Problem 3. Service times of a queuing system follow Exponential Distribution with an unknown parameter θ.
A sample of service times $X_1, X_2, \ldots, X_n$ is observed.

(a). Show that the $\text{Gamma}(\alpha, \lambda)$ family of prior distributions is conjugate.

Let $\theta \sim \text{Gamma}(\alpha, \lambda)$ so that we can define our prior as follows:

$$f_\Theta(\theta) = \frac{\lambda^\alpha \theta^{\alpha-1} e^{-\lambda\theta}}{\Gamma(\alpha)}$$

The data given the parameter θ have an exponential distribution, so $X_i|\theta \sim \exp(\theta)$

$$f_{X_i|\Theta}(x_i|\theta) = \theta e^{-\theta x_i}$$

We write the joint distribution for n independent, identically distributed samples by:

$$f(X|\theta) = \theta^n e^{-\theta \Sigma x_i}$$

The posterior is proportional to the product of the likelihood and the prior.

So

$$f_{\Theta|X}(\theta|x) \propto f_{X|\Theta}(x|\theta) * f_\Theta(\theta)$$

$$\propto \theta^n e^{-\theta \Sigma x_i} * \frac{\lambda^\alpha \theta^{\alpha-1} e^{-\lambda\theta}}{\Gamma(\alpha)}$$

$$\propto \theta^{n+\alpha-1} e^{-\theta(\Sigma x_i+\lambda)} \frac{\lambda^\alpha}{\Gamma(\alpha)} \leftarrow \text{Omit}$$

$$\propto \theta^{n+\alpha-1} e^{-\theta(\Sigma x_i+\lambda)}$$

The pdf of a Gamma Distribution is:

$$f(x; \alpha; \lambda) = \frac{x^{\alpha-1} e^{-\lambda x} \lambda^\alpha}{\Gamma(\alpha)} \quad \text{for } x > 0 \quad \alpha, \lambda > 0.$$

$$f(\theta; \alpha; \lambda) = \frac{\theta^{\alpha-1} e^{-\lambda\theta} \lambda^\alpha}{\Gamma(\alpha)} \approx \theta^{\alpha-1} e^{-\theta(\lambda)}$$

So the λ parameter is $\Sigma x_i + \lambda$
and α parameter is $n+\alpha$.

Thus,
$$\theta | X \sim \text{Gamma}(n+\alpha, \Sigma x_i + \lambda)$$

Since the product our prior and likelihood adhered to a Gamma distribution, we conclude that the Gamma family of priors is conjugate ∎

(b). Find the posterior parameters, posterior mean and variance (As functions of θ, α, λ, and X_i's)

From part (a), we have that

$\theta | X \sim Gamma(n+\lambda, \Sigma x_i + \lambda)$

$\mu = E[\theta | X] = (n+\lambda)(\Sigma x_i + \lambda)$

$\sigma^{2} = V[\theta | X] = (n+\lambda)(\Sigma x_i + \lambda)^2$

(c). Suppose that we can allow $\alpha = 0$ and consider a prior density $\pi(\theta) = \frac{1}{\theta}$ for $\theta > 0$. Find the Posterior distribution, its mean, and its variance.

Posterior $\propto$ Likelihood $*$ Prior

$\propto \theta^n e^{-\theta \Sigma x_i} * \frac{1}{\theta}$

$\propto \theta^{n-1} e^{-\theta \Sigma x_i}$

Now $\theta | X \sim Gamma(n, \Sigma x_i)$

$\mu = E[\theta | X] = n * \Sigma x_i$

$\sigma^{2} = V[\theta | X] = n * (\Sigma x_i)^2$