第九章　多元函数微分学

- 多变量函数的连续性
- 多变量函数的微分
- 隐函数定理和逆映射定理
- 空间曲线与曲面
- Taylor公式与极值
- **向量场的微商**
- 微分形式

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三维空间中，

数量场： $f(x,y,z)$,　　例如：温度场，高度场

向量场： $\vec{\mathbf{v}}(x,y,z)=P(x,y,z)\vec{\mathbf{i}}+Q(x,y,z)\vec{\mathbf{j}}+R(x,y,z)\vec{\mathbf{k}}$

位置场、速度场、力场、电场、磁场等

光滑向量场：P,Q,R 有连续的偏导数.

向量场的偏微商：

$$\frac{\partial\vec{\mathbf{v}}}{\partial x}=\frac{\partial P}{\partial x}\vec{\mathbf{i}}+\frac{\partial Q}{\partial x}\vec{\mathbf{j}}+\frac{\partial R}{\partial x}\vec{\mathbf{k}}$$

引进 $\nabla=\frac{\partial}{\partial x}\vec{\mathbf{i}}+\frac{\partial}{\partial y}\vec{\mathbf{j}}+\frac{\partial}{\partial z}\vec{\mathbf{k}}$，称为**Hamilton 算子**, 或 **Nabla 算子**.

$$\nabla u=\left(\frac{\partial}{\partial x}\vec{\mathbf{i}}+\frac{\partial}{\partial y}\vec{\mathbf{j}}+\frac{\partial}{\partial z}\vec{\mathbf{k}}\right)u=\frac{\partial u}{\partial x}\vec{\mathbf{i}}+\frac{\partial u}{\partial y}\vec{\mathbf{j}}+\frac{\partial u}{\partial z}\vec{\mathbf{k}}=\mathbf{grad}\,u.$$

Laplace算子： $\Delta=\nabla\cdot\nabla=\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

$\Delta u=0$ ⟶ Laplace方程.

在**直角坐标系**下，设有向量场

$$\vec{\mathbf{v}}(x,y,z)=P(x,y,z)\,\vec{\mathbf{i}}+Q(x,y,z)\,\vec{\mathbf{j}}+R(x,y,z)\,\vec{\mathbf{k}}$$

向量场 $\vec{\mathbf{v}}$ 的**散度**定义为:

$$\operatorname{div}\vec{\mathbf{v}}=\nabla\cdot\vec{\mathbf{v}}=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}.$$

向量场 $\vec{\mathbf{v}}$ 的**旋度**定义为:

$$\operatorname{rot}\vec{\mathbf{v}}=\nabla\times\vec{\mathbf{v}}=\begin{vmatrix}\vec{\mathbf{i}} & \vec{\mathbf{j}} & \vec{\mathbf{k}}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ P & Q & R\end{vmatrix}$$

$$=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{\mathbf{i}}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{\mathbf{j}}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{\mathbf{k}}$$

设 φ,ψ 为数量场，$\boldsymbol{a},\boldsymbol{b}$ 为向量场. 算符∇ 有下述运算规则：

$$(1)\ \nabla(\varphi+\psi)=\nabla\varphi+\nabla\psi;$$

$$(2)\ \nabla\cdot(\boldsymbol{a}+\boldsymbol{b})=\nabla\cdot\boldsymbol{a}+\nabla\cdot\boldsymbol{b};$$

$$(3)\ \nabla\times(\boldsymbol{a}+\boldsymbol{b})=\nabla\times\boldsymbol{a}+\nabla\times\boldsymbol{b};$$

$$(4)\ \nabla(\varphi\psi)=\varphi\nabla\psi+\psi\nabla\varphi;$$

$$(5)\ \nabla\cdot(\varphi\boldsymbol{a})=\varphi\nabla\cdot\boldsymbol{a}+\boldsymbol{a}\cdot(\nabla\varphi);$$

$$(6)\ \nabla\times(\varphi\boldsymbol{a})=\nabla\varphi\times\boldsymbol{a}+\varphi(\nabla\times\boldsymbol{a});$$

$$(7)\ \nabla\cdot(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{b}\cdot\nabla\times\boldsymbol{a}-\boldsymbol{a}\cdot\nabla\times\boldsymbol{b}.$$

$$(\bullet)\ \mathbf{rot}\,\mathbf{grad}\,\varphi=\nabla\times\nabla\varphi=0; \qquad \mathbf{div}\,\mathbf{rot}\,\boldsymbol{a}=\nabla\cdot(\nabla\times\boldsymbol{a})=0.$$

空间里的点，其位置可以用直角坐标 (x,y,z) 表示，也能用曲线坐标 (q_1,q_2,q_3) 来表示. 例如：

圆柱坐标系：

$$\begin{cases} x=\rho\cos\varphi \\ y=\rho\sin\varphi \\ z=z \end{cases} \qquad \begin{cases} \rho=\sqrt{x^2+y^2+z^2} \\ \varphi=\arctan\dfrac{y}{x} \\ z=z \end{cases}$$

球坐标系：

$$\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases} \qquad \begin{cases} r=\sqrt{x^2+y^2+z^2} \\ \theta=\arctan\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \varphi=\arctan\left(\dfrac{y}{x}\right) \end{cases}$$

曲线坐标系$\left(q_1,q_2,q_3\right)$:

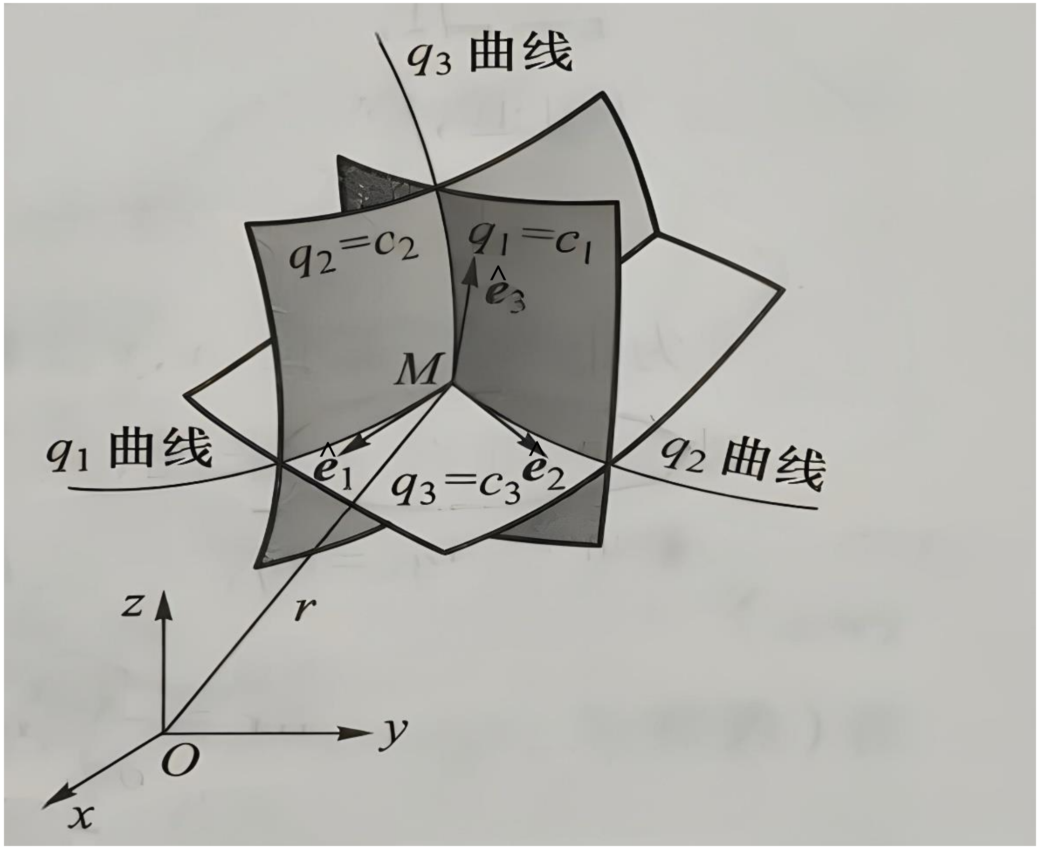
$$\vec{\mathbf{r}}=x\left(q_1,q_2,q_3\right)\vec{\mathbf{i}}+y\left(q_1,q_2,q_3\right)\vec{\mathbf{j}}+z\left(q_1,q_2,q_3\right)\vec{\mathbf{k}}$$

坐标曲线的切线量： $$\frac{\partial\vec{\mathbf{r}}}{\partial q_i}=\frac{\partial x}{\partial q_i}\vec{\mathbf{i}}+\frac{\partial y}{\partial q_i}\vec{\mathbf{j}}+\frac{\partial z}{\partial q_i}\vec{\mathbf{k}}$$

Lamé (拉梅)系数： $$H_i=\left|\frac{\partial\vec{\mathbf{r}}}{\partial q_i}\right|=\sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2+\left(\frac{\partial y}{\partial q_i}\right)^2+\left(\frac{\partial z}{\partial q_i}\right)^2}$$

$\hat{\mathbf{e}}_i=\dfrac{1}{H_i}\dfrac{\partial\vec{\mathbf{r}}}{\partial q_i}=\dfrac{1}{H_i}\left(\dfrac{\partial x}{\partial q_i}\vec{\mathbf{i}}+\dfrac{\partial y}{\partial q_i}\vec{\mathbf{j}}+\dfrac{\partial z}{\partial q_i}\vec{\mathbf{k}}\right)$ 为$\vec{\mathbf{r}}\left(q_1,q_2,q_3\right)$处坐标曲线的**单位**切向量.

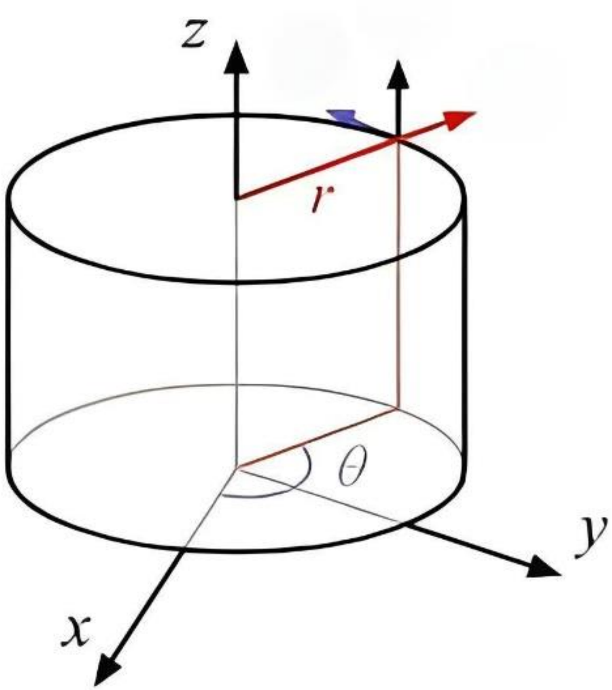
坐标曲线与坐标曲面:

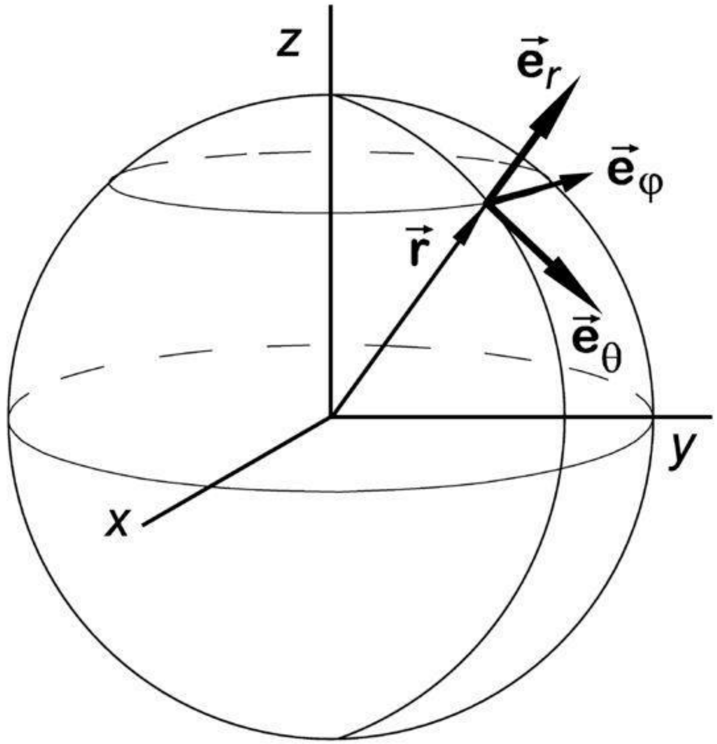


$(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ 依次是空间一点 $\boldsymbol{M}(q_1, q_2, q_3)$ 处的坐标曲线的单位切向量，且指向参数增加的方向.

若坐标曲线两两正交，即：$\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$，则称 $\left(q_1, q_2, q_3\right)$ 为**正交曲线坐标系.**

柱坐标系、球面坐标系都是正交曲线坐标系.





两组基矢量的关系：

$$H_i=\left|\frac{\partial\vec{\mathbf{r}}}{\partial q_i}\right|=\sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2+\left(\frac{\partial y}{\partial q_i}\right)^2+\left(\frac{\partial z}{\partial q_i}\right)^2},\quad \hat{\mathbf{e}}_i=\frac{1}{H_i}\frac{\partial\vec{\mathbf{r}}}{\partial q_i}=\frac{1}{H_i}\left(\frac{\partial x}{\partial q_i}\vec{\mathbf{i}}+\frac{\partial y}{\partial q_i}\vec{\mathbf{j}}+\frac{\partial z}{\partial q_i}\vec{\mathbf{k}}\right)$$

$$\Rightarrow \hat{\mathbf{e}}_i=\left(\vec{\mathbf{i}},\vec{\mathbf{j}},\vec{\mathbf{k}}\right)\cdot\frac{1}{H_i}\begin{pmatrix}\frac{\partial x}{\partial q_i}\\ \frac{\partial y}{\partial q_i}\\ \frac{\partial z}{\partial q_i}\end{pmatrix}\Rightarrow\left(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3\right)=\left(\vec{\mathbf{i}},\vec{\mathbf{j}},\vec{\mathbf{k}}\right)\cdot\begin{pmatrix}\frac{\partial x}{\partial q_1}&\frac{\partial x}{\partial q_2}&\frac{\partial x}{\partial q_3}\\ \frac{\partial y}{\partial q_1}&\frac{\partial y}{\partial q_2}&\frac{\partial y}{\partial q_3}\\ \frac{\partial z}{\partial q_1}&\frac{\partial z}{\partial q_2}&\frac{\partial z}{\partial q_3}\end{pmatrix}\begin{pmatrix}\frac{1}{H_1}&&\\&\frac{1}{H_2}&\\&&\frac{1}{H_3}\end{pmatrix}$$

$$\Rightarrow\left(\vec{\mathbf{i}},\vec{\mathbf{j}},\vec{\mathbf{k}}\right)=\left(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3\right)\begin{pmatrix}\frac{1}{H_1}&&\\&\frac{1}{H_2}&\\&&\frac{1}{H_3}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial q_1}&\frac{\partial y}{\partial q_1}&\frac{\partial z}{\partial q_1}\\ \frac{\partial x}{\partial q_2}&\frac{\partial y}{\partial q_2}&\frac{\partial z}{\partial q_2}\\ \frac{\partial x}{\partial q_3}&\frac{\partial y}{\partial q_3}&\frac{\partial z}{\partial q_3}\end{pmatrix}.$$

定理： $(\mathrm{d}s)^2=(\mathrm{d}x)^2+(\mathrm{d}y)^2+(\mathrm{d}z)^2=H_1{}^2(\mathrm{d}q_1)^2+H_2{}^2(\mathrm{d}q_2)^2+H_3{}^2(\mathrm{d}q_3)^2$

证明： $\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\\ \mathrm{d}z\end{pmatrix}=\begin{pmatrix}\frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3}\\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3}\\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3}\end{pmatrix}\begin{pmatrix}\mathrm{d}q_1\\ \mathrm{d}q_2\\ \mathrm{d}q_3\end{pmatrix}=\left(\vec{\mathbf{r}}_{q_1}'^T,\vec{\mathbf{r}}_{q_2}'^T,\vec{\mathbf{r}}_{q_3}'^T\right)\begin{pmatrix}\mathrm{d}q_1\\ \mathrm{d}q_2\\ \mathrm{d}q_3\end{pmatrix}$

$$\Rightarrow \mathrm{d}s^2=(\mathrm{d}x,\mathrm{d}y,\mathrm{d}z)\cdot\begin{pmatrix}\mathrm{d}x\\ \mathrm{d}y\\ \mathrm{d}z\end{pmatrix}=(\mathrm{d}q_1,\mathrm{d}q_2,\mathrm{d}q_3)\cdot\begin{pmatrix}\vec{\mathbf{r}}_{q_1}'\\ \vec{\mathbf{r}}_{q_2}'\\ \vec{\mathbf{r}}_{q_3}'\end{pmatrix}\left(\vec{\mathbf{r}}_{q_1}'^T,\vec{\mathbf{r}}_{q_2}'^T,\vec{\mathbf{r}}_{q_3}'^T\right)\begin{pmatrix}\mathrm{d}q_1\\ \mathrm{d}q_2\\ \mathrm{d}q_3\end{pmatrix}$$

$$=(\mathrm{d}q_1,\mathrm{d}q_2,\mathrm{d}q_3)\cdot\left(\vec{\mathbf{r}}_{q_i}'\cdot\vec{\mathbf{r}}_{q_j}'\right)_{3\times3}\begin{pmatrix}\mathrm{d}q_1\\ \mathrm{d}q_2\\ \mathrm{d}q_3\end{pmatrix}$$

$$=\left(\mathrm{d}q_1,\mathrm{d}q_2,\mathrm{d}q_3\right)\cdot\begin{pmatrix} H_1^2 & & \\ & H_2^2 & \\ & & H_3^2 \end{pmatrix}\cdot\begin{pmatrix}\mathrm{d}q_1\\ \mathrm{d}q_2\\ \mathrm{d}q_3\end{pmatrix}$$

$$=H_1^{\ 2}(\mathrm{d}q_1)^2+H_2^{\ 2}(\mathrm{d}q_2)^2+H_3^{\ 2}(\mathrm{d}q_3)^2.$$

弧微分公式刻画了两点 $\vec{\mathbf{r}}(q_1,q_2,q_3),\vec{\mathbf{r}}(q_1+\mathrm{d}q_1,q_2+\mathrm{d}q_2,q_3+\mathrm{d}q_3)$ (由勾股定理定义)的空间距离.

对坐标曲线 $q_i (1 \le i \le 3)$,

$$\Delta s_i = \left|\vec{\mathbf{r}}(\cdots, q_i + dq_i, \cdots) - \vec{\mathbf{r}}(\cdots, q_i, \cdots)\right| \doteq \left|\frac{\partial \vec{\mathbf{r}}}{\partial q_i}\right| \mathrm{d}q_i = H_i \mathrm{d}q_i .$$

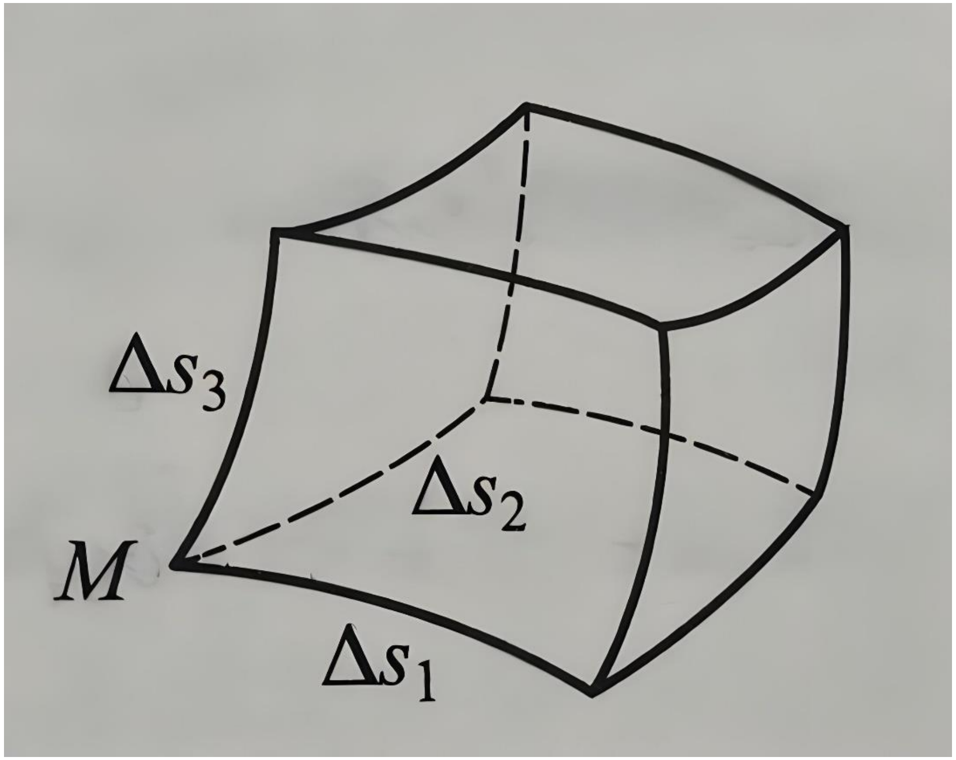
称 $\mathrm{d}s_i = H_i \mathrm{d}q_i$ 为坐标曲线 q_i 的**弧微分**.

类似可得坐标面**面积的微分**为：

$$\mathrm{d}\Sigma_{q_1} = \mathrm{d}s_2 \mathrm{d}s_3 = H_2 H_3 \mathrm{d}q_2 \mathrm{d}q_3$$

$$\mathrm{d}\Sigma_{q_2} = \mathrm{d}s_1 \mathrm{d}s_3 = H_1 H_3 \mathrm{d}q_1 \mathrm{d}q_3$$

$$\mathrm{d}\Sigma_{q_3} = \mathrm{d}s_1 \mathrm{d}s_2 = H_1 H_2 \mathrm{d}q_1 \mathrm{d}q_2$$



体积微元：

$$\mathrm{d}V = \mathrm{d}s_1 \cdot \mathrm{d}s_2 \cdot \mathrm{d}s_3 = H_1 H_2 H_3 \mathrm{d}q_1 \mathrm{d}q_2 \mathrm{d}q_3$$

定理： Nabla算子$\nabla=\frac{\partial}{\partial x}\vec{\mathbf{i}}+\frac{\partial}{\partial y}\vec{\mathbf{j}}+\frac{\partial}{\partial z}\vec{\mathbf{k}}$ 在正交曲线坐标系$(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3)$下的表示为 $\nabla=\frac{1}{H_1}\frac{\partial}{\partial q_1}\hat{\mathbf{e}}_1+\frac{1}{H_2}\frac{\partial}{\partial q_2}\hat{\mathbf{e}}_2+\frac{1}{H_3}\frac{\partial}{\partial q_3}\hat{\mathbf{e}}_3$

证明：

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial q_1}\cdot\frac{\partial q_1}{\partial x}+\frac{\partial f}{\partial q_2}\cdot\frac{\partial q_2}{\partial x}+\frac{\partial f}{\partial q_3}\cdot\frac{\partial q_3}{\partial x}=\left(\frac{\partial q_1}{\partial x},\frac{\partial q_2}{\partial x},\frac{\partial q_3}{\partial x}\right)\begin{pmatrix}\frac{\partial f}{\partial q_1}\\ \frac{\partial f}{\partial q_2}\\ \frac{\partial f}{\partial q_3}\end{pmatrix}$$

$$\Rightarrow\frac{\partial}{\partial x}=\left(\frac{\partial q_1}{\partial x},\frac{\partial q_2}{\partial x},\frac{\partial q_3}{\partial x}\right)\begin{pmatrix}\frac{\partial}{\partial q_1}\\ \frac{\partial}{\partial q_2}\\ \frac{\partial}{\partial q_3}\end{pmatrix}\Rightarrow\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}=\begin{pmatrix}\frac{\partial q_1}{\partial x}&\frac{\partial q_2}{\partial x}&\frac{\partial q_3}{\partial x}\\ \frac{\partial q_1}{\partial y}&\frac{\partial q_2}{\partial y}&\frac{\partial q_3}{\partial y}\\ \frac{\partial q_1}{\partial z}&\frac{\partial q_2}{\partial z}&\frac{\partial q_3}{\partial z}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial q_1}\\ \frac{\partial}{\partial q_2}\\ \frac{\partial}{\partial q_3}\end{pmatrix}$$

由 $$\left(\vec{\mathbf{i}},\vec{\mathbf{j}},\vec{\mathbf{k}}\right)=(\hat{\mathbf{e}}_1,\hat{\mathbf{e}}_2,\hat{\mathbf{e}}_3)\begin{pmatrix}\frac{1}{H_1}&&\\&\frac{1}{H_2}&\\&&\frac{1}{H_3}\end{pmatrix}\begin{pmatrix}\frac{\partial x}{\partial q_1}&\frac{\partial y}{\partial q_1}&\frac{\partial z}{\partial q_1}\\ \frac{\partial x}{\partial q_2}&\frac{\partial y}{\partial q_2}&\frac{\partial z}{\partial q_2}\\ \frac{\partial x}{\partial q_3}&\frac{\partial y}{\partial q_3}&\frac{\partial z}{\partial q_3}\end{pmatrix}$$

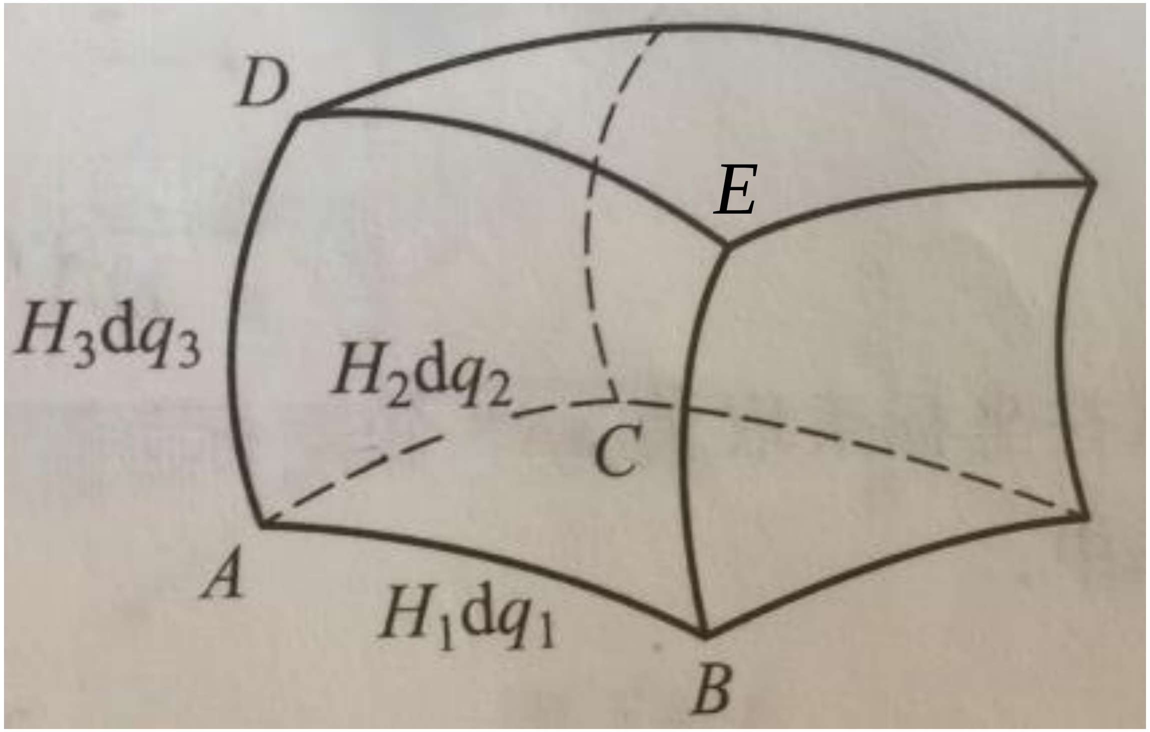
于是，$\nabla = \dfrac{\partial}{\partial x}\vec{\mathbf{i}} + \dfrac{\partial}{\partial y}\vec{\mathbf{j}} + \dfrac{\partial}{\partial z}\vec{\mathbf{k}} = \left(\vec{\mathbf{i}}, \vec{\mathbf{j}}, \vec{\mathbf{k}}\right)\cdot\begin{pmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} \end{pmatrix}$

$$= \left(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\right)\begin{pmatrix} \frac{1}{H_1} & & \\ & \frac{1}{H_2} & \\ & & \frac{1}{H_3} \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial y}{\partial q_1} & \frac{\partial z}{\partial q_1} \\ \frac{\partial x}{\partial q_2} & \frac{\partial y}{\partial q_2} & \frac{\partial z}{\partial q_2} \\ \frac{\partial x}{\partial q_3} & \frac{\partial y}{\partial q_3} & \frac{\partial z}{\partial q_3} \end{pmatrix}\cdot\begin{pmatrix} \frac{\partial q_1}{\partial x} & \frac{\partial q_2}{\partial x} & \frac{\partial q_3}{\partial x} \\ \frac{\partial q_1}{\partial y} & \frac{\partial q_2}{\partial y} & \frac{\partial q_3}{\partial y} \\ \frac{\partial q_1}{\partial z} & \frac{\partial q_2}{\partial z} & \frac{\partial q_3}{\partial z} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial q_1} \\ \frac{\partial}{\partial q_2} \\ \frac{\partial}{\partial q_3} \end{pmatrix}$$

$$= \left(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\right)\begin{pmatrix} \frac{1}{H_1} & & \\ & \frac{1}{H_2} & \\ & & \frac{1}{H_3} \end{pmatrix}\cdot I_3 \cdot\begin{pmatrix} \frac{\partial}{\partial q_1} \\ \frac{\partial}{\partial q_2} \\ \frac{\partial}{\partial q_3} \end{pmatrix}$$

$$= \frac{1}{H_1}\frac{\partial}{\partial q_1}\hat{\mathbf{e}}_1 + \frac{1}{H_2}\frac{\partial}{\partial q_2}\hat{\mathbf{e}}_2 + \frac{1}{H_3}\frac{\partial}{\partial q_3}\hat{\mathbf{e}}_3.$$

散度的定义： $\operatorname{div}\vec{\mathbf{A}} \triangleq \lim_{\Delta V \to 0} \dfrac{\oiint_S \vec{\mathbf{A}} \cdot d\mathbf{S}}{\Delta V}$



通过面ABDE以及其相对面的通量可以表示为

$$A_2 \cdot H_1 H_3 dq_1 dq_3\big|_{q_2+dq_2} - A_2 \cdot H_1 H_3 dq_1 dq_3\big|_{q_2} = \frac{\partial\left(A_2 H_1 H_3\right)}{\partial q_2} dq_1 dq_2 dq_3$$

同理，另外两个方向的通量为

$$\frac{\partial\left(A_1H_2H_3\right)}{\partial q_1}\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3,\ \frac{\partial\left(A_3H_1H_2\right)}{\partial q_3}\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3$$

体积微元的体积为： $H_1H_2H_3\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3$

故向量场 $\vec{\mathbf{A}}$ 在 $A\left(q_1,q_2,q_3\right)$ 的散度为：

$$\mathrm{div}\vec{\mathbf{A}} \triangleq \lim_{\Delta V\to 0}\frac{\oiint_S \vec{\mathbf{A}}\cdot\mathrm{d}\mathbf{S}}{\Delta V} = \lim_{\Delta V\to 0}\frac{\frac{\partial\left(A_1H_2H_3\right)}{\partial q_1}+\frac{\partial\left(A_2H_1H_3\right)}{\partial q_2}+\frac{\partial\left(A_3H_1H_2\right)}{\partial q_3}}{H_1H_2H_3}$$

$$=\frac{1}{H_1H_2H_3}\left[\frac{\partial(A_1H_2H_3)}{\partial q_1}+\frac{\partial(A_2H_3H_1)}{\partial q_2}+\frac{\partial(A_3H_1H_2)}{\partial q_3}\right]$$

散度的定义：$\mathrm{div}\vec{\mathbf{A}} \triangleq \lim\limits_{\Delta V \to 0} \dfrac{\oiint_S \vec{\mathbf{A}} \cdot \mathrm{d}\mathbf{S}}{\Delta V}$

定理：向量场$\vec{\mathbf{A}} = A_1(q_1, q_2, q_3)\hat{\mathbf{e}}_1 + A_2(q_1, q_2, q_3)\hat{\mathbf{e}}_2 + A_3(q_1, q_2, q_3)\hat{\mathbf{e}}_3$ 的散度为

$$\mathrm{div}\vec{\mathbf{A}} = \nabla \cdot \vec{\mathbf{A}} = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial(A_1 H_2 H_3)}{\partial q_1} + \frac{\partial(A_2 H_3 H_1)}{\partial q_2} + \frac{\partial(A_3 H_1 H_2)}{\partial q_3}\right]$$

若将直角坐标系下的散度公式当成定义，利用Nabla算子的特点,也可推导出正交曲线坐标下散度的计算公式.

引理： $$\nabla\cdot\left(\frac{\hat{\mathbf{e}}_1}{H_2H_3}\right)=\nabla\cdot\left(\frac{\hat{\mathbf{e}}_2}{H_3H_1}\right)=\nabla\cdot\left(\frac{\hat{\mathbf{e}}_3}{H_1H_2}\right)=0.$$

证明： 首先由 $$\nabla=\frac{1}{H_1}\frac{\partial}{\partial q_1}\hat{\mathbf{e}}_1+\frac{1}{H_2}\frac{\partial}{\partial q_2}\hat{\mathbf{e}}_2+\frac{1}{H_3}\frac{\partial}{\partial q_3}\hat{\mathbf{e}}_3 \Rightarrow \nabla q_1=\frac{\hat{\mathbf{e}}_1}{H_1}$$

$$\Rightarrow \nabla\times\left(\frac{\hat{\mathbf{e}}_1}{H_1}\right)=\nabla\times(\nabla q_1)=\vec{0}$$

$$\left(\mathbf{div\,rot}\,\boldsymbol{a}=\nabla\cdot(\nabla\times\boldsymbol{a})=\mathbf{0}\right)$$

于是 $$\nabla\cdot\left(\frac{\hat{\mathbf{e}}_1}{H_2H_3}\right)=\nabla\cdot\left(\frac{\hat{\mathbf{e}}_2}{H_2}\times\frac{\hat{\mathbf{e}}_3}{H_3}\right)=\nabla\times\left(\frac{\hat{\mathbf{e}}_2}{H_2}\right)\cdot\frac{\hat{\mathbf{e}}_3}{H_3}-\nabla\times\left(\frac{\hat{\mathbf{e}}_3}{H_3}\right)\cdot\frac{\hat{\mathbf{e}}_2}{H_2}$$

$$=\vec{0}\cdot\frac{\hat{\mathbf{e}}_3}{H_3}-\vec{0}\cdot\frac{\hat{\mathbf{e}}_2}{H_2}=0.$$

$$\textbf{(7)}\ \nabla\cdot(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{b}\cdot\nabla\times\boldsymbol{a}-\boldsymbol{a}\cdot\nabla\times\boldsymbol{b}.$$

计算 $\vec{\mathbf{A}}$ 的散度：

$$(5)\ \nabla\cdot(\varphi\boldsymbol{a})=\varphi\nabla\cdot\boldsymbol{a}+(\nabla\varphi)\cdot\boldsymbol{a}$$

$$\nabla\cdot\vec{\mathbf{A}}=\nabla\cdot\left(A_1\hat{\mathbf{e}}_1\right)+\nabla\cdot\left(A_2\hat{\mathbf{e}}_2\right)+\nabla\cdot\left(A_3\hat{\mathbf{e}}_3\right)$$

$$=\nabla\cdot\left[\left(H_2H_3A_1\right)\frac{\hat{\mathbf{e}}_1}{H_2H_3}\right]+\nabla\cdot\left[\left(H_3H_1A_2\right)\frac{\hat{\mathbf{e}}_2}{H_3H_1}\right]+\nabla\cdot\left[\left(H_1H_2A_3\right)\frac{\hat{\mathbf{e}}_3}{H_1H_2}\right]$$

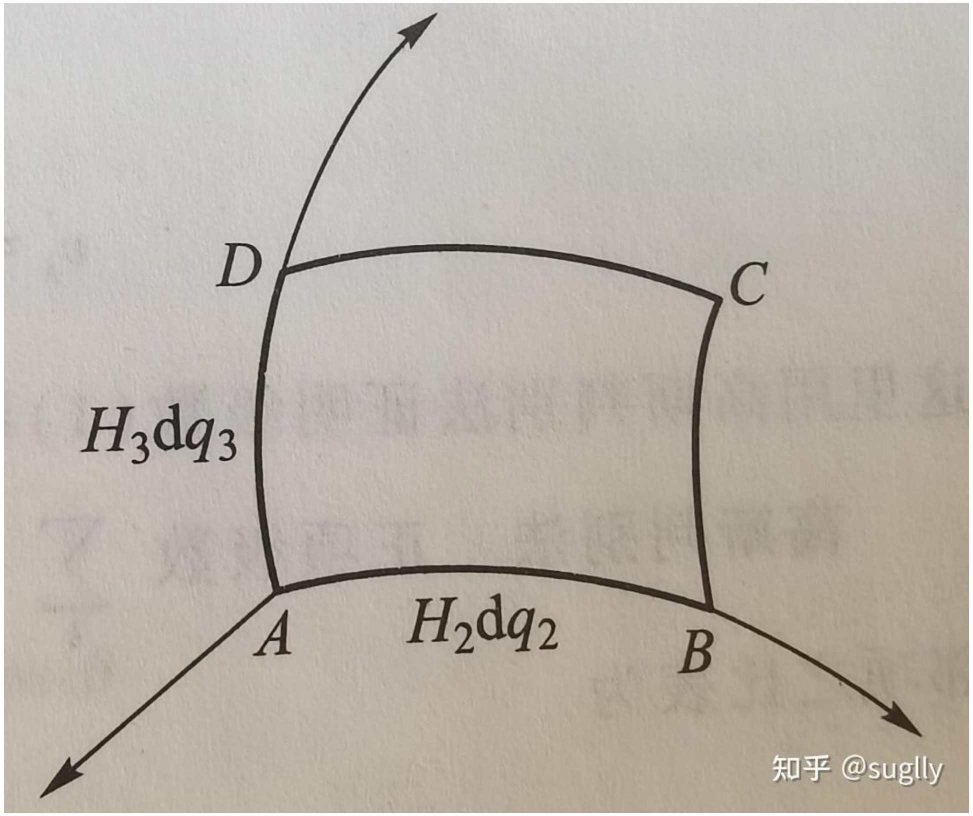
$$=\nabla\left(H_2H_3A_1\right)\cdot\frac{\hat{\mathbf{e}}_1}{H_2H_3}+\nabla\left(H_3H_1A_2\right)\cdot\frac{\hat{\mathbf{e}}_2}{H_3H_1}+\nabla\left(H_1H_2A_3\right)\cdot\frac{\hat{\mathbf{e}}_3}{H_1H_2}$$

$$=\left[\frac{\partial\left(H_2H_3A_1\right)\hat{\mathbf{e}}_1}{\partial q_1}+\cdots+\cdots\right]\cdot\frac{\hat{\mathbf{e}}_1}{H_2H_3}+\left[\cdots+\frac{\partial\left(H_3H_1A_2\right)}{\partial q_2}\frac{\hat{\mathbf{e}}_2}{H_2}+\cdots\right]\cdot\frac{\hat{\mathbf{e}}_2}{H_3H_1}$$

$$+\left[\cdots+\cdots\frac{\partial\left(H_1H_2A_3\right)}{\partial q_3}\frac{\hat{\mathbf{e}}_3}{H_3}\right]\cdot\frac{\hat{\mathbf{e}}_2}{H_1H_2}$$

$$=\frac{1}{H_1H_2H_3}\left[\frac{\partial(A_1H_2H_3)}{\partial q_1}+\frac{\partial(A_2H_3H_1)}{\partial q_2}+\frac{\partial(A_3H_1H_2)}{\partial q_3}\right]$$

旋度的定义： $\mathbf{rot}(\vec{\mathbf{A}})\cdot\vec{\mathbf{n}} \triangleq \lim\limits_{\Delta S\to 0}\dfrac{\oint_{\partial S}\vec{\mathbf{A}}\cdot \mathrm{d}\vec{\mathbf{r}}}{\Delta S}$



沿 $\hat{\mathbf{e}}_1$ 方向，先计算 $\overrightarrow{BC}$ 和 $\overrightarrow{DA}$ 线上的积分为：

$$A_3H_3\big|_{q_2+dq_2}\,\mathrm{d}q_3 - A_3H_3\big|_{q_2}\,\mathrm{d}q_3 = \frac{\partial\left(A_3H_3\right)}{\partial q_2}\mathrm{d}q_2\mathrm{d}q_3$$

同理，$\overrightarrow{AB}$ 和 $\overrightarrow{CD}$ 线上的积分为：

$$A_2H_2\Big|_{q_3}\mathrm{d}q_2 - A_2H_2\Big|_{q_3+\mathrm{d}q_3}\mathrm{d}q_2 = \frac{\partial\left(A_2H_2\right)}{\partial q_3}\mathrm{d}q_2\mathrm{d}q_3$$

两者相加，并除以 $ABCD$ 的面积得：

$$\mathbf{rot}\vec{\mathbf{A}}\cdot\widehat{\mathbf{e}}_1 = \frac{1}{H_2H_3}\left[\frac{\partial\left(A_3H_3\right)}{\partial q_2} - \frac{\partial\left(A_2H_2\right)}{\partial q_3}\right]$$

同理，可得 $\mathbf{rot}\vec{\mathbf{A}}\cdot\widehat{\mathbf{e}}_2 = \frac{1}{H_1H_3}\left[\frac{\partial\left(A_1H_1\right)}{\partial q_3} - \frac{\partial\left(A_3H_3\right)}{\partial q_1}\right]$

$$\mathbf{rot}\vec{\mathbf{A}}\cdot\widehat{\mathbf{e}}_3 = \frac{1}{H_1H_2}\left[\frac{\partial\left(A_2H_2\right)}{\partial q_1} - \frac{\partial\left(A_1H_1\right)}{\partial q_2}\right]$$

综合得：

$$\mathbf{rot}\vec{\mathbf{A}}=\frac{1}{H_2H_3}\left[\frac{\partial(A_3H_3)}{\partial q_2}-\frac{\partial(A_2H_2)}{\partial q_3}\right]\hat{\mathbf{e}}_1+\frac{1}{H_1H_3}\left[\frac{\partial(A_1H_1)}{\partial q_3}-\frac{\partial(A_3H_3)}{\partial q_1}\right]\hat{\mathbf{e}}_2$$
$$+\frac{1}{H_1H_2}\left[\frac{\partial(A_2H_2)}{\partial q_1}-\frac{\partial(A_1H_1)}{\partial q_2}\right]\hat{\mathbf{e}}_3$$

旋度的定义： $\mathbf{rot}(\vec{\mathbf{A}})\cdot\vec{\mathbf{n}}\triangleq\lim\limits_{\Delta S\to 0}\dfrac{\oint_{\partial S}\vec{\mathbf{A}}\cdot\mathrm{d}\vec{\mathbf{r}}}{\Delta S}$

定理： 向量场 $\vec{\mathbf{A}}=A_1(q_1,q_2,q_3)\hat{\mathbf{e}}_1+A_2(q_1,q_2,q_3)\hat{\mathbf{e}}_2+A_3(q_1,q_2,q_3)\hat{\mathbf{e}}_3$ 的旋度为：

$$\mathbf{rot}(\vec{\mathbf{A}})=\nabla\times\vec{\mathbf{A}}=\frac{1}{H_1H_2H_3}\begin{vmatrix} H_1\hat{\mathbf{e}}_1 & H_2\hat{\mathbf{e}}_2 & H_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ A_1H_1 & A_2H_2 & A_3H_3 \end{vmatrix}$$

若将直角坐标系下的旋度公式当成定义，利用Nabla算子的特点,也可推导出正交曲线坐标下旋度的计算公式.

$$\nabla\times\vec{\mathbf{A}} = \nabla\times\left[A_1\hat{\mathbf{e}}_1\right]+\nabla\times\left[A_2\hat{\mathbf{e}}_2\right]+\nabla\times\left[A_3\hat{\mathbf{e}}_3\right]$$

$$= \nabla\times\left[\left(H_1A_1\right)\frac{\hat{\mathbf{e}}_1}{H_1}\right]+\nabla\times\left[\left(H_2A_2\right)\frac{\hat{\mathbf{e}}_2}{H_2}\right]+\nabla\times\left[\left(H_3A_3\right)\frac{\hat{\mathbf{e}}_3}{H_3}\right]$$

$$= \nabla\left(H_1A_1\right)\times\frac{\hat{\mathbf{e}}_1}{H_1}+\nabla\left(H_2A_2\right)\times\frac{\hat{\mathbf{e}}_2}{H_2}+\nabla\left(H_3A_3\right)\times\frac{\hat{\mathbf{e}}_3}{H_3}$$

$$= \left[\cdots+\frac{\partial\left(H_1A_1\right)\hat{\mathbf{e}}_2}{\partial q_2}+\frac{\partial\left(H_1A_1\right)}{\partial q_3}\frac{\hat{\mathbf{e}}_3}{H_3}\right]\times\frac{\hat{\mathbf{e}}_1}{H_1}$$

$$+\left[\frac{\partial\left(H_2A_2\right)}{\partial q_1}\frac{\hat{\mathbf{e}}_1}{H_1}+\cdots+\frac{\partial\left(H_2A_2\right)\hat{\mathbf{e}}_3}{\partial q_3}\right]\times\frac{\hat{\mathbf{e}}_2}{H_2}$$

$$+\left[\frac{\partial\left(H_3A_3\right)}{\partial q_1}\frac{\hat{\mathbf{e}}_1}{H_1}+\frac{\partial\left(H_3A_3\right)}{\partial q_2}\frac{\hat{\mathbf{e}}_2}{H_2}+\cdots\right]\times\frac{\hat{\mathbf{e}}_3}{H_3}$$

$$
=\frac{1}{H_2H_3}\left[\frac{\partial\left(H_3A_3\right)}{\partial q_2}-\frac{\partial\left(H_2A_2\right)}{\partial q_3}\right]\hat{\mathbf{e}}_1
$$

$$
+\frac{1}{H_3H_1}\left[\frac{\partial\left(H_1A_1\right)}{\partial q_3}-\frac{\partial\left(H_3A_3\right)}{\partial q_1}\right]\hat{\mathbf{e}}_2
$$

$$
+\frac{1}{H_1H_2}\left[\frac{\partial\left(H_2A_2\right)}{\partial q_1}-\frac{\partial\left(H_1A_1\right)}{\partial q_2}\right]\hat{\mathbf{e}}_3
$$

$$
=\frac{1}{H_1H_2H_3}\begin{vmatrix} H_1\hat{\mathbf{e}}_1 & H_2\hat{\mathbf{e}}_2 & H_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ H_1A_1 & H_2A_2 & H_3A_3 \end{vmatrix}
$$

对于标量场，Laplace算符定义为 $\nabla^2 f = \nabla\cdot(\nabla f).$

$$\nabla^2 f = \nabla\cdot\left(\frac{1}{H_1}\frac{\partial f}{\partial q_1}\hat{\mathbf{e}}_1+\frac{1}{H_2}\frac{\partial f}{\partial q_2}\hat{\mathbf{e}}_2+\frac{1}{H_3}\frac{\partial f}{\partial q_3}\hat{\mathbf{e}}_3\right)$$

$$=\frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2H_3}{H_1}\frac{\partial f}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{H_1H_3}{H_2}\frac{\partial f}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{H_1H_2}{H_3}\frac{\partial f}{\partial q_3}\right)\right]$$

对于矢量场，Laplace算符定义为：对矢量的各个分量分别做Laplace运算，再组成一个矢量.

$$\nabla^2\vec{\mathbf{A}} = \nabla\left(\nabla\cdot\vec{\mathbf{A}}\right)-\nabla\times\left(\nabla\times\vec{\mathbf{A}}\right).$$

在正交曲线坐标系(q_1,q_2,q_3)下：

Lamé系数： $$H_i=\left|\frac{\partial\vec{\mathbf{r}}}{\partial q_i}\right|=\sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2+\left(\frac{\partial y}{\partial q_i}\right)^2+\left(\frac{\partial z}{\partial q_i}\right)^2}$$

坐标基矢： $$\hat{\mathbf{e}}_i=\frac{1}{H_i}\frac{\partial\vec{\mathbf{r}}}{\partial q_i}=\frac{1}{H_i}\left(\frac{\partial x}{\partial q_i}\vec{\mathbf{i}}+\frac{\partial y}{\partial q_i}\vec{\mathbf{j}}+\frac{\partial z}{\partial q_i}\vec{\mathbf{k}}\right)$$

弧长微元： $$(\mathrm{d}s)^2=H_1{}^2(\mathrm{d}q_1)^2+H_2{}^2(\mathrm{d}q_2)^2+H_3{}^2(\mathrm{d}q_3)^2$$

坐标曲线上的弧微元： $$\mathrm{d}s_i=H_i\mathrm{d}q_i$$

坐标平面上的面积微元： $$\mathrm{d}\Sigma_{q_i}=\prod_{j\neq i}\mathrm{d}s_j=\prod_{j\neq i}H_j\mathrm{d}q_j$$

体积微元： $$\mathrm{d}V=\mathrm{d}s_1\cdot\mathrm{d}s_2\cdot\mathrm{d}s_3=H_1H_2H_3\mathrm{d}q_1\mathrm{d}q_2\mathrm{d}q_3$$

在正交曲线坐标系$\left(q_1, q_2, q_3\right)$下：

梯度算子： $$\nabla = \frac{1}{H_1}\frac{\partial}{\partial q_1}\hat{\mathbf{e}}_1 + \frac{1}{H_2}\frac{\partial}{\partial q_2}\hat{\mathbf{e}}_2 + \frac{1}{H_3}\frac{\partial}{\partial q_3}\hat{\mathbf{e}}_3.$$

散度： $$\mathrm{div}\vec{\mathbf{A}} = \frac{1}{H_1H_2H_3}\left[\frac{\partial(A_1H_2H_3)}{\partial q_1} + \frac{\partial(A_2H_3H_1)}{\partial q_2} + \frac{\partial(A_3H_1H_2)}{\partial q_3}\right]$$

旋度： $$\mathbf{rot}(\vec{\mathbf{A}}) = \nabla\times\vec{\mathbf{A}} = \frac{1}{H_1H_2H_3}\begin{vmatrix} H_1\hat{\mathbf{e}}_1 & H_2\hat{\mathbf{e}}_2 & H_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ A_1H_1 & A_2H_2 & A_3H_3 \end{vmatrix}$$

Laplace： $$\nabla^2 = \frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2H_3}{H_1}\frac{\partial}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1H_3}{H_2}\frac{\partial}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{H_1H_2}{H_3}\frac{\partial}{\partial q_3}\right)\right].$$

在柱坐标曲线下，$\vec{\mathbf{r}}=\left(\rho\cos\theta,\rho\sin\theta,z\right)$. $H_\rho=1, H_\theta=\rho, H_z=1$

$$\nabla u=\frac{\partial u}{\partial\rho}\hat{e}_\rho+\frac{1}{\rho}\frac{\partial u}{\partial\theta}\hat{e}_\theta+\frac{\partial u}{\partial z}\hat{e}_z$$

$$\nabla\cdot\vec{\mathbf{A}}=\frac{1}{\rho}\left[\frac{\partial\left(\rho A_1\right)}{\partial\rho}+\frac{\partial A_2}{\partial\theta}+\frac{\partial\left(\rho A_3\right)}{\partial z}\right]$$

$$\nabla\times\vec{\mathbf{A}}=\frac{1}{\rho}\begin{vmatrix}\hat{e}_\rho & \hat{e}_\theta & \hat{e}_z\\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial z}\\ A_1 & \rho A_2 & A_3\end{vmatrix}$$

$$\nabla^2 u=\nabla\cdot\nabla u=\frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)+\frac{\partial}{\partial\theta}\left(\rho\frac{\partial u}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial u}{\partial z}\right)\right]$$

在球坐标曲线下：$\vec{\mathbf{r}}=\left(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta\right).$

$$H_r=1$$
$$H_\theta=r$$
$$H_\varphi=r\sin\theta$$

$$\nabla u=\frac{\partial u}{\partial r}\hat{e}_r+\frac{1}{r}\frac{\partial u}{\partial\theta}\hat{e}_\theta+\frac{1}{r\sin\theta}\frac{\partial u}{\partial\varphi}\hat{e}_\varphi$$

$$\nabla\cdot\vec{\mathbf{A}}=\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial\left(A_1\mathbf{r}^2\right)}{\partial r}+r\frac{\partial\left(A_2\sin\theta\right)}{\partial\theta}+r\frac{\partial A_3}{\partial\varphi}\right]$$

$$\nabla\cdot\vec{\mathbf{A}}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{e}_r & r\hat{e}_\theta & r\sin\theta\hat{e}_\varphi\\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\varphi}\\ A_1 & rA_2 & r\sin\theta A_3\end{vmatrix}$$

$$\nabla^2u=\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin\theta}\frac{\partial^2u}{\partial\varphi^2}\right]$$