Контролно 2 21.01.2021

$\boxed{1.}$ два червени и два сини зара

X - броят на падналите се четни числа върху червените

Y - броят на падналите се петици върху четирите зара

↳ $X \in \{0,1,2\}$

$Y \in \{0,1,2,3,4\}$

а) съвместното разпределение на X и Y

$\boxed{2.}$ $f(x,y) = \begin{cases} c(x+y)^2, & 0<x<y<1 \\ 0, & \text{иначе} \end{cases}$

а) константата c

$$1 = \iint_{\mathbb{R}} = c(x+y)^2\,dx\,dy = c\int_0^1 \left(\int_0^y (x+y)^2 dx\right) dy =$$

$$= c\int_0^1 \left(\int_0^y (x^2+2xy+y^2)\,dx\right) dy =$$

$\to \int 2xy\,dx = 2y\int x\,dx = 2y\frac{x^2}{\cancel{2}} = yx^2$

$\int y^2 dx = y^2\int 1\,dx = y^2 x$

$$= c\int_0^1 \left[\frac{x^3}{3} + x^2y + y^2x\right]_0^y dy =$$

$$= c\int_0^1 \frac{y^3}{3} + y^3 + y^3\,dy =$$

$\int_0^1 \frac{y^3}{3} = \frac{1}{3}\int_0^1 y^3 = \frac{1}{3}\left[\frac{y^4}{4}\right]_0^1 = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$

$$= c\int_0^1 \frac{1}{12} + \frac{y^4}{4} + \frac{y^4}{4} =$$

$$= c\left[\frac{1}{12} + \frac{y^4}{4} + \frac{y^4}{4}\right]_0^1 = c\left[\frac{1}{12} + \frac{1}{4} + \frac{1}{4}\right] = c\left(\frac{1}{12} + \frac{3}{12} + \frac{3}{12}\right) =$$

$$= c\cdot\frac{7}{12} \to \boxed{c = \frac{12}{7}}$$

б) да се намери $f_{X+Y}(x,y)$ и средната стойност $E(Y|X=\frac{1}{2})$

↳ Полагаме: $\begin{cases} Z_1 = X+Y \\ Z_2 = X \end{cases} \Rightarrow \begin{cases} X = Z_2 \\ Y = Z_1 - X = Z_1 - Z_2 \end{cases}$

$$|J| = \begin{vmatrix} \frac{\partial X(z_1,z_2)}{\partial z_1} & \frac{\partial X(z_1,z_2)}{\partial z_2} \\ \frac{\partial Y(z_1,z_2)}{\partial z_1} & \frac{\partial Y(z_1,z_2)}{\partial z_2} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = 0-1 = |-1| = 1$$

$$f_{Z_1,Z_2}(z_1,z_2) = f_{X,Y}(z_2, z_1-z_2)\cdot 1 = c(x+y)^2\cdot 1 =$$

$$= \frac{12}{7}(\cancel{z_2}+z_1-\cancel{z_2})^2\cdot 1 = \boxed{\frac{12}{7}\cdot z_1^2}$$

↳ $$f_X(x) = \int_R f_{X,Y}(x,w)\,dw = \int_x^1 c(x+w)^2\,dw =$$

$$= c\int_x^1 x^2+2xw+w^2\,dw = c\left[x^2w + \frac{\cancel{2}xw^2}{\cancel{2}} + \frac{w^3}{3}\right]_x^1 =$$

$$= c\left[x^2+x+\frac{1}{3}-x^3-x^3-\frac{x^3}{3}\right] = c\left[x^2+x+\frac{1}{3}-\frac{7}{3}x^3\right] =$$

$$= \frac{12}{7}x^2+\frac{12}{7}x+\frac{12}{7}\cdot\frac{1}{3}-\frac{12}{\cancel{7}}\cdot\frac{\cancel{7}}{\cancel{3}}x^3 =$$

$$= \frac{4}{7}+\frac{12}{7}x+\frac{12}{7}x^2-4x^3$$

$$f_X\left(\frac{1}{2}\right) = \frac{4}{7}+\frac{12}{7}\cdot\frac{1}{2}+\frac{12}{7}\cdot\left(\frac{1}{2}\right)^2-4\cdot\left(\frac{1}{2}\right)^3 =$$

$$= \frac{4}{7}+\frac{6}{7}+\frac{3}{7}-\frac{1}{2} = \frac{13}{7}-\frac{1}{2} = \frac{26}{14}-\frac{7}{14} = \frac{19}{14}$$

$$E\left[Y \mid X=\tfrac{1}{2}\right] = \frac{1}{f_X\left(\frac{1}{2}\right)} \int_R y\, c \left(\tfrac{1}{2}+y\right)^2 dy =$$

$$= \frac{14}{19} \int_{\frac{1}{2}}^{1} y\, \frac{12}{7} \left(\tfrac{1}{2}+y\right)^2 dy = \frac{\cancel{14}^{2}}{19} \cdot \frac{12}{\cancel{7}} \int_{\frac{1}{2}}^{1} y \left(\tfrac{1}{4}+y+y^2\right) dy =$$

$$= \frac{24}{19} \int_{\frac{1}{2}}^{1} \frac{y}{4}+y^2+y^3 \, dy = \frac{24}{19} \left[\frac{y^2}{8} + \frac{y^3}{3} + \frac{y^4}{4} \right]_{\frac{1}{2}}^{1} =$$

$$= \frac{24}{19} \left[\frac{1}{8} + \frac{1}{3} + \frac{1}{4} - \frac{1}{32} - \frac{1}{24} - \frac{1}{64} \right] =$$

$$= \frac{24}{19} \left[\frac{3+8+6-1}{24} - \frac{2-1}{64} \right] = \frac{24}{19} \left[\frac{16}{24} - \frac{3}{64} \right] =$$

$$= \frac{24}{19} \left[\frac{2}{3} - \frac{3}{64} \right] = \frac{24}{19} \cdot \frac{128-9}{192} = \frac{24}{19} \cdot \frac{119}{192} = \frac{7 \cdot 17}{8 \cdot 19} \approx \underline{0,782}$$