DERIVATIONS:

Estimating the Method of Moments Estimators for different Distributions.

1 Binomial Distribution:-

We know that for Binomial (n, p)

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $x = 1, \dots, n$.

Mean $= np$

Variance $= np(1-p)$

Sample mean [S.Mean] $= \frac{1}{N}\sum_{i=1}^{N} x(i)$, Sample Variance [S.variance] $= \frac{1}{N}\sum_{i=1}^{N} (x(i) - \bar{x})^2$

So,

$$np = \frac{1}{N}\sum_{i=1}^{N} x(i) \quad \text{—— } 1$$

$$np(1-p) = \frac{1}{N}\sum_{i=1}^{N} (x(i) - \bar{x})^2 \text{ —— } 2$$

2 divided by 1

$$\frac{2}{1} = \frac{np(1-p)}{np} \Rightarrow 1 - p = \frac{\sum_{i=1}^{N} (x(i) - \bar{x})^2}{\sum_{i=1}^{N} x(i)}$$

$$\hat{p} = 1 - \frac{\sum_{i=1}^{N} (x(i) - \bar{x})^2}{\sum_{i=1}^{N} x(i)} = \left[1 - \frac{\text{S.Variance}}{\text{S.mean}}\right]$$

therefore, Binomial distribution parameters - p value is $1 - \frac{\text{Var\_data}}{\text{mean\_data}}$ [In terms of R code Implementation].

$$1 - \frac{\text{S.Variance}}{\text{S.Mean}}$$

$$n = \frac{\text{S.Mean}}{1 - \frac{\text{S.Variance}}{\text{S.mean}}} = \frac{\text{S.mean}^2}{\text{S.Variance} - \text{S.Mean}}.$$

2) Bernoulli Distribution :-

we know that for Bernoulli (p)

$$\text{Mean} = P$$

$$\text{Variance} = P(1-P)$$

Probability density function is given as

$$P^x (1-P)^{1-x} \quad x \in \{0,1\}.$$

clearly, Method of moments Estimator of P is Mean 'M'.

$$\text{Sample Mean} = \binom{n}{k} P^k (1-P)^{n-k}$$

Therefore, Bernoulli distribution parameters. P value is Mean_data. [i.e Sample Mean]

3) Geometric Distribution (P) :-

$$\text{Here, Mean} = \frac{1}{P}$$

$$\text{Variance} = \frac{(1-P)}{P^2}, \quad P_X(x) = P(1-P)^{x-1}$$

clearly, Method of Moment Estimator P is $\frac{1}{\text{Mean}}$. $\left[\because \text{Mean} = \frac{1}{P} \quad P = \frac{1}{\text{Mean}}\right]$

4) Normal Distribution (μ, σ^2) :-

Here, $\text{Mean} = \mu$

$$\text{Variance} = \sigma^2$$

Probability density function $= f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$

$$\text{Sample Mean} = \frac{1}{N} \sum_{i=1}^{N} x_i .$$

Therefore, It is clear that
Method of Moment Estimator P for
Normal distribution is ~~Mean~~ Sample Mean.

⑤ Poisson distribution

$$P_X(x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \lambda \geq 0$$

Mean = λ

Variance = λ.

Therefore, Method of Moment Estimator
'λ' for Poisson distribution is Mean
(i.e Sample mean).

⑥ Point mass at a :

Here, Mean = a

Variance = 0

Sample mean = $\frac{1}{N}\sum_{i=1}^{N} x_i \Rightarrow a = \frac{1}{N}\sum_{i=1}^{N} x_i$.

Therefore Method of Moment Estimator
a is Sample Mean.

a

⑦ Uniform distribution.

Here, Mean = $\frac{a+b}{2}$

Variance = $\frac{(b-a)^2}{12}$.

$E[X] = \frac{a+b}{2}$

$$E[X^2] = \int_a^b x^2 f(x)\,dx = \int_a^b x^2 \cdot \frac{1}{b-a}\,dx$$

$$= \frac{a^2+b^2+ab}{3}$$

Sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$

Sample Second moment $\bar{x}_{(2)} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$

$\bar{x} = \frac{a+b}{2}$, $\bar{x}_{(2)} = \frac{a^2+b^2+ab}{3}$

$a+b = 2\bar{x}$

$a = 2\bar{x} - b$ $\qquad \bar{x}_{(2)} = \frac{(2\bar{x}-b)^2 + b^2 + (2\bar{x}-b)b}{3}$

$b = \bar{x} \pm \frac{1}{2}\sqrt{4\bar{x}^2 - 4\cdot 4\bar{x}^2 + 4\cdot 3\bar{x}_{(2)}}$

$b = \bar{x} + \sqrt{3(\bar{x}_{(2)} - \bar{x}^2)}$

$b = \bar{x} + \sqrt{3(\text{Variance})}$

$b = \text{Mean} + \sqrt{3(\text{Variance})}$

$= \bar{x} + \sqrt{3s^2}$

$a = 2\bar{x} - b = \bar{x} - \sqrt{3(\text{Variance})}$

$= \text{mean} - \sqrt{3(\text{variance})}$

$= \bar{x} - \sqrt{3s^2}$

Therefore, Method of Momement Estimators

$a = \text{Mean} - \sqrt{3(\text{Variance})}$, $b = \text{Mean} + \sqrt{3(\text{Variance})}$;

$a = \bar{x} - \sqrt{3s^2}$, $b = \bar{x} + \sqrt{3s^2}$.