Homework 3 for MATH5311

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Problem 1

Derive the modified equation of the box scheme and analyze its dispersion relation.

Solution. The box scheme can be written in the form

$$(1-a\nu)U_j^{n+1} + (1+a\nu)U_{j+1}^{n+1} = (1+a\nu)U_j^n + (1-a\nu)U_{j+1}^n,$$

where $\nu = \Delta t/\Delta x$. The corresponding modified equation is

$$u_t + au_x = \frac{a}{12}(1-a^2\nu^2)(\Delta x)^2 u_{xxx} = \epsilon u_{xxx}.$$

Using simplified notation, we then analyze its dispersion relation. We assume the solution is in the form $u(x,t) = \exp\{i(kx - wt)\}$. Plugging the solution into the original equation, we get the dispersion relation

$$w = ak + \epsilon k^3$$

and corresponding solution

$$u(x,t) = \exp\{ik(x-ct)\},$$

where $c = a + \epsilon k^2$ denotes wave speed.

The sign of ϵ depends on the values of a and ν, and there are four cases:

- for $a > 0$ and $0 < \nu < |1/a|$, $\epsilon > 0$;
- for $a > 0$ and $\nu > |1/a|$, $\epsilon < 0$;
- for $a < 0$ and $0 < \nu < |1/a|$, $\epsilon < 0$;
- for $a < 0$ and $\nu > |1/a|$, $\epsilon > 0$.

Note that when a and ϵ have the same sign, the high frequency waves run faster than the original wave; on the contrary, the high frequency waves are slower. Therefore, for $|a|\nu < 1$, the box scheme has a phase advance oscillation (regardless of whether a is positive or negative, though the sign of a determines the direction of wave propagation); for $|a|\nu > 1$, oscillation is behind.