Homework 2

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Problem 1

Prove that if $n = 1$ and $u \in W^{1,p}(0,1)$ for some $p \in [1, +\infty)$, then u is equal a.e. to an absolutely continuous function, and u' (which exists a.e.) belongs to $L^p(0,1)$.

Proof. Let $v(x) = \int_0^x u'(s)ds$. Then v is an absolutely continuous function. Therefore for any test function $\phi \in C_c^\infty(0,1)$,

$$\begin{aligned}\int_0^1 (v(x) - u(x))\phi'(x)dx &= \int_0^1 \int_0^x u'(s)ds\phi'(x)dx - \int_0^1 u(x)\phi'(x)dx \\ &= \int_0^1 \int_s^1 \phi'(x)dxu'(s)ds - \int_0^1 u(x)\phi'(x)dx \\ &= -\int_0^1 \phi(s)u'(s)ds + \int_0^1 u'(x)\phi(x)dx \\ &= -\int_0^1 \phi(x)u'(x)dx + \int_0^1 u'(x)\phi(x)dx = 0.\end{aligned}$$

As ϕ was chosen arbitrarily, we see that u is equal a.e. to an absolutely continuous function v as required. □

Problem 2

Suppose Ω is connected and $u \in W^{1,p}(\Omega)$ for some $p \in [1, +\infty)$ satisfies

$$Du = 0 \quad \text{a.e. in } \Omega$$

Prove that u is constant a.e. in Ω.

Proof. Let u_ε denote the standard mollification of u defined on the dialated domain Ω_ε and recall, for each $\varepsilon > 0$, that $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$ with

$$D(u_\varepsilon) = D(\eta_\varepsilon * u) = \eta_\varepsilon * Du.$$

Thus, we have that $D(u_\varepsilon) = 0$ in Ω_ε. Since Ω is connected and u_ε is smooth, for each $\varepsilon > 0$, we have $u_\varepsilon(x) = C_\varepsilon$ in Ω_ε for some constant C_ε. Also, $u_\varepsilon \to u$ a.e. as $\varepsilon \to 0$, implying u is constant a.e. in Ω. □

Problem 3

Verify that if $n > 1$, the unbounded function

$$u = \log\log\left(1 + \frac{1}{|x|}\right)$$

belongs to $W^{1,n}(B_1)$, where B_1 is the unit ball with the center at the origin.

Proof. Let us convert n-dimensional integrals into intergrals over spheres

$$\begin{aligned}\int_{B_1}|u(x)|^n dx &= \int_0^1\int_{\partial B_r(0)}\left|\log\log\left(1+\frac{1}{r}\right)\right|^n dSdr\\ &= |\partial B_1|\int_0^1\left|\log\log\left(1+\frac{1}{r}\right)\right|^n r^{n-1}dr\\ &\overset{t=\frac{1}{r}}{=} |\partial B_1|\int_1^\infty \frac{1}{t^n}\cdot\frac{|\log\log(1+t)|^n}{t}dt.\end{aligned}$$

Since $\lim_{t\to\infty}\frac{|\log\log(1+t)|^n}{t}=0$, there exists $T>0$ such that when $\frac{|\log\log(1+t)|^n}{t}<1$ for $t\geq T$. Thus

$$\begin{aligned}\int_{B_1}|u(x)|^n dx &= |\partial B_1|\int_1^\infty \frac{1}{t^n}\cdot\frac{|\log\log(1+t)|^n}{t}dt\\ &\leq |\partial B_1|\int_1^T \frac{1}{t^n}\cdot\frac{|\log\log(1+t)|^n}{t}dt + |\partial B_1|\int_1^\infty\frac{1}{t^n}dt<\infty.\end{aligned}$$

As for $u_{x_i}(x)$, we have

$$\begin{aligned}u_{x_i}(x) &= \frac{1}{\log(1+\frac{1}{|x|})}\cdot\frac{1}{1+\frac{1}{|x|}}\cdot(-\frac{1}{|x|^2})\cdot\frac{x_i}{|x|}\\ &= -\frac{1}{\log(1+\frac{1}{|x|})}\cdot\frac{1}{1+\frac{1}{|x|}}\cdot\frac{x_i}{|x|^3}\\ &= -\frac{1}{\log(1+\frac{1}{|x|})}\cdot\frac{1}{|x|+1}\cdot\frac{x_i}{|x|^2}.\end{aligned}$$

Thus the first derivative exists, and similarly, we have

$$\begin{aligned}\int_{B_1}\left|u_{x_i}\right|^n dx &\leq |\partial B_1|\int_0^1\left|\frac{1}{\log\left(1+\frac{1}{r}\right)}\cdot\frac{1}{r+1}\cdot\frac{1}{r^2}\right|^n r^{n-1}dr\\ &\overset{t=\frac{1}{r}}{=} |\partial B_1|\int_1^\infty\frac{1}{|\log(1+t)|^n}\cdot\frac{1}{(1+t)^n}\cdot t^{n-1}dt\\ &\overset{s=log(1+t)}{=} |\partial B_1|\int_{\log 2}^\infty\frac{1}{s^n}\cdot\frac{1}{e^{sn}}\cdot(e^s-1)^{n-1}e^s ds\\ &\leq |\partial B_1|\int_{\log 2}^\infty\frac{1}{s^n}\cdot\frac{1}{e^{sn}}\cdot e^{sn}ds<\infty.\end{aligned}$$

In conclusion, $u\in W^{1,n}(B_1)$. □

Problem 4

Fix $\alpha > 0$. Show that there exists a constant C depending only on n and α such that

$$\int_{B_1} u^2 \, \mathrm{d}x \leq C \int_{B_1} |Du|^2 \, \mathrm{d}x$$

for all $u \in H^1(B_1)$ satisfying

$$|\{x \in B_1 : u(x) = 0\}| \geq \alpha$$

where $|E|$ denotes the Lebesgue measure of the set E, and B_1 is the unit ball with the center at the origin.

Proof. We follow the analysis in Leoni's book. To begin with, we prove the general version of Poincare inequality. Ω denotes the unit ball B_1.

Lemma 1. *Let $1 \leq p < \infty$ and let $\Omega \subset \mathbb{R}^N$ be a connected extension domain for $W^{1,p}(\Omega)$ with finite measure. Let $E \subset \Omega$ be a lebesgue measureable set with positive measure. Then there exists a constant $C = C(p, \Omega, E) > 0$ such that for all $u \in W^{1,p}(\Omega)$,*

$$\int_\Omega |u(x) - u_E|^p \, dx \leq C \int_\Omega |Du(x)|^p \, dx$$

where

$$u_E = \frac{1}{|E|} \int_E u(x) \, dx.$$

Proof. Assume by contradiction that the result is false. Then there exists a sequence $\{u_n\}_{n=1}^\infty$ in $W^{1,p}(\Omega)$ such that

$$\|u_n - u_E\|_{L^p(\Omega)} > n \|Du_n\|_{L^p(\Omega)} . \tag{1}$$

Define

$$v_n := \frac{u_n - (u_n)_E}{\|u_n - (u_n)_E\|_{L^p(\Omega)}},$$

then $v_n \in W^{1,p}(\Omega)$ and by (1),

$$\|v_n\|_{L^p(\Omega)} = 1, \quad (v_n)_E = 0, \quad \|Dv_n\|_{L^p(\Omega)} \leq \frac{1}{n}.$$

By the Rellich compactness theorem, there exists a subsequence $\left\{v_{n_k}\right\}_k$ such that $v_{n_k} \to v$ in $L^p(\Omega)$ for some function $v \in L^p(\Omega)$ with $\|v\|_{L^p(\Omega)} = 1, v_E = 0$. Moreover, for every

$\phi \in C_c^1(\Omega)$ and $i = 1, 2, \cdots, N$, by Hölder's inequality,

$$\left|\int_\Omega v \frac{\partial \phi}{\partial x_i}\, dx\right| = \lim_{k\to\infty} \left|\int_\Omega v_{n_k} \frac{\partial \phi}{\partial x_i}\, dx\right| = \lim_{k\to\infty} \left|\int_\Omega \frac{\partial v_{n_k}}{\partial x_i} \phi\, dx\right|$$
$$\le \lim_{k\to\infty} \left\|Dv_{n_k}\right\|_{L^p(\Omega)} \|\phi\|_{L^{p'}(\Omega)} = 0,$$

and so $v \in W^{1,p}(\Omega)$ with $Dv = 0$ a.e. by definition. Since Ω is connected, this implies that v should be a constant. But $v_E = 0$ indicates $v = 0$. This contradicts the fact that $\|v\|_{L^p(\Omega)} = 1$ and completes the proof. □

Our last step is to set $E = \{x \in \Omega : u(x) = 0\}$. Notice that $|E| > 0$, then $u_E = 0$ and we done by Lemma 1. □

Problem 5

Assume $1 \le p \le +\infty$, $\Omega \subset \mathbb{R}^n$ is a bounded smooth open set, and $u \in W^{1,p}(\Omega)$.
(i). Prove that $|u| \in W^{1,p}(\Omega)$.
(ii). Denote $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$. Prove that $u^+, u^- \in W^{1,p}(\Omega)$. Furthermore,

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \le 0\} \end{cases}$$

and

$$Du^- = \begin{cases} -Du & \text{a.e. on } \{u < 0\} \\ 0 & \text{a.e. on } \{u \ge 0\} \end{cases}$$

(iii). Prove that

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

Proof. (ii). Following the remark, we denote

$$F_\varepsilon(z) = \begin{cases} \sqrt{z^2 + \varepsilon^2} - \varepsilon, & \text{if } z \ge 0, \\ 0, & \text{if } z < 0, \end{cases}$$

then $F_\varepsilon(z) \in C^1(\mathbb{R})$ and

$$(F_\varepsilon)'(z) = \begin{cases} \dfrac{z}{\sqrt{z^2 + \varepsilon^2}}, & \text{if } z > 0, \\ 0, & \text{if } z \le 0, \end{cases}$$

which implies that $\left\|(F_\varepsilon)'\right\|_{L^\infty(\mathbb{R})} \le 1$ for every $\varepsilon > 0$. Note that

$$z^+ = \lim_{\varepsilon\to 0} F_\varepsilon(z) = \begin{cases} z, & \text{if } z \ge 0, \\ 0, & \text{if } z < 0, \end{cases}$$

and

$$\lim_{\varepsilon \to 0} (F_\varepsilon)'(z) = \begin{cases} 1, & \text{on } \{z > 0\}, \\ 0, & \text{on } \{z \le 0\}, \end{cases}$$

for every $\phi \in C_c^\infty(\Omega)$, we apply the dominated convergence theorem

$$\begin{aligned} \int_\Omega u^+ \frac{\partial \phi}{\partial x_j}\, dx &= \int_\Omega \lim_{\varepsilon \to 0} F_\varepsilon(u) \frac{\partial \phi}{\partial x_j}\, dx \\ &= \lim_{\varepsilon \to 0} \int_\Omega F_\varepsilon(u) \frac{\partial \phi}{\partial x_j}\, dx \\ &= -\lim_{\varepsilon \to 0} \int_\Omega (F_\varepsilon)'(u) \frac{\partial u}{\partial x_j} \phi\, dx \\ &= -\int_\Omega \lim_{\varepsilon \to 0} (F_\varepsilon)'(u) \frac{\partial u}{\partial x_j} \phi\, dx \\ &= -\int_{u>0} \frac{\partial u}{\partial x_j} \phi\, dx, \end{aligned}$$

implying

$$Du^+ = \begin{cases} Du, & \text{a.e. on } \{u > 0\}, \\ 0, & \text{a.e. on } \{u \le 0\}. \end{cases}$$

Since $u^- = (-u)^+$, we can directly derive that

$$Du^- = \begin{cases} -Du, & \text{a.e. on } \{u < 0\} \\ 0, & \text{a.e. on } \{u \ge 0\}. \end{cases}$$

(i). Since $|u| = u^+ + u^-$, $|u| \in W^{1,p}(\Omega)$ by linearity.

(iii). On the set $\{u = 0\}$, we have

$$Du = Du^+ - Du^- = 0 \text{ a.e.}$$

□