Problem 2: The following data responses y are generated from a regular Poisson model with a single covariate variable x:

x	y
0.4	1
0.6	0
0.8	6
1.0	6
1.2	7
1.4	9

(a). Please write down the Poisson Model for this dataset, stating all requirements.

Mean Model: $\mu_i = E(y_i) = g(\beta_0 + \beta_1 x_i) = e^{\beta_0 + \beta_1 x_i}$

Distribution: $y_i \sim \text{Poisson}(\mu)$ has density $f(y|\mu) = \dfrac{e^{-\mu}\mu^y}{y!}$ for $y = 0, 1, 2, \ldots$

Likelihood function: $L(\beta|y) = \prod_{i=1}^{n} f(y_i)$

Log-likelihood function: $\ell(\beta|y) = \sum_{i=1}^{n} \log(f(y_i)) = \sum_{i=1}^{n} \{ y_i \log \mu_i - \mu_i - \log(y_i) \}$

Take Derivative and set to 0

$$\frac{\partial \ell(\beta|y)}{\partial \beta} = \sum_{i=1}^{n} \{ y_i \frac{\partial \log \mu_i}{\partial \beta} - \frac{\partial \mu_i}{\partial \beta}$$

(b). Calculate the MLE estimates $\hat{\beta}_0$ and $\hat{\beta}_1$ for β_1 and β_2, and then provide the variance estimates for $\hat{\beta}_0$ and $\hat{\beta}_1$.

Solve
$$\begin{cases} \sum_{i=1}^{n} \left(y_i - e^{\beta_0 + \beta_1 x_i}\right) = 0 & ① \\ \sum_{i=1}^{n} \left(y_i - e^{\beta_0 + \beta_1 x_i}\right) x_i = 0 & ② \end{cases}$$

Let us solve ①:

$$\left(1 - e^{\beta_0 + 0.4\beta_1}\right) + \left(0 - e^{\beta_0 + 0.6\beta_1}\right) + \left(6 - e^{\beta_0 + 0.8\beta_1}\right) + \left(6 - e^{\beta_0 + 1.0\beta_1}\right)$$
$$+ \left(7 - e^{\beta_0 + 1.2\beta_1}\right) + \left(9 - e^{\beta_0 + 1.4\beta_1}\right) = 0$$

$$29 = e^{\beta_0 + 0.4\beta_1} + e^{\beta_0 + 0.6\beta_1} + e^{\beta_0 + 0.8\beta_1} + e^{\beta_0 + 1.0\beta_1} + e^{\beta_0 + 1.2\beta_1}$$
$$+ e^{\beta_0 + 1.4\beta_1}$$

$$= e^{\beta_0}\left(e^{0.4\beta_1} + e^{0.6\beta_1} + e^{0.8\beta_1} + e^{1.0\beta_1} + e^{1.2\beta_1} + e^{1.4\beta_1}\right)$$

Now, let us solve ②:

$$(0.4-0.4e^{\beta_0+0.4\beta_1})+(0-0.6e^{\beta_0+0.6\beta_1})+(4.8-0.8e^{\beta_0+0.8\beta_1})$$

$$+(6-e^{\beta_0+\beta_1})+(8.4-1.2e^{\beta_0+1.2\beta_1})+(12.6-1.4e^{\beta_0+1.4\beta_1})=0$$

$$32.2=e^{\beta_0}\left(0.4e^{0.4\beta_1}+0.6e^{0.6\beta_1}+0.8e^{0.8\beta_1}+e^{\beta_1}+1.2e^{1.2\beta_1}+1.4e^{1.4\beta_1}\right)$$

Divide ① by ②

$$\frac{29}{32.2}=\frac{e^{0.4\beta_1}+e^{0.6\beta_1}+e^{0.8\beta_1}+e^{\beta_1}+e^{1.2\beta_1}+e^{1.4\beta_1}}{0.4e^{0.4\beta_1}+0.6e^{0.6\beta_2}+0.8e^{0.8\beta_1}+e^{\beta_1}}$$

$$32.2e^{0.4\beta_1}+32.2e^{0.6\beta_1}+32.2e^{0.8\beta_1}+32.2e^{\beta_1}+32.2e^{1.2\beta_1}+32.2e^{1.4\beta_1}=$$
$$29(0.4e^{0.4\beta_1})+29(0.6e^{0.6\beta_1})+29(0.8e^{0.8\beta_1})+29(e^{\beta_1})$$

$\Downarrow$

$$20.6e^{0.4\beta_1}+14.8e^{0.6\beta_2}+0.9e^{0.8\beta_1}+3.2e^{\beta_1}-2.6e^{1.2\beta_1}-8.4e^{1.4\beta_1}=0$$

$\Downarrow$

$$e^{\beta_1}\simeq 2.16604 \implies \boxed{\hat{\beta}_1=\log(7.16604)=1.969}$$

$29 = e^{\beta_0} * (43.8367)$

$\boxed{\hat{\beta}_0 = -0.413}$

Now write log likelihood

$$L(y|\mu) = \prod_{i=1}^{n} \frac{e^{-\mu} \mu_i^{y_i}}{y_i!}$$

$$\ell(y|\mu) = \sum_{i=1}^{n} \log\left(\frac{e^{-\mu} \mu_i^{y_i}}{y_i!}\right)$$

$$= \sum_{i=1}^{n} \log\left(\frac{e^{-\mu_i} \mu_i^{y_i}}{(y_i)!}\right)$$

$$= \sum_{i=1}^{n} -\mu_i + y_i \log \mu_i - \log(y_i!)$$

In our case, we have params β_0, β_1 and x

$$\ell(y_i | \beta_0, \beta_1, x_i) = \sum_{i=1}^{n} -e^{\beta_0 + \beta_1 x_i} + \sum_{i=1}^{n} y_i \cdot (\beta_0 + \beta_1 x_i) - \sum_{i=1}^{n} \log(y_i!)$$

$$\frac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{n} -e^{\beta_0 + \beta_1 x} + \sum_{i=1}^{n} y_i$$

Second Partial Derivative Now

$$\frac{\partial^2 \ell}{\partial \beta_0^2} = \sum_{i=1}^{n} -e^{\beta_0+\beta_1 x_i}$$

$$I_{\beta_0} = -E\left(\sum_{i=1}^{n} -e^{\beta_0+\beta_1 x_i}\right) = \sum_{i=1}^{n} e^{\beta_0+\beta_1 x_i} = e^{0.77}+e^{0.77}+e^{1.16}+e^{1.56}+e^{1.95}+e^{2.34}$$

$$= 28.967 \qquad Var(\hat{\beta}_0) = I_{\beta_0}^{-1} = 0.035$$

Now take derivative w.r.t β_1

$$\frac{\partial \ell}{\partial \beta_1} = \sum_{i=1}^{n} -x_i e^{\beta_0+\beta_1 x_i} + \sum_{i=1}^{n} x_i y_i$$

$$\frac{\partial^2 \ell}{\partial \beta_1^2} = \sum_{i=1}^{n} -x_i^2 e^{\beta_0+\beta_1 x_i}$$

$$I_{\beta_1} = -E\left(\sum_{i=1}^{n} -x_i^2 e^{\beta_0+\beta_1 x_i}\right) = \sum_{i=1}^{n} x_i^2 e^{\beta_0+\beta_1 x_i} =$$

$$(0.41)^2 e^{0.37} + (0.6)^2 e^{0.77} + (0.8)^2 e^{1.16} + e^{1.56} + (1.2)^2 e^{1.95} + (1.4)^2 e^{2.34}$$

Thus $I_{\beta_1} = 38.278$

$$Var(\hat{\beta}_1) = I_{\beta_1}^{-1} = 0.026$$

(c). The values of the linear predictor are 0.37, 0.77, 1.16, 1.56, 1.95, 2.34 for the 6 observations. Please compute the deviance residuals and draw the index plot of the deviance residuals.

$$r_i = sgn(y_i - \hat{\mu}_i) * \sqrt{2 * y_i * \log\left(\frac{y_i}{\hat{\mu}_i}\right) - (y_i - \hat{\mu}_i)}$$

$$r_1 = sgn(1 - 0.37) * \sqrt{2 * 1 * \log\left(\frac{1}{0.37}\right) - (1 - 0.37)}$$
$$= 1.165549$$

$$r_2 = sgn(0 - 0.77) * \sqrt{2 * 0 * \overset{0}{\cancel{\log}}\left(\frac{0}{0.77}\right) - (0 - 0.77)}$$
$$= -0.8774964$$

$$r_3 = sgn(6 - 1.16) * \sqrt{2 * 6 * \log\left(\frac{6}{1.16}\right) - (6 - 1.16)}$$
$$= 3.85747$$

$$r_4 = sgn(6 - 1.56) * \sqrt{2 * 6 * \log\left(\frac{6}{1.56}\right) - (6 - 1.56)}$$
$$= 3.424162$$

$$r_5 = sgn(7 - 1.95) * \sqrt{2 * 7 * \log\left(\frac{7}{1.95}\right) - (7 - 1.95)}$$
$$= 3.583731$$

$$r_6 = sgn(9 - 2.34) * \sqrt{2 * 9 * \log\left(\frac{9}{2.34}\right) - (9 - 2.34)}$$
$$= 4.193725$$

(d) Draw a partial residual plot to study the linearity of the covariate variable x (show your calculation)

$$r_i^k = r_i^D + B_1 x_i$$

$r_1^k = -0.394 + 1.969 * 0.4 = 0.3936$

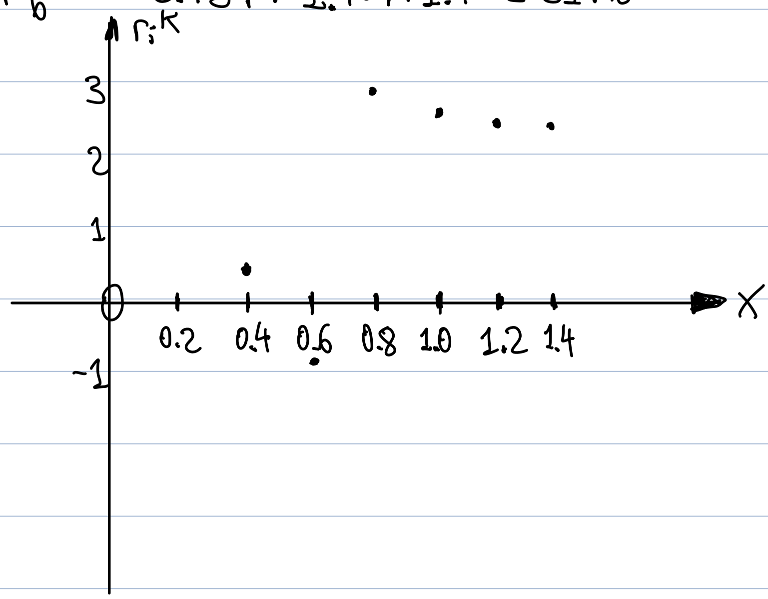
$r_2^k = -2.078 + 1.969 * 0.6 = -0.8966$

$r_3^k = 1.400 + 1.969 * 0.8 = 2.9752$

$r_4^k = 0.547 + 1.969 * 1.0 = 2.516$

$r_5^k = -0.011 + 1.969 * 1.2 = 2.3518$

$r_6^k = -0.439 + 1.969 * 1.4 = 2.3176$

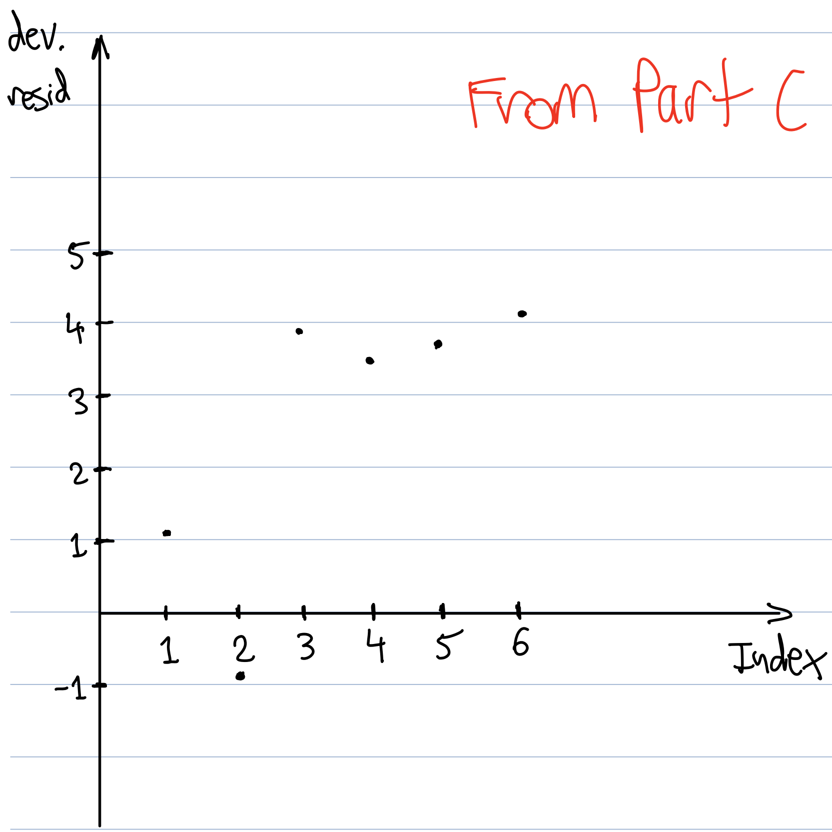


(e). $D_m(F) = (0.394)^2 + (2.078)^2 + (1.400)^2 + (0.547)^2 + (0.011)^2 + (0.439)^2 = 6.925$

$D_M(R)$

y	μ	r_i^D
1	$e^{-0.413}$	$2\left(1 * \log\left(\frac{1}{e^{-0.413}}\right) - 1 + e^{-0.413}\right)$
0	$e^{-0.413}$	$2\left(0 * \log\left(\frac{0}{e^{-0.413}}\right) - 0 + e^{-0.413}\right)$
6	$e^{-0.413}$	$2\left(6 * \log\left(\frac{6}{e^{-0.413}}\right) - 6 + e^{-0.413}\right)$
6	$e^{-0.413}$	$2\left(6 * \log\left(\frac{6}{e^{-0.413}}\right) - 6 + e^{-0.413}\right)$
7	$e^{-0.413}$	$2\left(7 * \log\left(\frac{7}{e^{-0.413}}\right) - 7 + e^{-0.413}\right)$
9	$e^{-0.413}$	$2\left(9 * \log\left(\frac{9}{e^{-0.413}}\right) - 9 + e^{-0.413}\right)$

$\sum r_i^D = 88.257$



Problem 4:

(c). $$\pi = \frac{e^{\beta_0+\beta_1 X_1+\beta_2 X_2}}{1+e^{\beta_0+\beta_1 X_1+\beta_2 X_2}}$$

$$= \frac{e^{-4.739309+0.06773256X_1+0.5986317X_2}}{1+e^{-4.739309+0.06773256X_1+0.5986317X_2}}$$

Problem 5: Given the following data points:

-1.43	-0.95	-0.19	0.02	0.14	0.83	1.35	1.46	2.62
1	2	3	4	5	6	7	8	9

Compute the kernel density estimate $\hat{f}(x)$ at point $x=0.05$. Use the rectangular kernel $K(t)$ with binwidth $h=0.22$. Here $K(t)=\frac{1}{2}$ if $|t|\leq 1$, and it equals 0 if $|t|>1$.

$$\hat{f}_N(x;h)=\frac{1}{nh}\sum_{i=1}^{n}K\left(\frac{x-X_i}{h}\right) \qquad K(z)=\frac{1}{2}\mathbb{1}_{\{-1<z<1\}}$$

$$\hat{f}_N(x=0.05;h=0.22)=\frac{1}{9*0.22}\left[K\left(\frac{0.05-(-1.43)}{0.22}\right)+K\left(\frac{0.05-(-0.95)}{0.22}\right)\right.$$

$$+K\left(\frac{0.05-(-0.19)}{0.22}\right)+K\left(\frac{0.05-0.02}{0.22}\right)+K\left(\frac{0.05-0.14}{0.22}\right)+$$

$$\left.K\left(\frac{0.05-0.83}{0.22}\right)+K\left(\frac{0.05-1.35}{0.22}\right)+K\left(\frac{0.05-1.46}{0.22}\right)+K\left(\frac{0.05-2.62}{0.22}\right)\right]$$

$$=\frac{1}{9*0.22}\left[K(6.73)+K(4.55)+K(1.09)+K(0.136)+K(-0.409)+K(-3.545)\right.$$

$$\left.+K(-5.91)+K(-6.409)+K(-11.682)\right]=0$$

$$=0.5050505\left[0+0+0+1+1+0+0+0+0\right]$$

$$=\boxed{1.01}$$