# Непрекъснати случайни величини

**85.** $f_X(x) = \begin{cases} c(x^2+2x), & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$

а) константата $c$

$$\int_{-\infty}^{\infty} f_X(x) = 1 = \int_0^1 c(x^2+2x)\,dx = c\left[\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^1 =$$

$$= c\left[\frac{1}{3} + 2\cdot\frac{1}{2}\right] = c\left(\frac{1}{3}+1\right) = c\left(\frac{4}{3}\right)$$

$$\Rightarrow c = \frac{3}{4}$$

б) $EX$ и $DX$

$$EX = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\cdot c\cdot(x^2+2x)\,dx =$$

$$= c\left[\frac{x^4}{4} + 2\frac{x^3}{3}\right]_0^1 = c\left(\frac{1}{4}+\frac{2}{3}\right) = c\cdot\frac{11}{12} = \frac{3}{4}\cdot\frac{11}{12} = \frac{11}{16}$$

$$DX = EX^2 - (EX)^2$$

$$EX^2 = \int_0^1 x^2\cdot f_X(x)\,dx = \int_0^1 x^2\cdot c(x^2+2x)\,dx =$$

$$= c\left[\frac{x^5}{5} + 2\frac{x^4}{4}\right]_0^1 = c\left(\frac{1}{5}+\frac{2}{4}\right) = c\left(\frac{1}{5}+\frac{1}{2}\right) = c\left(\frac{2+5}{10}\right) =$$

$$= \frac{3}{4}\cdot\frac{7}{10} = \frac{21}{40}$$

$$DX = \frac{21}{40} - \left(\frac{11}{16}\right)^2$$

3) $\mathbb{P}(X < \mathbb{E}X)$ т.е. $\mathbb{P}(X < 11/16)$

$$\mathbb{P}(X < 11/16) = \int_{\cancel{-\infty}\ 0}^{11/16} f_X(x)\,dx =$$

$$\mathbb{P}(X < \tfrac{11}{16}) = F_X(\tfrac{11}{16})$$

$F_X(y) = ?$

$$F_X(y) = \begin{cases} 0, & y<0 \\ c(\frac{y^3}{3}+y^2), & y \in [0,1] \\ 1, & y>1 \end{cases}$$

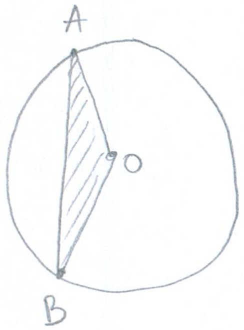
$$\mathbb{P}(X<y) = \int_0^y c(x^2+2x)\,dx = c\left[\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^y$$

$$\rightarrow c\left[\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^{11/16} \approx 0,455$$

4) $\mathbb{E}(X^2+3X) = ?$

$$\mathbb{E}(X^2+3X) = \mathbb{E}X^2 + 3\mathbb{E}X = \frac{21}{40} + 3\cdot\frac{11}{16} = \frac{21}{40} + \frac{33}{16}$$

**86.**



$\mathbb{E}S_{AOB} = ?$ → очакване на лицето на $\triangle AOB$

$\varphi \sim U(0,\pi)$

$$S_{AOB} = \frac{\sin\varphi}{2}$$

$$\mathbb{E}S_{AOB} = \mathbb{E}\frac{\sin\varphi}{2} = \int_0^{\pi} \frac{\sin x}{2}\cdot\frac{1}{\pi}\,dx = \frac{1}{2\pi}\left[-\cos x\right]_0^{\pi} = \frac{1}{\pi}$$

**87.** $X \sim U(0,7)$ – X – безотказно работи в години на даден апарат

В случай на дефект ще бъде заменен на същата година или преди това

Y – времето до смяна на апарата

$\mathbb{P}(Y<4)$, $\mathbb{E}Y$ и $DY$ = ?

↳ Считаме, че след дефект апаратът веднага бива заменен

$$Y(\omega) = \begin{cases} X(\omega), & X(\omega) < 5 \\ 5, & X(\omega) \ge 5 \end{cases}, \text{ т.е. } Y = X \cdot 1_{\{X<5\}} + 5 \cdot 1_{\{X \ge 5\}}$$

$\rightarrow \mathbb{P}(Y<4) = \mathbb{P}(X<4) = F_X(4) = \frac{4}{7}$

$* \; F_Y(x) = \mathbb{P}(U(0,7) < x) = \int_0^x \frac{1}{7}\,dy = \frac{x}{7}$.

↳ $\mathbb{E}Y = \mathbb{E}(X \cdot 1_{\{X<5\}} + 5 \cdot 1_{\{X \ge 5\}}) =$

$= \mathbb{E}(X \cdot 1_{\{X<5\}}) + \mathbb{E}(5 \cdot 1_{\{X \ge 5\}}) =$

$= \int_0^5 x \cdot f_X(x)\,dx + 5 \cdot \int_5^7 f_X(x)\,dx =$ $\frac{1}{7}$

$= \left[\frac{x^2}{2} \cdot \frac{1}{7}\right]_0^5 + 5 \cdot \left[\frac{2}{7}\right]_5^7 = \frac{25}{14} + 5 \cdot \frac{2}{7} = \frac{25}{14} + \frac{10}{7} = \frac{45}{14}$

$\boxed{\mathbb{E}1_A = \mathbb{P}(A)}$

$DY = \mathbb{E}Y^2 - (\mathbb{E}Y)^2$

$\mathbb{E}Y^2 = \int_0^5 x^2 \cdot \frac{1}{7}\,dx + 5^2 \int_5^7 \frac{2}{7}\,dx = \left[\frac{x^3}{3} \cdot \frac{1}{7}\right]_0^5 + 25\left[\frac{2}{7}\right]_5^7 =$

$= \frac{125}{21} + \frac{50}{7} = \frac{125}{21} + \frac{150}{21} = \frac{275}{21}$

$DY = \frac{271}{21} - \left(\frac{45}{14}\right)^2$

↳ $n = 1000$ апарата

$X$ = „да се промени преди петата година" $= \mathbb{P}(Y<5)$

$X \sim Ber\left(\frac{5}{7}\right)$

→ средно ще трябва да се променят $Bin\left(1000, \frac{5}{7}\right)$,

$1000 \cdot \frac{5}{7} = 714{,}28$

89.

$X_1 \sim Exp(\lambda_1)$  $X_2 \sim Exp(\lambda_2)$

$\mathbb{E}X_1 = 8 \min \Rightarrow \mathbb{E}X = \frac{1}{\lambda} \Rightarrow 8 = \frac{1}{\lambda} \rightarrow \lambda = \frac{1}{8}$

$\mathbb{E}X_2 = 5 \min \Rightarrow \lambda_2 = \frac{1}{5}$

↳ клиент избира по случаен начин опашка и чака < 4 минути
↳ A

I – избира $X_1$ – първата опашка
II – избира $X_2$ – втората опашка

$\mathbb{P}(I \mid A) = ?$

$$\mathbb{P}(I \mid A) = \frac{\mathbb{P}(I \cap A)}{\mathbb{P}(A)}$$

$$\mathbb{P}(I \mid A) = \frac{\mathbb{P}(A \mid I)\,\mathbb{P}(I)}{\mathbb{P}(A \mid I)\,\mathbb{P}(I) + \mathbb{P}(A \mid II)\,\mathbb{P}(II)} \rightarrow \mathbb{P}(A)$$

$$\mathbb{P}(A) = \mathbb{P}(X_1 < 4)\cdot\frac{1}{2} + \mathbb{P}(X_2 < 4)\cdot\frac{1}{2} =$$
$$= \left(1 - e^{-\lambda_1 \cdot 4} + 1 - e^{-\lambda_2 \cdot 4}\right)\cdot\frac{1}{2} =$$
$$= \left(1 - e^{-\frac{4}{8}\frac{1}{2}} + 1 - e^{-\frac{4}{5}}\right)\cdot\frac{1}{2} = \left(2 - e^{-\frac{1}{2}} - e^{\frac{4}{5}}\right)\cdot\frac{1}{2}$$

$$\mathbb{P}(A \mid I)\,\mathbb{P}(I) = \mathbb{P}(X_1 < 4)\cdot\frac{1}{2} = \left(1 - e^{-\lambda_1 \cdot 4}\right)\cdot\frac{1}{2} = \left(1 - e^{-\frac{1}{2}}\right)\cdot\frac{1}{2}$$

$$\mathbb{P}(I \mid A) = \frac{\left(1 - e^{-\frac{1}{2}}\right)\cdot\cancel{\frac{1}{2}}}{\left(2 - e^{-\frac{1}{2}} - e^{-\frac{4}{5}}\right)\cdot\cancel{\frac{1}{2}}} = \frac{1 - e^{-\frac{1}{2}}}{2 - e^{-\frac{1}{2}} - e^{-\frac{4}{5}}}$$

90. X – „Време на преглед на пациент"

$EX = 30 min \to \lambda = \frac{1}{30}$ или $EX = 30 min = \frac{1}{2}$

$\lambda = 2$

първият за преглед в 11:00

вторият в 11:30

↳ средно колко време ще прекара вторият в клиниката – ?


Y = „времето, което вторият пациент прекарва в клиниката"

$EY = ?$

$$Y = X_2 \cdot 1\{X_1 \le \tfrac{1}{2}\} + \left(X_2 + \left(X_1 - \tfrac{1}{2}\right)\right) \cdot 1\{X_1 > \tfrac{1}{2}\}$$

ако първия е завършил на време

в случай да така, прекараното време е него/вото $X_2$ + разликата между прегледа на първия – половин час

↳ $Y = X_2 + \left(X_1 - \frac{1}{2}\right) \cdot 1\{X_1 > \frac{1}{2}\}$

$$EY = EX_2 + E\left(X_1 - \tfrac{1}{2}\right) \cdot 1_{\{X_1 > \frac{1}{2}\}} =$$

$$= \frac{1}{2} + \int_{\frac{1}{2}}^{\infty} \left(x - \tfrac{1}{2}\right) \cdot 2e^{-2x}\,dx =$$ → може и до тук

$$= \frac{1}{2} - \int_{\frac{1}{2}}^{\infty} \left(x - \tfrac{1}{2}\right) de^{-2x} = \frac{1}{2} - \left[\left(x - \tfrac{1}{2}\right) e^{-2x}\right]_{\frac{1}{2}}^{\infty} + \int_{\frac{1}{2}}^{\infty} e^{-2x}\,dx =$$

$$= \frac{1}{2} - \frac{1}{2} e^{-2x} \Big|_{\frac{1}{2}}^{\infty} = \frac{1}{2}\left(1 + e^{-1}\right)$$

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* Центриране е да ги направим със средно 0

$P(\text{ези}) = \overset{p}{\frac{3}{4}}$, хвърля се $\overset{n}{2000}$ пъти

$P(\#$ падналите се езита да е между 1475 и 1535$) = ?$

→ нормално разпределение

$X_i = \begin{cases} 1, & \text{ези на } i\text{-то} \\ 0, & \text{иначе} \end{cases}$ $\quad X_i \sim Ber(p)$

$P(1475 \le X_1 + X_2 + \dots + X_{2000} \le 1535) = ?$

↳ прилагаме ЦГТ: $\quad$ D на Бернули $= p\cdot(1-p)$

$\frac{3}{4}\cdot\left(1-\frac{3}{4}\right) = \frac{3}{4}\cdot\frac{1}{4} = \frac{3}{16}$

$\sigma^2 = \frac{3}{16} \Rightarrow \sigma = \sqrt{\frac{3}{16}}$

$P(1475 \le X_1 + X_2 + \dots + X_n \le 1535) =$

$= P\left(\frac{1475 - \mu n}{\sigma\sqrt{n}} \le \frac{X_1 + \dots + X_n - \mu n}{\sigma\sqrt{n}} \le \frac{1535 - \mu n}{\sigma\sqrt{n}}\right) =$

$= P\left(\frac{1475 - \frac{3}{4}\cdot 2000}{\frac{\sqrt{3}}{4}\cdot\sqrt{2000}} \le \frac{X_1 + \dots + X_n - \mu n}{\sigma\sqrt{n}} \le \frac{1535 - \frac{3}{4}\cdot 2000}{\frac{\sqrt{3}}{4}\cdot\sqrt{2000}}\right) =$

$= P\left(\frac{-25}{19,3649} \le \frac{X_1 + \dots + X_n - \mu n}{\sigma\sqrt{n}} \le \frac{35}{19,3649}\right) =$

$n > 30,\ \sigma^2 < \infty$

$\approx P(-1,29 \le N(0,1) \le 1,83) = \int_{-1,29}^{1,83} \frac{1}{2\pi} e^{-\frac{x^2}{2}} dx$

• $\int e^{-x^2/2} dx$ → няма явен вид

$\Rightarrow P(N(0,1) \le x) = F_{N(0,1)}(x) = \Phi(x)$

$P(-1,29 \le N(0,1) \le 1,83) = \Phi(1,83) - \Phi(-1,29) =$

$= 0,9664 - (0,0985) = \underline{\underline{0,8679}}$

Контролно 2 СЕМ 21.01.2023

**1.** $f_{X,Y}(x,y) = \begin{cases} cx^2+1 & , \; x,y \ge 0 \, , \; x+2y \le 1 \\ 0 & \text{иначе} \end{cases}$

а) c, плътността на X и очакването на Y

$$\begin{cases} x,y \ge 0 \\ x \le 1-2y \iff y \le \frac{1-x}{2} \end{cases}$$

$$1 = \int_0^1 \int_0^{\frac{1-x}{2}} cx^2+1 \, dy \, dx = \int_0^1 cx^2+1 \, [y]_0^{\frac{1-x}{2}} \, dx =$$

$$= \int_0^1 (cx^2+1) \left( \frac{1-x}{2} \right) dx = \frac{1}{2} \int_0^1 (cx^2+1)(1-x) \, dx =$$

$$= \frac{1}{2} \int_0^1 cx^2 - cx^3 + 1 - x \, dx = \frac{1}{2} \left[ \frac{cx^3}{3} - \frac{cx^4}{4} + x - \frac{x^2}{2} \right]_0^1 =$$

$$= \frac{1}{2} \left[ \frac{c}{3} - \frac{c}{4} + 1 - \frac{1}{2} \right] = \frac{1}{2} \left[ \frac{4c}{12} - \frac{3c}{12} + \frac{2-1}{2} \right] =$$

$$= \frac{1}{2} \left[ \frac{c}{12} + \frac{1}{2} \right] = \frac{c}{24} + \frac{1}{4} = \frac{c}{24} + \frac{6}{24} = \frac{6c}{24}$$

$$\Rightarrow 6c = 24$$
$$c = 24 - 6$$
$$c = 18$$

→ ПЛЪТНОСТТА НА $X$

$$f_X(x) = \int_0^{\frac{1-x}{2}} cx^2+1\, dy = (cx^2+1)\left(\frac{1-x}{2}\right)$$

→ очакването на $Y$

$$EY = \int_0^1 \int_0^{\frac{1-x}{2}} y(cx^2+1)\, dy\, dx = \int_0^1 (cx^2+1)\left(\frac{1-x}{2}\right)^2 \cdot \frac{1}{2}\, dx =$$

$$= \int_0^1 (cx^2+1)\left(\frac{1-2x+x^2}{4}\right) \cdot \frac{1}{2}\, dx = \frac{1}{8}\int_0^1 (cx^2+1)(1-2x+x^2)\, dx =$$

$$= \frac{1}{8}\int_0^1 cx^2-2cx^3+cx^4+1-2x+x^2\, dx =$$

$$= \frac{1}{8}\left[\frac{cx^3}{3} - \frac{2cx^4}{4} + \frac{cx^5}{5} + x - \frac{2x^2}{2} + \frac{x^3}{3}\right]_0^1 =$$

$$= \frac{1}{8}\left[\frac{18}{3} - \frac{36}{4} + \frac{18}{5} + \cancel{1} - \frac{\cancel{2}}{2} + \frac{1}{3}\right] =$$

$$= \frac{1}{8}\left[\frac{19}{3} - \frac{36}{4} + \frac{18}{5}\right] = \frac{1}{8}\left[\frac{380-540+216}{60}\right] =$$

$$= \frac{1}{\cancel{8}}\left[\frac{\cancel{56}}{60}\right] = \frac{7}{60}$$

$cx^2+1$

б) $\mathbb{E}[Y \mid X = 1/2]$

$$f_{X|Y}(y \mid 1/2) = \frac{f_{X,Y}(\frac{1}{2}, y)}{f_X(\frac{1}{2})} = \frac{\cancel{c(\frac{1}{2})^2+1}}{\cancel{(c(\frac{1}{2})^2+1)}(\frac{1-\frac{1}{2}}{2})} =$$

$$= \frac{1}{\frac{\frac{1}{2}}{2}} = \frac{1}{\frac{1}{4}} = 4$$

$\Rightarrow$ за $y \in (0, \frac{1}{4}) \Rightarrow \mathbb{E}[Y \mid X = \frac{1}{2}] = \mathbb{E}[U(0, \frac{1}{4})] =$

$\Rightarrow \mathbb{E}\, U(a,b) = \frac{b-a}{2} \Rightarrow \frac{\frac{1}{4}-0}{2} = \frac{\frac{1}{4}}{2} = \frac{1}{8}$

в) плътността на $Z = X + 2Y$

$$\begin{vmatrix} Z = X + 2Y \\ W = X \end{vmatrix} \Rightarrow \begin{vmatrix} X = W \\ Y = \frac{Z-W}{2} \end{vmatrix}$$

$$|J| = \begin{vmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\Rightarrow f_{Z,W}(z,w) = f_{X,Y}(w, \tfrac{z-w}{2}) \cdot \frac{1}{2} = (cw^2+1) \cdot \frac{1}{2}$$

за $w, \frac{z-w}{2} \in (0,1)$ т.е. $z \in (0,1)$ и $w \in (0,z)$

$$\Rightarrow f_Z(t) = \int_0^t (cw^2+1) \cdot \frac{1}{2}\, dw = \frac{1}{2} \int_0^t cw^2+1\, dw =$$

$$= \frac{1}{2}\left[\frac{cw^3}{3} + w\right]_0^t = \frac{1}{2}\left[\frac{18t^3}{3} + t\right] = 3t^3 + \frac{t}{2}$$

за $t \in (0,1)$

2.] $f_{X,Y}(x,y) = \begin{cases} cx^3y, & x,y \geq 0,\ x+y \leq 1 \\ 0, & \text{иначе} \end{cases}$ $\quad x \leq 1-y$, $y \leq 1-x$

a) c, плътността на X и очакването на Y, cov(X,Y)

$$1 = \int_0^1 \int_0^{1-x} cx^3y \, dy\, dx = \int_0^1 cx^3 \left[\frac{(1-x)^2}{2}\right] dx =$$

$$= \frac{1}{2}\int_0^1 cx^3(1-x)^2\, dx = \frac{1}{2}\int_0^1 cx^3(1-2x+x^2)\, dx =$$

$$= \frac{1}{2}c\int_0^1 x^3-2x^4+x^5\, dx = \frac{1}{2}c\left[\frac{x^4}{4}-\frac{2x^5}{5}+\frac{x^6}{6}\right] =$$

$$= \frac{1}{2}c\left[\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right] = \frac{1}{2}c\left[\frac{30-48+20}{120}\right] =$$

$$= \frac{1}{2}c\cdot\frac{\cancel{2}}{120} = \frac{c}{120} \rightarrow \underline{c = 120}$$

→ плътността на X

$$f_X(x) = \int_0^{1-x} cx^3y\, dy = cx^3\frac{(1-x)^2}{2},\ x \in [0,1]$$

→ очакването на Y

$$f_Y(y) = \int_0^{1-y} cx^3y\, dx = c\frac{(1-y)^4}{4}y \rightarrow$$

$$EY = \int_0^1\int_0^{1-x} y\cdot cx^3y\, dy\, dx = c\int_0^1 x^3\frac{(1-x)^3}{3} = \frac{1}{3}c\int_0^1 x^3(1-x^3-3x+3x^2) =$$

$$= x^3-x^6-3x^4+3x^5$$

$$\mathbb{E}XY = \int_0^1 \int_0^x xy\,3x\,dy\,dx = \int_0^1 3x^2 \int_0^x y\,dy\,dx =$$

$$= \int_0^1 3x^2 \cdot \frac{x^2}{2}\,dx = \int_0^1 \frac{3}{2}x^4\,dx = \frac{3}{2}\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{2}\cdot\frac{1}{5} = \underline{\underline{\frac{3}{10}}}$$

$$\Rightarrow Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y =$$

$$= \frac{3}{10} - \frac{3}{4}\cdot\frac{3}{8} = \frac{3}{10} - \frac{9}{32} = \frac{96-90}{320} = \frac{6}{320} = \underline{\underline{\frac{3}{160}}}.$$

Контролно 2 ВиС 2022

$\boxed{1.}$ $f_{X,Y}(x,y) = \begin{cases} cx+1, & x,y \geq 0,\ x+y \leq 1 \\ 0, & \text{иначе} \end{cases}$ , $x \leq 1-y$, $y \leq 1-x$

а) $c$ и $Cov(X,Y) = ?$

$$1 = \int_0^1\int_0^{1-x} cx+1\,dy\,dx = \int_0^1 cx+1 \int_0^{1-x} dy\,dx = \int_0^1 (cx+1)(1-x)\,dx =$$

$$= \int_0^1 cx - cx^2 + 1 - x\,dx = \left[\frac{cx^2}{2} - \frac{cx^3}{3} + x - \frac{x^2}{2}\right]_0^1 =$$

$$= \left[\frac{c}{2} - \frac{c}{3} + 1 - \frac{1}{2}\right] = \left[\frac{3c-2c}{6} + \frac{1}{2}\right] = \frac{c}{6} + \frac{1}{2} = \frac{c+3}{6}$$

$$\Rightarrow \frac{c+3}{6} = c+3 = 6 \rightarrow c = 6-3 \rightarrow \underline{\underline{c=3}}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y$$

$$f_X(x) = \int_0^{1-x} cx+1 \, dy = (cx+1)(1-x)$$

$$\mathbb{E}X = \int_0^1 x(cx+1)(1-x)\,dx = \int_0^1 (cx^2+x)(1-x)\,dx =$$

$$= \int_0^1 cx^2 - cx^3 + x - x^2 \, dx = \left[\frac{cx^3}{3} - \frac{cx^4}{4} + \frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 =$$

$$= \left[\frac{c}{3} - \frac{c}{4} + \frac{1}{2} - \frac{1}{3}\right] = \frac{3}{3} - \frac{3}{4} + \frac{1}{2} - \frac{1}{3} = \frac{12-9+6-4}{12} = \frac{5}{12}$$

$$f_Y(y) = \int_0^{1-y} cx+1\,dx = c\left[\frac{x^2}{2} + x\right]_0^{1-y} =$$

$$= c\left[\frac{(1-y)^2}{2} + (1-y)\right] = \frac{c}{2}(1-y)^2 + (1-y)$$

$$\mathbb{E}Y = \frac{c}{2}\int_0^1 y(1-y)^2\,dy + \int_0^1 y(1-y)\,dy = \frac{c}{2}\int_0^1 y-2y^2+y^3\,dy + \int_0^1 y-y^2\,dy =$$

$$= \frac{c}{2}\left[\frac{y^2}{2} - \frac{2y^3}{3} + \frac{y^4}{4}\right]_0^1 + \left[\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1 =$$

$$= \frac{c}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] + \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{c}{2}\left[\frac{6-8+3}{12}\right] + \frac{1}{6} =$$

$$= \frac{c}{2}\cdot\frac{1}{12} + \frac{1}{6} = \frac{3}{2}\cdot\frac{1}{12} + \frac{1}{6} = \frac{1}{8} + \frac{1}{6} = \frac{6+8}{48} = \frac{14}{48} = \frac{7}{24}$$

$$E(XY) = \int_0^1 \int_0^{1-x} xy\, c(x+1)\,dy\,dx = \int_0^1 x\,(c(x+1)) \int_0^{1-x} y\,dy\,dx =$$

$$= \int_0^1 x\,(c(x+1)) \cdot \frac{(1-x)^2}{2}\,dx = \frac{1}{2}\int_0^1 (cx^2+x)\,(1-x)^2\,dx =$$

$$= \frac{1}{2}c\int_0^1 x^2(1-x)^2\,dx + \frac{1}{2}\int_0^1 x\,(1-x)^2\,dx =$$

$$= \frac{1}{2}c\int_0^1 x^2-2x^3+x^4\,dx + \frac{1}{2}\int_0^1 x-2x^2+x^3\,dx =$$

$$= \frac{1}{2}c\left[\frac{1}{3}-\frac{2}{4}\frac{1}{2}+\frac{1}{5}\right] + \frac{1}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] =$$

$$= \frac{3}{2}\left[\frac{10-15+6}{30}\right] + \frac{1}{2}\left[\frac{6-8+3}{12}\right] = \frac{3}{2}\cdot\frac{1}{30} + \frac{1}{2}\cdot\left(\frac{1}{12}\right) =$$

$$= \frac{1}{20} + \frac{1}{24} = \frac{6+5}{120} = \frac{11}{120}$$

$$\Rightarrow \mathrm{Cov}(X,Y) = \frac{11}{120} - \frac{5}{12}\cdot\frac{7}{24} = \frac{11}{120} - \frac{35}{288} = \frac{132-175}{1440} =$$

$$= -\frac{43}{1440}$$

5) $E(X \mid Y = 1/2)$

$cx+1$

$$f_{X,Y}(x \mid \tfrac{1}{2}) = \frac{f_{X,Y}(x, \frac{1}{2})}{f_Y(\frac{1}{2})} = \frac{cx+1}{\frac{c}{2}(1-y)^2 + (1-y)}$$

$$\frac{3}{2}\left(1-\frac{1}{2}\right)^2 + \left(1-\frac{1}{2}\right) = \frac{3}{2} \cdot \frac{1}{4} + \frac{1}{2} = \frac{3}{8} + \frac{4}{8} = \frac{7}{8}$$

$$\Rightarrow \quad f_{X,Y}(x \mid \tfrac{1}{2}) = \frac{cx+1}{\frac{7}{8}}, \quad \text{za} \quad x \in [0, \tfrac{1}{2}]$$

$$E[X \mid Y = \tfrac{1}{2}] = \int_0^{1/2} x \cdot \frac{cx+1}{\frac{7}{8}} \, dx =$$

$$= \frac{8}{7} \int_0^{1/2} cx^2 + x \, dx = \frac{8}{7}\left[c \cdot \frac{\frac{1}{8}}{3} + \frac{\frac{1}{4}}{2}\right] = \frac{8}{7}\left[c \cdot \frac{1}{24} + \frac{1}{8}\right] =$$

$$= \frac{8}{7}\left[\frac{3}{24} + \frac{3}{24}\right] = \frac{8}{7} \cdot \frac{6}{24} = \frac{6}{21} = \frac{2}{7}$$

**2.** $N(\mu, 10^2)$

ЦГТ

15% от консервите съдържат по-малко от 250gr грах

а) параметъра $\mu = ?$

$$\mathbb{P}(N(\mu, 10^2) < 250) = 15\%$$

$$\mathbb{P}\left(N(0,1) < \frac{250-\mu}{10^2}\right) = 15\%$$

От таблицата:

$$\frac{250-\mu}{10} \approx -1{,}04 \quad \to 15\%,\ 0{,}15$$

$$\to \frac{250-\mu}{10} = -1{,}04$$

$$250-\mu = -1{,}04 \cdot 10$$

$$250-\mu = -10{,}4$$

$$-\mu = -10{,}4-250 \quad (-1)$$

$$\mu = 10{,}4+250$$

$$\mu = 260{,}4$$

б) процентът консерви, които съдържат повече от 280gr

$$\mathbb{P}(N(\mu, 10^2) > 280) = \mathbb{P}\left(N(0,1) > \frac{280-\mu}{10}\right) \approx$$

$$1-\Phi(1{,}96) = \Phi(-1{,}96) = 0{,}0250 \approx 2{,}5\%$$

в) Намерете $\sigma$, ако знаем че 97% от консерви са съдържали между 230 и 270 гр грах

↳ $N(250, \sigma^2)$

$$\mathbb{P}(230 \le N(250, \sigma^2) \le 270) =$$

$$= 1 - 2\,\mathbb{P}(N(250, \sigma^2) < 230) =$$

$$= 1 - 2\,\mathbb{P}\left(N(0,1) < \frac{230-250}{\sigma}\right) \overset{\text{по условие}}{=} 95\%$$

↳ Търсим $\sigma$ така че $\Phi\left(\frac{-20}{\sigma}\right) = 0{,}015$

↳ $-\frac{20}{\sigma} \approx -2{,}17 \Rightarrow \sigma = 9{,}2166$

$\boxed{3}$ Контролно е съставено от 4 задачи. Вероятността да се реши всяка от тях е 80%, 70%, 60% и 40%.

↳ решена задача носи 10 точки и 0 иначе
оценката е по формула 2+ точки/10

↳ Ако явилите се са 100 души, пресметнете

а) Вероятността средният резултат на задача 1 да бъде под 8 точки.

↳ Нека $X_i$ е резултатът на задача 1 $\sim Ber\left(\frac{8}{10}\right)$ → 80%

$\mathbb{E}X_i = 10 \cdot \frac{8}{10} = 8$ и $DX = 100 \cdot \frac{8}{100} \cdot \frac{2}{10} = 16$

↳ Нека $S_n = X_1 + \dots + X_n$. От ЦГТ $\frac{S_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1)$

$$\mathbb{P}\left(\frac{S_{100}}{100} < 8\right) = \mathbb{P}(S_{100} < 800) = \mathbb{P}\left(\frac{S_{100} - 100 \cdot 8}{4\sqrt{100}} < \frac{8}{4\sqrt{100}}\right)$$

$\frac{8}{4 \cdot 10} = \frac{8}{40} = 0{,}2$

↳ $\Phi(0{,}2) = 0{,}5793 \approx 58\%$

б) очакваната средна оценка

Нека $Y_i$ = # точки на студент $i \sim X_i^{(1)} + X_i^{(2)} + X_i^{(3)} + X_i^{(4)}$,

където $X_i^{(1)} \sim 10 \cdot Ber\left(\frac{8}{10}\right)$, $X_i^{(2)} \sim 10 \cdot Ber\left(\frac{7}{10}\right)$, $X_i^{(3)} \sim 10 \cdot Ber\left(\frac{6}{10}\right)$

и $X_i^{(4)} \sim 10 \cdot Ber\left(\frac{4}{10}\right)$ са нез. сл. вел.

$$\mathbb{E} Y_i = 10\left(\frac{8}{10} + \frac{7}{10} + \frac{6}{10} + \frac{4}{10}\right) = 25$$

$$D Y_i = 100\left(\frac{8}{10} \cdot \frac{2}{10} + \frac{7}{10} \cdot \frac{3}{10} + \frac{6}{10} \cdot \frac{4}{10} + \frac{4}{10} \cdot \frac{6}{10}\right) = 100 \cdot \frac{16+21+24+24}{100} =$$

$$= 85$$

$\Rightarrow$ Оценката на $i$ е $Z_i := 2 \cdot \frac{Y_i}{4}$

$$\Rightarrow \mathbb{E} Z_i = \mathbb{E}\left[2 + \frac{Y_i}{10}\right] = 4,5$$

$$D Z_i = D\left[2 + \frac{Y_i}{10}\right] = \frac{DY_i}{100} = \frac{85}{100} = \frac{17}{20}$$

в) Вероятността средната оценка да е поне 1% по-висока от очакваната =?

$$\mathbb{P}\left(\frac{Z_1 + \dots + Z_{100}}{100} - 4,5 > \frac{1}{100} \cdot 4,5\right) =$$

$$\mathbb{P}\left(\frac{Z_1 + \dots + Z_{100} - 4,5 \cdot 100}{\sqrt{100 \cdot \frac{17}{20}}} > \frac{\frac{4,5}{100}}{\sqrt{100 \cdot \frac{17}{20}}}\right) \to \frac{0,045}{2,062} \sim 0,0218$$

Контролно 2 КН 2024

$\boxed{1.}$ $\mathbb{E}[X_i] = 5011$

$DX_i = 4000$

$n = ?$ $\mathbb{P}(X_1 + \dots + X_n > 10^6)$

$S_n = \sum_{i=1}^{n} X_i$ $\mathbb{P}\left( \frac{S_n - \mathbb{E}[S_n]}{\sqrt{D(S_n)}} > \frac{10^6 - 5011 \cdot n}{\sqrt{4000 \cdot n}} \right)$

$\overset{ЦГТ}{\approx} \mathbb{P}\left( N(0,1) > \frac{10^6 - 5011n}{\sqrt{4000n}} \right) = 99\%$

$\Rightarrow \frac{10^6 - 5011}{\sqrt{4000n}} < -2{,}33 \quad \to \underline{\underline{n \geq 200}}$

$\Rightarrow \mathbb{E}[S_n] = n \cdot 5011,$

$DS_n = n \cdot \sigma^2 = n \cdot 4000$

и искаме да намерим такова $n$, че

$\mathbb{P}(S_n \geq 10^6) \geq 99\% \ (0{,}99)$

$\Rightarrow \mathbb{P}\left| \frac{S_n - n\mu}{\sigma\sqrt{n}} \geq \frac{10^6 - n\mu}{\sigma\sqrt{n}} \right) \gtrsim 0{,}99$

$\downarrow$

$\mathbb{P}\left| N(0,1) \geq \frac{10^6 - n\mu}{\sigma\sqrt{n}} \right) \geq 0{,}99$

$\frac{10^6 - 5011n}{\sqrt{4000n}} \leq -2{,}33$

$$\frac{10^6 - 5011n}{\sqrt{4000n}} \le -2,33$$

$$10^6 - 5011n \le -2,33 \cdot \sqrt{4000n}$$

$$10^6 - 5011n \le -2,33 \cdot 63,25\sqrt{n}$$

$$10^6 - 5011n \le -147,36\sqrt{n}$$

$10^6 - 5011n + 147,36\sqrt{n} = 0$ , заместваме $\sqrt{n}$ с $x$

$$10^6 - 5011x^2 + 147,36x = 0$$

↳ решаваме квадратното уравнение

$$-5011x^2 + 147,36x + 10^6 = 0$$

$a = -5011$ , $b = 147,36$ , $c = 10^6$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

$$\sqrt{21714,97 + 20044 \cdot 10^6}$$

$$\sqrt{20044021714,97}$$

$$x = \frac{-147,36 \pm \sqrt{20044021714,97}}{-10022} \quad \sqrt{\ldots} \approx 141576,5109$$

$$x = \frac{-147,36 \pm 141576}{-10022}$$

$$x = \frac{-147,36 - 141576}{-10022} = \frac{-141723,36}{-10022} \approx 14,14$$

↳ връщаме се към $n = x^2$

$$n = (14,14)^2 \approx 199,9396 \approx 200$$

$$n \ge 200$$

б) $n = ?$ $\quad \mathbb{P}(\max X_i > 10^4) \geq 95\%$

$\leftrightarrow$ търсим такова $x$, че

$$\mathbb{P}(\max(X_1, \dots, X_n) \geq 10^4) \geq 99\%$$

$$\downarrow$$

$$\frac{10^4 - M}{\sqrt{4000}} = \frac{10^4 - 5011}{\sqrt{4000}} = \frac{4989}{63{,}25} \approx \underline{78{,}8}$$

→ по теорема

$\hookrightarrow \mathbb{P}(\max(X_1, X_2, \dots, X_n) < x) = (\mathbb{P}(X \leq x))^n$

$\hookrightarrow$ ние искаме $(\mathbb{P}(X \leq 10^4))^n \leq 0{,}01$

→ търсим $\mathbb{P}(X \leq 10^4)$

$$\mathbb{P}\left(N(0,1) \leq \frac{10^4 - 5011}{63{,}25}\right) \approx \mathbb{P}(N(0,1) \leq 78{,}88) \approx 1$$

$$\boxed{\mathbb{P}(\max(X_1, \dots, X_n) \geq 10^4) = 1 - (\mathbb{P}(X \leq 10^4))^n}$$

2. $X \sim Exp(2)$ и $Y \sim U(0,3)$

$$f_X(x) = \begin{cases} 2e^{-2x}, & \text{ако } x > 0 \\ 0, & \text{иначе} \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{1}{3}, & \text{ако } 0 < x < 3 \\ 0, & \text{иначе} \end{cases}$$

$Cor(X,Y) = ?$

$\mathbb{P}(X < Y) = ?$

$F_{XY}(x,y) = ?$

а) Вй тайо X и Y са независими $Cor(X,Y) = 0$

$\mathbb{E}[X] = \frac{1}{2}$, $DX = \frac{1}{2^2} = \frac{1}{4}$

$\mathbb{E}[Y] = \frac{a+b}{2} = \frac{3}{2}$, $DY = \frac{(b-a)^2}{12} = \frac{3^2}{12} = \frac{9}{12} = \frac{3}{4}$

~~$\mathbb{P}(X<Y) = \int_0^3 \int_0^y f_X(x) f_Y(y)\, dx\, dy$~~

~~$\int_0^y 2e^{-2x} dx = 1 - e^{-2y}$~~

~~$\int_0^3 1 - e^{-2y} \cdot \frac{1}{3}\, dy = \frac{1}{3} \int_0^3 (1 - e^{-2y})\, dy =$~~

$P(X < Y) = ?$

$$P(X<Y) = \iint f_{X,Y}(x,y)\,dx\,dy$$

и тъй като X и Y са независими, тяхната съвместна плътност е произведение от отделните плътности т.е.

$$f_{XY}(x,y) = f_X(x)\,f_Y(y)$$

$$P(X<Y) = \iint f_X(x)\,f_Y(y)\,dx\,dy = \int_0^3\left(\int_0^y 2e^{-2x}\,dx\right)\cdot\frac{1}{3}\,dy$$

$$\Rightarrow \int_0^y 2e^{-2x}\,dx = \left[-e^{-2x}\right]_0^y = 1-e^{-2y} \rightarrow \text{вътрешния интеграл}$$

↳ Външния

$$\int_0^3 (1-e^{-2y})\cdot\frac{1}{3}\,dy = \frac{1}{3}\int_0^3 1-e^{-2y}\,dy$$

$$\Rightarrow \int_0^3 1\,dy = 1\cdot(3-0) = 3$$

$$\int_0^3 e^{-2y}\,dy = \left[\frac{1}{2}e^{-2y}\right]_0^3 = -\frac{1}{2}e^{-2y}+\frac{1}{2} = -\frac{1}{2}e^{-6}+\frac{1}{2}$$

$$\Rightarrow P(X<Y) = \frac{1}{3}\left(3-\left(-\frac{1}{2}e^{-6}+\frac{1}{2}\right)\right) = \frac{1}{3}\left(3+\frac{1}{2}e^{-6}-\frac{1}{2}\right) =$$

$$= \frac{1}{3}\left(\frac{5}{2}+\frac{1}{2}e^{-6}\right) = \frac{1}{3}\left(\frac{5}{2}+\frac{e^{-6}}{2}\right) = \frac{5}{6}+\frac{e^{-6}}{6} = \frac{1}{6}\left(5+e^{-6}\right)$$

$\cancel{z} = \frac{X}{Y}$ ,

#TODO !

$$z = \frac{X}{Y}, \quad w = X \Rightarrow \begin{cases} X = w \\ Y = \frac{w}{z} \end{cases}$$

$$J = \begin{vmatrix} \frac{\partial X}{\partial w} & \frac{\partial X}{\partial z} \\ \frac{\partial Y}{\partial w} & \frac{\partial Y}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{z} & -\frac{w}{z^2} \end{vmatrix} = \left| -\frac{w}{z^2} \right|$$

$$f_{z,w}(z,w) = f_{X,Y}\left(w, \frac{w}{z}\right) \cdot |J|$$

$$\hookrightarrow f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = 2e^{-2x} \cdot \frac{1}{3} = \frac{2}{3} e^{-2x}$$

$$f_{z,w}(z,w) = \frac{2}{3} e^{-2w} \cdot \frac{w}{z^2} = \frac{2}{3} \cdot \frac{we^{-2w}}{z^2}$$

$$f_z(z) = \int_0^3 f_{z,w}(z,w)\, dw = \int_0^3 \frac{2}{3} \cdot \frac{we^{-2w}}{z^2}\, dw = \frac{2}{3z^2} \int_0^3 we^{-2w}\, dw$$

$\boxed{3.}$ $X \perp\!\!\!\perp Y$ $X \sim Exp\left(\frac{1}{\alpha}\right)$, $Y \sim Exp\left(\frac{1}{\beta}\right)$

#TODO

a) $\overline{X} := (X_1 + \dots X_n)/n$

$\mathbb{P}(|\alpha - \overline{X}| > c) \leq 1\%$

От ЦГТ за големи $n$:

$\frac{\overline{X} - \alpha}{\alpha \sqrt{n}} \approx \cancel{Z} \sim N(0,1)$. Заиобо обрсим $c$:

$\mathbb{P}\left(|Z| > \frac{c}{\alpha \sqrt{n}}\right) \leq 1\%$

$\Leftrightarrow \mathbb{P}\left(\epsilon > \frac{c}{\alpha \sqrt{n}}\right) \leq 0{,}5\%$.

$\Rightarrow \frac{c}{\alpha\sqrt{n}} \approx 2{,}57 \Rightarrow c \approx 2{,}57 \cdot \alpha \sqrt{n}$

Контролно 2 СЕМ 2023

$\boxed{1.}$ $f_{X,Y}(x,y) = cx^2+1$ , $x,y \geq 0$, $x+2y \leq 1$ и 0 иначе

a) c, плътността на X и очакването на Y

$$1 = \int_0^1 \int_0^{\frac{1-x}{2}} cx^2+1 \, dy\, dx = \int_0^1 (cx^2+1) \left( \frac{1-x}{2} \right) dx = \frac{1}{2}\int_0^1 (cx^2+1)(1-x)\, dx =$$

$$= \frac{1}{2}\int_0^1 cx^2 - cx^3 + 1 - x \, dx = \frac{1}{2}\left[ \frac{cx^3}{3} - \frac{cx^4}{4} + 1 - \frac{x^2}{2} \right]_0^1 =$$

$$= \frac{1}{2}\left[ \frac{c}{3} - \frac{c}{4} + 1 - \frac{1}{2} \right] = \frac{1}{2}\left[ \frac{4c-3c}{12} + \frac{2-1}{2} \right] =$$

$$= \frac{1}{2}\left[ \frac{c}{12} + \frac{1}{2} \right] = \frac{c}{24} + \frac{1}{4} = \frac{c+6}{24}$$

$$\Rightarrow c+6 = 24$$

$$c = 24-6$$

$$c = 18$$

↳ плътността на X

$$f_X(x) = \int_0^{\frac{1-x}{2}} cx^2+1 \, dy = [cx^2+1] \cdot \left( \frac{1-x}{2} \right)$$

$$EY = \int_0^1 \int_0^{\frac{1-x}{2}} y(cx^2+1)\,dy\,dx =$$

$$= \int_0^1 \int_0^{\frac{1-x}{2}} \frac{y^2}{2}(cx^2+1)\,dy\,dx = \int_0^1 (cx^2+1)\cdot \frac{\left(\frac{1-x}{2}\right)^2}{2}\,dx =$$

$$= \int_0^1 (cx^2+1)\cdot \frac{\frac{(1-x)^2}{4}}{\frac{2}{1}}\,dx = \int_0^1 (cx^2+1)\frac{(1-x)^2}{8}\,dx =$$

$$= \frac{1}{8}\int_0^1 (cx^2+1)(1-2x+x^2)\,dx = \frac{1}{8}\int_0^1 cx^2-2cx^3+cx^4+1-2x+x^2\,dx =$$

$$= \frac{1}{8}\left[\frac{cx^3}{3} - \frac{\cancel{2}cx^4}{\cancel{4}} + \frac{cx^5}{5} + 1 - \frac{\cancel{2}x^2}{\cancel{2}} + \frac{x^3}{3}\right]_0^1 =$$

$$= \frac{1}{8}\left[\frac{c}{3} - \frac{c}{2} + \frac{c}{5} + \cancel{1} - \cancel{1} + \frac{1}{3}\right]_0^1 =$$

$$= \frac{1}{8}\left[\frac{10c-15c+6c}{30} + \frac{1}{3}\right] = \frac{1}{8}\left[\frac{c}{30} + \frac{1}{3}\right] =$$

$$= \frac{1}{8}\left[\frac{18}{30} + \frac{1}{3}\right] = \frac{18}{240} + \frac{1}{24} = \frac{18+10}{240} = \frac{28}{240} = \frac{7}{60}$$

б) $\mathbb{E}[\cancel{X} \mid X = 1/2] = ?$

$$f_{Y|X}(y|\tfrac{1}{2}) = \cancel{\frac{f_{X,Y}(\frac{1}{2}, y)}{f_X(\frac{1}{2})}} = \cancel{\frac{c(\frac{1}{2})^2 + 1}{}} = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$f_X(x) = [cx^2+1] \cdot \left(\frac{1-x}{2}\right) = (18x^2+1)\left(\frac{1-x}{2}\right) = 9(x^2+1)(1-x)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{18\cancel{(x^2+1)}}{9\cancel{(x^2+1)}(1-x)} = \frac{2}{1-x}$$

и заместваме $x = \frac{1}{2}$

$$\Rightarrow f_{Y|X}(y|\tfrac{1}{2}) = \frac{2}{1-\frac{1}{2}} = \frac{2}{\frac{2-1}{2}} = \frac{2}{\frac{1}{2}} = \underline{\underline{4}}$$

$$\frac{1-x}{2} = \frac{1-\frac{1}{2}}{2} = \frac{\frac{2-1}{2}}{2} = \frac{\frac{1}{2}}{\frac{2}{1}} = \frac{1}{4}$$

$$\Rightarrow \mathbb{E}[Y \mid X = \tfrac{1}{2}] = \int_0^{\frac{1-\frac{1}{2}}{2}} y\, f_{Y|X}(y|\tfrac{1}{2})\, dy = \int_0^{\frac{1}{4}} y \cdot 4\, dy =$$

$$= 4\int_0^{1/4} y\, dy = 4 \cdot \left[\frac{y^2}{2}\right]_0^{1/4} = 4 \cdot \frac{(\frac{1}{4})^2}{2} = 4 \cdot \frac{\frac{1}{16}}{\frac{2}{1}} = 4 \cdot \frac{1}{16 \cdot 2} = \underline{\underline{\frac{1}{8}}}$$

# НЕПРЕКЪСНАТИ

## 4.1. Основни понятия

2024 / КН

$\boxed{4.1}$ $f_X(-x)=f_X(x), \quad x\in\mathbb{R}$

$F_X(0)=?$

$$\underset{\mathbb{P}(X\le 0)}{F_X(0)} = \int_{-\infty}^{0} f_X(x)\,dx = \int_{-\infty}^{\infty} f_X(x)\,dx - \int_{0}^{\infty} f_X(x)\,dx =$$

$$= \int_{-\infty}^{\infty} f_X(x)\,dx - \int_{-\infty}^{0} f_X(x)\,dx = 1 - F_X(0) \Rightarrow$$

$$F_X(0) = 1 - F_X(0)$$

$$2F_X(0) = 1$$

$$\underline{F_X(0) = 1/2}$$

$$\int_{0}^{\infty} f_X(x)\,dx \underset{\substack{u:=-x\\ du:=-dx}}{=} -\int_{0}^{-\infty} f_X(-u)\,du = \int_{-\infty}^{0} f_X(u)\,du = F_X(0)$$

$F_X(-x)=?$

$$F_X(-x) = \int_{-\infty}^{-x} f_X(t)\,dt \underset{\substack{u:=-t\\ du:=-dt}}{=} \int_{x}^{\infty} f_X(u)\,du = \underline{1-F_X(x)}$$

$\mathbb{P}(|X|<x)=?$

$$\mathbb{P}(|X|\le x) = \mathbb{P}(-x<X\le x) = F_X(x) - \underbrace{F_X(-x)}_{1-F_X(x)} = F_X(x) - (1-F_X(x)) =$$

$$= F_X(x) - 1 + F_X(x) = \underline{2F_X(x) - 1}$$

4.3. $X \sim Exp(\lambda)$

$X \sim f_X(x) = \lambda e^{-\lambda x} \cdot 1_{[0,\infty)}(x)$

a) $Y = -X$

б) $Y = 2X - 1$

в) $Y = \sqrt{X}$

a) $g(x) = -x$

$g^{-1}(y) = h(g) = -y$

$f_{-X}(y) = f_X(-y) \cdot |-1| = \lambda e^{\lambda x} \cdot 1_{[0,\infty)}(-y) = \lambda e^{\lambda x} \cdot 1_{(-\infty,0)}(y)$

б) $\underset{y}{g(x)} = \underset{g^{-1}(y)}{2x} - 1$

$y = 2g^{-1}(y) - 1$

$2g^{-1}(y) = y + 1$

$g^{-1}(y) = \frac{y+1}{2}$

$\frac{d}{dy} g^{-1}(y) = \frac{1}{2}$

$f_{2X-1}(y) = f_X(g^{-1}(y)) |(g^{-1}(y))'| = f_X\left(\frac{y+1}{2}\right) \cdot \left|\frac{1}{2}\right| =$

$= \frac{\lambda e^{-\lambda\left(\frac{y+1}{2}\right)}}{2} \cdot 1_{[0,+\infty)}\left(\frac{y+1}{2}\right) = \frac{\lambda}{2} e^{-\frac{\lambda}{2}(y+1)^2} \cdot 1_{[-1,\infty)}(y)$

$0 < \frac{y+1}{2} < \infty$

$2 \cdot 0 < y+1 < \infty$

$0 < y+1 < \infty$

$-1 < y < \infty$

б) $Y = \sqrt{X}$

$g(x) = \sqrt{x}$ , за $x > 0$

$g^{-1}(y) = y^2$ , за $y > 0$

$$f_{\sqrt{X}}(y) = \lambda e^{-\lambda y^2} \cdot 1_{(0,\infty)}(y^2) =$$
$$= \lambda e^{-\lambda y^2} \cdot |2y| \cdot 1_{(0,\infty)}(y) =$$
$$= \lambda e^{-\lambda^2} 2y \cdot 1_{(0,\infty)}(y)$$

---

а) $g(x) = -x$ , $g^{-1}(y) = -y$

~~$g^{-1}(y)$~~ $\frac{d}{dy} g^{-1}(y) = \frac{d}{dy}(-y) = -1$ производната на обратната ф-я

$$f_Y(y) = f_X(g^{-1}(y) \cdot |g^{-1}(y)|') = f_X(-y)|-1|$$

$$f_Y(y) = \lambda e^{-\lambda(-y)} \cdot 1 = \lambda e^{\lambda y} \text{ , за } y \leq 0$$

б)

б) $Y = 2X - 1$

1) намираме на обратната ф-я

~~$y = g(x) = 2x+1$~~

$Y = 2X + 1 \Rightarrow Y - 1 = 2X$

$X = \frac{Y+1}{2}$ ← изразяване $X$ чрез $Y$

$$\Rightarrow \boxed{g^{-1}(y) = \frac{y+1}{2}}$$

2) намираме производната на обратната ф-я

$\left(g^{-1}(y)\right)' = \frac{1}{2} \cdot (y+1)'$, $(y+1)'$ спрямо $y$ е 1, защото $y' = 1$, а $1' = 0$

$$\underline{\left(g^{-1}(y)\right)' = \frac{1}{2}}$$

3. заместваме във формулата за смяна на променливите

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left|\left(g^{-1}(y)\right)'\right| =$$

$$= f_X \cdot \frac{y+1}{2} \cdot \left|\frac{1}{2}\right|$$

$g^{-1}(y)$

$$\Rightarrow f_Y(y) = \lambda e^{-\lambda\left(\frac{y+1}{2}\right)} \cdot \frac{1}{2} = \frac{\lambda}{2} e^{\frac{-\lambda(y+1)}{2}}.$$

и тъй като $X \geq 0$, трябва $2X - 1 \geq -1$, $Y \geq -1$

$* \quad 0 < \frac{y+1}{2} < \infty$

$0 < y+1 < \infty$

$-1 < y < \infty$

б) $Y = \sqrt{X}$

$g^{-1}(y) = y^2$

$Y = \sqrt{X} \;/^2$

$Y^2 = X$

$\left(g^{-1}(y)\right)' = \left(y^2\right)' = 2y$

$\Rightarrow f_Y(y) = f_X\left(g^{-1}(y)\right) \cdot \left|\left(g^{-1}(y)\right)'\right| =$

$0 < y^2 < \infty \;/^2$

$0^2 < y < \infty^2$

$0 < y < \infty$

$= f_X(y^2) \cdot |2y| =$

$= \lambda e^{-\lambda y^2} \cdot 2y = 2\lambda y^{-\lambda y^2}$, за $(0,\infty)(y)$

4.5.

$$f_X(x) = \begin{cases} c(x^2+2x) & , x \in [0,1] \\ 0 & , x \notin [0,1] \end{cases}$$

a) $c$?

б) $E[X]$, $DX$

в) $P(X \le EX)$

г) $E[X^2+3X]$

a) $$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 c(x^2+2x)\,dx = c\left[\frac{x^3}{3} + \frac{2x^2}{2}\right]_0^1 =$$

$$= c\left(\frac{1}{3}+\frac{2}{2}\right) = c\left(\frac{2+6}{6}\right) = c\,\frac{8}{6} \rightarrow c \cdot \frac{4}{3}$$

$$1 = \frac{4}{3}c \rightarrow \frac{1}{c} = \frac{4}{3} \qquad c = \frac{3}{4}$$

б) $$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx = \int_0^1 x \cdot \frac{3}{4}(x^2+2x)\,dx =$$

$$\cancel{= \frac{3}{4}\int_0^1 x^3 + 2x^2\,dx = \frac{3}{4}\cdot 2\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_0^1 =}$$

$$\cancel{= \frac{3}{2}\cdot\left[\frac{1}{4}+\frac{1}{3}\right] = \frac{3}{2}\cdot\frac{3+4}{12} = \frac{3}{2}\cdot\frac{7}{12} = \frac{7}{8}}$$

$$= \frac{3}{4}\int_0^1 x^3 + 2x^2\,dx = \frac{3}{4}\int_0^1 x^3\,dx + 2\cdot\frac{3}{4}\int_0^1 x^2\,dx =$$

$$= \frac{3}{4}\cdot\left[\frac{x^4}{4}\right]_0^1 + \frac{3}{2}\cdot\left[\frac{x^3}{3}\right]_0^1 = \frac{3}{4}\cdot\frac{1}{4} + \frac{3}{2}\cdot\frac{1}{3} = \frac{3}{16} + \frac{1}{2} =$$

$$= \frac{3+8}{16} = \frac{11}{16}$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2 \cdot \frac{3}{4}(x^2+2x)\,dx =$$

$$= \frac{3}{4}\int_0^1 x^4 + 2x^3\,dx = \frac{3}{4}\int_0^1 x^4 dx + \frac{3}{4}\cdot 2\int_0^1 x^3 dx =$$

$$= \frac{3}{4}\cdot\frac{1}{5} + \frac{3}{2}\cdot\frac{1}{4} = \frac{3}{20} + \frac{3}{8} = \frac{6+15}{40} = \frac{21}{40}$$

$$DX = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{21}{40} - \left(\frac{11}{16}\right)^2$$

в) $\mathbb{P}(X \leq \mathbb{E}X) = \int_0^{\frac{11}{16}} \frac{3}{4}(x^2+2x)\,dx = \frac{3}{4}\int_0^{\frac{11}{16}} x^2 dx + 2\cdot\frac{3}{4}\int_0^{\frac{11}{16}} x\,dx = \ldots$

$\quad = F_X(x)$

г) $\mathbb{E}(X^2+3X) = \mathbb{E}X^2 + 3\,\mathbb{E}X = \ldots$ ✓

$\boxed{4.4}$ $X$ е н.с.в. с плътност $f_X$

$Y = X^2$, разпределение на $Y$?

$$\mathbb{P}(X^2 \leq x) = \int_{-\sqrt{x}}^{0} f_X(t)\,dt + \int_0^{\sqrt{x}} f_X(t)\,dt =$$

$X$ е н.с.в. с плотностью $f_X$

$Y = X^2$ , $\sim X^2$

$\varphi$ $g^{-1}(y) = \sqrt{y}$

$Y = x^2$

$x = \sqrt{y}$

$g^{-1}(y) = \pm\sqrt{y}$

$\Rightarrow f_Y(y) =$

$$\boxed{(g^{-1}(y))' = \frac{1}{2\sqrt{y}}}$$

$\boxed{4.4}$ $X$ е н.с.в. с плътност $f_X$

$Y = X^2$, $\sim Y$ ?

$\rightarrow$ $Y = X^2 \,/\sqrt{\ }$

$X = \pm\sqrt{y}$ $\rightarrow$ имаме $g_1^{-1}(y) = \sqrt{y}$

$g_2^{-1}(y) = -\sqrt{y}$

$\rightarrow$ $\left(g_1^{-1}(y)\right)' = \frac{1}{2\sqrt{y}}$

$\left(g_2^{-1}(y)\right)' = -\frac{1}{2\sqrt{y}}$

$\hookrightarrow$ $f_Y(y) = f_X\left(g_1^{-1}(y)\right)\left|\left(g_1^{-1}(y)\right)'\right| + f_X\left(g_2^{-1}(y)\right)\left|\left(g_2^{-1}(y)\right)'\right| =$

$= f_X(\sqrt{y}) \cdot \left|\frac{1}{2\sqrt{y}}\right| + f_X(-\sqrt{y}) \cdot \left|-\frac{1}{2\sqrt{y}}\right| =$

$= f_X(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}} + f_X(-\sqrt{y})\frac{1}{2\sqrt{y}}$

$= \frac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right)$, $y \geq 0$

# ~~Нормално~~ Равномерно разпределение

$X \sim Unif(a,b)$

$f_X(x) = \frac{1}{b-a} \cdot 1_{(a,b)}(x)$

$Y = \frac{X-a}{b-a}$ , $Y \sim ?$

$\hookrightarrow g(x) = \frac{x-a}{b-a}$

$y = \frac{x-a}{b-a}$

$y(b-a) = x-a$

$x = a+(b-a)y$

$g^{-1}(y) = a+(b-a)y$

$|(g^{-1}(y))'| = b-a$

$f_Y(y) = f_X(g^{-1}(y))\,|(g^{-1}(y))'| =$

$= \underbrace{f_X(a+(b-a)y)}_{\frac{1}{b-a}} \cdot (b-a) =$

$= \frac{1}{\cancel{b-a}} \cdot \cancel{b-a} = 1 \quad \sim Y \sim Unif(0,1)$

$a < a+(b-a)y < b$

$\Downarrow$

$0 < y < 1$

$\boxed{4.15}$ $X \sim Unif(0,7)$ → времето на безотказна работа на даден апарат в години

↳ мени се ~~на~~ петата година или преди ако има дефект

$Y =$ „Времето до смяна на апарата"

$$\mathbb{P}(Y<4) = \mathbb{P}(\min(X,5)<4) = \mathbb{P}(X<4)$$

↳ той като $X$ е равномерно разпределено в интервала $[0,7]$, то $\mathbb{P}(X<4) = \frac{4}{7}$

↳ $\mathbb{E}[Y]$

↳ Ако $X<5$, то $Y=X$

Ако $X \geq 5$, то $Y=5$

↳ $Y$ има следната плътност на разпределение:

$$f_Y(y) = \begin{cases} \frac{1}{7}, & 0 \leq y \leq 5 \\ 0, & \text{иначе} \end{cases}$$

$$\rightarrow \mathbb{E}[Y] = \int_0^5 y f_Y(y)\,dy + \int_5^7 5 f_Y(y)\,dy$$

↳ смятаме ги поотделно

$$\int_0^5 y f_Y(y)\,dy = \int_0^5 y \cdot \frac{1}{7}\,dy = \frac{1}{7}\int_0^5 y\,dy = \frac{1}{7} \cdot \left[\frac{y^2}{2}\right]_0^5 = \frac{1}{7} \cdot \frac{25}{2} = \frac{25}{14}$$

$$\int_5^7 5 f_Y(y)\,dy = \int_5^7 5 \cdot \frac{1}{7}\,dy = 5 \cdot \frac{1}{7}\int_5^7 1\,dy = 5 \cdot \frac{1}{7}(7-5) = \frac{10}{7}$$

просто слагаме границата, защото няма $y$

$$\Rightarrow \mathbb{E}[Y] = \frac{25}{14} + \frac{10}{7} = \frac{25+20}{14} = \frac{45}{14}$$

$$\mathbb{E}[Y^2] = \underbrace{\int_0^5 y^2 \cdot f_Y(y)\,dy}_{I} + \underbrace{\int_5 5^2 \cdot f_Y(y)\,dy}_{II} =$$

$$I = \int_0^5 y^2 \cdot \frac{1}{7}\,dy = \frac{1}{7} \cdot \left[\frac{y^3}{3}\right]_0^5 = \frac{1}{7} \cdot \frac{125}{3} = \frac{125}{21}$$

$$II = \int_5^7 25 \cdot \frac{1}{7}\,dy = \frac{25}{7}\int_5^7 1\,dy = \frac{25}{7} \cdot (7-5) = \frac{50}{7}$$

$$\mathbb{E}[Y^2] = \frac{125}{21} + \frac{50}{7} = \frac{125+150}{21} = \frac{275}{21}$$

$$DX = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{275}{21} - \left(\frac{45}{14}\right)^2 \approx 2{,}764$$

↳ Ако са продадени 1000 апарата, колко средно са променени преди 5-тата година?

средно $\sim Bin\left(1000, \frac{5}{7}\right)$

$$= 1000 \cdot \frac{5}{7} = \frac{5000}{7} = 714{,}28$$

**4.17.** Ако $F_X(x)$ е строго нарастваща и непрекъсната, това означава, че $F_X$ има обратна ф-я която също е непрекъсната и строго нарастваща

и $Y = F_X(x)$, т.е. $F_Y(y)$ е дефинирана като

$$F_Y(y) = \mathbb{P}(X \le y) = \mathbb{P}(F_X(x) \le y)$$

Тъй като $F_X$ е строго $\uparrow$ и непрекъсната, тя приема винаги стойности в интервала $(0,1) \Rightarrow F_Y(y) = \mathbb{P}(F_X(x) \le y) = \mathbb{P}(\cancel{X} \le F_X^{-1}(y))$

$$F_Y(y) = \mathbb{P}(X \le F_X^{-1}(y)) = F_X(F^{-1}_X(y))$$

за $y$ в интервала $(0,1)$ $F_X$ и $F_X^{-1}$ са обратни функции, така че $F_Y(y) = y$ . $Y \sim Unif(0,1)$

↳ Или по-кратко

$$\mathbb{P}(Y \le y) = \mathbb{P}(F_X(X) \le y) = \mathbb{P}(X \le F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y,$$

$$y \in (0,1) \rightarrow Y \sim Unif(0,1)$$

$\boxed{4.22.}$

$\mathbb{E}X_1 = 8 \Rightarrow \lambda_1 = \frac{1}{8}$

$\mathbb{E}X_2 = 5 \Rightarrow \lambda_2 = \frac{1}{5}$

$X_1 \sim Exp(\lambda_1)$ $\quad X_2 \sim Exp(\lambda_2)$

A = "клиентът чака < 4min"

I – избира $X_1$ опашка

II – избира $X_2$ опашка

$\mathbb{P}(I \mid A) = ?$

$$\mathbb{P}(I \mid A) = \frac{\mathbb{P}(A \mid I)\,\mathbb{P}(I)}{\mathbb{P}(A \mid I)\,\mathbb{P}(I) + \mathbb{P}(A \mid II)\,\mathbb{P}(II)} \rightarrow \mathbb{P}(A)$$

$\mathbb{P}(I) = 1/2$

$\mathbb{P}(II) = 1/2$

$$\mathbb{P}(A) = \mathbb{P}(X_1 < 4)\cdot\frac{1}{2} + \mathbb{P}(X_2 < 4)\cdot\frac{1}{2} =$$

$$= 1 - e^{-\lambda_1 4}\cdot\frac{1}{2} + 1 - e^{-\lambda_2 4}\cdot\frac{1}{2} =$$

$$= \frac{1}{2}\left(1 - e^{-4\lambda_1} + 1 - e^{-4\lambda_2}\right) = \frac{1}{2}\left(2 - e^{-\frac{4}{8}} - e^{-\frac{4}{5}}\right)$$

$$\mathbb{P}(A \mid I)\,\mathbb{P}(I) = \mathbb{P}(X_1 < 4)\cdot\frac{1}{2} = \frac{1}{2}\left(1 - e^{-\lambda_1 \cdot 4}\right) = \frac{1}{2}\left(1 - e^{-\frac{4}{8}}\right)$$

$$\Rightarrow \mathbb{P}(I \mid A) = \frac{\frac{1}{2}\left(1 - e^{-\frac{4}{8}}\right)}{\frac{1}{2}\left(2 - e^{-\frac{4}{8}} - e^{-\frac{4}{5}}\right)}$$

1.18. $X, Y \sim \text{Unif}(0,1)$

$X+Y \sim ?$

$\boxed{+}$ $X \sim N(\mu, \sigma^2)$

$$Y = \frac{X-\mu}{\sigma}, \quad Y \sim ?$$

↳ $E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{E[X]-\mu}{\sigma} = \frac{\mu-\mu}{\sigma} = 0$

$$DY = D\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} \cdot DX = \frac{1}{\cancel{\sigma^2}} \cdot \cancel{\sigma^2} = 1$$

$$\Rightarrow Y \sim N(0,1)$$

↳ със смяна на променливите ↓

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$Y = \frac{X-\mu}{\sigma}$$

$$g^{-1}(y) = \mu + \sigma y$$

$$y = \frac{x-\mu}{\sigma}$$

$$y\sigma = x - \mu$$

$$x = \mu + \sigma y$$

$$\left(g^{-1}(y)\right)' = \sigma$$

$$f_Y(y) = f_X\left(g^{-1}(y)\right) \left|\left(g^{-1}(y)\right)'\right| =$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{\frac{-(\sigma y+\mu-\mu)^2}{2\sigma^2}} \cdot \sigma =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{y^2}{2}} \Rightarrow Y \sim N(0,1)$$

# Теорема на Моавер-Лаплас

↳ позволява да се използва ~~[illegible]~~ нормалното разпределение за апроксимация на сумата от много случайни величини, които имат подобно разпределение.

$Z \sim N(0,1)$, $\quad X_n \sim Bin(n, p)$, $\quad p \in (0,1)$

$$\mathbb{P}\left( \frac{X_n - np}{\sqrt{np(1-p)}} \leq x \right) \xrightarrow[n \to \infty]{} \mathbb{P}(Z \leq x)$$