Homework 4 for MATH5311

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Problem 1

Show that $\{\phi_j, j = 0, 1, \cdots, N\}$ forms a basis of S_N. Compute a_{ij} for uniform grid on $[0, 1]$ and compute the rank of the matrix $A = (a_{ij})$.

Solution. We define $\phi_j(x)$ in the form

$$\phi_j(x) = \begin{cases} \frac{x-x_{j-1}}{x_j-x_{j-1}} & \text{if } x_{j-1} \leq x \leq x_j; \\ \frac{x-x_{j+1}}{x_j-x_{j+1}} & \text{if } x_j < x \leq x_{j+1}; \\ 0 & \text{else,} \end{cases}$$

for $j = 1, 2, \cdots, N-1$,

$$\phi_0(x) = \begin{cases} \frac{x-x_1}{x_0-x_1} & \text{if } x_0 \leq x \leq x_1; \\ 0 & \text{else} \end{cases}$$

and

$$\phi_N(x) = \begin{cases} \frac{x-x_{N-1}}{x_N-x_{N-1}} & \text{if } x_{N-1} \leq x \leq x_N; \\ 0 & \text{else.} \end{cases}$$

If $c_0\phi_0 + c_1\phi_1 + \cdots + c_N\phi_N = 0$, then we have $c_j = 0$ for all $j = 0, 1, \cdots, N$, by letting $x = x_j$ in the equation. Thus $\{\phi_j(x)\}$ are linearly independent. For all $v \in S_N$, if we take $c_j = v(x_j)$, then $v(x) = \sum_{j=0}^{N} c_j\phi_j(x)$, since both sides of the equation are piecewise linear and have the same function value at the grid points, indicating they are equal everywhere.

We divide $[0, 1]$ into N equal parts, then $x_j = j/N$. By computing $a_{ij} = \int_0^1 \phi_i'\phi_j' dx$, the matrix A is in the form

$$A = N \begin{pmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{pmatrix}_{(N+1)\times(N+1)}.$$

Let

$$P = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ \vdots & & \ddots & & \\ 1 & & & 1 & \\ 1 & & & & 1 \end{pmatrix},$$

then

$$PAQ = N \begin{pmatrix} 0 & -1 & & & & \\ 0 & 2 & -1 & & & \\ 0 & -1 & 2 & -1 & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Thus $\mathrm{rank}(A) = \mathrm{rank}(PAQ) = N$.