Упр. записи Минчев

→ $\mathbb{P}(X \in [a,b]) = \mathbb{P}(X \in (a,b)) = \dots = \int_a^b f_X(x)\, dx$

↳ pdf → probability distribution function

↳ не винаги съществува → плътност

→ това което винаги съществува е cdf - cumulative distribution function

→ функция на разпределение

$$F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(x)\, dx$$

$$F'_X(x) = f_X(x)$$

→ $X \sim Unif(a,b)$

- $f_X(x) = \frac{1}{b-a} \cdot 1_{\{x \in [a,b]\}}$
- $F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx = \int_0^x \frac{1}{b-a}\, dt = \frac{x}{b-a}$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$\boxed{\begin{aligned} \mathbb{E}X &= \frac{b-a}{2} \\ DX &= \frac{(b-a)^2}{12} \end{aligned}}$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(y) \cdot f_X(y)\, dy$$

→ Експоненциално - използва се за събития случващи се във времето, когато ще чакаме, колко време отнема до обслужим някого, да дойде някаква заявка итн.

$F_X(x) = \mathbb{P}(X < x)$

$F'_X(x) = f_X(x)$

$\boxed{1.}$ $\mathbb{P}(\text{ези}) = \overset{p}{3/4}$, $\overset{n}{2000}$ пъти

$\mathbb{P}($между 1475 и 1535 ези$) = ?$

↳ $X_i = \begin{cases} 1, & \text{ези на } i\text{-тия} \\ 0, & \text{иначе} \end{cases}$ $\quad X_i \sim Ber(p)$

$\mathbb{P}(1475 \le X_1 + \dots + X_n \le 1535)$

ЦГТ:

→ какви са разпределенията няма значение

Нека $X_1, X_2, \dots$ нез. и ед. разп. сл. вел. (iid) и

$\mu = \mathbb{E}X_1$ и $\sigma^2 = DX_1 < \infty$. Тогава

$$\frac{X_1 + \dots + X_n}{n} - \mu \xrightarrow{п.с.} 0$$ → закона за големите числа

$$\frac{\frac{X_1 + \dots + X_n}{n} - \mu}{\sigma/\sqrt{n}} \xrightarrow[n\to\infty]{d} N(0,1)$$

↓

$$\frac{X_1 + \dots + X_n - n\mu}{\sigma\sqrt{n}}$$

↳ ще получаваме числа, които са някъде около нулата

→ Нормално разпределение

$X \sim N(\mu, \sigma^2)$, ако $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

* Свойства:

1. $X_1 \sim N(\mu_1, \sigma_1^2)$
2. $X_2 \sim N(\mu_2, \sigma_2^2)$
3.

1. $cX_1 \sim N(c \cdot \mu_1, c^2\sigma_1^2)$
2. $c + X_1 \sim N(c+\mu_1, \sigma_1^2)$
3. $X_1 \perp\!\!\!\perp X_2 \quad X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

→ Връщаме се на задачата

$$\mathbb{P}\left(\frac{1475-\mu n}{\sigma\sqrt{n}} \le \frac{X_1+\dots+X_n-\mu n}{\sigma\sqrt{n}} \le \frac{1535-\mu n}{\sigma\sqrt{n}}\right) =$$

$\mu = 3/4$, $n = 2000$, $\sigma^2 = 3/16$

$$\mathbb{P}\left(\frac{-25}{\frac{\sqrt{3}}{4}\cdot\sqrt{2000}} \le \frac{X_1+\dots+X_n-\mu n}{\sigma\sqrt{n}} \le \frac{1535-\mu n}{\sigma\sqrt{n}}\right) \approx \quad n>30,\ \sigma^2<\infty$$

$$\mathbb{P}(-1{,}29 \le N(0,1) \le 1{,}83)$$

* $P(N(0,1) \le x) = F_{N(0,1)}(x) = \Phi(x)$

$$\mathbb{P}(-1{,}29 \le N(0,1) \le 1{,}83) = \Phi(1{,}83) - \Phi(-1{,}29) =$$

$= 0{,}9664 - (0{,}0985) =$ число...

$\boxed{\mathcal{F}}$ $f_{X,Y}(x,y) = \begin{cases} cxy & , 0 \le x \le y \le 1 \\ 0 & , \text{инаге} \end{cases}$

1. $c = ?$

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* $\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$

$P(X \in (a,b)) = \int\limits_{a}^{b} f_X(x)\, dx$

$P((X,Y) \in A) = \iint\limits_{A} f_{X,Y}(x,y)\, dx$

$EXY = \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\, dx\, dy$

$F_{X,Y}(x,y) = P(X \le x, Y \le y)$

$$\frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} = \frac{\partial^2 F_{X,Y}(x,y)}{\partial y \partial x} = f_{X,Y}(x,y)$$

$\Rightarrow \int\limits_{0}^{1}\left[\int\limits_{x}^{1} cxy\, dy\right] dx$

$\downarrow$

$$\int\limits_{x}^{1} cxy\,dy = cx\int\limits_{x}^{1} y\,dy = cx\left[\frac{y^2}{2}\right]_x^1 = cx\left(\frac{1}{2} - \frac{x^2}{2}\right)$$

$\rightarrow \int_0^1 cx\left(\frac{1}{2}-\frac{x^2}{2}\right)dx = \frac{c}{2}\int_0^1 x(1-x^2)\,dx \ \ldots$

2. Маргинални плътности

$f_X(x) = \int_x^1 cxy\,dy$

граници на y
+ по dy

$\mathbb{E}X = \int_0^1 x \cdot f_X(x)\,dx \ \ldots$

3. $\mathbb{P}(X<\frac{3}{4},\ Y-X<\frac{1}{6}) = \mathbb{P}(X<\frac{3}{4},\ Y<X+\frac{1}{6}) =$

$= \int_0^{3/4}\int_{x}^{x+\frac{1}{6}} cxy\,dy\,dx = c\int_0^{3/4}\left[\frac{y^2}{2}\right]_x^{x+\frac{1}{6}}dx = \frac{c}{2}\int_0^{3/4} x\left((x+\frac{1}{6})^2-x^2\right)dx.$

4. $\mathbb{E}(Y-X \mid X=\frac{1}{4}) = \mathbb{E}(Y-\frac{1}{4} \mid X=\frac{1}{4}) =$

$= \int_{-\infty}^{\infty} (y-\frac{1}{4})\, f_{Y|X}(y|\frac{1}{4})\,dy$;

$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$ $\rightarrow$ съвместна плътност; $\rightarrow$ маргиналната

4.2. Колекция от случайни величини

$\boxed{4.6.}$ X – пали лампите
Y – гаси лампите

$$f_{X,Y}(x,y) = cxy, \quad 0 < x < y < 1$$

а) константата $c = ?$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot 1_{D_{X,Y}}(x,y)\,dy\,dx =$$

$$= c\int_0^1 x\left(\int_x^1 y\,dy\right)dx = c\int_0^1 x\left[\frac{y^2}{2}\right]_x^1 dx = c\int_0^1 x\left(\frac{1}{2} - \frac{x^2}{2}\right)dx =$$

$$= \frac{c}{2}\int_0^1 x - x^3\,dx = \frac{c}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{c}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{c}{2}\left[\frac{2}{4} - \frac{1}{4}\right] =$$

$$= \frac{c}{2} \cdot \frac{1}{4} = \frac{c}{8} \Rightarrow 1 = \frac{c}{8} \rightarrow \boxed{c = 8}$$

б) маргиналните плътности и математическите очаквания на X и Y

$\rightarrow$ Като позваме, че $1_{D_{X,Y}}(x,y) = 1_{(0,1)}(x) \cdot 1_{(x,1)}(y) \equiv 1_{(0,y)}(x) \cdot 1_{(0,1)}(y)$

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = 8x\left(\int_x^1 y\,dy\right) \cdot 1_{(0,1)}(x) =$$

$$= 8x\left[\frac{y^2}{2}\right]_x^1 \cdot 1_{(0,1)}(x) = 8x\left[\frac{1}{2} - \frac{x^2}{2}\right] \cdot 1_{(0,1)}(x) =$$

$$= 4x(1-x^2) \cdot 1_{(0,1)}(x)$$ $\rightarrow$ защо до тук ?

$$\rightarrow f_Y(y) = 8y\left(\int_0^1 x\,dx\right) \cdot 1_{(0,1)}(y) = 8y\left[\frac{x^2}{2}\right]_0^1$$

и как се стига до $4y^3 \cdot 1_{(0,1)}(y)$

$\to \mathbb{E}[X] = \int_0^1 x \cdot \underbrace{f_X(x)}_{4x(1-x^2)} dx = 4\int_0^1 x \cdot x(1-x^2)\, dx = 4\int_0^1 x^2(1-x^2)\, dx =$

$= 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 4 \cdot \frac{5-3}{15} = 4 \cdot \frac{2}{15} = \frac{8}{15}$

$\to \mathbb{E}[Y] = \int_0^1 y \cdot \underbrace{f_Y(y)}_{4y^3} dy = 4\int_0^1 y \cdot y^3\, dy = 4\left[\frac{y^5}{5}\right]_0^1 =$

или $x, 1$?

$= 4 \cdot \frac{1}{5} = \frac{4}{5}$

в) Вероятността пациент да бъдат заспали преди 45-тата минута и да светят по-малко от 10 минути

$\to$ Нека $B = \{(x,y) \in \mathbb{R}^2 : x < \frac{3}{4}, y < x + \frac{1}{6}\}$. Тогава

$P(X < \frac{3}{4}, Y - X < \frac{1}{6}) = P((X,Y) \in B) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 1_B(x,y) f_{X,Y}(x,y)\, dy\, dx =$

$= 8\int_0^{\frac{3}{4}} x\left(\int_x^{x+\frac{1}{6}} y\, dy\right) dx = 8\int_0^{\frac{3}{4}} x \left[\frac{y^2}{2}\right]_x^{x+\frac{1}{6}} dx \; (=)$

$\hookrightarrow \frac{(x+\frac{1}{6})^2}{2} - \frac{x^2}{2} = \frac{x^2 + \frac{2x}{6} + \frac{1}{36}}{2} - \frac{x^2}{2} = \frac{\cancel{x^2} + \frac{x}{3} + \frac{1}{36}}{2} - \frac{\cancel{x^2}}{2} =$

$= \frac{\frac{x}{3} + \frac{1}{36}}{2} = \frac{\frac{12x+1}{36}}{\frac{2}{1}} = \frac{12x+1}{36 \cdot 2} = \boxed{\frac{x+1}{6}}$

$(=)\; 8\int_0^{\frac{3}{4}} x \cdot \frac{(x+1)}{6} dx = \frac{8}{6}\int_0^{\frac{3}{4}} x^2 + x\, dx = \frac{8}{6} \cdot \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_0^{\frac{3}{4}} = \frac{8}{6} \cdot \left[\frac{(\frac{3}{4})^3}{3} + \frac{(\frac{3}{4})^2}{2}\right] =$

$= \frac{8}{6}\left[\frac{\frac{27}{64}}{3} + \frac{\frac{9}{16}}{2}\right] = \frac{8}{6}\left[\frac{2 \cdot \frac{27}{64} + 3 \cdot \frac{9}{16}}{6}\right] = \frac{\frac{27}{32} + \frac{27}{16}}{6} = \frac{\frac{27+54}{32}}{6} = \frac{\frac{81}{32}}{\frac{6}{1}} = \frac{81}{32 \cdot 6}$

$\Rightarrow \frac{8}{6} \cdot \frac{27}{32 \cdot 2} = \frac{27}{48} \approx 0{,}5625$?

→ като използваме наблюдението, че

$$\mathbb{1}_{D_{X,Y}}(x,y)\cdot \mathbb{1}_{B}(x,y) = \mathbb{1}_{(0,\frac{3}{4})}(x)\cdot \mathbb{1}_{(x,x+\frac{1}{6})}(y)$$

г) вероятността лампите да светят по-малко от 20 минути

Нека $C=\{(x,y)\in\mathbb{R}^2 : y<x+\frac{1}{3}\}$. Тогава

$$\mathbb{P}(Y-X<\tfrac{1}{3}) = \mathbb{P}((X,Y)\in C) =$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbb{1}_C(x,y)\, f_{X,Y}(x,y)\,dy\,dx = 8\int_{0}^{\frac{2}{3}}\int_{x}^{x+\frac{1}{3}} xy\,dy\,dx + 8\int_{\frac{2}{3}}^{1}\int_{x}^{1} xy\,dy\,dx$$

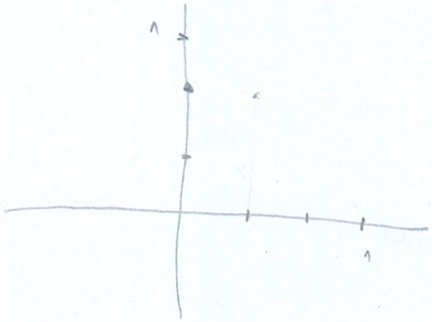
↳ защо?

$$\to \mathbb{1}_{D_{X,Y}}(x,y)\cdot \mathbb{1}_{C}(x,y) = \mathbb{1}_{(0,1)}(x)\cdot \mathbb{1}_{(x,\min\{x+\frac{1}{3},1\})}(y) =$$

?

$$= \mathbb{1}_{(0,\frac{2}{3})}(x)\cdot \mathbb{1}_{(x,x+\frac{1}{3})}(y) + \mathbb{1}_{(\frac{2}{3},1)}(x)\cdot \mathbb{1}_{(x,1)}(y)$$

#?



$\boxed{2.7}$ $f_{X,Y}(x,y) = ax^2 + bxy$, за $x \in (0,1)$ и $y \in (0,2)$ и 0 иначе

↳ a и b = ?

$\mathbb{E}[X] = \frac{13}{18}$ и $P(X+Y \geq 1) = ?$

$$\Rightarrow 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx = \int_0^1\int_0^2 (ax^2+bxy)\,dy\,dx = \int_0^1 ax^2+bx \int_0^2 y\,dy\,dx =$$

$$= 2\int_0^1 (ax^2+bx)\,dx = 2a+b\left[\frac{x^3}{3}+\frac{x^2}{2}\right]_0^1 = 2a+b\left[\frac{1}{3}+\frac{1}{2}\right] = 2a+b\cdot\frac{2+3}{6} =$$

$= \frac{5}{3}a+b$ → добито е $\frac{2}{3}a+b$ → ??

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^2 (ax^2+bxy)\,dy \cdot 1_{(0,1)}(x) = 2(ax^2+bx)\cdot 1_{(0,1)}(x)$$

↳ от друго знаем $\mathbb{E}[X] = \frac{13}{18}$

$$\frac{13}{18} = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = 2\int_0^1 x(ax^2+bx)\,dx = 2\cdot\left[\frac{ax^4}{4}+\frac{bx^3}{3}\right]_0^1 =$$

$$= 2\cdot\left(\frac{a}{4}+\frac{b}{3}\right) = \frac{1}{2}a+\frac{2}{3}b$$

$$\Rightarrow \begin{cases} \frac{2}{3}a+b = 1 \\ \frac{1}{2}a+\frac{2}{3}b = \frac{13}{18} \end{cases}$$ → система за да намерим a и b = ?

↳ $a = 1$ и $b = \frac{1}{3}$

$\Rightarrow$ Нека $C = \{ (x,y) \in \mathbb{R} : x+y \geq 1 \}$. Тогава $1_C(x,y) = 1_{(1-x,\infty)}(y)$ и

$$\mathbb{P}(X+Y \geq 1) = \mathbb{P}((X,Y) \in C) = \int_0^1 \int_0^2 1_C(x,y)(x^2 + \frac{1}{3}xy)\,dy\,dx =$$

$$= \int_0^1 \int_{1-x}^2 (x^2 + \frac{1}{3}xy)\,dy\,dx = \dots$$

$\downarrow$

защо нова граница ?

$\boxed{4.8.}$ $F_{X,Y}$

Изразете ф-ция на разпределението на $(\max\{X,Y\}, \min\{X,Y\})$ през $F_{X,Y}$

За всички $s,t \in \mathbb{R}: s<t$, имаме

$$F_{\max\{X,Y\},\min\{X,Y\}}(t,s) = \mathbb{P}(\max\{X,Y\} \le t, \min\{X,Y\} \le s) =$$

$$= \mathbb{P}(\max\{X,Y\} \le t) - \mathbb{P}(\max\{X,Y\} \le t, \min\{X,Y\} > s) =$$

$$= \mathbb{P}(X \le t, Y \le t) - \mathbb{P}(X \le t, Y \le t, X > s, Y > s) =$$

$$= \mathbb{P}(X \le t, Y \le t) - \mathbb{P}(s < X \le t, s < Y \le t) =$$

$$= F_{X,Y}(t,t) - (F_{X,Y}(t,t) - F_{X,Y}(t,s) - F_{X,Y}(s,t) + F_{X,Y}(s,s)) =$$

$$= F_{X,Y}(t,s) + F_{X,Y}(s,t) - F_{X,Y}(s,s)$$

От друга страна: $F_{\max\{X,Y\}, \min\{X,Y\}}(t,s) = \mathbb{P}(\max\{X,Y\} \le t) = \mathbb{P}_{X,Y}(t,t)$, за $s,t \in \mathbb{R}: s \ge t$

$\boxed{4.9.}$ X и Y са нез. и едн. разп. iid с плътност $f(x) = \mathbb{1}_{(0,1)}(x)$

$Var(Z) = ?$ $\quad Z := |X-Y|$

По условие $f_{X,Y}(x,y) = \mathbb{1}_{(0,1)}(x) \cdot \mathbb{1}_{(0,1)}(y)$, откъдето

$$\mathbb{E}[|X-Y|] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y| f_{X,Y}(x,y)\, dy\, dx =$$

$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)\cdot \mathbb{1}_{(0,x)}(y)\, f_{X,Y}(x,y)\, dy\, dx =$$

от къде тази 2ка?

$$= 2\int_0^1\int_0^x (x-y)\, dy\, dx = 2\int_0^1 x - \left(-\int_0^x -y\, dy\right) dx = 2\int_0^1 x - \frac{x^2}{2}\, dx =$$

$= 2\Big[\ ^x$ .... ?