TO............ FROM............ DATE............ NO............

Consider 1-d problem

$$\begin{cases} -u'' = f \\ u(0) = u(1) = 0 \end{cases}$$ on a uniform grid

and $S_N = \{ v \mid v$ is piecewise quardratic function with $v(0) = v(1) = 0 \}$

$S_N \subset H_0^1(\Omega)$

(1) write down the weak formulation of the problem and its finite element discretization.

[redacted]

(2) construct the basis functions of S_N.

(3) compute the stiffness matrix A.

(1) Weak formulation:

$$\int_\Omega \nabla u \nabla v = \int_\Omega f v \quad , \forall v \in H_0^1(\Omega)$$

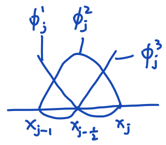
Discretization:

Find a finite dimensional subspace $S_N \subset H_0^1(\Omega)$ with a basis $\{\phi_1, \phi_2, \cdots, \phi_N\}$

Assume $u = \sum_i u_i \phi_i$, then we have $\sum_i u_i \int_\Omega \nabla \phi_i \nabla \phi_j - \int_\Omega \phi_j f = 0.$

Linear system: $\sum_i a_{ij} u_i - b_j = 0$, $j = 1, 2, \cdots, N.$

(2) We need three points to determine a quadratic function.



base function ϕ_j^1 , ϕ_j^2 , ϕ_j^3

$$\phi_j^1(x_{j-1}) = 1, \quad \phi_j^1(x_{j-\frac{1}{2}}) = \phi_j^1(x_j) = 0$$

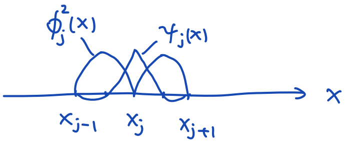
$$\phi_j^2(x_{j-\frac{1}{2}}) = 1, \quad \phi_j^2(x_{j-1}) = \phi_j^2(x_j) = 0$$

$$\phi_j^3(x_j) = 1, \quad \phi_j^3(x_{j-1}) = \phi_j^3(x_{j-\frac{1}{2}}) = 0$$

$$\Rightarrow \quad \phi_j^1(x) = \frac{2}{h^2}(x - x_{j-\frac{1}{2}})(x - x_j) \quad , \quad x \in [x_{j-1}, x_j]$$

$$\phi_j^2(x) = -\frac{4}{h^2}(x - x_{j-1})(x - x_j), \quad x \in [x_{j-1}, x_j]$$

$$\phi_j^3(x) = \frac{2}{h^2}(x - x_{j-1})(x - x_{j-\frac{1}{2}}), \quad x \in [x_{j-1}, x_j]$$



let $$\psi_j(x) = \begin{cases} \phi_j^3(x), & x \in [x_{j-1}, x_j) \\ \phi_{j+1}^1(x), & x \in [x_j, x_{j+1}] \\ 0, & \text{elsewhere} \end{cases} \quad j = 1, 2, \cdots, N-1.$$

$$\phi_j(x) = \begin{cases} \phi_j^2(x), & x \in [x_{j-1}, x_j] \\ 0, & \text{elsewhere} \end{cases} \quad j = 1, 2, \cdots, N$$

Then $S_h = \{\phi_j(x)\}_{j=1}^{N} \cup \{\psi_j(x)\}_{j=1}^{N-1}$

(3)

$$A = \frac{1}{3h}\begin{pmatrix} 8 & -8 & & & \\ -8 & 14 & -8 & & \\ & -8 & 8 & -8 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}$$