MSDS 596 Regression & Time Series

Lecture 13 Seasonal Time Series

Department of Statistics
Rutgers University

Dec 3, 2020

- **Quiz 2**: 6:45pm today, 75 minutes, Same rules as Quiz 1.
- **Project report**: due Tuesday Dec 8 at 4:30pm. Refer to syllabus and project guideline for requirements.
- **Final evaluation**: Thursday Dec 10, 3:40pm - 9:40pm
    - One-on-one, 8-10 minutes each.
    - We'll use the usual lecture Zoom link. You need to turn on **both your audio and visual** for this.
    - **Time slot sign-up** on Canvas under People/Groups. Choose your desired 20-minute window (2 people per window).
        - If you have a strong time preference (e.g. non-US time zone, working) please sign up by the end of Fri Dec 4.
        - Otherwise, please wait and sign up beginning Sat Dec 5.
        - All please sign up no later than next Wed Dec 9.
    - You'll be examined both on mastery of course material and your project. Prepare to give a 2-minute overview of your project and answer questions about your report.

Please fill out the Student Instructional Ratings Survey!

The SIRS can be accessed from your Canvas navigation tab. **Your input will be entirely anonymous & very much appreciated.**

A message from the Statistics Department Undergraduate Co-Directors:

> *The Statistics Department and its professors take the Student Instructional Rating Surveys very seriously. All data is reported anonymously without any identifying information. The results help individual professors improve their teaching by indicating their specific strengths and weaknesses. Answers to the open-ended questions help professors improve both their teaching and their courses. The department takes the survey results into account when considering promotion, while students can use the numerical summaries to help with their course and section selection. You will do future students a great service by letting your opinions be known.*

Schedule

Week	Date	Topic
1	9/3	Intro to linear regression (JF1,2)
2	9/10	Estimation (JF2)
3	9/17	Inference I (JF3)
4	9/24	Inference II (JF3)
5	10/1	Inference and prediction (JF4,5)
6	10/8	Explanation; model diagnostics (JF6-8)
7	10/15	Transformation and model selection (JF9-10)
8	10/22	Shrinkage methods (JF11)
9	10/29	Time series exploratory analysis (CC2,3)
10	11/5	Linear time series: ARIMA models (CC4-5)
11	11/12	Model specification and estimation (CC6,7)
12	11/19	Diagnostics and forecasting (CC8,9)
	11/26	(no class)
13	12/3	Seasonal models (CC10)
14	week of 12/7	Project & final evaluation

Seasonal ARIMA models

Example. Quarterly series:

$$Y_1 \quad Y_2 \quad Y_3 \quad Y_4 \quad Y_5 \quad Y_6 \quad Y_7 \quad Y_8 \quad Y_9 \quad Y_{10} \quad Y_{11} \quad Y_{12} \quad Y_{13} \quad Y_{14} \quad Y_{15} \quad Y_{16}$$

- Assume *common* AR(1) model for each sub-series

$$Y_1, Y_5, Y_9, \ldots \quad \text{and}$$

$$Y_2, Y_6, Y_{10}, \ldots \quad \text{etc.}$$

- We have that

$$Y_t = \Phi_1 Y_{t-4} + e_t \quad \text{or} \quad (1 - \Phi_1 B^4) Y_t = e_t$$

  where $\{e_t\} \sim \mathrm{WN}(0, \sigma^2)$.
- Note that $(Y_t, Y_{t-1}, Y_{t-2}, Y_{t-3})$ are independent.

Seasonal AR models

- The seasonal period s: $s = 4$ (quarterly), $s = 12$ (monthly), etc.
- Seasonal AR$(P)_s$ model

$$(1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps}) Y_t = e_t,$$

  with seasonal AR characteristic polynomial

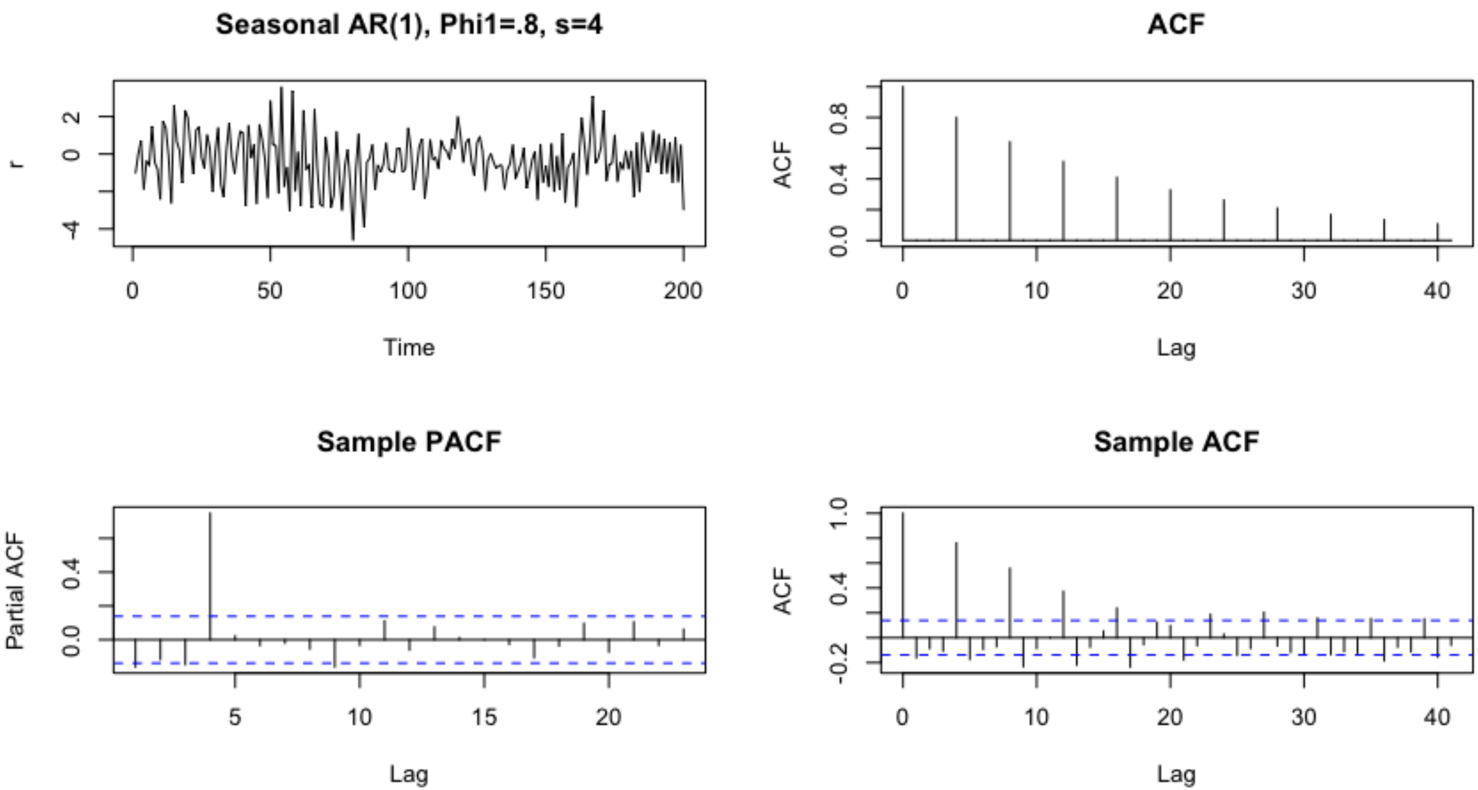
$$\Phi(z) = 1 - \Phi_1 z^s - \Phi_2 z^{2s} - \ldots - \Phi_P z^{Ps};$$

- Stationarity $\Leftrightarrow$ all roots to $\Phi(z) = 0$ are larger than 1 in modulus.
- Example: $Y_t = \Phi Y_{t-12} + e_t$, with $|\Phi| < 1$.
  - This is a regular AR(12) model with $\phi_{12} = \Phi$ and all other ϕ_j's zero.
  - $\rho_{12k} = \Phi^k$ for $k = 1, 2, \ldots$, and $\rho_j = 0$ for j not a multiple of $s = 12$.
- Generally, a seasonal AR$(P)_s$ is a regular AR(—) model?

  Ps

Seasonal AR$(1)_4$ model

AR(4)



Seasonal MA models

- Seasonal MA$(Q)_s$ model

$$Y_t = (1 - \Theta_1 B^s - \Theta_2 B^{2s} - \ldots - \Theta_Q B^{Qs})e_t,$$

with seasonal MA characteristic polynomial

$$\Theta(z) = 1 - \Theta_1 z^s - \Theta_2 z^{2s} - \ldots - \Theta_Q z^{Qs};$$

- Invertibility $\Leftrightarrow$ all roots to $\Theta(z) = 0$ are larger than 1 in modulus.
- Generally, a seasonal MA$(Q)_s$ is a regular MA$(\underline{})$ model?

Qs

Multiplicative seasonal ARMA models

ARMA$(p, q) \times (P, Q)_s$: the multiplicative seasonal ARMA model with regular order (p, q), seasonal order (P, Q) and seasonal period s is written as

$$\phi(B)\Phi(B)Y_t = \theta(B)\Theta(B)e_t,$$

with AR characteristic polynomial $\phi(B)\Phi(B)$,
and MA characteristic polynomial $\theta(B)\Theta(B)$, where

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p, \qquad \theta(z) = 1 - \theta_1 z - \ldots - \theta_q z^q,$$
$$\Phi(z) = 1 - \Phi_1 z^s - \ldots - \Phi_P z^{Ps}, \quad \text{and} \quad \Theta(z) = 1 - \Theta_1 z^s - \ldots - \Theta_Q z^{Qs}.$$

MA(13)

Example. A multiplicative seasonal ARMA$(0, 1) \times (0, 1)_{12}$ model:

$$Y_t = (1 - \theta B)\left(1 - \Theta B^{12}\right) e_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}.$$

Note. Multiplicative seasonal ARMA$(p, q) \times (P, Q)_s$ models are regular ARMA$(p + Ps, q + Qs)$ models. They have only $p + P + q + Q$ parameters, thus are more parsimonious ($p + Ps + q + Qs$ parameters in the regular ARMA model).

Seasonal ARIMA Models

Recall the seasonal difference of period s is defined as

$$\nabla_s Y_t = (1 - B^4) Y_t = Y_t - Y_{t-s},$$

to be distinguished from the d-th regular difference: $\nabla^d Y_t = (1 - B)^d Y_t$.

$\{Y_t\}$ follows a multiplicative seasonal ARIMA model, denoted ARIMA$(p, d, q) \times (P, D, Q)_s$, with regular order (p, d, q), seasonal order (P, D, Q) and seasonal period s, if

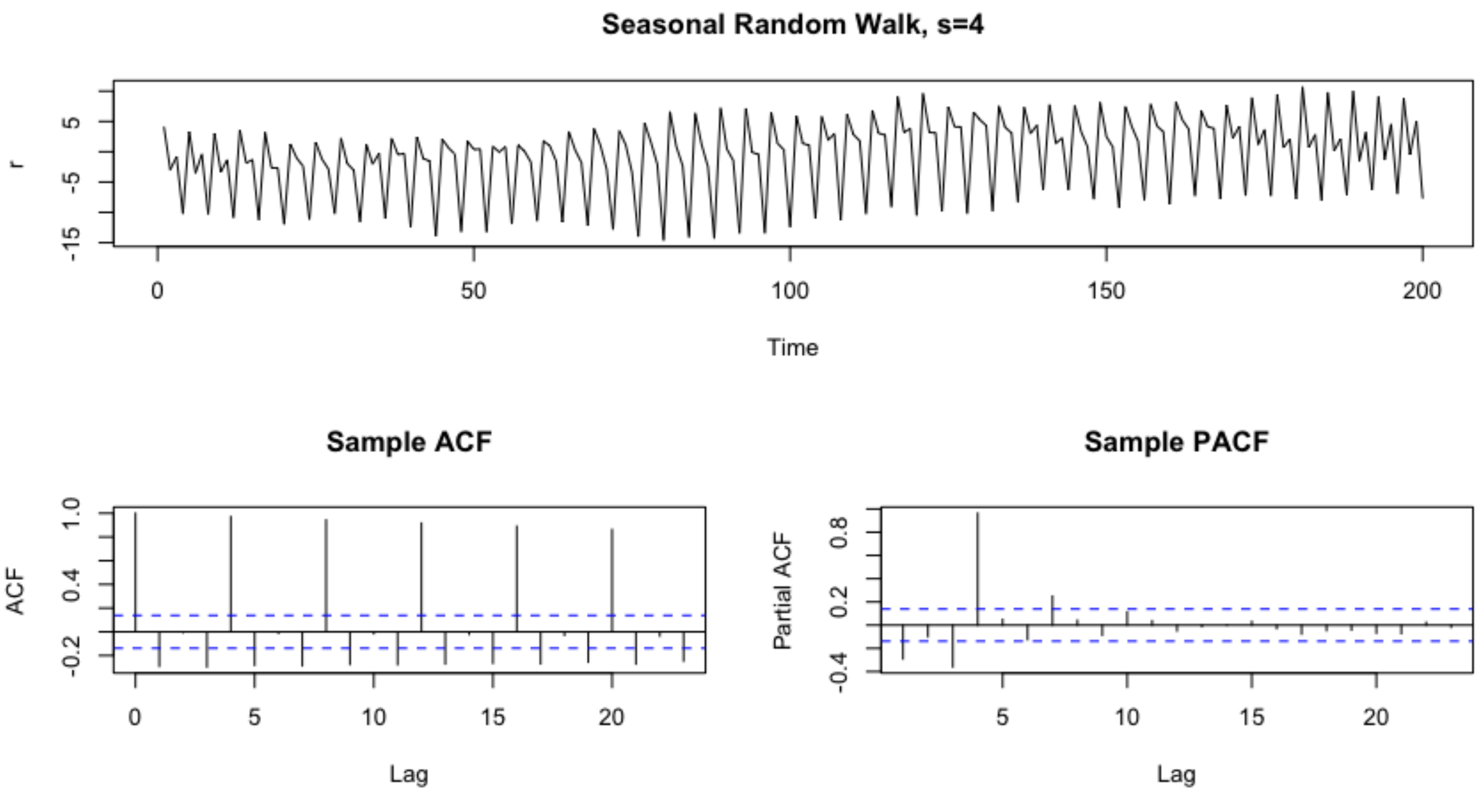
$$W_t = \nabla^d \nabla_s^D Y_t$$

follows a multiplicative seasonal ARMA$(p, q) \times (P, Q)_s$ model.

Examples.

- Seasonal random walk: $Y_t = Y_{t-s} + e_t$ or $\nabla_s Y_t = e_t$.
- The airline model: ARIMA$(0, 1, 1) \times (0, 1, 1)_s$ (Box & Jenkins '76)

Seasonal random walk



Airline model

The ARIMA$(0, 1, 1) \times (0, 1, 1)_s$ model, or airline model, is widely applicable to many seasonal time series.

Example. For monthly series ($s = 12$), the airline model takes the form

$$(1 - B)(1 - B^{12})Y_t = (1 - \theta_1 B)(1 - \Theta_1 B^{12})e_t.$$

In other words,

$$W_t = \nabla\nabla_{12} Y_t = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13} \sim \text{ARMA}(0, 1) \times (0, 1)_{12}.$$

See R for illustrations.