中国科学技术大学
University of Science and Technology of China

第九章　多元函数微分学

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- **多变量函数的微分**
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一元函数的微分

对一元函数 $y = f(x)$，若$\Delta y = A\Delta x + o(\Delta x)$，则称$f(x)$在 x 处可微，其中$A\Delta x$ 称为微分，记为$dy = A\Delta x$. $f(x)$在 x 处可微的充要条件是 $f(x)$在该点可导，且

$$\mathrm{d}y = f'(x)\Delta x = f'(x)\mathrm{d}x$$

问题：那么对于二元函数来说，如何推广微分定义，对应的微分和导数是否能延续这样的关系?

一元函数从变化率的研究引入了导数的概念：

$$f'(x_0)=\frac{\mathrm{d}y}{\mathrm{d}x}\bigg|_{x=x_0}=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

几何意义是曲线 $y=f(x)$ 在点 $\left(x_0, f(x_0)\right)$ 处切线的斜率：

$k=\tan\alpha=f'(x_0).$

定义： 设函数 $z = f(x, y)$ 在点 (x_0, y_0) 的某一邻域内有定义. 当 y 固定在 y_0 而 x 在 x_0 处有增量 Δx 时，相应的函数有增量

$$f(x_0 + \Delta x, y_0) - f(x_0, y_0)$$

如果 $\lim\limits_{\Delta x \to 0} \dfrac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$ 存在，则称此极限为 $z = f(x, y)$ 在点 (x_0, y_0) 处**对 x 的偏导数**，记为

$$\left.\frac{\partial z}{\partial x}\right|_{(x_0, y_0)}, \quad \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)}, \quad \left. z_x^{'} \right|_{(x_0, y_0)} \text{ 或 } f_x^{'}(x_0, y_0)\text{，即}$$

$$f_x^{'}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

类似地，函数 $z = f(x, y)$ 在点 (x_0, y_0) 处**对 y 的偏导数**为

$$f_y'(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

记为

$$\left.\frac{\partial z}{\partial y}\right|_{(x_0, y_0)},\quad \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)},\quad \left.z_y'\right|_{(x_0, y_0)} \text{ 或 } f_y'(x_0, y_0).$$

二元函数偏导数的定义可以类推到三元及三元以上的函数.

由定义知：**求多元函数对某个自变量的偏导数，只要把其它自变量都当成常数，把该函数当成此自变量的一元函数求导即可.**

推论：若 $\frac{\partial f}{\partial x}\equiv 0$，则 $f(x,y)=\varphi(y)$；若 $\frac{\partial f}{\partial y}\equiv 0$，则 $f(x,y)=\psi(x)$.

对于分段函数在分段点的偏导数，**要利用偏导数的定义来求.**

Examples：

1. 设函数 $f(x,y)=x^2+(y-1)\arcsin\sqrt{\frac{y}{x}}$，求 $f_x'(2,1), f_y'(2,1)$.

2. $f(x,y)$ 满足 $\begin{cases}\dfrac{\partial f}{\partial x}=-\sin y+\dfrac{1}{1-xy}\\ f(1,y)=\sin y\end{cases}$，求 $f(x,y)$.

3. 二元函数

$$f(x,y)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

在点(0,0)的偏导数为?

$$f_x'(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{0}{\Delta x}=0,$$

$$f_y'(0,0)=\lim_{\Delta y\to 0}\frac{f(0,0+\Delta y)-f(0,0)}{\Delta y}=\lim_{\Delta x\to 0}\frac{0}{\Delta y}=0$$

注：偏导存在，未必连续.

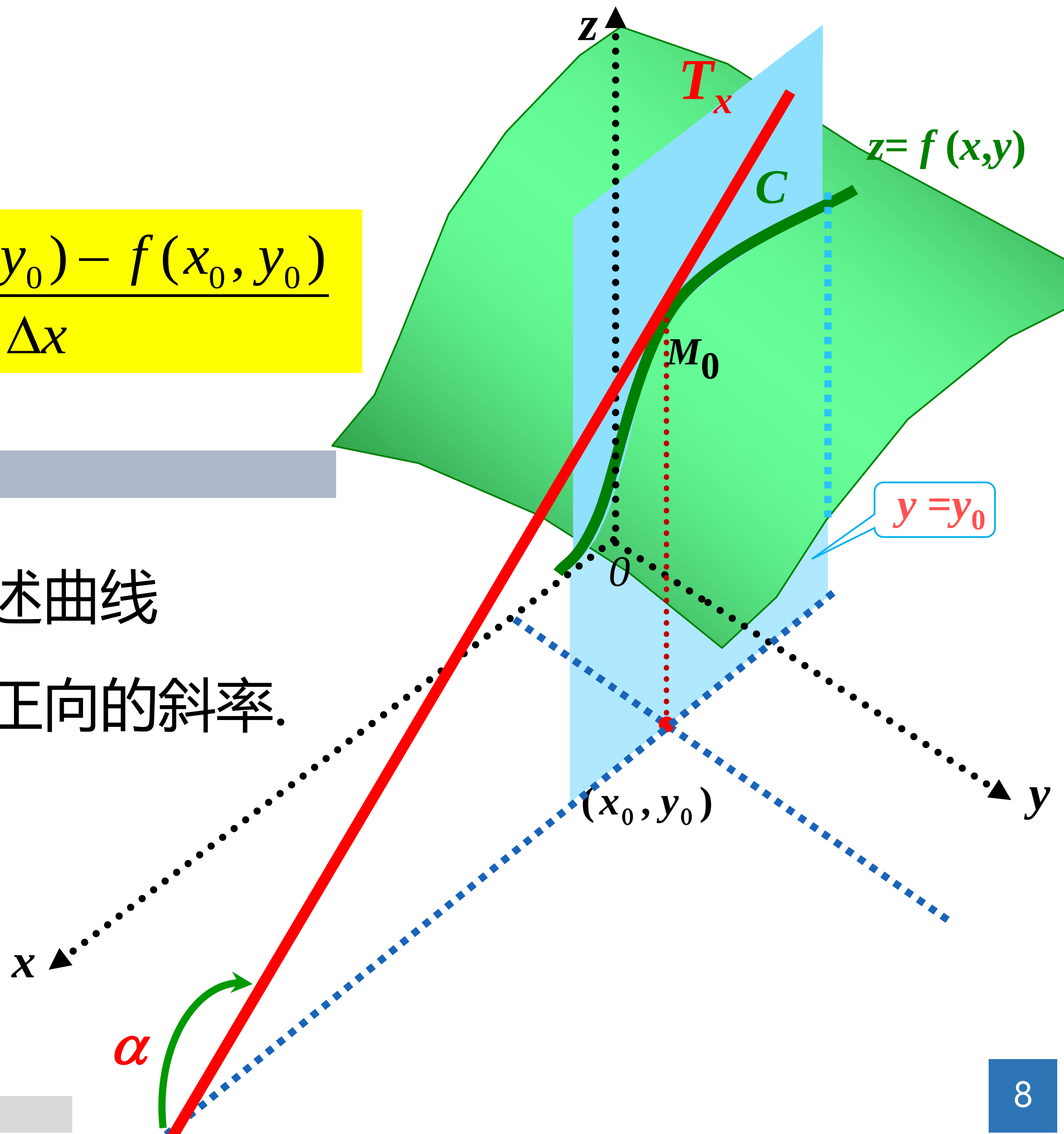
偏导数的几何意义

C 的方程 $\begin{cases} z = f(x, y_0) \\ y = y_0 \end{cases}$

$$f_x'(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

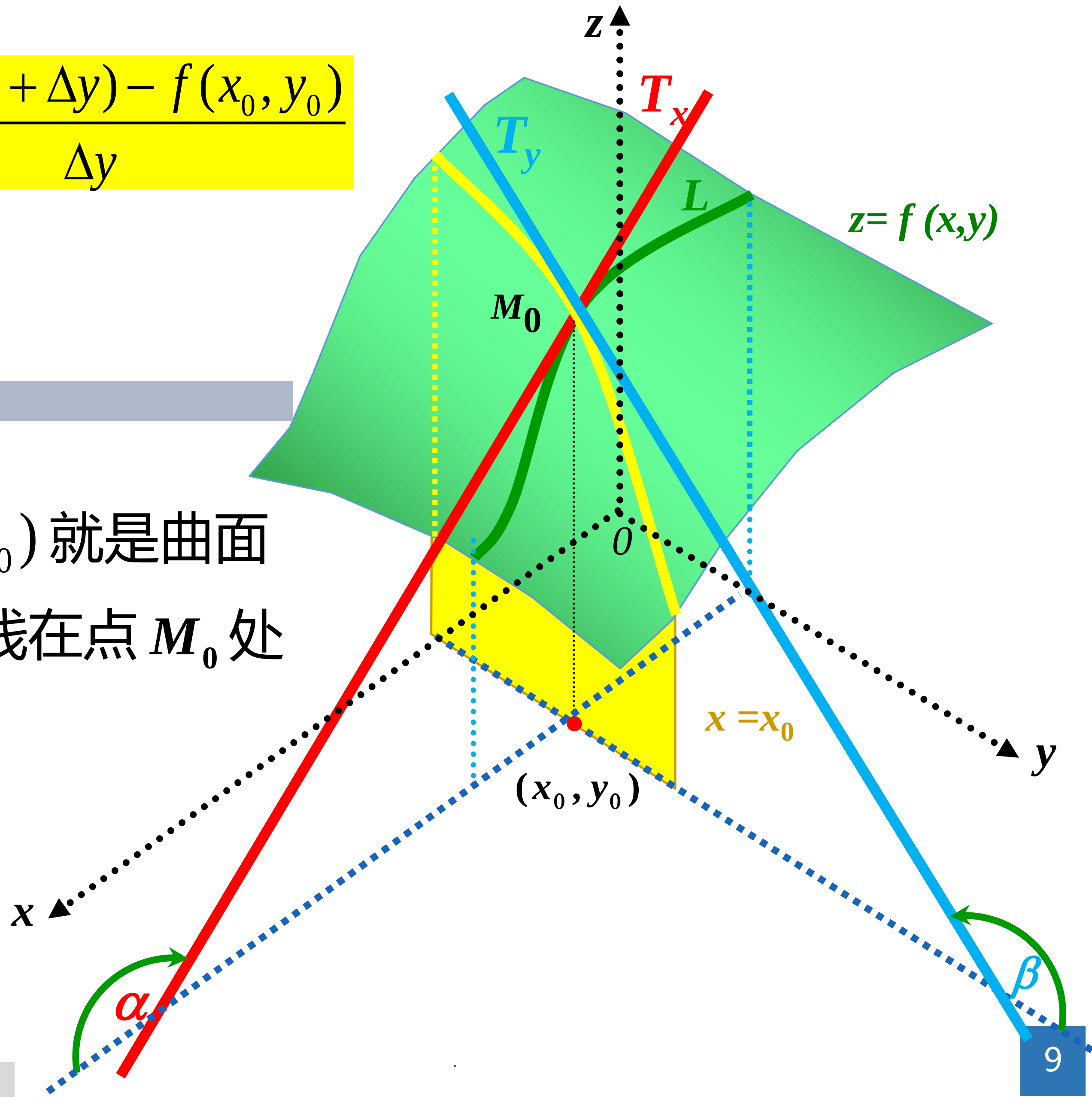
偏导数 $f_x'(x_0, y_0)$表示上述曲线

在M_0处的切线T_x对x轴正向的斜率.



$$f_y'(x_0,y_0)=\lim_{\Delta y\to 0}\frac{f(x_0,y_0+\Delta y)-f(x_0,y_0)}{\Delta y}$$

同理，偏导数 $f_y'(x_0,y_0)$ 就是曲面被平面 $x=x_0$ 所截曲线在点 M_0 处 T_y 对 y 轴正向的斜率.



设函数 $z = f(x,y)$ 在区域 D 内具有偏导数

$$\frac{\partial z}{\partial x} = f_x'(x,y), \qquad \frac{\partial z}{\partial y} = f_y'(x,y)$$

如果这两个函数的偏导数存在，则称它们是函数 $z = f(x,y)$ 的二阶偏导数．按照对变量求导次序的不同，共有四个二阶偏导数：

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}''(x,y) = f_{11}'', \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}''(x,y) = f_{12}''$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}''(x,y) = f_{21}'', \qquad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}''(x,y) = f_{22}''$$

例： 设 $z = x^3y^2 - 3xy^3 - xy + 1$，求 $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}, \frac{\partial^3 z}{\partial x^3}$.

解： $\frac{\partial z}{\partial x} = 3x^2y^2 - 3y^3 - y$, $\quad \frac{\partial z}{\partial y} = 2x^3y - 9xy^2 - x$;

$\frac{\partial^2 z}{\partial x^2} = 6xy^2 \quad \frac{\partial^3 z}{\partial x^3} = 6y^2$.

$\frac{\partial^2 z}{\partial y \partial x} = 6x^2y - 9y^2 - 1 \quad \frac{\partial^2 z}{\partial x \partial y} = 6x^2y - 9y^2 - 1$

定理：$z = f(x, y)$ 的两个二阶混合偏导函数 $\frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}$ 在 D 内连续，则两者相等，即偏导次序可以交换.

若二阶偏导不连续，不能保证偏导可交换次序.

例如：$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$ 则：

$$f''_{xy}(0,0)=\lim_{y\to 0}\frac{f'_x(0,y)-f'_x(0,0)}{y}=-1 \neq f''_{yx}(0,0)=\lim_{x\to 0}\frac{f'_y(x,0)-f'_x(0,0)}{x}=1$$

$$f''_{xy}(x,y)=\begin{cases} \dfrac{\left(x^2-y^2\right)\left(x^4+10x^2y^2+y^4\right)}{\left(x^2+y^2\right)^3}, & x^2+y^2\neq 0 \\ -1, & x^2+y^2=0 \end{cases}$$

$$f''_{yx}(x,y)=\begin{cases} \dfrac{\left(x^2-y^2\right)\left(x^4+10x^2y^2+y^4\right)}{\left(x^2+y^2\right)^3}, & x^2+y^2\neq 0 \\ 1, & x^2+y^2=0 \end{cases}$$

可验证，$f(x,y)$ 的所有二阶偏导在$(0,0)$都存在，但都不连续.

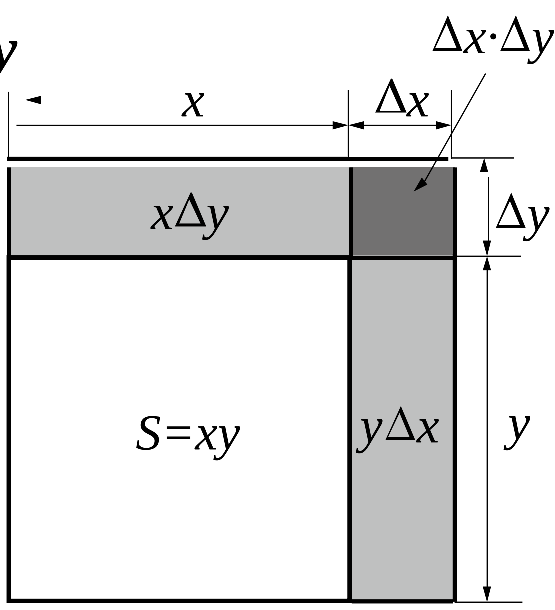
在实际问题中，有时需要研究多元函数中各个自变量都取得增量时因变量所获得的增量，即所谓全增量的问题.

例： 设矩形的长和宽分别为 x, y，则其面积为 $S = xy$. 若边长 x 有增量 Δx，y 有增量 Δy 时， 面积 S 相应的增量为?

$$\Delta S = (x+\Delta x)(y+\Delta y) - xy = y\Delta x + x\Delta y + \Delta x \cdot \Delta y$$

$$= y\Delta x + x\Delta y + o(\rho)\left(\rho = \sqrt{(\Delta x)^2 + (\Delta y)^2}\right)$$

$$\approx y\Delta x + x\Delta y.$$



$$0 \le \frac{|\Delta x \cdot \Delta y|}{\rho} = \frac{|\Delta x \cdot \Delta y|}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} \le \frac{1}{2}\sqrt{(\Delta x)^2 + (\Delta y)^2} \to 0$$

单变元函数的微分公式：$\Delta y = A\Delta x + o(\Delta x)$

一般地，若函数 $z = f(x, y)$ 在点 (x, y) 的某邻域内有定义. 称

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x, y)$$

为函数在点 (x, y) 处对应于自变量增量 $(\Delta x, \Delta y)$ 的增量.

与一元情形类似，我们希望利用关于自变量增量 $\Delta x, \Delta y$ 的线性函数来近似增量 Δz.

由此引入二元函数微分的定义.

设 $z=f(x,y)$ 为区域 D 上的二元函数，$M_0=(x_0,y_0)\in D$. 记 $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$. 若存在常数 A,B 使得当 $\rho\to 0$ 时有：

$$f\left(x_0+\Delta x,y_0+\Delta y\right)-f\left(x_0,y_0\right)=A\Delta x+B\Delta y+o(\rho)$$

则称 f 在 M_0 处**可微**，$A\Delta x+B\Delta y$ 为 f 在 M_0 处的**微分**，记为 $df\Big|_{M_0}$

显然，若 f 可微，则：

$$A=\lim_{\Delta x\to 0}\frac{f\left(x_0+\Delta x,y_0\right)-f\left(x_0,y_0\right)}{\Delta x}=\frac{\partial f}{\partial x}\left(x_0,y_0\right)$$

$$B=\lim_{\Delta y\to 0}\frac{f\left(x_0,y_0+\Delta y\right)-f\left(x_0,y_0\right)}{\Delta y}=\frac{\partial f}{\partial y}\left(x_0,y_0\right)$$

可微必可偏导

全微分的几何意义

$$z = f(x, y)$$

$M(x_0, y_0, z_0)$

$N(x_0+\Delta x, y_0+\Delta y, z_0+\Delta z)$

$\Delta z = AN$：曲面的增量

过点M的切平面：

$f'_x(x_0,y_0)(x-x_0)+f'_y(x_0,y_0)(y-y_0)-(z-z_0)=0$ 即：

$$dz = f'_x(x_0,y_0)\Delta x + f'_y(x_0,y_0)\Delta y$$

$$= z - z_0 = AB$$

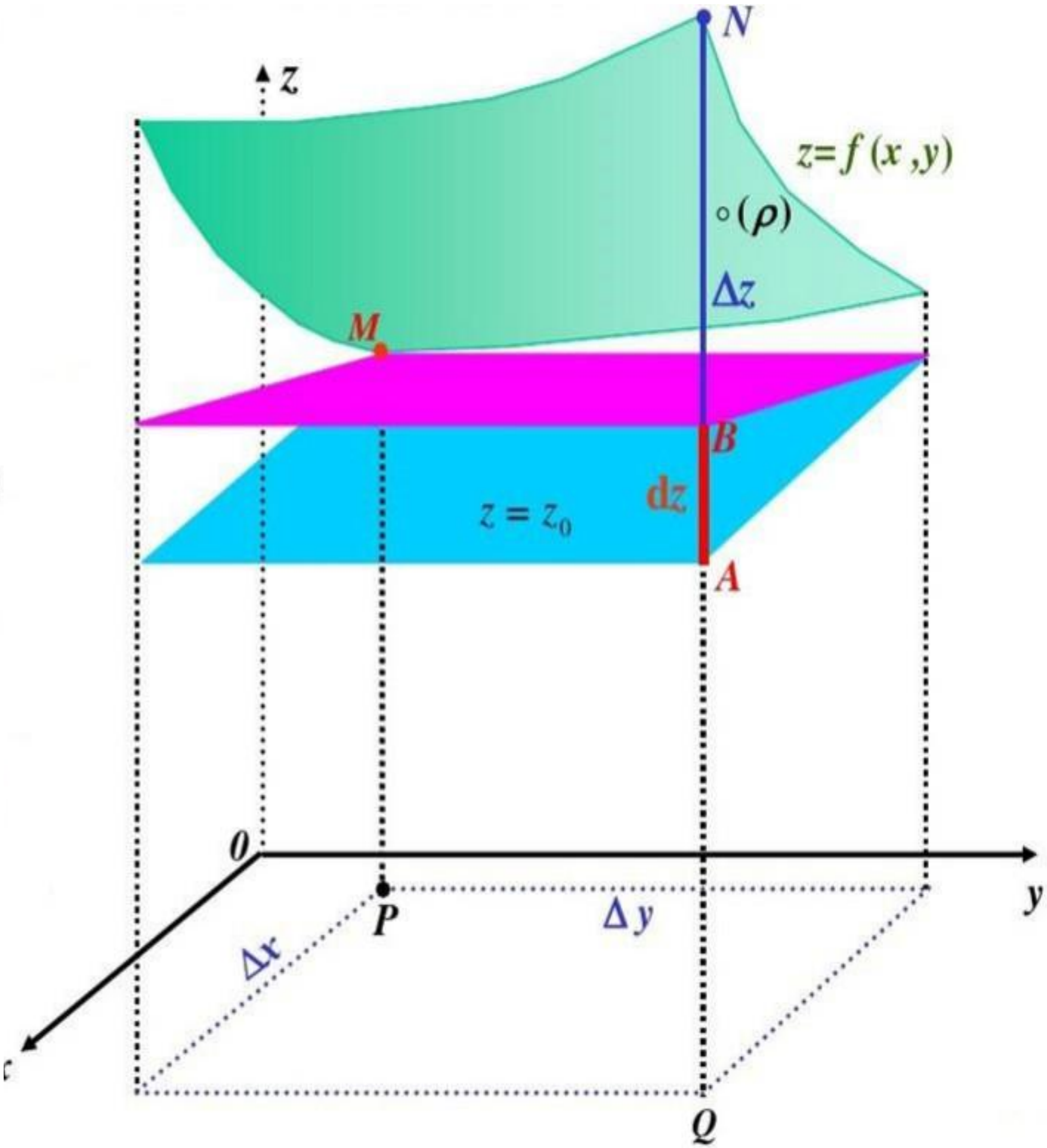
$dz = AB$：切平面的增量

$$\Delta z = dz + o(\sqrt{\Delta x^2 + \Delta y^2})$$

$$= AB + BN$$

当$\Delta x, \Delta y$很小时

$$\Delta z \approx dz$$



定理：①若 $f(x,y)$ 的两个偏导 $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ 在 (x_0,y_0) 的某邻域内存在且有界，则 $f(x,y)$ 在 (x_0,y_0) 处连续；

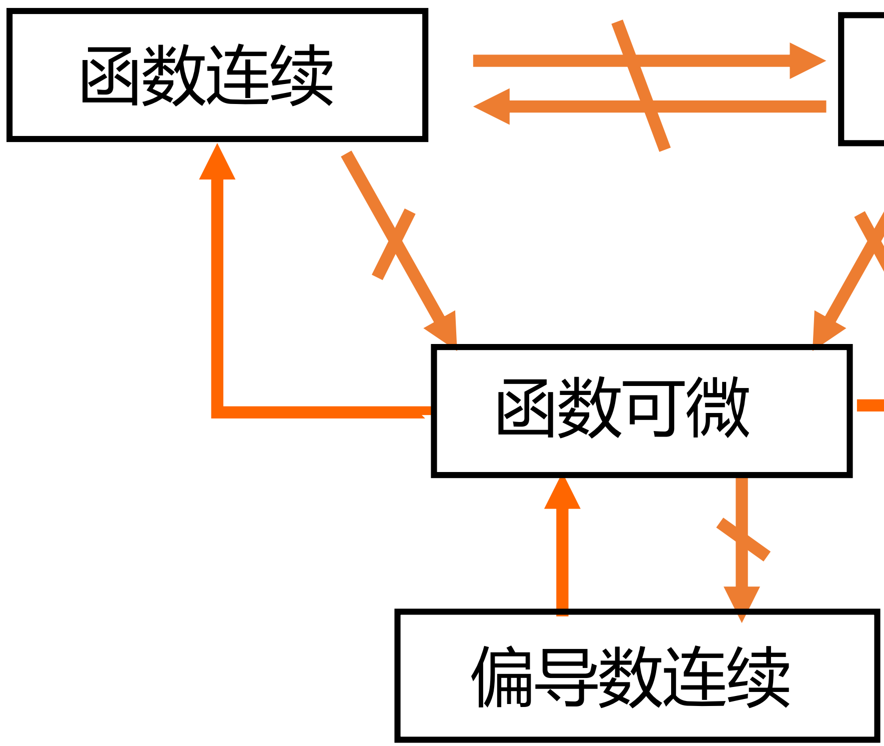
②若 $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ 在 D 内存在且有界，则 $f(x,y)$ 在 D 内一致连续.

定理：若 $f(x,y)$ 的两个偏导 $\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}$ 在 (x_0,y_0) 的某邻域内存在且在该点连续，则 $f(x,y)$ 在 (x_0,y_0) 处可微.

即：**函数的各偏导存在且连续** $\Rightarrow$ **可微.**

连续、可偏导与可微的关系

$$f(x,y)\text{ 可微}\Leftrightarrow f\left(x_0+\Delta x,y_0+\Delta y\right)-f\left(x_0,y_0\right)=A\Delta x+B\Delta y+o(\rho)$$



Examples：

1. 讨论下面函数在 $(0,0)$ 点处的连续性及偏导数的存在性.

(1) $f(x,y)=\sqrt{x^2+y^2}$. (2) $f(x,y)=\begin{cases} x^2+y^2, & xy=0 \\ 1, & xy\neq 0 \end{cases}$

2. 讨论 $f(x,y)=\sqrt{|xy|}$ 在点 $(0,0)$ 处的连续性、一阶偏导数存在性及可微性.

3. 讨论 $f(x,y)$ 在点 $(0,0)$ 处的连续性、一阶偏导数存在性和连续性、可微性.

$$f(x,y)=\begin{cases} \left(x^2+y^2\right)\sin\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0 \\ 0, & x^2+y^2=0 \end{cases}$$

例： 设二元函数

$$f(x,y)=\begin{cases}(x+y)^n\sin\dfrac{1}{\sqrt{x^2+y^2}}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$$

其中n是正整数，讨论n为何值时，函数$f(x,y)$在原点$(0,0)$处

(1)连续；(2)一阶偏导数存在；(3)可微；(4)一阶偏导数连续.

Answer: (1) $n\geq 1$ (2) $n\geq 2$ (3) $n\geq 2$ (4) $n\geq 3$

- 设 $z=f(x,y)$ 为区域 D 上的二元函数，若 f 在 D 上每一点都

可微，则称 f 在 D 上可微，$\mathrm{d}z=\mathrm{d}f=\dfrac{\partial f}{\partial x}\mathrm{d}x+\dfrac{\partial f}{\partial y}\mathrm{d}y=\dfrac{\partial z}{\partial x}\mathrm{d}x+\dfrac{\partial z}{\partial y}\mathrm{d}y$

称为 f 在 D 上的**微分**.

- 二元函数的可微和偏导、微分可推广到 n 元函数 $y=f(x_1,x_2,\cdots,x_n)$

$$\mathrm{d}f=\frac{\partial f}{\partial x_1}\mathrm{d}x_1+\frac{\partial f}{\partial x_2}\mathrm{d}x_2+\cdots+\frac{\partial f}{\partial x_n}\mathrm{d}x_n$$

- n 元**向量函数** $F(x_1,x_2,\cdots,x_n)=\big(f_1(x_1,\cdots,x_n),f_2(x_1,\cdots,x_n),\cdots,f_m(x_1,\cdots,x_n)\big)$

的微分定义为：

$$\mathrm{d}F=(df_1,df_2,\cdots,df_m)=\begin{pmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n}\end{pmatrix}_{m\times n}\begin{pmatrix}\mathrm{d}x_1\\ \mathrm{d}x_2\\ \vdots\\ \mathrm{d}x_n\end{pmatrix}$$

Jacobi矩阵

设二元函数$z=f(x,y)$在点(x,y)可微，当$|\Delta x|,|\Delta y|$都很小时，有

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)\approx f_x(x,y)\Delta x+f_y(x,y)\Delta y.$$

可得近似公式：

$$f(x+\Delta x,y+\Delta y)\approx f(x,y)+f_x(x,y)\Delta x+f_y(x,y)\Delta y$$

例： 计算$(1.04)^{2.02}$的近似值.

解： 设函数 $f(x,y)=x^y$.　$x=1,y=2,\Delta x=0.04,\Delta y=0.02.$

$$f(1,2)=1,f_x(x,y)=yx^{y-1},f_y(x,y)=x^y\ln x,f_x(1,2)=2,f_y(1,2)=0$$

由二元函数全微分近似计算公式得

$$(1.04)^{2.02}\approx 1+2\times 0.04+0\times 0.02.$$

平面上的一个方向 $\boldsymbol{e}$ 可由唯一的**单位**向量描述. 设它与 x,y 轴正向的夹角分别为 α,β, 则 $\boldsymbol{e}=(\cos\alpha,\cos\beta)$ 称为 $\boldsymbol{e}$ 的**方向余弦**表示.

例： 方向 (a,b) 的方向余弦为 $\left(\dfrac{a}{\sqrt{a^2+b^2}},\dfrac{b}{\sqrt{a^2+b^2}}\right)$.

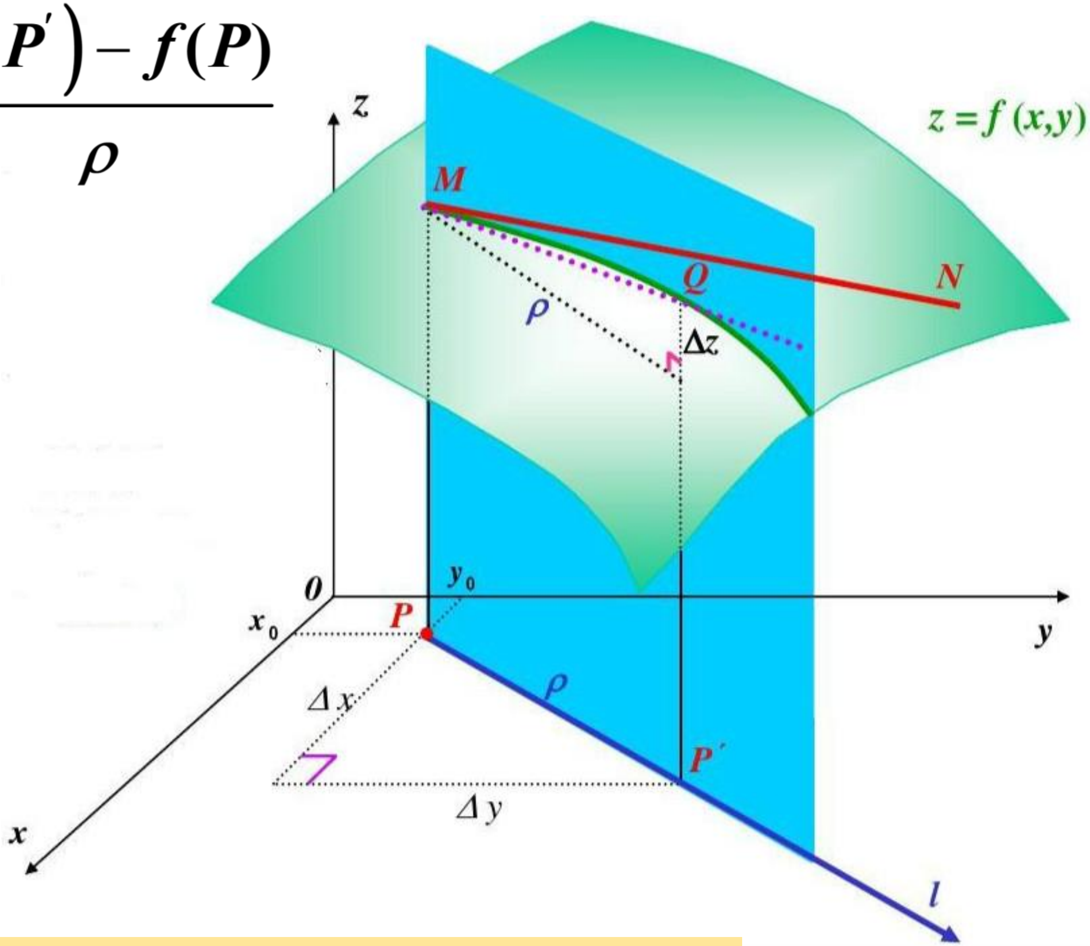
设 $f(x,y)$ 在 D 上有定义，$P(x_0,y_0)\in D$, $\boldsymbol{e}=(\cos\alpha,\cos\beta)$.

定义： 若 $\lim\limits_{\rho\to 0}\dfrac{f\left(x_0+\rho\cos\alpha,y_0+\rho\cos\beta\right)-f\left(x_0,y_0\right)}{\rho}$ 存在，则

称之为 f 在点 P 沿方向 $\boldsymbol{e}$ 的**方向导数**，记为 $\left.\dfrac{\partial f}{\partial \boldsymbol{e}}\right|_P$.

$$\left.\frac{\partial z}{\partial e}\right|_P = \lim_{\rho\to 0}\frac{f\left(x_0+\rho\cos\alpha, y_0+\rho\cos\beta\right)-f\left(x_0, y_0\right)}{\rho}$$

$$=\lim_{\rho\to 0}\frac{\Delta z}{\rho}=\lim_{\rho\to 0}\frac{f\left(P'\right)-f(P)}{\rho}$$



方向导数 $\left.\frac{\partial z}{\partial e}\right|_P$ 是曲面在点 M 处沿方向 l 的变化率.

定理： 设 $f(x,y)$ 是平面区域 D 上的可微函数，则 f 在 D 中任一点 $M_0(x_0,y_0)$，沿任一方向 $e=(\cos\alpha,\cos\beta)$ 的方向导数都存在，且

$$\left.\frac{\partial f}{\partial e}\right|_{M_0}=\left.\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta\right)\right|_{M_0}=\left.\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\right|_{M_0}\cdot e$$

证明： 可微 $\Rightarrow \Delta z=f\left(x_0+\Delta x,y_0+\Delta y\right)-f\left(x_0,y_0\right)=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\cdot\Delta x+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\cdot\Delta y+o(\rho).$

$$\left.\frac{\partial z}{\partial e}\right|_{P}=\lim_{\rho\to0}\frac{f\left(x_0+\rho\cos\alpha,y_0+\rho\cos\beta\right)-f\left(x_0,y_0\right)}{\rho}\quad\left(\xrightarrow{\Delta x=\rho\cos\alpha,\Delta y=\rho\sin\alpha}\right)$$

$$=\lim_{\rho\to0}\frac{\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}\cdot\rho\cos\alpha+\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}\cdot\rho\sin\alpha+o(\rho)}{\rho}=\left.\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta\right)\right|_{M_0}$$

定理： 设 $f(x,y)$ 是平面区域 D 上的可微函数，则 f 在 D 中任一点 $M_0(x_0,y_0)$，沿任一方向 $e=(\cos\alpha,\cos\beta)$ 的方向导数都存在且

$$\left.\frac{\partial f}{\partial e}\right|_{M_0}=\left.\left(\frac{\partial f}{\partial x}\cos\alpha+\frac{\partial f}{\partial y}\cos\beta\right)\right|_{M_0}=\left.\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\right|_{M_0}\cdot e$$

- 函数 f 可微，则 f 沿各方向的方向导数存在；反之不然.

定义： $grad(f)\triangleq\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)$ $(\triangleq\nabla f)$ 称为函数 $f(x,y)$ 的**梯度**.

- 沿梯度方向，方向导数最大，函数增长最快；增长率为梯度向量的模长；沿负梯度方向，方向导数最小，函数下降最快；下降率为梯度向量的模长.

方向导数和梯度定义可推广到n元函数. 以 $u = f(x,y,z)$ 为例:

$$\left.\frac{\partial f}{\partial e}\right|_{M_0} = \lim_{\rho\to 0}\frac{f(x_0+\rho\cos\alpha, y_0+\rho\cos\beta, z_0+\rho\cos\gamma) - f(x_0,y_0,z_0)}{\rho};$$

其中 $(\cos\alpha, \cos\beta, \cos\gamma)$ 是方向 e 的方向余弦.

$$grad\left(f(x,y,z)\right) \triangleq \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \nabla f.$$

在 f 可微时仍有：方向导数

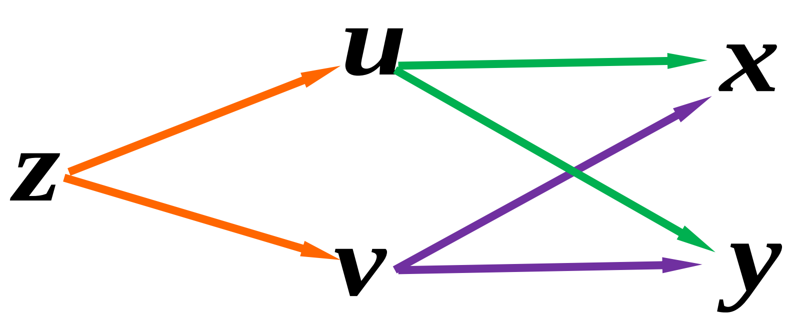
$$\frac{\partial f}{\partial e} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma = grad(f)\cdot e;$$

即方向导数 $\frac{\partial f}{\partial e}$ 是梯度向量 $grad(f)$ 在方向 e 上的投影.

定理： 设 $z=f(u,v)$ 在区域 $\widetilde{D}$ 上可微，$u=\varphi(x,y), v=\psi(x,y)$ 在区域 D 上可微，则复合函数 $f\left(\varphi(x,y),\psi(x,y)\right)$ 在 D 上可微，且其偏导为：

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

链式法则如图示：



例： 设 $z = e^u \sin v$，而 $u = xy$，$v = x + y$. 求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$.

解：
$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$
$$= e^u \sin v \cdot y + e^u \cos v \cdot 1$$
$$= e^u (y \sin v + \cos v),$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = e^u \sin v \cdot x + e^u \cos v \cdot 1$$
$$= e^u (x \sin v + \cos v).$$

注： 也可以先化为 $z = e^{xy} \sin(x + y)$，再求偏导数.

Examples:

1. 设 $z=f\left(\frac{y}{x}\right)$，其中 f 可微，求 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$.

2. $u=f(x-y,y-z,z-x)$, 其中 f 可微，求 $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}$.

3. $z=f(u,v)$, 其中 f 可微，且 $u=e^{t^2}+t, v=\sin\sqrt{t}+\ln t$. 求 $\frac{\mathrm{d}z}{\mathrm{d}t}$.

Answers: 1: 0; 2: 0; $3: f_u'(2te^{t^2}+1)+f_v'(\frac{1}{2\sqrt{t}}\cos\sqrt{t}+\frac{1}{t})$

$z=f(u,v),\ u=\varphi(x,y),\ v=\psi(x,y)$, z,u,v 均可微, u,v 为中间变量.

$$\begin{aligned}
\mathrm{d}z &= \frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v \\
&= \frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right) \\
&= \left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)dy \\
&= \frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y
\end{aligned}$$

可微函数 z 的微分，不管用中间变量，或是最终自变量计算，结果是一样的------这称为**一阶微分的形式不变性**.

设 $u=u(x,y)$, $v=v(x,y)$ 为可微的多元函数，则

(1) $\mathrm{d}(u\pm v)=\mathrm{d}u\pm\mathrm{d}v$ (2) $\mathrm{d}(u\cdot v)=v\mathrm{d}u+u\mathrm{d}v$ (3) $\mathrm{d}\left(\dfrac{u}{v}\right)=\dfrac{v\mathrm{d}u-u\mathrm{d}v}{v^2}(v\neq 0)$

例： 设函数 $u=f\left(\dfrac{x}{y},\dfrac{y}{z}\right)$，其中 f 具有二阶连续偏导，求 $du,\dfrac{\partial^2 u}{\partial x\partial y},\dfrac{\partial^2 u}{\partial z\partial y}$.

解： 由一阶微分形式不变性，和全微分的四则运算法则得：

$$\mathrm{d}u=f_1'\mathrm{d}\left(\frac{x}{y}\right)+f_2'\mathrm{d}\left(\frac{y}{z}\right)=f_1'\frac{y\mathrm{d}x-x\mathrm{d}y}{y^2}+f_2'\frac{z\mathrm{d}y-y\mathrm{d}z}{z^2}=\frac{1}{y}f_1'\mathrm{d}x+\left(\frac{1}{z}f_2'-\frac{x}{y^2}f_1'\right)\mathrm{d}y-\frac{y}{z^2}f_2'\mathrm{d}z$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{f_{12}''}{yz}-\frac{x}{y^3}f_{11}''-\frac{f_1'}{y^2}.\qquad \frac{\partial^2 u}{\partial z\partial y}=\frac{x}{yz^2}f_{12}''-\frac{y}{z^3}f_{22}''-\frac{f_2'}{z^2}.$$

1. 设函数$z=f\left(\dfrac{x}{g(y)},y\right)=xg(y), f,g$ 都是可微函数，且 $g(y)\neq 0,$ 求

$\dfrac{\partial f}{\partial x}(x,y),\dfrac{\partial f}{\partial y}(x,y)$ 及 $\dfrac{\partial^2 f}{\partial x\partial y}(x,y).$

2.设函数$f(x,y)$ 具有连续偏导数，且 $f\left(x,x^2\right)\equiv 1.$

(1) 若 $f_x'\left(x,x^2\right)=x,$ 求 $f_y'\left(x,x^2\right);$

(2) 若 $f_y'(x,y)=x^2+2y,$ 求$f(x,y).$

3.设函数 $u=u(x,y)$ 具有二阶连续偏导数，且满足方程 $\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}=0$ 及条件 $u(x,2x)=x, u_x'(x,2x)=x^2$，求 $u_{xx}''(x,2x), u_{xy}''(x,2x), u_{yy}''(x,2x)$.

4.设变换 $u=x-2y, v=x+ay$ 可把方程(其中二阶偏导均连续)

$$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0$$

简化为 $\frac{\partial^2 z}{\partial u\partial v}=0$，求 a.

一元向量值函数 $r(t)=\left(x(t),y(t),z(t)\right)$ 的导数与微分定义为：

$$\mathbf{r}'(t)=\left(x'(t),y'(t),z'(t)\right),\quad \mathrm{d}\mathbf{r}(t)=(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z)=\mathbf{r}'(t)\mathrm{d}t$$

特别地，$y=f(x)$ 可写为 $r(t)=\left(x,f(x),0\right)$.

二元向量值函数 $r(u,v)=\left(x(u,v),y(u,v),z(u,v)\right)$ 的偏导数、微分：

$$\mathbf{r}_u'=\frac{\partial \mathbf{r}}{\partial u}=\left(\frac{\partial x}{\partial u},\frac{\partial y}{\partial u},\frac{\partial z}{\partial u}\right),\quad \mathbf{r}_v'=\frac{\partial \mathbf{r}}{\partial v}=\left(\frac{\partial x}{\partial v},\frac{\partial y}{\partial v},\frac{\partial z}{\partial v}\right).$$

$$\mathrm{d}\mathbf{r}(u,v)=\left(\mathrm{d}x(u,v),\mathrm{d}y(u,v),\mathrm{d}z(u,v)\right)=\mathbf{r}_u'\mathrm{d}u+\mathbf{r}_v'\mathrm{d}v$$

特别地，$z=f(x,y)$ 可写为 $r(x,y)=\left(x,y,f(x,y)\right)$.

$$\mathrm{d}r(x,y)=\left(1,0,f_x'\right)\mathrm{d}x+\left(0,1,f_y'\right)\mathrm{d}y.$$

$D\subset\mathbb{R}^n, f:D\to\mathbb{R}^m$为$n$元$m$维向量值函数. 记 $x=(x_1,x_2,\cdots,x_n)^T$

$$f(x)=\left(f_1(x),f_2(x),\cdots,f_m(x)\right)^T$$

设每个分量函数 $f_i(x)$ 都是可微的，则称映射f可微，且 f 的微分定义为

$$\mathrm{d}f=\begin{pmatrix}\mathrm{d}f_1\\ \vdots\\ \mathrm{d}f_m\end{pmatrix}=\begin{pmatrix}\dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n}\\ \vdots & & \vdots\\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n}\end{pmatrix}\begin{pmatrix}\mathrm{d}x_1\\ \vdots\\ \mathrm{d}x_n\end{pmatrix}.$$

$$J f \triangleq \begin{pmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{pmatrix}$$

称为向量值函数 f 的**Jacobi矩阵.**

当 $m = n$ 时，Jacobi矩阵的**行列式**记为 $\det J_x f = \dfrac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)}$,

几何意义是体积微元放大的倍数.

例： $f:\mathbb{R}^2\to\mathbb{R}^2$ 为**极坐标变换** $f(r,\theta)=\big(x(r,\theta),y(r,\theta)\big)=(r\cos\theta,r\sin\theta)$

$$Jf=\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{pmatrix}=\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix},\quad \frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=r$$

例： $f:\mathbb{R}^3\to\mathbb{R}^3$ 为**球坐标变换**

$$f(r,\theta,\varphi)=\big(x,y,z\big)=(r\sin\theta\cos\varphi,r\sin\theta\sin\varphi,r\cos\theta)$$

$$Jf=\begin{pmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi}\end{pmatrix},\quad \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi}\end{vmatrix}=r^2\sin\theta.$$

设 D, D' 分别为 $\mathbb{R}^n$, $\mathbb{R}^m$ 中区域，$f: D \to \mathbb{R}^m, g: D' \to \mathbb{R}^\ell$. 若 $f(D) \subset D'$,

则存在复合映射 $g \circ f: D \to \mathbb{R}^l$，$(x_1, \cdots, x_n) \xrightarrow{f} (y_1, \cdots, y_m) \xrightarrow{g} (z_1, \cdots, z_\ell)$.

类似于复合函数链式求导定理的证明可得

$$\frac{\partial z_i}{\partial x_j} = \frac{\partial z_i}{\partial y_1} \cdot \frac{\partial y_1}{\partial x_j} + \frac{\partial z_i}{\partial y_2} \cdot \frac{\partial y_2}{\partial x_j} + \cdots + \frac{\partial z_i}{\partial y_m} \cdot \frac{\partial y_m}{\partial x_j}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \cdots & \frac{\partial z_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_\ell}{\partial x_1} & \cdots & \frac{\partial z_\ell}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial z_1}{\partial y_1} & \cdots & \frac{\partial z_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_\ell}{\partial y_1} & \cdots & \frac{\partial z_\ell}{\partial y_m} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}.$$

$$\Rightarrow J(g \circ f)(x) = Jg(f(x)) \cdot Jf(x).$$