Homework 1

Yan Bokai
20909484

THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

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Problem 1

Let u be a positive harmonic function in the whole space $\mathbb{R}^n$. Prove that u is constant.

Proof. Since u is smooth, then $\Delta(D_{x_i}u) = 0$, that is, $D_{x_i}u$ is also harmonic in $\mathbb{R}^n$. Hence $D_{x_i}u$ satisfies the mean value property. By the divergence theorem we have

$$D_{x_i}u(x_0) = \frac{1}{|B_1|r^n}\int_{B_r(x_0)} D_{x_i}u(y)\,dy = \frac{1}{|B_1|r^n}\int_{\partial B_r(x_0)} u(y)\nu_i\,dS_y,$$

which implies

$$|D_{x_i}u(x_0)| \leq \frac{1}{|B_1|r^n}\int_{\partial B_r(x_0)} u(y)\,dS_y = \frac{n}{r}u(x_0).$$

Let $r \to \infty$ and we conclude that $Du(x_0) = 0$ for any $x_0 \in \mathbb{R}^n$. Then u is constant. $\square$

Problem 2

Let Ω be an open set in $\mathbb{R}^n$. Suppose $u \in C(\Omega)$ (just continuous!) and u satisfies the mean value property

$$u(x) = \fint_{\partial B_r(x)} u(y)dS_y$$

for every ball $B_r(x) \subset \Omega$. Prove that u is harmonic in Ω.

Proof. Let φ be a standard mollifier, and recall that $\varphi \in C_0^\infty(B_1(0))$ with $\int_{B_1(0)} \varphi dx = 1$, and φ is a radial function. Define $\psi(|x|) = \varphi(x)$, we have $\int_0^1 |\partial B_r|\psi(r)dr = 1$. For each $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$, and let $u^\varepsilon = \varphi_\varepsilon * u$ in $\Omega_\varepsilon = \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon\}$, then $u^\varepsilon \in C^\infty(\Omega_\varepsilon)$. Now for each $x \in \Omega_\varepsilon$ we have

$$\begin{aligned} u^\varepsilon(x) &= \int_\Omega \varphi_\varepsilon(x-y)u(y)dy = \frac{1}{\varepsilon^n}\int_{B_\varepsilon(x)} \varphi\left(\frac{x-y}{\varepsilon}\right)u(y)dy \\ &= \frac{1}{\varepsilon^n}\int_0^\varepsilon \psi(\frac{r}{\varepsilon})(\int_{\partial B_r(x)} u(y)dS_y)dr \\ &= \frac{1}{\varepsilon^n}u(x)\int_0^\varepsilon |\partial B_r|\psi(\frac{r}{\varepsilon})dr = u(x). \end{aligned}$$

Hence we get $u(x) = u^\varepsilon(x)$ for any $x \in \Omega_\varepsilon$. Then $u \in C^\infty(\Omega_\varepsilon)$ for any $\varepsilon > 0$, that is, $u \in C^\infty(\Omega)$.

If $\Delta u \neq 0$, there exists a ball $B_r(x) \subset \Omega$ such that, without loss of generality, $\Delta u > 0$ within the ball $B_r(x)$. Let

$$\phi(r) = \fint_{\partial B_r(x)} u(y)dS_y,$$

then

$$\phi'(r) = \frac{r}{n}\fint_{B_r(x)} \Delta u(y)dy > 0,$$

which is a contradiction with the condition. Hence $\Delta u = 0$ in Ω, *i.e.*, u is harmonic in Ω. $\square$

Problem 3

Let $\{u_k\}$ be a sequence of harmonic functions in an open set Ω. Assume that u_k converges to a function u uniformly over Ω.
(i) Prove that u is also harmonic in Ω.
(ii) Prove that ∇u_k converges to ∇u uniformly over every compact subset of Ω.

Proof. (i) For each ball $B_r(x) \subset \Omega$,

$$u(x) = \lim_{k\to\infty} u_k(x) = \lim_{k\to\infty} \fint_{B_r(x)} u_k(y)\, dy = \fint_{B_r(x)} \lim_{k\to\infty} u_k(y)\, dy = \fint_{B_r(x)} u(y)\, dy,$$

where the limit and integral are interchangeable because of uniform convergence. Then by the converse to mean value property, u is harmonic.

(ii) For each compact subset $V \subset \Omega$, define $r = \operatorname{dist}(V, \partial\Omega)$. Note that $(u_k)_{x_i} - u_{x_i} (i = 1, \cdots, n)$ is harmonic, we have

$$\begin{aligned}
|(u_k)_{x_i}(x) - u_{x_i}(x)| &= \left| \fint_{B_{r/2}(x)} (u_k)_{x_i} - u_{x_i}\, dy \right| \\
&= \left| \frac{2^n}{|B_1| r^n} \int_{\partial B_{r/2}(x)} (u_k - u)\nu_i\, dS \right| \\
&\le \frac{2n}{r} \|u_k - u\|_{L^\infty(\partial B_{r/2}(x))} \\
&\le \frac{2n}{r} \|u_k - u\|_{L^\infty(\Omega)}.
\end{aligned}$$

Then $\sup_{x\in V} |(u_k)_{x_i}(x) - u_{x_i}(x)| \le \frac{2n}{r}\|u_k - u\|_{L^\infty(\Omega)} \to 0$ as $k \to \infty$, that is, $(u_k)_{x_i} \rightrightarrows u_{x_i}$. Hence ∇u_k converges to ∇u uniformly over V. □

Problem 4

Let B_1 be the unit ball with center at the origin, and B_1^+ be the open half-ball $\{x \in \mathbb{R}^n : |x| < 1, x_n > 0\}$. Assume $u \in C(\overline{B_1^+})$ is harmonic in B_1^+, and $u = 0$ on $\partial B_1^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \ge 0 \\ -u(x_1, \cdots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in B_1$. Prove that v is harmonic in B_1.

Proof. By the definition of v, we can see that $v \in C(\overline{B_1})$. Note that by the Poisson's formula for the ball, we define

$$w(x) = \frac{1 - |x|^2}{n|B_1|} \int_{\partial B_1(0)} \frac{v(y)}{|x - y|^n}\, dS_y \text{ for } x \in B_1$$

and let $w = v$ on ∂B_1. By symmetry, $w = 0$ when $x_n = 0$, so $w \equiv v$ on $\partial B_1^+ \cap \{x_n = 0\}$. Notice that v is harmonic in B_1^+, by uniqueness of solution of Laplace equation, $w \equiv u \equiv v$ in $\overline{B_1^+}$. Similarly, note that v is also harmonic in $B_1 \backslash \overline{B_1^+}$ and $w \equiv v$ on $\partial\left(B_1 \backslash \overline{B_1^+}\right)$, $w \equiv v$ in $\overline{B_1} \backslash B_1^+$ by uniqueness.

Since $w \equiv v$ in B_1 and w is harmonic, we conclude v is harmonic. □

Problem 5

Let Ω be an open set in $\mathbb{R}^n$ with $n \geq 3$, and u be a harmonic function in Ω. Let $\xi \in \mathbb{R}^n$ and $\lambda > 0$. Define

$$u_{\xi,\lambda}(x) := \frac{\lambda}{|x-\xi|^{n-2}} u\left(\xi + \frac{\lambda^2(x-\xi)}{|x-\xi|^2}\right).$$

Prove that $u_{\xi,\lambda}$ is harmonic in its domain.

Proof. Let $g(x) = \frac{\lambda}{|x-\xi|^{n-2}}$, $f(x) = \xi + \frac{\lambda^2(x-\xi)}{|x-\xi|^2}$, then $v(x) = g(x)u(f(x)) = u_{\xi,\lambda}(x)$. Denote $v_i = v_{x_i}$, $v_{ij} = v_{x_i x_j}$, $g_i = g_{x_i}$, $g_{ij} = g_{x_i x_j}$, $f = (f^1, \ldots, f^n)^t$, $f_i = f_{x_i}$, $f_{ij} = f_{x_i x_j}$, $f_i^k = f_{x_i}^k$, $f_{ij}^k = f_{x_i x_j}^k$. Now compute

$$v_i = g_i u(f) + g \sum_{k=1}^n u_k(f) f_i^k = g_i u(f) + g Du \cdot f_i,$$

then

$$\begin{aligned} v_{ii} &= g_{ii} u(f) + 2g_i \sum_{k=1}^n u_k(f) f_i^k + g \sum_{k=1}^n u_k(f) f_{ii}^k + \sum_{k=1}^n (\sum_{j=1}^n u_{kj}(f) f_i^j) f_i^k \\ &= g_{ii} u(f) + 2g_i Du \cdot f_i + g Du \cdot f_{ii} + g f_i D^2 u f_i^t, \end{aligned}$$

and hence

$$\Delta v = \sum_{i=1}^n v_{ii} = \Delta g u(f) + 2DuDf(Dg)^t + gDu \cdot \Delta f + gTr((Df)^t D^2 u Df). \tag{1}$$

The convention is that f is a column vector, Dg and Du are row vectors. Since $g(x+\xi)$ is a mutiple of the fundamental solution, $\Delta g = 0$ for $x \neq \xi$. Note that

$$f^k = \frac{x_k - \xi_k}{|x-\xi|^2} \quad f_i^k = \frac{\delta_{ik}}{|x-\xi|^2} - \frac{2(x_i - \xi_i)(x_k - \xi_k)}{|x-\xi|^4}$$

where δ_{ij} is 1 if $i = j$ and 0 otherwise, so

$$Df = \frac{1}{|x-\xi|^2}(I - \frac{2(x-\xi)(x-\xi)^t}{|x-\xi|^2})$$

where $x - \xi$ is the column vector. Now compute,

$$(Df)^t Df = |x-\xi|^{-4}(I - 4|x-\xi|^{-2}(x-\xi)(x-\xi)^t + 4|x-\xi|^{-4}(x-\xi)(x-\xi)^t(x-\xi)(x-\xi)^t) = |x-\xi|^{-4} I.$$

Thus, we have $(Df)^t = |x-\xi|^{-4}Df^{-1}$, then

$$Tr((Df)^t D^2uDf) = |x-\xi|^{-4}Tr((Df)^{-1}D^2uDf) = |x-\xi|^{-4}Tr(D^2u) = |x-\xi|^{-4}\Delta u = 0.$$

Since

$$f^k_{ij} = -2|x-\xi|^{-4}(\delta_{ij}(x_k-\xi_k)+\delta_{ik}(x_j-\xi_j)+\delta_{jk}(x_i-\xi_i))+8|x-\xi|^{-6}(x_k-\xi_k)(x_i-\xi_i)(x_j-\xi_j),$$

we have

$$\begin{aligned}\Delta f^k &= \sum_{i=1}^n f^k_{ii} = \sum_{i=1}^n(-2|x-\xi|^{-4}((x_k-\xi_k)+2\delta_{ik}(x_i-\xi_i))+8|x-\xi|^{-6}(x_k-\xi_k)(x_i-\xi_i)^2)\\ &= -2(n+2)|x-\xi|^{-4}(x_k-\xi_k)+8|x-\xi|^{-4}(x_k-\xi_k)\\ &= 2(2-n)|x-\xi|^{-4}(x_k-\xi_k).\end{aligned}$$

Therefore, $\Delta f = 2(2-n)(x-\xi)|x-\xi|^{-4}$, and then

$$gDu\cdot\Delta f = 2(2-n)|x-\xi|^{-n-2}\lambda Du\cdot x.$$

On the other hand,

$$g_i = -(n-2)\lambda(xi-\xi_i)|x-\xi|^{-n} \quad (Dg)^t = (2-n)\lambda(x-\xi)|x-\xi|^{-n}$$

Thus, refer to (1), we compute

$$\begin{aligned}2DuDf(Dg)^t &= 2Du\frac{1}{|x-\xi|^2}(I-\frac{2(x-\xi)(x-\xi)^t}{|x-\xi|^2})(2-n)\lambda(x-\xi)\frac{1}{|x-\xi|^n}\\ &= -2(2-n)|x-\xi|^{-n-2}\lambda Du\cdot x.\end{aligned}$$

Apply those equation to (1), we have $\Delta v = 0$, so $u_{\xi,\lambda}$ is harmonic in its domain. □

Problem 6

Let $\Omega\subset\mathbb{R}^n$ be a bounded smooth open set, and $u\in C^2(\bar\Omega)$ solve the equation

$$\begin{cases}-\Delta u = f & \text{in } \Omega,\\ u = g & \text{on } \partial\Omega.\end{cases}$$

Prove that there exists a positive constant C , which depends only on n and Ω, such that

$$\max_{\bar\Omega}|u| \le C\left(\max_{\bar\Omega}|f| + \max_{\partial\Omega}|g|\right).$$

Proof. We say $v\in C^2(\bar\Omega)$ is *subharmonic* if $-\Delta v \le 0$ in Ω. We first prove for subharmonic

v that

$$v(x) \le \fint_{B_r(x)} v\,dy \text{ for all } B_r(x) \subset \Omega.$$

As in the proof in the mean value theorem, we define

$$\Phi(r) = \frac{1}{n|B_1|r^{n-1}} \int_{\partial B_r(x)} v(y)\,dS_y,$$

then

$$\Phi'(r) = \frac{1}{n|B_1|r^{n-1}} \int_{B_r(x)} \Delta v(y)\,dy \ge 0.$$

Therefore $\Phi(r) \ge \Phi(\varepsilon)$ for all $r \ge \varepsilon > 0$, and so

$$v(x) = \lim_{\varepsilon \to 0} \Phi(\varepsilon) \le \Phi(r) = \fint_{\partial B_r(x)} v(y)\,dS_y.$$

Employing polar coordinates, we have

$$\fint_{B_r(x)} v(y)\,dy = \frac{1}{|B_1|r^n} \int_0^r \left(\int_{\partial B_s(x)} v(y)\,dS_y \right) ds \ge v(x) \frac{n}{r^n} \int_0^r s^{n-1} ds = v(x).$$

We then prove $\max_{\bar{\Omega}} v = \max_{\partial\Omega} v$ for subharmonic v. Define $v_\varepsilon = v + \varepsilon|x|^2$ for $\varepsilon > 0$. Then for each $x \in \Omega$, $\Delta v_\varepsilon = \Delta v + 2n\varepsilon \ge 2n\varepsilon > 0$ since v is subharmonic and $\varepsilon > 0$. Assume there exists $x_0 \in \Omega$ such that $v_\varepsilon(x_0) = \max_{\bar{\Omega}} v_\varepsilon$, then $\Delta v_\varepsilon(x_0) = Tr\left(D^2 v_\varepsilon(x_0)\right) \le 0$ by the property of the maximum point, which is a contradiction. Therefore, v_ε cannot attain its maxima within Ω. We conclude that

$$\max_{\bar{\Omega}} v_\varepsilon = \max_{\partial\Omega} v_\varepsilon.$$

Since we know that Ω is bounded, we assume $\Omega \subset B_R(0)$ for some $R > 0$. Then

$$\max_{\bar{\Omega}} v \le \max_{\bar{\Omega}} v_\varepsilon = \max_{\partial\Omega} v_\varepsilon \le \max_{\partial\Omega} v + \max_{\partial\Omega} \varepsilon|x|^2 \le \max_{\partial\Omega} v + \varepsilon R^2$$

Let $\varepsilon \to 0$, then $\max_{\bar{\Omega}} v \le \max_{\partial\Omega} v$. Since $\partial\Omega \subset \bar{\Omega}$, we conclude $\max_{\bar{\Omega}} v = \max_{\partial\Omega} v$.

In the end, we estimate $|u|$. Since $u \in C^2(\bar{\Omega})$, $f = -\Delta u \in C(\bar{\Omega})$. Let $\lambda = \max_{\bar{\Omega}} |f|$. Note that $\Delta\left(u + \frac{\lambda}{2n}|x|^2\right) = \Delta u + \lambda \ge 0$, $u + \frac{\lambda}{2n}|x|^2$ is subharmonic. Using our last proposition, we have

$$\begin{aligned}
\max_{\bar{\Omega}} u &\le \max_{\bar{\Omega}} \left(u + \frac{\lambda}{2n}|x|^2 \right) = \max_{\partial\Omega} \left(u + \frac{\lambda}{2n}|x|^2 \right) \\
&\le \max_{\partial\Omega} |u| + \frac{\lambda}{2n} \max_{\partial\Omega} |x|^2 \\
&\le \max_{\partial\Omega} |g| + \frac{\operatorname{diam}(\bar{\Omega})^2}{2n} \max_{\bar{\Omega}} |f| \\
&\le C \left(\max_{\partial\Omega} |g| + \max_{\bar{\Omega}} |f| \right).
\end{aligned}$$

Similarly, since $-u + \frac{\lambda}{2n}|x|^2$ is also subharmonic, we conclude that

$$\max_{\bar{\Omega}} |u| \leq C \left(\max_{\bar{\Omega}} |f| + \max_{\partial\Omega} |g| \right).$$

□