Homework 2 for MATH5311

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Problem 1

Calculate the truncation error of the ADI method and verify that it is second order in both time and space.

Solution. The ADI method consists of two alternate updates: odd steps implicit in x and explicit in y, and even steps implicit in y and explicit in x. Specifically, we have

$$\begin{aligned}\left(1 - \tfrac{1}{2}\nu_x\delta_x^2\right)U^{n+\frac{1}{2}} &= \left(1 + \tfrac{1}{2}\nu_y\delta_y^2\right)U^n, \\ \left(1 - \tfrac{1}{2}\nu_y\delta_y^2\right)U^{n+1} &= \left(1 + \tfrac{1}{2}\nu_x\delta_x^2\right)U^{n+\frac{1}{2}},\end{aligned}$$

where $\nu_x = (\Delta t)/(\Delta x)^2$, $\nu_y = (\Delta t)/(\Delta y)^2$, $\delta_x^2 U = (U_{i+1,j} - 2U_{i,j} + U_{i-1,j})$ and $\delta_y^2 U = (U_{i,j+1} - 2U_{i,j} + U_{i,j-1})$. We may combine the two iterations into one equivalent step:

$$\left(1 - \frac{1}{2}\nu_x\delta_x^2\right)\left(1 - \frac{1}{2}\nu_y\delta_y^2\right)U^{n+1} = \left(1 + \frac{1}{2}\nu_x\delta_x^2\right)\left(1 + \frac{1}{2}\nu_y\delta_y^2\right)U^n.$$

For the Crank-Nicolson scheme, we have

$$\left(1 - \frac{1}{2}\nu_x\delta_x^2 - \frac{1}{2}\nu_y\delta_y^2\right)U^{n+1} = \left(1 + \frac{1}{2}\nu_x\delta_x^2 + \frac{1}{2}\nu_y\delta_y^2\right)U^n.$$

We have known that the truncation error of Crank-Nicolson scheme is $O\left((\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2\right)$.

Since $\nu_x\nu_y\delta_x^2\delta_y^2\delta_t u^{n+\frac{1}{2}} = (\Delta t)^3 u_{xxyyt}^{n+\frac{1}{2}} + O\left((\Delta t)^5 + (\Delta t)^3\left((\Delta x)^2 + (\Delta y)^2\right)\right)$, the truncation error of ADI scheme is $O\left((\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2\right)$. So it is second order in both time and space.