(3) $$\int_0^{+\infty} \frac{1-e^{-ax^2}}{x^2}\,dx = \int_0^{+\infty}\int_0^a \frac{x^2 e^{-ux^2}}{x^2}\,du\,dx$$

$$= \int_0^a \int_0^{+\infty} e^{-ux^2}\,dx\,du \overset{y=\sqrt{u}x}{=} \int_0^a \frac{1}{\sqrt{u}} \int_0^{+\infty} e^{-y^2}\,dy\,du$$

$$= \frac{\sqrt{\pi}}{2}\int_0^a \frac{1}{\sqrt{u}}\,du = \sqrt{\pi a}$$

(5) $$\int_0^{+\infty} \frac{\arctan ax}{x(1+x^2)}\,dx = \int_0^{+\infty}\int_0^a \frac{x}{1+(ux)^2}\,\frac{1}{x(1+x^2)}\,du\,dx$$

$$= \int_0^a \int_0^{+\infty} \frac{1}{1+u^2x^2}\cdot\frac{1}{1+x^2}\,dx\,du$$

$$= \frac{\pi}{2}\int_0^a \frac{1}{1+u}\,du = \frac{\pi}{2}\ln(1+a)$$

注：题中出现的含参变量积分的一致收敛性需验证。

Q8 (1), (2), (4), (6)

解: (1) 令 $y = \frac{x-a}{\sqrt{2}\sigma}$

$$原式 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} (\sqrt{2}\sigma y + a) e^{-y^2} dy$$

$$= \frac{a}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-y^2} dy = a$$

(2) $y = \frac{x-a}{\sqrt{2}\sigma}$

$$原式 = \int_{-\infty}^{+\infty} \frac{2\sigma^2 y^2}{\sigma\sqrt{2\pi}} e^{-y^2} \sqrt{2}\sigma \, dy$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} y^2 e^{-y^2} dy$$

$$\overset{分部}{=} \frac{\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-y^2} dy$$

$$= \sigma^2$$

(4) $$\int_0^{+\infty} \frac{\sin^2 x}{x^2} dx = \int_0^{+\infty} \sin^2 x \, d(-\tfrac{1}{x})$$

$$= \int_0^{+\infty} \frac{2\sin x \cos x}{x} dx$$

$$= \int_0^{+\infty} \frac{\sin 2x}{2x} d(2x) = \frac{\pi}{2}$$

(6) $$\int_0^{+\infty} \frac{\sin^4 x}{x^2} dx \overset{分部}{=} \int_0^{+\infty} \frac{4\sin^3 x \cos x}{x} dx$$

$$= \int_0^{+\infty} \frac{(3\sin x - \sin 3x)\cos x}{x} dx$$

$$= \int_0^{+\infty} \frac{1}{x}\left(-\frac{1}{2}\sin 4x + \sin 2x\right) dx$$

$$= \frac{\pi}{4}.$$

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Q1

Pf: $$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt \overset{t=x^2}{=} 2\int_0^{+\infty} x^{2s-1} e^{-x^2} dx$$

$$\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt \overset{t=ax}{=} a^s \int_0^{+\infty} x^{s-1} e^{-ax} dx$$

Q2

PF: $B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dt$

$$\overset{x=\sin^2 t}{=} 2\int_0^{\frac{\pi}{2}} \sin^{2p-1} t \cos^{2q-1} t\,dt$$

Q3 (2),(4),(6),(9),(10)

解: (2) 不妨设 $a>0$

$$\int_0^a x^2\sqrt{a^2-x^2}\,dx \overset{x^2=a^2y}{=} \int_0^1 a^3 y\sqrt{1-y}\,\frac{a}{2\sqrt{y}}\,dy$$

$$= \frac{1}{2}a^4 B\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{1}{16}a^4\pi$$

(4) $\int_0^1 x^{n-1}(1-x^m)^{q-1}\,dx \overset{y=x^m}{=} \int_0^1 y^{\frac{n-1}{m}}(1-y)^{q-1}\frac{1}{m}y^{\frac{1}{m}-1}\,dy$

$$= \frac{1}{m}B\left(\frac{n}{m}, q\right)$$

(6) $\int_0^{\frac{\pi}{2}} \tan^\alpha x\,dx = \int_0^{\frac{\pi}{2}} \sin^\alpha x \cos^{-\alpha} x\,dx = \frac{1}{2}B\left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right)$

(9) $\lim\limits_{n\to\infty}\int_1^2 (x-1)^2\sqrt[n]{\frac{2-x}{x-1}}\,dx \overset{x=1+t}{=} \lim\limits_{n\to\infty}\int_0^1 t^2\left(\frac{1-t}{t}\right)^{\frac{1}{n}}\,dt$

$$= \lim_{n\to\infty} B\left(3-\frac{1}{n}, 1+\frac{1}{n}\right)$$

$$= B(3,1) = \frac{1}{3}$$

(10) $\lim\limits_{n\to\infty}\int_0^{+\infty}\frac{1}{1+x^n}\,dx \overset{y=x^n}{=} \lim\limits_{n\to\infty}\int_0^{+\infty}\frac{1}{1+y}\frac{1}{n}y^{\frac{1}{n}-1}\,dy$

$$= \lim_{n\to\infty}\frac{1}{n}B\left(1-\frac{1}{n}, \frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\Gamma\left(1-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\frac{\pi}{\sin\frac{\pi}{n}} = 1$$

4. 解: $\lim\limits_{\alpha\to\infty}\sqrt{\alpha}\int_0^1 x^{\frac{3}{2}}(1-x^5)^\alpha\,dx \overset{y=x^5}{=} \lim\limits_{\alpha\to\infty}\sqrt{\alpha}\int_0^1 y^{\frac{3}{10}}(1-y)^\alpha\frac{1}{5}y^{\frac{1}{5}-1}\,dy$

$$= \lim_{\alpha\to\infty}\frac{\sqrt{\alpha}}{5}B\left(\frac{1}{2}, \alpha+1\right) = \lim_{\alpha\to\infty}\frac{\sqrt{\pi}}{5}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}\sqrt{\alpha} = \frac{\sqrt{\pi}}{5}$$

$$\Gamma(x) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$$

$$\Rightarrow \frac{\Gamma(\alpha+1)\sqrt{\alpha}}{\Gamma(\alpha+\frac{3}{2})} \sim \frac{\left(\frac{\alpha+1}{e}\right)^{\alpha+1}\sqrt{\alpha}}{\left(\frac{\alpha+\frac{3}{2}}{e}\right)^{\alpha+\frac{3}{2}}} = \frac{(\alpha+1)^{\alpha+1}\sqrt{\alpha}}{(\alpha+\frac{3}{2})^{\alpha+\frac{3}{2}}}\sqrt{e}$$

$$= \frac{\left(\frac{1}{\alpha}+1\right)^{\alpha\cdot\frac{1+\alpha}{\alpha}}}{\left(\frac{1}{\frac{2\alpha}{3}}+1\right)^{\frac{2\alpha}{3}\cdot\frac{3}{2\alpha}\cdot(\frac{3}{2}+\alpha)}}\sqrt{e} \sim \frac{e\sqrt{e}}{e^{\frac{3}{2}}} = 1$$

Q5.

解: $S = \int_0^a (a^n - x^n)^{\frac{1}{n}}\,dx \overset{x^n = a^n t}{=} \int_0^1 a(1-t)^{\frac{1}{n}}\, a\frac{1}{n} t^{\frac{1}{n}-1}\,dt$

$$= \frac{a^2}{n} B\left(\frac{1}{n}, \frac{1}{n}+1\right)$$