Контролно 2 21.01.2023

$\boxed{1.}$ $f_{X,Y}(x,y) = \begin{cases} cx^2+1 & , x,y \geq 0, x+2y \leq 1 \\ 0 & \text{иначе} \end{cases}$

а) c, плътността на X и очакването на Y

$\begin{cases} x,y \geq 0 \\ x \leq 1-2y \iff y \leq \frac{1-x}{2} \end{cases}$

$$1 = \int_0^1 \int_0^{\frac{1-x}{2}} cx^2+1 \, dy \, dx = \int_0^1 cx^2+1 \, [y]_0^{\frac{1-x}{2}} \, dx =$$

$$= \int_0^1 (cx^2+1)\left(\frac{1-x}{2}\right) dx = \frac{1}{2}\int_0^1 (cx^2+1)(1-x) \, dx =$$

$$= \frac{1}{2}\int_0^1 cx^2 - cx^3 + 1 - x \, dx = \frac{1}{2}\left[\frac{cx^3}{3} - \frac{cx^4}{4} + x - \frac{x^2}{2}\right]_0^1 =$$

$$= \frac{1}{2}\left[\frac{c}{3} - \frac{c}{4} + 1 - \frac{1}{2}\right] = \frac{1}{2}\left[\frac{4c}{12} - \frac{3c}{12} + \frac{2-1}{2}\right] =$$

$$= \frac{1}{2}\left[\frac{c}{12} + \frac{1}{2}\right] = \frac{c}{24} + \frac{1}{4} = \frac{c}{24} + \frac{6}{24} = c+6 = 24$$

$\Rightarrow c+6 = 24$

$c = 24-6$

$\boxed{c = 18}$

плътността на X

$$f_X(x) = \int_0^{\frac{1-x}{2}} cx^2+1 \, dy = (cx^2+1)\left(\frac{1-x}{2}\right)$$

очакването на Y

$$EY = \int_0^1 \int_0^{\frac{1-x}{2}} y(cx^2+1)\,dy\,dx = \int_0^1 (cx^2+1)\left(\frac{1-x}{2}\right)^2 \cdot \frac{1}{2}\,dx =$$

$$= \int_0^1 (cx^2+1)\left(\frac{1-2x+x^2}{4}\right)\cdot\frac{1}{2}\,dx = \frac{1}{8}\int_0^1 (cx^2+1)(1-2x+x^2) =$$

$$= \frac{1}{8}\int_0^1 cx^2 - 2cx^3 + cx^4 + 1 - 2x + x^2 \, dx =$$

$$= \frac{1}{8}\left[\frac{cx^3}{3} - \frac{2cx^4}{4} + \frac{cx^5}{5} + x - \frac{2x^2}{2} + \frac{x^3}{3}\right]_0^1 =$$

$$= \frac{1}{8}\left[\frac{18}{3} - \frac{2\cdot 18}{4} + \frac{18}{5} + \cancel{1} - \cancel{1} + \frac{1}{3}\right] =$$

$$= \frac{1}{8}\left[6 - 9 + \frac{18}{5} + \frac{1}{3}\right] = \frac{1}{8}\left[-3 + \frac{54}{15} + \frac{5}{15}\right] =$$

$$= \frac{1}{8}\left[-\frac{45}{15} + \frac{54}{15} + \frac{5}{15}\right] = \frac{1}{8}\left[-\frac{45}{15} + \frac{59}{15}\right] = \frac{1}{\cancel{8}_4}\cdot\frac{\cancel{14}^7}{15} = \boxed{\frac{7}{60}}$$

б) $E(Y|X=\frac{1}{2})$

$$f_{Y|X}(y|\tfrac{1}{2}) = \frac{f_{X,Y}(\frac{1}{2},y)}{f_X(\frac{1}{2})} = \frac{\cancel{c(\frac{1}{2})^2+1}}{\cancel{(c(\frac{1}{2})^2+1)}\left(\frac{1-\frac{1}{2}}{2}\right)} = \frac{1}{\frac{\frac{1}{2}}{\frac{2}{1}}} =$$

$$= \frac{1}{\frac{1}{4}} = \boxed{4} \rightarrow \text{за } y \in (0,\tfrac{1}{4}) \Rightarrow E[Y|X=\tfrac{1}{2}] = E[U(0,\tfrac{1}{4})]$$

$$\Rightarrow E\,U(a,b) = \frac{b-a}{2} \Rightarrow \frac{\frac{1}{4}-0}{2} = \frac{1}{8}$$

в) плътността на $Z = X + 2Y$

$$\begin{cases} Z = X + 2Y \\ W = X \end{cases} \Rightarrow \begin{cases} X = W \\ Y = \frac{Z-W}{2} \end{cases}$$

$$|J| = \begin{vmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial W} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$f_{Z,W}(z,w) = f_{X,Y}\left(w, \frac{z-w}{2}\right) \cdot \frac{1}{2} = (cw^2+1)\cdot\frac{1}{2}$$

за $w, \frac{z-w}{2} \in (0,1)$ т.е. $z \in (0,1)$ и $w \in (0,z)$

$$\Rightarrow f_Z(t) = \int_0^t (cw^2+1)\cdot\frac{1}{2}\,dw = \frac{1}{2}\int_0^t cw^2+1\,dw =$$

$$= \frac{1}{2}\left[\frac{cw^3}{3}+w\right]_0^t = \frac{1}{2}\left[\frac{18t^3}{3}+t\right] = \frac{18t^3}{6}+\frac{t}{2} = \boxed{3t^3+\frac{t}{2}}$$

за $t \in (0,1)$

→ другиот начин:

$$\mathbb{P}(X+2Y \le t) = \mathbb{P}\left(Y \le \frac{t-X}{2}\right)$$

$$\int_0^{\frac{t}{2}}\int_0^{t-2y} cx^2+1\,dx\,dy = \int_0^{\frac{t}{2}} c\left[\frac{x^3}{3}+x\right]_0^{t-2y} dy =$$

$$= \int_0^{t/2} c\left[\frac{(t-2y)^3}{3} + t-2y\right] dy \quad \underline{t-2y=w}$$

$$= \frac{1}{2}\int_0^{t} c\,\frac{w^3}{3}+w\,dw = \frac{1}{2}\left[c\frac{w^4}{12}+\frac{w^2}{2}\right]_0^{t} =$$

$$= \frac{1}{2}\left[\frac{c\,t^4}{12}+\frac{t^2}{2}\right] = \frac{1}{2}\left[\frac{18\,t^4}{12}+\frac{t^2}{2}\right] =$$

$$= \frac{1}{2}\cdot\frac{1}{2}\left[18\cdot\frac{t^4}{6}+t^2\right] = \frac{1}{4}(3t^4+t^2)$$

$$f_Z(t) = 3t^3+\frac{1}{2}t \quad \text{за} \quad t\in(0,1)$$

не го разбирам

научи си зако бината ???

$\boxed{2}$ Магазинът работи от 10:00 до 18:00

Нека времената между клиентите под 30 год. са

$X_1, X_2, \ldots \sim Exp\left(\frac{1}{5}\right)$ и аналогично клиентите над 30 год.

$Y_1, Y_2, \ldots \sim Exp\left(\frac{1}{10}\right)$

$\rightarrow \mathbb{E}Y = \frac{1}{\lambda} \rightarrow 10 = \frac{1}{\lambda} \rightarrow \lambda = \frac{1}{10}$

а) Каква е вероятността магазинът да бъде посетен от поне 100 клиенти под 30 год. за 1 ден?

$\boxed{\mathbb{E}X_1 = 5}$

$DX_1 = \frac{1}{\left(\frac{1}{5}\right)^2} = \frac{1}{\frac{1}{25}} = 25 \rightarrow \boxed{DX_1 = 25}$

100 клиенти по 5 min = 500min

от 10:00 до 18:00 има $8 \cdot 60 = 480 \min$

$\mathbb{P}(X_1 + \ldots + X_{100} < 480) =$

$$= \mathbb{P}\left(\frac{X_1 + \ldots + X_{100} - 100 \cdot 5}{\sqrt{25}\sqrt{100}} < -\frac{20}{\sqrt{25}\sqrt{100}}\right) \overset{ЦГТ}{\approx}$$

$$-\frac{20}{5 \cdot 10} = -\frac{20}{50} = -\frac{2}{5} = -0,4$$

$= \Phi(-0,4) = 0,3446 \approx \boxed{34,45\%}$

$$= \frac{\ldots}{\frac{122,4}{3}} = \frac{\ldots}{40,8} = -0,49$$

$\boxed{3.}$ а) $\mathbb{P}($след 1000 стъпки x-координатата $> 10) = ?$

↳ Нека X_i е стъпката абсцисата и
Y_i е стъпката ординатата

$$X_i, Y_i = \begin{cases} (1,0) \\ (-1,0) \\ (0,1) \\ (0,-1) \end{cases} \quad \text{с вероятност } \frac{1}{4}$$

$$\mathbb{E}X_1 = \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot -1 + 0\cdot\frac{1}{2} = 0$$

$$\mathbb{E}X_1^2 = \frac{1}{4}\cdot 1^2 + \frac{1}{4}\cdot 1^2 = \frac{1}{2} \Rightarrow DX_1 = \frac{1}{2}$$

$$\mathbb{P}(X_1 + \dots + X_{1000} > 10) = \mathbb{P}\left(\frac{X_1 + \dots + X_{1000}}{\sqrt{\frac{1}{2}}\sqrt{1000}} > \frac{10}{\sqrt{500}}\right) \overset{ЦГТ}{\approx} \Phi(-0,45) \approx 32,636\ \%$$

в) Намерете разпределението на времето до първия клиент за деня.

$$Z_1 = \min(X_1, Y_1) \sim Exp\left(\frac{1}{5} + \frac{1}{10}\right) = Exp\left(\frac{3}{10}\right)$$

в) Ако Z_i е времето между $(i-1)$-ви и i-ти човек, тогава

$$Z_i \sim Exp\left(\frac{3}{10}\right) \Rightarrow EZ_i = \frac{1}{\frac{3}{10}} = \frac{10}{3}$$

$$DZ_i = \left(\frac{10}{3}\right)^2$$

$$P(Z_1 + \dots + Z_{150} \leq 480) = P\left(\frac{Z_1 + \dots Z_{150} - \overset{50}{\cancel{150}} \cdot \frac{10}{\cancel{3}}}{\sqrt{\left(\frac{10}{3}\right)^2} \cdot \sqrt{150}} \leq - \frac{20}{\sqrt{\left(\frac{10}{3}\right)^2}\sqrt{150}}\right)$$

$= 500$

$$\frac{-20}{\frac{10}{3} \cdot 12{,}24} =$$

ЦГТ

$$\approx \Phi(-0{,}49) \approx 0{,}3121 \approx \underline{31{,}21\ \%}$$