Контролно 2 ТН 2020/2021

1. $X \in [0,1]$ се полни

$Y \to$ се уплаши ребята

$$f_{X,Y}(x,y) = \begin{cases} 3x, & 0 \le y \le x \le 1 \\ 0, & \text{иначе} \end{cases}$$

а) $P(X < \frac{1}{2}, Y > \frac{1}{4}) = ?$

$$P(X < \tfrac{1}{2}, Y > \tfrac{1}{4}) = \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{\frac{1}{4}}^{x} 3x \, dy \, dx = \int_{\frac{1}{4}}^{\frac{1}{2}} 3x\left(x - \tfrac{1}{4}\right) dx =$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} 3x^2 - \frac{3x}{4} \, dx = \left[\frac{3x^3}{3} - \frac{3x^2}{8}\right]_{\frac{1}{4}}^{\frac{1}{2}} =$$

$$= \left(\frac{1}{2}\right)^3 - \frac{3 \cdot \left(\frac{1}{2}\right)^2}{8} - \left(\frac{1}{4}\right)^3 + \frac{3 \cdot \left(\frac{1}{4}\right)^2}{8} =$$

$$= \frac{1}{8} - \frac{3}{32} - \frac{1}{64} + \frac{3}{128} = \frac{16}{128} - \frac{12}{128} - \frac{2}{128} + \frac{3}{128} = \boxed{\frac{5}{128}}$$

б) $Cov(X,Y) \to Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X\mathbb{E}Y$

За $x \in [0,1]$

$$f_X(x) = \int_0^x 3x \, dy = 3x^2 \Rightarrow \mathbb{E}X = \int_0^1 x \, 3x^2 \, dx = \left[\frac{3x^4}{4}\right]_0^1 = \boxed{\frac{3}{4}}$$

За $y \in [0,1]$

$$f_Y(y) = \int_y^1 3x \, dx = \left[\frac{3x^2}{2}\right]_y^1 = \frac{3}{2} - \frac{3y^2}{2} = \frac{3}{2}(1-y^2)$$

$$\Rightarrow \mathbb{E}Y = \frac{3}{2}\int_0^1 y(1-y^2) \, dy = \frac{3}{2}\int_0^1 y - y^3 \, dy = \frac{3}{2}\left[\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \cdot \frac{3}{2} =$$

$$= \boxed{\frac{3}{8}}$$

$$\mathbb{E}XY = \int_0^1 \int_0^x xy\, 3x\, dy\, dx = \int_0^1 3x^2 \int_0^x y\, dy\, dx =$$

$$= \int_0^1 3x^2 \cdot \frac{x^2}{2}\, dx = \int_0^1 \frac{3}{2} x^4\, dx = \frac{3}{2}\left[\frac{x^5}{5}\right]_0^1 = \frac{3}{2} \cdot \frac{1}{5} = \boxed{\frac{3}{10}}$$

$$\Rightarrow \quad Cov(X,Y) = \mathbb{E}XY - \mathbb{E}X\,\mathbb{E}Y =$$

$$= \frac{3}{10} - \frac{3}{4} \cdot \frac{3}{8} = \frac{3}{10} - \frac{9}{32} = \frac{48}{160} - \frac{45}{160} = \boxed{\frac{3}{160}}$$

2. $10 \times 100 = 1000$ диода Вадим

Времето на изправяване при произволна диода $\sim Exp\left(\frac{1}{10}\right)$, $\mathbb{E}X = 10$

$X \sim Exp\left(\frac{1}{10}\right)$

$$P(X > 3) = \int_3^\infty \frac{1}{10} e^{-\frac{1}{10}x}\, dx = \left[-e^{-\frac{1}{10}x}\right]_3^\infty = e^{-3/10} = p$$

$\Rightarrow \quad i = 1, 2, \ldots, 1000$

$X_i = 1$ диод i работи след 3 часа годишно $= \begin{cases} 1, & \text{работи} \\ 0, & \text{иначе} \end{cases}$

ЦГТ

$$P(Y_1 + \ldots + Y_{1000} < 750) = P\left(\frac{Y_1 + \ldots + Y_{1000} - np}{\sqrt{np(1-p)}} < \frac{750 - np}{\sqrt{np(1-p)}}\right) =$$

$\downarrow$ D

$$\approx P(0{,}1 < 0{,}66) = \Phi(0{,}66) \approx 74{,}54\%.$$

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2. $f_{X,Y}(x,y) = \begin{cases} c(x^2 + e^y x), & 0 < x, y < 1 \\ 0, & \text{иначе} \end{cases}$

а) c, $\mathrm{Cov}(X,Y)$

$$1 = c \int_0^1 \int_0^1 (x^2 + e^y x)\, dx\, dy = c \int_0^1 \frac{1}{3} + \frac{1}{2} e^y \, dy =$$

$$= c\left[\frac{1}{3} + \frac{1}{2}(e-1)\right] = \frac{3(e-1)}{6}$$

$$\Rightarrow c = \frac{6}{3(e-1)}, \text{ за } x \in (0,1)$$

2. средна температура $-0{,}5\,^\circ C$

X = „средна температура през януари в София в градуси по Целзий" $\sim N(0,1)$

$P(X \geq 20) = \frac{1}{100} = 1\,\%$

Нека σ^2, $\sigma \geq 0$ е дисперсията на X, $X \sim N(-0{,}5, \sigma^2)$

$$\frac{1}{100} = P(X \geq 20) = P\left(\frac{X + 0{,}5}{\sigma} \geq \frac{20{,}5}{\sigma}\right) = P\left(N(0,1) \geq \frac{20{,}5}{\sigma}\right) = 1 - \Phi\left(\frac{20{,}5}{\sigma}\right)$$

Ако знаем, че $\Phi(x) = P(N(0,1) \leq x)$

$\Phi(z) = 0{,}99$

$$\Rightarrow \frac{20{,}5}{\sigma} \approx 2{,}33 \quad \text{и} \quad \sigma \approx 8{,}80$$

$$\Rightarrow P(X \leq -15) = P\left(\frac{X + 0{,}5}{8{,}80} \leq \frac{-15{,}5}{8{,}80}\right) \approx P(N(0,1) \leq -1{,}65) \approx 0{,}0495$$

$$\approx 4{,}95\,\%$$