Bayesian Analysis of Functional Coefficient Conditional Autoregressive Range model

Zhongxin Ni[1] Yixin Qian[1]

[1]School of Economics, Shanghai University

2017-11-10

Outline

Introduction

GARCH Model

- Volatility analysis is one of the central theme in empirical finance cause it is an important factor in asset pricing and risk management.
  Hence, numerous approaches had emerged to analyze the volatility, with the generalized autoregressive conditional heteroskedasticity (GARCH) models (see Bollerslev, 1986; Engle, 1982) being the most widely used.
- due to the recent research, these return-based volatility models may not be the best choice to analysis the volatility because, under many circumstance, the range of time series is a more efficient measure of volatility (see Alizadeh, Brandt and Diebold, 2002; Andersen, Torben and Bollerslev, 1998; Parkinson, 1980). Consequently, using financial range data to model volatility got its momentum in recent decades.

Introduction

- Volatility analysis is one of the central theme in empirical finance cause it is an important factor in asset pricing and risk management. Hence, numerous approaches had emerged to analyze the volatility, with the generalized autoregressive conditional heteroskedasticity (GARCH) models (see Bollerslev, 1986; Engle, 1982) being the most widely used.
- due to the recent research, these return-based volatility models may not be the best choice to analysis the volatility because, under many circumstance, the range of time series is a more efficient measure of volatility (see Alizadeh, Brandt and Diebold, 2002; Andersen, Torben and Bollerslev, 1998; Parkinson, 1980). Consequently, using financial range data to model volatility got its momentum in recent decades.

FCARR Model

- Following the definition of by Chou (2005), let be the logarithmic price of a specific asset and then the range series can be defined as

$$R_t \equiv Max\{P_\tau\} - Min\{P_\tau\} \tag{1}$$

$$\tau = t-1, t-1+\frac{1}{n}, t-1+\frac{2}{n}, ..., t \tag{2}$$

The parametric n is the number of intervals used in measuring the price within each range-measured interval. In this paper, we only consider an arch type FCARR model for convenience and lower model complexity. To capture the nonlinearity we assume the range series are generated from the following model,

FCARR Model

$$R_t = \lambda_t \epsilon_t \tag{3}$$

$$\lambda_t = \alpha_0(R_{t-d}) + \alpha_1(R_{t-d})R_{t-1} + \cdots + \alpha_p(R_{t-d})R_{t-p} \tag{4}$$

$$\epsilon_t \mid I_{t-1} \sim f(1, \xi_t) \tag{5}$$

where λ_t is the conditional mean of range with a non-linear arch type structure. The positive number d is the delay parameter, p is the ARCH order so that we denote the model defined in (3), (4) and (5) as FCARR(p, d). $\alpha_j(\cdot)$s are unknown smoothing functions satisfying $\alpha_0(\cdot) > 0$ and $\alpha_j(\cdot) \geq 0$ when $j \geq 1$ for stationary.

FCARR Model

A natural choice for the distribution item ϵ_t is the exponential with unit mean as it has non-negative support, which suits the characteristic of range series. It has been proved that is the distribution item ϵ_t, or the normalized range $\epsilon_t = R_t/\lambda_t$ is $i.i.d.$, then the conditional variance of the range is proportional to the square of its conditional expectation, see Engle (2002).

FCARR Model

The proposed FCARR model combine the advantages of CARR model in Chou (2005) and FARCH model in Song et al (2014).

1. the information contained in FCARR is of greater amount than that in FARCH model cause the range series inflects the whole price path in the interval in computing process while the log-return series only uses the closing prices.

2. likely to capture the non-linearity of the auto-correlation of range series when compared with the fixed coefficient CARR model, which means it provides more flexibility to trace the real data generating process behind the financial market.

3. able to investigate how historical volatility influence the future volatility dynamically according to the lagged range.

Bayesian Analysis of FCARR Model

Nonparametric Modeling

Inspired by Eilers and Marx (1996) and Lang and Brezger (2004), we introduce the Bayesian P-splines approach to estimate the functional coefficients. Following the principle of B-splines (De Boor, 2001), $\alpha_j(R_{t-d})$ in (4) can be approximated by

$$\alpha_j(R_{t-d}) = \sum_{k=1}^{K_\gamma} \gamma_{jk} B_k^\gamma(R_{t-d}) = \gamma_j^T \mathbf{B}^\gamma(R_{t-d}) \tag{6}$$

where K_γ is the number of spines determined by the number of knots, $\gamma_j = (\gamma_{j1}, \cdots, \gamma_{jk_\gamma})^T$ is the vector of unknown parameters, $B^\gamma(R_{t-d}) = (B_1^\gamma(R_{t-d}), \cdots, B_{k_\gamma}^\gamma(R_{t-d}))^T$, and the functions $B_k^\gamma(\cdot)$ are B-splines basis. In practice, $B_k^\gamma(\cdot)$ is often chosen to be cubic B-splines and K_γ ranging from 10 to 30 provides sufficient flexibility for modeling coefficients.

Bayesian Analysis of FCARR Model

Likelihood Function

In addition, we impose the constraints: $\gamma_0^T B^\gamma(R_{t-d}) > 0$, $\gamma_j^T B^\gamma(R_{t-d}) \geq 0, j = 1, \cdots, p$ for $t = d+1, \cdots, T$. For convenience, let $\Gamma = (\gamma_0^T, \cdots, \gamma_p^T)^T$, $B_{R_j}(R_{t-d}) = B^\gamma(T_{t-d})R_{t-j}$ for $j = 1, \cdots, p$, and $B_{R_0}^\gamma = B^\gamma(R_{t-d})$. So far, the conditional log-likelihood function can be written as

$$
\begin{aligned}
l(\Gamma) &= -\sum_{t=s+1}^{T} [ln(\lambda_t) + \frac{R_t}{\lambda_t}] \\
&= -\sum_{t=s+1}^{T} [ln(\sum_{j=0}^{p} B_{R_j}^\gamma) + \frac{R_t}{\sum_{j=0}^{p} B_{R_j}^\gamma(R_{t-d})}]
\end{aligned}
\tag{7}
$$

Where $s = max\{p, d\}$

Bayesian Analysis of FCARR Model

Over-Fitting

In order to attenuate the over-fitting phenomenon, smoothness tuning parameters $\rho_{\gamma j}$, which determine the penalty degree, for controlling the amount are considered into the log-likelihood function, denote M_γ as penalty matrices derived from the specified difference penalty, the penalized log-likelihood can be written as:

$$l_p = l(\Gamma) - \sum_{j=0}^{p} \rho_{\gamma j} \gamma_j^T M_\gamma \gamma_j \tag{8}$$

The flexibility and smoothness trade-off is tuned by the parameter $\rho_{\gamma j}$, leaving finding the its optimal values to be a non-trivial step in the maximum likelihood method.

Bayesian Analysis of FCARR Model

Problems with ML Framework

In the context of ML estimation, these smoothing parameters are chosen via a cross-validation procedure.

However, the computational burden for determining the optimal values of $\rho_{\gamma j}$ is heavy when the number of smooth functions in the model is large.

Therefore, for the proposed FARCH model, the optimal values of $\rho_{\gamma j}$ is difficult to obtain using the ML-based methods.

Bayesian Analysis of FCARR Model

Bayesian Framework

In the full Bayesian framework, the coefficients γ are regard as random, consequently we can assign Gaussian prior to these coefficients so that the posterior distribution would have the same form of penalized likelihood. We can set the prior as follow:

$$p(\gamma_0 \mid \tau_{\gamma_0}) = (\frac{1}{2\pi\gamma_0})^{(K^*_\gamma)} exp\{-\frac{1}{2\tau_{\gamma_0}}\gamma_0^T M_\gamma \gamma_j\} I(\gamma_0^T B^\gamma > 0) \quad (9)$$

$$p(\gamma_j \mid \tau_{\gamma_j}) = (\frac{1}{2\pi\gamma_j})^{(K^*_\gamma)} exp\{-\frac{1}{2\tau_{\gamma_j}}\gamma_j^T M_\gamma \gamma_j\} I(\gamma_j^T B^\gamma \geq 0),$$
$$j = 1, \cdots, p \quad (10)$$

where $K^*_\gamma = rank(M_\gamma)$, $B^\gamma = (B^\gamma(R_1), \cdots, , B^\gamma(R_{T-d}))$, and the additional variance parameters $\tau_{\gamma 0}$ and $\tau_{\gamma 1}$ can play the role of $\rho_{\gamma j}$, and $I(\cdot)$ is an indicator function.

Posterior Inference

Let $R = \{R_1, \cdots, R_T\}$ be the set of observed range series, $\tau_\gamma = \{\tau_0, \cdots, \tau_p\}$ and $\Theta = \{\Gamma, \tau_\gamma\}$ include all unknown parameters in the model. After assuming all the priors of parameters, the posterior distribution which combine the information of both prior and likelihood function can be deduce easily:

$$p(\theta \mid R) \propto exp\left\{ l(\Gamma) - \sum_{j=0}^{p} [\frac{1}{2\tau_{\lambda_j}^2} \gamma_j^T M_\gamma \gamma_j - \frac{K_\gamma^*}{2} ln(\tau_{\gamma_j}^2) - (\alpha_\gamma + 1) ln(\tau_{\gamma_j}^2) - \right. \tag{11}$$

MCMC Method

Despite the fact that this posterior distribution is intractable, we can apply some modern sampling method to get the numerical estimation of each parameter. Gibbs sampler (Geman and Geman, 1984) algorithm has been applied to draw each component of Θ given others from its full conditional distribution iteratively. Since the model is of highly nonlinearity and complexity, some full conditional distributions are nonstandard, making the sampling process not straightforwardly. To solve this problem, we use the Metropolis-Hasting (MH) algorithm (Metropolis et al., 1953; Hastings, 1970) to simulate observations from the nonstandard distributions

Summary

- The first main message of your talk in one or two lines.
- The second main message of your talk in one or two lines.
- Perhaps a third message, but not more than that.

- Outlook
  - Something you haven't solved.
  - Something else you haven't solved.

For Further Reading I

A. Author.
Handbook of Everything.
Some Press, 1990.

S. Someone.
On this and that.
Journal of This and That, 2(1):50–100, 2000.