1. (a) Similar to Gaussian elimination, we define

$$A^{(1)} = A = \begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ \vdots & \vdots & & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{pmatrix},$$

$U_1 = I - u_1 e_n^T$, where $u_1 = \left(a_{1n}^{(1)}/a_{nn}^{(1)}, \cdots, a_{n-1,n}^{(1)}/a_{nn}^{(1)}, 0 \right)^T$.

Then

$$A^{(2)} = U_1 A^{(1)} = \begin{pmatrix} a_{11}^{(2)} & \cdots & a_{1,n-1}^{(2)} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n-1,1}^{(2)} & \cdots & a_{n-1,n-1}^{(2)} & 0 \\ a_{n1}^{(1)} & \cdots & a_{n,n-1}^{(1)} & a_{nn}^{(1)} \end{pmatrix}.$$

Repeating the process and letting $U_k = I - u_k e_{n+1-k}^T$,

$$U_{n-1} U_{n-2} \cdots U_2 U_1 A = L$$

$$\begin{aligned} \Rightarrow \quad A &= U_1^{-1} U_2^{-1} \cdots U_{n-1}^{-1} L \\ &= (I - u_1 e_n^T)^{-1} \cdots (I - u_{n-1} e_2^T)^{-1} L \\ &= (I + u_1 e_n^T) \cdots (I + u_{n-1} e_2^T) L \\ &= (I + u_1 e_n^T + u_2 e_{n-1}^T + \cdots + u_{n-1} e_2^T) L \\ &= UL. \end{aligned}$$

In summary, we can compute $A = UL$ by

```
for k = 1:1:n-1
    A(1:n-k, n+1-k) = A(1:n-k, n+1-k)/A(n+1-k, n+1-k)
    A(1:n-k, 1:n-k) = A(1:n-k, 1:n-k) - A(1:n-k, n+1-k)·A(n+1-k, 1:n-k)
end
```

(b) ① Compute $A = UL$.

② Solve $Uy = b$.

```
for k = n:-1:1
```

$b(k,1) = b(k,1) - A(k, k+1:n)\, b(k+1:n, 1)$

end

③ Solve $Lx = y$.

for $k = 1:1:n-1$

$b(k,1) = b(k,1) - A(k, 1:k-1)\, b(1:k-1, 1)$

$b(k,1) = b(k,1) / A(k,k)$

end

2.(a)

$$L_k \cdots L_1 A = \begin{pmatrix} a_{11}^{(1)} & \cdots & a_{1k}^{(1)} & a_{1,k+1}^{(1)} & \cdots & a_{1n}^{(1)} \\ & \ddots & \vdots & \vdots & & \vdots \\ & & a_{kk}^{(k)} & a_{k,k+1}^{(k)} & \cdots & a_{kn}^{(k)} \\ & & 0 & a_{k+1,k+1}^{(k+1)} & \cdots & a_{k+1,n}^{(k+1)} \\ & & \vdots & \vdots & & \vdots \\ & & 0 & a_{n,k+1}^{(k+1)} & \cdots & a_{nn}^{(k+1)} \end{pmatrix}$$

$$= L_{k+1}^{-1} \cdots L_{n-1}^{-1} U$$

$$= \left(I + l_{k+1} e_{k+1}^T + l_{k+2} e_{k+2}^T + \cdots + l_{n-1} e_{n-1}^T \right) U$$

$$= \begin{pmatrix} I & 0 \\ 0 & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$$

$$= \begin{pmatrix} U_{11} & U_{12} \\ 0 & L_{22} U_{22} \end{pmatrix}$$

Since

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = A = \begin{pmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$$

$$= \begin{pmatrix} L_{11} U_{11} & L_{11} U_{12} \\ L_{21} U_{11} & L_{21} U_{12} + L_{22} U_{22} \end{pmatrix},$$

$$A_{22} - A_{21} A_{11}^{-1} A_{12}$$

$$= L_{21} U_{12} + L_{22} U_{22} - L_{21} U_{11} (L_{11} U_{11})^{-1} L_{11} U_{12}$$

$$= L_{22} U_{22}$$

$$= (L_k \cdots L_1 A)[k+1:n,\ k+1:n] .$$

(b)

$$L_k \cdots L_1 A = \begin{pmatrix} a_{11}^{(1)} & \cdots & a_{1k}^{(1)} & a_{1,k+1}^{(1)} & \cdots & a_{1n}^{(1)} \\ & \ddots & \vdots & \vdots & & \vdots \\ & & a_{kk}^{(k)} & a_{k,k+1}^{(k)} & \cdots & a_{kn}^{(k)} \\ & & 0 & a_{k+1,k+1}^{(k+1)} & \cdots & a_{k+1,n}^{(k+1)} \\ & & \vdots & \vdots & & \vdots \\ & & 0 & a_{n,k+1}^{(k+1)} & \cdots & a_{nn}^{(k+1)} \end{pmatrix}$$

① $a_{11}^{(1)} \neq 0$ since $A[1,1]$ is invertible.

② If $a_{ii}^{(i)} \neq 0$, $i = 1, 2, \cdots, k$, note that

$$L_k[1:k+1, 1:k+1] \cdots L_1[1:k+1, 1:k+1] A[1:k+1, 1:k+1]$$

$$= \begin{pmatrix} a_{11}^{(1)} & & * \\ & \ddots & \\ & a_{kk}^{(k)} & \\ & & a_{k+1,k+1}^{(k+1)} \end{pmatrix}$$

$$\det(LHS) = \det A[1:k+1, 1:k+1] \neq 0$$

$$\shortparallel$$

$$\det(RHS) = \prod_{i=1}^{k+1} a_{ii}^{(i)}$$

$$\Longrightarrow a_{k+1,k+1}^{(k+1)} \neq 0$$

By induction, the pivot in LU is always non-zero.

(c) Note that

$$\begin{pmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} I & -A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{pmatrix},$$

$\begin{pmatrix} A_{11} & \\ & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{pmatrix}$ is SPD, implying $A_{22}-A_{21}A_{11}^{-1}A_{12}$ is SPD too.

(d) Let $A = LL^T$, where L is a lower triangular matrix.

Since $0 < \det A = (\det L)^2 = \prod_{k=1}^{n} l_{kk}^2$, $l_{kk} \neq 0$.

Thus $a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2 = l_{kk}^2 > 0$

3. (a) $$A = \begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ \vdots & \vdots & \ddots & \\ l_{n1} & l_{n2} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \cdots & l_{n1} \\ & l_{22} & \cdots & l_{n2} \\ & & \ddots & \vdots \\ & & & l_{nn} \end{pmatrix}$$

$\Rightarrow l_{11}^2 = 2 \Rightarrow l_{11} = \sqrt{2}$ (We choose $l_{11} > 0$)

$l_{21} = -\frac{1}{l_{11}} = -\frac{1}{\sqrt{2}}$

$l_{31} = l_{41} = \cdots = l_{n1} = 0$

$l_{22} = \sqrt{\frac{3}{2}}$

$\vdots$

In summary,

$$l_{ij} = \begin{cases} \sqrt{1+\frac{1}{i}} & i=j \\ -\sqrt{\frac{i-1}{i}} & i=j+1 \end{cases}$$

$$\begin{cases} \quad 0 \quad \text{else} \end{cases}$$

$$\det A = \prod_{i=1}^{n} l_{ii}^2 = n+1 .$$

(b) Referring to 2, there exists U_1 and L such that $A = LU_1$.

Let $D = \operatorname{diag} U_1$, then $A = LDU$, where U is an upper triangular matrix with diagonal ones.

Since $LDU = A = A^T = U^T D L^T$, $D^{-1}(U^T)^{-1} L D = L^T U^{-1}$.

A matrix is upper triangular as well as lower triangular, and therefore diagonal.

$$L^T U^{-1} = I \Rightarrow U = L^T \Rightarrow A = LDL^T$$

4. (a) $Ax = b$, $A\tilde{x} = \tilde{b}$

$$A(\tilde{x} - x) = \tilde{b} - b \Rightarrow \tilde{x} - x = A^{-1}(\tilde{b} - b)$$

$$\Rightarrow \|\tilde{x} - x\| \le \|A^{-1}\| \, \|\tilde{b} - b\|$$

and $\|b\| \le \|A\| \, \|x\|$

$$\frac{\|\tilde{x} - x\|}{\|x\|} \le \|A^{-1}\| \, \|A\| \frac{\|\tilde{b} - b\|}{\|b\|}$$

(b) $1 = \|A A^{-1}\| \le \|A\| \, \|A^{-1}\| = k(A)$

$\Rightarrow k(A) \ge 1$