$X$ е сл. вел. $cX$ сл. вел ?

↪ $\{cX \le t\} \overset{?}{\in} \mathcal{A}$ → дали е измеримо

$\{cX \le t\} = \{X \le \frac{t}{c}\} \in \mathcal{A}$

↓

$X$ е сл. вел $\Rightarrow cX$ е сл. вел.

↪ $\{X+Y < t\} = \{X < t-Y\} =$

$= \bigcup_{q \in \mathbb{Q}} \{X < q < t-Y\} =$

↓

някакво $q$ рационално число

$= \bigcup_{q \in \mathbb{Q}} \{X < q\} \cap \{q < t-Y\} \in \mathcal{A}$

→ от някое свойство на алгебрата въвеждаме тогава $q$, че с него доказваме, че двете събития са от $\mathcal{A} \Rightarrow \bigcup$ на тяхното $\cap \in \mathcal{A}$.

Доказателство, че $1_H$ е сл. вел.

↪ $\{1_H = 1\} = \{\omega \in \Omega : 1_H(\omega) = 1\} = H \in \mathcal{A}$

$\{1_H = 0\} = \{\omega \in \Omega : 1_H(\omega) = 0\} = H^c \in \mathcal{A}$

$\Rightarrow 1_H$ е сл. величина.

1\. $X$ – # ези  + хвърля се 2 пъти монетата

$Y$ – # тури

$\rho_{X,Y} = ?$

$Y = 2 - X$

$|\rho_{XY}| = 1$

$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, 2-X) = \mathrm{Cov}(X,2) - \mathrm{Cov}(X,X) =$$

$$= 0 - DX = -DX$$

$$\rho_{X,Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{DX}\,\sqrt{DY}} = \frac{-DX}{\sqrt{DX}\,\sqrt{DY}} = -1$$

4\. $X \sim Ge(p)$  $X \perp\!\!\!\perp Y$

$Y \sim Ge(p)$

$$\hookrightarrow P(\min\{X,Y\} \geq k) = P(X \geq k, Y \geq k) =$$

$$= (1-p)^k (1-p)^k = (1-2p+p^2)^k = (1-(2p-p^2))^k$$

$$\sim Ge(2p-p^2)$$

$$\hookrightarrow P(\max\{X,Y\} < k) = P(X<k, Y<k) = P(1-P(X \geq k))P(1-P(Y \geq k)) =$$

$$= (1-(1-p)^k)(1-(1-p)^k) = \{1-(1-p)^k-(1-p)^k+(1-p)^{2k} =$$

$$= 1-2(1-p)^k+(1-p)^{2k} =$$

$$= (1-(1-p)^k)+(1-p)^{2k} = \underline{\underline{(1-(1-p)^k)^2}}$$

$X_1, \dots, X_n$ i.i.d $\sim$ Gen

$X_{max} := \max\{X_1, \dots, X_n\}$

$X_{min} := \min\{X_1, \dots, X_n\}$

$\hookrightarrow F_{X_{max}}(x) = \mathbb{P}(X_{max} \le x) =$

$= \mathbb{P}(X_1 \le x, X_2 \le x, \dots, X_n \le x) = \mathbb{P}(X_1 \le x)\,\mathbb{P}(X_2 \le x) \cdots \mathbb{P}(X_n \le x)$

$= \left(\mathbb{P}(X_1 \le x)\right)^n$

$\hookrightarrow F_{X_{min}}(x) = \mathbb{P}(X_{min} \le x) =$

$= 1 - \mathbb{P}(X_{min} > x) = 1 - \mathbb{P}(X_1 > x, \dots, X_n > x) =$

$= 1 - \left(\mathbb{P}(X_1 > x)\right)^n = 1 - \left(1 - F_{X_1}(x)\right)^n$

→ Ковариацията на случайна величина и константа е 0, защото константите не варират
⇒ $\mathrm{cov}(X, 2) = 0$

→ $\mathrm{cov}(X, X) = DX$ → ковариацията на сл. вел. със самата себе си е дисперсията на тази величина

→ линейност на ковариацията
$$\mathrm{cov}(X + Z, Y) = \mathrm{cov}(X, Y) + \mathrm{cov}(Z, Y)$$

→ доказателство на Коши-Шварц

→ $|\rho(X,Y)| \leq 1$

↳ ще работим с центриране и нормиране моменти

$$0 \leq E[(\bar{X} \pm \bar{Y})^2] = E\bar{X}^2 \pm 2E\bar{X}\bar{Y} + E\bar{Y}^2 =$$
$$= 1 + 2\rho(X,Y) + 1 = 2 + 2\rho(X,Y) = 2(1 \pm \rho(X,Y))$$

$$\Rightarrow 0 \leq 2(1 \pm \rho(X,Y))$$
$$0 \cdot 2 \leq 1 \pm \rho(X,Y)$$
$$1 \geq |\rho(X,Y)|$$

→ $|\rho(X,Y)| = 1$

$$1 \leq E[(\bar{X} \pm \bar{Y})^2] = E\bar{X}^2 \pm 2E\bar{X}\bar{Y} + EY^2 = 1 \pm 2\rho(X,Y) + 1 =$$
$$= 2 \pm 2\rho(X,Y) = 2(1 \pm \rho(X,Y))$$

$$\Rightarrow 1 = 2(1 \pm \rho(X,Y)) =$$
$$2 = 1 \pm \rho(X,Y)$$
$$2 - 1 = |\rho(X,Y)| \Rightarrow 1 = |\rho(X,Y)|$$

(4) $X$ = „брой успешни опита"

$N \sim Poi(\lambda)$

$X \mid N = n \sim Bin(n, p)$

пораждаща

$$g_X(s) = \mathbb{E}\left[\mathbb{E}[s^X \mid N]\right] = \sum_{k=0}^{\infty} \underbrace{\mathbb{E}[s^k \mid N = n]}_{Bin(n,p)} \underbrace{\mathbb{P}(N = n)}_{Poi(\lambda)} =$$

$$= \sum_{k=0}^{M} (sp + (1-p))^k \cdot \frac{e^{-\lambda} \lambda^k}{k!} = e^{-\lambda + \lambda(sp - (1-p))} =$$

$$= e^{\lambda p(s-1)}$$

(4) $N \sim Bin(M, q)$

$X \mid N = n \sim Bin(n, p)$

пораждаща

$$g_X(s) = \mathbb{E}\left[\mathbb{E}[s^X \mid N]\right] = \sum_{k=0}^{\infty} \underbrace{\mathbb{E}[s^k \mid N = n]}_{Bin(n,p)} \underbrace{\mathbb{P}(N = n)}_{Bin(M,q)} =$$

$n \quad p$

$$= \sum_{k=0}^{M} (sp + (1-p))^k \cdot \binom{M}{k} \cdot q^k \cdot (1-q)^{M-k} =$$

$$= \left((sp + (1-p)) \cdot q \cdot (1-q)\right)^M = (spq + (1-pq))^M$$

$X_1, \dots, X_n$ са i.i.d сл. вел.

пример от ръководство

$$X_{max} := \max\{X_1, \dots, X_n\}$$
$$X_{min} := \min\{X_1, \dots, X_n\}$$

Изразете: $F_{X_{max}}$ и $F_{X_{min}}$ през $F_{X_1}$

$$\underline{F_{X_{max}}(x)} = \mathbb{P}(X_{max} \le x) =$$
$$= \mathbb{P}(X_1 \le x, \dots, X_n \le x) = \mathbb{P}(X_1 \le x)\mathbb{P}(X_2 \le x) \dots \mathbb{P}(X_n \le x) =$$
$$= (\mathbb{P}(X_1 \le x))^n$$

$$\underline{F_{X_{min}}(x)} = \mathbb{P}(X_{min} \le x) =$$
$$= 1 - \mathbb{P}(X_{min} > x) =$$
$$= 1 - \mathbb{P}(X_1 > x, \dots, X_n > x) = 1 - (\mathbb{P}(X_1 > x))^n =$$
$$= 1 - (1 - F_{X_1}(x))^n$$

$X_1 \sim Bin(n_1, p)$
$X_2 \sim Bin(n_2, p)$
$X_1 \perp\!\!\!\perp X_2$
$X_1 + X_2 \sim ?$ → разпределението на сумата?

$$g_{X_1+X_2}(s) = g_{X_1}(s)\, g_{X_2}(s) = (q + sp)^{n_1}(q + sp)^{n_2} =$$
$$= (q+sp)^{n_1+n_2} \to$$ която е пораждащата функция на $Bin(n_1+n_2, p)$

$X \sim Ge(p)$ $\quad X \perp\!\!\!\perp Y$

$Y \sim Ge(p)$

a) $\min\{X, Y\}$

б) $\max\{X, Y\}$

$\hookrightarrow$ $\mathbb{P}(\min\{X,Y\} \geq k) = \mathbb{P}(X \geq k, Y \geq k) =$

$= (1-p)^k (1-p)^k = [(1-p)(1-p)]^k = (1-p-p+p^2)^k =$

$= (1-2p+p^2)^k = (1-(2p-p^2))^k \Rightarrow \min(X,Y) \sim Ge(2p-p^2)$

$\hookrightarrow$ $\mathbb{P}(\max\{X,Y\} < k) = \mathbb{P}(X<k, Y<k) =$

$= (1-\mathbb{P}(X\geq k))(1-\mathbb{P}(Y \geq k)) =$

$= (1-(1-p)^k)(1-(1-p)^k) =$

$= 1-(1-p)^k-(1-p)^k+(1-p)^{2k} =$

$= 1-2(1-p)^k+(1-p)^{2k} =$

$= (1-(1-p)^k)+(1-p)^{2k} =$

$= ((1-(1-p)^k)^2$

4. $X_1 \sim Po(\lambda_1)$

$X_2 \sim Po(\lambda_2)$

$X_1 \perp\!\!\!\perp X_2$

$X_1+X_2 \sim$ → разпределението на сумата

$$\hookrightarrow g_{X_1+X_2}(s) = g_{X_1}(s)\, g_{X_2}(s) = e^{\lambda_1(s-1)}\, e^{\lambda_2(s-1)} = e^{(\lambda_1+\lambda_2)(s-1)} \Rightarrow$$

$$X_1+X_2 \sim Po(\lambda_1+\lambda_2)$$

стр. 17 от ЗРР + пример за $f(x,y)$???