СЕИ h.sariev@math.bas.bg 6/10/22

сем. упр. 1 събития по вероятност и статистика АС

Оценяване

Задачи | 2 контролни > средно 4.5 => освобождаване или изпит

домашни 0,5 бонус

презентация на тема - report / ml / math тема

Преговор

$M = \{a_1, \dots, a_n\},\ n \in \mathbb{N};\ |M| = n$

мн-во от индекси, защото не ни интересуват стойностите на елем.

$\{1, \dots, n\}$

Пермутации

$P_n = \{(a_1, \dots, a_n)$ - редица $\}$ $\underbrace{i_j \neq i_l,\ j \neq l}_{\text{индексите са различни}}$

$$\boxed{|P_n| = P_n = n!} = n\cdot(n-1)\cdot(n-2)\cdot\ldots$$

Д-во |

$\{a_1\} \longrightarrow (a_1)$ | $\{1\} \longrightarrow (1)$ | $P_1 = 1$

$\{a_1, a_2\} \to$ | $\{1, 2\} \longrightarrow (1,2), (2,1)$ | $P_2 = 2$

$P_n = n!$ | $\{1,2,3\} \longrightarrow (1,2,3), (1,3,2), \dots$ | $P_3 = 6 = 3\cdot 2 = 3!$

$$P_{n+1} = \sum_{i=1}^{P_1} |B_i^{(n)}|$$

$(\_1\_2) = \{(312), (132), (123)\} = B_1$

$(\_2\_1) = \{(321), (231), (213)\} = B_2$

$$P_3 = \sum_{i=1}^{P_2} |B|$$

Вариация (редът има значение)

$V_n^k = \{(a_{i_1}, \dots, a_{i_k}),\ i_j \neq i_l,\ j \neq l\},\ k = 0, 1, \dots, n$

$$\boxed{|V_n^k| = V_n^k = \frac{n!}{(n-k)!}}$$

интуиция: $(a_{i_1}, \dots a_{i_k}, \underbrace{a_{i_{k+1}} \dots a_{i_n}}_{\text{пермутиране на } (n-k)!}) \longrightarrow (a_{i_1} \dots a_{i_k})$

$$P_n = P_{n-k} \cdot V_n^k \iff V_n^k = \frac{P_n}{P_{n-k}}$$

Комбинация

- редът няма значение
- различни елементи

$$C_n^k = \{\{a_{i_1} \dots a_{i_k}\}, i_j \neq i_l, j \neq l\}, k = \overline{0,n}$$

$$\boxed{|C_n^k| = C_n^k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}}$$

черт. | $(a_{i_1} \dots a_{i_k}) \to \{a_{i_1} \dots a_{i_k}\}$

$$V_n^k = P_k \cdot C_n^k \iff C_n^k = \frac{V_n^k}{P_k}$$

Пермутация с повторение $(k_1, \dots, k_n)$

- може да повтаряме

$P = \{(a_{i_1}, \dots, a_{i_k})\}$ $(k; k_1 \dots k_n)$

$$\begin{array}{ccccc} 1 & 2 & \dots & n & \\ \downarrow & \downarrow & & \downarrow & \\ k_1 & +k_2 & + \dots & +k_n & = k, \; k \in \mathbb{N} \end{array}$$

$$\boxed{|P(k; k_1 \dots k_n)| = P(k; k_1 \dots k_n) = \frac{k!}{k_1!\,k_2! \dots k_n!}}$$

- действаме елемент по елемент

$$P(k; k_1 \dots k_n) = C_k^{k_1} \cdot \underbrace{C_{k-k_1}^{k_2} \cdot \dots \cdot C_{k-k_1-\dots-k_{n-1}}^{k_n}}_{P(n-k_1; k_2 \dots k_n)}$$

Комбинация с повторение

$$\{\underbrace{1, 1, \dots 1}_{k_1}, \underbrace{2, 2 \dots 2}_{k_2}, \dots, \underbrace{n, n, \dots, n}_{k_n}\}$$

- една такава комбинация

Вариация с повторение

$$V(n,k) = \{(a_{i_1}, \dots, a_{i_k})\}^{(*)}, k \in \mathbb{N}$$

$$\boxed{|V(n,k)| = V(n,k) = n^k}$$

броене на декартово мн-во

$$|A_1 \times \ldots \times A_m| = |A_1|\,|A_2| \ldots |A_m|$$

(*)

$$= \underbrace{M \times M \times M \times \ldots \times M}_{n}$$

доказателство по индукция

Комбинация на елементи n, със степен k

- подредба няма значение

$$C(n,k) = \{\{a_{i_1} \ldots a_{i_k}\}\}, \quad k \in \mathbb{N}$$

$$|C(n,k)| = C(n,k) = \cancel{\frac{n^k}{k!}} = C^k_{n+k-1}$$

$$V_n^k = P_k \cdot C_n^k = \sum_{i=1}^{C_n^k} |D_i|$$

без повторени $D_i = \{(\underbrace{a_{1i}, \ldots, a_{ki}}) \}$ (* *)

$k!$

с повт.:

$$V(n,k) = \sum_{i=1}^{C(n,k)} P(k; k_{1i}, \ldots, k_{ni})$$

(* *)

$$\begin{matrix} 1 & 2 & \ldots & n \\ \downarrow & \downarrow & & \downarrow \\ k_{1,i} & k_{2i} & \ldots & k_{ni} \end{matrix}$$

трик: $\{a_{i_1}, \ldots, a_{i_k}\}$, $1 \le i_1 \le i_2 \le \ldots \le i_k \le n$
$\qquad\qquad\qquad +0 \quad +1 \quad\quad +k-1$

$\{a_{i_1}, a_{i_2+1}, \ldots, a_{i_k+(k-1)}\}$ $\quad i_1 < i_2+1 < i_3+2 < \ldots < i_k+k-1$

$M^* = \{1, \ldots, n, n+1, \ldots, n+k-1\}$

$\{j_1, \ldots, j_k\}$ $\quad j_i \in M^* = \{1, \ldots, n+k-1\}$

Зад 1 a) k елементи от n $\Rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$ (вариацията може да разделим по $k!$ начина $\Leftarrow$ наредбата няма значение)

б) $(a_1, \ldots, a_k)$ от различни (наредба има значение) $= n \cdot (n-1) \cdot \ldots \cdot (n-k+1) \Rightarrow V_n^k = \frac{n!}{(n-k)!}$

* $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ $\Rightarrow$ брой подмн-ва с k ел.

в) k ел. от n, с повторения, $k \in \mathbb{N}_0$ $\Rightarrow n^k$

Задача ~~4~~ 2

$$x_1 + \dots + x_k = n$$

а) $x_i \in \mathbb{N}$, $k \le n$

$M = \{1, \dots, k\}$

$\{\underbrace{1, \dots, k,}_{n-k} i_1, \dots, i_{n-k}\}$

$C(k, n-k) = C_{n-1}^{k-1}$

stars & bars

n звезди, n-1 между звезди, k-1 прегради $\Rightarrow \binom{n-1}{k-1}$

~~k на брой кутии за n на брой топки~~

$x_1 + x_2 + x_3 = 4$

$\underbrace{\bullet}_{x_1} | \underbrace{\bullet}_{x_2} | \underbrace{\bullet\bullet}_{x_3}$

$1 + 1 + 2 = 4$

$1 + 2 + 1 (= 2$ $\bullet | \bullet\bullet | \bullet$

б) $x_i \in \mathbb{N}_0$

$\{i_1, \dots, i_n\}$ $\quad C(k, n) = \binom{n+k-1}{n}$

k кутии

n елем.

* различни случаи събираме

добавяме на $\forall x_i$ 1ца

$(x_1 + 1) + (x_2 + 1) + \dots + (x_k + 1) = n + k$

$x_1^* + x_2^* + \dots + x_k^* = n + k$

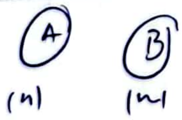
$x_i^* \in \mathbb{N}$

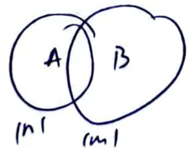
$\Rightarrow \binom{n+k-1}{k-1}$

$n = 5 \quad k = 8$

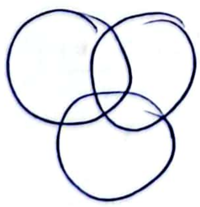
$\{1, 2, 1, 1, 2\}$

~~3.~~ Задача 3 Принцип за включване и изключване





$|A \cup B| = |A| + |B|$ $\quad |A \cup B| = |A| + |B| - |A \cap B|$



$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C|$
$- |B \cap C| + |A \cap B \cap C|$

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| - \dots$$
$$\dots + (-1)^{n-1} |A_1 \cap \dots \cap A_n|$$

Задача 4

$\underbrace{\sqcup \ \sqcup \ \dots \ \sqcup}_{n \text{ клетки}}$

k топчи

$M = \{1, \dots, n\}$

$A = \{$клетка i е празна$\}$

$\bigcup_{i=1}^{n} A = \{$поне 1 празна$\}$

различни	неразлични	
$V_n^k = \frac{n!}{(n-k)!}$ $(i_1, \dots, i_k) i_j \neq i_l, j \neq l \leq n$	$C_n^k = \binom{n}{k}, k > n : \binom{n}{k} = 0$ $\{i_1, \dots, i_k\}, i_j = i_l, j \neq l \leq n$	в ∀ клетка ≤ 1
$V(n,k) = n^k$ $(i_1, \dots, i_k)$ $k \in \mathbb{N}$	$C(n,k) = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$ $x_1 + \dots + x_n = k, x_i \in \mathbb{N}_0$ $x_i = \#$ частици в клетка i $\{i_1, \dots, i_n\}$	неогр.
$n^k - \lvert \bigcup_{i=1}^{n} A_i \rvert$ $=$ $\#$ 1-ва клетка е празна $= (n-1)^k$ "A_1" $k \geq n$	$C(n, k-n)$ $\binom{k-1}{n-1}$ $x_1 + \dots + x_n = k, x_i \in \mathbb{N}$ $k \geq n$	≥ 1 $k \geq n$

$A_i = \{\forall$ конфигурации, т.ч. клетка i е празна$\}$

$\lvert \bigcup_{i=1}^{n} A_i \rvert = \lvert\{$поне 1 празна$\}\rvert$

$= \lvert A_1 \rvert \dots + \lvert A_n \rvert$

$- \lvert A_1 \cap A_2 \rvert - \dots$

$+ \lvert A_1 \cap A_2 \cap A_3 \rvert \dots =: S$

$\lvert A_1 \cap A_2 \rvert = (n-2)^k$

$\lvert A_1 \cap \dots \cap A_j \rvert = (n-j)^k$

$\Rightarrow S = n \cdot (n-1)^k - \binom{n}{2}(n-2)^k + \dots$

$= \sum_{i=1}^{n} \binom{n}{i} \cdot (n-i)^k (-1)^{i+1}$

$\Rightarrow n^k - S = \sum_{i=0}^{n} \binom{n}{i} (n-i)^k (-1)^i$

* $\binom{n}{k} = \binom{n}{n-k}$

заради симетричност на Паскал

$\Rightarrow$ да изберем ел. от A означава да не изберем ел. от A^c

Задача 5

$M = \{1, 2, 3, 4, 5\}$

~~а) $\binom{3}{5}$~~

Задача 6

$M = \{1, 2, \dots, 12\}$

а) C_{12}^4

б) $\{A, \_, \_, \_\}$ C_{10}^3

$\{B, \_, \_, \_\}$ C_{10}^3

$\{\_, \_, \_, \_\}$ C_{10}^4

Задача 7

$M = \{A, B, C\}$

а) 2^5 (вар.)

б) $2^5 - 2$

в) $3(2^5 - 2)$

г) $3(2^5 - 2) + 3$

д) $3^5 - 2$

reflection principle

Сем
упр. 2

13/10/22

- $\binom{n}{k} = \binom{n}{n-k}$

$\{a_{i_1} \dots a_{i_k}\} \longleftrightarrow \{$остатъка$\}$

- $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$

$\{a_{i_1} \dots a_{i_k}\}$

има a_n / \ няма a_n $\{a_1 \dots a_{n-1}\}$

$\{a_{i_1}, \dots, a_{i_{k-1}}, a_n\}$

- $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n \quad : M = \{a_1, \dots, a_n\}$

$\left\{ \{\emptyset\}, \{a_1\} \vdots \{a_n\}, \{a_i, a_j\}, \dots, \{a_1, \dots, a_n\} \right\}$ мн-во от всички подмн-ва

$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n$

$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k 1^{n-k} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \dots + (-1)^n \binom{n}{n}$

$\binom{n}{0} = \binom{n}{1} - \binom{n}{2} + \dots + (-1)^{n-1} \binom{n}{n}$

-> триъгълник на Паскал

```
       1        --+-->  1 = 2^0
      1 1       --+-->  2 = 2^1
     1 2 1      --+-->  4 = 2^2
    1 3 3 1     --+-->  8 = 2^3
   1 4 6 4 1
```

! Запиши за суму и разлику

$1 \binom{0}{0}$

$1 \binom{1}{0} \quad 1 \binom{1}{1}$

$1 \binom{2}{0} \quad 2 \binom{2}{1} \quad 1 \binom{2}{2}$

$\dots$

$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

\# подмн-ва с k елемента + специален (*) който сме избрали

$C(n,k) = C(n,k+1) + C(n-1,k)$

$C(1,0)$

$C(2,0) \quad C(1,1)$

$C(3,0) \quad C(2,1) \quad C(1,2)$

$C(4,0) \quad C(3,1) \quad C(2,2) \quad C(1,3)$

$\Omega = \{w_1, \dots, w_n\}$ мн-во от всички елем. събития и в изходи

$\mathbb{P}(\{w_1\}) \dots \mathbb{P}(\{w_n\})$

$\mathbb{P}(\{w_1\}) + \dots + \mathbb{P}(\{w_n\}) = 1$

$\mathbb{P}(\{w_i\}) \geq 0$

$\mathbb{P}(\{w_1\}) = \dots = \mathbb{P}(\{w_n\}) = \frac{1}{n}$

$\mathbb{P}(A) = \frac{m}{n} = \frac{|A|}{|\Omega|} = \frac{\#\text{благоприятни } \{w_i\} \text{ изходи}}{\#\text{всички изходи } (n)}$

$A = \{w_{i_1}, \dots, w_{i_m}\}$ $m<n$

→ нечестна монета

$\mathbb{P}(\{\text{ези}\}) > \mathbb{P}(\{\text{тура}\})$ - едната страна тежи повече от другата

$A \subseteq \Omega \; \forall A \in \mathcal{P}(\Omega) / \cancel{2^{\Omega}} \equiv A \in 2^{\Omega}$

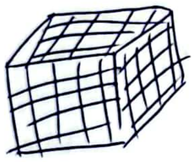
$A = \{w_{i_1}, \dots, w_{i_m}\}$ $m<n$ $\quad A = \bigcup_{j=1}^{m} \{w_{i_j}\}$

$\mathbb{P}(A) = \mathbb{P}(\{w_{i_1}\}) + \dots + \mathbb{P}(\{w_{i_m}\})$

$\mathbb{P}: \mathcal{P}(\Omega) \to [0,1]$

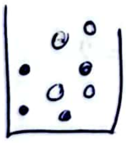
$\mathcal{A} \subseteq \mathcal{P}(\Omega)$

Зад. 11:



$\mathbb{P}(\text{червени точно } 2) = \frac{\#}{\#\{\} = 10^3} = \frac{12 * 8}{10^3} = \dots$

чегова зад:



M - черни

$N-M$ - бели

$n \leq N$

$k \leq M$

$n-k \leq N-M$

$\mathbb{P}(\{k \text{ черни топки}\}) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$

$\{\underbrace{ч_1, \dots, ч_m, б_1, \dots, б_{N-M}}_{N}\}$

$\{\underbrace{ч, ч, \dots, ч}_{M}, \underbrace{б, \dots, б}_{N-M}\}$

$\downarrow$

$\{\underbrace{ч, ч, ч, \dots, ч}_{k}, \underbrace{б, \dots, б}_{n-k}\}$

$= \frac{\binom{3}{2}\binom{4}{1}}{\binom{7}{3}} = \frac{12}{35}$

$\frac{\cancel{3 \cdot 2} \cdot 4}{\dots}$

$\frac{\frac{3!}{1! \cdot 2!} \cdot \frac{4!}{3! \cdot 1!}}{\frac{7!}{4! \cdot 3!}} = \frac{3 \cdot 4}{7 \cdot 5} = \frac{12}{35}$

пример: $\{\bullet\bullet\bullet\circ\circ\circ\circ\}$

$\{\bullet\bullet\circ\}$

$\{\bullet\bullet\circ\}$

$\{\bullet\circ\circ\}$

$\{\circ\circ\circ\}$

$= \frac{1}{4}$

$M=3$ - черни топки ~~кои избиране~~

$N=7$ - всичко топки

$n=3$ - колко топки изваждаме

$k=2$ - вероятността 2 топки да са черни

! исключај упр.
вземи задаче
найди 12 курс
извади до ори.

$$\mathbb{P}((ч,ч,б,\dots)) = \frac{M}{N}\cdot\frac{M-1}{N-1}\cdot\frac{N-M}{N-2}\dots = \cancel{M(M-1)\dots(M-k+1)(N-M)}$$

k ч.
n-k б.

първа топка е черна — втора черна — трета бяла

$$= \frac{M(M-1)\dots(M-k+1)(N-M)\dots(N-M-\overset{-(n+k)+1}{(n-k-1)})}{N(N-1)\dots(N-n+1)}$$

$$\mathbb{P}(\{k \text{ черни}\}) = \binom{n}{k}\cdot\frac{M!}{(M-k)!}\cdot\frac{(N-M)!}{(N-M-(n-k))!}\cdot\frac{(N-n)!}{N!}$$

$$= \frac{n!}{k!\,(n-k)!}\cdot\frac{M!}{(M-k)!}\cdot\frac{(N-M)!}{(N-M-(n-k))!}\cdot\frac{(N-n)!}{N!}$$

$$= \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

Зад.

~~кой от заровете~~
различни цветове на заровете

а) 6,6 различни — всички изходи са еднакво вероятни $\left(\frac{\text{благоприятни}}{\text{всички}}\right)$

б) 6,6 неразличими — " —

$$\Omega_{(а)} = \{(i,j) : i,j \in \{1,\dots,6\}\} \qquad |\Omega_{(а)}| = 36$$

$$\Omega_{(б)} = \{\{i,j\} : i,j \in \{1,\dots,6\}\} \qquad |\Omega_{(б)}| = 21$$

комб. с пов. от 2 ред
на 1-6

$$C(6,2) = \binom{6+2-1}{2}$$

$$= \frac{7\cdot\overset{3}{\cancel{6}}\cdot\cancel{5!}}{\cancel{5!}\cdot\cancel{2}} = 21$$

$$\mathbb{P}_{а}((6,6)) = \frac{1}{36} \qquad \mathbb{P}_{а}(\underset{(6,6)}{\{6,6\}}) = \frac{1}{36} = \cancel{\frac{1}{36}} + \frac{1}{36} \qquad \mathbb{P}_{а}((1,6)) = \frac{1}{36}$$

$$\mathbb{P}_{а}(\{1,6\}) = \frac{1}{18}$$

$$\mathbb{P}(\{(1,6)\}\cup\{(6,1)\})$$

$$\mathbb{P}_{б}(\{6,6\}) = \frac{1}{21} \qquad \mathbb{P}_{б}(\underset{\{6,1\}}{\{1,6\}})$$

Зад. 12 $\Omega = \{(a,b,c,d) : a,b,c,d \in \{0,1,\dots,9\}\}$, $|\Omega| = 10^4$

$\frac{\#\text{благо}}{\#\text{общо}}$

а) да не съдържа = цифри

$\frac{10 \cdot 9 \cdot 8 \cdot 7}{10^4}$

б) да има точно 2 = цифри

$\frac{10\binom{4}{2} 9 \cdot 8}{10^4} \Leftrightarrow \binom{10}{3} \frac{4!}{2!1!1!} \cdot 3$

$\{0, \dots, 9\}$

избиране 3 цифри $\hookrightarrow \{a, b, c\}$

$\binom{10}{3} \cdot \binom{4}{2,1,1} \cdot 3$

$P(4; 2,1,1)$

$\frac{4!}{2!1!1!}$

в) да има точно 3 = цифри

$\frac{10 \cdot 9 \cdot \binom{4}{3}}{10^4}$

$\{a, b\}$

$2 \cdot \binom{10}{2} \cdot \frac{4!}{3!1!}$

$P(4; 3,1)$

г)

$\frac{\quad}{10^4} \Leftrightarrow \frac{\binom{10}{2} \frac{4!}{2! \cdot 2!}}{10^4}$

д) $\{a+b = c+d\} \cap \{= 7\}$

0 7
1 6
2 5
3 4
4 3
5 2
6 1
7 0

$((a,b)(c,d))$

$8 \cdot 8$

СЕМ 20/10/22
упр. 3
сем

1. [③ ③] [③ ③] [③ ③] като на пр. с карти

$$\mathbb{P}(2_3 | 1_3) = \frac{\mathbb{P}(2_3 \cap 1_3)}{\mathbb{P}(1_3)} = \frac{1/3}{1/2} = 2/3$$

Зад. 14 от упр. moodle

$\Omega = \{(a_1, \dots, a_{52}) : a_i \in \{1, \dots, 52\}, a_i \neq a_j\}, |\Omega| = 52!$

7С преди 2Аса

$(7C, \dots)$

$(\dots A, \dots, 7C, \dots, A \dots)$

$- 7C - A_1 - A_2 - A_3 - A_4$

$- A_1 - 7C - A_2 - A_3 - A_4$

$$\frac{2 \cdot \binom{52}{2} \cdot 4! \cdot 47!}{52!} = 2/5$$

(A, 7, A, A, A)
(7, A, A, A, A) – тея не интересуват ⇒ 2/5
(A, A, 7, A, A)
(A, A, A, 7, A)
(A, A, A, A, 7)

Зад. 16 от упр. moodle

$\Omega = \{(a_1, \dots, a_{10}) : a_i \in \{1, \dots, 6\}\} \quad |\Omega| = 6^{10}$

$\#1 = \#6$

$\#1 = \{0, \dots, 5\}$

$i \in \{0, \dots, 5\}$

i 6ци

i 1ци

$10 - 2i \to 2, 3, 4, 5$

$$\frac{\sum_{i=0}^{n} \binom{10}{i} \binom{10-i}{i} 4^{10-2i}}{6^{10}}$$

$\{6_1, \dots, 6_{10-2i}\}$

$\{j_1 \dots j_i\} - \binom{10}{i} \quad |\cup A_i| = \sum |A_i|$

Зад 15 от упр. moodle

① ② ... ⓝ

а) връщане
б) не връщане

$k \leq n$

$\Omega = \{(a_1, \ldots, a_k) : a_i \in \{1, \ldots, n\}\}$ а) $|\Omega| = n^k$ б) $a_i \neq a_j$ $|\Omega| = V_n^k = \frac{n!}{(n-k)!}$

$\mathbb{P}(a_1 < a_2 < \ldots < a_k) =$ а) $\frac{\binom{n}{k}}{n^k}$

$(\quad) \xleftarrow{1} \{b_1, \ldots, b_k\}$ б) $\frac{\binom{n}{k}}{V_n^k}$

комбинация без пов. от ред k

Зад. 21 от упр. moodle ① $\Omega_1 = \{(a_1, \ldots, a_{20}) : a_i \in \{M, Ж\}\}$ $|\Omega_1|$

$\# M = 10$
$\# Ж = 10$

② $\Omega_2 = \{(a_1, \ldots, a_{20}) : a_i \in \{M_1, \ldots, M_{10}, Ж_1, \ldots, Ж_{10}\}\}$ $|\Omega_2| = 20!$

② II начин $\rightarrow$ $2 \cdot \frac{10 \cdot 10 \cdot 9 \cdot 9 \cdot \ldots}{20!}$

① I начин $\rightarrow$

$\frac{2}{\frac{20!}{10! \cdot 10!}}$ $(M, Ж, M, Ж \ldots)$
$(Ж, M, Ж, M, \ldots)$

Зад. 1013 от упр. moodle $\Omega = \{(a_1, \ldots, a_n) : a_i \in \{1, \ldots, n\}, a_i \neq a_j\}$, $|\Omega| = n!$

$x, y \in \{1, \ldots, n\} : x \neq y$

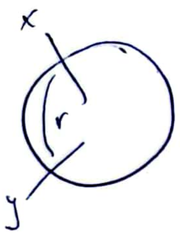
$(a_1, \ldots, a_n) \longleftrightarrow$

$(x/y, \overbrace{b_1, \ldots, b_r}^{\text{между } x \text{ и } y}, x/y)$
$+$
$(c_1, \ldots, c_{n-r-2})$ - останали хора
$+$
място къде вмъкваме

$b_j \neq b_l$

$$\frac{\dfrac{\overset{x\cdots y}{\underset{y\cdots x}{}} - 2\,(n-2)!\,\cancel{(n-(r+2))!}\cdot(n-r-1)}{\cancel{(n-r-2)!}}}{n!} \qquad \{1,\dots,n\}\setminus\underbrace{\{x,y,b_1,\dots,b_r\}}_{r+2}$$

$$= \frac{2\,(n-r-1)}{n(n-n)}$$



d) $\dfrac{2(n-r-1)}{n(n-1)} + \dfrac{\dfrac{2\cdot(n-2)!\,(n-r-2)!\,(r+1)}{(n-r-2)!}}{n!}$

a) $\dfrac{2(n-r-1)}{n(n-1)}$

$A, B \quad \mathbb{P}(B)>0$ $\qquad (\Omega, \mathcal{A}, \mathbb{P}_0)$

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)}, \qquad \downarrow$$

$$(B, \mathcal{A}\cap B, \mathbb{P}_B(\cdot \cap B))$$

$$\mathbb{P}_B(B)=1$$

$\mathbb{P} \to \mathbb{P}(\cdot|B)$

$\mathbb{P}(A|B) = 1-\mathbb{P}(A^c|B)$

$\mathbb{P}(A\cup C|B) = \mathbb{P}(A|B)+\mathbb{P}(C|B)$

$A\cap C=\emptyset$

que $\mathbb{P}(A)>0 \Rightarrow \mathbb{P}(A|B) = \mathbb{P}(B|A)\dfrac{\mathbb{P}(A)}{\mathbb{P}(B)}$

$\longrightarrow \mathbb{P}(A\cap B) = \mathbb{P}(A|B)\,\mathbb{P}(B) = \mathbb{P}(B|A)\cdot\mathbb{P}(A)$

$A_1\dots A_n : \mathbb{P}(A_1\cap\dots\cap A_{n-1})>0$

$\longrightarrow \mathbb{P}(A_1\cap\dots A_{r-1}\cap A_n) = \mathbb{P}(A_1)\cdot\mathbb{P}(A_2|A_1)\cdot\mathbb{P}(A_3|A_1\cap A_2)$

$\dots\ \mathbb{P}(A_r|A_r\cap\ ?$

$\mathbb{P}(A\cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B)$

$\Updownarrow$

$\overrightarrow{\mathbb{P}(B)>0} \quad \mathbb{P}(A|B) = \mathbb{P}(A)$

Зад. 18 от упр. moodle

	>1	$=1$	
не ходи	51	32	83
ходи	9	8	17
	60	40	

$\frac{32}{40} = 4/5$

$15\% \cdot 60 = 9$

4 т. =>

$B = \{>1\}$

$A = \{\text{хирург}\}$

$60\% = \mathbb{P}(B) = 1 - \mathbb{P}(B^c)$

$17\% = \mathbb{P}(A)$

$15\% = \mathbb{P}(A \mid B)$

$\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B)$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$

$(A \cup B)^c = A^c \cap B^c$

$$\mathbb{P}(A^c \mid B^c) = \frac{\mathbb{P}(A^c \cap B^c)}{\mathbb{P}(B^c)} = \frac{\mathbb{P}((A \cup B)^c)}{\mathbb{P}(B^c)} = \frac{1 - \mathbb{P}(A \cup B)}{1 - \mathbb{P}(B)}$$

Зад. 20 от упр. moodle

$A = \{\text{дама}\}$

$B = \{\text{пика}\}$

$\mathbb{P}(A) = 4/52$

$\mathbb{P}(B) = 13/52$

$\mathbb{P}(A \cap B) = 1/52 \overset{?}{=} \frac{4}{52} \cdot \frac{13}{52}$

събитията са независими

Зад. 22

$$1 - \frac{365!}{335!\,365^{30}} = 0.7$$

Зад. 23

$$\mathbb{P}(\text{Играч I печели}) = \underset{E}{\tfrac{1}{2}} + \underset{T}{\tfrac{1}{2}} \cdot \underset{T}{\tfrac{1}{2}} \cdot \underset{E}{\tfrac{1}{2}} + \underset{T}{\tfrac{1}{2}} \cdot \underset{T}{\tfrac{1}{2}} \cdot \underset{E}{\tfrac{1}{2}}$$

$$\mathbb{P}(E) + \mathbb{P}(TTE) + \mathbb{P}(TTTTE) \ldots$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{2} \cdot \frac{1}{1 - 1/4}$$

$$\mathbb{P}(\text{U } \mathrm{I}) = \tfrac{1}{2} + \underbrace{\tfrac{1}{2}\,\tfrac{1}{2}\,\mathbb{P}(\text{U } \mathrm{I})}_{1-\mathbb{P}(\text{U } \mathrm{II})}$$

<u>Зад. 27</u> ~~го бен има 2 деца кой~~

от упр. moodle

A = {по-м момче}

B = {по-с момче}

$\mathbb{P}(A) = \mathbb{P}(B) = 1/2$

$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

$\mathbb{P}(A \mid B) = \mathbb{P}(A) = 1/2$

2-те са момчета   поне 1 момче

$$\mathbb{P}(A \cap B \mid A \cup B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A \cup B)} = \frac{1/4}{3/4} = 1/3$$

(м, м)

(н, м)   поне 1 момче

(м, н)

(н, н)   двете са момчета

27/10/2

Зад. 24

$\{1, \ldots, n\}$

I начин $(a_1, a_2, \ldots, a_n)$

$(n-1)$

II $A_5 = \{(a_1, a_2, a_3, 5, a_4, \ldots, a_{n-1}), \ a_i \in \{1, \ldots, 4, 6, \ldots, n\}\}$

$|A_5| = (n-1)!$

$A_5 \cap A_6 = \{(a_1, \ldots, 5, 6, \ldots), \ a_i \in \{1, \ldots, 4, 7, \ldots n\}\}, \ |A_5 \cap A_6| = (n-2)!$

$\bigcup_{i=1}^{n} A_i = \{$поне 1 да получи правилно$\}$

$\left(\bigcup_{i=1}^{n} A_i\right)^c = \{$никой не е получил$\}$

$$\mathbb{P}(\text{никой}) = 1 - \mathbb{P}(\text{поне } 1)$$

$$= 1 - \frac{|\bigcup_{i=1}^{n} A_i|}{n!}$$

$$= 1 - \frac{\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)!}{n!}$$

$$= \frac{n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}}{n!}$$

$$= \frac{!n}{n!}$$

Зад 28

$a_1, a_2, a_3, \ldots, a_k, \ldots, a_{n-1}, a_n$

$a_1 \to \{1, 2\}$, $1/2$

$a_2 \to \{1, 2, 3\}$, $1/3$

$a_{n-1} \to \{1, \ldots, n-1\}$

$a_n \to \{1, \ldots, n\}$

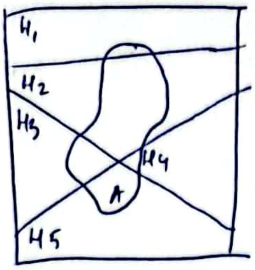
$1/n$ - да се изтегли

$\frac{1}{\cancel{n-1}} \cdot \frac{\cancel{n-1}}{n} = \frac{1}{n}$

да се изтегли

да не се изтегли, след това

//



$H_1, \dots, H_n \in \mathcal{A}: H_i \cap H_j = \emptyset,\ i \neq j$

$\bigcup_{i=1}^{n} H_i = \Omega$

$$\mathbb{P}(A) = \sum_{i=1}^{n} \mathbb{P}(A \cap H_i) = \sum_{i=1}^{n} \mathbb{P}(A \mid H_i)\,\mathbb{P}(H_i)$$

формула за пълна вероятност

$\mathbb{P}(H_i) > 0$

$\mathbb{P}(\text{ези} \mid \text{честна монета}) = 1/2$

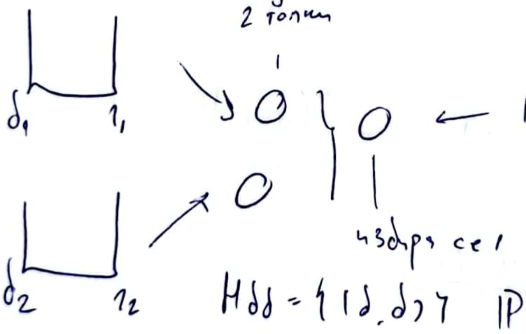
$\mathbb{P}(\text{тура} \mid \text{честна}) = 1/2$

$$\mathbb{P}(H_i \mid A) = \frac{\mathbb{P}(A \mid H_i)\,\mathbb{P}(H_i)}{\underbrace{\sum_{j=1}^{n} \mathbb{P}(A \mid H_j)\,\mathbb{P}(H_j)}_{\mathbb{P}(A)}}$$

шанс монета да е честна с хвърлени 5 езита

Зад. 2 урни

вадят се 2 топки



$A = \{\text{бяла}\}$

$\mathbb{P}(\text{бяла топка})$?

избира се 1

$H_{бб} = \{(б, б)\}$

$H_{чб} = \{(ч, б)\}$

$H_{бч} = \{(б, ч)\}$

$H_{чч} = \{(ч, ч)\}$

$$\mathbb{P}(A) = \mathbb{P}(A \mid H_{бб})\,\mathbb{P}(H_{бб}) + \mathbb{P}(A \mid H_{бч})\,\mathbb{P}(H_{бч}) + \mathbb{P}(A \mid H_{чб})\,\mathbb{P}(H_{чб}) + \mathbb{P}(A \mid H_{чч})\,\mathbb{P}(H_{чч})$$

$\mathbb{P}(H_{бб}) = \frac{б_1}{б_1 + ч_1} \cdot \frac{б_2}{б_2 + ч_2}$

независими => умножаваме

$\mathbb{P}(H_{чб}) = \frac{ч_1}{б_1 + ч_1} \cdot \frac{б_2}{б_2 + ч_2}$

$\mathbb{P}(H_{бч}) = \frac{б_1}{б_1 + ч_1} \cdot \frac{ч_2}{б_2 + ч_2}$

$\mathbb{P}(H_{чч}) = \frac{ч_1}{б_1 + ч_1} + \frac{ч_2}{б_2 + ч_2}$

$\mathbb{P}(A \mid H_{бб}) = 1$

$\mathbb{P}(A \mid H_{чб}) = \mathbb{P}(A \mid H_{бч}) = 1/2$

$\mathbb{P}(A \mid H_{чч}) = 0$

Зад. 34

$\boxed{c}\ \boxed{c}\ \boxed{c}\ \boxed{6}$

$\boxed{?}\ \boxed{?}\ \boxed{?}\ | \longrightarrow H = \{3$ стан. зара$\}$ $\quad P(H) = \dfrac{\binom{3}{3}}{\binom{4}{3}} = 1/4$

$H^c = \{2$ стан$\}$ $\quad P(H^c) = \dfrac{\binom{3}{2}\cdot\binom{1}{1}}{\binom{4}{3}} = 3/4$

а) P(3 бухи)

б) P(различни числа)

в) P(последователни)

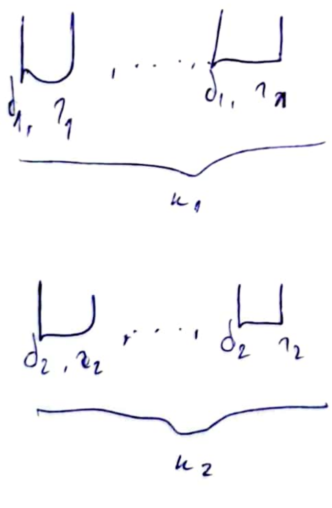
$P(a)|H) = \dfrac{1}{6^3}$, $P(a)|H^c) = \dfrac{1}{6^2}$

$P(б)|H) = \dfrac{\cancel{6}\cdot 5\cdot 4}{\cancel{6}\cdot 6\cdot 6} = P(б)|H^c) = \dfrac{5\cdot 4}{6\cdot 6}$

еднакви са защото в I случай следващите 2 трябва да се съобразяват с първия зар

$P(в)|H^c) = \dfrac{2}{6^2}$ $\qquad P(в)|H) = \dfrac{\cancel{6}\cdot 3!}{6^3}$

Зад.



$H_1 = \{1$ гр$\}$

$H_2 = \{2$ гр.$\} = H_1^c$

$A = \{$бяла$\}$

$$P(H_1|A) = \frac{P(A|H_1)P(H_1)}{P(A|H_1)P(H_1) + P(A|H_2)P(H_2)}$$

$P(H_1) = \dfrac{k_1}{k_1+k_2} = 1 - P(H_2)$

$P(A|H_1) = \dfrac{б_1}{б_1+ч_1}$

Зад. 35

P(синя кола) $= 15\%$

P(зелена кола) $= 85\%$

$\xrightarrow{?} P$(синя к. | A) видяна синя

P(зелена к. | A)

P(A | синя) $= 80\%$

P(A | зел.) $= 20\%$

CamScanner

$$\mathbb{P}(C|A) = \frac{\mathbb{P}(A|C)\,\mathbb{P}(C)}{\mathbb{P}(A|C)\,\mathbb{P}(C) + \mathbb{P}(A|З)\,\mathbb{P}(З)} = 41\% = \frac{12}{29}$$

$$100 \begin{cases} \to 85_З & \;\; 17_С \quad (0{,}2 \cdot 85) \\ \to 15_С & \;\; 12_С \quad (0.8 \cdot 15) \end{cases} \quad \frac{12}{29_С}$$

Зад. 29

	А		Б
малки	$\mathbb{P}(У_А \mid м) = 93\%$	$>$	$\mathbb{P}(У_Б \mid м) = 87\%$
големи	$\mathbb{P}(У_А \mid г) = 73\%$	$>$	$\mathbb{P}(У_Б \mid г) = 69\%$
	$\mathbb{P}(У_А) = 78\%$	$<$	$\mathbb{P}(У_Б) = 83\%$

Зад. 36

$\mathbb{P}(+ \mid \text{болен}) = 99\%$

$\mathbb{P}(- \mid \text{здрав}) = 99\%$

$\mathbb{P}(\text{болен}) = 0{,}5\%$

$\mathbb{P}(Б \mid +) \approx 33\%$

false pos $\mathbb{P}(З \mid +) = 66\%$

false neg $\mathbb{P}(Б \mid -) = \frac{0{,}01 \cdot 0{,}005}{0{,}01 \cdot 0{,}005 + 0{,}99 \cdot 0{,}995}$

Зад. 39

$\frac{б}{б}$ $\frac{б}{ч}$ $\frac{ч}{ч}$

$$\mathbb{P}\left(\tfrac{б}{б} \mid \text{виждам бяла}\right) \longrightarrow = \frac{\mathbb{P}(\text{-"-} \mid \tfrac{б}{б})\,\mathbb{P}(\tfrac{б}{б})}{\mathbb{P}(\text{-"-} \mid \tfrac{б}{б})\,\mathbb{P}(\tfrac{б}{б}) + \mathbb{P}(\text{-"-} \mid \tfrac{б}{ч})\,\mathbb{P}(\tfrac{б}{ч})}$$

$\mathbb{P}(\tfrac{б}{б}) = \mathbb{P}(\tfrac{ч}{б}) = \mathbb{P}(\tfrac{ч}{ч}) = 1/3$

$\mathbb{P}(\text{вижд бяла} \mid \tfrac{б}{б}) = 1$

$\mathbb{P}(\text{-"-} \mid \tfrac{ч}{ч}) = 0$

$\mathbb{P}(\text{-"-} \mid \tfrac{б}{ч}) = 1/2$

$$= \frac{1/3 \cdot 1}{1/3 \cdot 1 + 1/3 \cdot 1/2} = \frac{1/\cancel{3}}{3/\cancel{6}\,2} = 2/3$$

Зад. 25

$\mathbb{P}(I$ пет $|$ отворена вр. 3$)$

$\mathbb{P}(I$ пет$) = \mathbb{P}(II$ пет$) = \mathbb{P}(III$ пет$) = 1/3$

$\mathbb{P}($отв $3 \mid III$ пет$) = 0$

$\mathbb{P}($отв. $3 \mid II) = 1$

$\mathbb{P}($отв. $3 \mid I) = p \in [0, 1] = 0$

$\frac{1}{p+1}$, за $p = 1/2 \Rightarrow 2/3$

$p = 1 \Rightarrow 1/2$

Ден 3/11/22
упр. 5.

33.

4 new
3 old

ако използваме веднъж топка, вече не е нова

$$P(H_3) = \frac{\binom{4}{3}}{\binom{7}{3}}$$

теглим 3 нови топки в 1-ва игра

$A = \{3 \text{ new } 2\text{ра}\}$

$P(A) = ?$

$H_i = \{i \text{ new}, 1\text{-ва}\}$ - колко нови топки са изтеглени в първата игра

$$P(H_i) = \frac{\binom{4}{i}\binom{3}{3-i}}{\binom{7}{3}}$$

i на нови топки; или остатъка стари

$$P(A) = \sum_{i=0}^{3} P(A|H_i)\,P(H_i)$$

пълна вероятност

$P(A|H_3) = 0$

$P(A|H_2) = 0$

$P(A|H_1) = \frac{\overset{=\binom{3}{3}}{1}}{\binom{7}{3}}$

$P(A|H_0) = \frac{\binom{4}{3}}{\binom{7}{3}}$

Зад 43 от упр.

55 m - 0.4

45 f - 0.7

$P(ee\text{т}) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$

$P(2 \text{ ези}, 1 \text{ тура}) = 3\cdot\frac{1}{8}$

$= P(ee\text{т}) + P(e\text{т}e) + P(\text{т}ee)$

$A = \{2 \text{ успешни}, 1 \text{ неуспешен}\}$

$H_i = \{i \text{ на } f, 3-i \text{ на } m\}$

$i = \overline{0,3}$

$$P(H_3|A) = \frac{P(A|H_3)\cdot P(H_3)}{\sum_{j=0}^{3} P(A|H_j)\,P(H_j)}$$

$P(A|H_3) = 3\cdot 0{,}7^2\cdot 0{,}3$

f_1	f_2	f_3
✓	✓	✗
✓	✗	✓
✗	✓	✓

$P(A|H_0) = 3\cdot 0{,}4^2\cdot 0{,}6$

$P(A|H_2) = 0{,}7^2\cdot 0{,}6 + 2\cdot 0{,}7\cdot 0{,}3\cdot 0{,}4$

$P(A|H_1) = 2\cdot 0{,}7\cdot 0{,}4\cdot 0{,}6 + 0{,}3\cdot 0{,}4^2$

f_1	f_2	m
✓	✓	✗
✓	✗	✓
✗	✓	✓

f_1	m_1	m_2
✓	✓	✗
✓	✗	✓
✗	✓	✓

$$P(H_3) = \frac{\binom{45}{3}}{\binom{100}{3}} = \frac{45}{100}\cdot\frac{44}{99}\cdot\frac{43}{98}$$

$P(fff) = P(f)\,P(ff|f)\cdot P(fff|ff)$

$$P(H_i) = \frac{\binom{45}{i}\binom{55}{3-i}}{\binom{100}{3}}$$

<u>Зад. 4.</u>

A = {удът от точно 1 куршум}

H_i = {i-тият ловец уцелва}

$\mathbb{P}(H_1) = 0.2$

$\mathbb{P}(H_2) = 0.4$

$\mathbb{P}(H_3) = 0.6$

$$\mathbb{P}(H_1|A) = \frac{\mathbb{P}(A|H_1)\,\mathbb{P}(H_1)}{\sum_{j=1}^{3} \mathbb{P}(A|H_j)\,\mathbb{P}(H_j)}$$

$\downarrow$

$\Sigma\, \mathbb{P}(H_i) = \mathbb{P}(\Omega) = 1$

$H_i \cap H_j = \emptyset$

$\bigcup_{i=1} H_i = \Omega$

$H_1, H_2, H_3 \to \{H_1, H_1^c\}$

$\downarrow$

$\mathbb{P}(A H_1^c) = 0.4 \cdot 0.4 + 0.6 \cdot 0.6$

$\{H_1 \cap H_2 \cap H_3,$
$H_1 \cap H_2 \cap H_3^c,$
$H_1 \cap H_2^c \cap H_3,$
$\dots$
$H_1^c \cap H_2^c \cap H_3^c\}$

$A = \{H_1 \cap H_2^c \cap H_3^c\} \cup \{H_1^c \cap H_2 \cap H_3^c\}$
$\cup \{H_1^c \cap H_2^c \cap H_3\}$

$A \cap H_1 = H_1 \cap H_2^c \cap H_3^c$

$$\mathbb{P}(H_1|A) = \frac{\mathbb{P}(A \cap H_1)}{\mathbb{P}(A)} = \frac{\mathbb{P}(H_1 \cap H_2^c \cap H_3^c)}{\mathbb{P}(H_1 \cap H_2^c \cap H_3^c) + \mathbb{P}(\ ) + \mathbb{P}(}$$

$\mathbb{P}(H_1 \cap H_2^c \cap H_3^c) = 0.2 \cdot 0.6 \cdot 0.4$

Λ_1	Λ_2	Λ_3
✓	×	×

<u>Зад 25</u>

n бели
m зелени
l червени

а) с връщане
б) без връщане

$\Omega = \{б, з, чб, чз, ччб, ччз, \dots, чч\dots$

възможни изходи

A = {бяло преди зелено}

а) $\mathbb{P}(A) = \underbrace{\frac{n}{n+m+l}}_{\text{бяло на 1-во}} + \underbrace{\frac{l}{n+m+l}}_{\text{червено}} \cdot \mathbb{P}(A)$

$\mathbb{P}(A) = \frac{n}{n+m}$

1:04:20

$\mathbb{P}(A) = \mathbb{P}(б) + \mathbb{P}(чб) + \ldots$

$H_0 = \{\text{само ч}\}$

$H_k^б = \{\underbrace{ч\ldots ч}_{(k-1)} б \text{ бяла на } k\text{-ти вход}\}$

$H_k^з = \{\text{зелени на } k\text{-ти вход}\}$

$\mathbb{P}(A \cap H_0) = 0$ - само черни

$\mathbb{P}(A \cap H_k^з) = 0$ $\qquad H_0 \cup \bigcup_k H_k^б \cup \bigcup_k H_k^з$

$$\mathbb{P}(A) = \cancel{\sum_{k=1}^{\infty} \mathbb{P}(}\ \mathbb{P}(A \cap \Omega)$$

$$= \mathbb{P}\left(A \cap \left(\bigcup_{k=1}^{\infty} H_k^б\right)\right)$$

$$= \mathbb{P}\left(\bigcup_{k=1}^{\infty} A \cap H_k^б\right)$$

$H_k^б \subseteq A$

$$= \sum_{k=1}^{\infty} \mathbb{P}(A \cap H_k^б)$$

$$= \sum_{k=1}^{\infty} \mathbb{P}(H_k^б)$$

$$= \mathbb{P}(б) + \mathbb{P}(чб) + \ldots = \frac{n}{n+m+l} \sum_{k=0}^{\infty} \left(\frac{l}{n+m+l}\right)^k$$

без връщане

$$= \frac{n}{n+m+l} \cdot \frac{1}{1 - l/(n+m+l)}$$

$$= \frac{n}{n+m}$$

геометрична прогресия (за изпита стига до тук)

$$\mathbb{P}(H_k^б) = \mathbb{P}(\underbrace{ч\ldots ч}_{k-1} б) = \left(\frac{l}{n+m+l}\right)^{k-1} \cdot \frac{n}{n+m+l}$$

$$\mathbb{P}(A) = \underbrace{\mathbb{P}(A \cap H_1^б)}_{\mathbb{P}(H_1^б) = \frac{n}{n+m+l}} + \underbrace{\mathbb{P}(A \cap H_1^з)}_{0} + \underbrace{\mathbb{P}(A \cap \underbrace{(H_1^б \cup H_1^з)^c}_{G})}_{\mathbb{P}(A|G)\,\mathbb{P}(G)}$$

$$\mathbb{P}(G) = \frac{l}{n+m+l}$$

$$\mathbb{P}(A) = \sum_{k=1}^{\infty} \mathbb{P}(H_k^б)$$

$$\mathbb{P}(A|G) = \sum_{k=1}^{\infty} \mathbb{P}(H_k^б | G) = \sum_{k=1}^{\infty} \mathbb{P}(H_k^б) = \mathbb{P}(A)$$

първа бяла

изтеглени черни

$\mathbb{P}(H_1^б | G) = 0$

$\mathbb{P}(H_2^б | G) = \mathbb{P}(чб | ч) = \mathbb{P}(H_1^б) = \mathbb{P}(б)$

$\mathbb{P}(H_3^б | G) = \mathbb{P}(ччб | ч) = \mathbb{P}(чб) = \mathbb{P}(H_2^б)$

б) $\Omega = \{б, з, чб, чз, \dots, \underbrace{ч \dots ч}_{k} б, \underbrace{ч \dots ч}_{k} з\}$

$H_1^б, \dots, H_{l+1}^б, H_1^з, \dots, H_{l+1}^з$

$$A \cap H_k^з = \emptyset$$

$$P(A) = \sum_{k=1}^{l+1} P(H_k^б) = \frac{n}{n+m+l} \qquad P(A^c) = \frac{m}{n+m+l} \ ?$$

$$P(H_k^б) = P(\underbrace{ч, ч, \dots, ч}_{k-1}, б) = P(ч)\, P(чч|ч)\, P(ччч|чч) \dots P(бч|ч)$$

$$\frac{l}{l+m+n} \cdot \frac{l-1}{l+m+n-1} \cdots \frac{l-(k-2)}{n+m+l-(k-2)} \cdot \frac{n}{n+m+l-(k}$$

$A^c = \{$зелено преди бяло$\}$

$$\frac{P(A)}{P(A^c)} = \frac{n}{m} \qquad P(A) + P(A^c) = 1$$

$$\Rightarrow P(A) = \frac{n}{n+m}$$

Зад. пълна вер.

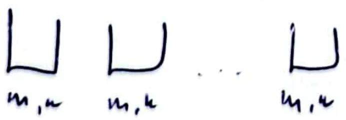
$$P(A) \overset{*}{=} P^*(A|C)\, P^*(C) + P^*(A|C^c)\, P^*(C^c)$$

$$P(A|B) = P(A|B\cap C)\ \cancel{P(B\cap C)\ P(B)}\ P^*(C|B) + P(A|B\cap C^c)\, P(C^c|B)$$

$$P(\cdot|B) = P^*(\cdot) \qquad P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A\cap B\cap C)}{P(B)} + \frac{P(A\cap B\cap C^c)}{P(B)} \cdot \frac{P(B\cap C^c)}{P(B\cap C^c)}$$

$$\frac{P(B\cap C)}{P(B\cap C)}$$

Зад. 32



m - бели топки
k - черни топки
n - урни

$A_j = \{j$-тата урна бяла$\}$

$$P(A_n) = P(A_n|A_{n-1}) \cdot P(A_{n-1}) + P(A_n|A_{n-1}^c)\, P(A_{n-1}^c) = \frac{m}{m+k}$$

$$P(A_n|A_{n-1}) = \frac{m+1}{m+k+1} \qquad P(A_n|A_{n-1}^c) = \frac{m}{m+k+1}$$

$$\mathbb{P}(A_1) = \frac{m}{m+n}$$

$$\mathbb{P}(A_2) = \mathbb{P}(A_2 \mid A_1)\, \mathbb{P}(A_1) = \mathbb{P}(A_2 \mid A_1^c)\, \mathbb{P}(A_1^c)$$

$$= \frac{m+1}{m+n+1} \cdot \frac{m}{m+n} + \frac{m}{m+n+1} \cdot \frac{n}{m+n}$$

$$= \frac{m}{m+n}$$

stat 110

probability and sets

S - sample space = rolling a dice twice (Ω)

~~s - possible outcome of 4 = one dice showing a 4, $s \in S$~~

A - event = both dices showing 4; $A \subseteq S$ ($A \subseteq \Omega$)

s - possible outcome - true or false

A occurs - $s_0 \in A$

A или B = $A \cup B$

A и B - $A \cap B$

не A = A^c

A и B са mutually exclusive - $A \cap B = \emptyset$

A имплицира B - $A \subseteq B$

вероятност за A = $\mathbb{P}(A)$

A и B са независими $\Rightarrow \mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$

пр. 1. A: card is a heart = 13

$\mathbb{P}(A) = \frac{13}{52} = \frac{1}{4}$

B: - " - = 13

C: $\frac{4}{52} = \frac{1}{13}$ = 4

D: $\frac{26}{52} = \frac{1}{2}$ = 26

$A \cap C$: a card is a heart and an ace $\Rightarrow \frac{1}{52}$

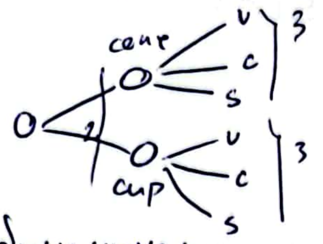
$A \cup B$: a card is a heart or a diamond $\Rightarrow$

~~Math review~~

Lecture 1

Counting

експеримент



$\Rightarrow 2 \cdot 3 = 3 \cdot 2$

биномиален коеф.

$$\binom{n}{k} = \begin{cases} \frac{n!}{(n-k)!\,k!}, & \text{ако } 0 \le k \le n \\ 0, & \text{ако } k > n \end{cases}$$

избиране под-мн-во от k-ти ред от n елементи, без значение на ред

	ordered	unordered
w/ replacem.	n^k	$\binom{n+k-1}{k}$
w/o replacem.	$n(n-1)\ldots(n-k+1)$	$\binom{n}{k}$

Homework 1

1) 1. (а) 4 ~~нод~~ зара >

~~(б)~~ sum = 21 sum = 22

6 6 6 3	6 6 6 4
6 6 5 4	6 6 5 5
6 5 5 5	~~6 5~~
~~6 5~~	

(б) ~~28~~ ~~4~~ =

веровятно ради факт дека с'с 2 букве има само 28 опције (дадено ~~англиску~~ латиничну азбуку), а с'с 3 букве има + 28 пермутације

2. (а) flush $\cancel{\dfrac{\frac{13}{5}\binom{5}{13}}{\binom{5}{52}} = 5!}$

$$\frac{\cancel{\binom{13}{5}}\,4\cdot\binom{13}{5} - 1}{\binom{52}{5}} = \frac{4\cdot\frac{13!}{8!\,5!} - 1}{\frac{52!}{47!\,5!}} = \frac{4\cdot\frac{13\cdot12\cdot11\cdot10\cdot9}{5\cdot4\cdot3\cdot2\cdot1} - 1}{\frac{52\cdot51\cdot50\cdot49\cdot48}{5\cdot4\cdot3\cdot2\cdot1}}$$

- royal flush

зашото има 4 suits

$$= \frac{4\cdot13\cdot11\cdot9 - 1}{52\cdot51\cdot5\cdot49\cdot4}$$

$$= \frac{5147}{2598960}$$

$= 0.00198 \quad |\cdot100$

$= 0{,}198\%$

(в) 2 ~~р~~ пара б flush

$$\frac{\binom{13}{2}\cdot\binom{4}{2}^2\cdot 44}{\binom{52}{5}}$$

- extra карта

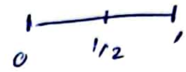
Геометрична вероятност

Ω - тяло от геометрични обекти

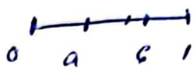
μ - мярка, обем

$\mu(\Omega) < \infty \Rightarrow \mu$ е крайно

$$\mathbb{P}(A) = \frac{\mu(A)}{\mu(\Omega)}$$



$\mathbb{P} = \frac{1}{2}$

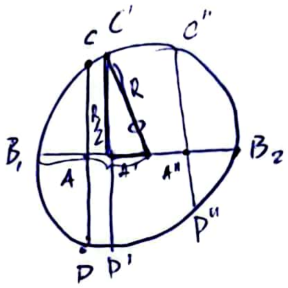


$$\frac{\mu([a,b])}{\mu([0,1])} = \frac{b-a}{1-0}$$

$\Omega \subseteq \mathbb{R}^d$

$\boxed{\mu([a,b]) = b-a}$

Зад. 44 кръг с $r = R$



$C'D' = R = C''D''$

$R^2 = \frac{R^2}{4} + (OA')^2$

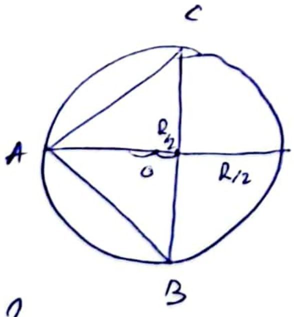
$\Rightarrow OA' = \frac{R\sqrt{3}}{2}$

$C'A' = R/2$ - по условие

$$\mathbb{P}([B_1, A'] \cup [A'', B_2]) = 2 \cdot \frac{\mu([B_1, A'])}{\mu([B_1, B_2])} = \cancel{2} \cdot \frac{R(1 - \frac{\sqrt{3}}{2})}{\cancel{2}R}$$

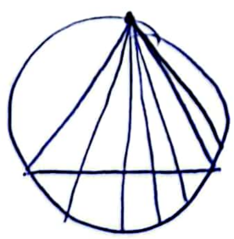
Зад. 52

Същото като горе: зад. 44



μ_1 $1/2$

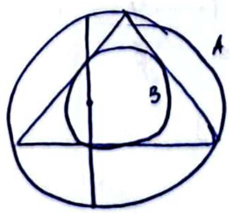
Реш. 2



μ_2

$1/3$

Реш. 3



$B = 1/4 A$

μ_3

Зад. 45

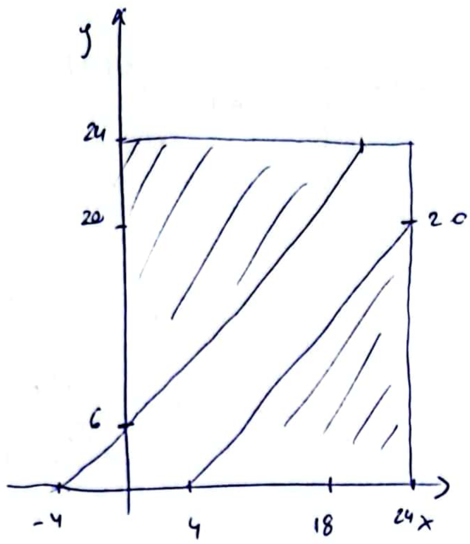
x - I

y - II

$x, y \in [0, 24]$

$x + 6 < y$

$y + 4 < x$



$(x, y): x + 6 < y$

$x + 6 = y$

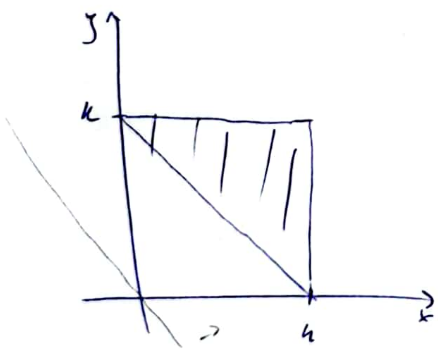
$x - 4 = y$

$$\mathbb{P}(\triangle + \triangle) = \frac{\frac{18^2}{2} + \frac{20^2}{2}}{24^2}$$

Зад. 47

$x, y \in [0, k]$

$(x, y, k) \Leftrightarrow x + y > k$

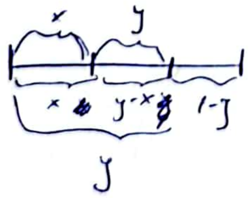


$y = x$

$y = -x$

$\mathbb{P} = 1/2$

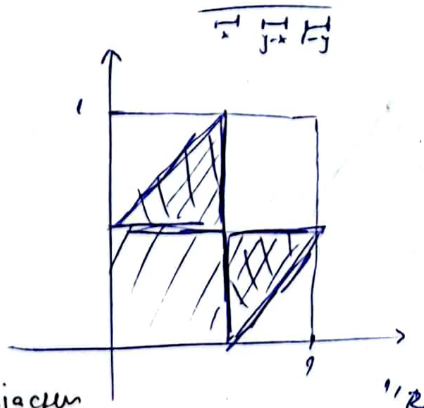
Зад. 51



$x < y$

$x, y \in [0, 1]$

$y < x$



a) $x < y$

$\cancel{x} + y - \cancel{x} > 1 - y$

$x + 1 - y > y - x$

$y - x + 1 - y > x$

$y > 1/2$

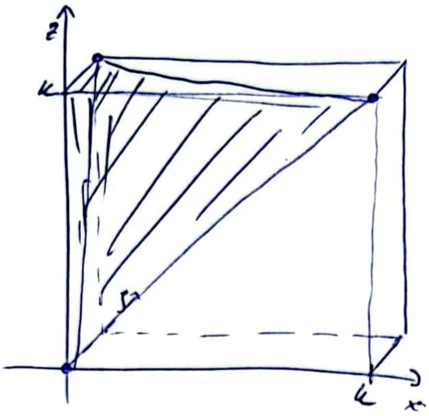
$y - x = < 1/2$

$x < 1/2$

$y = 1/2 + x$

разяснен

зад. 48 $(x, y, z) \in \mathbb{R}^3$



$x, y, z \in [0, k]$

$x+y > z$

$x+z > y$

$y+z > x$

$$k^3 - 3\frac{\frac{k^2}{2}k}{3} = \frac{\frac{k^3}{2}}{k^3} = 1/2$$

$x + y = z$

0	0	0
k	0	k
0	k	k

II фиксираме най-голямата (z) и отиваме в 47 зад.

$H_1 = \{z$ - най-гол.$\}$

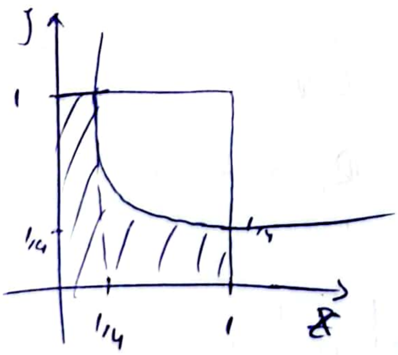
$H_2 = \{x$ - най-гол.$\}$

$H_3 = \{y$ - ॥ - $\}$

$$\mathbb{P}(A) = \underbrace{\mathbb{P}(A|H_1)}_{1/2} \cdot \underbrace{\mathbb{P}(H_1)}_{1/3} + \ldots + \ldots = 1/2$$

A - да изберем x, y, z които могат да образуват триъгълник

зад. 50



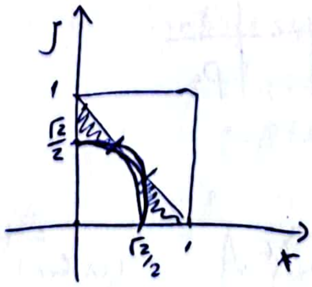
1) $(x, y) \in [0, 1]^2 : xy \le 1/4$

$y = 1/4x$

$$\int_{1/4}^{1} 1/4x\,dx + \int_{0}^{1/4} 1\,dx$$

$$\int_{1/4}^{1} 1/4x\,dx + \int_{0}^{1/4} 1\,dx$$

2)



$x+y \le 1$

$x^2+y^2 \geq 1/2$

$(x-a)^2 + (y-b)^2 = r^2$

$a = 0$

$b = 0$

$r = \frac{\sqrt{2}}{2}$

$x+y = 1$

$x^2+y^2 = 1/2$

$x^2 + (1-x)^2 = 1/2$

$2x^2 - 2x + 1/2 = 0$

$x = 1/2$

$y = 1/2$

Случайна вероятност

$(\Omega, \mathcal{A}, \mathbb{P})$

$\mathbb{P}: \mathcal{A} \to [0,1]$

$X: \Omega \to \mathbb{R}$

$$\Omega = \{\underbrace{(T,T)}_{0}, \underbrace{(E,T)}_{1}, \underbrace{(T,E)}_{1}, \underbrace{(E,E)}_{2}\}$$

$$\mathbb{P}(X=2) = \mathbb{P}(\{\omega \in \Omega : X(\omega) = 2\})$$
$$= \mathbb{P}(X^{-1}(\{2\}))$$
$$= \mathbb{P}(\{(E,E)\})$$

$X^{-1}: \mathcal{B}(\mathbb{R}) \to \mathcal{P}(\Omega)$

$\mathcal{B}$ - борелови : σ алгебра дефинирана от интервали

$\mathcal{B}(\mathbb{R}) = \bigcap \mathcal{F}$

$\mathcal{F}$ е σ алг. в/у $\mathbb{R}$

$\mathcal{F} \supseteq \{$от в. инт.$\}$

$\{X=2\} \subseteq \Omega$ $\qquad$ $\{X=2\} = \{\omega \in \Omega : X(\omega) = 2\} = X^{-1}(\{2\}) = \{E$

~~$\{\omega \in \Omega\}$~~ $\{X=2\} \in \mathcal{A}$

отговоре в ?
от recording

! измерваме неща които са в $\mathcal{A}$

$$(\Omega, \mathcal{A}, \mathbb{P}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$$

-> дискретни

X	x_1	x_2	...	x_n
	p_1	p_2	...	p_n

X	0	1	2
	$\mathbb{P}(X=0)$	$\mathbb{P}(X=1)$	$\mathbb{P}(X=2)$
	$\parallel$ $\frac{1}{4}$	$\parallel$ $\frac{1}{2}$	$\parallel$ $\frac{1}{4}$

$\{a \le X \le b\} \in \mathcal{A}$

$\parallel$

$\forall \{X \in [a,b]\} \in \mathcal{A}$

$\to \{X \in A\} \in \mathcal{A}, \ \forall A \in \mathcal{B}(\mathbb{R})$

Деф: $X: \Omega \to \mathbb{R}$: $\forall a,b \in \mathbb{R}$

(сл. вел.) $\{a \le X \le b\} \in \mathcal{A}$

$\{a \le X < b\} = X \in [a,b) \in \mathcal{F}$ в/у $\mathbb{R}$

$\mathcal{B}(\mathbb{R}) = \bigcap \mathcal{F}$

$\mathcal{F}$ в/у $\mathbb{R}$

$\forall a,b, [a,b) \in \mathcal{F}$

СЕМ
упр. 7

Дискретни сл. вел

17/11/22

$(\Omega, \mathcal{A}, \mathbb{P})$ $\quad X: \Omega \to \mathbb{R}^d \quad | \quad X: \Omega \to \mathcal{X} \subseteq \mathbb{R}^d$

$\mathbb{P}: \mathcal{A} \to [0,1]$

(i) $\mathcal{X} = \{x_1, x_2, \dots\}, \ x_n \in \mathbb{R}^d$

(ii) $\{X = x_n\} = \{\omega \in \Omega : X(\omega) = x_n\} \in \mathcal{A}$

$\hookrightarrow \mathbb{P}(X = x_n) = p_n$

X	x_1	x_2	$\dots$	x_n	$\dots$
$\mathbb{P}_X$	p_1	p_2	$\dots$	p_n	$\dots$

печалба, брой

$\mathcal{P}(\mathcal{X})$
$\parallel$

$(\mathcal{X}, 2^{\mathcal{X}}, \mathbb{P}_X)$ $\quad 2^{\mathcal{X}} = \sigma(\mathcal{X}) = \bigcap_{\mathcal{T} \supseteq \{x_1, \dots\}} \mathcal{T}$

измеримо пр-во

сигма алгебра

$\Omega = \{\omega_1, \omega_2, \dots\}$

$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\})$

$\mathbb{P}_X(\cdot) = \sum_n p_n \delta_{x_n}(\cdot)$

$\hookrightarrow = \sum_{x \in A} p_x$

$\delta_x(A) = \begin{cases} 0, & x \notin A \\ 1, & x \in A \end{cases}$

$x \in \mathcal{X}$ $\quad \mathbb{P}_X : 2^{\mathcal{X}} \to [0,1]$ $\quad \delta_x(A) = \Delta_A(x)$

$A \subseteq \mathcal{X} \ (A \in 2^{\mathcal{X}})$

Не говорим за мета събития, а за конкретни кои ни интересуват.

$A = \bigcup_{x \in A} \{x\} \in \sigma(X)$

Зад 53 $\quad X$ # ези

X	0	1	$\dots$	k	$\dots$	10
$\mathbb{P}_X$	$\frac{1}{2^{10}}$	$\binom{10}{1}\frac{1}{2^{10}}$	$\dots$	$\binom{10}{k}\frac{1}{2^{10}}$	$\dots$	$\binom{10}{10}\frac{1}{2^{10}}$

$\Omega = \{(a_1, \dots, a_{10}) : a_i \in \{E, T\}\}$

$\mathbb{P}(X=0) = \mathbb{P}((T,T,\dots,T)) = \frac{1}{2} \cdot \frac{1}{2} \cdot \ldots \cdot \frac{1}{2} = \frac{1}{2^{10}}$

$\mathbb{P}(X=1) = \mathbb{P}(\underbrace{(E,T,\dots,T)}_{\omega_1}, \underbrace{(T,E,\dots,T)}_{\omega_2} \dots) = \frac{1}{2} \cdot \frac{1}{2^9} = \frac{1}{2^{10}}$

$\parallel \binom{10}{1}$

$\sum \mathbb{P}(\omega_i) = \binom{10}{1} \cdot \frac{1}{\ }$

а) $\mathbb{P}(X=5) = \binom{10}{5} \frac{1}{2^{10}}$

б) $\mathbb{P}(\{X=4\} \cup \{X=6\})$

$= \mathbb{P}(X=4) + \mathbb{P}(X=6)$

$= 2 \cdot \mathbb{P}(X=4) = 2 \cdot \binom{10}{4} \frac{1}{2^{10}}$

$\parallel$
$\binom{10}{6}$

в) $\{X = 7,5\} = \emptyset$

Зад. 60 X # разрушени части

$$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X=0) - \mathbb{P}(X=1)$$

X	0	1	$\dots$	j	$\dots$
$\mathbb{P}_X$	$(1-p)^n$				

$$\binom{m}{1} \underbrace{p \cdot (1-p)^{n-1}}_{\text{торпедо}} \frac{1}{m}$$

избрани отсек

$$\binom{m}{1} \sum_{k=1}^{n} \binom{n}{k} \underbrace{p^k (1-p)^{n-k}}_{\text{няколко торпеда}} \left(\frac{1}{m}\right)^k = \mathbb{P}(X=1)$$

избран отсек

$$\mathbb{P}(X=j) = \binom{m}{j} \sum_{k=j}^{n} \binom{n}{k} p^k (1-p)^{n-k} \left(\frac{j}{m}\right)^k \cdot \frac{C(j,k-j)}{C(j,k)}$$

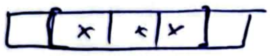
$\{a_1, \dots, a_k\}$ $a_i \in \{1,2,\dots$

$C(j,k) = \binom{j+k-1}{k}$

$\{1,2,\dots,j, a_1^k, \dots, a_{k-j}^k\}$ $a_i^k \in$

$/ C(j,k-j)$

по един изстрел в всяка кутийка



- да осигури да има поне 1 изстрел в всяка част

$$\mathbb{E}[X] = \sum_n x_n p_n = \sum_n x_n \mathbb{P}(X = x_n) = \sum_n x_n \mathbb{P}_X(\{x_n\})$$

Зад. 54 X - # шестици от 2 хвърляния

X	0	1	2
$\mathbb{P}_X$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

$Y: X \to Y$ - печалба

$2 \to 95$

$1 \to 0$

$0 \to -5$

Y	-5	0	95
$\mathbb{P}_Y$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

$$-5 \cdot \frac{25}{36} + 95 \cdot \frac{1}{36} = -\frac{125}{36} + \frac{95}{36} = -\cancel{\frac{30}{36}} \ \frac{5}{6}$$

$\mathbb{E}[Y] = -\frac{5}{6}$

не е справедлива

! зад 55 X - печалба

-

X	$2-A$	2^2-A	$\dots$	2^n-A
$\mathbb{P}_X$	$1/2$	$1/4$	$\dots$	$\frac{1}{2^n}$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \frac{(2^n-A)}{2^n} = \sum_{n=1}^{\infty}\left(1-\frac{A}{2^n}\right) = \infty$$

-

X	$2A-A$	$4A-(A+2A)$	$\dots$	$2^nA-(A+2A+\dots+2^{n-1}A)$
$\mathbb{P}_X$	$\frac{1}{2}$	$1/4$	$\dots$	$1/2^n$

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} \frac{2^nA-(A+2A+\dots+2^{n-1}A)}{2^n}$$

$$1+2(S_{n-1}) = S_n = S_{n-1}+2^n$$

$$= A\sum_{n=1}^{\infty} \frac{(2^n-(\overbrace{1+2+\dots+2^{n-1}}^{(2^n-A)}))}{2^n}$$

$$= A\sum_{n=1}^{\infty}\frac{1}{2^n} = A$$

$X: \Omega \to \mathcal{X}\ \{x_1, x_2, \dots\}$

$Y: \Omega \to \mathcal{Y}\ \{y_1, y_2, \dots\}$

$\mathbb{P}(\{X=x_i, Y=y_j\}) = p(x_i, y_j)$

сечение

$Y \backslash X$	x_1	x_2	$\dots$	x_i
y_1	$p(x_1,y_1)$	$p(x_2,y_1)$	$\dots$	$p(x_i,y_1)$
y_2	$p(x_1,y_2)$	$p(x_2,y_2)$	$\dots$	$p(x_i,y_2)$
$\vdots$				
y_j	$p(x_1,y_j)$	$p(x_2,y_j)$	$\dots$	$p(x_i,y_j)$
	$p(x_1)$	$p(x_2)$	$\dots$	$p(x_i)$

$$\sum_j \mathbb{P}(X=x_i, Y=y_j) = \mathbb{P}\left(\bigcup_j \{X=x_i, Y=y_j\}\right) = \mathbb{P}\left(\{X=x_i\} \cap \underbrace{\{\bigcup_j \{Y=y_j\}\}}_{=\Omega}\right) = \mathbb{P}(X=x_i)$$

$\{\{Y=y_1\}, \{Y=y_2\}, \dots\}$

Y	y_1	y_2	$\dots$	y_n
	$p(y_1)$	$p(y_2)$	$\dots$	$p(y_n)$

-> независимост

$\mathbb{P}(X=x_i, Y=y_j) = \mathbb{P}(X=x_i)\mathbb{P}(Y=y_j)$

$X \perp\!\!\!\perp Y \quad \forall i,j$

3ag 57

X # езч A
Y # езч B

$X \perp\!\!\!\perp Y$

$Y \backslash X$	0	1	2	3
0	$\frac{1}{8}\cdot\frac{1}{4}$	$\frac{3}{8}\cdot\frac{1}{4}$	$\frac{3}{8}\cdot\frac{1}{4}$	$\frac{1}{8}\cdot\frac{1}{4}$
1	$\frac{1}{8}\cdot\frac{2}{4}$	$\frac{3}{8}\cdot\frac{2}{4}$	$\frac{3}{8}\cdot\frac{2}{4}$	$\frac{1}{8}\cdot\frac{2}{4}$
2	$\frac{1}{8}\cdot\frac{1}{4}$	$\frac{3}{8}\cdot\frac{1}{4}$	$\frac{3}{8}\cdot\frac{1}{4}$	$\frac{1}{8}\cdot\frac{1}{4}$

1
1 2 1
1 3 3 1

$\mathbb{P}(X>Y) = \mathbb{P}((X,Y)\in\{(1,0),(2,0)\ldots$

Анегели $= 1/2$

$$\mathbb{P}(Y=1 \mid X>Y) = \frac{\mathbb{P}(Y=1, X>Y)}{\mathbb{P}(X>Y)}$$

$$= \frac{\mathbb{P}(\{(2,1)\}\cup\{(3,1)\})}{\mathbb{P}(X>Y)}$$

$$= \frac{\mathbb{P}(2,1)+\mathbb{P}(3,1)}{\mathbb{P}(X>Y)}$$

$= 1/2$

Z	$(5-3)$	$(0-3)$
	$\mathbb{P}(X>Y) = 1/2$	$1/2$

$\mathbb{E}[Z^A] = \frac{2}{2} - \frac{3}{2} = -1/2$

$\mathbb{E}[Z^B] = 1/2 - 1/2 = 0$

zero-sum game

Сем
упр. 8 24/11/22

X	x_1	$\dots$	x_n
	$\frac{1}{n}$		$\frac{1}{n}$

$\mathbb{E}[X]$, $Var(X)$

а) $x_i = \frac{i-1}{n-1}$

б) $x_i = a + (b-a)\frac{i-1}{n-1}$, $b > a$

$\mathbb{E}[X] = \sum_{i=1}^{n} x_i \, \mathbb{P}(X = x_i)$ - очакване

$D(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$ - дисперсия

$\Updownarrow$

$Var(X) = \sum_{i=1}^{n} (x_i - \mathbb{E}[X])^2 \, \mathbb{P}(X = x_i)$

$\mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$\boxed{\sum_n \mathbb{P}(X = x_n) = 1}$

(+) $a, b \in \mathbb{R}$ $\mathbb{E}[aX+b] = \sum_n (a x_n + b)\,\mathbb{P}(X = x_n)$

$= a \underbrace{\sum_n x_n \mathbb{P}(X = x_n)}_{\mathbb{E}[X]} + b \underbrace{\sum_n \mathbb{P}(X = x_n)}_{1}$

$= a\,\mathbb{E}[X] + b$

(+) $Var(aX+b) = \sum_n ((a x_n + b) - \mathbb{E}[aX+b])^2 \, p(x_n)$

$= \sum_n a^2 (x_n - \mathbb{E}[X])^2 p(x_n)$

$= a^2 \, Var(X)$

а) $\mathbb{E}[\cancel{\frac{i-1}{n-1}X}] = \sum_{i=1}^{n} \frac{i-1}{n-1} \mathbb{P}(X = \cancel{\frac{i-1}{n-1}} x_i)$

$\cancel{= \frac{i-1}{n-1} \sum_{i=1}^{n} \mathbb{P}(X = x_i)}$

$\cancel{= \frac{i-1}{n-1}}$

$= \frac{1}{n} \sum_{i=1}^{n} \frac{i-1}{n-1} = \frac{1}{n(n-1)} \sum_{i=1}^{n-1} i = \frac{1}{2}$

$\boxed{\begin{aligned} &\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \\ &\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \end{aligned}}$

$\cancel{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$

$\cancel{\sum_{i=1}^{n} i^2 = n(n+1)}$

$$\mathbb{E}[X^2] = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-1}{n-1}\right)^2$$
$$= \frac{1}{n(n-1)^2}\sum_{i=1}^{n}(i-1)^2$$
$$= \frac{1}{n(n-1)^2}\sum_{i=1}^{n-1} i^2$$
$$= \frac{1}{n(n-1)^2}\,\frac{n(n-1)(2n-1)}{6}$$
$$= \frac{2n-1}{6(n-1)}$$

$$Var(X) = \frac{n+1}{12(n-1)}$$

d) $x_i = a + (b-a)\frac{i-1}{n-1}$, $b > a$

$$X_2 = (b-a)X_1 + a$$

Зад. 62 1. X_1 точки на I зар

X_2 точки на II зар, $\left.\begin{matrix} X_1 \perp\!\!\!\perp X_2 \\ X_1 \overset{d}{=} X_2 \end{matrix}\right\}$ i.i.d.

$X = X_1 + X_2$

$$\mathbb{E}[X] = \mathbb{E}[X_1 + X_2]$$
$$= \mathbb{E}[X_1] + \mathbb{E}[X_2]$$

$$Var(X) = Var(X_1 + X_2)$$

свойство!:

$\mathbb{E}[X+Y]$
$$= \sum_{x,y}(x+y)\,\mathbb{P}(X=x, Y=y)$$
$$= \sum_{x,y} x\,p(x,y) + \sum_{x,y} y\,p(x,y)$$
$$= \sum_x x \underbrace{\sum_y p(x,y)}_{p(x)} + \sum_y y \underbrace{\sum_x p(x,y)}_{p(y)}$$
$$= \sum_x x\,p(x) + \sum_y y\,p(y)$$
$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

$\sum_y p(x,y) = p(x)$

$$(+)\ Var(aX+bY) = \mathbb{E}\left[\left((aX+bY) - \mathbb{E}[aX+bY]\right)^2\right]$$
$$= \mathbb{E}\left[\left(a(X-\mathbb{E}[X]) + b(Y-\mathbb{E}[Y])\right)^2\right]$$
$$= a^2 Var(X) + b^2 Var(Y) + 2ab\,\underbrace{\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]}_{cov(X,Y)\ =\ \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]}$$

$X \perp\!\!\!\perp Y \Rightarrow \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$

$X \perp\!\!\!\perp Y \quad \mathbb{P}(X=x_i, Y=y_j) = \mathbb{P}(X=x_i)\mathbb{P}(Y=y_j)$

$\forall x_i \in X$, $y_j \in Y$

$$\mathbb{E}[XY] = \sum_{x,y} xy\,\underbrace{\mathbb{P}(X=x, Y=y)}_{\mathbb{P}(X=x)\mathbb{P}(Y=y)}$$
$$= \sum_x x\,p(x)\,\underbrace{\Big(\sum_y y\,p(y)\Big)}_{\mathbb{E}[Y]}$$
$$= \mathbb{E}[Y]\sum_x x\,p(x)$$
$$= \mathbb{E}[Y]\,\mathbb{E}[X]$$

зад $X \perp\!\!\!\perp Y$

X	-1	1
	1/2	1/2

Y	1	3	5
	1/2	1/4	1/4

пр.

X	1	1
	1/2	1/2

$Var(X) = 0$

разпр. $\mathbb{E}$, Var

а) $2X+Y+1$

б) XY

а) $Z = 2X+Y+1$
$f(x,y)$

$Y \backslash X$	-1	1
1	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{8}$	$\frac{1}{8}$
5	$\frac{1}{8}$	$\frac{1}{8}$

всеки + с всеки
x 6 израза

Z	0	2	4	6	8
	(-1,1)	(1,1) (-1,3)	(1,3) (-1,5)		
	$\frac{1}{4}$	$\frac{1}{4}+\frac{1}{8} = \frac{3}{8}$	$\frac{1}{8}+\frac{1}{8} = 1/4$		

Разпределение: Z

Z	0	2	4	6	8
	(-1,1)	(-1,3)	(-1,5) (1,1)	(1,3)	(1,5)
	1/4	1/8	$\frac{1}{8}+\frac{1}{8} = 1/4$	1/8	1/8

Зад

$Y \backslash X$	-1	0	1	
0	$1/10$ +	$1/5$ +	0	$= 3/10$
1	$1/5$ +	$3/10$ +	$1/5$	$= 7/10$
	$3/10$ +	$5/10$ +	$2/10$	$= 1$

разпр. на X и Y

$\mathbb{E}$, Var, cov

$\rho_{x,y}$

$Z = X^2 + 2Y$

$\mathbb{E}$, Var

$\mathbb{E}[X^2 + 2Y]$

$= \mathbb{E}[X^2] + 2\mathbb{E}[$

$= Var(X) + (\mathbb{E}[X])^2 + 2$

$Var(X^2 + 2Y)$

$=$

$$\rho_{xy} = \frac{cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

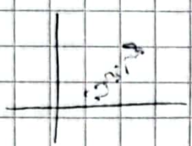
коефициент на корелация

$\sigma = \sqrt{Var(X)}$

стандартно отклонение

$-1 \le \rho_{xy} \le 1$

$\rho_{xy} = -1$ $\qquad$ $\rho_{xy} = 1$



$\rho_{xy} = 0$

→ Разпределение на Бернули

p → успех → 1

$1-p$ → неуспех → 0

$X \in \{0, 1\}$ сл. вел.

$\mathbb{P}(X=1) = p = 1 - \mathbb{P}(X=0)$

X	0	1
	$(1-p)$	p

$\mathbb{E}[X] = p$

$Var(X) = p(1-p)$

$\mathbb{E}[X^2] = p$

X^2	0^2	1^2
	$(1-p)$	p

X	-1	0	1
	$1/4$	$1/2$	$1/4$

X^2	0	1
	$1/2$	$1/4 + 1/4 = 1/2$

$\mathbb{P}(X^2 = 1) = \mathbb{P}(\{X=1\} \cup \{X=-1\})$ $\qquad \{f(X) = x\} = \{X \in f^{-1}(\{x\})\}$

$= \mathbb{P}(X=1) + \mathbb{P}(X=-1)$

$= 1/2$

→ Биномно разпределение

$X \in \{0, 1, \dots, n\}$

$n \in \mathbb{N},\ p \in [0,1]$

$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$X_1, \dots, X_n \overset{iid}{\sim} Ber(p)$

$\mathbb{P}(X_1 + \dots + X_n = k) = \mathbb{P}(X=k)$

$$X \overset{d}{=} \sum_{i=1}^{n} X_i$$

$X_1 \overset{d}{=} X_2$

$\mathbb{P}(X_1 = x) = \mathbb{P}(X_2 = x)$

$\mathbb{E}[f(X_1)] = \mathbb{E}[f(X_2)]$

$\sum_x f(x)\mathbb{P}(X_1 = x) = \sum_x f(x)\mathbb{P}(X_2 = x)$

$\mathbb{E}[X] = \sum_{k=0}^{n} k\binom{n}{k} p^k (1-p)^{n-k}$

$= \sum_{k=1}^{n} k\binom{n}{k} p^k (1-p)^{n-k}$ (*)

$k\binom{n}{k} = \cancel{k}\frac{n\,(n-1)!}{\cancel{k}(k-1)!\,(n-k)!}$

$= n\binom{n-1}{k-1}$ → 2 пъти → $\mathbb{D}(X)$

(*) $n\sum_{k=1}^{n}\binom{n-1}{k-1}p^k(1-p)^{n-k}$

$= np\sum_{j=0}^{n-1}\binom{n-1}{j}p^{j}(1-p)^{(n-1)-j}$

$= np\,(p + (1-p))^{n-1}$

$= np$

$\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n} X_i\right]$

$= \sum_{i=1}^{n}\mathbb{E}[X_i]$

$= np$

41 15/12/22

.11 **Зад. 77**

а) I X_i i-то хвърляне $i = \overline{1, n}$ $X_1, \dots, X_n$ са i.i.d - еднакво разпр.

$$Z = \sum_{i=2}^{n-1} X_i$$

$$X = Z + X_1$$

$$Y = Z + X_n$$

$$\mathbb{E}[X_i] = \frac{1}{6}\sum_{i=1}^{6} i$$

$$\operatorname{Cov}(X, Y) = \operatorname{Cov}(Z + X_1, Z + X_n)$$
$$= \underbrace{\operatorname{Cov}(Z, Z)}_{Var(Z) = (n-2)Var(X_1)} + \underbrace{\operatorname{Cov}(Z, X_1)}_{\sum_{i=2}^{n-1} Cov(X_1, X_i) = 0} + \underbrace{\operatorname{Cov}(Z, X_n)}_{0} + \underbrace{\operatorname{Cov}(X_1, X_n)}_{0}$$

$$Var(X) = Cov(X, X) = (n-1)Var(X_1) = Var(Y)$$

$$\rightarrow \rho_{XY} = \frac{n-2}{n-1}$$

II $\mathbb{E}(X_1)$ $N = (N_1, \dots, N_6)$

$X_1 \mid N = ()$ $\mathbb{P}(N = (i_1, \dots, i_6)) = 1/6^6$ $i_1, \dots, i_6 \in \{1, \dots, 6\}$

разгъвая на зара $\mathbb{P}(N_1 = i_1, \dots, N_6 = i_6)$

спалване: без знание стои по вероятност

$$\mathbb{E}[X_1] = \mathbb{E}[\mathbb{E}[X_1 \mid N]]$$
$$= \mathbb{E}\left[\frac{1}{6}\sum_{i=1}^{6} N_i\right]$$
$$= 1/6 \sum_{i=1}^{6} \mathbb{E}[N_i]$$
$$= \mathbb{E}[N_1] = 3.5$$

N_1	1	2	...	6
	1/6	1/6	...	1/6

$$\mathbb{P}(X_1 = k) = 1/6$$

$k = 1, \dots, 6$

б I не са независими $X_1, \dots, X_n$ и не бяха

условно знаем разгъвката

$$I)\ \mathbb{E}[X_1 X_2] = \mathbb{E}[\mathbb{E}[X_1 X_2 \mid N]]$$

независими когато знаем разгъвката

$$= \mathbb{E}\left[\sum_{k=1}^{6}\sum_{l=1}^{6} N_k N_l \frac{1}{36}\right]$$
$$= 1/36 \sum_{k=1}^{6}\sum_{l=1}^{6} \mathbb{E}[N_k N_l]$$
$$= \underbrace{1/36 \sum_{k=1}^{6} \mathbb{E}[N_k^2]}_{1/6\, \mathbb{E}[N_1^2]} + 1/36 \sum_{k \neq l} \mathbb{E}[N_k]\mathbb{E}[N_l]$$
$$= 1/6\, \mathbb{E}[N_1^2] + \frac{30}{36}(\mathbb{E}[N_1])^2$$

$Cov(X_1 X_2) = \mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\mathbb{E}[X_2] = 0,486 > 0$
не са независими

зад 78 Snacks с n различни играчки
колко средно трябва да купим за да съберем всички

$X_i \sim Geo\left(\frac{n-i+1}{n}\right)$ брой успех до неуспех

$X_1 \sim Geo(1)$, p

$\mathbb{P}(X_1 = 0) = 1$
брой до първа уникална

Успех = уникална играчка
брой неуникални до уникална

$X = \underbrace{X_1 + X_2 + \dots + X_n}_{\text{неуспехи}} + n$ брой пакети

$\mathbb{E}[X] = \sum_{i=1}^{n} [X_i] + n = n \sum_{k=1}^{n} 1/k$

зад. 58 Берuни $\mathbb{P}(r^{ти}$ успех на $(k+r)^{тия}$ опит$) = ?$

У, p — НУ, $1-p$

$X \sim NB(r, p)$, последното е фиксирано $k+r$

$\mathbb{P}(X = k) = \binom{r+k-1}{k} (1-p)^k p^r$
k = неуспехи

ляв - $X_1 \sim NB(n+1, 1/2)$ - пръв празен

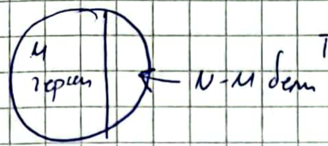
десен - $X_2 \sim NB(n+1, 1/2)$

$Y = 1$ празно е ..., 0 празно е ...

$\mathbb{P}(\{X_1 = n-k\} \cup \{X_2 = n-k\})$ $\mathbb{P}(Y=1)$

$\mathbb{P}(X = n-k) = \mathbb{P}(X = n-k \mid Y = 1) + \mathbb{P}(X = n-k \mid Y = 0)\,\mathbb{P}(Y=$

Hypergeometric



Теглиме n топки с връщане

$X \sim Bin\left(n, \frac{M}{N}\right)$

$\mathbb{P}(чч) = \frac{M}{N} \cdot \frac{M}{N}$

→ без връщане

$\mathbb{P}(ч)\,\mathbb{P}(ч|ч) = \mathbb{P}(чч) = \frac{M}{N} \cdot \frac{M-1}{N-1}$

$X \sim HG(N, M, n)$, $M<N$, $n<N$, $n<N-M$

$\mathbb{P}(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$, $k = 0, 1, \dots, \min(M, n)$

$\binom{n}{k} \frac{M}{N} \cdot \frac{M-1}{N-1} \cdot \ldots \cdot \frac{M-k+1}{N-k+1} \cdot \frac{N-M}{N-k} \cdot \ldots \cdot \frac{N-M-(n-k)+1}{N-n+1}$

$$\sum_{k=0}^{\min(M,n)} \binom{M}{k}\binom{N-M}{n-k} = \binom{N}{n}$$

$$\mathbb{E}[X] = \sum_{k=0}^{\min(M,n)} k \cdot \mathbb{P}(X=k)$$

$$= \sum_{k=1}^{\min(M,n)} k \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

$$= n \cdot \frac{M}{N} \sum_{k=1}^{\min(M,n)} \frac{\binom{M-1}{k-1}\binom{N-M}{n-k}}{\binom{N-1}{n-1}} = 1$$

$k\binom{M}{k} = M\binom{M-1}{k-1}$

$\binom{N}{n} = \frac{N}{n}\binom{N-1}{n-1}$

1.65 $X \sim HG(7, 3, 4)$

Поасон

$X \sim P_o(\lambda),\ \lambda > 0$

$P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$ · брой на събития в даден интервал от време

$k = 0, 1, 2, \dots$

$n \to \infty$ при Bi когато взимаме повече опити вероятността е все по-малка

$p(n) \to \lambda$

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot \frac{e^{-\lambda}\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!}$$

$$= e^{-\lambda} \sum_{j=0}^{\infty} \frac{\lambda^{j+1}}{j!}$$

$$= \lambda e^{-\lambda} \underbrace{\sum_{j=0}^{\infty} \frac{\lambda^j}{j!}}_{e^{\lambda}}$$

$$= \lambda$$

$X_1 \sim P_o(\lambda_1)$ $\quad X_1 \perp\!\!\!\perp X_2;\ \lambda_1, \lambda_2 \geq 0$

$X_2 \sim P_o(\lambda_2)$

$$\mathbb{P}(X_1 + X_2 = k) = \sum_{i=0}^{k} \mathbb{P}(X_1 + X_2 = k,\ X_1 = i)$$

$$= \sum_{i=0}^{k} \mathbb{P}(X_1 = i)\,\mathbb{P}(X_2 = k - i)$$

$$= \sum_{i=0}^{k} \frac{e^{-\lambda_1}\lambda_1^i}{i!} \cdot \frac{e^{-\lambda_2}\lambda_2^{k-i}}{(k-i)!} \cdot \frac{k!}{k!}$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!} \underbrace{\sum_{i=0}^{k} \binom{k}{i} \lambda_1^i \lambda_2^{k-i}}_{(\lambda_1+\lambda_2)^k}$$ биномна ф-ла

$$X_1 + X_2 \sim P_o(\lambda_1 + \lambda_2)$$

Зад. 66 $\quad X \sim P_o(\lambda) \qquad X_1 \perp\!\!\!\perp X_2 \perp\!\!\!\perp X_3$

$\mathbb{E}(X) = 2 = \lambda \qquad X_1 \sim P_o(2) \qquad Y = X_1 + X_2 + X_3 \sim P_o(6)$

$X \sim P_o(2) \qquad X_2 \sim P_o(2) \qquad \mathbb{P}(Y < 4) \overset{?}{=}$

$X_3 \sim P_o(2)$

Полиномно разпределение

всеки опит има r различни несъвместими изхода

събития: $\{1, \ldots, r\}$
$\uparrow$ k_1 пъти, $\uparrow$ k_r пъти

перм. с повторение $= \dfrac{n!}{k_1! \cdot \ldots \cdot k_r!}$

$$k_1 + \ldots + k_r = n$$

$$\mathbb{P}\{ = \underbrace{\frac{n!}{k_1! \cdot \ldots \cdot k_r!}}_{\text{подредици}} \underbrace{p_1^{k_1} p_2^{k_2} \cdot \ldots \cdot p_r^{k_r}}_{\text{една редица}}$$

$$X = (X_1, \ldots, X_r) \sim Multi(n; p_1, \ldots, p_r)$$

$X \to \mathbb{N}_0$, $X_1 \to \mathbb{N}_0$, $X_r \to \mathbb{N}_0$

$$\mathbb{P}(X = (k_1, \ldots, k_r)) = \frac{n!}{k_1! \cdot \ldots \cdot k_r!} p_1^{k_1} \cdot \ldots \cdot p_r^{k_r}$$

$$k_1, \ldots, k_r \in \mathbb{N}_0 : k_1 + \ldots + k_r = n$$

$r = 2$ $\quad \mathbb{P}(X = (k, n-k)) = \dfrac{n!}{k!(n-k)!} p_1^{k} (1-p_1)^{n-k}$

биномно $\quad \mathbb{P}(Y = k)$

$$X \sim Multi(n, p_1, (1-p_1))$$

$$Y \sim Bin(n, p_1)$$

$$\circledcirc \sum_{\substack{k_1, \ldots, k_n: \\ k_1 + k_2 + \ldots + k_r = n}} \frac{n!}{k_1! \cdot \ldots \cdot k_r!} p_1^{k_1} \cdot \ldots \cdot p_r^{k_r} = (p_1 + \ldots + p_r)^n$$

$$= 1^n$$

$$= 1$$

① имаме 9 топки и 3 клетки 1) $\mathbb{P}($във всяка клетка по 3 топки$) = ?$

$$X \sim Multi(9; 1/3, 1/3, 1/3)$$

$$\mathbb{P}(X = (3, 3, 3))$$

2) $\mathbb{P}($в ~~клетка~~ една клетка 4 топки, в друга 3, в останалата 2$) = ?$

$$\mathbb{P}(X = (4,3,2), (4,2,3), (3,2,4), (3,4,2), (2,3,4), (2,4,3))$$

$$= \underset{\text{колко разл. номинации}}{3!} \cdot \frac{9!}{4!3!2!} \left(\frac{1}{3}\right)^9$$

(1) Урна има бяла, червена и зелена топка

$n = 5$ с връщане

2 цвята => биномно
3+ цвята => multi;

$\mathbb{P}$(поне 2 бели и 2 червени)

$X \sim Multi(5; \overset{б}{\tfrac{1}{3}}, \overset{ч}{\tfrac{1}{3}}; \overset{з}{\tfrac{1}{3}})$

$\mathbb{P}(X = (2,1,2), (3,0,2), (2,0,3))$ - несъвместими