Контролно ВиС 12.06.2022

$\boxed{1.}$ $f_{X,Y}(x,y) = \begin{cases} cx+1 & , x,y \geq 0, x+y \leq 1 \\ 0 & , \text{иначе} \end{cases}$ $\rightarrow y \leq 1-x$

a) c, $cov(X,Y)$

$$1 = \int_0^1 \int_0^{1-x} cx+1 \, dy \, dx = \int_0^1 cx+1 \int_0^{1-x} dy \, dx =$$

$$= \int_0^1 (cx+1)(1-x) \, dx = \int_0^1 cx - cx^2 + 1 - x \, dx =$$

$$= \left[c\frac{x^2}{2} - c\frac{x^3}{3} + x - \frac{x^2}{2} \right]_0^1 = \left[\frac{c}{2} - \frac{c}{3} + 1 - \frac{1}{2} \right] =$$

$$= \frac{3c}{6} - \frac{2c}{6} + \frac{1}{2} = \frac{c}{6} + \frac{1}{2} = \frac{c}{6} + \frac{3}{6} = \frac{3+c}{6}$$

$$\Rightarrow 3+c = 6 \rightarrow c = 6-3$$

$$\boxed{c=3}$$

$$EXY = \int_0^1 \int_0^{1-x} (xy)(cx+1) \, dy \, dx = \int_0^1 x(cx+1) \int_0^{1-x} y \, dy \, dx =$$

$$= \int_0^1 x(cx+1) \left[\frac{y^2}{2} \right]_0^{1-x} dx = \int_0^1 x(cx+1) \frac{(1-x)^2}{2} \, dx =$$

$$= \frac{1}{2} c \int_0^1 x^2 (1-x)^2 \, dx + \frac{1}{2} \int_0^1 x(1-x)^2 \, dx =$$

$$= \frac{1}{2} c \int_0^1 x^2 (1-2x+x^2) \, dx + \frac{1}{2} \int_0^1 x(1-2x+x^2) \, dx =$$

$$= \frac{1}{2} c \left[\frac{x^3}{3} - 2\frac{x^4}{4} + \frac{x^5}{5} \right]_0^1 + \frac{1}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1 =$$

$$= \frac{3}{2} \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right] + \frac{1}{2} \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right] = \frac{1}{2} \cdot \frac{3}{20} + \frac{1}{2} \cdot \frac{2}{24} = \frac{1}{20} + \frac{1}{24} = \frac{6}{120} + \frac{5}{120} = \boxed{\frac{11}{120}}$$

$-\frac{5}{10} + \frac{2}{10}$

→ За X: за $x \in [0,1]$: $f_X(x) = \int_0^{1-x} cx+1 \, dy = (cx+1)(1-x)$

$$\mathbb{E}X = \int_0^1 x(cx+1)(1-x)\, dx = \int_0^1 (cx^2+x)(1-x)\, dx =$$

$$= \int_0^1 cx^2 - cx^3 + x - x^2 \, dx = \left[\frac{cx^3}{3} - \frac{cx^4}{4} + \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 =$$

$$= \left[\frac{3}{3} \underbrace{- \frac{3}{4} + \frac{1}{2}}_{-\frac{3}{4} + \frac{2}{4}} - \frac{1}{3} \right] = \frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \boxed{\frac{5}{12}}$$

→ $\mathbb{E}X^2 = \int_0^1 x^2(cx+1)(1-x)\, dx = \int_0^1 (cx^3+x^2)(1-x)\, dx =$

$$= \int_0^1 cx^3 - cx^4 + x^2 - x^3 \, dx = \left[\frac{cx^4}{4} - \frac{cx^5}{5} + \frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 =$$

$$= \left[\frac{3}{4} - \underbrace{\frac{3}{5} + \frac{1}{3}}_{-\frac{9}{15} + \frac{5}{15} = -\frac{4}{15}} - \frac{1}{4}\right] = \frac{2}{4} - \frac{4}{15} = \frac{1}{2} - \frac{4}{15} = \frac{15}{30} - \frac{8}{30} = \boxed{\frac{7}{30}}$$

$$\to DX = ĒX^2 - (ĒX)^2 = \frac{7}{30} - \left(\frac{5}{12}\right)^2 = \frac{7}{30} - \frac{25}{144} = \frac{43}{720} \approx \underline{\underline{0.0597}}$$

→ За Y: за $y \in [0,1]$: $f_Y(y) = \int_0^{1-y} cx+1 \, dx = c\left[\frac{x^2}{2}+x\right]_0^{1-y} =$

$= c\left[\frac{(1-y)^2}{2}+(1-y)\right] = \frac{c}{2}(1-y)^2+1-y$

⇒ $\mathbb{E}Y = \frac{c}{2}\int_0^1 y(1-y)^2 \, dy + \int_0^1 y(1-y) \, dy =$

$= \frac{c}{2}\int_0^1 y(1-2y+y^2) \, dy + \int_0^1 y-y^2 \, dy =$

$= \frac{c}{2}\left[\frac{y^2}{2}-\frac{2y^3}{3}+\frac{y^4}{4}\right]_0^1 + \left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1 =$

$= \frac{c}{2}\left[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right] + \left[\frac{1}{2}-\frac{1}{3}\right] =$

$- \frac{3}{2}\cdot\frac{2}{3} + \frac{3}{2}\left[\frac{2}{4}+\frac{1}{4}\right] + \left[\frac{3}{6}-\frac{2}{6}\right] =$

$= -1 + \frac{9}{8} + \frac{1}{6} = -1 + \frac{27}{24} + \frac{4}{24} = -1 + \frac{31}{24} = \frac{-24}{24} + \frac{31}{24} = \boxed{\frac{7}{24}}$

→ $\mathbb{E}Y^2 = \frac{c}{2}\int_0^1 y^2(1-y)^2 \, dy + \int_0^1 y^2(1-y) \, dy =$

$= \frac{c}{2}\int_0^1 y^2-2y^3+y^4 + \int_0^1 y^2-y^3 \, dy =$

$= \frac{c}{2}\left[\frac{y^3}{3}-\frac{2y^4}{4}+\frac{y^5}{5}\right]_0^1 + \left[\frac{y^3}{3}-\frac{y^4}{4}\right]_0^1 =$

$= \frac{3}{2}\left[\frac{1}{3}-\frac{2}{4}+\frac{1}{5}\right] + \left[\frac{1}{3}-\frac{1}{4}\right] = \frac{1}{2}-\frac{9}{20}+\frac{1}{12} = \frac{30-27+5}{60} = \frac{8}{60} =$

$\frac{3}{2}\left(-\frac{10}{20}+\frac{4}{20}\right) = \frac{3}{2}\cdot -\frac{6}{20}$ $= \boxed{\frac{2}{15}}$

⇒ $DY = \mathbb{E}Y^2 - (\mathbb{E}Y)^2 = \frac{2}{15} - \left(\frac{7}{24}\right)^2 = \frac{139}{2880} \approx 0,0483$

⇒ $Cor(X,Y) = \frac{\mathbb{E}XY - \mathbb{E}X\mathbb{E}Y}{\sqrt{DX DY}} = \frac{\frac{11}{120}-\frac{5}{12}\cdot\frac{7}{24}}{\sqrt{\frac{43}{720}\cdot\frac{139}{2880}}} \approx \underline{\underline{0,5562}}$

б) $E(X|Y=1/2)$

$$f_{X,Y}(x|\tfrac{1}{2}) = \frac{f_{X,Y}(x,\frac{1}{2})}{f_Y(\frac{1}{2})} = \frac{cx+1}{\frac{7}{8}} \quad \text{за} \quad x \in [0, \tfrac{1}{2}]$$

$$f_Y(y) = \frac{c}{2}(1-y^2) + 1 - y = \frac{c}{2}(1-2y+y^2) + 1 - \frac{1}{2} =$$

$$= \frac{c}{2}\left(1 - 2\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2\right) + \frac{1}{2} =$$

$$= \frac{3}{2}\cdot\frac{1}{4} + \frac{1}{2} = \frac{3}{8} + \frac{1}{2} = \frac{3}{8} + \frac{4}{8} = \boxed{\frac{7}{8}}$$

$$E[X|Y=\tfrac{1}{2}] = \int_0^{\frac{1}{2}} x \cdot \frac{cx+1}{7/8}\,dx = \frac{8}{7}\int_0^{\frac{1}{2}} cx^2 + x \,dx =$$

$$= \frac{8}{7}\left[c\frac{x^3}{3} + \frac{x^2}{2}\right]_0^{\frac{1}{2}} = \frac{8}{7}\left[c\cdot\frac{(\frac{1}{2})^3}{3} + \frac{(\frac{1}{2})^2}{2}\right] =$$

$$= \frac{8}{7}\left[c\cdot\frac{\frac{1}{8}}{3} + \frac{\frac{1}{4}}{2}\right] = \frac{8}{7}\left[c\cdot\frac{1}{24} + \frac{1}{8}\right] = \frac{8}{7}\left[3\cdot\frac{1}{24} + \frac{1}{8}\right] =$$

$$= \frac{8}{7}\cdot\frac{2}{8} = \boxed{\frac{2}{7}} \approx 0,2857$$

$\boxed{2.}$ $N(\mu, \sigma^2) \to N(\mu, 10^2)$

$\to$ 15% от консерв. със съдържат по-малко от 250g грах

а) параметъра μ

$$\mathbb{P}(N(\mu, 10^2) < 250) = 15\%$$

$$\mathbb{P}\left(N(0,1) < \frac{250-\mu}{10^2}\right) = 15\%$$

$15\% = 0,15$

От таблицата: $\dfrac{250-\mu}{10^2} \approx -1,04$

$$\Rightarrow \frac{250-\mu}{10} = -1,04$$

$$250-\mu = -1,04 \cdot 10$$

$$-\mu = -1,04 \cdot 10 - 250 \quad |(-1)$$

$$\mu = 1,04 \cdot 10 + 250 = 250 + 10,4 = \boxed{260,4}$$

б) процента консерви които съдържат повече от 280g грах.

$$\mathbb{P}(N(\mu, 10^2) > 280) = \mathbb{P}\left(N(0,1) > \frac{280-\mu}{10}\right) \approx$$

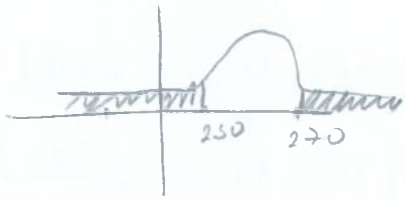
$$\approx 1 - \Phi(1,96) = \Phi(-1,96) = 0,0250 \approx \underline{2,5\%}$$

$$\frac{280-260,4}{10} = \frac{19,6}{10} = 1,96$$

След промяна да допуснем, че моделът е $N(250, \sigma^2)$

в) Намерете σ, ако знаете, че 97% от консервите съдържат между 230 и 270 грама грах.

$$P(230 \le N(250, \sigma^2) \le 270) =$$



$$= 1 - 2P(N(250, \sigma^2) < 230) =$$

$$= 1 - 2P\left(N(0,1) < \frac{230-250}{\sigma}\right) \overset{\text{по условие}}{=} 95\%$$

$\Rightarrow$ Търсим σ, така че $\Phi\left(-\frac{20}{\sigma}\right) = 0{,}015$

$$\Rightarrow -\frac{20}{\sigma} \approx -2{,}17 \Rightarrow \boxed{\sigma = 9{,}2166}$$

???

3. 4 задачи → 80%, 70%, 60%, 40%

↳ 10 точки решено и 0 иначе

оценка $2 + \text{точки}/10$

100 души

а) $\mathbb{P}$(средният резултат на първа задача да е под 8 т) = ?

→ Нека X_i е резултатът на зад 1 $\sim 10 \cdot Ber\left(\frac{8}{10}\right) \to 80\%$

$\mathbb{E}X_i = 10 \cdot \frac{8}{10} = 8$ и $DX = 100 \cdot \frac{8}{10} \cdot \frac{2}{10} = 16$

Нека $S_n = X_1 + \dots + X_n$. От ЦГТ $\frac{S_n - n \cdot 8}{4\sqrt{n}} \xrightarrow{d} N(0,1)$

$$\mathbb{P}\left(\frac{S_{100}}{100} < 8\right) = \mathbb{P}(S_{100} < 800) = \mathbb{P}\left(\frac{S_{100} - 100 \cdot 8}{4\sqrt{100}} < 0\right) \overset{ЦГТ}{\approx}$$

$$\mathbb{P}(N(0,1) < 0) = \frac{1}{2}$$

б) Очакваната средна оценка

Нека $Y_i = \#$точки на студент $i \sim X_i^{(1)} + X_i^{(2)} + X_i^{(3)} + X_i^{(4)}$, където $X_i^{(1)} \sim 10 \cdot Ber\left(\frac{8}{10}\right)$, $X_i^{(2)} \sim 10 \cdot Ber\left(\frac{7}{10}\right)$, $X_i^{(3)} \sim 10 \cdot Ber\left(\frac{6}{10}\right)$, $X_i^{(4)} \sim 10 \cdot Ber\left(\frac{4}{10}\right)$, са нез. сл. вел., които кореспондират с точките на i по съответните задачи.

$$\mathbb{E}Y_i = 10\left(\frac{8}{10} + \frac{7}{10} + \frac{6}{10} + \frac{4}{10}\right) = 25$$

$$DY_i = 100\left(\frac{8 \cdot 2 + 7 \cdot 3 + 6 \cdot 4 + 4 \cdot 6}{100}\right) = 85$$

Оценката на i е $Z_i := 2 + \frac{Y_i}{40}$ → защото са 4 задачи

$$\Rightarrow \mathbb{E}Z_i = \mathbb{E}\left[2 + \frac{Y_i}{10}\right] = 4,5$$

$$DZ_i = D\left[2 + \frac{Y_i}{10}\right] = \frac{DY_i}{100} = \frac{17}{20}$$

в) Вероятността средната оценка да е поне 1% по-висока от очакваната

↳ от ЦГТ: $\dfrac{\dfrac{\sum_{i=1}^{n} z_i}{n} - 4,5}{\sqrt{17/20}\,/\sqrt{n}} \xrightarrow[n\to\infty]{d} N(0,1)$

$\Rightarrow \mathbb{P}\left(\dfrac{z_1 + \dots + z_{100}}{100} - 4,5 > \dfrac{1}{100} \cdot 4,5 \right) =$

$= \mathbb{P}\left(\dfrac{\dfrac{\sum_{i=1}^{100} z_i}{100} - 4,5}{\sqrt{17/2000}} > \dfrac{4,5/100}{\sqrt{17/2000}} \right) \overset{ЦГТ}{\approx} \mathbb{P}(N(0,1) > 0,49) \approx \underline{\underline{31,21\,\%}}$