Homework 3

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Problem 1

Let

$$Lu = -\sum_{i,j=1}^{n} \partial_j \left(a_{ij}\partial_i u\right) + cu.$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided

$$c(x) \geq -\mu \quad \forall x \in \Omega.$$

Proof. The corresponding bilinear form $B[\cdot,\cdot]$ is defined as

$$B[u,v] = \int_\Omega \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + cuv\,dx, \quad u,v \in H_0^1(\Omega),$$

then

$$|B[u,v]| \leq \sum_{i,j=1}^{n} \left\|a^{ij}\right\|_{L^\infty} \int_\Omega |Du||Dv|\,dx + \|c\|_{L^\infty} \int_\Omega |u||v|\,dx \leq \alpha \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

for some positive appropriate constant α. Assume $c(x) \geq -\mu$, where μ is a fixed constant to be assigned later. Since L is uniformly elliptic, there exists a constant $\theta > 0$ such that $\sum_{i,j=1}^{n} a^{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2$ for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, and

$$\begin{aligned}
\theta \int_\Omega |Du|^2\,dx &\leq \int_\Omega \sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j}\,dx \\
&= B[u,u] - \int_\Omega cu^2\,dx \\
&\leq B[u,u] + \mu \int_\Omega u^2\,dx \\
&\leq B[u,u] + C\mu \int_\Omega |Du|^2\,dx,
\end{aligned}$$

where in the last inequality we used Poincaré inequality with a constant $C > 0$. Let $\mu = \frac{\theta}{2C} > 0$, then we have

$$\frac{\theta}{2} \int_\Omega |Du|^2 dx \leq B[u,u].$$

On the other hand,

$$\begin{aligned}
\|u\|_{H_0^1(\Omega)}^2 &= \|u\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2 \\
&\leq (1+C)\|Du\|_{L^2(\Omega)}^2 \\
&\leq \frac{2}{\theta}(1+C)B[u,u].
\end{aligned}$$

Therefore, $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram Theorem. $\square$

Problem 2

A function $u \in H_0^2(\Omega)$ is a weak solution of this boundary value problem for the biharmonic equation

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

if

$$\int_\Omega \Delta u \Delta v \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x$$

for all $v \in H_0^2(\Omega)$. Given $f \in L^2(\Omega)$, prove that there exists a unique weak solution of (1).

Proof. Define

$$B[u,v] = \int_\Omega \Delta u \Delta v \, dx = \int_\Omega \sum_{i,j=1}^n u_{x_i x_i} v_{x_j x_j} \, dx, \quad u, v \in H_0^2(\Omega).$$

For $u, v \in H_0^2(\Omega)$,

$$|B[u,v]| = \left| \int_\Omega \sum_{i,j=1}^n u_{x_i x_i} v_{x_j x_j} \, dx \right| \le \sum_{i,j=1}^n \|D^2 u\|_{L^2(\Omega)} \|D^2 v\|_{L^2(\Omega)} \le \alpha \|u\|_{H_0^2(\Omega)} \|v\|_{H_0^2(\Omega)}$$

for some $\alpha > 0$. Using integration by parts and the boundary condition, for $u, v \in H_0^2(\Omega)$, we have

$$\begin{aligned}
\int_\Omega \left|D^2 u\right|^2 \, dx &= \int_\Omega \sum_{i,j=1}^n u_{x_i x_j} u_{x_i x_j} \, dx \\
&= -\int_\Omega \sum_{i,j=1}^n u_{x_j} u_{x_i x_j x_j} \, dx + \int_{\partial\Omega} \sum_{i,j=1}^n u_{x_j} u_{x_i x_j} \nu^i \, dS \\
&= \int_\Omega \sum_{i,j=1}^n u_{x_i x_i} u_{x_j x_j} \, dx + \int_{\partial\Omega} \sum_{i,j=1}^n u_{x_j} u_{x_i x_j} \nu^i \, dS \\
&= \int_\Omega \sum_{i,j=1}^n u_{x_i x_i} u_{x_j x_j} \, dx = \int_\Omega |\Delta u|^2 \, dx
\end{aligned}$$

and

$$\begin{aligned}
\|Du\|_{L^2(\Omega)}^2 &= \int_\Omega \sum_{i=1}^n u_{x_i} u_{x_i} \, dx = -\int_\Omega \sum_{i=1}^n u u_{x_i x_i} \, dx + \int_{\partial\Omega} \sum_{i=1}^n u u_{x_i} \nu \, dS \\
&= -\int_\Omega u \Delta u \, dx \le \varepsilon \|u\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon} \|\Delta u\|_{L^2(\Omega)}^2 \\
&\le \varepsilon C \|Du\|_{L^2(\Omega)}^2 + \frac{1}{4\varepsilon} \|\Delta u\|_{L^2(\Omega)}^2,
\end{aligned}$$

where in the last inequality we used Poincaré inequality and $C > 0$ is a constant. Let

$\varepsilon = \frac{1}{2C}$, then

$$\|Du\|_{L^2(\Omega)}^2 \leq C\|\Delta u\|_{L^2(\Omega)}^2.$$

Therefore,

$$\begin{aligned}\|u\|_{H_0^2(\Omega)}^2 &= \|u\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2 + \left\|D^2 u\right\|_{L^2(\Omega)}^2 \\ &\leq (1+C)\|Du\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2 \\ &\leq \left(1+C+C^2\right)\|\Delta u\|_{L^2(\Omega)}^2 \\ &= \left(1+C+C^2\right) B[u,u].\end{aligned}$$

Thus,

$$\frac{1}{1+C+C^2}\|u\|_{H_0^2(\Omega)}^2 \leq B[u,u], \quad u \in H_0^2(\Omega).$$

By Lax-Milgram theorem, there exists a unique elements $u_f \in H_0^2(\Omega)$ such that

$$B[u_f, v] = \int_\Omega fv\,dx$$

for all $v \in H_0^2(\Omega)$. $\square$

Problem 3

Assume Ω is connected. A function $u \in H^1(\Omega)$ is a weak solution of Neumann's problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2}$$

if

$$\int_\Omega \nabla u \nabla v \;\mathrm{d}x = \int_\Omega fv \;\mathrm{d}x$$

for all $v \in H^1(\Omega)$. Suppose $f \in L^2(\Omega)$. Prove that (2) has a weak solution if and only if

$$\int_\Omega f \;\mathrm{d}x = 0$$

Proof. If the Neumann's problem (2) has a weak solution, let $v = 1$, then $\int_\Omega f\,dx = 0$.

If $\int_\Omega f\,dx = 0$, we define

$$B[u,v] = \int_\Omega Du \cdot Dv\,dx, \quad u, v \in H^1(\Omega).$$

Consider the subspace of $H^1(\Omega)$ that

$$U = \left\{u \in H^1(\Omega) \mid \int_\Omega u\,dx = 0\right\}.$$

Let $l(u) = \int_\Omega u\,dx$, then l is a continuous linear functional on $H^1(\Omega)$ since the Ω is bounded, then $U = l^{-1}(\{0\})$ is closed in $H^1(\Omega)$, indicating U is a Hilbert space with the same norm defined in $H^1(\Omega)$. Note that

$$|B[u,v]| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} = \|u\|_U \|v\|_U$$

and by Poincaré inequality

$$\|u\|_{L^2(\Omega)}^2 = \int_\Omega |u - (u)_\Omega|^2 \, dx \leq C \|Du\|_{L^2(\Omega)}^2,$$

we have

$$\|u\|_U^2 = \|u\|_{L^2(\Omega)}^2 + \|Du\|_{L^2(\Omega)}^2 \leq (1+C)\|Du\|_{L^2(\Omega)}^2 = (1+C)B[u,u].$$

By Lax-Milgram theorem, given $f \in L^2(\Omega)$ with $\int_\Omega f \, dx = 0$, there exists a unique $u_f \in U$ such that

$$B\left[u_f, v\right] = (f, v)_{L^2(\Omega)}$$

for all $v \in U$.

Now let $v \in H^1(\Omega)$, then $v - (v)_\Omega \in U$. Then we have

$$\begin{aligned} \int_\Omega f v \, dx &= \int_\Omega f (v)_\Omega \, dx + \int_\Omega f \left(v - (v)_\Omega\right) \, dx \\ &= (v)_\Omega \int_\Omega f \, dx + B\left[u_f, v - (v)_\Omega\right] \\ &= B\left[u_f, v\right] = \int_\Omega Du_f \cdot Dv \, dx. \end{aligned}$$

Therefore, $u_f \in U \subset H^1(\Omega)$ is a weak solution. In general, $u_f + C$ is always a weak solution for any constant C. $\square$

Problem 4

Let u be a smooth solution of $Lu = -\sum_{i,j=1}^n a^{ij} u_{ij} = 0$ in Ω. Set

$$v := |\nabla u|^2 + \lambda u^2$$

Show that

$$Lv \leq 0 \text{ in } \Omega \text{ if } \lambda \text{ is large enough.}$$

Deduce

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\left(\|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\right).$$

Proof. Since $Lu = -\sum_i \sum_j a^{ij} u_{ij} = 0$, taking the partial derivative with respect to x_k on both sides, we have

$$\sum_i \sum_j a_k^{ij} u_{ij} + \sum_i \sum_j a^{ij} u_{ijk} = 0$$

for all $k \in \{1, \cdots, n\}$. Through direct calculation, we obtain

$$\begin{aligned}
-Lv &= \sum_i \sum_j a^{ij} v_{ij} \\
&= 2 \sum_i \sum_j \sum_k a^{ij}(u_{ik}u_{jk} + u_k u_{ijk}) + 2\lambda \sum_i \sum_j a^{ij}(u_i u_j + u u_{ij}) \\
&= 2 \sum_i \sum_j \sum_k a^{ij} u_{ik} u_{jk} - 2 \sum_k u_k \sum_i \sum_j a_k^{ij} u_{ij} + 2\lambda \sum_i \sum_j a^{ij} u_i u_j \\
&\geq 2\theta |D^2 u|^2 + 2\lambda\theta |Du|^2 - \left(2n \max_{i,j} \|Da^{ij}\|_{L^\infty(\Omega)}\right) |Du||D^2u|,
\end{aligned}$$

where the last inequality holds because

$$\sum_i \sum_j \sum_k a^{ij} u_{ik} u_{jk} \geq \theta \sum_k \sum_i u_{ik}^2 = \theta |D^2 u|^2$$

and

$$\begin{aligned}
\sum_k u_k \sum_i \sum_j a_k^{ij} u_{ij} &= \sum_i \sum_j u_{ij} \sum_k u_k a_k^{ij} \\
&\leq \sum_i \sum_j |u_{ij}| \sum_k |u_k||a_k^{ij}| \\
&\leq \sum_i \sum_j |u_{ij}||Du||Da^{ij}| \\
&\leq n \max_{i,j} \|Da^{ij}\|_{L^\infty(\Omega)} |Du||D^2u|.
\end{aligned}$$

If $Du \neq 0$, we set λ large enough such that

$$\lambda \geq \frac{\left(\frac{n}{\theta} \max_{i,j} \|Da^{ij}\|_{L^\infty(\Omega)}\right) |Du||D^2u| - |D^2u|^2}{|Du|^2},$$

then $Lv \leq 0$ in Ω; if $Du = 0$, we also have $Lv \leq 0$. We conclude that $Lv \leq 0$ in Ω if λ is large enough.

By the maximum principle, we have

$$\begin{aligned}
\|Du\|_{L^\infty(\Omega)}^2 &\leq \left\| |Du|^2 + \lambda u^2 \right\|_{L^\infty(\Omega)} = \left\| |Du|^2 + \lambda u^2 \right\|_{L^\infty(\partial\Omega)} \\
&\leq \|Du\|_{L^\infty(\partial\Omega)}^2 + \lambda \|u\|_{L^\infty(\partial\Omega)}^2 \\
&\leq \left(\|Du\|_{L^\infty(\partial\Omega)} + \sqrt{\lambda} \|u\|_{L^\infty(\partial\Omega)} \right)^2,
\end{aligned}$$

i.e. $\|Du\|_{L^\infty(\Omega)} \leq C \left(\|Du\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)} \right)$.

□

Problem 5

Assume Ω is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary value problem

$$\begin{cases} -\Delta u = 0 \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial \Omega \end{cases}$$

are constant functions.

Proof. For (a), multiplying both sides by u and integrating, we have

$$0 = -\int_{\Omega} u \Delta u \, dx = \int_{\Omega} |Du|^2 \, dx - \int_{\partial \Omega} \frac{\partial u}{\partial \nu} u \, dS = \int_{\Omega} |Du|^2 \, dx,$$

indicating $Du = 0$ a.e. in Ω. Since Ω is connected, the only solutions are constant functions.

For (b), assume by contradiction that u is not a constant. Then there exists a point $x_0 \in \bar{\Omega}$ such that u attains its maximum at x_0. If $x_0 \in \partial \Omega$, that is, $u(x_0) > u(x)$ for all $x \in \Omega$, then Hopf's lemma implies

$$\frac{\partial u}{\partial \nu}(x_0) > 0,$$

which is contradictory to the boundary condition $\frac{\partial u}{\partial \nu} = 0$ on $\partial \Omega$. Thus, u must obtain its maximum inside Ω and the strong maximum principle implies that $u \equiv C$. □