Final Exam for Math5281

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Problem 1

Suppose $u \in L^1_{loc}(\Omega)$ is a very weak solution of the Laplacian equation in the sense that

$$\int_\Omega u(x)\Delta\varphi(x)\mathrm{d}x = 0 \text{ for all } \varphi \in C_c^\infty(\Omega).$$

Prove that (up to redefinition on a set of measure zero) u is smooth in Ω and satisfies $\Delta u = 0$ pointwisely in Ω.

Proof. Let u_ε denote the standard mollification of u defined on the dialated domain Ω_ε, and $u_\varepsilon \to u$ a.e. as $\varepsilon \to 0$. For each $\varepsilon > 0$ and each multiindex α, we have $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ and

$$D^\alpha(u_\varepsilon) = D^\alpha(\eta_\varepsilon * u) = D^\alpha\eta_\varepsilon * u.$$

Then we have

$$\Delta u_\varepsilon = \Delta\eta_\varepsilon * u = \int_\Omega \Delta\eta_\varepsilon(x-y)u(y)\,dy.$$

For any fixed $x \in \Omega_\varepsilon$, let $\varphi_x(y) = \eta_\varepsilon(x-y)$, then φ_x is bounded support on $B(x,\varepsilon)$ in Ω as a function of y, and $\varphi_x \in C_c^\infty(\Omega)$. Since u is a very weak solution and by definition, we have

$$\Delta u_\varepsilon(x) = \int_\Omega \Delta\varphi_x(y)u(y)\,dy = 0$$

for $x \in \Omega_\varepsilon$. Now choose $\varepsilon' > 0$ and define

$$u_{\varepsilon\varepsilon'} := \eta_{\varepsilon'} * u_\varepsilon.$$

Note that $\eta_{\varepsilon'}(x-y)$ is supported on Ω_ε whenever $x \in \Omega_{\varepsilon+\varepsilon'}$; in particular, for a fixed $x \in \Omega_{\varepsilon+\varepsilon'}$, $\eta_{\varepsilon'}(x-y)$ is supported on $B(x,\varepsilon')$. Since $\eta_{\varepsilon'}$ is radial and u_ε is harmonic in $\Omega_{\varepsilon+\varepsilon'}$ and hence satisfies the mean value properties there, for such x

fixed we compute in polar coordinates

$$\begin{aligned} u_{\varepsilon\varepsilon'}(x) &= \int_{\Omega_\varepsilon} \eta_{\varepsilon'}(x-y)u_\varepsilon(y)dy \\ &= \frac{1}{\varepsilon'^n}\int_{B(x,\varepsilon')} \eta\left(\frac{|x-y|}{\varepsilon'}\right)u_\varepsilon(y)dy \\ &= \frac{1}{\varepsilon'^n}\int_0^{\varepsilon'} \eta\left(\frac{r}{\varepsilon'}\right)\left(\int_{\partial B(x,r)} u_\varepsilon(y)dS(y)\right)dr \\ &= \frac{u_\varepsilon(x)n\alpha(n)}{\varepsilon'^n}\int_0^{\varepsilon'} r^{n-1}\eta\left(\frac{r}{\varepsilon'}\right)dr \\ &= \frac{u_\varepsilon(x)}{\varepsilon'^n}\int_{B(x,\varepsilon)} \eta\left(\frac{x-y}{\varepsilon'}\right)dy \\ &= u_\varepsilon(x)\int_{B(0,\varepsilon)} \eta_\varepsilon(y)dy \\ &= u_\varepsilon(x). \end{aligned}$$

Since convolution is associative, we find that for $x \in \Omega_{\varepsilon+\varepsilon'}$,

$$u_{\varepsilon'\varepsilon}(x) = u_{\varepsilon\varepsilon'}(x) = u_\varepsilon(x) \to u(x) \text{ a.e. as } \varepsilon \to 0$$

and

$$u_{\varepsilon'\varepsilon}(x) \to u_{\varepsilon'}(x) \text{ a.e. as } \varepsilon \to 0,$$

hence

$$u(x) = u_{\varepsilon'}(x) \quad \text{a.e.}$$

Since $u_{\varepsilon'}$ is smooth and harmonic in Ω_ε, so is u. Sending $\varepsilon' \to 0$, up to redefinition on a set of zero, u is smooth in Ω and satisfies $\Delta u = 0$ pointwisely in Ω. $\square$

Problem 2

Let $B_1 \subset \mathbb{R}^n$ with $n \geq 3$, and $u \in C^2\left(\overline{B_1}\backslash\{0\}\right)$ be a bounded harmonic function in $B_1\backslash\{0\}$. Prove that there exists a harmonic function v in B_1 such that

$$v \equiv u \quad \text{in } B_1\backslash\{0\}.$$

Proof. Let v be the solution of the Dirichlet problem

$$\begin{cases} -\Delta v = 0 & \text{in } B_1 \\ v(x) = u(x) & \text{on } \partial B_1. \end{cases}$$

Such a function v exists as it's given by the Poisson's formula. Then v is bounded in B_1 (since by maximum principle $|v| \leq \max_{|x|=1} u(x)$) and harmonic in B_1. Our goal is to show that $v = u$ in $B_1 \backslash \{0\}$.

Let $w = u - v$, then w is harmonic in $B_1 \backslash \{0\}$. Moreover, since w is bounded and $|x|^{2-n} \to \infty$ as $x \to 0$, we have

$$\lim_{x \to 0} \frac{w(x)}{|x|^{2-n} - 1} = 0.$$

So for fixed $x_* \in B_1 \backslash \{0\}$ and every $\varepsilon > 0$, there exists δ such that $0 < \delta < |x_*|$, and for all x satisfying $|x| \leq \delta$,

$$|w(x)| \leq \varepsilon \left(|x|^{2-n} - 1 \right).$$

Note that $\varepsilon \left(|x|^{2-n} - 1 \right)$ is a harmonic function and takes a value of 0 on ∂B_1, $|w(x)| \leq \varepsilon \left(|x|^{2-n} - 1 \right)$ on ∂B_1 and ∂B_δ. Again, weak maximum principle applied in the set $\delta \leq |x| \leq 1$ gives that

$$|w(x)| \leq \varepsilon \left(|x|^{2-n} - 1 \right)$$

for all $\delta \leq |x| \leq 1$. This is also true for x_*, *i.e.*

$$|w(x_*)| \leq \varepsilon \left(|x_*|^{2-n} - 1 \right).$$

We conclude by the arbitrariness of ε that $w(x_*) = 0$. And then by the arbitrariness of $x_* \in B_1 \backslash \{0\}$, we have $w = 0$ in $B_1 \backslash \{0\}$, or equivalently, $v = u$ in $B_1 \backslash \{0\}$. □

Problem 3

Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$.

Proof. Let $R > 0$ be such that

$$\mathrm{supp}(u) \subset B_R(0).$$

Define weak solution as following: $u \in H^1(\mathbb{R}^n)$ is a weak solution if

$$\int_{\mathbb{R}^n} Du \cdot Dv + c(u)v\,dx = \int_{\mathbb{R}^n} fv\,dx \tag{1}$$

for all $v \in H^1(\mathbb{R}^n)$. We choose $h \neq 0$ with $|h| \leq 1, k \in \{1, \ldots, n\}$ and define

$$v = -D_k^{-h} D_k^h u.$$

Then $v \in H^1(\mathbb{R}^n)$ and has compact support. Using this v in (1), using integration by parts and the difference quotients integration by parts formula, and using that u has compact support, we get

$$\int_{\mathbb{R}^n} \left(\left| D_k^h Du \right|^2 + D_k^h(c(u)) D_k^h u \right) \, dx = -\int_{\mathbb{R}^n} f \left(D_k^{-h} D_k^h u \right) \, dx.$$

By direct calculation,

$$\left(D_k^h(c(u))\right)(x) = \frac{c\left(u\left(x + he_k\right)\right) - c(u(x))}{h} = \eta_h(x) \frac{u\left(x + he_k\right) - u(x)}{h} = \eta_h(x) D_k^h u(x),$$

where

$$\eta_h(x) = \int_0^1 c'(ta + (1-t)b)\,dt,$$

and $a = u\left(x + he_k\right), b = u(x)$. Thus $c' \geq 0$ implies $\eta_h(x) \geq 0$, and we get

$$D_k^h(c(u)) D_k^h u = \eta_h(x) \left(D_k^h u(x) \right)^2 \geq 0.$$

Also, using that $0 < |h| \leq 1$ and applying Theorem 3 in §5.8.2 in the region $B_{R+1}(0)$, we get

$$\left| \int_{\mathbb{R}^n} f \left(D_k^{-h} D_k^h u \right) dx \right| = \left| \int_{B_{R+1}(0)} f \left(D_k^{-h} D_k^h u \right) dx \right| \leq \varepsilon \int_{\mathbb{R}^n} \left| D D_k^h u \right|^2 dx + \frac{C}{\varepsilon} \int_{\mathbb{R}^n} f^2 dx.$$

From these inequalities, we obtain

$$\int_{\mathbb{R}^n} \left| D_k^h Du \right|^2 dx \leq \varepsilon \int_{\mathbb{R}^n} \left| D_k^h Du \right|^2 dx + \frac{C}{\varepsilon} \int_{\mathbb{R}^n} f^2 dx.$$

Choosing $\varepsilon = \frac{1}{2}$,

$$\int_{\mathbb{R}^n} \left| D_k^h Du \right|^2 dx \leq C \int_{\mathbb{R}^n} f^2 dx.$$

This is true for each $k \in \{1, \ldots, n\}, 0 < |h| < 1$. Thus, applying Theorem 3 in §5.8.2 again in the region $B_{R+1}(0)$, we get

$$\int_{B_{R+1}(0)} \left|D^2 u\right|^2 dx \leq C \int_{\mathbb{R}^n} f^2 dx,$$

thus

$$\int_{\mathbb{R}^n} \left|D^2 u\right|^2 dx = \int_{B_{R+1}(0)} \left|D^2 u\right|^2 dx \leq C \int_{\mathbb{R}^n} f^2 dx.$$

Combining this with $u \in H^1\left(\mathbb{R}^n\right)$, we obtain $u \in H^2\left(\mathbb{R}^n\right)$. $\square$