (a_1)_{zy} - (a_3)_{xy} + (a_2)_{xz} - (a_1)_{yz} = 0$

10.1.2 (1) 原式 $= \int_0^1 dx \int_0^1 \frac{y}{(1+x^2+y^2)^{\frac{3}{2}}} dy = \int_0^1 -(y^2+x^2+1)^{-\frac{1}{2}} \Big|_{y=0}^{y=1} dx$

$= \int_0^1 \frac{1}{\sqrt{x^2+1}} dx - \int_0^1 \frac{1}{\sqrt{x^2+2}} dx = \left(\frac{1}{2} \ln \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}-x} - \frac{1}{2} \ln \frac{\sqrt{x^2+2}+x}{\sqrt{x^2+2}-x}\right)\Big|_0^1 = \ln \frac{(2+\sqrt{2})(\sqrt{3}-1)}{2}$

(2) 原式 $= \int_0^\pi dy \int_0^\pi \sin(x+y)\, dx = \int_0^\pi 2\cos y\, dy = 0$

10.1.4 由 $\varphi \in R[a,b]$ 知其不连续点所成集合 A 为 R^1 中零测集，从而 $F(x,y) = \varphi(x)$ 在 R^2 中不连续点所成集合为 $A \times [0,1]$，仍为 R^2 中零测集

故 $F \in R([a,b]\times[c,d])$，同理 $G(x,y) = \psi(y) \in R([a,b]\times[c,d])$

从而 $f(x,y) = F(x,y)\cdot G(x,y) \in R(D)$，进而由 Fubini thm 知

$\iint_D f(x,y)\,dx\,dy = \int_c^d \psi(y)\,dy \cdot \int_a^b \varphi(x)\,dx.$

10.1.6 原式 $= \int_c^d dy \int_a^b \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right) dx = \int_c^d \frac{\partial f}{\partial y}(b,y) - \frac{\partial f}{\partial y}(a,y)\, dy$

$= f(b,y)\Big|_c^d - f(a,y)\Big|_c^d = f(b,d) + f(a,c) - f(b,c) - f(a,d).$

10.1.1 (1) 原式 $= \int_0^1 dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\, dx$　(3) 原式 $= \int_0^1 dy \int_y^{2-y} f(x,y)\, dx$

(2) 原式 $= \int_0^{2a} dx \int_0^{\sqrt{2ax-x^2}} f(x,y)\, dy$　　Rmk: 画图！

10.1.2 (5) 原式 $= \int_a^{3a} dy \int_{y-a}^{y} (x+y-1)\, dx = \int_a^{3a} 2ay - \frac{1}{2}a^2 - a\, dy = 7a^3 - 2a^2$

(8) 原式 $= \int_0^{\frac{\pi}{4}} dy \int_y^{\frac{\pi}{2}-y} \cos(x+y)\, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} dx \int_{\frac{\pi}{2}-x}^{x} -\cos(x+y)\, dy$

$= \int_0^{\frac{\pi}{4}} (1-\sin 2y)\,dy + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\sin 2x)\, dx = y + \frac{1}{2}\cos 2y \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1$

10.1.3 (1) 原式 $= 4\int_0^1\int_0^1 (x^2+y^2)\,dx\,dy = 8\int_0^1\int_0^1 x^2\,dx\,dy = \frac{8}{3}$

(2) 原式 $= 0$　被积函数关于 x 为奇函数，被积区域关于 y 轴对称

10.1.5 $\int_0^a dx\int_0^x f(x)f(y)\,dy=\frac{1}{2}\left(\int_0^a dx\int_0^x f(x)f(y)dy+\int_0^a dy\int_0^y f(x)f(y)\,dx\right)$

$=\frac{1}{2}\left(\int_0^a dx\int_0^x f(x)f(y)\,dy+\int_0^a dx\int_x^a f(x)f(y)\,dy\right)=\frac{1}{2}\int_0^a\int_0^a f(x)f(y)\,dxdy$

$=\frac{1}{2}\left(\int_0^a f(x)\,dx\right)^2$; $\int_0^a dx\int_0^x f(y)\,dy=\int_0^a dy\int_y^a f(y)\,dx=\int_0^a (a-y)f(y)\,dy$

10.1.7 由 $f\in C((0,0))$ 可得 $\forall\varepsilon>0,\exists r>0$, st $|f(x,y)-f(0,0)|<\varepsilon\quad\forall(x,y)\in B_r(0)$

故 $\left|\frac{1}{\pi r^2}\iint_{B_r(0)}f(x,y)\,dxdy-f(0,0)\right|\leq\frac{1}{\pi r^2}\iint_{B_r(0)}|f(x,y)-f(0,0)|\,dxdy<\varepsilon$

令 $\varepsilon\to0^+$ 即得 $\lim\limits_{r\to0}\frac{1}{\pi r^2}\iint_{B_r(0)}f(x,y)\,dxdy=f(0,0)$

10.2.1.(1) 原式 $\xlongequal[y=r\sin\theta]{x=r\cos\theta}\int_0^{\frac{\pi}{2}}d\theta\int_0^R \ln(1+r^2)\cdot r\,dr=\frac{\pi}{4}\left[\ln(1+R^2)\cdot(1+R^2)-R^2\right]$

(3) 原式 $=\int_0^\pi \sin(x+y)\Big|_0^\pi dy=\int_0^\pi -2\sin y\,dy=-4.$

(5) 原式 $=\iint_{S_{扇形}}\left(1+\frac{y^2}{x^2}\right)dxdy\xlongequal[y=r\sin\theta]{x=r\cos\theta}\int_0^{\arctan R}d\theta\int_0^R(1+\tan^2\theta)\,r\,dr$

$=\frac{1}{2}R^2\int_0^{\arctan R}(1+\tan^2\theta)\,d\theta=\frac{1}{2}R^3$

10.2.2 (2) 原式 $\xlongequal[y=br\sin\theta]{x=ar\cos\theta}\int_0^{\arctan\left|\frac{a}{b}\right|}d\theta\int_0^2 r\cdot abr\,dr=\frac{8}{3}ab\arctan\left|\frac{a}{b}\right|$

(4) 原式 $\xlongequal[v=\frac{x^2}{y}]{u=\frac{y^2}{x}}\int_b^a du\int_n^m\frac{1}{3}\,dvdu=\frac{1}{3}(a-b)(m-n)$

(6) 原式 $\xlongequal[y=\sqrt{r\sin\theta}]{x=\sqrt{r\cos\theta}}\int_0^{\frac{\pi}{2}}d\theta\int_0^1 4r\sqrt{\sin\theta\cos\theta}\,\frac{1}{4\sqrt{\sin\theta\cos\theta}}\,dr=\frac{\pi}{4}$

(8) 原式 $\xlongequal[y=v]{x+y=u}\int_0^1 du\int_0^u \sin\frac{v}{u}\,dv=\int_0^1(-u\cos1+u)\,du=\frac{1-\cos1}{2}$

10.2.3 (1) $S=2\int_1^{\sqrt2}dx\int_{\frac{1}{x}}^{\sqrt{\frac{3-x^2}{2}}}dy=2\int_1^{\sqrt2}\sqrt{\frac{3-x^2}{2}}-\frac{1}{x}\,dx$

$=\sqrt2\left(\frac{x}{2}\sqrt{3-x^2}-\frac{3}{2}\arccos\frac{x}{\sqrt3}\right)\Big|_1^{\sqrt2}-2\ln x\Big|_1^{\sqrt2}=\frac{3\sqrt2}{2}\left(\arccos\frac{1}{\sqrt3}-\arccos\frac{\sqrt2}{\sqrt3}\right)-\ln2$

(2) $S=\int_{-a}^{a}dx\int_{x-\sqrt{a^2-x^2}}^{x+\sqrt{a^2-x^2}}dy=2\int_{-a}^{a}\sqrt{a^2-x^2}\,dx=S_{圆}=\pi a^2$

10.2.5 由 Cauchy Inequality，左 $\geq\left(\int_0^1\sqrt{e^f\cdot e^{-f}}\,dx\right)^2=1$

10.2.6 左 $\xlongequal[v=x-y]{u=x+y}\int_{-1}^{1}dv\int_{-1}^{1}e^{f(u)}\frac{1}{2}du=\int_{-1}^{1}e^{f(u)}du.$

$=\int_0^1 e^{f(u)}+e^{f(-u)}du=\int_0^1 e^{f(u)}+e^{-f(u)}du\geq\int_0^1 2\,du=2.$

10.2.7 左 $\xlongequal[y=v]{x-y=u}\int_{-\frac{A}{2}}^{\frac{A}{2}}dv\int_{-\frac{A}{2}-v}^{\frac{A}{2}-v}f(u)\,du=\int_0^A du\int_{-\frac{A}{2}}^{\frac{A}{2}-u}f(u)\,dv$

$+\int_{-A}^{0}\int_{-\frac{A}{2}-u}^{\frac{A}{2}}f(u)\,dv=\int_0^A f(u)(A-|u|)\,du+\int_{-A}^{0}(A-|u|)f(u)\,du$

$=\int_{-A}^{A}f(t)(A-|t|)\,dt.$

习题课提要

1. Cauchy 不等式的证明 $<$ "Δ"法 / 重积分

2. 设 f 是定义在 $B_1(0)$ 上的三阶连续可微函数且 $f(0,0)=0$

(1) 证明、存在 $B_1(0)$ 上 2 阶连续可微函数 g_1, g_2 满足

$$f(x,y)=x\,g_1(x,y)+y\,g_2(x,y)$$

(2) 又设 $\nabla f(0,0)=0$ 且 $\det\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2}\end{pmatrix}(0,0)<0$

证明、在原点的一个邻域内存在变换 $x=x(u,v)$, $y=y(u,v)$

使得 $f(x(u,v),y(u,v))=u^2-v^2$.

3. $x,y,z\geq 0$, $x+y+z=1$，求 $\max x^a y^b z^c$ $(a,b,c>0)$