1. Let $u \neq 0$ be the eigenvector of A, and λ the associated eigenvalue. Then $Au = \lambda u$ implies that

$$-u_{j-1} + 2u_j - u_{j+1} = \lambda u_j, \quad j = 1, 2, \cdots, n$$

with boundary condition

$$\begin{cases} u_0 - u_1 = 0 \\ u_{n+1} - u_n = 0 \end{cases}.$$

This is a discrete difference equation of 2nd order, whose solution is in the form of $u_j = c_1 \alpha_1^j + c_2 \alpha_2^j$, where α_1, α_2 are the roots of characteristic polynomial

$$-1 + 2\alpha - \alpha^2 = \lambda \alpha \quad \Rightarrow \begin{cases} \alpha_1 + \alpha_2 = 2 - \lambda \\ \alpha_1 \alpha_2 = 1 \end{cases}$$

And c_1, c_2 are determined by

$$\begin{cases} (c_1 + c_2) - (c_1 \alpha_1 + c_2 \alpha_2) = 0 \\ (c_1 \alpha_1^{n+1} + c_2 \alpha_2^{n+1}) - (c_1 \alpha_1^n + c_2 \alpha_2^n) = 0 \end{cases}.$$

$$\Rightarrow \begin{cases} c_1(1 - \alpha_1) + c_2(1 - \alpha_2) = 0 \\ c_1(\alpha_1^{n+1} - \alpha_1^n) + c_2(\alpha_2^{n+1} - \alpha_2^n) = 0 \end{cases}$$

$$\det \begin{pmatrix} 1-\alpha_1 & 1-\alpha_2 \\ \alpha_1^{n+1} - \alpha_1^n & \alpha_2^{n+1} - \alpha_2^n \end{pmatrix} = 0 \quad \Rightarrow \left(\frac{\alpha_1}{\alpha_2}\right)^n = 1$$

$$\Rightarrow \frac{\alpha_1}{\alpha_2} = e^{i\frac{2\pi}{n}k}, \quad k = 1, 2, \cdots, n$$

Then by condition $\begin{cases} \alpha_1 + \alpha_2 = 2 - \lambda \\ \alpha_1 \alpha_2 = 1 \end{cases}$, we have

$$\alpha_1 = e^{i\frac{\pi}{n}k}, \quad \alpha_2 = e^{-i\frac{\pi}{n}k}$$

$$\lambda = 2-(\alpha_1+\alpha_2) = 2-2\,\mathrm{Re}\left(e^{i\frac{\pi}{n}k}\right)$$
$$= 2\left(1-\cos\frac{\pi k}{n}\right), \quad k=1,2,\cdots,n$$

For c_1 and c_2, we can choose

$$c_1 = 1-\alpha_2, \quad c_2 = \alpha_1 - 1,$$

then
$$u_j = (1-\alpha_2)\alpha_1^j + (\alpha_1-1)\alpha_2^j$$
$$= e^{\frac{i\pi kj}{n}} - e^{\frac{i\pi k(j-1)}{n}} + e^{-\frac{i\pi k(j-1)}{n}} - e^{-\frac{i\pi kj}{n}}$$
$$= 2i\left(\sin\frac{\pi kj}{n} - \sin\frac{\pi k(j-1)}{n}\right)$$
$$= 4i\sin\frac{\pi k}{2n}\cos\frac{\pi k(2j-1)}{2n}, \quad j=1,2,\cdots,n.$$

Above all, the eigenvalues of A are

$$\lambda_k = 2\left(1-\cos\frac{\pi k}{n}\right), \quad k=1,2,\cdots,n,$$

and the corresponding eigenvector is

$$u_k = \begin{pmatrix} \cos\frac{\pi k}{2n} \\ \vdots \\ \cos\frac{\pi k(2j-1)}{2n} \\ \vdots \\ \cos\frac{\pi k(2n-1)}{2n} \end{pmatrix}$$

2. (a) i) "$\Rightarrow$"

$\sum_{k=0}^{\infty} A^k$ is convergent, then $\lim_{k\to\infty} A^k = 0$.

implying $\rho(A) < 1$.

ii) "$\Leftarrow$" If $\rho(A) < 1$, we choose a norm $\|\cdot\|_\varepsilon$,

where $\varepsilon = \delta(1-\rho(A))$ with $\delta < 1$.

such that $\|A\|_\varepsilon \le \rho(A) + \delta(1-\rho(A)) \le c < 1$.

Then since $\|A^k\|_\varepsilon \leq \|A\|_\varepsilon^k \to 0$ as $k \to \infty$,

$$\left\| \sum_{k=0}^{\infty} A^k \right\|_\varepsilon \leq \sum_{k=0}^{\infty} \|A\|_\varepsilon^k \leq \sum_{k=0}^{\infty} c^k = \frac{1}{1-c},$$

indicating $\sum_{k=0}^{\infty} A^k$ is convergent.

(b) Since $(I + A + A^2 + \cdots + A^n)(I - A) = I - A^{n+1}$,

letting $n \to \infty$, we have

$$\left(\sum_{k=1}^{\infty} A^k \right)(I - A) = I - \lim_{n\to\infty} A^{n+1} = I.$$

By definition, $\sum_{k=1}^{\infty} A^k = (I-A)^{-1}$

(c) We choose the norm $\|\cdot\|_\varepsilon$ defined in (a).

$$(I-A)^{-1} - \sum_{k=0}^{K} A^k = \sum_{k=0}^{\infty} A^k - \sum_{k=0}^{K} A^k$$

$$= \sum_{k=K+1}^{\infty} A^k$$

$$\left\| (I-A)^{-1} - \sum_{k=0}^{K} A^k \right\|_\varepsilon \leq \sum_{k=K+1}^{\infty} \|A^k\|_\varepsilon$$

$$\leq \sum_{k=K+1}^{\infty} \|A\|_\varepsilon^k$$

$$= \frac{\|A\|_\varepsilon^{K+1} \left(1 - \lim_{k\to\infty} \|A\|_\varepsilon^k\right)}{1 - \|A\|_\varepsilon}$$

$$= \frac{\|A\|_\varepsilon^{K+1}}{1 - \|A\|_\varepsilon}.$$

3. (a) $A = D - E - F$,

where $D = \mathrm{diag}(a_{11}, a_{22}, \cdots, a_{nn})$

$$E = \begin{pmatrix} 0 & & & \\ -a_{21} & 0 & & \\ \vdots & \vdots & \ddots & \\ -a_{n1} & -a_{n2} & \cdots & 0 \end{pmatrix}$$

$$F = \begin{pmatrix} 0 & -a_{12} & \cdots & -a_{1n} \\ & 0 & \cdots & -a_{2n} \\ & & \ddots & \vdots \\ & & & 0 \end{pmatrix}$$

① backward Gauss - Seidel

We update $\xi_i^{(k+1)}$, $\xi_j^{(k+1)}$ $j>i$ are available.

$$(Ax-b)_i = 0$$

$$\Rightarrow a_{ii}\xi_i^{(k+1)} + \sum_{j>i} a_{ij}\xi_j^{(k+1)} + \sum_{j<i} a_{ij}\xi_j^{(k)} = \beta_i$$

$$\xi_i^{(k+1)} = \left(\beta_i - \sum_{j>i} a_{ij}\xi_j^{(k+1)} - \sum_{j<i} a_{ij}\xi_j^{(k)} \right) / a_{ii}$$

Using matrix formulation

$$x_{k+1} = D^{-1}(b + Fx_{k+1} + Ex_k)$$

$$\Rightarrow Dx_{k+1} = b + Fx_{k+1} + Ex_k$$

$$\Rightarrow x_{k+1} = (D-F)^{-1}Ex_k + (D-F)^{-1}b$$

$$\Rightarrow x_{k+1} = [I - (D-F)^{-1}A]x_k + (D-F)^{-1}b$$

$$\Rightarrow G = I - (D-F)^{-1}A$$

② symmetric Gauss - Seidel

i) We update $\xi_i^{(k+\frac{1}{2})}$, $\xi_j^{(k+\frac{1}{2})}$ $j<i$ are available

$$x_{k+\frac{1}{2}} = D^{-1}(b + Ex_{k+\frac{1}{2}} + Fx_k)$$

$$\Rightarrow x_{k+\frac{1}{2}} = (D-E)^{-1}Fx_k + (D-E)^{-1}b$$

ii) We update $\xi_i^{(k+1)}$, $\xi_j^{(k+1)}$ $j>i$ are available

$$x_{k+1} = D^{-1}(b + Ex_{k+\frac{1}{2}} + Fx_{k+1})$$

$$\Rightarrow x_{k+1} = (D-F)^{-1}Ex_{k+\frac{1}{2}} + (D-F)^{-1}b$$

$$= (I-(D-F)^{-1}A)(I-(D-E)^{-1}A)x_k$$

$$+ (I-(D-F)^{-1}A)(D-E)^{-1}b + (D-F)^{-1}b$$

$$\Rightarrow \quad G = (I - (D-F)^{-1}A)(I - (D-E)^{-1}A)$$

(b) ① $G = I - (D-F)^{-1}A$

It suffices to prove $\rho(G) < 1$.

Let λ be an eigenvalue of G and $u \neq 0$ the corresponding eigenvector. Since $A^T = A$ and

$$G = I - (D-F)^{-1}A = (D-F)^{-1}E,$$

we have $(D-F)^{-1}Eu = \lambda u$, $E = F^T$.

Suppose $u^* D u = \delta > 0$, $u^* F u = \alpha + i\beta$, then

$$u^* F^T u = (u^* F u)^* = (\alpha + i\beta)^* = \alpha - i\beta$$

Since $F^T u = \lambda (D-F) u$,

$$\alpha - i\beta = u^* F^T u = \lambda u^* (D-F) u = \lambda(\delta - (\alpha + i\beta)).$$

then $\lambda = \dfrac{\alpha - i\beta}{\delta - (\alpha + i\beta)}$

$$|\lambda|^2 = \frac{\alpha^2 + \beta^2}{(\delta - \alpha)^2 + \beta^2}$$

Since A is SPD,

$$u^* A u = u^* (D - E - F) u = \delta - 2\alpha > 0.$$

$$(\delta - \alpha)^2 + \beta^2 = \alpha^2 + \beta^2 + \delta(\delta - 2\alpha) > \alpha^2 + \beta^2.$$

thus $|\lambda| < 1$ for any eigenvalue of G. $\Rightarrow \rho(G) < 1$

② $G = (I - (D-F)^{-1}A)(I - (D-E)^{-1}A)$

When A is SPD, we have shown that

$$\rho(I - (D-F)^{-1}A) < 1, \quad \rho(I - (D-E)^{-1}A) < 1,$$

then $\rho(G) \leq \rho(I - (D-F)^{-1}A)\,\rho(I - (D-E)^{-1}A) < 1$.

implying the convergence.

4. ① block Jacobi

Let $\tilde{A} = \begin{pmatrix} A+2I & -I & & \\ -I & A+2I & -I & \\ & & \ddots & \\ & & -I & A+2I \end{pmatrix}$

Assume $\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$, where

$$\tilde{D} = \begin{pmatrix} A+2I & & \\ & \ddots & \\ & & A+2I \end{pmatrix}$$

$$\tilde{L} = -\begin{pmatrix} 0 & -I & & \\ & 0 & -I & \\ & & \ddots & -I \\ & & & 0 \end{pmatrix}$$

$$\tilde{U} = \tilde{L}^T.$$

Thus $G = \tilde{D}^{-1}(\tilde{L}+\tilde{U})$.

Our goal is to show $\rho(G) < 1$.

Let λ be an eigenvalue of G and x the corresponding eigenvector. Since G is symmetric, λ is a real number. By $Gx = \lambda x$, we have

$$(\tilde{L}+\tilde{U})x = \lambda \tilde{D} x$$

$$\Rightarrow (\lambda\tilde{D} - \tilde{L} - \tilde{U})x = 0$$

Denote $T_\lambda = (\lambda\tilde{D} - \tilde{L} - \tilde{U}) = \begin{pmatrix} \lambda(A+2I) & -I & & \\ -I & \lambda(A+2I) & & \\ & & \ddots & -I \\ & & -I & \lambda(A+2I) \end{pmatrix}$

i) If $\lambda = 1$, $T_\lambda = A$.

Since A is invertible (all the eigenvalues of A

are not 0), $T_\lambda x = 0$ only has zero solution, which contradicts that x is an eigenvector.

ii) If $\lambda > 1$, T_λ is SDD

$\Rightarrow$ T_λ is invertible

It also contradicts to the condition.

Therefore, $\lambda < 1$ for all eigenvalues of G, implying $\rho(G) < 1$.

② block Gauss-Seidel

$G = (\tilde{D} - \tilde{L})^{-1} \tilde{U}$

Using the same idea, assume $Gx = \lambda x$.

$\Rightarrow$ $(\lambda D - \lambda L - U) x = 0$

Let $T_\lambda = \lambda D - \lambda L - U$.

i) If $|\lambda| > 1$, T_λ is SDD

$\Rightarrow$ T_λ is invertible

$\Rightarrow$ contradiction

ii) If $|\lambda| = 1$, T_λ is diagonally dominant.

T_λ is also irreducible

$\Rightarrow$ T_λ is invertible

$\Rightarrow$ contradiction

Thus, $|\lambda| < 1$ for all λ

$\Rightarrow \rho(G) < 1$