Q 10.1

①: "$\Leftarrow$" : $\|f_n - f\|_p \geq \max\{\|f_n\|_p - \|f\|_p,\ \|f\|_p - \|f_n\|_p\}$

so ~~$\|f_n\|$~~ $\|f_n - f\|_p \to 0 \Rightarrow \|f_n\|_p - \|f\|_p \to 0$

as $n \to \infty$

②: "$\Rightarrow$"

We have known: $|f_n - f|^p \leq 2^p(\cancel{|f_n|} |f_n|^p + |f|^p)$

We want to $\boxed{\text{apply DCT.}}$ so first. let $h_n = 2^p \cdot |f_n|^p + 2^p \cdot |f|^p$.

1): $f_n \to f$ a.e $\Rightarrow h_n \to 2^{p+1}|f|^p$ a.e

2): $\|f_n\|_p \to \|f\|_p \Rightarrow \lim_{n\to\infty} \int h_n = \lim_{n\to\infty} \int (2^p |f_n|^p) + (2^p |f|^p) \cdot d\mu$.

$= 2^p \cdot \lim_{n\to\infty} \left(\int |f_n|^p + \int |f|^p\right) = 2^p \cdot 2\|f\|_p^p < \infty$

$= \int 2^{p+1} \cdot |f|^p$.

So Apply DCT on $|f_n - f|^p$:

$\int \lim_{n\to\infty} |f_n - f|^p = \lim_{n\to\infty} \int |f_n - f|^p = 0$.

So $\|f_n - f\|_p \to 0$.

2.

① Let $\mathcal{F} = \{ \sum_{i=1}^{n} c_i I_{\prod_{j=1}^{d} (a_{ij}, b_{ij})} : n \in \mathbb{Z}^+, c_i$ is rational vector. $a_{ij}, b_{ij} \in \mathbb{Q} \}$

② Claim: $C_c(\mathbb{R}^d)$ is dense in $L_p(\mathbb{R}^d)$

Proof: we know simple functions are dense in $L_p(\mathbb{R}^d)$

so just need to show:

$\forall \varepsilon > 0, A \subseteq \mathbb{R}^d, \exists h \in C_c(\mathbb{R}^d)$. s.t. $\|h - I_A\|_p < \varepsilon$

Lebesgue measure tells me:

$\exists K \subseteq A \subseteq O$, k is compact. O is open.

$\mu(K) < \mu(O) + \varepsilon^p$.

$\mathbb{R}^n$ is locally compact and Hausdorff.

By Urysohn: $\exists h \in C(\mathbb{R}^d, [0,1])$, s.t.

$I_K \leq h \prec O$

So such $h \in C_c(\mathbb{R}^d)$, and

$$\|h - I_A\|_p^p = \int_{\mathbb{R}^d} |h - I_A|^p \cdot d\mu$$

$$= \int_K |h - I_A|^p d\mu + \int_{O \setminus K} |h - I_A|^p d\mu + \int_{O^c} |h - I_A|^p d\mu$$

$$= \int_{O \setminus K} |h - I_A|^p d\mu$$

$$\leq \mu(O \setminus K) = \varepsilon^p.$$

Let $T := \{ \sum_{1}^{n} c_i \, \mathbb{1}_{(a_{ij}, b_{ij})} \quad n \in \mathbb{Z}^+, \ a_{ij}, b_{ij} \in \mathbb{Q} \}$.

③ Claim: T is dense on $C_c(\mathbb{R}^d)$

Proof: $\forall f \in C_c(\mathbb{R}^d)$ let A = a cube with rational coordinate containing support (f).

Now we $\boxed{\text{split } A}$ into countable small cubes. $Q_1, \cdots, Q_n, \cdots$

these $\{Q_n\}$ s.t
1) the coordinates of Q_n are rational
2) $\{Q_i\}$ are disjoint
3) $\sum_{1}^{\infty} \mu(Q_i) - \mu(A) < \varepsilon$.
4) $\sup_{x \in Q_i} f(x) - \inf_{x \in Q_i} f(x) < \delta$
5): $\bigsqcup_{1}^{\infty} Q_n \supset A$

(Here actually I'm not spliting A. I'm spliting an apprximation of A)

Let $g(x) := a_i$ when $x \in Q_i$. $a_i \in [\inf_{x \in Q_i} f(x), \sup_{x \in Q_i} f(x)]$

$$\therefore \|f-g\|_p^p = \int_{\mathbb{R}} |f-g|^p \, d\mu \le \int_{\bigsqcup_1 Q_n} |f-g|^p \, d\mu$$

$$\le \int_{\bigsqcup_1^{\infty} Q_n} \delta^p \cdot d\mu \quad \le \delta^p \cdot (\mu(A) + \varepsilon)$$

Pick δ and ε properly, we can get:

For $\forall \lambda > 0$. $\exists g \in T$ s.t $\|f-g\|_p \le \lambda$.

$\therefore$ ✓

④ F is dense on T. & F is countable.

□ !!!!

10.3

a): $\|fg\|_1 = \int |fg| \cdot d\mu$

$\leq \int |f| \cdot \|g\|_\infty \, d\mu$

$= \|g\|_\infty \cdot \|f\|_1$

Obviously, equality holds iff when $|g(x)| = \|g\|_\infty$, for $f \in L^1, g \in L^\infty$

b. $\forall f$

1): $\|f\|_\infty = 0 \iff \mu(\{x \mid |f(x)| > 0\}) = 0$

$\iff x = 0$ a.e

2): If $\|f\|_\infty = A$, $\|g\| = B$

then,

$\mu(\{x: |f(x)| > A\}) = 0$

$\mu(\{x: |g(x)| > B\}) = 0$

$\mu(\{x: |f(x)+g(x)| > A+B\})$

$\leq \mu(\{x: |f(x)|+|g(x)| > A+B\})$

$= 1 - \mu(\{x: |f(x)| + |g(x)| \leq A+B\})$

$\mu\{x: |f(x)| + |g(x)| \leq A+B\})$

$\geq \mu\{x: |f(x)| \leq A, |g(x)| \leq B\}$

$= 1$

$\therefore \mu(\{x: |f(x)+g(x)| > A+B\}) = 0$

$A+B \geq \|f+g\|_\infty$

(C):

$\Rightarrow$: $\mu(\{x: |f_n(x) - f(x)| > 0\}) \to 0$

$\Rightarrow f_n \to f$ uniformly on E, $\mu(E^c) = 0$.

$\Leftarrow$:

$\|f - f_n\|_\infty = \inf\{A < \infty, \mu(\{|f-f_n| > A\}) = 0\}$

$= \inf\{A < \infty, \mu(\{|f-f_n| > A\} \cap E) = 0\}$.

$= 0$ ~~of~~ since $f \to f_n$ uniformly on E

(d):

let $\{f^{(n)}\}$ be a Cauchy seq in l^∞.

$\{f^n\}$ is Cauchy $\Leftrightarrow \forall \varepsilon>0, \exists N$. s.t $\|f^n - f^m\|_\infty < \varepsilon$

$\Leftrightarrow \forall \varepsilon>0, \exists N$. s.t.

$\mu(\{x: |f^n(x) - f^m(x)| > \varepsilon\}) = 0$

$\therefore \forall x, \{f^n(x)\}$ is Cauchy, thus converge.

Assuing that $f^n(x) \to f(x)$ as $n\to\infty$. almost surely

So $f^n \to f$ uniformly a.s.

so $\|f_n - f\|_\infty \to 0$

3

(e):

$f \in L^\infty \Rightarrow \forall \varepsilon>0, \exists N \in \mathbb{Z}^+$. s.t $(N-1)\varepsilon \le \|f\|_\infty < N\varepsilon$

let $g_\varepsilon(x) = \sum_{i=0}^{N-1} i\varepsilon \cdot I_{i\varepsilon \le f(x) < (i+1)\varepsilon}$.

then g_ε is simple,

$\|f-g\|_\infty \le \varepsilon$

$\therefore$ Simple function are dense in L^∞

10.4 Let $M = \|f\|_\infty$

①: $\forall q \geq p$:

$$\int |f|^p \, d\mu = \int |f|^q \cdot |f|^{p-q} \cdot d\mu$$
$$\leq \int |f|^q \cdot M^{p-q} \cdot d\mu$$
$$\leq M^{p-q} \cdot \|f\|_q^q \quad < \infty$$

$\therefore f \in L^p$. $\|f\|_p \leq M^{1-\frac{q}{p}} \cdot \left(\|f\|_q^q\right)^{\frac{1}{p}}$.

Since $\|f\|_q^q < \infty$. as $p \to \infty$. we get:

$$\|f\|_p \leq M$$

②: From the definition of $\|\cdot\|_\infty$, we know:

For any $\varepsilon > 0$. we have:

~~$\mu(\{f \geq \|f\|_\infty - \varepsilon\}) > 0$~~ $\mu(\{x : f(x) \geq \|f\|_\infty - \varepsilon\}) > 0$

Let $E = \{x : f(x) \geq \|f\|_\infty - \varepsilon\}$.

$= \{x : f(x) \geq M - \varepsilon\}$

$\therefore \int_X |f|^p \, d\mu \geq \int_E |f|^p \, d\mu \geq \mu(E) \cdot |M - \varepsilon|^p$.

$\therefore \|f\|_p \geq (\mu(E))^{\frac{1}{p}} \cdot |M - \varepsilon|$

Fix ε. then $\mu(E)$ also fixed.

as $p \to \infty$, we have:

$\|f\|_p \geq |M - \varepsilon|$. for any $\varepsilon > 0$

$\therefore$ ① + ② $\Rightarrow$ $\|f\|_p = M$ as $p \to \infty$

10.5

① T is a tpo of X^*

Pf: 1): ϕ is open and compact $\subseteq X$, $X^* \setminus X^* = \phi$

so $\phi \in T$, $X^* \in T$

2): For any $\bigcup_{\alpha \in A} O_\alpha$ (O_α is open).

If: $\forall \alpha$, $\infty \notin O_\alpha$. then $\bigcup_{\alpha \in A} O_\alpha$ is a open set $\subseteq X$.

so $\bigcup_{\alpha \in A} O_\alpha \in T$

If: $\exists \alpha^0 \in A$. $\infty \in O_{\alpha^0}$. $O_{\alpha^0} \in T \Rightarrow X^* \setminus O_{\alpha^0}$ is compact.

$\left(\bigcup_{\alpha \in A} O_\alpha\right)^c = \bigcap_{\alpha \in A} O_\alpha^c \subseteq O_{\alpha^0}^c$ is compact

So $\bigcup_{\alpha \in A} O_\alpha \in T$

3): For any $\bigcap_{i=1}^N O_i$. where O_i is open in X^*.

If $\cancel{\forall}\exists i$, $\infty \notin O_i$. then $\bigcap_{i=1}^N O_i$ is open in X

$\therefore \bigcap_1^N O_i \in T$

Else: suppose $\cancel{\exists i_0}\ \forall i \in \{1 \cdots n\}$. ~~st~~ $\infty \in \cancel{O_{i_0}}\ O_i$

then $\left(\bigcap_1^N O_i\right)^c = \bigcup_1^N O_i^c = \bigcup_{i=1}^N$ (compact set)

$=$ compact set

$\therefore \bigcap_1^N O_i \in T$.

② (X^*, T) is compact.

Pf: For any open covering $\{O_\alpha\}_{\alpha \in A}$ of X^*.

$\exists \alpha_1$ s.t $O_{\alpha_1} \ni \infty$.

Considering $\{O_\alpha\}_{\alpha \in A, \alpha \neq \alpha_1}$: It's an open covering of $O_{\alpha_1}^c$

$O_{\alpha_1}^c$ is compact $\Rightarrow \exists$ finite subcover $O_1, O_2, \cdots O_N$

So $O_{\alpha_1}, O_1, O_2 \cdots O_N$ ~~is~~ are finite subcover of X^*

③ (X^*, T) is Hausdorff

$\forall x, y \in X^*.\ x \neq y$

1) $x, y \in X$ ✓ (Because X is LCH, open sets in X $\in T$).

2): If, WLOG, $x = \infty$.

$\begin{cases} X \text{ is LCH} \\ y \in X \end{cases} \Rightarrow \exists\ O$ is open in X. s.t. $y \in O$, $X \setminus O$ is compact

let $A =$ such O, $B = (X \setminus O) \cup \{\infty\}$. $X^* \setminus B$ is compact

So both A, B is open in X^*. $\begin{cases} A \cap B = \phi. \\ x \in B \\ y \in A \end{cases}$ ✓

④: Inclusion map $i: X \to X^*$ is an embedding

Let $T_1 = \{U \cap i(X) : U \in T\} = \{U \cap X : U \in T\}$ $(i(X) = X)$

~~和~~ T_1 is the topology of X

So i is ~~an~~ an embedding

⑤: extends continuously $\Rightarrow f = g + c$: $g \in C_0(X)$. $f(\infty) = c$.

Let $f_1(x) = \begin{cases} f(x) & \text{if } x \in X. \\ c & \text{if } x = \infty,\ c \text{ is a constant.} \end{cases}$

Let $g = f_1 - \cancel{f}c$.

$\forall \varepsilon > 0$. $\{x \in X : |g(x)| \geq \varepsilon\} = \{x \in X : |f(x) - c| \geq \varepsilon\}$

$= X \setminus \{x \in X : |f(x) - c| < \varepsilon\}$

f_1 is continuous + $B(c, \varepsilon)$ is open $\Rightarrow$ $\{x \in X^* : |f_1(x) - c| < \varepsilon\}$ is open

And this open set contains ∞, so $X^* \setminus \{x \in X^* : |f_1(x) - c| < \varepsilon\}$

is compact, so $X \setminus \{x \in X : |f(x) - c| < \varepsilon\}$ is cpt.

So $\{x \in X : |g(x)| \geq \varepsilon\}$ is compact. $g \in C_0(X)$.

⑥ $f = g + c \Rightarrow f_1$ is continuous on X^*

For any open $V \in \mathbb{C}$.

$f_1^{-1}(\check{V}) = \{\infty\} \cup f^{-1}(V) = \cancel{g^{-1}(V) \cup \{\infty\}}.\ g^{-1}(V-c) \cup \{\infty\}.$

$g \in C_0(X) \Rightarrow g^{-1}(V-c)$ is open in X

$\Leftrightarrow g^{-1}\left((V-c)^c\right)$ is closed in X.

$\Leftrightarrow \left(g^{-1}(V-c)\right)^c$ is closed in X.

$g^{-1}\left((V-c)^c\right) = \{x \in X : g(x) \in (V-c)^c\}$

1) If ~~Since~~ $c \in V$. so $\exists$ a open ball $B(0,\delta) \subset V-c$.

$\therefore g^{-1}\left((V-c)^c\right) \subseteq \{x \in X : g(x) \in B(0,\delta)^c\} = \{x \in X : |g(x)| \geq \varepsilon\}.$

$g \in C_0(X) \Rightarrow \{x \in X : |g(x)| \geq \varepsilon\}$ is cpt

$\Rightarrow g^{-1}\left((V-c)^c\right)$ is cpt.

$\Rightarrow \cancel{g^{-1}(V-c) \text{ is }}\ f_1^{-1}(V)$ is open in X^*.

2) If $c \notin V$. $f_1^{-1}(V) = f^{-1}(V)$ is open in X. ✓

⑦ Without LC, we can still show ~~Hausdorff~~ compact (see ②)

but we use LC when showing ~~compact~~. Hausdorff (see ③).

10.6

① Let $x^0=(\underbrace{0,0,0\cdots 0}_{n\text{ zeros}},1)$

We construct $f: S^n\setminus\{x^0\}\to\mathbb{R}^n$, where

$$f((x_1,x_2,\cdots x_{n+1}))=\frac{1}{1-x_{n+1}}(x_1,x_2,\cdots x_n)$$

Obviously f is well-defined.

② f is injective

Proof: $\forall x,y\in S^n\setminus\{x^0\}$ if $x\neq y$ but $f(x)=f(y)$

then
$$\begin{cases}\dfrac{x_1}{1-x_{n+1}}=\dfrac{y_1}{1-y_{n+1}}\\ \dfrac{x_2}{1-x_{n+1}}=\dfrac{y_2}{1-y_{n+1}}\\ \vdots\\ \dfrac{x_n}{1-x_{n+1}}=\dfrac{y_n}{1-y_{n+1}}\end{cases}\Rightarrow\begin{cases}x_1=\dfrac{1-x_{n+1}}{1-y_{n+1}}y_1\\ \vdots\\ x_n=\dfrac{1-x_{n+1}}{1-y_{n+1}}y_n\end{cases}$$

let $t:=\dfrac{1-x_{n+1}}{1-y_{n+1}}$. then $x_i=ty_i$ $(i\in\{1,2,\cdots n\})$

$$\text{so }\sum_{i=1}^{n+1}x_i^2=x_{n+1}^2+\sum_{i=1}^{n}t^2y_i^2=x_{n+1}^2+t^2\cdot\sum_1^n y_i^2$$
$$=x_{n+1}^2+t^2(1-y_{n+1}^2).$$

$$\text{Since }1=\sum_1^{n+1}x_i^2=x_{n+1}^2+\frac{(1-x_{n+1})^2}{(1-y_{n+1})^2}\cdot(1-y_{n+1}^2)$$
$$=x_{n+1}^2+\frac{(1-x_{n+1})^2}{1-y_{n+1}}\cdot(1+y_{n+1})$$
$$=\frac{x_{n+1}^2-x_{n+1}^2y_{n+1}}{1-y_{n+1}}+\frac{x_{n+1}^2-2x_{n+1}+1+x_{n+1}^2y_{n+1}-2x_{n+1}y_{n+1}+y_{n+1}}{1-y_{n+1}}$$

So $x_{n+1}^2-x_{n+1}-x_{n+1}y_{n+1}+y_{n+1}=0$

So $(x_{n+1}-y_{n+1})(x_{n+1}-1)=0$

And $x_{n+1}\neq 0$ So: $x_{n+1}=y_{n+1}$, so $t=1$, so $x=y$.

A contradiction! So f is injective

③ f is surjective.

$\forall y\in\mathbb{R}^n$. suppose $y=(y_1,\cdots y_n)$. And we now find whether there's a solution for $f(x)=y$.

$$f(x)=y\Leftrightarrow\begin{cases}t\,x_i=y_i, & i=1,2,\cdots n\\ t=\dfrac{1}{1-x_{n+1}}\end{cases}$$

1) If $\sum_1^n y_i^2 = 1$. then let $x = (y_1, y_2 \cdots y_n, 0)$

then $f(x) = y$ ✓

2): If $\sum_1^n y_i^2 \neq 1$. let $x_{n+1} = \dfrac{\sum_{i=1}^n y_i^2 - 1}{\sum_1^n y_i^2 + 1}$.

so $0 < x_{n+1} < 1$.

$\forall i \in \{1, 2, \cdots n\}$. let $x_i = y_i \cdot (1 - x_{n+1})$

so such x satisfies $f(x) = y$. And:

$$\sum_{i=1}^n x_i^2 + x_{n+1}^2 = (1 - x_{n+1})^2 \cdot \sum_{i=1}^n y_i^2 + x_{n+1}^2.$$

$$= \sum_{i=1}^n y_i^2 - 2x_{n+1} \sum y_i^2 + x_{n+1}^2 (\sum y_i^2 + 1)$$

$$= \sum_1^n y_i^2 - \frac{2(\sum y_i^2)^2 - 2\sum y_i^2}{\sum y_i^2 + 1} + \frac{(\sum_{i=1}^n y_i^2 - 1)^2}{\sum y_i^2 + 1}.$$

$$= 1.$$

so $x \in S \setminus \{x^0\}$

④ f is continuous.

It suffices to show f_i is continuous for all $i \in \{1, 2, \cdots n\}$. where f_i is i-th coordinate of f.

$$f_i^{-1}(a, b) = \{x_i \in \mathbb{R} : a < \frac{x_i}{1 - x_{n+1}} < b\} \cap S^n.$$

so if (a, b) is a open interval of $\mathbb{R}$, then $f_i^{-1}(a, b)$ is also open

⑤. Obviously, $\forall \{x^n\}$ in $S^n \setminus \{x^0\}$.

if $x^n \to x^0$. then $f(x^n) \to +\infty$.

so $f(x^0) = +\infty$, and thus we get the function.

$\uparrow$

($f(x^0)$ can be written as $+\infty$ and doesn't affect continuity).

10.7

For any open $E \subseteq U^*$

① $\infty \notin E$ then $\phi^{-1}(E) = E$ is open

② $\infty \in E$ then $\phi^{-1}(E) = X \setminus \phi^{-1}(U^* \setminus E)$

$\infty \in E \Rightarrow U^* \setminus E$ is compact.

Prob 10.5 $\Rightarrow$ U^* is Hausdorff $\Rightarrow$ $U^* \setminus E$ is closed.

$\phi^{-1}(U^* \setminus E) = U^* \setminus E$ is closed

(since $U^* \setminus E \subseteq U$).

So $X \setminus \phi^{-1}(U^* \setminus E)$ is open

$\phi^{-1}(E)$ is open.